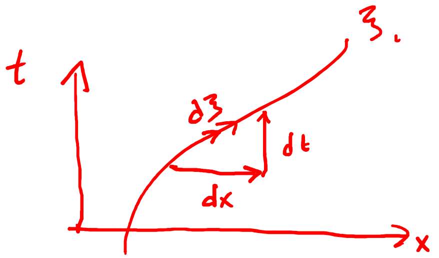
M3N10/M4N10/M5N10: 2020-2021

Computational Partial Differential Equations (CPDEs : 2020-2021)

Lecture 13-14-15 : Hyperbolic PDEs (part a)

Consider the linear advection equation for *u(x,t)* :

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{for} \quad t > 0, \quad \text{in} \quad u(x,0) = f(x).$$



where c is a **positive** constant. The characteristics of this equation are $dx/dt = c$ or $x - ct =$ constant. Furthermore, u is constant along each characteristic and so the solution is $u = f(x - ct)$. This represents a wave travelling in the positive x-direction with speed c, **without change of size or shape** as shown in Figure 14.1 – we emphasis that the exact solution of (14.1) says that the initial function $f(x)$ propagates unaltered in form in the positive x-direction as time progresses.

$$\frac{du}{d\xi} = \frac{\partial u}{\partial t}\frac{dt}{d\xi} + \frac{\partial u}{\partial x}\cdot\frac{dx}{d\xi} = 0$$

equivalent: $\frac{dt}{d\xi} = 1$; $\frac{dx}{d\xi} = c$ $\Rightarrow \frac{dx}{dt} = c$.

$$\frac{du}{d\xi} = 0 \leftarrow$$

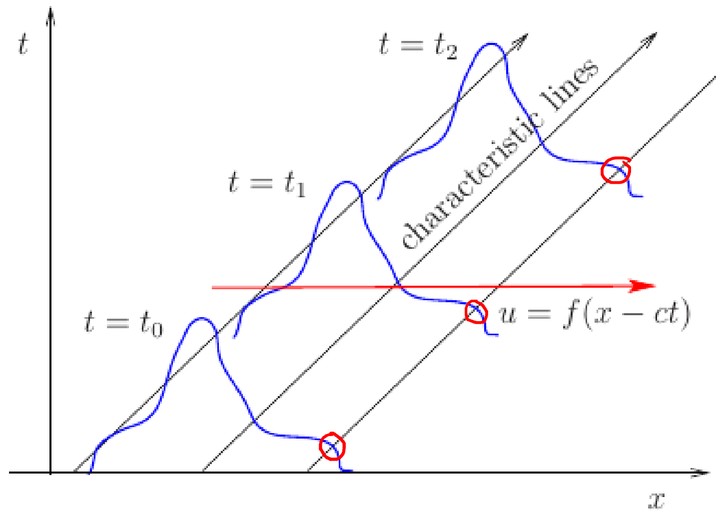
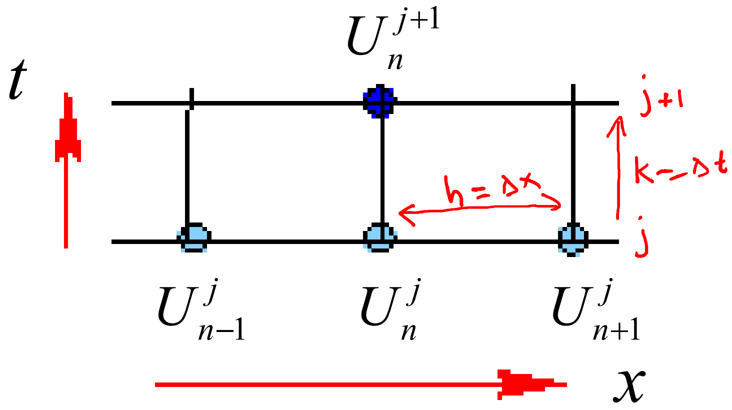


Figure 14.1: Propagation of a function $u = f(x)$ with time t.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{for} \quad t > 0, \quad \text{in} \quad u(x,0) = f(x).$$

$$\frac{\partial u}{\partial t} \approx \frac{U_n^{j+1} - U_n^{j}}{k} + O(k)$$



A key parameter:
Courant number, or
Courant-Friedrichs-Lewy (CFL) number:

$$\text{CFL} = \frac{c\Delta t}{\Delta x} = \frac{ck}{h} = \nu.$$

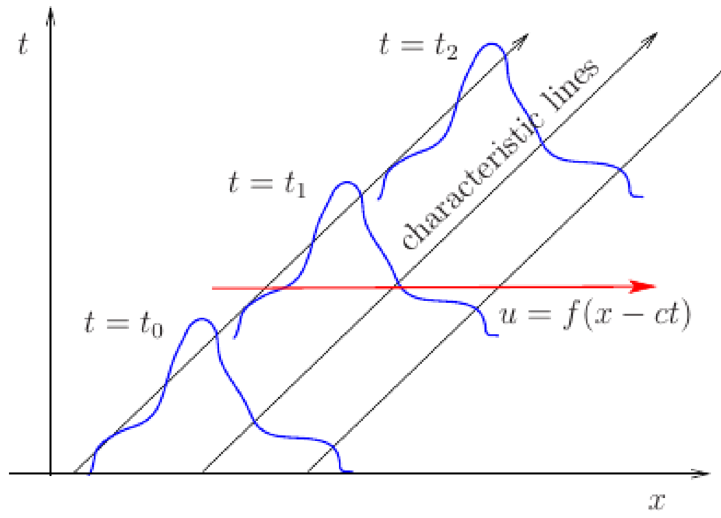


Figure 14.1: Propagation of a function $u = f(x)$ with time t.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

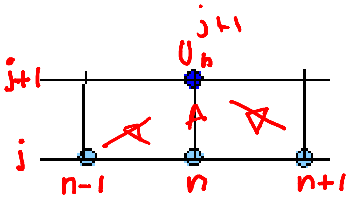
$$A: \quad \frac{U_n^{j+1} - U_n^j}{k} = -\frac{c}{2h}\left\{ U_{n+1}^j - U_{n-1}^j \right\}$$

$$\boxed{q = ck/h}$$

The real solution therefore has a growth factor $\lambda = \exp(-i\xi q)$. We note that $|\lambda| = 1$ so that there is no growth or decay. We can model (14.1) in many different ways. We shall use a forward difference for u_t and consider 6 schemes (**Schemes-6**) :

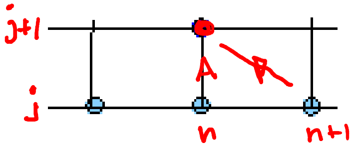
$$\frac{U_n^{j+1} - U_n^j}{-q} = \begin{cases} \mathbf{A}: & \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \mathbf{B}: & U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \mathbf{C}: & U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \mathbf{D}: & \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \mathbf{E}: & \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \mathbf{F}: & \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

$$A: \quad U_n^{j+1} = U_n^j + \frac{q}{2}(U_{n-1}^j - U_{n+1}^j)$$



Explicit, centred. $\sim O(h^2)$.

