状态空间大小为 3 的马氏链的环流的大偏差

在该问题中有下面 8 种环流：$c_1 : 1 \to 1, c_2 : 2 \to 2, c_3 : 3 \to 3$
$c_4 : 1 \to 2 \to 1, c_5 : 1 \to 3 \to 1, c_6 : 2 \to 3 \to 2$
$c_7 : 1 \to 2 \to 3 \to 1, c_8 : 1 \to 3 \to 2 \to 1$
在有 3 个状态的图中，考虑所有的 n 步欧拉回路。为了方便，令起始位置的状态为 1。n 步的欧拉回路可以分解为上述的 8 个环路。而且给定环路的完成顺序可以唯一确定一个欧拉回路。

xi 表示一条 n 步欧拉回路，且在这条回路中，环流 c_i 出现 k_i 次。由于这条回路可以唯一分解为上述环流，则每种边出现的次数可以得到，则

$$k_1 + k_2 + k_3 + 2(k_{12} + k_{13} + k_{23}) + 3(k_{123} + k_{132}) = n$$

$$\mathbb{P}(\xi) = p_{11}^{k_1} p_{22}^{k_2} p_{33}^{k_3} p_{12}^{k_{12}+k_{123}} p_{21}^{k_{12}+k_{132}} p_{13}^{k_{13}+k_{132}} p_{31}^{k_{13}+k_{123}} p_{23}^{k_{23}+k_{123}} p_{32}^{k_{23}+k_{132}}$$

只需计算 A 这种欧拉回路有多少种。由初始状态是 1，考虑 c_1, c_4, c_5, c_7, c_8 (状态 1 开始，状态 1 结束)。对于这几种环流，只有当一个结束，才会有另一个开始。也就是说在 c_7 开始，但没完成之前，不会先完成 c_5。由这几种环流的不同排列，可以有

$$\binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1, \, k_{12}, \, k_{13}, \, k_{123}, \, k_{132}}$$

种不同的可能。先把环流 c_6 嵌入进去，其中 c_6 可以嵌入在状态 2，也可以嵌入在状态 3 上，所以会有这些种可能：

$$\binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1}{k_{23}}$$

再把环流 c_2 和 c_3 嵌入在上述可能的欧拉路径中，依次有这些中嵌入方法
$c_2 : \binom{k_{12}+k_{123}+k_{132}+k_{23}+k_2-1}{k_2}$,
$c_3 : \binom{k_{13}+k_{123}+k_{132}+k_{23}+k_3-1}{k_3}$。

不能先插入 $2->2$ 和 $3->3$，再插入 $2->3->2$ 比如在环 $1->3->2->1$ 中先插入 $2->2$，即使规定在 $1->3->2->1$ 中，不能在 2 位置插入 $2->3->2$，但是得到的 $1->3->2->2->1$

的第二个 2 中，插入 $2->3->2$，也会产生类似的效果。并且我们无法准确清楚有多少 $2->2$ 已经插入在环 $1->3->2->1$ 上。

以状态 1 开始，有下面多种情况：

$$A_1 =$$
$$\binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1,\ k_{12},\ k_{13},\ k_{123},\ k_{132}} \binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1}{k_{23}}$$
$$\binom{k_{12} + k_{123} + k_{132} + k_{23} + k_2 - 1}{k_2} \binom{k_{13} + k_{123} + k_{132} + k_{23} + k_3 - 1}{k_3}$$

以状态 2 开始，有下面多种情况：

$$A_2 =$$
$$\binom{k_2 + k_{12} + k_{23} + k_{123} + k_{132}}{k_2,\ k_{12},\ k_{23},\ k_{123},\ k_{132}} \binom{k_{12} + k_{23} + k_{123} + k_{132} + k_{13} - 1}{k_{13}}$$
$$\binom{k_{12} + k_{123} + k_{132} + k_{13} + k_1 - 1}{k_1} \binom{k_{23} + k_{123} + k_{132} + k_{13} + k_3 - 1}{k_3}$$

