1 三状态环流大偏差的 rate function:

三状态环流大偏差的 rate function:

$$
\begin{aligned}
I(\nu) = & \sum_{i\in I}\left(-\nu^i \log\frac{\nu^i}{w^i} + (\nu^i - \nu_i)\log(\nu^i - \nu_i)\right) \\
& - (\tilde{\nu} - \sum_{i\in I}\nu_i)\log(\tilde{\nu} - \sum_{i\in I}\nu_i) + \sum_{t\in C_\infty}\nu_t \log\nu_t \\
& - (\nu_1\log w_1 + \nu_2\log w_2 + \nu_3\log w_3) \\
& - (\nu_{12} + \nu_{123})\log(w_{12} + w_{123}) \\
& - (\nu_{13} + \nu_{132})\log(w_{13} + w_{132}) \\
& - (\nu_{12} + \nu_{132})\log(w_{12} + w_{132}) \\
& - (\nu_{23} + \nu_{123})\log(w_{23} + w_{123}) \\
& - (\nu_{13} + \nu_{123})\log(w_{13} + w_{123}) \\
& - (\nu_{23} + \nu_{132})\log(w_{23} + w_{132})
\end{aligned}
$$

其中 $I = \{1, 2, 3\}$ 是马氏链的状态空间。

$$
\mathcal{C}_\infty = \{(1), (2), (3), (1, 2), (2, 3), (1, 3), (1, 2, 3), (1, 32)
$$

表示所有可能出现环的集合。ν_c 表示环 c 出现的频率，w_c 表示环 c 的环流。

且 $\nu^i = \sum_{J_{c_s}(i)=1}\nu_{c_s}$，例如 $\nu^1 = \nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}$。$\tilde{\nu} = \nu_1 + \nu_2 + \nu_3 + \nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}$。

w^i, w_i 表示类似的含义。

由 $p_{ii}=0$ 情况的想法，可以得：

$$
\begin{aligned}
I(\nu) &= \nu_1\log(\frac{\nu_1}{w_1}/\frac{\nu^1}{w^1})+\nu_2\log(\frac{\nu_2}{w_2}/\frac{\nu^2}{w^2})+\nu_3\log(\frac{\nu_3}{w_3}/\frac{\nu^3}{w^3})\\
&+\nu_{12}\log(\frac{1}{p_{12}p_{21}}\frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^2-\nu_2}{\nu^2})\\
&+\nu_{13}\log(\frac{1}{p_{13}p_{31}}\frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^3-\nu_3}{\nu^3})\\
&+\nu_{23}\log(\frac{1}{p_{23}p_{32}}\frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^3-\nu_3}{\nu^3}\frac{\nu^2-\nu_2}{\nu^2})\\
&+\nu_{123}\log(\frac{1}{p_{12}p_{23}p_{31}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^2-\nu_2}{\nu^2}\frac{\nu^3-\nu_3}{\nu^3}))\\
&+\nu_{132}\log(\frac{1}{p_{13}p_{32}p_{21}}\frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^2-\nu_2}{\nu^2}\frac{\nu^3-\nu_3}{\nu^3}))
\end{aligned}
$$

数值上已经验证：当 $\nu=w$ 时，有：

$$
\frac{1}{p_{12}p_{21}}\frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^2-\nu_2}{\nu^2}=1
$$

$$
\frac{1}{p_{12}p_{23}p_{31}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}\frac{\nu^1-\nu_1}{\nu^1}\frac{\nu^2-\nu_2}{\nu^2}\frac{\nu^3-\nu_3}{\nu^3}=1
$$

2 $p_{13}=0$ 的情形

rate function:

$$
\begin{aligned}
I(\nu) &= -(\nu^1\log(\frac{\nu^1}{w^1})+\nu^2\log(\frac{\nu^2}{w^2})+\nu^3\log(\frac{\nu^3}{w^3}))\\
&+\nu_1\log(\frac{\nu_1}{w_1})+\nu_2\log(\frac{\nu_2}{w_2})+\nu_3\log(\frac{\nu_3}{w_3})\\
&+(\nu_{12}+\nu_{123})\log(\frac{\nu_{12}+\nu_{123}}{w_{12}+w123})+(\nu_{23}+\nu_{123})\log(\frac{\nu_{23}+\nu_{123}}{w_{23}+w123})\\
&+\nu_{12}\log(\frac{\nu_{12}}{w_{12}})+\nu_{23}\log(\frac{\nu_{23}}{w_{23}})+\nu_{123}\log(\frac{\nu_{123}}{w_{123}})
\end{aligned}
$$

