Lecture 18. Dissipation and Dispersion

Diffusion refers to a phenomenon where some quantity spreads out in space as time goes on. Dissipation is used to refer to loss of energy. The effects are related. If you consider the heat equation

$$T_t = \mu \nabla^2 T \tag{18.1}$$

and start with an initial condition with high temperature in some small region, then the temperature distribution spreads out. This would be called diffusion. If you calculate

$$\int_\Omega T^2(x,t)dx, \tag{18.2}$$

this quantity will decrease with time. So there is a loss of energy. (This is not really energy in physical sense if T is temperature, you can substitute velocity here). Both are caused by the Laplacian term. If something spreads out, its square integral will probably decrease.

Dispersion is related to wave phenomena. For a linear wave equation

$$u_t + cu_x = 0, \qquad \text{or} \qquad u_{tt} = c^2 u_{xx} \tag{18.3}$$

you can resolve the solution into Fourier modes. Then each mode will travel at the same speed. We say such equations are non-dispersive. These two equations are also non-dissipative. For smooth solutions with periodic boundary conditions, they conserve all integrals of the form

$$\int_\Omega |u(x,t)|^p dx. \tag{18.4}$$

An equation like

$$u_t + cu_x = \nu u_{xxx} \tag{18.5}$$

would have solutions whose Fourier modes travel at different speed depending on ν and wave number. We say these equations exhibit dispersive behaviour. We observe these phenomena at the level of PDE and we can map them to the behaviour of numerical schemes.

If we solve a non-dissipative and non-dispersive PDE with a numerical scheme, we want the numerical solutions to be non-dissipative and non-dispersive too. Sadly this is not possible since numerical solutions will exhibit these behaviours even if the exact solution does not. Then we say that the numerical scheme is dissipative and/or dispersive and the interesting thing is to design discretisation schemes which at least minimise these effects which are not present in the exact solution.

When solving PDES analytically a common approach is to assume that the solution $u(x,t)$ has a wave form

$$u(x,t) = \hat{u} e^{i(\omega t + \alpha x)}, \tag{18.6}$$

where ω is a frequency and α the wavenumber with wavelength λ given by $\lambda = 2\pi/\alpha$. Consider the equation

$$u_t = \nu u_{xx}, \tag{18.7}$$

using (18.6) gives

$$i\omega + \nu\alpha^2 = 0, \tag{18.8}$$

from which it is clear that (18.6) can only be satisfied if $\omega = i\nu\alpha^2$, we call (18.8) the **dispersion relationship**. The solution to (18.7) thus becomes

$$u(x,t) = \hat{u}e^{-\nu\alpha^2 t}e^{i\alpha x}. \tag{18.9}$$

This indicates that the wave does not move and decays with time. Likewise for the linear advection equation

$$u_t + cu_x = 0, \quad \text{with } c \text{ a constant value.} \tag{18.10}$$

We get the dispersion relationship $\omega = -c\alpha$, and hence

$$u(x,t) = \hat{u}e^{i\alpha(x-ct)}. \tag{18.11}$$

In this case, assuming ω was real, the solution propagates with speed c and with no decay in amplitude. Also note that the speed of propagation is independent of the frequency ω.

The decay (or growth) and propagation of Fourier modes is an important aspect in the behaviour of solutions to PDEs. With finite difference (F-D) discretisation of the PDE an aspect to consider is upon obtaining the discretised solution, is how well do they decay and the F-D derived solution propagation characteristics match those of the original PDE. The numerical scheme will be unstable if some of the Fourier modes grow without bound. In what follows, **dissipation** in solutions of PDEs is when the Fourier modes do not grow with time and at least one mode decays. A PDE is **non-dissipative** if they neither grow nor decay, while we define **dispersion** when solutions of PDEs exhibit differing Fourier mode wavelengths propagating at different speeds. In this respect, observe that (18.9) is dissipative for $\nu > 0$ for all wave-numbers $\alpha \neq 0$, while (18.11) is neither dissipative nor dispersive.
Next consider

$$u_t + cu_{xxx} = 0, \tag{18.12}$$

for which the dispersion relationship is

$$\omega = \alpha^3 c, \tag{18.13}$$

and hence the Fourier mode representing a solution to (18.12) is

$$u(x,t) = \hat{u}e^{i\alpha(x+\alpha^2 ct)}. \tag{18.14}$$

There are two aspects here, (1) unlike (18.11) the Fourier mode propagates in the opposite direction with speed $\alpha^2 c$, and (2) Fourier modes with different wave-numbers (α) propagate with different speeds ($\alpha^2 c$). So (18.12) is dispersive but non-dissipative (*i.e.* the amplitude neither decays nor grows). With some thought, one should be able to see that PDEs containing even ordered $x-$derivatives will be dissipative. PDEs containing only odd derivatives in x will be non-dissipative and involve propagating waves, and be dispersive when the order is greater than one.

