Lecture 20. The Navier-Stokes equations in two-dimensions

An important application of many of the techniques of this course is to the equations of Fluid Dynamics. The motion of an incompressible Newtonian fluid is defined by its velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ which satisfy the equations

$$\nabla \cdot \mathbf{u} = 0, \tag{20.1}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}, \tag{20.2}$$

where $\mathbf{F}$ is a body force such as gravity g (say). These equations are tremendously important with very wide application. Here ν is a constant parameter, the inverse of the Reynolds number if the variables have been made non-dimensional. It measures the relative strength of diffusion and advection. The character of the equations is very different if ν is large or small because ν multiplies the highest derivative in the equation.

Here, we shall only consider two-dimensional flows, of the form

$$\mathbf{u} = (u(x, y, t),\ v(x, y, t), 0) \qquad p = p(x, y, t). \tag{20.3}$$

Flows of this form can be represented by a single scalar function, $\psi(x, y, t)$. The incompressibility condition (20.1) can be satisfied by writing

$$\mathbf{u} = \nabla \times (0, 0, \psi) \qquad u = \psi_y, \quad v = -\psi_x. \tag{20.4}$$

Then the vorticity, $\nabla \times \mathbf{u} = (0, 0, \omega)$ where

$$\omega = -\nabla^2 \psi. \tag{20.5}$$

Now taking the curl of (20.2), we can eliminate the pressure field. The resulting vorticity equation only has a z-component, which is

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + G, \tag{20.6}$$

where $G = \hat{z} \cdot \nabla \times \mathbf{F}$. We could if we wish substitute for ω to obtain a single equation in ψ.

We see that ν measures the relative importance of diffusion to advection. We also note that ν multiplies the highest derivative in the equation, so we require an extra boundary condition when $\nu \neq 0$.

What boundary conditions shall we impose? As we have not included an external force, we will only get motion if it is driven by a moving boundary. A typical problem is the Driven Cavity flow. Suppose we have a rectangle, $0 < x < 1$ and $0 > y > -L$. On the three walls $x = 0$, $x = 1$ and $y = -L$ we impose that the velocity is zero, $\psi_x = \psi_y = 0$. On the top wall $y = 0$, we impose a constant horizontal flow, so that $\psi_y = 1$ on $y = 0$, but otherwise ψ, ψ_x and ψ_y are all zero on the boundary. We could envisage ourselves caught in a storm. We dig ourselves a hole to sit in to shelter from the driving wind. But does the flow penetrate into the hole?

Nearly inviscid flow, $\nu \to 0$, High Reynolds Number

If ν is small, we are nearly at the inviscid Euler flow we considered last lecture, where ω was advected around, being stretched but not dissipated. If anything, adding a small amount of diffusion would probably make the flow better behaved, by preventing high gradients from developing. If ν is small, we might consider using an explicit representation of the diffusion, as the stability constraint $\nu k/h^2 < 1/2$ might not be too severe.

Stokes flow: $\nu \to \infty$, Low Reynolds Number

If the flow is very viscous, or slow, the parameter $\nu >> 1$. In these circumstances the equations become

$$\nabla^2\omega = 0, \qquad \nabla^2\psi = -\omega. \tag{20.7}$$

This type of flow is known as Stokes flow or creeping flow which is of importance in many applications, such as the flow in MEMS (micro-electromechanical systems), flow in polymers and material processing, and flow in and around micro-biological systems. Inertial effects in these systems can often be ignored leading to a simplified set of equations. How might we solve this equation numerically? Those two Laplacians look very tempting. Can we devise a multigrid code? Effectively, we are solving the **biharmonic equation** $\nabla^2(\nabla^2\psi) = 0$.

We have a slight difficulty. We are missing a boundary condition on ω but instead we have two conditions on ψ. How can we nevertheless use two multigrid processes?

Suppose we knew ω. Then we could clearly calculate ψ using the Dirichlet boundary condition $\psi = 0$ only. The solution would not in general satisfy the Neumann boundary condition. However, if we could use it and the newly found ψ to define a boundary condition on ω, we would have the basis of an iterative scheme.

