

状态空间大小为 3 的马氏链的环流的大偏差

在该问题中有下面 8 种环流: $c_1 : 1 \rightarrow 1, c_2 : 2 \rightarrow 2, c_3 : 3 \rightarrow 3$
 $c_4 : 1 \rightarrow 2 \rightarrow 1, c_5 : 1 \rightarrow 3 \rightarrow 1, c_6 : 2 \rightarrow 3 \rightarrow 2$
 $c_7 : 1 \rightarrow 2 \rightarrow 3 \rightarrow 1, c_8 : 1 \rightarrow 3 \rightarrow 2 \rightarrow 1$

在有 3 个状态的图中, 考虑所有的 n 步欧拉回路。为了方便, 令起始位置的状态为 1。 n 步的欧拉回路可以分解为上述的 8 个环路。而且给定环路的完成顺序可以唯一确定一个欧拉回路。

x_i 表示一条 n 步欧拉回路, 且在这条回路中, 环流 c_i 出现 k_i 次。由于这条回路可以唯一分解为上述环流, 则每种边出现的次数可以得到, 则

$$k_1 + k_2 + k_3 + 2(k_{12} + k_{13} + k_{23}) + 3(k_{123} + k_{132}) = n$$

$$\mathbb{P}(\xi) = p_{11}^{k_1} p_{22}^{k_2} p_{33}^{k_3} p_{12}^{k_{12}+k_{123}} p_{21}^{k_{12}+k_{132}} p_{13}^{k_{13}+k_{132}} p_{31}^{k_{13}+k_{123}} p_{23}^{k_{23}+k_{123}} p_{32}^{k_{23}+k_{132}}$$

只需计算 A 这种欧拉回路有多少种。由初始状态是 1, 考虑 c_1, c_4, c_5, c_7, c_8 (状态 1 开始, 状态 1 结束)。对于这几种环流, 只有当一个结束, 才会有另一个开始。也就是说在 c_7 开始, 但没完成之前, 不会先完成 c_5 。由这几种环流的不同排列, 可以有

$$\binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1, k_{12}, k_{13}, k_{123}, k_{132}}$$

种不同的可能。先把环流 c_6 嵌入进去, 其中 c_6 可以嵌入在状态 2, 也可以嵌入在状态 3 上, 所以会有这些种可能:

$$\binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1}{k_{23}}$$

再把环流 c_2 和 c_3 嵌入在上述可能的欧拉路径中, 依次有这些中嵌入方法

$$c_2 : \binom{k_{12}+k_{123}+k_{132}+k_{23}+k_2-1}{k_2},$$

$$c_3 : \binom{k_{13}+k_{123}+k_{132}+k_{23}+k_3-1}{k_3}。$$

不能先插入 $2 \rightarrow 2$ 和 $3 \rightarrow 3$, 再插入 $2 \rightarrow 3 \rightarrow 2$ 比如在环 $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ 中先插入 $2 \rightarrow 2$, 即使规定在 $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ 中, 不能在 2 位置插入 $2 \rightarrow 3 \rightarrow 2$, 但是得到的 $1 \rightarrow 3 \rightarrow 2 \rightarrow 2 \rightarrow 1$

的第二个 2 中，插入 $2- > 3- > 2$ ，也会产生类似的效果。并且我们无法准确清楚有多少 $2- > 2$ 已经插入在环 $1- > 3- > 2- > 1$ 上。

以状态 1 开始，有下面多种情况：

$$A_1 = \begin{pmatrix} k_1 + k_{12} + k_{13} + k_{123} + k_{132} \\ k_1, k_{12}, k_{13}, k_{123}, k_{132} \end{pmatrix} \begin{pmatrix} k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1 \\ k_{23} \end{pmatrix} \\ \begin{pmatrix} k_{12} + k_{123} + k_{132} + k_{23} + k_2 - 1 \\ k_2 \end{pmatrix} \begin{pmatrix} k_{13} + k_{123} + k_{132} + k_{23} + k_3 - 1 \\ k_3 \end{pmatrix}$$

