MATH 317 MIDTERM

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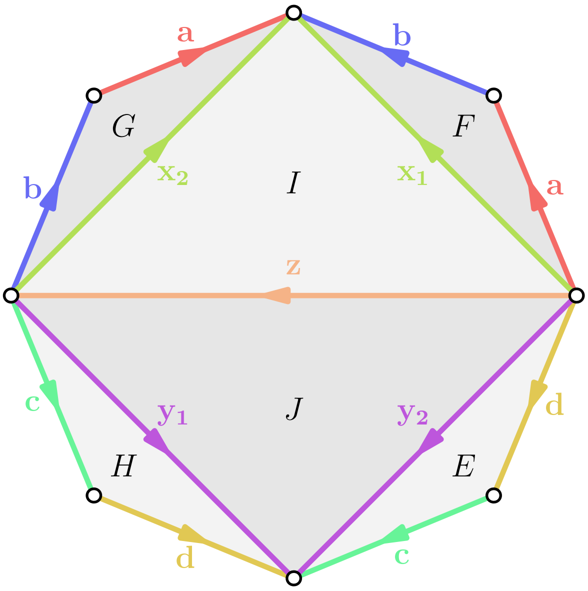
Problem 1: Let Σ_2 be the surface of genus 2. Give a construction of Σ_2 as a simplicial complex and use this to find $\pi_1(\Sigma_2)$ and $H_\bullet(\Sigma_2)$. Describe the Abelianization map from $\pi_1(\Sigma_2) \to H_1(\Sigma_2)$.

Proof. Σ_2 can be constructed as a simplicial complex with nine 1-simplices $\{a, b, c, d, x_1, x_2, y_1, y_2, z\}$, six 2-simplices $\{E, F, G, H, I, J\}$, and a single 0-simplex, as below:



The fundamental group can be derived via Van Kampen's Theorem: it is generated by the 1-simplices quotiented by the attaching maps of the 2-simplices, yielding

$$\pi_1(\Sigma_2) = \frac{\langle a, b, c, d, x_1, x_2, y_1, y_2, z\rangle}{\langle bax_2^{-1},\ abx_1^{-1},\ zx_2x_1^{-1},\ cdy_1^{-1},\ dcy_2^{-1},\ zy_1y_2^{-1}\rangle} = \frac{\langle a, b, c, d\rangle}{\langle aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle}$$

as $x_1 = ab,\ x_2 = ba,\ y_1 = cd,\ y_2 = dc,\ z = y_1y_2^{-1} = x_2x_1^{-1} = bab^{-1}a^{-1} = cdc^{-1}d^{-1}$.

Now for the homology groups: there is only one 0-simplex, so $Z_0 = \mathbb{Z}$. $B_0 = 0$, as the boundary of any 1-simplex is trivial, again because there is only one 0-simplex. Thus $H_0 = \mathbb{Z}$.

Everything in C_1 is a cycle because there is only one 0-simplex, so $Z_1 = C_1 = \mathbb{Z}^9$. B_1 is generated by the six relations given by the 2-cells:

$$B_1 = \langle a + b - x_1,\ b + a - x_2,\ c + d - y_1,\ d + c - y_2,\ x_1 - x_2 - z,\ y_1 - y_2 + z\rangle$$

In the quotient Z_1/B_1, x_1, x_2, y_1, y_2, z are all generated by a, b, c, d, as

$$x_1 = x_2 = a + b, \quad y_1 = y_2 = c + d, \quad z = x_1 - x_2 = y_2 - y_1 = 0,$$

so the homology group is just

$$H_1 = Z_1/B_1 = \langle a, b, c, d\rangle = \mathbb{Z}^4.$$

Z_2 is generated by a single element. To see this, take an arbitrary element $p \in C_2$

$$p := p_E \cdot E + p_F \cdot F + p_G \cdot G + p_H \cdot H + p_I \cdot I + p_J \cdot J.$$