Suppose at $x = 0$ we have $\psi = 0$ and $\psi_x = f(y)$. Thus one step away from the wall, we have $\psi(h) = h\psi_x + \frac{1}{2}h^2\psi_{xx}(0)$. Now $-\omega = \psi_{xx} + \psi_{yy} = \psi_{xx}$ on $x = 0$. We can therefore identify

$$\omega(0, y) = \frac{2f(y)}{h} - \frac{2\psi(h, y)}{h^2} + O(h), \tag{20.8}$$

So we can envisage an iterative scheme where we use ψ to find ω and then ω to find ψ. In place of (20.8) we could derive an $O(h^2)$ approximation using two points.

$$\begin{aligned} \psi(h) &= h\psi' + \tfrac{1}{2}h^2\psi'' + \tfrac{1}{6}h^3\psi''' + O(h^4) \\ \psi(2h) &= 2h\psi' + 2h^2\psi'' + \tfrac{4}{3}h^3\psi''' + O(h^4) \end{aligned} \tag{20.9}$$

Eliminating h''' we have $8\psi(h) - \psi(2h) = 6h\psi' + 2h^2\psi'' + O(h^4)$ and so

$$\omega(0, y) = \frac{3f(y)}{h} - \frac{8\psi(h, y) - \psi(2h, y)}{2h^2} + O(h^2), \tag{20.10}$$

Using such schemes we can solve Dirichlet problems for ω and ψ in turn.

The general case – intermediate Reynolds number

If both advection and diffusion are important, we would like somehow to marry our ideas for dealing with each process. Ideally, we would like to treat the advection explicitly but the diffusion implicitly, hopefully using a multigrid algorithm. One idea we might try is an **Operator Splitting** approach. In the first part of the algorithm we advect ω ignoring the diffusion, while in the 2nd part we diffuse the new ω.

Rayleigh-Benard Convection

An interesting fluid dynamical is known as **Convection**. Convection occurs because fluids expand when heated. Their density decreases and they become buoyant, in accordance with Archimedes' principle. If T denotes the excess temperature The body force $\mathbf{F} = -\beta T\mathbf{g}$ where $\mathbf{g}$ is the gravitational acceleration and β is the coefficient of expansion. Gravity can then drive fluid motion, which then advects heat around. Suitably non-dimensionalised, the governing equations for the temperature T and velocity $\mathbf{u}$ are

$$\left.\begin{aligned} T_t + \mathbf{u}\cdot\nabla T &= \nabla^2 T \\ P^{-1}\left(\omega_t + \mathbf{u}\cdot\nabla\omega\right) &= RT_x + \nabla^2\omega \\ \omega &= -\nabla^2\psi. \end{aligned}\right\} \tag{20.11}$$

There are two parameters in the problem: The Prandtl number, $P = \nu/\kappa$ is the ratio of the fluid kinematic viscosity to the thermal diffusivity. The Rayleigh number R is a measure of the thermal driving force. In deriving (20.11), it is assumed that the density decreases linearly with temperature, and that the fluid speed is much less than the speed of sound, permitting the Boussinesq approximation. Gravity acts in the y-direction.

The problem is known as Rayleigh-Benard convection. The equations above model a variety of flow with industrial relevance such as ventilation of rooms, flow in solar energy collectors, crystal growth in liquids, cooling of electronic parts or flow in a heat exchanger. For example, a typical example would be that of applying uniform heat to the bottom of a box (or the computational domain) while keeping the top and side walls at a constant and lower temperature. For small temperature gradients, measured by the Rayleigh number R, the velocity field is zero and the thermal equilibrium is given by the conductive state. There is a purely conductive solution, with $\mathbf{u} = 0$ and $T = T(y)$. What we expect to happen is that for $R \leq R_c$ the only solution is the conductive solution, but for $R > R_c$ this solution is unstable, and a flow (convection) develops. We want to investigate the critical value of R, and the convection patterns formed for different values of P, R. As the temperature gradient, or Rayleigh number, is increased beyond a critical value, a circular motion sets in that transports hot fluid from the lower wall in exchange for cooler fluid from the top wall. After a transient period, a steady number of rolls appear. As the Rayleigh number is increased further, the initially two rolls split and four rolls appear, with additional bifurcations observed on increasingly higher Rayleigh numbers.

Our solution plan is to use operator splitting. If T, ψ and ω are known at time t, then we:

1. Advect T;
2. Diffuse T;
3. Advect ω;

4. Diffuse ω;

5. Find new ψ and hence new velocity;

6. Derive new boundary condition on ω from new ψ;

7. Repeat.

The equations (20.11) written out explicitly, for a slightly more generalised form is :

$$\left.\begin{aligned} T_t + uT_x + vT_y &= T_{xx} + T_{yy} \\ \omega_t + u\omega_x + v\omega_y &= P\left(\omega_{xx} + \omega_{yy}\right) + RP\left(T_x \cos\phi + T_y \sin\phi\right) \\ \psi_{xx} + \psi_{yy} &= -\omega, \end{aligned}\right\} \tag{20.12}$$

with ϕ representing an inclination angle, with (R, P) defined as follows:

$$R = \frac{g\beta D^3(T_h - T_c)}{\nu\alpha}, \qquad P = \frac{\nu}{\alpha},$$

where D is a characteristic box size, β the thermal expansion coefficient, (T_c, T_h) temperatures of the cold and hot wall respectively, α is the thermal diffusivity and g the gravity.