

ALGEBRA I NOTES

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1. INTRODUCTION

In this class we'll be interested in the representations of matrix groups. Something like $\mathrm{GL}(V)$ or $\mathrm{SO}(V)$ clearly acts on V , but it can also act on other interesting spaces. One relevant case of this for us will be when G acts on polynomials in x_1, \dots, x_n . Let

$$P_d \subseteq \mathbb{C}[x_1, \dots, x_n]$$

be the subspace of homogeneous degree- d polynomials in n variables. This space has a basis given by the monomials

$$\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\}$$

and hence is finite-dimensional. P_d is stable under action by GL_n . This is because linear transformation does not affect the degree of monomials (every x_j is sent to a linear combination of x_1, x_2, \dots, x_n).

Consider the case of $G = \mathrm{O}_n$, the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by R is an intertwining map $P_d \rightarrow P_{d+2}$, meaning $R \circ g^* f = g^*(R \circ f)$.

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This Δ is an O_n -intertwining operator.

We call a function f *harmonic* if it has $\Delta(f) = 0$. The space of harmonic polynomials in n variables of degree d is denoted $H_d \subseteq P_d$. For $d \in \{0, 1\}$, $H_d = P_d$, but for $d \geq 2$ H_d is strictly smaller. Note that H_d is stable under orthogonal transformations.

We will now work toward showing that H_d is an irreducible SO_n -representation for $n \geq 3$.

A representation $\rho : G \mapsto \mathrm{GL}(V)$ is *unitary* if G always acts as a unitary operator (i.e. preserves Hermitian inner product) on V . We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} e^{-|x|^2} dx$$

where dx is the Lebesgue measure. Action of SO_n on P_d preserves this inner product.

Alternatively, we could put an inner product on P_d (or on all functions) from integration over S^{n-1} (the sphere). And polynomials in P_d are determined by their behavior on S^{n-1} .

Proposition: If V is a finite-dimensional vector space with an inner product, then any *unitary* action of G on V is completely reducible. Specifically, if $W \subseteq V$ is a G -stable subspace, then one can decompose the action into $V = W \oplus W^\perp$.

Proof. The thing that we need to prove is that if W is G -stable then W^\perp is as well. Let $x \in W^\perp$ and $w \in W$. Because g acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since $g^{-1} \cdot w \in W$ by G -stability of W . □

Key Lemma: If $F \subseteq C(S^{n-1})$ is any subspace stable under SO_n , then it has an element fixed by SO_{n-1} .

Proof. Let $N := (0, 0, \dots, 0, 1) \in S^{n-1}$. We have the evaluation map $\alpha : C(S^{n-1}) \rightarrow \mathbb{C}$ given by evaluating functions at N . We have an inner product on F given by

$$\langle f, g \rangle := \int_{S^{n-1}} f \bar{g}$$

which is clearly fixed by SO_n , thus F is a unitary representation of SO_n . By Riesz representation theorem, $\alpha(f) \equiv \langle f, \varphi \rangle$ for some $\varphi \in F$. For any $g \in \mathrm{SO}_{n-1}$, g fixes N , thus

$$\langle f, \varphi \rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi \rangle = \langle f, g\varphi \rangle$$

and because this is true for arbitrary $f \in F$ and $g \in \mathrm{SO}_{n-1}$, φ is fixed by SO_n . Now it remains to show that $\varphi \neq 0$. We can get this by assuming that some function in F takes a nonzero value on N (we can move N to some point where this is true, since F contains a nonzero function). \square

We can apply this key lemma to P_d or H_d as F .

Consider $P_d^{\mathrm{SO}_{n-1}}$, the homogeneous polynomials fixed by SO_{n-1} . On homework we showed that this is a subspace of $\mathbb{C}\langle x_n, R \rangle$ (where $R := x_1^2 + \cdots + x_n^2$). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \dots \rangle$$

thus $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$.

A very important fact about P_d is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2H_{d-4} \oplus R^3H_{d-6} \oplus \cdots \end{aligned}$$

(we will show this later). This allows us to deduce the dimension of H_d from P_d :

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1.$$

Likewise, we can decompose $P_d^{\mathrm{SO}_{n-1}}$ the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of $H_d^{\mathrm{SO}_{n-1}}$ as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each H_d is an *irreducible* representation of SO_n ; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an SO_n -representation, P_d decomposes exactly into the sequence H_{d-2j} for $2j \leq d$.