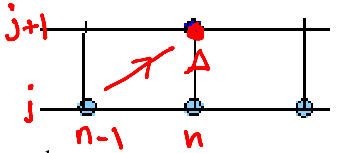
$$B: \quad U_n^{j+1} = U_n^j - q(U_{n+1}^j - U_n^j)$$



Explicit, forward Scheme

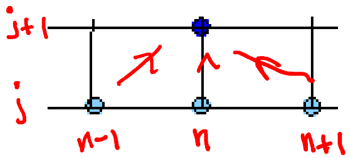
$O(h)$

$$C: \quad U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$$

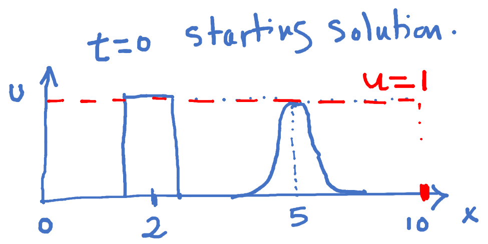


"Up-Winded" explicit.

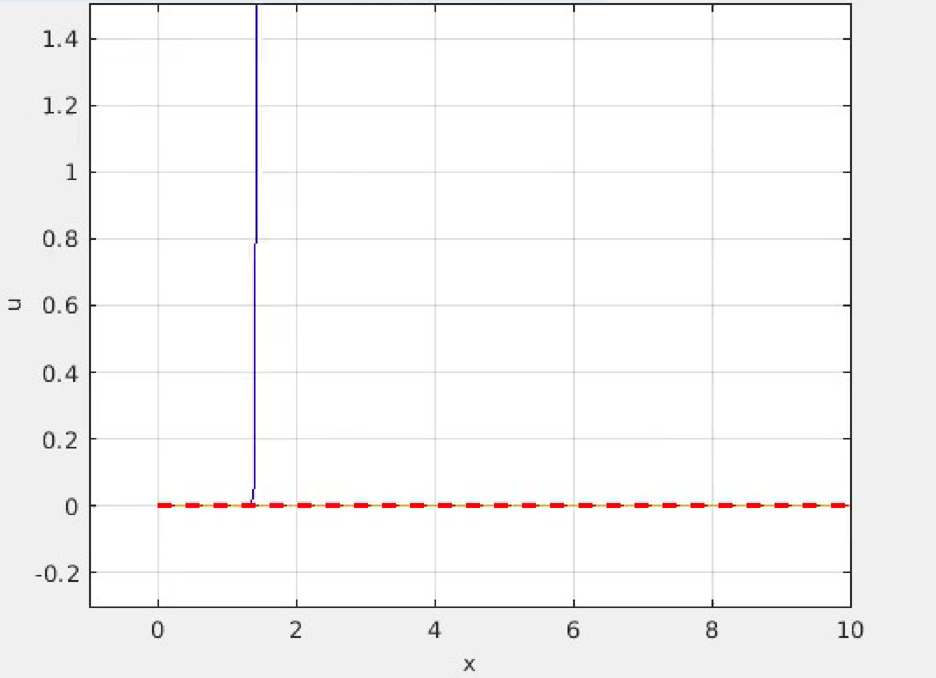
$$F: \quad U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$



$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,$$



Expected solution with time :



The real solution therefore has a growth factor $\lambda = \exp(-i\xi q)$. We note that $|\lambda| = 1$ so that there is no growth or decay. We can model (14.1) in many different ways. We shall use a forward difference for u_t and consider 6 schemes (**Schemes-6**) :

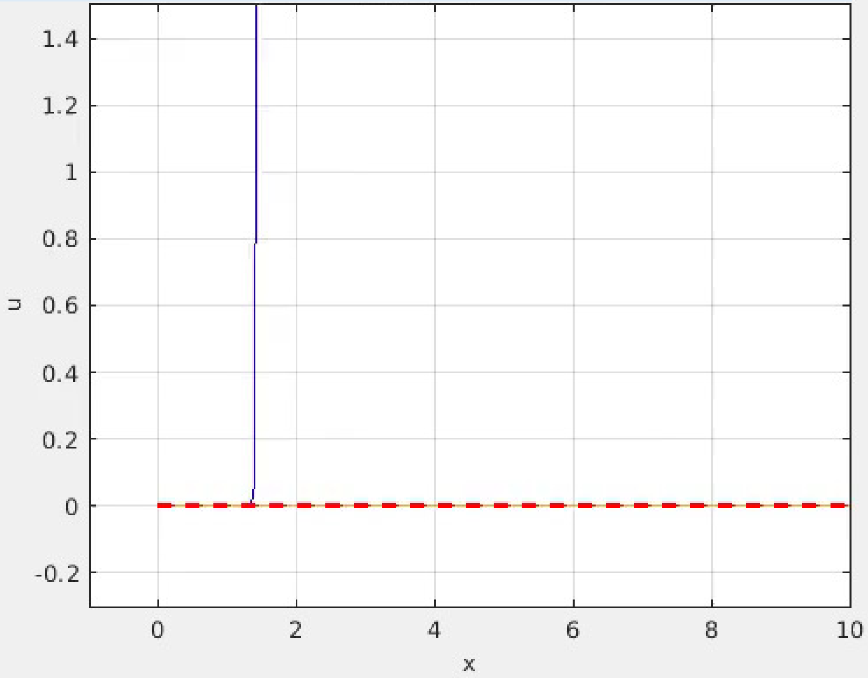
$$\frac{U_n^{j+1}-U_n^j}{-q} = \begin{cases} \mathbf{A:} & \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \mathbf{B:} & U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \mathbf{C:} & U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \mathbf{D:} & \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \mathbf{E:} & \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \mathbf{F:} & \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

CFL = q = (k c/h) = 1.0

C: $U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$

Case C: CFL=1.0



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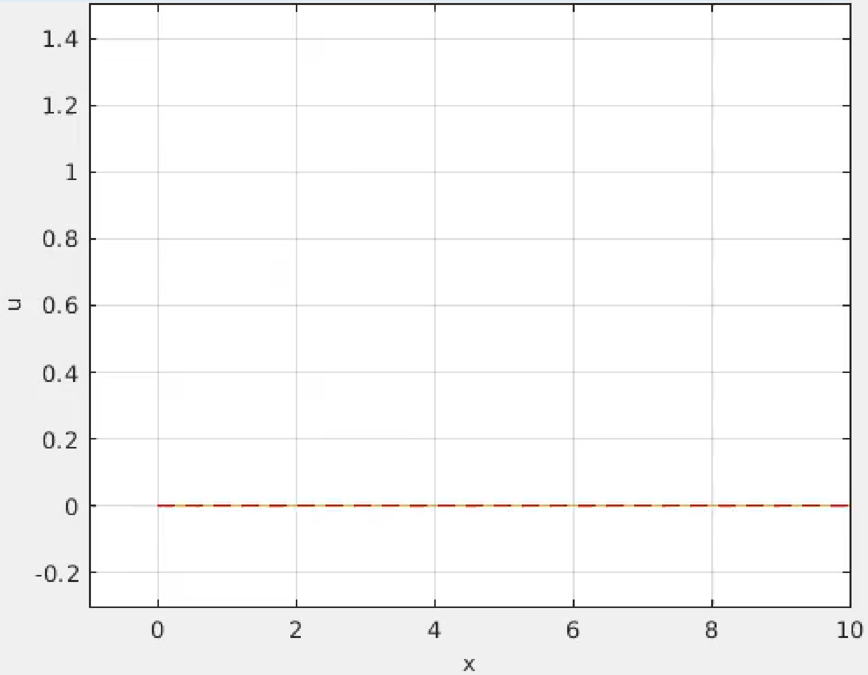
$$\text{F:}\quad U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$