以状态 3 开始，有下面多种情况：

$$A_3 =$$
$$\binom{k_3 + k_{13} + k_{23} + k_{123} + k_{132}}{k_3,\ k_{13},\ k_{23},\ k_{123},\ k_{132}} \binom{k_{13} + k_{23} + k_{123} + k_{132} + k_{12} - 1}{k_{12}}$$
$$\binom{k_{13} + k_{123} + k_{132} + k_{12} + k_1 - 1}{k_1} \binom{k_{23} + k_{123} + k_{132} + k_{12} + k_2 - 1}{k_2}$$

$$A_1 =$$
$$\binom{k_{12} + k_{13} + k_{123} + k_{132}}{k_{12},\ k_{13},\ k_{123},\ k_{132}}$$
$$\binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1} \binom{k_{12} + k_{13} + k_{123} + k_{132} + k_{23} - 1}{k_{23}}$$
$$\binom{k_{12} + k_{123} + k_{132} + k_{23} + k_2 - 1}{k_2} \binom{k_{13} + k_{123} + k_{132} + k_{23} + k_3 - 1}{k_3}$$

由

$$\frac{1}{n}\log C_{n-1}^m = \frac{1}{n}\log\frac{n(n-1)(n-2)\cdots(m+1)}{m(m-1)\cdots 1}, \quad n\to\infty$$

$$= \frac{1}{n}\log\frac{(n-1)(n-2)\cdots(n-m)}{m(m-1)\cdots 1}, \quad n\to\infty$$

$$= \frac{1}{n}\log\frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 1} + O(\frac{\log n}{n}), \quad n\to\infty$$

$$= \frac{1}{n}\log C_n^m + O\left(\frac{\log n}{n}\right), \quad n\to\infty$$

和 stirling 公式：

$$\frac{1}{n}A_1 =$$

$$\frac{1}{n}\binom{k_{12}+k_{13}+k_{123}+k_{132}}{k_{12},\ k_{13},\ k_{123},\ k_{132}}$$

$$\binom{k_1+k_{12}+k_{13}+k_{123}+k_{132}}{k_1}\binom{k_{12}+k_{13}+k_{123}+k_{132}+k_{23}}{k_{23}}$$

$$\binom{k_2+k_{12}+k_{123}+k_{132}+k_{23}}{k_2}\binom{k_3+k_{13}+k_{123}+k_{132}+k_{23}}{k_3} + O(\frac{\log n}{n})$$

$$= h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) - [h(\nu_{12})+h(\nu_{13})+h(\nu_{123})+h(\nu_{132})]$$

$$+ h(\nu_1+\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) - [h(\nu_1)+h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})]$$

$$+ h(\nu_{12}+\nu_{13}+\nu_{23}+\nu_{123}+\nu_{132}) - [h(\nu_{23})+h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})]$$

$$+ h(\nu_2+\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23}) - [h(\nu_2)+h(\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23})]$$

$$+ h(\nu_3+\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23}) - [h(\nu_3)+h(\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23})]$$

$$= h(\nu_{12}+\nu_{13}+\nu_{23}+\nu_{123}+\nu_{132})$$

$$+ h(\nu_1+\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132})$$

$$+ h(\nu_2+\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23})$$

$$+ h(\nu_3+\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23})$$

$$- [h(\nu_1)+h(\nu_2)+h(\nu_3)+h(\nu_{12})+h(\nu_{13})+h(\nu_{23})+h(\nu_{123})+h(\nu_{132})]$$

$$- \Big(h(\nu_{12}+\nu_{13}+\nu_{123}+\nu_{132}) + h(\nu_{12}+\nu_{123}+\nu_{132}+\nu_{23})$$

$$+ h(\nu_{13}+\nu_{123}+\nu_{132}+\nu_{23}) \Big)$$

$$\mathbb{P}(\xi) = p_{11}^{k_1}p_{22}^{k_2}p_{33}^{k_3}p_{12}^{k_{12}+k_{123}}p_{21}^{k_{12}+k_{132}}p_{13}^{k_{13}+k_{132}}p_{31}^{k_{13}+k_{123}}p_{23}^{k_{23}+k_{123}}p_{23}^{k_{23}+k_{132}}$$

