Lecture 22. Perfectly Matched Layers

Solutions of the wave equation can move large distances without dissipating. As a result, correct treatment of *"infinity"* is crucial - if outgoing waves are allowed to reflect back towards the region of interest, they may eventually completely invalidate the solution. Last lecture we derived an approximate boundary condition which greatly reduced spurious reflections at the edge of the computational domain, but we would like to do better.

How can we accurately impose non-reflecting boundary conditions? In real life we might clad a concert hall ceiling with absorbing material – curtains and carpets are well known to muffle echoes in houses. So imagine cladding the domain in which we are interested with a region which causes waves to decay. The difficulty is to do this in a manner which does not encourage reflections. Waves may bounce off many inhomogeneities (think of mirages, for example). Techniques to model the waves behaviour at the boundaries of a finite domain can be dealt with by attempting to solve or approximate the exact dispersion relation for waves in the far-field and impose the corresponding conditions on the boundary of the computational domain such that this boundary appears transparent to outgoing waves.

Techniques which *artificially* manipulate the waves behaviour at the boundaries of a finite domain is an alternative means to avoid issues with reflections at the boundaries. In this, the idea is to surround the computational domain by additional grid points on which a highly dissipative equation can be solved that damps the outgoing waves. A switching function is used to blend from the equations governing the flow evolution on the inside of the domain (i.e., the physically relevant domain) to the equations causing the dissipation of the flow in the absorbing layer. Even though this is a simple and appealing concept, the proper implementation in a numerical simulation is far from trivial.

Specifically, on a boundary at, say, $x = 0$ we want a solution which behaves like $\exp(ik(x - ct))$ in $x < 0$ to match with something which decays exponentially with x in $x > 0$. One way of doing that is to add an imaginary part to x. We may think of this as following the same solution along a contour in the complex x-plane other than the x-axis. Let's consider a variable $X = x + if(x)$ where $f > 0$ and $f = 0$ for $x < 0$. Then the derivatives transform as

$$\frac{\partial}{\partial X} = \frac{1}{1 + if'(x)} \frac{\partial}{\partial x}, \tag{22.1}$$

The idea is to solve the equation in X-space and hope that the resultant solution has the desired decaying properties in x-space.

A more general wave equation

A generalised form of the wave equation for $p(\mathbf{x}, t)$ is

$$p_{tt} = b\nabla \cdot (\nabla p), \tag{22.2}$$

where $a(\mathbf{x})$ and $b(\mathbf{x})$ are given, positive functions of position, and can represent an inhomogeneous medium through which the waves travel. We assume they do not vary with x at the boundary $x = 0$, though they may do elsewhere. If both $a$and b are constant,

we have the standard wave equation. Now (22.2) can be written as a system of first order equations if we introduce the vector $\mathbf{v}$ such that

$$\mathbf{v}_t = a\nabla p, \quad \text{and} \quad p_t = b\nabla \cdot \mathbf{v}. \tag{22.3}$$

We will start with the 1-D wave equation in terms of X and t, so that

$$v_t = ap_X, \quad \text{and} \quad p_t = bv_X. \tag{22.4}$$

Assuming an $\exp(-i\omega t)$ behaviour and transforming from X to x, we have

$$-i\omega v \equiv v_t = \frac{ap_x}{1+if'} \quad \text{and} \quad -i\omega p \equiv p_t = \frac{bv_x}{1+if'}, \tag{22.5}$$

or multiplying up

$$-i\omega p + \omega f' p = bv_x \quad \text{and} \quad -i\omega v + \omega f' v = ap_x. \tag{22.6}$$

If we introduce $\sigma(x) = \omega f'$ and re-insert the time derivatives, we have the system

$$p_t = bv_x - \sigma p \quad \text{and} \quad v_t = ap_x - \sigma v. \tag{22.7}$$

We can choose a function $\sigma(x)$ which is zero for $x < 0$ and implement this scheme easily. Common choices for $\sigma(x)$ range from piece-wise constant functions to piece-wise linear functions to smooth, monotonically increasing functions. An assortment of the most commonly used is given in Figure 22.1.