Case F Lax-Wendroff : CFL=1.0



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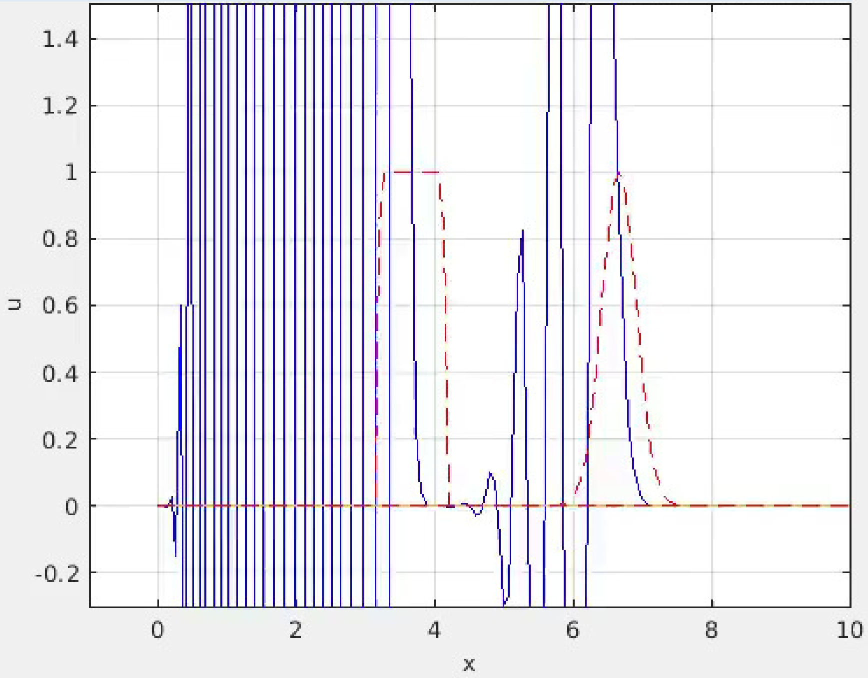
Case A: CFL=1.0



$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,$$

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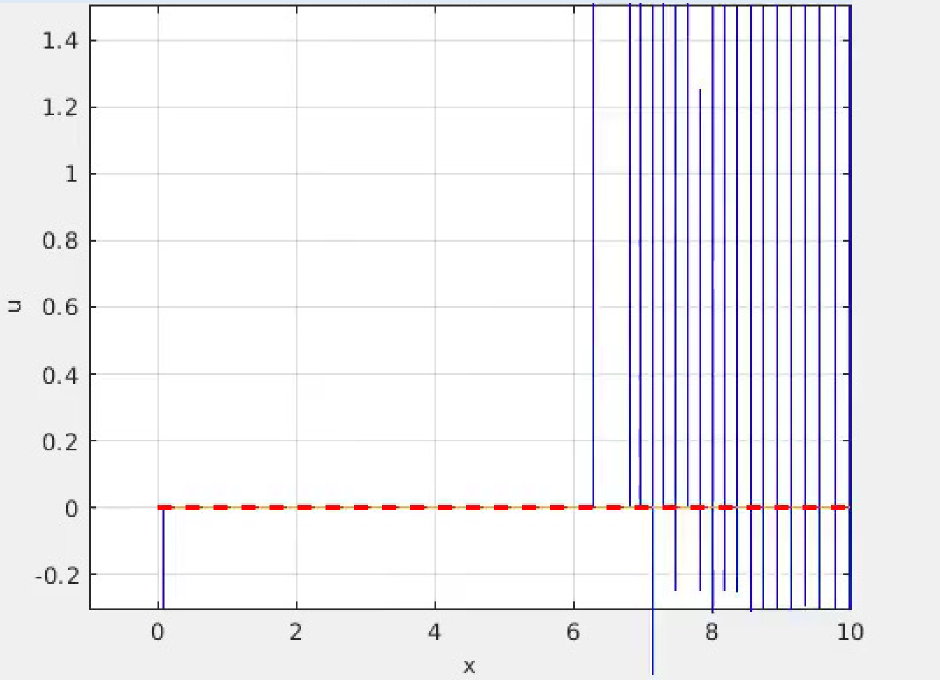
B: $U_n^{j+1} = U_n^j - q(U_{n+1}^j - U_n^j)$