以状态 2 开始，有下面多种情况：

$$A_2 = \begin{pmatrix} k_2 + k_{12} + k_{23} + k_{123} + k_{132} \\ k_2, k_{12}, k_{23}, k_{123}, k_{132} \end{pmatrix} \begin{pmatrix} k_{12} + k_{23} + k_{123} + k_{132} + k_{13} - 1 \\ k_{13} \end{pmatrix} \\ \begin{pmatrix} k_{12} + k_{123} + k_{132} + k_{13} + k_1 - 1 \\ k_1 \end{pmatrix} \begin{pmatrix} k_{23} + k_{123} + k_{132} + k_{13} + k_3 - 1 \\ k_3 \end{pmatrix}$$

以状态 3 开始，有下面多种情况：

$$A_3 = \begin{pmatrix} k_3 + k_{13} + k_{23} + k_{123} + k_{132} \\ k_3, k_{13}, k_{23}, k_{123}, k_{132} \end{pmatrix} \begin{pmatrix} k_{13} + k_{23} + k_{123} + k_{132} + k_{12} - 1 \\ k_{12} \end{pmatrix} \\ \begin{pmatrix} k_{13} + k_{123} + k_{132} + k_{12} + k_1 - 1 \\ k_1 \end{pmatrix} \begin{pmatrix} k_{23} + k_{123} + k_{132} + k_{12} + k_2 - 1 \\ k_2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} k_{12} + k_{13} + k_{123} + k_{132} \\ k_{12}, k_{13}, k_{123}, k_{132} \end{pmatrix} \\ \begin{pmatrix} k_1 + k_{12} + k_{13} + k_{123} + k_{132} \\ k_1 \end{pmatrix} \begin{pmatrix} k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1 \\ k_{23} \end{pmatrix} \\ \begin{pmatrix} k_{12} + k_{123} + k_{132} + k_{23} + k_2 - 1 \\ k_2 \end{pmatrix} \begin{pmatrix} k_{13} + k_{123} + k_{132} + k_{23} + k_3 - 1 \\ k_3 \end{pmatrix}$$

由

$$\begin{aligned}
\frac{1}{n} \log C_{n-1}^m &= \frac{1}{n} \log \frac{n(n-1)(n-2) \cdots (m+1)}{m(m-1) \cdots 1}, \quad n \rightarrow \infty \\
&= \frac{1}{n} \log \frac{(n-1)(n-2) \cdots (n-m)}{m(m-1) \cdots 1}, \quad n \rightarrow \infty \\
&= \frac{1}{n} \log \frac{n(n-1) \cdots (n-m+1)}{m(m-1) \cdots 1} + O\left(\frac{\log n}{n}\right), \quad n \rightarrow \infty \\
&= \frac{1}{n} \log C_n^m + O\left(\frac{\log n}{n}\right), \quad n \rightarrow \infty
\end{aligned}$$

和 stirling 公式:

$$\begin{aligned}
\frac{1}{n} A_1 &= \frac{1}{n} \binom{k_{12} + k_{13} + k_{123} + k_{132}}{k_{12}, k_{13}, k_{123}, k_{132}} \\
&\quad \binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1} \binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23}}{k_{23}} \\
&\quad \binom{k_2 + k_{12} + k_{123} + k_{132} + k_{23}}{k_2} \binom{k_3 + k_{13} + k_{123} + k_{132} + k_{23}}{k_3} + O\left(\frac{\log n}{n}\right) \\
&= h(\nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) - [h(\nu_{12}) + h(\nu_{13}) + h(\nu_{123}) + h(\nu_{132})] \\
&\quad + h(\nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) - [h(\nu_1) + h(\nu_{12} + \nu_{13} + \nu_{123} + \nu_{132})] \\
&\quad + h(\nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}) - [h(\nu_{23}) + h(\nu_{12} + \nu_{13} + \nu_{123} + \nu_{132})] \\
&\quad + h(\nu_2 + \nu_{12} + \nu_{123} + \nu_{132} + \nu_{23}) - [h(\nu_2) + h(\nu_{12} + \nu_{123} + \nu_{132} + \nu_{23})] \\
&\quad + h(\nu_3 + \nu_{13} + \nu_{123} + \nu_{132} + \nu_{23}) - [h(\nu_3) + h(\nu_{13} + \nu_{123} + \nu_{132} + \nu_{23})] \\
&= h(\nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}) \\
&\quad + h(\nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) \\
&\quad + h(\nu_2 + \nu_{12} + \nu_{123} + \nu_{132} + \nu_{23}) \\
&\quad + h(\nu_3 + \nu_{13} + \nu_{123} + \nu_{132} + \nu_{23}) \\
&\quad - [h(\nu_1) + h(\nu_2) + h(\nu_3) + h(\nu_{12}) + h(\nu_{13}) + h(\nu_{23}) + h(\nu_{123}) + h(\nu_{132})] \\
&\quad - \left(h(\nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) + h(\nu_{12} + \nu_{123} + \nu_{132} + \nu_{23}) \right. \\
&\quad \left. + h(\nu_{13} + \nu_{123} + \nu_{132} + \nu_{23}) \right) \\
\mathbb{P}(\xi) &= p_{11}^{k_1} p_{22}^{k_2} p_{33}^{k_3} p_{12}^{k_{12}+k_{123}} p_{21}^{k_{12}+k_{132}} p_{13}^{k_{13}+k_{132}} p_{31}^{k_{13}+k_{123}} p_{23}^{k_{23}+k_{123}} p_{23}^{k_{23}+k_{132}}
\end{aligned}$$

