

# The rate function for circulation of three state Markov chains

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## 1 The rate function for circulation of two state Markov chains

$$I(\nu) = \nu_1 \log\left(\frac{\nu_1}{\nu_1 + \nu_{12}} / \frac{w_1}{w_1 + w_{12}}\right) + \nu_2 \log\left(\frac{w_2}{w_1 + w_{12}}\right) + 2\nu_{12} \log\left(\frac{\nu_{12}}{\nu_1 + \nu_{12}} / \frac{w_{12}}{w_1 + w_{12}}\right)$$

## 2 The rate function for circulation of three state Markov chains

$$\begin{aligned} I(\nu) = & \sum_{i \in I} \left( -\nu^i \log \frac{\nu^i}{w^i} + (\nu^i - \nu_i) \log(\nu^i - \nu_i) \right) \\ & - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) + \sum_{t \in C_\infty} \nu_t \log \nu_t \\ & - (\nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3) \\ & - (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\ & - (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\ & - (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\ & - (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\ & - (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\ & - (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132}) \end{aligned}$$

where  $I = \{1, 2, 3\}$  is the state space for Markov chains

$$C_\infty = \{(1), (2), (3), (1, 2), (2, 3), (1, 3), (1, 2, 3), (1, 3, 2)\}$$

is the class of all cycles occurring.  $\nu_c$  is the frequency of  $c$  occurring,  $w_c$  is the circulation of  $c$ .

And  $\nu^i = \sum_{J_{c_s}(i)=1} \nu_{c_s}$ , such as  $\nu^1 = \nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}$   $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3 + \nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}$

$w^i, w_i$  has the similar definition.

Refer to the result of  $p_{ii} = 0$ , we can get

$$\begin{aligned}
I(\nu) = & \nu_1 \log\left(\frac{\nu_1}{w_1} / \frac{\nu^1}{w^1}\right) + \nu_2 \log\left(\frac{\nu_2}{w_2} / \frac{\nu^2}{w^2}\right) + \nu_3 \log\left(\frac{\nu_3}{w_3} / \frac{\nu^3}{w^3}\right) \\
& + \nu_{12} \log\left(\frac{1}{p_{12}p_{21}} \frac{\nu_{12}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^2 - \nu_2}{\nu^2}\right) \\
& + \nu_{13} \log\left(\frac{1}{p_{13}p_{31}} \frac{\nu_{13}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^3 - \nu_3}{\nu^3}\right) \\
& + \nu_{23} \log\left(\frac{1}{p_{23}p_{32}} \frac{\nu_{23}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^3 - \nu_3}{\nu^3} \frac{\nu^2 - \nu_2}{\nu^2}\right) \\
& + \nu_{123} \log\left(\frac{1}{p_{12}p_{23}p_{31}} \frac{\nu_{123}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^2 - \nu_2}{\nu^2} \frac{\nu^3 - \nu_3}{\nu^3}\right) \\
& + \nu_{132} \log\left(\frac{1}{p_{13}p_{32}p_{21}} \frac{\nu_{132}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^2 - \nu_2}{\nu^2} \frac{\nu^3 - \nu_3}{\nu^3}\right)
\end{aligned}$$

So we need to valid following proposition. if  $\nu = w$ , then

$$\begin{aligned}
& \frac{1}{p_{12}p_{21}} \frac{\nu_{12}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^2 - \nu_2}{\nu^2} = 1 \\
& \frac{1}{p_{12}p_{23}p_{31}} \frac{\nu_{123}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \frac{\nu^1 - \nu_1}{\nu^1} \frac{\nu^2 - \nu_2}{\nu^2} \frac{\nu^3 - \nu_3}{\nu^3} = 1
\end{aligned}$$

We use

$$\begin{aligned}
\frac{w^1 - w_1}{w^1} &= \frac{w_{12} + w_{13} + w_{123} + w_{132}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}} \\
&= p_{12} + p_{13} \\
&= p_{21} + p_{31} \\
&= 1 - p_{11} \\
&= D(\{2, 3\}^c)
\end{aligned}$$

and

$$\begin{aligned}
w_{12} &= p_{12}p_{21} \frac{D(\{1, 2\}^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{13} &= p_{13}p_{31} \frac{D(\{1, 3\}^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{23} &= p_{23}p_{32} \frac{D(\{2, 3\}^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{123} &= p_{12}p_{23}p_{31} \frac{D(\{1, 2, 3\}^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{132} &= p_{13}p_{32}p_{21} \frac{D(\{1, 2, 3\}^c)}{\sum_{i \in I} D(\{i\}^c)}
\end{aligned}$$

substitute  $w$ , the above proposition can be valid.

