状态空间大小为 3 的马氏链的环流的大偏差

在该问题中有下面 8 种环流：$c_1 : 1 \to 1, c_2 : 2 \to 2, c_3 : 3 \to 3$
$c_4 : 1 \to 2 \to 1, c_5 : 1 \to 3 \to 1, c_6 : 2 \to 3 \to 2$
$c_7 : 1 \to 2 \to 3 \to 1, c_8 : 1 \to 3 \to 2 \to 1$
在有 3 个状态的图中，考虑所有的 n 步欧拉回路。为了方便，令起始位置的状态为 1。n 步的欧拉回路可以分解为上述的 8 个环路。而且给定环路的完成顺序可以唯一确定一个欧拉回路。
xi 表示一条 n 步欧拉回路，且在这条回路中，环流 c_i 出现 k_i 次。由于这条回路可以唯一分解为上述环流，则每种边出现的次数可以得到，则

$$k_1 + k_2 + k_3 + 2(k_{12} + k_{13} + k_{23}) + 3(k_{123} + k_{132}) = n$$

$$\mathbb{P}(\xi) = p_{11}^{k_1} p_{22}^{k_2} p_{33}^{k_3} p_{12}^{k_{12}+k_{123}} p_{21}^{k_{12}+k_{132}} p_{13}^{k_{13}+k_{132}} p_{31}^{k_{13}+k_{123}} p_{23}^{k_{23}+k_{123}} p_{32}^{k_{23}+k_{132}}$$

只需计算 A 这种欧拉回路有多少种。由初始状态是 1，考虑 c_1, c_4, c_5, c_7, c_8 (状态 1 开始，状态 1 结束)。对于这几种环流，只有当一个结束，才会有另一个开始。也就是说在 c_7 开始，但没完成之前，不会先完成 c_5。由这几种环流的不同排列，可以有

$$\binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1,\ k_{12},\ k_{13},\ k_{123},\ k_{132}}$$

种不同的可能。先把环流 c_6 嵌入进去，其中 c_6 可以嵌入在状态 2，也可以嵌入在状态 3 上，所以会有这些种可能：

$$\binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1}{k_{23}}$$

再把环流 c_2 和 c_3 嵌入在上述可能的欧拉路径中，依次有这些中嵌入方法
$c_2 : \binom{k_{12}+k_{123}+k_{132}+k_{23}+k_2-1}{k_2}$,
$c_3 : \binom{k_{13}+k_{123}+k_{132}+k_{23}+k_3-1}{k_3}$。

不能先插入 $2- > 2$ 和 $3- > 3$，再插入 $2- > 3- > 2$ 比如在环 $1- > 3- > 2- > 1$ 中先插入 $2- > 2$，即使规定在 $1- > 3- > 2- > 1$ 中，不能在 2 位置插入 $2- > 3- > 2$，但是得到的 $1- > 3- > 2- > 2- > 1$

的第二个 2 中，插入 $2->3->2$，也会产生类似的效果。并且我们无法准确清楚有多少 $2->2$ 已经插入在环 $1->3->2->1$ 上。

以状态 1 开始，有下面多种情况：

$$A_1 = \binom{k_1+k_{12}+k_{13}+k_{123}+k_{132}}{k_1,\ k_{12},\ k_{13},\ k_{123},\ k_{132}}\binom{k_{12}+k_{13}+k_{123}+k_{132}+k_{23}-1}{k_{23}}\binom{k_{12}+k_{123}+k_{132}+k_{23}+k_2-1}{k_2}\binom{k_{13}+k_{123}+k_{132}+k_{23}+k_3-1}{k_3}$$

以状态 2 开始，有下面多种情况：

$$A_2 = \binom{k_2+k_{12}+k_{23}+k_{123}+k_{132}}{k_2,\ k_{12},\ k_{23},\ k_{123},\ k_{132}}\binom{k_{12}+k_{23}+k_{123}+k_{132}+k_{13}-1}{k_{13}}\binom{k_{12}+k_{123}+k_{132}+k_{13}+k_1-1}{k_1}\binom{k_{23}+k_{123}+k_{132}+k_{13}+k_3-1}{k_3}$$