Case B: CFL=1.0



Non-optimal CFL number, CFL > 1

Case C: upwinded, **CFL= 1.05**

$$\text{C:}\;\; U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$$

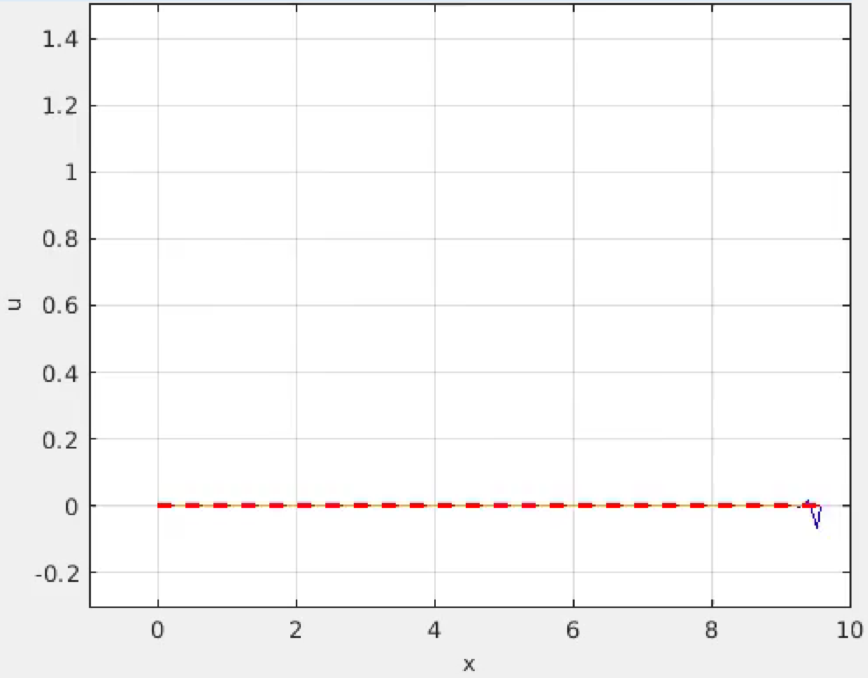
$$N = 150$$

$$\Delta x = 10/150$$

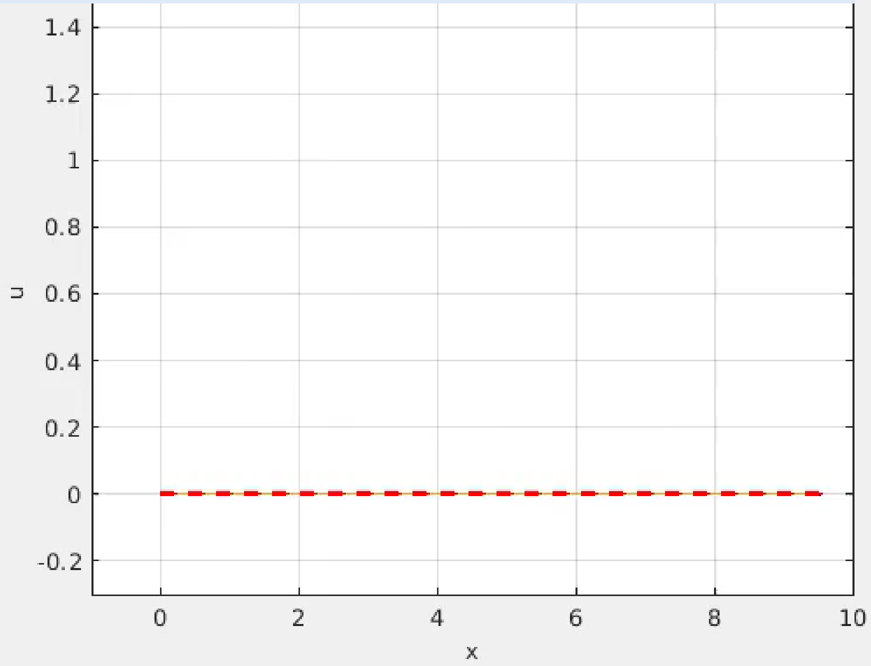
Case F: Lax-Wendroff, **CFL= 1.05**

$$\text{F:}\;\; U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$

Case C: CFL=1.05



Case F Lax-Wendroff : CFL=1.05



Non-optimal CFL number, CFL < 1

Case C: upwinded, **CFL= 0.95**

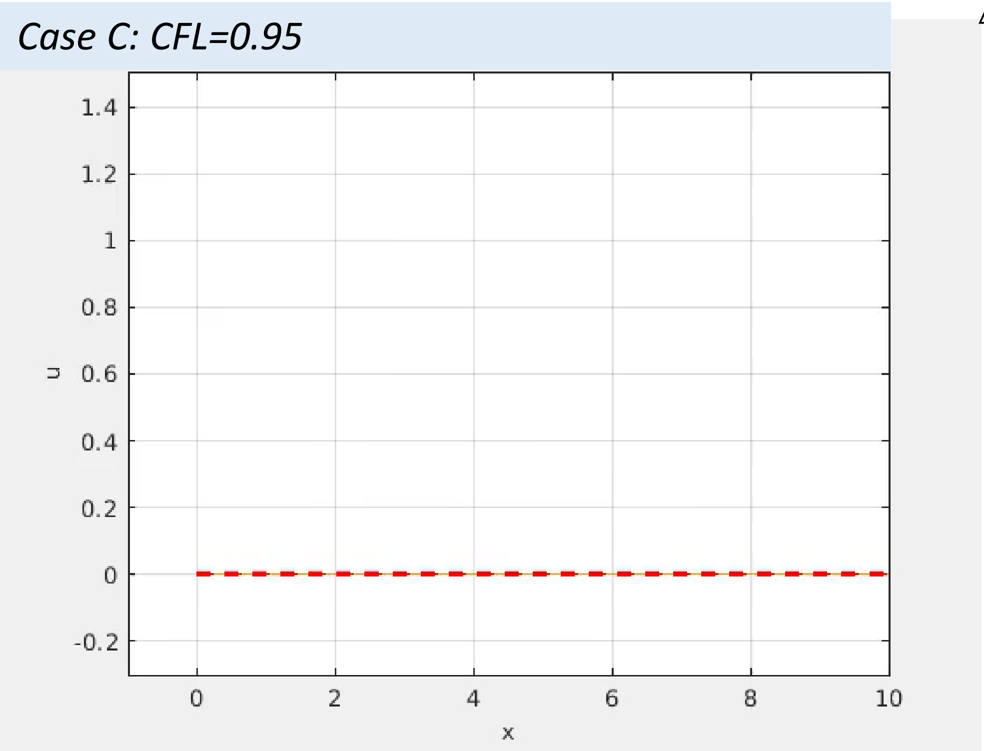
$$\text{C:}\ \ U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$$

