Lecture 21. The 2D Wave Equation

We return now to pure Hyperbolic equations, such as the one-dimensional wave equation for $u(x,t)$,

$$u_{tt} = c^2 u_{xx} \qquad \text{with } u(x,0) \text{ and } u_t(x,0) \text{ known.} \tag{21.1}$$

We know we could write this as two first order equations, but suppose we treat it directly, introducing a grid (h,k) and seek an approximation U_n^j to $u(nh,jk)$ in the familiar way. Using explicit centred differences in time and space we have the two-step second-order accurate in space and time scheme

$$U_n^{j+1} - 2U_n^j + U_n^{j-1} = q^2\left(U_{n+1}^j - 2U_n^j + U_{n-1}^j\right), \qquad \text{where } q = \frac{ck}{h}. \tag{21.2}$$

The scheme is often referred to as the **leapfrog scheme**.

It being a second order equation in t, we need two initial conditions on U_n^0 and U_n^1 and then we can easily step to find U_n^j . We analyse the stability with three Fourier methods, looking at solutions $U_n^j = (\lambda)^j \exp(in\xi)$ and we find

$$\lambda^2 - 2\lambda + 1 = \lambda q^2(2\cos\xi - 2) \tag{21.3}$$

or

$$\lambda^2 - \left(2 - 4q^2\sin^2(\frac{1}{2}\xi)\right)\lambda + 1 = 0. \tag{21.4}$$

We need $|\lambda| \leq 1$ for stability. Now without calculating λ explicitly, we can see from the constant term that the two roots have product 1. If the roots are real and distinct this means that one of the roots will have modulus greater than 1, and so the scheme will be unstable. Thus we require complex or double roots, or

$$\left(2 - 4q^2\sin^2(\frac{1}{2}\xi)\right)^2 \leq 4 \quad \rightarrow \quad -2 \leq 2 - 4q^2\sin^2(\frac{1}{2}\xi) \leq 2. \tag{21.5}$$

As usual, the worst case is $\xi = \pi$ and we require the Courant-Friedrichs-Lewy condition $q \leq 1$ for stability, as we might have expected.

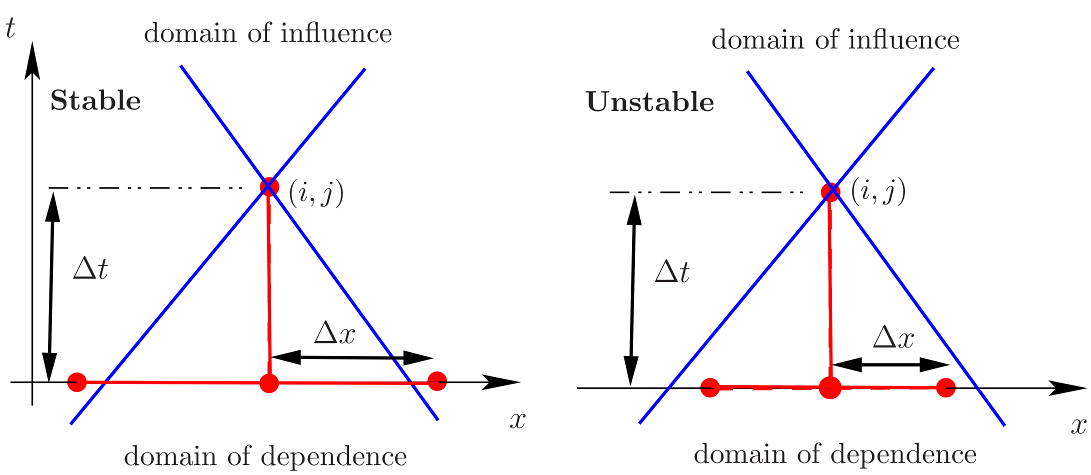

Figure 21.1: Domains of dependence and influnce, CFL-condition. Stable configuration (left), unstable configuration (right).

The scheme (21.2) is conservative and models left and right travelling waves, $u = f(x-ct)+g(x+ct)$. If we have a boundary, say at $x=0$, where we impose $u=0$, then we must have $f=g$. Thus if a leftwards travelling wave encounters such a boundary, it will reflect, giving rise to a rightwards travelling wave. So if we want to have a **non-reflecting boundary**, we would need to impose the condition

$$cu_x = +u_t \qquad \text{on } \; x=0. \tag{21.6}$$

This is easy enough to implement. We can combine (21.2) on the boundary $n=0$ with

$$q(U_1^j - U_{-1}^j) = U_n^{j+1} - U_n^{j-1} \tag{21.7}$$

to eliminate the ghost points at $n=-1$, keeping 2nd order accuracy. Can we extend this scheme to two and three dimensions?

2D Wave equation: non reflecting boundary conditions

Suppose now $u(x,y,t)$ satisfies

$$u_{tt} = c^2\nabla^2 u = c^2(u_{xx}+u_{yy}). \tag{21.8}$$

We can model this explicitly on a rectangular grid (h_x, h_y) by

$$\frac{U_{mn}^{j+1} - 2U_{mn}^j + U_{mn}^{j-1}}{c^2k^2} = \frac{1}{h_x^2}\delta_x^2 U_{mn}^j + \frac{1}{h_y^2}\delta_y^2 U_{mn}^j. \tag{21.9}$$

For simplicity, we take $h_x = h_y = h$. We analyse the stability with $U_{mn}^j = (\lambda)^j \exp(im\xi + in\eta)$ and as before we find stability provided λ is complex for all ξ and η, which requires $q \le 1/\sqrt{2}$. Similarly, in 3D, the CFL condition would be $q \le 1/\sqrt{3}$.

In 2 or 3 dimensions it is quite likely that we will want to include artificial boundaries to the computational domain which do not reflect waves, but this is harder to do. Suppose we are in $y>0$ with a boundary at $y=0$. Taking Fourier transforms, the general solution to (21.1) can be written as a superposition of the waves $\exp(ilx+imy+i\omega t)$, where l, m and ω are related by

$$c^2(l^2+m^2) = \omega^2. \tag{21.10}$$

On $y=0$, we wish to impose that only waves travelling in the negative y direction are permitted, in other words, assuming $\omega > 0$ then we must have $m>0$, so that

$$mc = +\sqrt{\omega^2 - c^2l^2}. \tag{21.11}$$

The 1-D condition (21.6), is obtained by putting $l=0$ in this formula, but this is not physically correct for waves which hit the boundary obliquely. The exact condition is difficult to implement because of the square root. A better approximation for small cl/ω is obtainable by expanding (21.11) to next order:

$$mc = \omega - \frac{c^2l^2}{2\omega}. \tag{21.12}$$

This is equivalent to the unusual boundary condition

$$cu_{yt} = u_{tt} - \frac{1}{2}c^2u_{xx} \qquad \text{on } \; y=0. \tag{21.13}$$

This can be implemented on the boundary and greatly reduces unwanted reflections. But still it is imperfect. An important technique for avoiding unphysical reflections is known as **PML**, which stands for **Perfectly Matched Layer**.