以状态 3 开始，有下面多种情况：

$$A_3 = \binom{k_3+k_{13}+k_{23}+k_{123}+k_{132}}{k_3,\ k_{13},\ k_{23},\ k_{123},\ k_{132}}\binom{k_{13}+k_{23}+k_{123}+k_{132}+k_{12}-1}{k_{12}}\binom{k_{13}+k_{123}+k_{132}+k_{12}+k_1-1}{k_1}\binom{k_{23}+k_{123}+k_{132}+k_{12}+k_2-1}{k_2}$$

$$A_1 = \binom{k_{12}+k_{13}+k_{123}+k_{132}}{k_{12},\ k_{13},\ k_{123},\ k_{132}}\binom{k_1+k_{12}+k_{13}+k_{123}+k_{132}}{k_1}\binom{k_{12}+k_{13}+k_{123}+k_{132}+k_{23}-1}{k_{23}}\binom{k_{12}+k_{123}+k_{132}+k_{23}+k_2-1}{k_2}\binom{k_{13}+k_{123}+k_{132}+k_{23}+k_3-1}{k_3}$$

由

$$
\begin{aligned}
\frac{1}{n}\log C_{n-1}^{m} &= \frac{1}{n}\log\frac{n(n-1)(n-2)\cdots(m+1)}{m(m-1)\cdots 1}, \quad n\to\infty \\
&= \frac{1}{n}\log\frac{(n-1)(n-2)\cdots(n-m)}{m(m-1)\cdots 1}, \quad n\to\infty \\
&= \frac{1}{n}\log\frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 1} + O(\frac{\log n}{n}), \quad n\to\infty \\
&= \frac{1}{n}\log C_{n}^{m} + O\left(\frac{\log n}{n}\right), \quad n\to\infty
\end{aligned}
$$

和 stirling 公式：

$$
\begin{aligned}
&\frac{1}{n}A_1 = \\
&\quad \frac{1}{n}\binom{k_{12}+k_{13}+k_{123}+k_{132}}{k_{12},\ k_{13},\ k_{123},\ k_{132}} \\
&\quad \binom{k_1+k_{12}+k_{13}+k_{123}+k_{132}}{k_1}\binom{k_{12}+k_{13}+k_{123}+k_{132}+k_{23}}{k_{23}} \\
&\quad \binom{k_2+k_{12}+k_{123}+k_{132}+k_{23}}{k_2}\binom{k_3+k_{13}+k_{123}+k_{132}+k_{23}}{k_3} + O(\frac{\log n}{n}) \\
&\quad = h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) - [h(\nu_{12})+h(\nu_{13})+h(\nu_{123})+h(\nu_{132})] \\
&\quad + h(\nu_1+\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) - [h(\nu_1)+h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})] \\
&\quad + h(\nu_{12}+\nu_{13}+\nu_{23}+\nu_{123}+\nu_{132}) - [h(\nu_{23})+h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})] \\
&\quad + h(\nu_2+\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23}) - [h(\nu_2)+h(\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23})] \\
&\quad + h(\nu_3+\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23}) - [h(\nu_3)+h(\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23})] \\
&\quad = h(\nu_{12}+\nu_{13}+\nu_{23}+\nu_{123}+\nu_{132}) \\
&\quad + h(\nu_1+\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) \\
&\quad + h(\nu_2+\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23}) \\
&\quad + h(\nu_3+\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23}) \\
&\quad - [h(\nu_1)+h(\nu_2)+h(\nu_3)+h(\nu_{12})+h(\nu_{13})+h(\nu_{23})+h(\nu_{123})+h(\nu_{132})] \\
&\quad - \Big(h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) + h(\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23}) \\
&\quad + h(\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23})\Big)
\end{aligned}
$$

$$
\mathbb{P}(\xi) = p_{11}^{k_1}p_{22}^{k_2}p_{33}^{k_3}p_{12}^{k_{12}+k_{123}}p_{21}^{k_{12}+k_{132}}p_{13}^{k_{13}+k_{132}}p_{31}^{k_{13}+k_{123}}p_{23}^{k_{23}+k_{123}}p_{23}^{k_{23}+k_{132}}
$$

环流分解

$$\pi_1 p_{11} = w_1$$

$$\pi_2 p_{22} = w_2$$

$$\pi_3 p_{33} = w_3$$

$$\pi_1 p_{12} = w_{12} + w_{123}$$

$$\pi_1 p_{13} = w_{13} + w_{132}$$

$$\pi_2 p_{21} = w_{12} + w_{123}$$

$$\pi_2 p_{23} = w_{23} + w_{132}$$

$$\pi_3 p_{31} = w_{13} + w_{123}$$

$$\pi_3 p_{23} = w_{23} + w_{132}$$

可得

$$\begin{cases} (w_{12} + w_{123}) p_{11} - w_1 p_{12} = 0 \\ (w_{13} + w_{132}) p_{11} - w_1 p_{13} = 0 \\ p_{11} + p_{12} + p_{13} = 1 \end{cases}$$