Theorem: If $n \geq 3$, then for each $d \geq 0$, the representation of SO_n in H_d is irreducible, and moreover the representations are all distinct for different d .¹

Proof. To show that the representations are distinct, we can use a homework problem which shows that the dimension of H_d is always increasing in d for any $n \geq 3$. \square

1.1. Differential Algebra. Let W be a vector space over k with basis w_1, \dots, w_n , and let x_1, \dots, x_n be a dual basis for W^* . We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over W , where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \rightarrow k[W]_{j-1}$$

which acts on $k[W]$ in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \dots, \partial_{w_n}]$$

¹In the case $n = 3$ this gives *all* the irreps. In general you miss $\Lambda^2(\mathbb{C}^n)$, but when $n = 3$ this is just \mathbb{C}^3 , which you get from H_1 .

acting on $k[W]$. There is a natural correspondence between $k[W]$ and $\mathcal{D}(W)$, if one assumes that k is characteristic 0. We have a k -bilinear pairing

$$\mathcal{D}(W) \times k[W] \rightarrow k$$

by $\langle u, f \rangle \mapsto u(f)(0)$. This is a *perfect pairing*. And in general we can do the same thing with

$$\text{Sym}^j(W) \times \text{Sym}^j(W^*) \rightarrow k.$$

Lemma: Let $\xi \in W$ and $f \in k[W]$. Then

$$\langle \xi^m, f \rangle = m!f(\xi).$$

In particular, if $f = \varphi \in W^*$, $\langle \xi^m, \varphi^m \rangle = m!\varphi^m(\xi)$.

Proof. We will show this for homogeneous f first, and the general result will follow from expressing f as a sum of homogeneous polynomials. Let the degree of f be d . Then by Taylor expansion,

$$f(\xi) = \sum_{k \geq 0} \frac{1}{k!} (\partial_\xi^k f)(0).$$

But note that only the d th term of this is nonzero, since $\partial_\xi^j f = 0$ unless $j = d$ (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other j both sides are 0. □

We can use this pairing to get another inner product on polynomials in $k[W]$ given by

$$\langle p, q \rangle := p(\partial)(q)(0)$$

where $p(\partial)$ is the corresponding element to p in $\mathcal{D}(W)$.² For this inner product, we have that multiplication by p is *adjoint* to $p(\partial)$, i.e.

$$\langle r, p(\partial)q \rangle = \langle pr, q \rangle.$$

With this fact, we can finally show why $P_d = H_d \oplus RP_{d-2}$:

$$W = \ker(\Delta) \oplus \text{im}(\Delta^*) = \ker(\Delta) \oplus \text{im}(R) = H_d \oplus RP_{d-2}.$$

Another application of this pairing: Let V be a finite dimensional vector space and $A \subseteq V$ a subset of V (not necessarily subspace). Let $\text{span}^d(A) \subseteq \text{Sym}^d(V)$ be generated over \mathbb{C} by a^d for $a \in A$. If A is dense in V then $\text{span}^d(A) = \text{Sym}^d(V)$. We will show this by using the pairing.

Assume for contradiction that $\text{span}^d(A) \neq \text{Sym}^d(V)$. Then there is some nonzero linear functional $F : \text{Sym}^d(V) \rightarrow \mathbb{C}$ which vanishes on $\text{span}^d(A)$. Then F corresponds to some differential polynomial f , and $\partial_a^d f(0) = 0$ for all $a \in A$. But $\partial_a^d f(0) = d!f(a)$, so $f(a) = 0$. But then A is dense, so $f = 0$.

1.2. Representation Theory Basics. If G acts on sets X and Y , then G can also act on the space of maps $X \rightarrow Y$ via conjugation:

$$g : f \mapsto g \circ f \circ g^{-1}.$$

We can ask about the space of maps which commute with this G -action. Or, equivalently, the maps which are fixed by the G -action. We call these *intertwining operators*. The set of such operators is denoted $\text{Hom}_G(X, Y)$.

We are usually interested in the case where X, Y are vector spaces and $\text{Hom}(X, Y)$ is the space of linear maps.