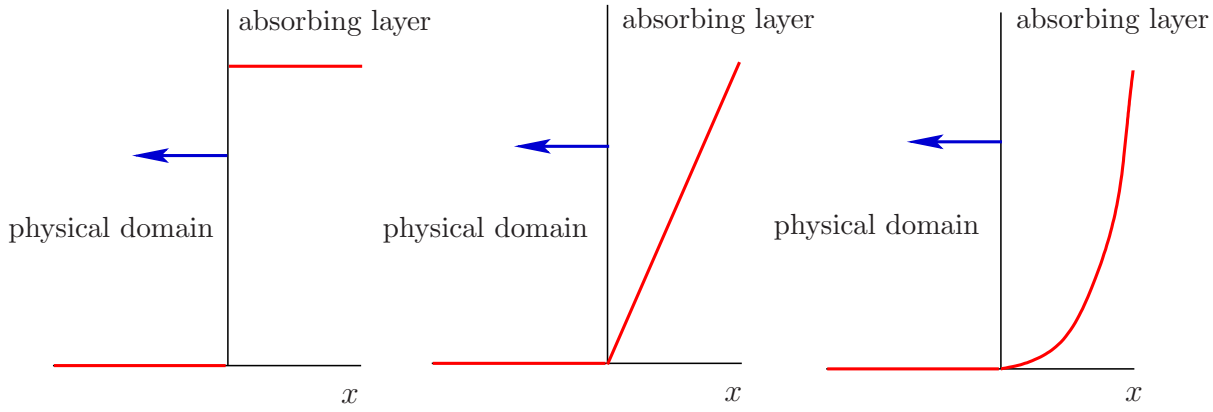

Figure 22.1: Choices for the absorption coefficient $\sigma(x)$.

For the 1-D problem σ appears in a simple manner, but we saw last time there is no difficulty in imposing a non-reflecting boundary condition. What happens in 2D?

A perfectly matched layer in 2D

Now let's write $\mathbf{v} = (v, w)$. Our system takes the form

$$-i\omega p \equiv p_t = b\nabla \cdot \mathbf{v} \equiv bv_X = bw_y, \tag{22.8}$$

$$-i\omega v \equiv v_t = ap_X \quad \text{and} \quad -i\omega w \equiv w_t = ap_y. \tag{22.9}$$

Now (22.9b) does not involve X. Substituting for X in (22.9a) and (22.8) and multiplying up by $(1 + i\sigma/\omega)$, we have

$$bv_x + bw_y(1 + i\sigma/\omega) = -i\omega p + \sigma p, \tag{22.10}$$

$$ap_x = -i\omega v + \sigma v. \tag{22.11}$$

Equation (22.11) easily translates to time derivatives, but (22.10) involves $1/(i\omega)$ which corresponds to time integration rather than differentiation. So we introduce a function $\psi(x, y, t)$, such that

$$-i\omega\psi = b\sigma w_y \text{ with } \psi = 0 \text{ at } t = 0. \tag{22.12}$$

Then (22.10) reads $bv_x + bw_y + \psi = -i\omega p + \sigma p$, and we can write the system as

$$\left.\begin{aligned} p_t &= b\nabla \cdot \mathbf{v} - \sigma p + \psi \\ v_t &= ap_x - \sigma v \\ w_t &= ap_y \\ \psi_t &= b\sigma w_y. \end{aligned}\right\} \tag{22.13}$$

Where $\sigma = 0$, these equations reduce to (22.5). In the layer where $\sigma > 0$, we have decay with no reflection. We can place a boundary a little distance into the layer, secure in the knowledge that any reflections from this boundary will have decayed exponentially and should not cause significant errors in the region of computational interest. Naturally, it is easy to modify these equations for other boundaries.