Assuming that $p \in Z_2$, we have

$$\begin{aligned} 0 &= \partial p \\ &= p_G(a+b-x_2) + p_F(a+b-x_1) + p_I(z+x_2-x_1) + \dots \\ &= (a+b)(p_G+p_F) + (c+d)(p_H+p_E) + x_1(p_F+p_I) + x_2(p_I-p_G) + \dots \end{aligned}$$

which implies that $p_G = p_I = p_E = -p_J = -p_H = -p_F$. Thus Z_2 is generated by a single element,

$$Z_2 = \langle E - F + G - H + I - J \rangle$$

and B_2 is trivial because there are no 3-simplices. So $H_2 = Z_2 = \mathbb{Z}$.

For any $n \geq 3$, $H_n = 0$ trivially because there are no n-simplices in Σ_2.

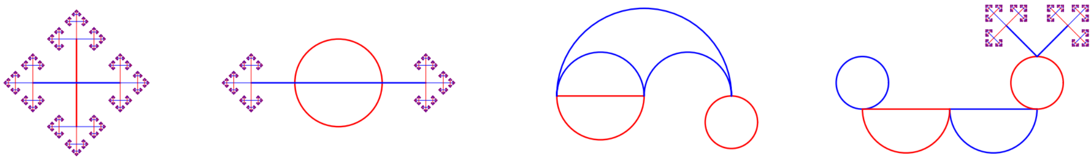
The Abelianization map $\pi_1(\Sigma_2) \to H_1(\Sigma_2)$ is just a quotient by the commutator subgroup, which is generated by $aba^{-1}b^{-1}$ and $cdc^{-1}d^{-1}$. In terms of this simplicial complex presentation, it is the quotient by z. Indeed, $z \equiv 0$ in $H_1(\Sigma_2)$. $\square$

Problem 2: Prove that $\mathbb{C}P^n$ does not non-trivially cover any other space.

Proof. $\mathbb{C}P^n$ can be constructed as a simplicial complex with one 0-cell S^0, one 2-cell S^2, ..., and one $2n$-cell S^{2n}, as we've seen in class.

Suppose $p : \mathbb{C}P^n \to X$ is a degree $d < \infty$ covering map. Give X a CW structure with one $2j$-cell $p(S^{2j})$ for $0 \leq j \leq n$. None of the cells can be collapsed to a lower dimension because the degree is supposed to be finite. But then X has the exact same CW structure as $\mathbb{C}P^n$, with the same attaching maps. Restricted to each cell, $p|_{S^{2j}}$ must be a degree-1 map because $\chi(S^{2j}) = 1$. So this covering can only be the identity. $\square$

Problem 3: Let $X = S^1 \vee S^1$. What is $\pi_1(X)$? For each of the following covering spaces of X, identify the corresponding subgroup of $\pi_1(X)$ and say whether the covering is regular (i.e. normal) or not.



Proof. $\pi_1(X)$ is the free group on two generators, i.e. $\mathbb{Z} * \mathbb{Z}$. Let a represent red and b blue.

The subgroups corresponding to each covering space can be calculated by taking representatives of edges outside a maximal spanning tree. To test whether a covering is normal, it is equivalent to test whether or not the subgroup depends on the basepoint or not (if it is the same for two given basepoints then that implies the existence of a deck transformation between them).

(a): This is the universal cover. Its fundamental group is trivial as there are no loops. This is a regular cover because 1 is clearly normal in $\mathbb{Z} * \mathbb{Z}$.

(b): Take the basepoint to be one of the two middle points and the maximal spanning tree as all edges except the red circle. This yields the subgroup

$$\langle ba,\ ba^{-1}\rangle.$$

But ba is not a loop anywhere other than that basepoint, so the cover is not normal.

(c): Take the basepoint to be the middle point among the three points and the maximal spanning tree to be the two blue edges attached to it. The resulting subgroup is

$$\langle ab, a^{-1}b, b^3, b^{-1}ab\rangle.$$

The subgroup one gets from choosing the left point as the basepoint is the same, but the right point has a loop a so it's not the same group. Thus, this is not a normal cover.