环流分解

$$\pi_1 p_{11} = w_1$$

$$\pi_2 p_{22} = w_2$$

$$\pi_3 p_{33} = w_3$$

$$\pi_1 p_{12} = w_{12} + w_{123}$$

$$\pi_1 p_{13} = w_{13} + w_{132}$$

$$\pi_2 p_{21} = w_{12} + w_{123}$$

$$\pi_2 p_{23} = w_{23} + w_{132}$$

$$\pi_3 p_{31} = w_{13} + w_{123}$$

$$\pi_3 p_{32} = w_{23} + w_{132}$$

可得

$$\begin{cases} (w_{12} + w_{123})p_{11} - w_1 p_{12} = 0 \\ (w_{13} + w_{132})p_{11} - w_1 p_{13} = 0 \\ p_{11} + p_{12} + p_{13} = 1 \end{cases}$$

$$\begin{aligned} p_{11} &= \frac{w_1}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}} \\ p_{12} &= \frac{w_{12} + w_{123}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}} \\ p_{13} &= \frac{w_{13} + w_{132}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}} \\ p_{22} &= \frac{w_2}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}} \\ p_{21} &= \frac{w_{12} + w_{132}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}} \\ p_{23} &= \frac{w_{23} + w_{123}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}} \\ p_{33} &= \frac{w_3}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}} \\ p_{31} &= \frac{w_{13} + w_{123}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}} \\ p_{32} &= \frac{w_{23} + w_{132}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}} \end{aligned}$$

$$\begin{aligned}
\frac{1}{n} \log \mathbb{P}(\xi) &= \frac{1}{n} \log p_{11}^{k_1} p_{22}^{k_2} p_{33}^{k_3} p_{12}^{k_{12}+k_{123}} p_{21}^{k_{12}+k_{132}} p_{13}^{k_{13}+k_{132}} p_{31}^{k_{13}+k_{123}} p_{23}^{k_{23}+k_{123}} p_{23}^{k_{23}+k_{132}} \\
&= \nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3 \\
&\quad + (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\
&\quad + (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\
&\quad + (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\
&\quad + (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\
&\quad + (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\
&\quad + (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132}) \\
&\quad - (\nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) \log(w_1 + w_{12} + w_{13} + w_{123} + w_{132}) \\
&\quad - (\nu_2 + \nu_{12} + \nu_{23} + \nu_{123} + \nu_{132}) \log(w_2 + w_{12} + w_{23} + w_{123} + w_{132}) \\
&\quad - (\nu_3 + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}) \log(w_3 + w_{13} + w_{23} + w_{123} + w_{132})
\end{aligned}$$

令 $\nu^i = \frac{k_i}{n}$, $i \in I$, 即 $\nu^i = \sum_{J_{c_s}(i)=1} \nu_{c_s}$. $\tilde{\nu} = \sum_{c \in \mathcal{C}_\infty} \nu_c$. 其中 $h(x) = x \log x$ 则:

$$\begin{aligned}
\frac{1}{n} \log \mathbb{P}(J^{c_s} = \frac{k_s}{n}, s = 1, 2, \dots, 8) &= \\
&\sum_{i \in I} (\nu^i \log \nu^i - (\nu^i - \nu_{(i)}) \log(\nu^i - \nu_{(i)})) \\
&+ (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) - \sum_{t \in \mathcal{C}_\infty} \nu_t \log \nu_t \\
&+ \mathbb{P}(\xi) + O\left(\frac{\log n}{n}\right)
\end{aligned}$$

rate function:

$$\begin{aligned}
I(\nu) = & \sum_{i \in I} \left(-\nu^i \log \frac{\nu^i}{w^i} + (\nu^i - \nu_{(i)}) \log(\nu^i - \nu_{(i)}) \right) \\
& - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) + \sum_{t \in C_\infty} \nu_t \log \nu_t \\
& - (\nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3) \\
& - (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\
& - (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\
& - (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\
& - (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\
& - (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\
& - (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132}) + O\left(\frac{\log n}{n}\right)
\end{aligned}$$