Schur-Weyl Duality: Let W be a finite-dimensional vector space over \mathbb{C} . $\text{GL}(W)$ can act on $W^{\otimes d}$ with g acting as $g^{\otimes d}$. S_d also acts on $W^{\otimes d}$ by permutation. It is not too hard to see that these two actions commute. But moreover, action by $\text{GL}(W)$ *spans* the space of S_d -intertwiners on $W^{\otimes d}$.

²In the homework, we establish that on H_d , this is actually *equivalent* to the inner product from integrating over S^{n-1} !

Proof. Let $\Phi : (\text{End}(W))^{\otimes d} \rightarrow \text{End}(W^{\otimes d})$ be given by

$$\Phi : a_1 \otimes \cdots \otimes a_d \mapsto (w_1 \otimes \cdots \otimes w_d \mapsto a_1(w_1) \otimes \cdots \otimes a_d(w_d)).$$

Φ is an invertible linear map with inverse

$$\Phi^{-1}f \mapsto f|_{W_1} \otimes \cdots \otimes f|_{W_d}$$

where W_j is $0 \otimes \cdots \otimes W \otimes \cdots \otimes 0$ with the W in the j th spot. Note also that Φ commutes with the action of S_d . By using Φ , we see that

$$\text{Sym}^d(\text{End } W) = ((\text{End } W)^{\otimes d})^{S_d} \xrightarrow{\Phi^{-1}} \text{End}_{S_d}(W^{\otimes d}).$$

So we only need to understand $\text{Sym}^d(\text{End } W)$. But $\text{GL}(W)$ is dense in $\text{End}(W)$, so by a previous lemma, we see that $\text{span}^d(\text{GL}(W)) = \text{Sym}^d(\text{End } W)$. \square

1.3. Spectral Theorem. Let A be a k -algebra with $a \in A$. We have an evaluation map

$$\text{ev}_a : k[t] \rightarrow A \quad \text{ev}_a : p \mapsto p(a).$$

Let $A_a := \text{im}(\text{ev}_a)$, i.e. the subalgebra of A generated by a . The kernel $\ker(\text{ev}_a)$ is an ideal of $k[t]$, and it is a principal ideal since $k[t]$ is a PID. Thus, in the case that ev_a is non-injective, there is a unique *minimal polynomial* of a , p_a , which divides every polynomial which vanishes at a .

Lemma: a is algebraic iff A_a is finite-dimensional.

Proof. If A_a is finite-dimensional then there is a relation between $1, a, a^2, \dots, a^n$ for some n , i.e. a polynomial that a solves. Conversely if a solves a polynomial of degree n then every linear combination of powers of a can be expressed by the first n powers of a . \square

We define the *spectrum* of a , denoted $\text{Spec}(a)$, as

$$\text{Spec}(a) := \{\lambda \in k : (a - \lambda) \text{ is not invertible}\}.$$

So for example, if A is a function algebra, $\text{Spec}(a)$ denotes the values that a can take. In the case that A is the matrix algebra $M_n(k)$, $\text{Spec}(a)$ is the set of eigenvalues of a .

The Spectral Theorem: Let A be a k -algebra of one of the following types:

- A is finite-dimensional over k and k is algebraically closed.
- A is countable-dimension and k is uncountable.

Then,

- (i) $\text{Spec}(a)$ is nonempty.
- (ii) a is nilpotent iff $\text{Spec}(a) = \{0\}$.
- (iii) If A is a division algebra then $A = k$.³

Proof. Lemma: If $\lambda_1, \dots, \lambda_n \notin \text{Spec}(a)$, i.e. $(a - \lambda_j)$ is invertible for each j , then if

$$\sum_j c_j (a - \lambda_j)^{-1} = 0$$

for some $c_j \in k$ then a is algebraic (proof is by clearing denominators). We will use this fact.

(i): We will split into two cases: if a is algebraic then $\text{Spec}(a)$ is finite but nonempty and if a is not algebraic then $\text{Spec}(a)$ is uncountable (and the converses to both of these are true).

If a is algebraic, then $\text{Spec}(a)$ is the roots of the minimal polynomial (HW), and particular this means $\text{Spec}(a)$ is finite and nonempty because k is algebraically closed.

If a is not algebraic, then by the Lemma, there is no linear relation between any finitely-many $(a - \lambda)^{-1}$ for $\lambda \notin \text{Spec}(a)$. We assumed that $\dim(A)$ is at most countable, and it has an independent set of size $|k \setminus \text{Spec}(a)|$, so $\text{Spec}(a)$ must be uncountable (because k is).

