

Lecture 16. The Lax-Wendroff method and conservation equations

One explicit method which attempts to get the second order terms correct is due to Lax and Wendroff. We know that

$$\frac{u_n^{j+1} - u_n^j}{k} = [u_t + \frac{1}{2}ku_{tt}]_n^j + O(k^2) .$$

So if $u_t = -cu_x$ with c constant, then $u_{tt} = -cu_{xt} = c^2u_{xx}$, and we may write

$$\frac{u_n^{j+1} - u_n^j}{k} = -c \frac{\Delta u_n^j}{2h} + \frac{kc^2}{2h^2} \delta^2 u_n^j + O(k^2, h^2)$$

This leads to the centred scheme (**Scheme F**) on lectures sheets 14-15.

$$U_n^{j+1} = U_n^j - \frac{1}{2}q\Delta U_n^j + \frac{1}{2}q^2\delta^2 U_n^j \quad (16.1)$$

Note that this formula requires c to be constant. However, a similar result can be derived for the **conservation** equation $u_t + [f(u, x)]_x = 0$. In general, an equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \{\mathbf{b}\} = Q$$

can be interpreted as a conservation law for a physical quantity of density ρ , for which Q is a source and \mathbf{b} is its *flux*. Over any fixed volume V bounded by the surface ∂V , normal $\hat{\mathbf{n}}$,

$$\frac{d}{dt} \int_V \rho dV = \int_V Q dV - \int_{\partial V} \mathbf{b} \cdot \hat{\mathbf{n}} dS ,$$

so that the rate of increase of the physical quantity in V is equal to the total rate of generation within V less the amount that flows out of V across the boundary.

Lax-Wendroff schemes for simultaneous conservation equations.

More generally, let \mathbf{u} and $\mathbf{f}[\mathbf{u}]$ be p -vectors satisfying

$$\mathbf{u}_t + (\mathbf{f}[\mathbf{u}])_x = 0. \quad (16.2)$$

Equation (16.2) represents the conservation of p physical quantities. Ideally, we would like our FDM to conserve them also. For example, we could write

$$\mathbf{U}_n^{j+1} - \mathbf{U}_n^j = -\frac{1}{2}s\Delta \mathbf{F}_n^j \quad \text{where} \quad \mathbf{F}_n^j = \mathbf{f}[\mathbf{U}_n^j] \quad \text{and} \quad s = k/h . \quad (16.3)$$

This scheme is **conservative** because if we sum over all values of n the right-hand-side all cancels (apart possibly from boundary terms if we are on a finite region of x), so that $\sum_n \mathbf{U}_n^{j+1} = \sum_n \mathbf{U}_n^j$. However, we have seen that (16.3) may be unstable. Alternatively we might rewrite (16.2) in the form

$$(u_l)_t + \sum_{m=1}^p A_{lm}(u_m)_x = 0 \quad \text{for} \quad l = 1 \dots p \quad \text{where} \quad A_{lm} = \frac{\partial(f_l)}{\partial(u_m)} , \quad (16.4)$$

and u_m is the m -th component of \mathbf{u} . If $A[\mathbf{u}]$ is the matrix whose (l, m) th element is A_{lm} , we could then approximate

$$\mathbf{U}_n^{j+1} - \mathbf{U}_n^j = -\frac{1}{2}sA(\Delta \mathbf{U}_n^j) \quad (16.5)$$

but this would **not** be conservative unless A is constant. The Lax-Wendroff idea applied to (16.2) gives

$$\mathbf{u}_{tt} = -(\mathbf{f}[\mathbf{u}])_{xt} = -(A\mathbf{u}_t)_x = (A(\mathbf{f})_x)_x$$

so that

$$\left. \begin{aligned} (\mathbf{u}_{tt})_n^j &= \frac{1}{h} \left[A_{n+1/2}^j (\mathbf{f}_x)_{n+1/2}^j - A_{n-1/2}^j (\mathbf{f}_x)_{n-1/2}^j \right] + O(h^2) \\ &= \frac{1}{h^2} \left[A_{n+1/2}^j (\mathbf{f}_{n+1}^j - \mathbf{f}_n^j) - A_{n-1/2}^j (\mathbf{f}_n^j - \mathbf{f}_{n-1}^j) \right] + O(h^2). \end{aligned} \right\} \quad (16.6)$$