$$p_{11} = \frac{w_1}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{12} = \frac{w_{12} + w_{123}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{13} = \frac{w_{13} + w_{132}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{22} = \frac{w_2}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{21} = \frac{w_{12} + w_{132}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{23} = \frac{w_{23} + w_{123}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{33} = \frac{w_3}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$p_{31} = \frac{w_{13} + w_{123}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$p_{32} = \frac{w_{23} + w_{132}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$
\begin{aligned}
\frac{1}{n}\log\mathbb{P}(\xi) =& \frac{1}{n}\log p_{11}^{k_1}p_{22}^{k_2}p_{33}^{k_3}p_{12}^{k_{12}+k_{123}}p_{21}^{k_{12}+k_{132}}p_{13}^{k_{13}+k_{132}}p_{31}^{k_{13}+k_{123}}p_{23}^{k_{23}+k_{123}}p_{23}^{k_{23}+k_{132}} \\
=& \nu_1\log w_1 + \nu_2\log w_2 + \nu_3\log w_3 \\
&+ (\nu_{12}+\nu_{123})\log(w_{12}+w_{123}) \\
&+ (\nu_{13}+\nu_{132})\log(w_{13}+w_{132}) \\
&+ (\nu_{12}+\nu_{132})\log(w_{12}+w_{132}) \\
&+ (\nu_{23}+\nu_{123})\log(w_{23}+w_{123}) \\
&+ (\nu_{13}+\nu_{123})\log(w_{13}+w_{123}) \\
&+ (\nu_{23}+\nu_{132})\log(w_{23}+w_{132}) \\
&- (\nu_1+\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})\log(w_1+w_{12}+w_{13}+w_{123}+w_{132}) \\
&- (\nu_2+\nu_{12}+\nu_{23}+\nu_{123}+\nu_{132})\log(w_2+w_{12}+w_{23}+w_{123}+w_{132}) \\
&- (\nu_3+\nu_{13}+\nu_{23}+\nu_{123}+\nu_{132})\log(w_3+w_{13}+w_{23}+w_{123}+w_{132})
\end{aligned}
$$

对于 n 状态的情况:

$$
\begin{aligned}
I(\nu) &= \sum_{i,j\in I}\left(\sum_{c\in\mathcal{C}_\infty, J_c(i,j)=1}\nu_c\right)\log\left(\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1} w_c \sum_{c\in\mathcal{C}_\infty,J_c(i)=1}\nu_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1}\nu_c \sum_{c\in\mathcal{C}_\infty,J_c(i)=1} w_c}\right) \\
&= \sum_{i,j\in I}\left(\sum_{c\in\mathcal{C}_\infty, J_c(i,j)=1}\nu_c\right)\log\left(\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1} w_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i,j)=1}\nu_c}\Big/\frac{\sum_{c\in\mathcal{C}_\infty,J_c(i)=1} w_c}{\sum_{c\in\mathcal{C}_\infty,J_c(i)=1}\nu_c}\right)
\end{aligned}
$$

令 $\nu^i = \frac{k_i}{n}$, $i\in I$，即 $\nu^i = \sum_{J_{c_s}(i)=1}\nu_{c_s}$. $\tilde{\nu} = \sum_{c\in\mathcal{C}_\infty}\nu_c$. 其中 $h(x) =$

$x\log x$ 则：

$$
\begin{aligned}
\frac{1}{n}\log\mathbb{P}(J^{c_s}=\frac{k_s}{n},s=1,2,\dots 8) = \\
&\sum_{i\in I}\big(\nu^i\log\nu^i-(\nu^i-\nu_{(i)})\log(\nu^i-\nu_{(i)})\big)\\
&+(\tilde{\nu}-\sum_{i\in I}\nu_i)\log(\tilde{\nu}-\sum_{i\in I}\nu_i)-\sum_{t\in C_\infty}\nu_t\log\nu_t\\
&+\mathbb{P}(\xi)+O(\frac{\log n}{n})
\end{aligned}
$$

rate function:

