Lecture 14-15 : Hyperbolic PDEs
The One-dimensional Advection Qquation and Upwinding

Consider the problem for $u(x,\, t)$

$$u_t + c\, u_x = 0 \qquad \text{for} \quad t > 0, \qquad \text{in} \quad u(x,\, 0) = f(x), \tag{14.1}$$

where c is a **positive** constant. The characteristics of this equation are $dx/dt = c$ or $x - ct =$ constant. Furthermore, u is constant along each characteristic and so the solution is $u = f(x - ct)$. This represents a wave travelling in the positive x-direction with speed c, **without change of size or shape** as shown in Figure 14.1 – we emphasis that the exact solution of (14.1) says that the initial function $f(x)$ propagates unaltered in form in the positive x-direction as time progresses.

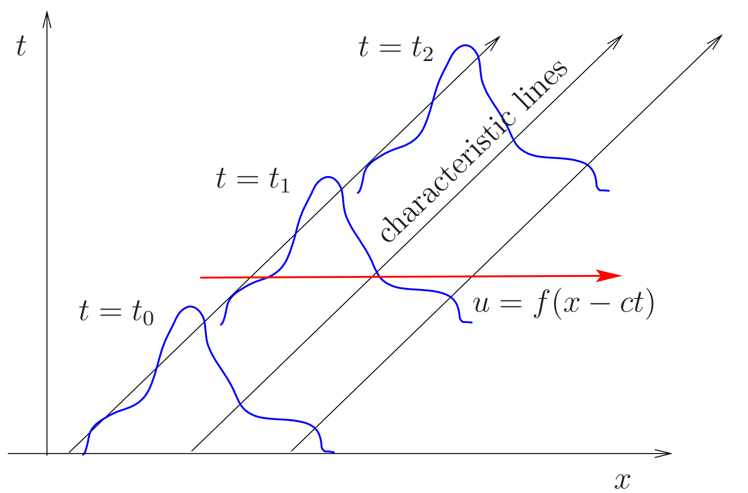

Figure 14.1: Propagation of a function $u = f(x)$ with time t.

Can we develop finite-difference methods (FDMs) with these conservation properties? For future reference, we observe that for a Fourier mode with $f(x) = \exp(ix\xi/h) \equiv e^{in\xi}$

$$u_n^{j+1} \equiv u(nh,\, (j+1)k) = u_n^j \exp(-i\xi q) \qquad \text{where} \quad q = ck/h > 0 \tag{14.2}$$

is the **Courant number** (also referred to as the **Courant-Friedrichs-Lewy** (**CFL**)-condition).

The real solution therefore has a growth factor $\lambda = \exp(-i\xi q)$. We note that $|\lambda| = 1$ so that there is no growth or decay. We can model (14.1) in many different ways. We shall use a forward difference for u_t and consider 6 schemes (**Schemes-6**) :

$$\frac{U_n^{j+1} - U_n^j}{-q} = \begin{cases} \mathbf{A:} \quad \frac{1}{2}\Delta U_n^j \equiv \frac{1}{2}(U_{n+1}^j - U_{n-1}^j) & \text{Explicit, centred} \\ \mathbf{B:} \quad U_{n+1}^j - U_n^j & \text{Explicit, forwards} \\ \mathbf{C:} \quad U_n^j - U_{n-1}^j & \text{Explicit, backwards} \\ \mathbf{D:} \quad \frac{1}{2}\Delta U_n^{j+1/2} & \text{Two-step, centred} \\ \mathbf{E:} \quad \frac{1}{2}\left[\frac{1}{2}\Delta U_n^{j+1} + \frac{1}{2}\Delta U_n^j\right] & \text{Crank-Nicolson} \\ \mathbf{F:} \quad \frac{1}{2}\Delta U_n^j - \frac{1}{2}q\delta^2 U_n^j & \text{Lax-Wendroff} \end{cases}$$

(We shall also later consider Keller's 'Box scheme'). We use the Fourier method to investigate the stability of all of these schemes, looking for a solution $U_n^j = \lambda^j \exp(in\xi)$. We find (**Table I**) :

$$\lambda = \begin{cases} \mathbf{A}: & 1 - iq\sin\xi \\ \mathbf{B}: & 1 - q(e^{i\xi} - 1) \\ \mathbf{C}: & 1 - q(1 - e^{-i\xi}) \\ \mathbf{D}: & \frac{1}{4}\left[iq\sin\xi \pm \sqrt{4 - q^2\sin^2\xi}\right]^2 \qquad \text{(two steps)} \\ \mathbf{E}: & \dfrac{2 - iq\sin\xi}{2 + iq\sin\xi} \\ \mathbf{F}: & 1 - iq\sin\xi - 2q^2\sin^2\frac{1}{2}\xi \end{cases}$$