(d): Take the basepoint to be the point where the two semicircles touch, and take the maximal spanning tree to be the two flat segments connected to it and one half of the red circle on the right, along with the entire free tree attached to it. This yields the subgroup

$$\langle aba^{-1}, a^2, b^2, ba^2b^{-1}\rangle$$

If we started from the leftmost point as the basepoint, b would be in the group, but it is not, so this is also not a normal cover.

$\square$

Problem 4: Let Σ_n be the closed, oriented surface of genus n.

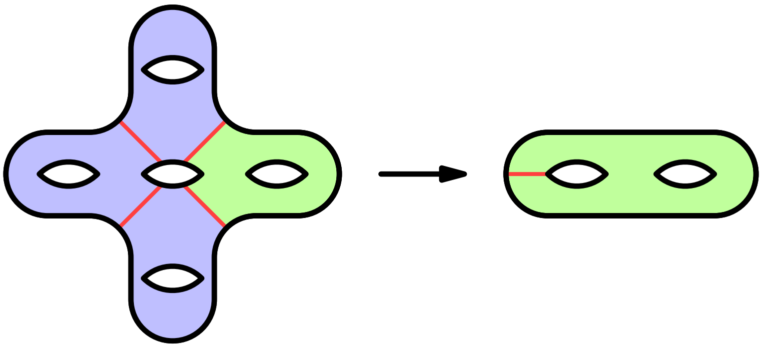
(a) Show that there is a degree p cover $\Sigma_m \to \Sigma_n$ iff $m = pn - p + 1$.
(b) If $m = pn - p + 1$, show that there is a *regular* cover $\Sigma_m \to \Sigma_n$ with deck group $\mathbb{Z}/p\mathbb{Z}$. Is there more than one?
(c) In the case $n = 2$ and $p = 4$, compute the action of the deck group on the homology of Σ_5 for the explicit covering constructed in part (b).

Proof. (a): Suppose a degree-p projection $q : \Sigma_m \to \Sigma_n$ exists. We know that such a projection multiplies the Euler characteristic by p. And we know that $\chi(\Sigma_g) = 2 - 2g$. So

$$2 - 2m = \chi(\Sigma_m) = p \cdot \chi(\Sigma_n) = p(2 - 2n) \implies m = p(n-1) + 1.$$

This proves the "only if" direction. For the "if" direction, we construct such a covering explicitly in part (b).

(b): Consider the cover $\Sigma_5 \to \Sigma_2$ shown below:



The red circles are identified to close up the green sector, forming another hole. Each of the other three "handles" is mapped in the same way. The deck group is $\mathbb{Z}/4\mathbb{Z}$, which acts by rotating the Σ_5. Since this is transitive on the fiber of any basepoint, it is certainly a regular covering space.

(c): $H_0(\Sigma_5) = \mathbb{Z}$, as all points are equivalent mod boundaries (Σ_5 is path-connected). So $\mathbb{Z}/4\mathbb{Z}$ acts trivially on $H_0(\Sigma_5)$.

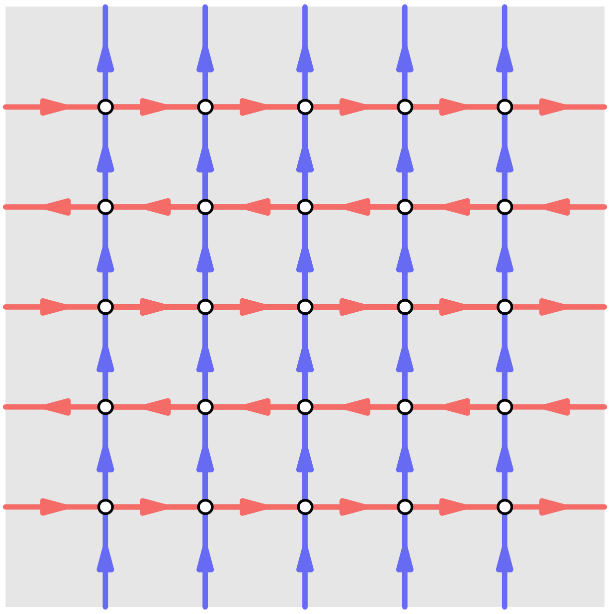
$H_1(\Sigma_5) = \mathbb{Z}^{10}$, generated by two loops for each hole. If these loops are $\langle a_1, b_1, \ldots, a_5, b_5 \rangle$ then $\mathbb{Z}/4\mathbb{Z}$ acts on these by rotating the outer loops and fixing the inner one.