即

$$
\begin{aligned}
I(\nu) &= \sum_{i,j\in I}\left(\sum_{c\in\mathcal{C}_\infty, J_c(i,j)=1}\nu_c\right)\log\left(\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1} w_c \sum_{c\in\mathcal{C}_\infty,J_c(i)=1}\nu_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1}\nu_c \sum_{c\in\mathcal{C}_\infty,J_c(i)=1} w_c}\right) \\
&= \sum_{i,j\in I}\left(\sum_{c\in\mathcal{C}_\infty, J_c(i,j)=1}\nu_c\right)\log\left(\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1} w_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1}\nu_c}\Big/\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i)=1} w_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i)=1}\nu_c}\right)
\end{aligned}
$$

3 $p_{ii}=0$ 的情形

rate function:

$$
\begin{aligned}
I(\nu) = &\sum_{i\in I}(\nu^i-\nu_i)\log(w^i-w_i) - (\tilde{\nu}-\sum_{i\in I}\nu_i)\log(\tilde{\nu}-\sum_{i\in I}\nu_i) \\
&+\nu_{12}\log\nu_{12}+\nu_{23}\log\nu_{23}+\nu_{13}\log\nu_{13}+\nu_{123}\log\nu_{123}+\nu_{132}\log\nu_{132} \\
&-(\nu_{12}+\nu_{123})\log(w_{12}+w_{123}) \\
&-(\nu_{13}+\nu_{132})\log(w_{13}+w_{132}) \\
&-(\nu_{12}+\nu_{132})\log(w_{12}+w_{132}) \\
&-(\nu_{23}+\nu_{123})\log(w_{23}+w_{123}) \\
&-(\nu_{13}+\nu_{123})\log(w_{13}+w_{123}) \\
&-(\nu_{23}+\nu_{132})\log(w_{23}+w_{132})
\end{aligned}
$$

化简得：

$$
\begin{aligned}
I(\nu) = {} & \nu_{12}\log(\frac{w_{12}+w_{13}+w_{123}+w_{132}}{w_{12}+w_{123}}\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{12}+w_{132}}\\
& \frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{13}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{132}}\\
& \frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{23}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{132}}\\
& \frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{123}\log(\frac{w_{12}+w_{13}+w_{123}+w_{132}}{w_{12}+w_{123}}\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{123}}\\
& \frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{132}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{132}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{132}}\\
& \frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{12}+w_{132}}\frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& =\nu_{12}\log(\frac{1}{p_{12}p_{21}}\frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{13}\log(\frac{1}{p_{13}p_{31}}\frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{23}\log(\frac{1}{p_{23}p_{32}}\frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{123}\log(\frac{1}{p_{12}p_{23}p_{31}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})\\
& +\nu_{132}\log(\frac{1}{p_{13}p_{32}p_{21}}\frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})
\end{aligned}
$$

可进一步考虑联系环流的表达式

$$
\begin{aligned}
w_{12} &= p_{12}p_{21}\frac{D(1,2^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{13} &= p_{13}p_{31}\frac{D(1,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{23} &= p_{23}p_{32}\frac{D(2,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{123} &= p_{12}p_{23}p_{31}\frac{D(1,2,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{132} &= p_{13}p_{32}p_{21}\frac{D(1,2,3^c)}{\sum_{i\in I} D(\{i\}^c)}
\end{aligned}
$$

只需证：

$$
\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132} = 1/\sum_{i\in I} D(\{i\}^c)
$$

即证（$p_{ii}=0$）：

$$
p_{12}p_{21}+p_{13}p_{31}+p_{23}p_{32}+p_{12}p_{23}p_{31}+p_{13}p_{32}p_{21}=1
$$

用 p_{12}, p_{21}, p_{31} 带换其他变量，可证。