**Computational Partial Differential Equations 2020-2021**
**MATH96021/MATH97030/MATH97138**
*Released : 19 March 2021*
*The project mark, will be weighted to comprise 30% of the overall Module.*

You are required to investigate the problem below and summarıse your findings in form of a well written project report – on which you will be assessed.

*Please name your files in following way:*

- Technical report : **Proj3_CID.pdf** – limit your report to 20 pages or less (including plots). **Anything beyond the 20 page limit will NOT be marked!**
- All your code(s), label as follows :
  **Proj3_part_1of_X_CID.m** (Matlab scripts example) or
  **Proj3_part_1of_X_CID.py** (Python scripts).
- Zip all program files and your technical report **Proj3_CID.pdf** into one zipped file **Proj3_CID.zip** and upload onto Blackboard.

Where in the above **CID** will be your College ID number.

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**Notes:**

1. Marking will consider both the correctness of your code as well as the soundness of your analysis *and* clarity and legibility of the technical report.
2. All figures created by your code should be well-made and properly labelled.
3. In order to assign partial credit, comment/annotate your matlab (or Python) scripts to indicate steps being undertaken or what is being attempted (SHORT COMMENTS!).
4. You are allowed to discuss general aspects of Matlab/Python with each other, however you are trusted not to discuss your code or analysis with other students.

Dr M. S. Mughal
19 March, 2021

## CW 3: Hyperbolic Systems

**Part A: (20 Marks)**

The one-dimensional wave equation for $u(x,t)$ is given by

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where $c$ is a positive valued wave speed, $x$ represents a spatial coordinate and $t$ the time. At $t = 0$

$$u(x,0) = \cos(\frac{\pi x}{2\delta}) \; ; \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{for } \; (-\delta \leq x \leq \delta); \tag{2}$$

while elsewhere, i.e. $x > \pm\delta$ and $t = 0$, you may assume $u(x,0) = \partial u/\partial t = 0$. You are required to solve this equation numerically for the domain $x \leq \pm 2$, where $\delta$ is an arbitrary parameter with values $\delta = (0.01,\ 0.1,\ 0.5)$.

1. You are required to investigate how the solution to Eqn. 1 evolves for $t > 0$ and $c = 4$. Discretise the equation based on your lecture notes, such that it is second-order accurate in time and space using the so-called Leapfrog scheme:

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = c^2 \left( \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{(\Delta x)^2} \right). \tag{3}$$

At the computational end points, namely $x = \pm 2$, investigate the treatment of boundary conditions which satisfy the following:

(a) Minimal numerical reflections off the outer boundaries; i.e. the waves pass through the boundaries.

(b) The solid wall condition.

Through appropriate numerical experiments investigate, discuss and demonstrate the accuracy of your numerical solution, and any dissipation and dispersive effects that you observe through appropriate plots, showing key features. You should include a discussion on the "modified PDE" that Eqn.(3) represents, and on the optimal choice of discretisation parameters giving you the most exact solution.

Undertake a discrete dissipation-dispersion analysis to explain your numerical observations.

**Part B: (10 Marks)**
The two-dimensional (2D) wave equation for $u(x, y, t)$ is given by

$$u_{tt} = u_{xx} + u_{yy}, \tag{4}$$

where $(x, y)$ represent spatial coordinates. At $t = 0$

$$u(x, y, 0) = \cos(\frac{\pi r}{2\delta}) \ ; \quad \frac{\partial u}{\partial t}(x, y, 0) = 0, \quad \text{for } (r \leq \pm\delta), \tag{5}$$

where $r = \sqrt{x^2 + (y - y_o)^2}$. Elsewhere, i.e. $r > \pm\delta$ and $t = 0$, you may assume $u(x, y, 0) = \partial u/\partial t = 0$. You are required to solve this equation with the 2D version of the method used in Part A, for the domain $(x, y) \leq \pm 2$, where $\delta = 0.2$ and $y_o = 0.5$ only (see Figure 1). A variable in the problem is the gap width $\pm d$ shown in the figure.

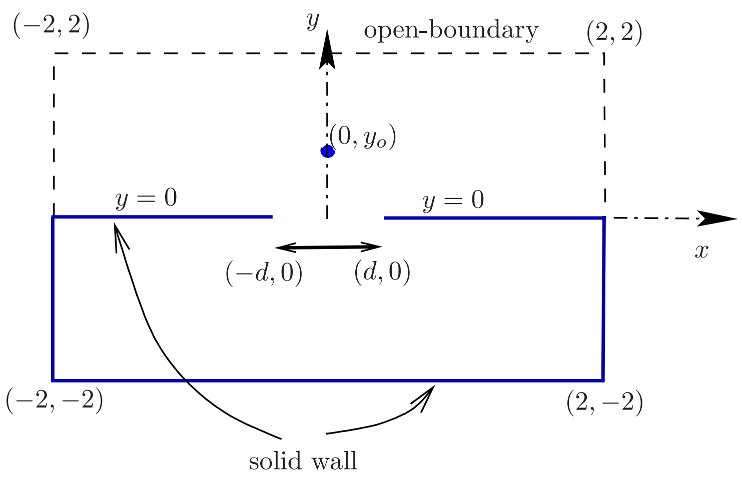


**Figure 1:** Problem description.

At the open boundary (given by the dashed lines), the wave solutions pass through the computational boundaries with zero reflection. On the blue solid lines, a solid wall condition is to be applied.

1. Setting $d = 1/2$ discretise the equation and compute solutions for large enough time such that the wave structure convects through the open boundaries of your computational domain. Show with appropriate plots how your solution evolves with time along the $(x, y = 0.2)$, $(x, y = -0.2)$ and $(x = 0, y)$ data planes. The plots should include sufficient detail and resolution of whether your treatment of the boundary-conditions is effective. Your report should describe the precise details of your finite-difference discretisation at the boundaries.

2. Investigate the effect on your solution accuracy for values of $d = (0.2, 0.1)$ and discuss the results that you compute. You should report

on any additional issues which arise as $d$ varies which affects accuracy of your numerical result and steps you undertake to overcome these. These should be by appropriate plots (contour plots are instructive) and a discussion of the main physical effects your code predicts as $d$ is varied,

For both Parts A and B, you should report how you ascertain that your results are grid independent (*subject to some computational tolerance you specify*).