环流分解

$$\pi_1 p_{11} = w_1$$

$$\pi_2 p_{22} = w_2$$

$$\pi_3 p_{33} = w_3$$

$$\pi_1 p_{12} = w_{12} + w_{123}$$

$$\pi_1 p_{13} = w_{13} + w_{132}$$

$$\pi_2 p_{21} = w_{12} + w_{123}$$

$$\pi_2 p_{23} = w_{23} + w_{132}$$

$$\pi_3 p_{31} = w_{13} + w_{123}$$

$$\pi_3 p_{23} = w_{23} + w_{132}$$

可得

$$\begin{cases} (w_{12} + w_{123})p_{11} - w_1 p_{12} = 0 \\ (w_{13} + w_{132})p_{11} - w_1 p_{13} = 0 \\ p_{11} + p_{12} + p_{13} = 1 \end{cases}$$

$$p_{11} = \frac{w_1}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{12} = \frac{w_{12} + w_{123}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{13} = \frac{w_{13} + w_{132}}{w_1 + w_{12} + w_{13} + w_{123} + w_{132}}$$

$$p_{22} = \frac{w_2}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{21} = \frac{w_{12} + w_{132}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{23} = \frac{w_{23} + w_{123}}{w_2 + w_{12} + w_{23} + w_{123} + w_{132}}$$

$$p_{33} = \frac{w_3}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$p_{31} = \frac{w_{13} + w_{123}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$p_{32} = \frac{w_{23} + w_{132}}{w_3 + w_{23} + w_{13} + w_{123} + w_{132}}$$

$$\frac{1}{n}\log \mathbb{P}(\xi) = \frac{1}{n}\log p_{11}^{k_1}p_{22}^{k_2}p_{33}^{k_3}p_{12}^{k_{12}+k_{123}}p_{21}^{k_{12}+k_{132}}p_{13}^{k_{13}+k_{132}}p_{31}^{k_{13}+k_{123}}p_{23}^{k_{23}+k_{123}}p_{23}^{k_{23}+k_{132}}$$

$$= \nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3$$

$$+ (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123})$$

$$+ (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132})$$

$$+ (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132})$$

$$+ (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123})$$

$$+ (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123})$$

$$+ (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132})$$

$$- (\nu_1 + \nu_{12} + \nu_{13} + \nu_{123} + \nu_{132}) \log(w_1 + w_{12} + w_{13} + w_{123} + w_{132})$$

$$- (\nu_2 + \nu_{12} + \nu_{23} + \nu_{123} + \nu_{132}) \log(w_2 + w_{12} + w_{23} + w_{123} + w_{132})$$

$$- (\nu_3 + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132}) \log(w_3 + w_{13} + w_{23} + w_{123} + w_{132})$$

是否会有这样公式？

$$A_1 =$$

$$e^{O(log(n))} \binom{k_1 + k_{12} + k_{13} + k_{123} + k_{132}}{k_1, \; k_{12} + k_{123}, \; k_{13} + k_{132}} \binom{k_2 + k_{23} + k_{12} + k_{123} + k_{132}}{k_2, \; k_{12} + k_{132}, \; k_{23} + k_{123}}$$

$$\binom{k_3 + k_{13} + k_{123} + k_{23} + k_{132}}{k_3, \; k_{13} + k_{123} \; k_{23} + k_{132}}$$

不会的！！！！！！

A_1 表示从状态出发，在 n 步的回路中，使得每种环按照规定频数出现的路径数量。

对于 n 状态的情况:

$$I(\nu) = \sum_{i,j \in I} \left(\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log\left(\frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c}\right)$$

$$= \sum_{i,j \in I} \left(\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c \right) \log\left(\frac{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i,j)=1} \nu_c} \Big/ \frac{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} w_c}{\sum_{c \in \mathcal{C}_\infty, J_c(i)=1} \nu_c}\right)$$