We recall that if $|\lambda| > 1 + O(k)$ the method is unstable; if $|\lambda| < 1$ it is dissipative, while if $|\lambda| = 1$ it is conservative. We see therefore that (**Table II**)

$$|\lambda|^2 = \begin{cases} \mathbf{A}: & 1 + q^2\sin^2\xi & & \text{stable only if } q^2 = O(k) \ i.e.\ k \sim h^2 \\ \mathbf{B}: & 1 + 2q(1+q)(1-\cos\xi) & \geqslant 1 \quad \forall\, q, \xi & \text{hopelessly unstable} \\ \mathbf{C}: & 1 - 2q(1-q)(1-\cos\xi) & \leqslant 1 \quad \forall\, q \leqslant 1 & \text{stable, dissipative} \\ \mathbf{D}: & 1 & & \text{provided } q^2\sin^2\xi \leqslant 4 \quad \text{or} \quad \frac{1}{2}q \leqslant 1 \\ \mathbf{E}: & 1 & \forall\, q & \text{stable and conservative} \\ \mathbf{F}: & 1 - 4q^2(1-q^2)\sin^4\frac{1}{2}\xi & \leqslant 1 \text{ if } q \leqslant 1 & \text{dissipative, but less than } \mathbf{C} \end{cases}$$

We may also be interested in $\arg\lambda$, which determines the *phase* of each mode. In case **E**, for example,

$$\arg\lambda = -2\tan^{-1}(\tfrac{1}{2}q\xi) \approx -q\xi \qquad \text{when } \xi \text{ is small.}$$

All the above schemes agree well with the exact solution, for which $\arg\lambda = -q\xi$, when ξ is small. The waves with higher values of ξ are dispersive; their phase velocity varies with frequency. A computer demonstration will illustrate the effects of this. Although the amplitude of each wave component may be conserved, phase differences develop which alter the shape of the whole.

We can see from the above the importance of the CFL condition, $q \leqslant 1$. The Courant or CFL number is a non-dimensional number that plays a central role in the numerical solution of hyperbolic equations. If c can be thought of a speed, $q = ck/h \equiv c\Delta t/\Delta x$ can be thought of a distance measured in grid points that a particle or information reaches to, in an increment of time $k \equiv \Delta t$.

Another very important conclusion we can draw is that backwards, or **Upwind** differences are a good idea. Case **C** is stable provided the CFL condition holds, whereas **B** is unconditionally unstable. If $c < 0$, the stable scheme is **B**. In general, the **Upwind** scheme can be written

$$U_n^{j+1} = U_n^j - sc\Delta U_n^j + s|c|\delta^2 U_n^j \qquad \text{where} \quad s = k/2h. \tag{14.3}$$

The need for upwind differences can be interpreted in terms of the characteristics of (14.1), $x - ct =$ constant which must pass through the Numerical Domain of Dependence.

Above we have a very simple equation with constant coefficients (*i.e. the* c). In this very special case, setting $q = 1$ for some of the discretsation schemes above, the numerical scheme reproduces the exact solution with no error. However in more complex hyperbolic systems with varying coefficients maintaining this *exactness* is not that straightforward and should not be expected, unless one goes to considerable effort towards honouring the true physical propagation paths or characteristics of the equations set.

Upwinding for simultaneous equations

Let $\mathbf{u}(x,\,t)$ be a p-dimensional vector satisfying the equation $\mathbf{u}_t + A\mathbf{u}_x = \mathbf{d}$ where A is a $p \times p$ matrix, which for simplicity we shall assume to be constant. We shall investigate how the upwinding method generalises to this problem, by **diagonalising** the matrix A, which we assume to have p distinct eigenvalues $\lambda_1 \ldots \lambda_p$ and corresponding eigenvectors. We make the linear transformation $\mathbf{u} = S\mathbf{v}$ where S is the matrix whose columns are the eigenvectors of A, so that

$$S^{-1}AS = D \equiv \mathrm{diag}(\lambda_1,\,\lambda_2,\ldots,\lambda_p) \qquad \text{and so} \quad A = SDS^{-1}\ .$$

Then v_i, the i-th component of $\mathbf{v}$, satisfies the equation

$$(v_i)_t + \lambda_i (v_i)_x = (S^{-1}\mathbf{d})_i\ .$$

We have thus separated the problem into p separate ones, each with its own characteristic family, $dx/dt = \lambda_i$, and we can use **upwinding** on each one in turn. Note that if any of the λ_i are complex, then some of the p equations are elliptic, and we cannot use a time-stepping approach for them. We assume here that all the λ_i are real. From (**Schemes-6**), the upwind scheme for $\mathbf{v}$ is

$$\mathbf{V}_n^{j+1} = \mathbf{V}_n^j - sD\Delta\mathbf{V}_n^j + sD^+\delta^2\mathbf{V}_n^j + kS^{-1}\mathbf{d}_n^j\ ,$$

where $D^+ \equiv \mathrm{diag}(|\lambda_1|,\,|\lambda_2|,\ldots,|\lambda_p|)$, and $\mathbf{V}$ is the FDA to $\mathbf{v}$. Then transforming back, defining $\mathbf{U} = S\mathbf{V}$ so that $\mathbf{V} = S^{-1}U$, we find

$$\mathbf{U}_n^{j+1} = \mathbf{U}_n^j - sA\Delta\mathbf{U}_n^j + sA^+\delta^2\mathbf{U}_n^j + k\mathbf{d}_n^j \qquad \text{where} \quad A^+ = SD^+S^{-1}\ . \tag{14.4}$$

This is the required generalisation of upwinding for p simultaneous equations. If all the eigenvalues of A are positive, then $D^+ = D$ and $A^+ = A$, while $A^+ = -A$ if they are all negative. Otherwise, A^+ bears no simple relation to A, and we must be very careful how we define "**up**" when upwinding. The stability condition for (14.4) is

$$\frac{k}{h}\max_{i=1\ldots p}|\lambda_i| \leqslant 1\ . \tag{14.5}$$