³For a counterexample of this when k is not algebraically closed, take the Quaternions over \mathbb{R} .

(ii): If $a^n = 0$, then $0 \in \text{Spec}(a)$ because a is not invertible, but all other $(a - \lambda)$ are invertible:

$$(a - \lambda)(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}) = a^n - \lambda^n = -\lambda^n.$$

so

$$(a - \lambda)^{-1} = -\lambda^{-n}(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}).$$

Conversely, suppose that $\text{Spec}(a) = \{0\}$. $\{0\}$ is a finite set, so by part (i), a is algebraic, but its minimal polynomial only has root $a = 0$, so $a^n = 0$ for some n .

(iii): Assume for contradiction that A is a division algebra yet $\exists a \in A \setminus k$. Then $(a - \lambda)$ is invertible for all $\lambda \in k$, but then $\text{Spec}(a)$ would be empty, contradicting (i). \square

1.4. Modules. A module M over ring A is called *simple* if it is nonzero and has no proper non-trivial submodules (i.e. its only submodules are 0 and M).

Schur's Lemma: If $f : M \rightarrow N$ is an A -linear map between *simple* A -modules M and N , then f is either 0 or an isomorphism.

Proof. $\ker(f)$ is a submodule of M and $\text{im}(f)$ is a submodule of N . By simplicity, both must be either trivial or the full module. This implies that f is either injective or 0, and either surjective or 0. \square

As a corollary, we see that $\text{End}_A M$ is a division ring.

Schur's Lemma for Algebras: If A is a k -algebra and M a simple A -module either

- k is algebraically closed and either A or M is finite-dimensional over k .
- $k = \mathbb{C}$ and either A or M is countable-dimension over k .

Then, $\text{End}_A M = k \text{id}_M$.

Proof. On HW we showed that $\dim(\text{End}_A M) \leq \dim_A M$. The lemma can be proven by applying the spectral theorem to the algebra $\text{End}_A M$. \square

If A satisfies the hypotheses of the Spectral Theorem and M is a simple A -module, then the center Z of A acts in M by scalars, as $z \cdot am = az \cdot m$ for $z \in Z, a \in A, m \in M$. And in particular if A is commutative then $\dim_k M = 1$ because every subspace of M is A -stable.

Schur's Lemma for Group Representations: If V, W are representations of a group G over a field k ,

- (i) If V, W are irreducible then all intertwiners are either 0 or isomorphisms.
- (ii) If $\dim_k(V)$ is finite and k is algebraically closed or $k = \mathbb{C}$ and $|G| = \aleph_0$, then

$$\text{End}_G V = k \cdot \text{id}_V.$$

Proof. This follows from applying Schur's Lemma for Algebras. A representation of G corresponds to a module over the group algebra $A := kG$. Note that $\dim_k(A) = |G|$. \square

1.5. Representations of S_n . S_n acts on \mathbb{R}^n by permuting the coordinates. We have the sign representation given by taking the determinant.

S_n also acts on P_d by permuting the variables:

$$\sigma(f)(x_1, \dots, x_n) := f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}).$$

Motivated by this action, we can consider the symmetric polynomials P^{S_n} .

A *partition* of n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_k$ whose sum is n . Let the set of partitions of n be \mathcal{P}_n . Corresponding to a partition, we have a decomposition of $[1, n]$ into I_1, I_2, \dots, I_k of length $|I_j| = \lambda_j$.

The *Vandermonde Determinant* is the polynomial

$$\Delta_n := \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

which can also be written as the determinant

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix}$$

Corresponding to a given partition λ , define

$$\Delta(I_m) := \prod_{i < j \in I_m} (x_j - x_i)$$

and

$$\Delta_\lambda := \prod_m \Delta(I_m).$$

Δ_λ is a homogeneous polynomial, and its degree is

$$d := \sum_m \frac{\lambda_m(\lambda_m - 1)}{2}$$

so $\Delta_\lambda \in P_d$.