$H_2(\Sigma_5) = \mathbb{Z}$, generated by the single 2-cell (or a linear combination of all 2-simplices). $\mathbb{Z}/4\mathbb{Z}$ acts trivially.

□

Problem 5: Show that the universal cover of the Klein bottle is homeomorphic to a plane. Deduce that the Klein bottle is a $K(\pi, 1)$ and conclude that its fundamental group is torsion-free.

Proof. $\mathbb{R}^2$ covers the Klein bottle via the covering map depicted below:



The plane is clearly contractible, so this is the universal cover and the Klein bottle is a $K(\pi, 1)$.

As we showed in class, if a $K(\pi, 1)$ has finite dimension then π is torsion-free. It follows that the Klein bottle's fundamental group is torsion-free as well.

Alternatively, one could explicitly state the multiplication rules for π and see that it is torsion-free. We have

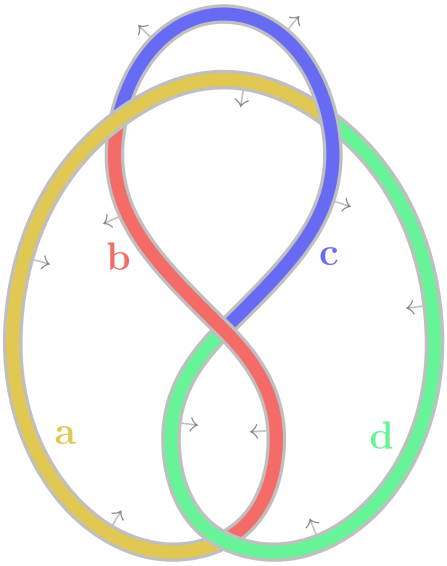
$$\pi := \langle a, b \mid ab = ba^{-1} \rangle$$

so every element in π can be written uniquely as $a^n b^m$ for some $n, m \in \mathbb{Z}$, with multiplication given by

$$(a^n b^m) \cdot (a^p b^q) = \begin{cases} a^{n-p} b^{m+q} & m \text{ is odd} \\ a^{n+p} b^{m+q} & m \text{ is even} \end{cases}$$

so as one can see, powers of any element $a^n b^m$ with $m \neq 0$ can never be 1, and clearly powers of a^n can never be 1 either. $\square$

Problem 6: The figure 8 knot is the embedded circle in the 3-sphere shown in the figure below. Compute the fundamental group and the homology groups of the complement of the knot in S^3.



Let this knot be $K \subset S^3$, with the orientation indicated by the arrows. We can use the Wirtinger presentation of $\pi_1(S^3 \setminus K)$ as described in a homework problem. It is generated by four elements a, b, c, d corresponding to unbroken knot-segments when viewed from below, which denote 1-cells crossing those segments in the direction given by the knot's orientation (here I've depicted the knot from above, though it shouldn't affect the fundamental group). The relations between a, b, c, d are given by conjugation at the crossing-points:

$$\begin{aligned}\pi_1(S^3 \setminus K) &= \langle a, b, c, d \mid aba^{-1} = c,\ \ dbd^{-1} = a,\ \ cdc^{-1} = a,\ \ bdb^{-1} = c\rangle \\ &= \langle b, d \mid bdb^{-1}dbd^{-1}b^{-1}db^{-1}d^{-1} = 1\rangle\end{aligned}$$

In the Abelianization of this group, this becomes $\langle b, d \mid b^{-1}d = 1\rangle = \mathbb{Z}$. Thus, $H_1(S^3 \setminus K) = \mathbb{Z}$.