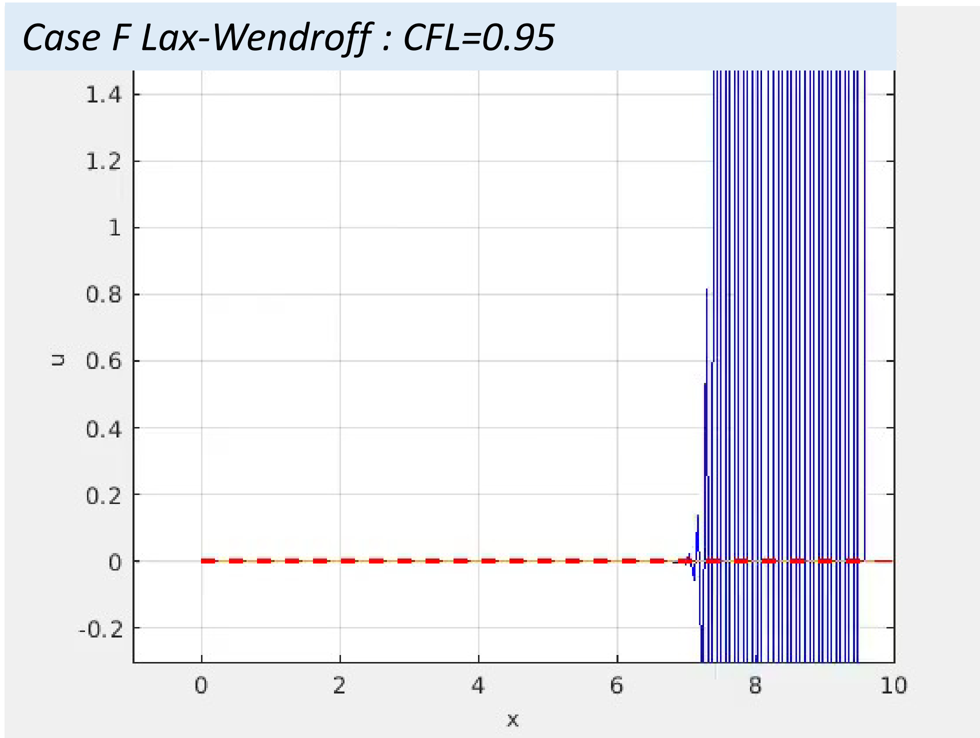
$$N = 150$$

$$\Delta x = 10/150$$

Case F: Lax-Wendroff, **CFL= 0.95**

$$\text{F:}\ \ U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$





Non-optimal CFL number, CFL < 1

Case C: upwinded, **CFL= 0.90**

$$\text{C:}\quad U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$$

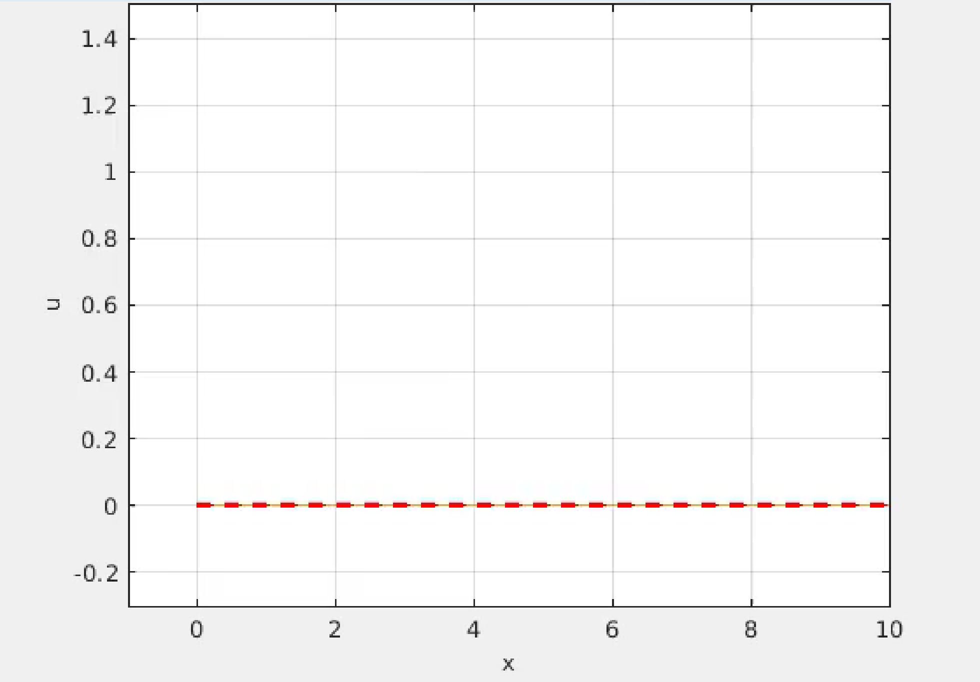
Case F: Lax-Wendroff, **CFL= 0.90**

$$\text{F:}\quad U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$

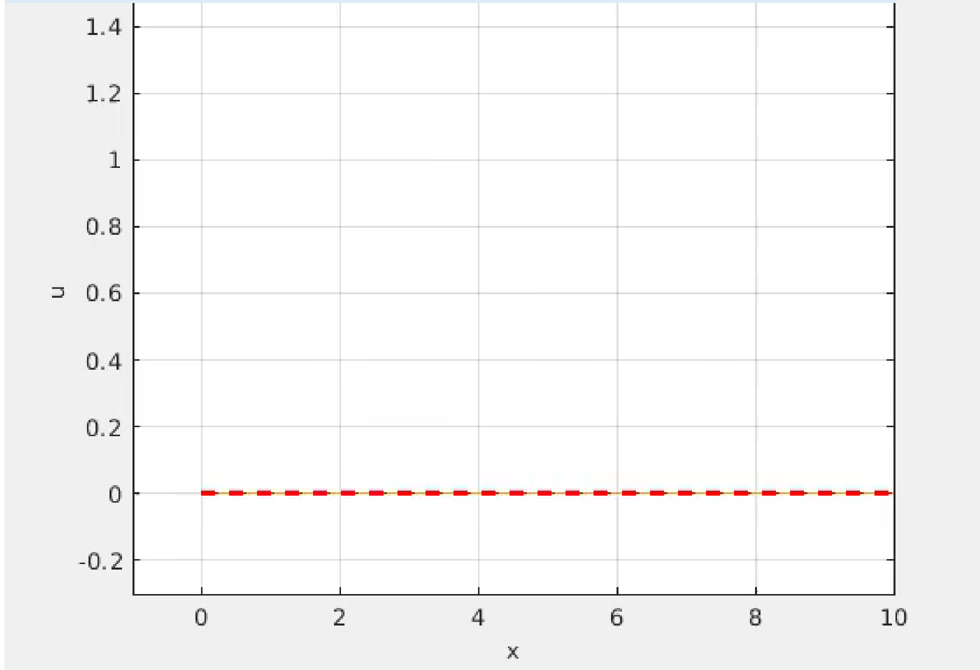
$$N = 150$$

$$\Delta x = 10/150$$

Case C: CFL=0.90



Case F Lax-Wendroff : CFL=0.90



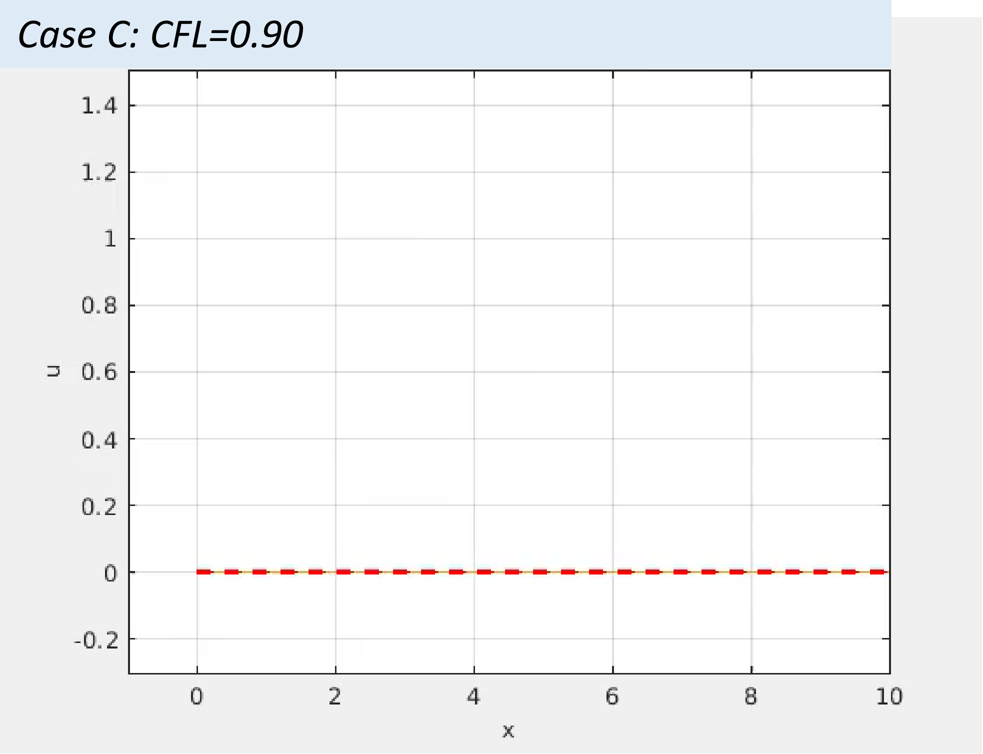
Non-optimal CFL number, CFL < 1, Spatial resolution effect

Case C: upwinded, **CFL= 0.90** C: $U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$