By the above validation, we also know:

$$\begin{aligned} w_{12} + w_{13} + w_{23} + w_{123} + w_{132} &= \frac{(1 - p_{11})(1 - p_{22})(1 - p_{33})}{\sum_{i \in I} D(\{i\}^c)} \\ &= \frac{\Pi_{i,j} D(\{i, j\}^c)}{\sum_{i \in I} D(\{i\}^c)} \end{aligned}$$

$$\begin{aligned} I(\nu) &= \sum_{c \in \mathcal{C}_\infty} \nu_c \log \frac{\nu_c}{w_c} + \sum_{i \in I} (\nu^i - \nu_i) \log \frac{\nu^i - \nu_i}{w^i - w_i} \\ &\quad - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log \left( \frac{\tilde{\nu} - \sum_{i \in I} \nu_i}{\tilde{w} - \sum_{i \in I} w_i} \right) \\ &\quad - \sum_{i \in I} \nu^i \log \left( \frac{\nu^i}{w^i} \right) \\ I(\nu) &\geq \sum_{i \in I} \nu_i \log \frac{\nu_i}{w_i} + \sum_{i \in I} (\nu^i - \nu_i) \log \frac{\nu^i - \nu_i}{w^i - w_i} - \sum_{i \in I} \nu^i \log \left( \frac{\nu^i}{w^i} \right) \\ &\quad + \sum_{c \in \mathcal{C}_\infty, c \neq \nu_i} \nu_c \log \frac{\nu_c}{w_c} - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log \left( \frac{\tilde{\nu} - \sum_{i \in I} \nu_i}{\tilde{w} - \sum_{i \in I} w_i} \right) \\ &\geq \sum_{i \in I} \nu^i \log \left( \frac{\nu^i}{w^i} \right) - \sum_{i \in I} \nu^i \log \left( \frac{\nu^i}{w^i} \right) \\ &\quad + \sum_{c \in \mathcal{C}_\infty, c \neq \nu_i} \frac{\nu_c}{\tilde{\nu} - \sum_{i \in I} \nu_i} \log \left( \frac{\nu_c}{\tilde{\nu} - \sum_{i \in I} \nu_i} / \frac{w_c}{\tilde{w} - \sum_{i \in I} w_i} \right) \\ &\geq 0 \end{aligned}$$

### 3 $p_{13} = 0$

rate function:

$$\begin{aligned}
I(\nu) &= -(\nu^1 \log(\frac{\nu^1}{w^1}) + \nu^2 \log(\frac{\nu^2}{w^2}) + \nu^3 \log(\frac{\nu^3}{w^3})) \\
&\quad + \nu_1 \log(\frac{\nu_1}{w_1}) + \nu_2 \log(\frac{\nu_2}{w_2}) + \nu_3 \log(\frac{\nu_3}{w_3}) \\
&\quad + (\nu_{12} + \nu_{123}) \log(\frac{\nu_{12} + \nu_{123}}{w_{12} + w_{123}}) + (\nu_{23} + \nu_{123}) \log(\frac{\nu_{23} + \nu_{123}}{w_{23} + w_{123}}) \\
&\quad + \nu_{12} \log(\frac{\nu_{12}}{w_{12}}) + \nu_{23} \log(\frac{\nu_{23}}{w_{23}}) + \nu_{123} \log(\frac{\nu_{123}}{w_{123}}) \\
&= -(\nu^1 \log(\frac{\nu^1}{w^1}) + \nu^2 \log(\frac{\nu^2}{w^2}) + \nu^3 \log(\frac{\nu^3}{w^3})) \\
&\quad + \nu_1 \log(\frac{\nu_1}{w_1}) + \nu_2 \log(\frac{\nu_2}{w_2}) + \nu_3 \log(\frac{\nu_3}{w_3}) \\
&\quad + (\nu_{12} + \nu_{123}) \log(\frac{\nu_{12} + \nu_{123}}{w_{12} + w_{123}}) + (\nu_{23} + \nu_{123}) \log(\frac{\nu_{23} + \nu_{123}}{w_{23} + w_{123}}) \\
&\quad + \nu_{12} \log(\frac{\nu_{12}}{w_{12}}) + \nu_{23} \log(\frac{\nu_{23}}{w_{23}}) + \nu_{123} \log(\frac{\nu_{123}}{w_{123}})
\end{aligned}$$