Finally consider the equation

$$u_t + au_x - \nu u_{xx} + cu_{xxx} = 0, \tag{18.15}$$

for which the dispersion relationship is

$$\omega = -a\alpha + i\nu\alpha^2 + c\alpha^3 \tag{18.16}$$

and hence the solution for the α-mode will be

$$u(x,t) = \hat{u} e^{-\nu\alpha^2 t} e^{i\alpha\left[x-(a-c\alpha^2)t\right]}. \tag{18.17}$$

The dissipation term is $e^{-\nu\alpha^2 t}$, while the propagating term is

$$e^{i\alpha\left[x-(a-c\alpha^2)t\right]}, \tag{18.18}$$

thus the PDE is dissipative and dispersive due to the dependence of the term $(a - c\alpha^2)$ term above on α^2, and becomes larger the more larger α is.

Dispersion and Dissipation in Discretised Equations

The discrete analogue of the Fourier mode (18.6) is

$$u_k^n = \hat{u} e^{i(\omega n\Delta t + \alpha k\Delta x)}, \tag{18.19}$$

and thus information about dispersion and dissipation for all wavelengths we consider $\alpha\Delta x$ in the range $0 \le \alpha\Delta x \le \pi$. Furthermore for what follows, the dispersion relation $\omega = \omega(\alpha)$ will in general be complex, hence we set $\omega = a + ib$, with (a, b) considered real only. Substituting $\omega = a + ib$ in (18.19) and a rewrite gives

$$u_k^n = \hat{u} e^{-bn\Delta t} e^{i(na\Delta t + k\alpha\Delta x)} = \hat{u}\left(e^{-b\Delta t}\right)^n e^{i\alpha(k\Delta x - (-a/\alpha)n\Delta t)}. \tag{18.20}$$

From this we see that

- if $b > 0$ for some α, the difference equation is dissipative;
- if $b < 0$ for some α, the solution grows without bound, and thus the scheme will be unstable;
- if $b = 0$ for all α, the scheme will be non-dissipative.

Furthermore

- if $a = 0$ for all α, there is no wave propagation.
- if $a \ne 0$ for some α, wave propagation with velocity $(-a/\alpha)$ occurs;
- if a/α is a non-trivial function of α, dispersive effects will arise.

Let us consider (18.3) (with c positive) discretised with an up-winded scheme, namely :

$$U_j^{n+1} = U_j^j + q\left(U_j^n - U_{j-1}^n\right), \tag{18.21}$$

where $q = ck/h$. Substitution of (18.19) into the above and simplification gives

$$e^{i\omega k} = 1 - q + q\left[\cos(\alpha h) + i\sin(\alpha h)\right]. \tag{18.22}$$

We next write $\omega = a + ib$ and hence

$$e^{iak}e^{-bk} = 1 - q + q\left[\cos(\alpha h) + i\sin(\alpha h)\right]. \tag{18.23}$$

Which thus gives

$$e^{-bk} = \sqrt{(1-q)^2 + q^2 + 2q(1-q)\cos(\alpha h)}. \tag{18.24}$$

Setting $q = 1$ thus gives $b = 0$, which implies that the scheme will be non-dissipative. Next setting $q = 1 - \varepsilon$, with $\varepsilon << 1$, it can be shown that

$$e^{-bk} = \sqrt{(1-\varepsilon)^2 + \varepsilon^2 + 2(\varepsilon - \varepsilon^2)\cos(\alpha h)}, \tag{18.25}$$

which suggests the scheme will be dissipative for q values just less than 1. Whether the scheme is dispersive or not we consider the imaginary part of (18.23), namely

$$e^{iak} = \cos(ak) + i\sin(ak) = \frac{1 - q + q\left[\cos(\alpha h) + i\sin(\alpha h)\right]}{(1-q)^2 + q^2 + 2q(1-q)\cos(\alpha h)}, \tag{18.26}$$

hence

$$\tan(ak) = \frac{q\sin(\alpha h)}{1 - q + q\cos(\alpha h)}, \tag{18.27}$$

i.e. hence

$$a = \frac{1}{k}\tan^{-1}\left(\frac{q\sin(\alpha h)}{1 - q + q\cos(\alpha h)}\right) = \frac{1}{k}\tan^{-1}\left(\frac{q\sin(\alpha h)}{1 - 2q\sin^2(\alpha h/2)}\right). \tag{18.28}$$

This is in general dispersive since a is a nonlinear function of α. This expression then requires a careful examination to ascertain behaviour of the low frequency (αh small) and high frequency (αh near π) waves.

Further details see, chapter 7 of Thomas, J. W. "Numerical Partial Differential Equations – Finite Difference Methods"