To compute the other homology groups, note that (as described in this same recent homework problem) $S^3 \setminus K$ can be deformation retracted to a 2-dimensional CW complex X (or a 2-dimensional simplicial complex). Thus, the homology groups of $S^3 \setminus K$ are the same as those of X.

$H_0(S^3 \setminus K) = \mathbb{Z}$, as X is connected, so every 0-cell is identified when taking the quotient by boundaries.

$H_1(S^3 \setminus K) = \mathbb{Z}$ as already shown above.

$H_2(S^3 \setminus K) = Z_2(X)$, as $B_2(X)$ is trivial (there are no 3-simplices in X hence no 2-boundaries). When X is constructed as a CW complex, the 2-cells (corresponding to the squares covering crossings) have boundaries

$$a + b - a - c,\ d + b - d - a,\ c + d - c - a,\ b + d - b - c.$$

If the coefficients of these in some 2-cycle p are p_1, p_2, p_3, p_4, then

$$\partial p = p_1(b-c)+p_2(b-a)+p_3(d-a)+p_4(d-c) = a(-p_2-p_3)+b(p_2+p_1)+c(-p_1-p_4)+d(p_3+p_4).$$

If p is a cycle, then $\partial p = 0$, so

$$p_1 = -p_2 = p_3 = -p_4$$

so $Z_2(X)$ is spanned by a single 2-cycle with coefficients in this proportion. Thus $H_2(S^3 \setminus K) = \mathbb{Z}$.

$H_n(S^3 \setminus K) = 0$ for $n \geq 3$, as there are no 3-cells in X.

Problem 7: State and prove the Five Lemma.

Proof. Given an exact sequence of five Abelian groups A, B, C, D, E with group homomorphisms $p_\bullet$ to another exact sequence A', B', C', D', E' such that the below diagram commutes, we would like to show that if p_A, p_B, p_D, p_E are isomorphisms, then p_C is as well.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, leftrightarrow, "p_A"'] & B \arrow[r, "f_2"] \arrow[d, leftrightarrow, "p_B"'] & C \arrow[r, "f_3"] \arrow[d, "p_C"] & D \arrow[r, "f_4"] \arrow[d, leftrightarrow, "p_D"] & E \arrow[d, leftrightarrow, "p_E"] \\
A' \arrow[r, "g_1"'] & B' \arrow[r, "g_2"'] & C' \arrow[r, "g_3"'] & D' \arrow[r, "g_4"'] & E'
\end{tikzcd}$$

We'll construct the inverse of p_C. Given $y \in C'$, we have $g_3(y) \in D'$ and $z := p_D^{-1}(g_3(y)) \in D$. By exactness of the bottom sequence, $g_3(y) \in \ker(g_4)$, so by the commutativity of the rightmost square,

$$f_4(z) \in \ker(p_E) \implies f_4(z) = 0 \implies z \in \operatorname{im}(f_3)$$

using the fact that p_E is an isomorphism. So take some $x \in C$ such that $f_3(x) = z$. It is clear by the commutativity of the C, D, C', D' square that $p_C(x) = y$. Now it remains to show that this x is unique.

It suffices to show that if $p_C(x) = 0$ then $x = 0$, since if $p_C(x) = p_C(x')$ then

$$p_C(x) - p_C(x') = p_C(x - x') = 0 \implies x = x'.$$

So suppose $p_C(x) = 0$. Then $f_3(x) = 0$ because p_D is an isomorphism, so $x \in \operatorname{im}(f_2)$. Let $w \in B$ such that $f_2(w) = x$. Again by commutativity, $g_2(p_B(w)) = 0$, so $p_B(w) \in \operatorname{im}(g_1)$, and thus because p_A is an isomorphism $w \in \operatorname{im}(f_1)$. But this implies that $w \in \ker(f_2)$, so $0 = f_2(w) = x$. $\square$