The *Specht Module* associated with λ , denoted $V(\lambda)$ is the k -span of Δ_λ under the action of S_n . It is clearly stable under the action of S_n .⁴

Examples:

- Let $\lambda = (1, 1, 1, \dots, 1)$. Then $\Delta_\lambda = 1$, and $V(\lambda) = P_0$, the constant polynomials. The action of S_n on $V(\lambda)$ is trivial. Thus, this λ represents the trivial representation.
- Let $\lambda = (n)$. Then $\Delta_\lambda = \Delta_n$, the entire Vandermonde determinant. Since this is just a determinant whose columns are permuted by the action of S_n , the action scales by the sign of the permutation. This makes $V(\lambda) = k\Delta_n$, which is one-dimensional. It is the sign representation of S_n .

Note that in all of these cases $V(\lambda)$ is irreducible. This is actually true in general:

Theorem: Assuming that the underlying field k is characteristic 0, the Specht module is always an irreducible representation. Moreover, all irreps of S_n can be expressed as $V(\lambda)$ for some partition λ .⁵

Proof. The proof has three steps. The first step will be to show that $V(\lambda)$ is irreducible, which we do on homework. Step 2 is that the number of irreducible representations of S_n is equal to the number of partitions of n . Step 3 will show that the modules $V(\lambda)$ are pairwise non-isomorphic for different λ , and hence we have a bijection.

Step 3: In homework (it is fairly clear I think) we showed that if $d_\mu \neq d_\lambda$ then $V(\mu) \not\cong V(\lambda)$, so it remains to show this for μ, λ that have equal degree.

Notation: for $\nu \in \mathbb{Z}_{\geq 0}^n$ (note: may have repeats!), S_n acts on ν in the natural way. Let $m_j(\nu)$ denote the number of elements of ν that are equal to j (this is invariant under S_n). Let $\nu(\lambda)$ be

$$\nu(\lambda) := (1, 2, \dots, \lambda_1, 1, 2, \dots, \lambda_2, \dots, 1, 2, \dots, \lambda_n).$$

Then $m_j(\nu(\lambda))$ is the length of the j th column in the Young diagram of λ , $D(\lambda)$. Or equivalently, the m_j form another partition corresponding to the transposed Young diagram $D^T(\lambda)$.

⁴This is not the most common way to construct the Specht module of λ .

⁵It is also true that $V(\lambda)$ is a subspace of the S_n -harmonic polynomials (as defined on HW) and the index is $\dim(V(\lambda))$.

Similarly, we can apply all this to polynomials in n variables. The monomial x^ν is

$$x^\nu := \prod_{j=1}^n x_j^{\nu_j}$$

so that for a partition λ ,

$$x^{\nu(\lambda)} := \prod_{i=1}^k \prod_{j=1}^{\lambda_i} x_{\lambda_i+j}^j.$$

Recall that Δ_λ is defined in terms of Vandermonde determinants, which are expressed as

$$\Delta_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1}.$$

And for Δ_λ , we have a similar thing but with Young subgroups:

$$\Delta_\lambda = \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1} = \frac{1}{x_1 x_2 \cdots x_n} \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x^{s(\nu(\lambda))}.$$

Now, if $V(\mu) = V(\lambda)$, then $\Delta_\mu \in V(\lambda)$, i.e. it is a linear combination of permutations of Δ_λ . So for example, the monomial $x^{\nu(\mu)}$ appears as $x^{\sigma(\nu(\lambda))}$ for some σ . But we can show that this fails (*fill in later*). \square

The *Young Subgroup* of S_n corresponding to λ consists of all permutations which preserve all the pieces I_m . It is denoted S_λ and is isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$.

Define P^{S_λ} to be the polynomials fixed by S_λ , and $P^{\text{sgn}(\lambda)}$ the polynomials on which S_λ acts in an anti-symmetric way. We can see that $P^{\text{sgn}(\lambda)}$ is stable under scaling by P^{S_λ} , i.e. it is a P^{S_λ} -submodule.

Lemma: Let $F : V(\lambda) \rightarrow P_d$ be an S_n -intertwiner. Then

- (1) If $d = d_\lambda$, then F acts by scaling.
- (2) If $d < d_\lambda$, then F is trivial.

Proof. For $s \in S_\lambda$, $s(F(\Delta_\lambda)) = F(s(\Delta_\lambda))$, and $s(\Delta_\lambda) = \text{sgn}_\lambda(s) \cdot \Delta_\lambda$, so F acts by scaling. *fill in later* \square

In general, we can see that $V(\lambda^t) = V(\lambda) \otimes \text{sgn}$.