The rate function for circulation of three state Markov chain

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Abstract

1 The rate function for circulation of three state Markov chain

$$
\begin{aligned}
I(\nu) = &\sum_{i \in I} \left(-\nu^i \log \frac{\nu^i}{w^i} + (\nu^i - \nu_i) \log(\nu^i - \nu_i) \right) \\
&- (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) + \sum_{t \in C_\infty} \nu_t \log \nu_t \\
&- (\nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3) \\
&- (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\
&- (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\
&- (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\
&- (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\
&- (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\
&- (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132})
\end{aligned}
$$

where $I = \{1, 2, 3\}$ is the state space for Markov chain

$$
\mathcal{C}_\infty = \{(1), (2), (3), (1, 2), (2, 3), (1, 3), (1, 2, 3), (1, 32)
$$

is the class of all cycles occurring. ν_c is the frequence of c occurring, w_c is the circulation of c. And $\nu^i = \sum_{J_{c_s}(i)=1} \nu_{c_s}$, such as $\nu^1 = \nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}$ $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3 + \nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}$
w^i, w_i has the similar definition.

Refer to the result of $p_{ii}=0$, we can get

$$
\begin{aligned}
I(\nu) = {} & \nu_1 \log(\frac{\nu_1}{w_1}/\frac{\nu^1}{w^1}) + \nu_2 \log(\frac{\nu_2}{w_2}/\frac{\nu^2}{w^2}) + \nu_3 \log(\frac{\nu_3}{w_3}/\frac{\nu^3}{w^3}) \\
& + \nu_{12} \log(\frac{1}{p_{12}p_{21}} \frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^2-\nu_2}{\nu^2}) \\
& + \nu_{13} \log(\frac{1}{p_{13}p_{31}} \frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^3-\nu_3}{\nu^3}) \\
& + \nu_{23} \log(\frac{1}{p_{23}p_{32}} \frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^3-\nu_3}{\nu^3} \frac{\nu^2-\nu_2}{\nu^2}) \\
& + \nu_{123} \log(\frac{1}{p_{12}p_{23}p_{31}} \frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^2-\nu_2}{\nu^2} \frac{\nu^3-\nu_3}{\nu^3})) \\
& + \nu_{132} \log(\frac{1}{p_{13}p_{32}p_{21}} \frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^2-\nu_2}{\nu^2} \frac{\nu^3-\nu_3}{\nu^3}))
\end{aligned}
$$

So we need to valid following proposition. if $\nu = w$, then

$$\frac{1}{p_{12}p_{21}} \frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^2-\nu_2}{\nu^2} = 1$$

$$\frac{1}{p_{12}p_{23}p_{31}} \frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}} \frac{\nu^1-\nu_1}{\nu^1} \frac{\nu^2-\nu_2}{\nu^2} \frac{\nu^3-\nu_3}{\nu^3} = 1$$

We use

$$
\begin{aligned}
\frac{w^1-w_1}{w^1} &= \frac{w_{12}+w_{13}+w_{123}+w_{132}}{w_1+w_{12}+w_{13}+w_{123}+w_{132}} \\
&= p_{12}+p_{13} \\
&= p_{21}+p_{31} \\
&= 1-p_{11} \\
&= D(\{2,3\}^c)
\end{aligned}
$$

and

$$
\begin{aligned}
w_{12} &= p_{12}p_{21} \frac{D(\{1,2\}^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{13} &= p_{13}p_{31} \frac{D(\{1,3\}^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{23} &= p_{23}p_{32} \frac{D(\{2,3\}^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{123} &= p_{12}p_{23}p_{31} \frac{D(\{1,2,3\}^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{132} &= p_{13}p_{32}p_{21} \frac{D(\{1,2,3\}^c)}{\sum_{i\in I} D(\{i\}^c)}
\end{aligned}
$$

substitute w, the above proposition can be valid.