Less dissipation

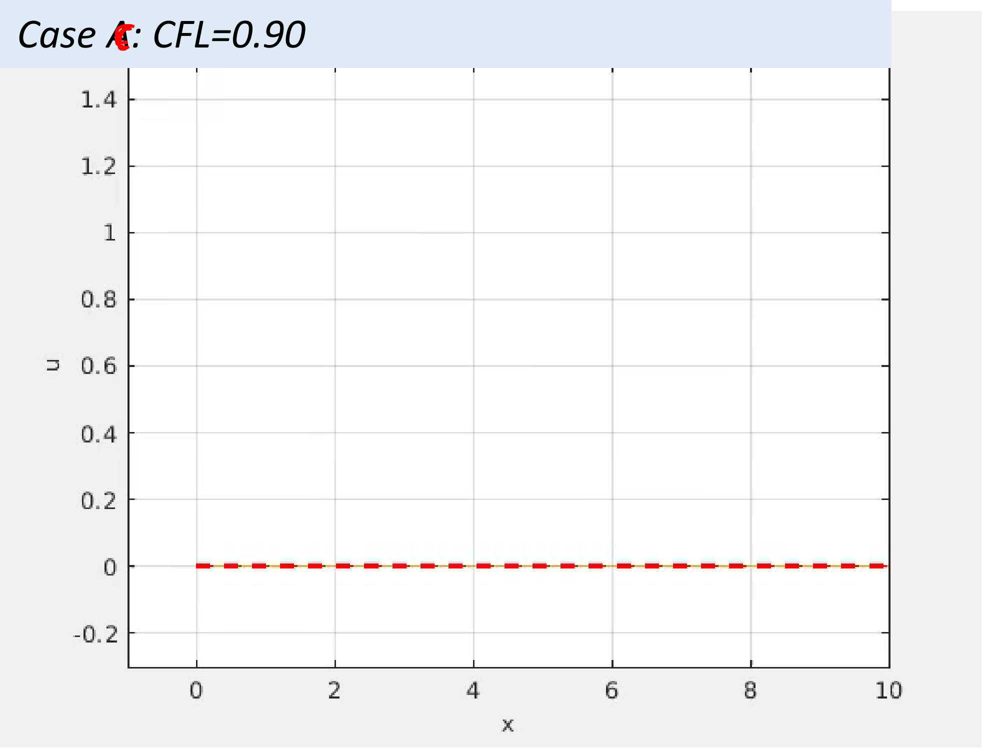
$N = 150$

$\Delta x = 10/150$



$N = 400$

$\Delta x = 10/400$



Non-optimal CFL number, CFL < 1, Spatial resolution effect

Case F: Lax-Wendroff, **CFL= 0.90**

$$\text{F:}\quad U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$

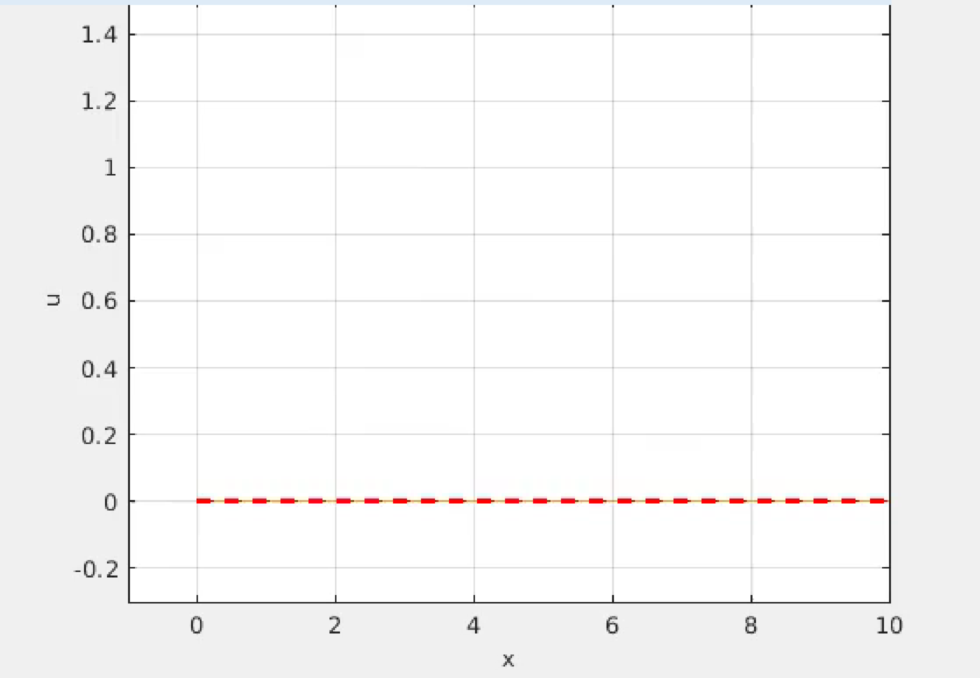
Even lesser *dissipation*.
Dispersive effect more localised

$$N = 150$$
$$\Delta x = 10/150$$

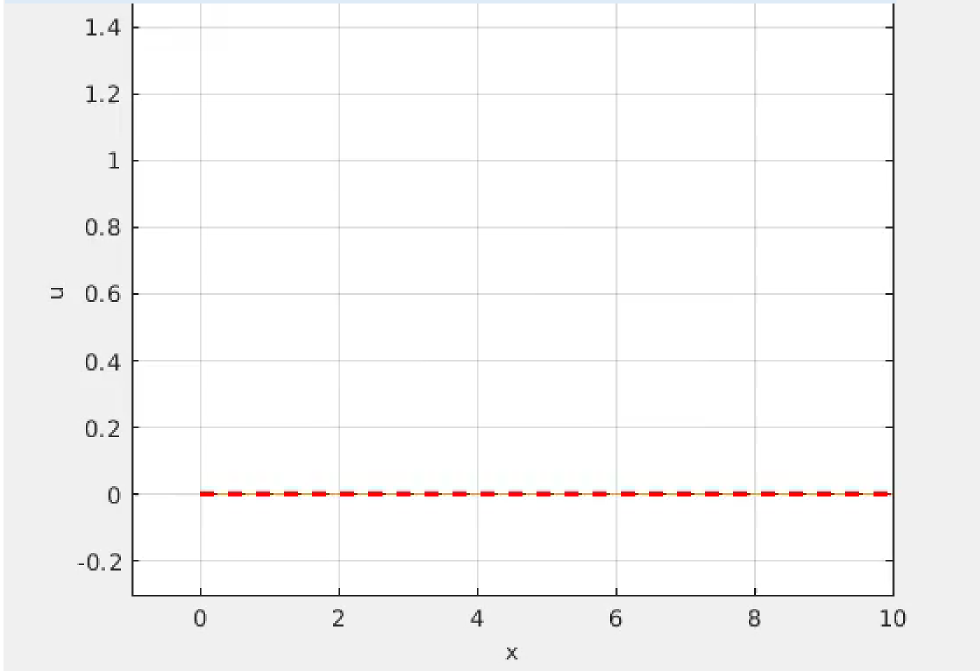
Less *dissipation* compared with C – but *dispersive* effects!

$$N = 400$$
$$\Delta x = 10/400$$

Case F Lax-Wendroff : CFL=0.90



Case F Lax-Wendroff : CFL=0.90



Summary of findings through numerical experiments:

- CFL > 1: all methods examined tend to blow up (A,B,C & F) (i.e. unstable numerically).
- CFL=1: A, B unstable, C & F give exact result.
- CFL < 1: C & F display dispersive, dissipative behaviour and out of phase with exact solution result. Results exasperated at regions where sudden changes, or large gradients in the u-field arise.
- CFL < 1: Greater resolution in grid, reduces the dispersive, dissipative behaviour of C & F.   Dissipation effects of F less than C. Dispersive behaviour in C not evident; F displays relatively weaker Dissipation but significant Dispersive behaviour.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