In the above, to the same order of accuracy we can approximate

$$A_{n+1/2}^j \equiv A[\mathbf{u}_{n+1/2}^j] = A \left[\frac{1}{2}(\mathbf{u}_{n+1}^j + \mathbf{u}_n^j) \right] + O(h^2) .$$

Using the estimate (16.6) for \mathbf{u}_{tt} , we obtain

$$\mathbf{U}_n^{j+1} = \mathbf{U}_n^j - \frac{1}{2}s\Delta\mathbf{F}_n^j + \frac{1}{2}s^2 \left[A_{n+1/2}^j (\mathbf{F}_{n+1}^j - \mathbf{F}_n^j) - A_{n-1/2}^j (\mathbf{F}_n^j - \mathbf{F}_{n-1}^j) \right] . \quad (16.7)$$

However while (16.7) is a second order scheme, it is still non-conservative if A depends on \mathbf{u} , and moreover it is necessary to calculate $A_{n+1/2}^j$. A superior two-step version of the Lax-Wendroff scheme is due to Richtmyer. The system

$$\left. \begin{aligned} \mathbf{U}_{n+1/2}^{j+1/2} &= \frac{1}{2} (\mathbf{U}_n^j + \mathbf{U}_{n+1}^j) - \frac{1}{2}s (\mathbf{F}_{n+1}^j - \mathbf{F}_n^j) \\ \mathbf{U}_n^{j+1} &= \mathbf{U}_n^j - s (\mathbf{F}_{n+1/2}^{j+1/2} - \mathbf{F}_{n-1/2}^{j+1/2}) \end{aligned} \right\} \quad (16.8)$$

has truncation error $O(k^2 + h^2)$, is **conservative** and stable if the Courant condition holds. When A is constant, so that $\mathbf{F} = A\mathbf{U}$, it reduces to (16.7). For the one-dimensional case $p = 1$, with $f = cu$, we obtain the two-step scheme (**Scheme D**) with a different step-size.

The two-step Lax-Wendroff scheme (16.8) may be extended to two space dimensions without difficulty. Consider the equation for $\mathbf{u}(x, y, t)$

$$\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = 0 \quad \text{and take} \quad h_x = h_y = h .$$

We define \mathbf{U} at the half time-levels for $(p, q) = (m, n + 1/2)$ or $(m + 1/2, n)$ by

$$\mathbf{U}_{pq}^{j+1/2} = \frac{1}{4} [\mathbf{U}_{p+,q}^j + \mathbf{U}_{p-,q}^j + \mathbf{U}_{p,q+}^j + \mathbf{U}_{p,q-}^j] - \frac{1}{2}s [\mathbf{F}_{p+,q}^j - \mathbf{F}_{p-,q}^j + \mathbf{G}_{p,q+}^j - \mathbf{G}_{p,q-}^j] ,$$

where $p\pm$ denotes $p \pm 1/2$ and similarly $q\pm$. At the integer time levels, for $(p, q) = (m, n)$ or $(m + 1/2, n + 1/2)$,

$$\mathbf{U}_{pq}^{j+1} = \mathbf{U}_{pq}^j - s \left[\mathbf{F}_{p+,q}^{j+1/2} - \mathbf{F}_{p-,q}^{j+1/2} + \mathbf{G}_{p,q+}^{j+1/2} - \mathbf{G}_{p,q-}^{j+1/2} \right] . \quad (16.9)$$

The resulting scheme is second order and conservative.