By above validation, we also know:

$$
\begin{aligned}
w_{12}+w_{13}+w_{23}+w_{123}+w_{132} &= \frac{(1-p_{11})(1-p_{22})(1-p_{33})}{\sum_{i\in I} D(\{i\}^c)} \\
&= \frac{\Pi_{i,j} D(\{i,j\}^c)}{\sum_{i\in I} D(\{i\}^c)}
\end{aligned}
$$

2 $p_{13} = 0$

rate function:

$$
\begin{aligned}
I(\nu) = &-(\nu^1 \log(\frac{\nu^1}{w^1}) + \nu^2 \log(\frac{\nu^2}{w^2}) + \nu^3 \log(\frac{\nu^3}{w^3})) \\
&+ \nu_1 \log(\frac{\nu_1}{w_1}) + \nu_2 \log(\frac{\nu_2}{w_2}) + \nu_3 \log(\frac{\nu_3}{w_3}) \\
&+ (\nu_{12} + \nu_{123}) \log(\frac{\nu_{12} + \nu_{123}}{w_{12} + w123}) + (\nu_{23} + \nu_{123}) \log(\frac{\nu_{23} + \nu_{123}}{w_{23} + w123}) \\
&+ \nu_{12} \log(\frac{\nu_{12}}{w_{12}}) + \nu_{23} \log(\frac{\nu_{23}}{w_{23}}) + \nu_{123} \log(\frac{\nu_{123}}{w_{123}})
\end{aligned}
$$

That is

$$
\begin{aligned}
I(\nu) &= \sum_{i,j \in I} \left(\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log(\frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c}) \\
&= \sum_{i,j \in I} \left(\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log(\frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c} / \frac{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c})
\end{aligned}
$$

3 $p_{ii} = 0$

rate function:

$$
\begin{aligned}
I(\nu) = &\sum_{i \in I} (\nu^i - \nu_i) \log(w^i - w_i) - (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) \\
&+ \nu_{12} \log \nu_{12} + \nu_{23} \log \nu_{23} + \nu_{13} \log \nu_{13} + \nu_{123} \log \nu_{123} + \nu_{132} \log \nu_{132} \\
&- (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123}) \\
&- (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132}) \\
&- (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132}) \\
&- (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123}) \\
&- (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123}) \\
&- (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132})
\end{aligned}
$$

We can simply it to following formula:

$$
\begin{aligned}
I(\nu) = \nu_{12}\log(&\frac{w_{12}+w_{13}+w_{123}+w_{132}}{w_{12}+w_{123}}\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{12}+w_{132}} \\
&\frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{13}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{132}} \\
&\frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{23}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{132}} \\
&\frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{123}\log(\frac{w_{12}+w_{13}+w_{123}+w_{132}}{w_{12}+w_{123}}\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{123}} \\
&\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{132}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{132}\log(\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{13}+w_{123}}\frac{w_{13}+w_{23}+w_{123}+w_{132}}{w_{23}+w_{132}} \\
&\frac{w_{12}+w_{23}+w_{123}+w_{132}}{w_{12}+w_{132}}\frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&=\nu_{12}\log(\frac{1}{p_{12}p_{21}}\frac{\nu_{12}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{13}\log(\frac{1}{p_{13}p_{31}}\frac{\nu_{13}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{23}\log(\frac{1}{p_{23}p_{32}}\frac{\nu_{23}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{123}\log(\frac{1}{p_{12}p_{23}p_{31}}\frac{\nu_{123}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}}) \\
&+\nu_{132}\log(\frac{1}{p_{13}p_{32}p_{21}}\frac{\nu_{132}}{\nu_{12}+\nu_{23}+\nu_{13}+\nu_{123}+\nu_{132}})
\end{aligned}
$$

Associationg the expression of circulation, and we can further considered

$$
\begin{aligned}
w_{12} &= p_{12}p_{21}\frac{D(1,2^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{13} &= p_{13}p_{31}\frac{D(1,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{23} &= p_{23}p_{32}\frac{D(2,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{123} &= p_{12}p_{23}p_{31}\frac{D(1,2,3^c)}{\sum_{i\in I} D(\{i\}^c)} \\
w_{132} &= p_{13}p_{32}p_{21}\frac{D(1,2,3^c)}{\sum_{i\in I} D(\{i\}^c)}
\end{aligned}
$$

We only need to valid

$$w_{12}+w_{23}+w_{13}+w_{123}+w_{132} = 1/\sum_{i\in I} D(\{i\}^c)$$

Using p to express w, we can get ($p_{ii}=0$)

$$p_{12}p_{21}+p_{13}p_{31}+p_{23}p_{32}+p_{12}p_{23}p_{31}+p_{13}p_{32}p_{21} = 1$$

And we can valid it by calculation