That is

$$\begin{aligned}
I(\nu) &= \sum_{i,j \in I} \left( \sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log \left( \frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c} \right) \\
&= \sum_{i,j \in I} \left( \sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log \left( \frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c} / \frac{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c} \right)
\end{aligned}$$

#### 4 $p_{ii} = 0$

rate function:

$$\begin{aligned}
I(\nu) = & \sum_{i \in I} (\nu^i - \nu_i) \log(w^i - w_i) - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) \\
& + \nu_{12} \log \nu_{12} + \nu_{23} \log \nu_{23} + \nu_{13} \log \nu_{13} + \nu_{123} \log \nu_{123} + \nu_{132} \log \nu_{132} \\
& - (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\
& - (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\
& - (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\
& - (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\
& - (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\
& - (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132})
\end{aligned}$$

We can simply it to following formula:

$$\begin{aligned}
I(\nu) &= \nu_{12} \log\left(\frac{w_{12} + w_{13} + w_{123} + w_{132}}{w_{12} + w_{123}} \frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{12} + w_{132}} \right. \\
&\quad \left. \frac{\nu_{12}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \right) \\
&\quad + \nu_{13} \log\left(\frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{13} + w_{123}} \frac{w_{13} + w_{23} + w_{123} + w_{132}}{w_{13} + w_{132}} \right. \\
&\quad \left. \frac{\nu_{13}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \right) \\
&\quad + \nu_{23} \log\left(\frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{23} + w_{123}} \frac{w_{13} + w_{23} + w_{123} + w_{132}}{w_{23} + w_{132}} \right. \\
&\quad \left. \frac{\nu_{23}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \right) \\
&\quad + \nu_{123} \log\left(\frac{w_{12} + w_{13} + w_{123} + w_{132}}{w_{12} + w_{123}} \frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{23} + w_{123}} \right. \\
&\quad \left. \frac{w_{13} + w_{23} + w_{123} + w_{132}}{w_{13} + w_{132}} \frac{\nu_{123}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \right) \\
&\quad + \nu_{132} \log\left(\frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{13} + w_{123}} \frac{w_{13} + w_{23} + w_{123} + w_{132}}{w_{23} + w_{132}} \right. \\
&\quad \left. \frac{w_{12} + w_{23} + w_{123} + w_{132}}{w_{12} + w_{132}} \frac{\nu_{132}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}} \right) \\
&= \nu_{12} \log\left(\frac{1}{p_{12}p_{21}} \frac{\nu_{12}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}}\right) \\
&\quad + \nu_{13} \log\left(\frac{1}{p_{13}p_{31}} \frac{\nu_{13}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}}\right) \\
&\quad + \nu_{23} \log\left(\frac{1}{p_{23}p_{32}} \frac{\nu_{23}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}}\right) \\
&\quad + \nu_{123} \log\left(\frac{1}{p_{12}p_{23}p_{31}} \frac{\nu_{123}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}}\right) \\
&\quad + \nu_{132} \log\left(\frac{1}{p_{13}p_{32}p_{21}} \frac{\nu_{132}}{\nu_{12} + \nu_{23} + \nu_{13} + \nu_{123} + \nu_{132}}\right)
\end{aligned}$$

Associating the expression of circulation, and we can further considered

$$\begin{aligned}
w_{12} &= p_{12}p_{21} \frac{D(1, 2^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{13} &= p_{13}p_{31} \frac{D(1, 3^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{23} &= p_{23}p_{32} \frac{D(2, 3^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{123} &= p_{12}p_{23}p_{31} \frac{D(1, 2, 3^c)}{\sum_{i \in I} D(\{i\}^c)} \\
w_{132} &= p_{13}p_{32}p_{21} \frac{D(1, 2, 3^c)}{\sum_{i \in I} D(\{i\}^c)}
\end{aligned}$$

We only need to valid

$$w_{12} + w_{23} + w_{13} + w_{123} + w_{132} = 1 / \sum_{i \in I} D(\{i\}^c)$$

Using  $p$  to express  $w$ , we can get ( $p_{ii} = 0$ )

$$p_{12}p_{21} + p_{13}p_{31} + p_{23}p_{32} + p_{12}p_{23}p_{31} + p_{13}p_{32}p_{21} = 1$$

And we can valid it by calculation