令 $\nu^i = \frac{k_i}{n}$, $i \in I$，即 $\nu^i = \sum_{J_{c_s}(i)=1} \nu_{c_s}$. $\tilde{\nu} = \sum_{c \in \mathcal{C}_\infty} \nu_c$. 其中 $h(x) = x \log x$ 则:

$$\frac{1}{n} \log \mathbb{P}(J^{c_s} = \frac{k_s}{n}, s = 1, 2, \ldots 8) =$$
$$\sum_{i \in I} \left(\nu^i \log \nu^i - (\nu^i - \nu_{(i)}) \log(\nu^i - \nu_{(i)}) \right)$$
$$+ (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) - \sum_{t \in C_\infty} \nu_t \log \nu_t$$
$$+ \mathbb{P}(\xi) + O(\frac{\log n}{n})$$

rate function:

$$I(\nu) = \sum_{i \in I} \left(-\nu^i \log \frac{\nu^i}{w^i} + (\nu^i - \nu_{(i)}) \log(\nu^i - \nu_{(i)}) \right)$$
$$- (\tilde{\nu} - \sum_{i \in I} \nu_i) \log(\tilde{\nu} - \sum_{i \in I} \nu_i) + \sum_{t \in C_\infty} \nu_t \log \nu_t$$
$$- (\nu_1 \log w_1 + \nu_2 \log w_2 + \nu_3 \log w_3)$$
$$- (\nu_{12} + \nu_{123}) \log(w_{12} + w_{123})$$
$$- (\nu_{13} + \nu_{132}) \log(w_{13} + w_{132})$$
$$- (\nu_{12} + \nu_{132}) \log(w_{12} + w_{132})$$
$$- (\nu_{23} + \nu_{123}) \log(w_{23} + w_{123})$$
$$- (\nu_{13} + \nu_{123}) \log(w_{13} + w_{123})$$
$$- (\nu_{23} + \nu_{132}) \log(w_{23} + w_{132}) + O(\frac{\log n}{n})$$

状态空间大小为 4 的马氏链的环流的大偏差

在所有的回路中使得，初始状态为 1，环流 c_t 有 k_t 的路径数量：

$A_1 =$

$$\binom{k_{12} + k_{14} + k_{1234} + k_{1432}}{k_{12}, k_{14}, k_{1234}, k_{1432}}$$

$$\binom{k_1 + k_{12} + k_{14} + k_{1234} + k_{1432}}{k_{12} + k_{14} + k_{1234} + k_{1432}} \binom{k_2 + k_{12} + k_{23} + k_{1234} + k_{1432} - 1}{k_{12} + k_{23} + k_{1234} + k_{1432} - 1}$$

$$\binom{k_3 + k_{23} + k_{34} + k_{1234} + k_{1432} - 1}{k_{23} + k_{34} + k_{1234} + k_{1432} - 1} \binom{k_4 + k_{14} + k_{34} + k_{1234} + k_{1432} - 1}{k_{14} + k_{34} + k_{1234} + k_{1432} - 1}$$

$$\left(\sum_{k_{23}^{12} + k_{23}^{14} = k_{23}} \sum_{k_{34}^{12} + k_{34}^{14} = k_{34}} \binom{k_{23}^{12} + k_{12} + k_{1234} - 1}{k_{23}^{12}} \binom{k_{34}^{12} + k_{1234} + k_{23}^{12} - 1}{k_{34}^{12}} \right.$$

$$\left. \binom{k_{34}^{14} + k_{14} + k_{1432} - 1}{k_{34}^{14}} \binom{k_{23}^{14} + k_{34}^{14} + k_{1432} - 1}{k_{23}^{14}} \right)$$

其中 k_{23}^{12} 和 k_{34}^{12} 分别表示 k_{23}, k_{34} 嵌入到 k_{1234} 和 k_{12}（从 1 出发，第二步为 2 的环）的数量。k_{23}^{14} 和 k_{34}^{14} 分别表示 k_{23}, k_{34} 嵌入到 k_{1432} 和 k_{14}（从 1 出发，第二步为 4 的环）的数量。（$k_{23}^{12}, k_{23}^{14}, k_{34}^{12}, k_{34}^{14} \geq 0$）

由马氏链的常返性，对于足够大的 n，路径中必然包含所有状态，则：

$$\frac{1}{n} A_j \leq A_i \leq n A_j$$

故：

$$\frac{1}{n} \sqrt[4]{A_1, A_2, A_3, A_4} \leq A_i \leq n \sqrt[4]{A_1, A_2, A_3, A_4}$$

从而

$$\frac{1}{n} \log A_1 = \frac{1}{4n} \log(A_1, A_2, A_3, A_4) + O(\frac{\log n}{n}) \tag{1}$$