$\Rightarrow$ $\begin{cases} \dfrac{dx}{dt} = c \;;\; \dfrac{dU}{d\xi} = 0 \text{ or unchanged} \\ \text{CFL}: \; ck/h = q \end{cases}$

The real solution therefore has a growth factor $\lambda = \exp(-i\xi q)$. We note that $|\lambda| = 1$ so that there is no growth or decay. We can model (14.1) in many different ways. We shall use a forward difference for u_t and consider 6 schemes (**Schemes-6**) :

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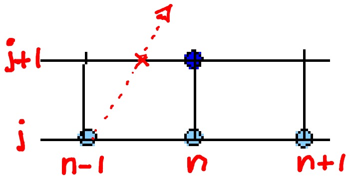
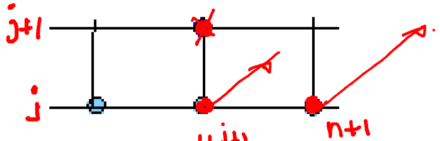
A: $U_n^{j+1} = U_n^j + \dfrac{q}{2}(U_{n-1}^j - U_{n+1}^j)$



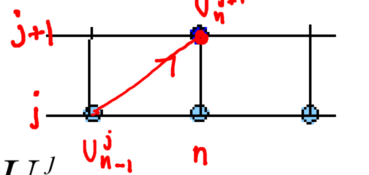
$\Rightarrow$ unphysical scenario.

B: $U_n^{j+1} = U_n^j - q(U_{n+1}^j - U_n^j)$



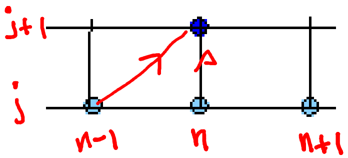
unphysical $\Rightarrow$ unstable numerical method.

C: $U_n^{j+1} = \cancel{U_n^j} - q(\cancel{U_n^j} - U_{n-1}^j)$ ✓



$\underline{U_n^{j+1} = U_{n-1}^j}$

F: $U_n^{j+1} = (1 - \cancel{q^2})U_n^j + \dfrac{q}{2}(\cancel{q-1})U_{n+1}^j + \dfrac{q}{2}(q+1)U_{n-1}^j$

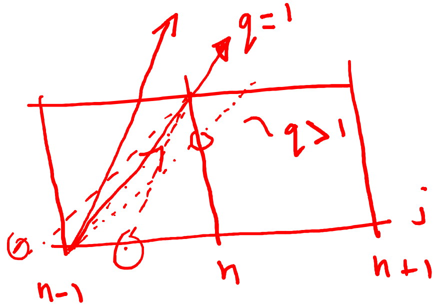


$U_n^{j+1} = U_{n-1}^j$

$\Big\}\; q = 1$

$q > 1$

$q < 1$



remain in Domain of influence - "respect Physics!"

Numerical solutions exhibit:

1. Advection
2. Dissipation
3. Dispersion
4. Phase Differences

2) Example of Dissipation (or Diffusion) :

The parabolic one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{with} \quad u(x,0) = f(x)$$

The solution can be written as

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x - \xi^2 t} d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/4t} ds. \end{aligned}$$

which can be derived using Fourier analysis. The variable $\hat{f}(\xi)$ denotes the Fourier transform of the initial data with ξ as the spatial wavenumber. Physically, the above solution asserts that the oscillatory components in the initial data of wavenumber ξ decay at a rate of $e^{-\xi^2 t}$. The solution will then be composed of increasingly smooth wave components

2) Example of Dispersion :

The one-dimensional Schroedinger equation

$$\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2}$$

The solution can be written as

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x - i\xi^2 t} d\xi \\ &= \frac{1}{\sqrt{4i\pi t}}\int_{-\infty}^{\infty} f(s) e^{-i(x-s)^2/4t} ds. \end{aligned}$$

Dispersion is observed in the form of solutions that (rather than decaying as t → ∞, as for the heat equation) break into oscillatory wave packets.

Numerical solutions exhibit:

1. Advection
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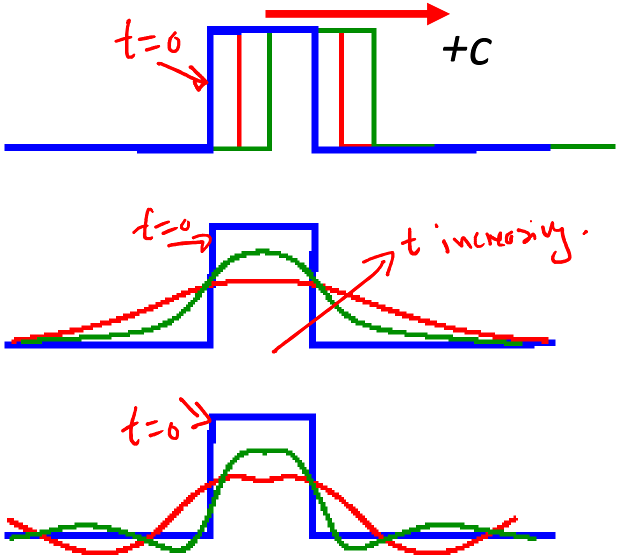

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$ **Advection**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$ **Dissipation**

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2}$$ **Dispersion**

The three mechanisms of advection, diffusion, and dispersion are central to the behaviour of partial differential equations and their discrete models, and together account for most linear phenomena. The study of numerical methods for these equations reveals many of the issues that come up repeatedly in more difficult problems.

The derivative terms in the advection $(\partial / \partial x)$, diffusion $(\partial^2 / \partial x^2)$ and dispersion $i(\partial^2 / \partial x^2)$ operators play a crucial role, both in terms of accuracy and in terms of the stability of the discretized scheme.

Modified Equation :

Solutions of finite-difference schemes applied to partial differential equations are thought of as approximations to the exact solutions of the continuous problem. Our finite difference solutions to the linear advections equation display dispersive, dissipative effects in addition to displaying phase difference in the evolution of the solution with time, especially when CFL < 1. Why is this? Since the linear advection equation $u_t + cu_x = 0$ does NOT have either the $(\partial^2 / \partial x^2)$ or $i(\partial^2 / \partial x^2)$ operators!