$$
\begin{aligned}
I(\nu)=&\sum_{i\in I}\left(-\nu^i\log\frac{\nu^i}{w^i}+(\nu^i-\nu_{(i)})\log(\nu^i-\nu_{(i)})\right)\\
&-(\tilde{\nu}-\sum_{i\in I}\nu_i)\log(\tilde{\nu}-\sum_{i\in I}\nu_i)+\sum_{t\in C_\infty}\nu_t\log\nu_t\\
&-(\nu_1\log w_1+\nu_2\log w_2+\nu_3\log w_3)\\
&-(\nu_{12}+\nu_{123})\log(w_{12}+w_{123})\\
&-(\nu_{13}+\nu_{132})\log(w_{13}+w_{132})\\
&-(\nu_{12}+\nu_{132})\log(w_{12}+w_{132})\\
&-(\nu_{23}+\nu_{123})\log(w_{23}+w_{123})\\
&-(\nu_{13}+\nu_{123})\log(w_{13}+w_{123})\\
&-(\nu_{23}+\nu_{132})\log(w_{23}+w_{132})+O(\frac{\log n}{n})
\end{aligned}
$$

状态空间大小为 4 的马氏链的环流的大偏差

在所有的回路中使得，初始状态为 1，环流 c_t 有 k_t 的路径数量：

$$
\begin{aligned}
&A_1 = \\
&\binom{k_{12}+k_{14}+k_{1234}+k_{1432}}{k_{12},k_{14},k_{1234},k_{1432}} \\
&\binom{k_1+k_{12}+k_{14}+k_{1234}+k_{1432}}{k_{12}+k_{14}+k_{1234}+k_{1432}}\binom{k_2+k_{12}+k_{23}+k_{1234}+k_{1432}-1}{k_{12}+k_{23}+k_{1234}+k_{1432}-1} \\
&\binom{k_3+k_{23}+k_{34}+k_{1234}+k_{1432}-1}{k_{23}+k_{34}+k_{1234}+k_{1432}-1}\binom{k_4+k_{14}+k_{34}+k_{1234}+k_{1432}-1}{k_{14}+k_{34}+k_{1234}+k_{1432}-1} \\
&\Bigg(\sum_{k_{23}^{12}+k_{23}^{14}=k_{23}}\ \sum_{k_{34}^{12}+k_{34}^{14}=k_{34}}\binom{k_{23}^{12}+k_{12}+k_{1234}-1}{k_{23}^{12}}\binom{k_{34}^{12}+k_{1234}+k_{23}^{12}-1}{k_{34}^{12}} \\
&\binom{k_{34}^{14}+k_{14}+k_{1432}-1}{k_{34}^{14}}\binom{k_{23}^{14}+k_{34}^{14}+k_{1432}-1}{k_{23}^{14}}\Bigg)
\end{aligned}
$$

其中 k_{23}^{12} 和 k_{34}^{12} 分别表示 k_{23},k_{34} 嵌入到 k_{1234} 和 k_{12}（从 1 出发，第二步为 2 的环）的数量。k_{23}^{14} 和 k_{34}^{14} 分别表示 k_{23},k_{34} 嵌入到 k_{1432} 和 k_{14}（从 1 出发，第二步为 4 的环）的数量。（$k_{23}^{12}, k_{23}^{14}, k_{34}^{12}, k_{34}^{14} \geq 0$）

由马氏链的常返性，对于足够大的 n，路径中必然包含所有状态，则：

$$\frac{1}{n}A_j \leq A_i \leq nA_j$$

故：

$$\frac{1}{n}\sqrt[4]{A_1, A_2, A_3, A_4} \leq A_i \leq n\sqrt[4]{A_1, A_2, A_3, A_4}$$

从而

$$\frac{1}{n}\log A_1 = \frac{1}{4n}\log(A_1, A_2, A_3, A_4) + O(\frac{\log n}{n}) \tag{1}$$

$$
\begin{aligned}
\frac{1}{n}\log \mathbb{P}(\xi) = &\sum_{i\in I}\left(\nu^i \log p_{ii} - (\nu^i - \nu_{(i)})\log w_i\right)\\
&+ (\nu_{12}+\nu_{1234})\log(w_{12}+w_{1234})\\
&+ (\nu_{23}+\nu_{1234})\log(w_{23}+w_{1234})\\
&+ (\nu_{34}+\nu_{1234})\log(w_{34}+w_{1234})\\
&+ (\nu_{41}+\nu_{1234})\log(w_{41}+w_{1234})\\
&+ (\nu_{14}+\nu_{1432})\log(w_{14}+w_{1432})\\
&+ (\nu_{43}+\nu_{1432})\log(w_{43}+w_{1432})\\
&+ (\nu_{32}+\nu_{1432})\log(w_{32}+w_{1432})\\
&+ (\nu_{21}+\nu_{1432})\log(w_{21}+w_{1432}) + O(\frac{\log n}{n})
\end{aligned}
$$