状态空间大小为 4 的马氏链的环流的大偏差

在所有的回路中使得, 初始状态为 1, 环流 c_t 有 k_t 的路径数量:

$$\begin{aligned}
A_1 = & \binom{k_{12} + k_{14} + k_{1234} + k_{1432}}{k_{12}, k_{14}, k_{1234}, k_{1432}} \\
& \binom{k_1 + k_{12} + k_{14} + k_{1234} + k_{1432}}{k_{12} + k_{14} + k_{1234} + k_{1432}} \binom{k_2 + k_{12} + k_{23} + k_{1234} + k_{1432} - 1}{k_{12} + k_{23} + k_{1234} + k_{1432} - 1} \\
& \binom{k_3 + k_{23} + k_{34} + k_{1234} + k_{1432} - 1}{k_{23} + k_{34} + k_{1234} + k_{1432} - 1} \binom{k_4 + k_{14} + k_{34} + k_{1234} + k_{1432} - 1}{k_{14} + k_{34} + k_{1234} + k_{1432} - 1} \\
& \left(\sum_{k_{23}^{12} + k_{23}^{14} = k_{23}} \sum_{k_{34}^{12} + k_{34}^{14} = k_{34}} \binom{k_{23}^{12} + k_{12} + k_{1234} - 1}{k_{23}^{12}} \binom{k_{34}^{12} + k_{1234} + k_{23}^{12} - 1}{k_{34}^{12}} \right) \\
& \binom{k_{34}^{14} + k_{14} + k_{1432} - 1}{k_{34}^{14}} \binom{k_{23}^{14} + k_{34}^{14} + k_{1432} - 1}{k_{23}^{14}}
\end{aligned}$$

其中 k_{23}^{12} 和 k_{34}^{12} 分别表示 k_{23}, k_{34} 嵌入到 k_{1234} 和 k_{12} (从 1 出发, 第二步为 2 的环) 的数量。 k_{23}^{14} 和 k_{34}^{14} 分别表示 k_{23}, k_{34} 嵌入到 k_{1432} 和 k_{14} (从 1 出发, 第二步为 4 的环) 的数量。 ($k_{23}^{12}, k_{23}^{14}, k_{34}^{12}, k_{34}^{14} \geq 0$)

由马氏链的常返性，对于足够大的 n ，路径中必然包含所有状态，则：

$$\frac{1}{n}A_j \leq A_i \leq nA_j$$

故：

$$\frac{1}{n}\sqrt[4]{A_1, A_2, A_3, A_4} \leq A_i \leq n\sqrt[4]{A_1, A_2, A_3, A_4}$$

从而

$$\frac{1}{n} \log A_1 = \frac{1}{4n} \log(A_1, A_2, A_3, A_4) + O\left(\frac{\log n}{n}\right) \quad (1)$$

$$\begin{aligned} \frac{1}{n} \log \mathbb{P}(\xi) = & \sum_{i \in I} (\nu^i \log p_{ii} - (\nu^i - \nu_{(i)}) \log w_i) \\ & + (\nu_{12} + \nu_{1234}) \log(w_{12} + w_{1234}) \\ & + (\nu_{23} + \nu_{1234}) \log(w_{23} + w_{1234}) \\ & + (\nu_{34} + \nu_{1234}) \log(w_{34} + w_{1234}) \\ & + (\nu_{41} + \nu_{1234}) \log(w_{41} + w_{1234}) \\ & + (\nu_{14} + \nu_{1432}) \log(w_{14} + w_{1432}) \\ & + (\nu_{43} + \nu_{1432}) \log(w_{43} + w_{1432}) \\ & + (\nu_{32} + \nu_{1432}) \log(w_{32} + w_{1432}) \\ & + (\nu_{21} + \nu_{1432}) \log(w_{21} + w_{1432}) + O\left(\frac{\log n}{n}\right) \end{aligned}$$