Let us consider, the upwinded scheme C:

$$U_n^{j+1} = U_n^j - q(U_n^j - U_{n-1}^j)$$

$$u(x,t+k) - u(x,t) + q(u(x,t) - u(x-h,t)) = 0$$

Taylor series expand around $u(x,t) = u$ (say), gives

$$u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + \ldots - u + q\left(u - (u - hu_x + \frac{h^2}{2}u_{xx} - \frac{h^3}{6}u_{xxx} + \ldots)\right) = 0, \text{ simplify and note } q = ck/h, \text{ so}$$

$$u_t + \frac{k}{2}u_{tt} + \frac{k^2}{6}u_{ttt} + c\left(u_x - \frac{h}{2}u_{xx} + \frac{h^2}{6}u_{xxx}\right) = 0, \text{ hence:}$$

$$u_t + cu_x = \frac{1}{2}(chu_{xx} - ku_{tt}) - \cancel{\frac{1}{6}(k^2u_{ttt} + ch^2u_{xxx})}. \text{ Next use exact eqn: } u_t = -cu_x, \text{ hence: } u_{tt} = -cu_{xt} \text{ and } u_{tt} = c^2u_{xx}.$$

$$u_t + cu_x = \frac{1}{2}(chu_{xx} - kc^2u_{xx}) = \frac{ch}{2}\left(1 - \frac{ck}{h}\right)u_{xx} \rightarrow$$

$$u_t + cu_x = \frac{ch}{2}\left(1 - \frac{ck}{h}\right)u_{xx}$$

$\frac{ck}{h} = q$: $q = 1 \Rightarrow u_t + cu_x = 0$; $q < 1 \Rightarrow u_t + cu_x = +|\,|\,u_{xx}$

$q > 1 \Rightarrow u_t + cu_x = -|\,|\,u_{xx}$ negative dissipation

We discretised $\dfrac{\partial u}{\partial t}+c\dfrac{\partial u}{\partial x}=0$ with **up-winded scheme C:** $U_n^{j+1}=U_n^j-q(U_n^j-U_{n-1}^j)$ but in actual fact we are solving numerically the modified equation:

$$u_t+cu_x=\frac{ch}{2}\left(1-\frac{ck}{h}\right)u_{xx}$$

We discretised $\dfrac{\partial u}{\partial t}+c\dfrac{\partial u}{\partial x}=0$ with **up-winded scheme C:** $U_n^{j+1}=U_n^j-q(U_n^j-U_{n-1}^j)$ but in actual fact we are solving numerically the modified equation:

$$u_t+cu_x=\frac{ch}{2}\left(1-\frac{ck}{h}\right)u_{xx}$$ ← ~ O(h)

In a similar manner, **Lax-Wendroff scheme F :** $U_n^{j+1}=(1-q^2)U_n^j+\dfrac{q}{2}(q-1)U_{n+1}^j+\dfrac{q}{2}(q+1)U_{n-1}^j$

The modified equations is:

$$u_t+cu_x=-\frac{ch^2}{6}\left(1-\left(\frac{ck}{h}\right)^2\right)u_{xxx}$$ ← $q=1$ ⇒ exact advection eqn ⇒ exact numerical sol.

$q<1$ ⇒ dispersion effects.

- U_{xxx} term leads to ***dispersive*** behaviour, rather than ***dissipation***.
- Magnitude of error smaller than up-winded scheme **C**. Since **F** is a higher-order method.
- Dispersive term gives rise to oscillating solution and also a shift in main peak location, i.e. ***phase error***.

Further reading: Leveque chapter 10

Summary of findings through numerical experiments:

- CFL > 1: all methods examined tend to blow up (A,B,C & F) (i.e. unstable numerically).
- CFL=1: A, B unstable, C & F give exact result.
- CFL < 1: C & F display dispersive, dissipative behaviour and out of phase with exact solution result. Results exasperated at regions where sudden changes, or large gradients in the u-field arise.
- CFL < 1: Greater resolution in grid, reduces the dispersive, dissipative behaviour of C & F. Dissipation effects of F less than C. Dispersive behaviour in C not evident; F displays relatively weaker Dissipation but significant Dispersive behaviour.

Modified equations, schemes C & F

$$u_t + cu_x = \frac{ch}{2}\left(1 - \frac{ck}{h}\right)u_{xx}$$

$6 \Rightarrow$

$$u_t + cu_x = -\frac{ch^2}{6}\left(1 - \left(\frac{ck}{h}\right)^2\right)u_{xxx}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

The real solution therefore has a growth factor $\lambda = \exp(-i\xi q)$. We note that $|\lambda| = 1$ so that there is no growth or decay. We can model (14.1) in many different ways. We shall use a forward difference for u_t and consider 6 schemes (**Schemes-6**) :

$$\frac{U_n^{j+1} - U_n^j}{-q} = \begin{cases} \mathbf{A:} & \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \mathbf{B:} & U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \mathbf{C:} & U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \mathbf{D:} & \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \mathbf{E:} & \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \mathbf{F:} & \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

A: $U_n^{j+1} = U_n^j + \frac{q}{2}(U_{n-1}^j - U_{n+1}^j)$

B: $U_n^{j+1} = U_n^j - q(U_{n+1}^j - U_n^j)$

C: $U_n^{j+1} = U_n^j + q(U_n^j - U_{n-1}^j)$

F: $$U_n^{j+1} = (1-q^2)U_n^j + \frac{q}{2}(q-1)U_{n+1}^j + \frac{q}{2}(q+1)U_{n-1}^j$$

Von-Neumann Fourier Stability analysis:

$$\lambda = \begin{cases} \mathbf{A:} & 1 - iq\sin\xi \\ \mathbf{B:} & 1 - q(e^{i\xi} - 1) \\ \mathbf{C:} & 1 - q(1 - e^{-i\xi}) \\ \mathbf{D:} & \frac{1}{4}\left[iq\sin\xi \pm \sqrt{4 - q^2\sin^2\xi}\right]^2 & \text{(two steps)} \\ \mathbf{E:} & \dfrac{2 - iq\sin\xi}{2 + iq\sin\xi} \\ \mathbf{F:} & 1 - iq\sin\xi - 2q^2\sin^2\frac{1}{2}\xi \end{cases}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

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Von-Neumann Fourier Stability analysis:

We recall that if $|\lambda| > 1 + O(k)$ the method is unstable; if $|\lambda| < 1$ it is dissipative, while if $|\lambda| = 1$ it is conservative. We see therefore that (**Table II**)

$$|\lambda|^2 = \begin{cases} \mathbf{A:} & 1 + q^2\sin^2\xi & & \text{stable only if } q^2 = O(k) \text{ i.e. } k \sim h^2 \\ \mathbf{B:} & 1 + 2q(1+q)(1-\cos\xi) & \geqslant 1 \quad \forall\, q, \xi & \text{hopelessly unstable} \\ \mathbf{C:} & 1 - 2q(1-q)(1-\cos\xi) & \leqslant 1 \quad \forall\, q \leqslant 1 & \text{stable, dissipative} \\ \mathbf{D:} & 1 & & \text{provided } q^2\sin^2\xi \leqslant 4 \quad \text{or} \quad \frac{1}{2}q \leqslant 1 \\ \mathbf{E:} & 1 & \forall\, q & \text{stable and conservative} \\ \mathbf{F:} & 1 - 4q^2(1-q^2)\sin^4\frac{1}{2}\xi & \leqslant 1 \text{ if } q \leqslant 1 & \text{dissipative, but less than } \mathbf{C} \end{cases}$$

$$= \begin{cases} \mathbf{A:} & 1 - iq\sin\xi \\ \mathbf{B:} & 1 - q(e^{i\xi} - 1) \\ \mathbf{C:} & 1 - q(1 - e^{-i\xi}) \\ \mathbf{D:} & \frac{1}{4}\left[iq\sin\xi \pm \sqrt{4 - q^2\sin^2\xi}\right]^2 & \text{(two steps)} \\ \mathbf{E:} & \dfrac{2 - iq\sin\xi}{2 + iq\sin\xi} \\ \mathbf{F:} & 1 - iq\sin\xi - 2q^2\sin^2\frac{1}{2}\xi \end{cases}$$