MATH 325 HW 7

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Problem 1: Let $\mathcal{U}(e,h,f)$ be an associative $\mathbb{C}$-algebra with generators e,h,f with relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h.$$

Let V be a $\mathcal{U}(e,h,f)$-module and $v \in V$ a nonzero element such that $h(v) = \lambda v$ where $\lambda \in \mathbb{C}$.

(a) Find an explicit formula for $h(f^i(v))$ as a function of λ.
(b) Assume that $e(v) = 0$. Find explicit formulas for $e(f^i(v))$.
(c) Show that if V is finite-dimensional then there is some nonzero $v \in V$ and nonnegative $d \in \mathbb{Z}$ such that $e(v) = 0$ and $h(v) = dv$.
(d) Classify all simple finite-dimensional $\mathcal{U}(e,h,f)$-modules up to isomorphism.

Proof. (a): We know $hf - fh = -2f$, so

$$hf(v) = (fh - 2f)(v) = (\lambda - 2)v.$$

I claim that in general $h(f^i v) = (\lambda - 2i)f^i v$. This can be seen inductively:

$$\begin{aligned} hf^i v &= hf(f^{i-1}v) \\ &= (fh - 2f)(f^{i-1}v) \\ &= f(hf^{i-1}v) - 2f^i(v) \\ &= f(\lambda - 2(i-1))f^{i-1}v - 2f^i v \\ &= (\lambda - 2i)f^i v. \end{aligned}$$

Similarly $h(e^i v) = (\lambda + 2i)e^i v$.

(b): Because $ef - fe = h$,

$$ef(v) = (fe + h)(v) = f(0) + \lambda v = \lambda v.$$

For general i, we have $ef^i(v) = \lambda f^i v + ((i-1)\lambda - i(i+1))f^{i-1}v$ by induction:

$$\begin{aligned} ef^i(v) &= ef(f^{i-1}v) \\ &= (fe + h)(f^{i-1}v) \\ &= f(ef^{i-1}v) + hf^{i-1}v \\ &= (\lambda f^i v + ((i-2)\lambda - i(i-1))f^{i-1}v) + (\lambda - 2(i-1))f^{i-1}v \\ &= \lambda f^i v + ((i-1)\lambda - i(i+1))f^{i-1}(v) \end{aligned}$$

(c): Let v be any nonzero vector in V. By part (a), we see that $e^i v$ is an eigenvector of h for all i, and they all have different eigenvalues, so $e^i v = 0$ for all but finitely-many $i \in \mathbb{N}$. Thus let i be maximal such that $e^i v \neq 0$. Then $e(e^i v) = 0$ but $h(e^i v) = (\lambda + 2i)(e^i v)$. *to do: why is λ an integer.*

(d): Let V be a simple $\mathcal{U}(e,h,f)$-module. By (c), there is some v with $e(v) = 0$ and $h(v) = dv$. By (a) and (b), the subspace $V' = \langle v, f(v), f^2(v), \ldots \rangle$ is closed under multiplication by h and e, and clearly f as well, so V' is a submodule. Since V is simple and $V' \neq 0$ (as $v \neq 0$), $V' = V$.

As before, $f^i(v)$ is an h-eigenvector for all i, and they all have different eigenvalues, which implies that the nonzero $f^i(v)$ are all distinct and linearly independent. So because V is finite-dimensional, there must be some minimal n for which $f^n(v) = 0$. Then V has basis

$$V := \langle v, fv, f^2v, \ldots, f^n v \rangle$$

By (a) and (b), it's already determined how h, e, f act on this basis. So there is exactly one simple $\mathcal{U}(e, f, h)$-module of dimension n up to isomorphism, for each n.

$\square$

Problem 2:

(a) Check that the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form an $\mathbb{R}$-basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and that these matrices satisfy the relations in problem 1.

Proof. $\mathfrak{sl}_2(\mathbb{R})$ is exactly the matrices with trace 0, i.e. those of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH + bE + cF$$

for some $a, b, c \in \mathbb{R}$. Thus E, F, H are a basis for $\mathfrak{sl}_2(\mathbb{R})$.

It is easy to check the three relations by just doing the matrix multiplications. In fact, we can check that $HE = E, EH = -E$, and and $HF = -F, FH = F$, and

$$EF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad FE = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

The relations follow. $\square$

Problem 3: Let $1_{ij} \in M_n(\mathbb{R})$ denote the matrices with 1 in the (i, j) place and 0 elsewhere.

(a) Check that for any $i < j$, the matrices $e = 1_{ij}, h = 1_{ii} - 1_{jj}, f = 1_{ji}$ satisfy the relations in problem 1.

(b) Let $\phi : M_n(\mathbb{R}) \to \mathrm{End}_{\mathbb{C}}(V)$ be a Lie algebra representation, where $M_n(\mathbb{R}) = \mathrm{Lie}(\mathrm{GL}_n(\mathbb{R}))$ is viewed as a Lie algebra wrt the commutator and V is a finite-dimensional complex vector space. Prove (without using arguments from class) that there is a nonzero $v \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that
- $\phi(1_{ii})(v) = \lambda_i v$ for all i.
- $\phi(1_{ij})(v) = 0$ for all $i < j$.
- $\lambda_i - \lambda_{i+1}$ is a nonnegative integer for all $1 \leq i \leq n - 1$.

Proof. (a) Just as in problem 2, we can check that $he = e, eh = -e, hf = -f, fh = f$. And $ef = 1_{ii}, fe = 1_{jj}$, which shows $ef - fe = h$.

(b): $\square$

The group $\mathrm{GL}_2(k)$ acts on k^2 in the usual way. For $k = \mathbb{R}$ this action induces an action on $C^\infty(\mathbb{R}^2)$ by $g : p \mapsto g^* p$. For any $p \in C^\infty(\mathbb{R}^2)$ and $\chi \in \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R})) = M_2(\mathbb{R})$, the Lie derivative $L_\xi(p)$ is a function on $\mathbb{R}^2$ defined by

$$(L_\xi(p))(x, y) = \frac{\mathrm{d}(e^{t\xi})^*(p) \cdot (x, y)}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}(p(e^{-t\xi}(x, y)))}{\mathrm{d}t}\Big|_{t=0}.$$

Problem 4: For $\chi = E, H, F$ as in problem 2, find an explicit formula for $L_\xi(p)$ in terms of the partials of the function p and check that the operators L_H, L_E, L_F satisfy relations in problem 1.

Proof. Let

$$e^{-t\xi} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}.$$

Note that

$$\xi = \begin{bmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{bmatrix}.$$

We can calculate $L_\xi(p)$ as

$$\begin{aligned} L_\xi(p) &= \frac{\mathrm{d}(p(e^{-t\xi}(x, y)))}{\mathrm{d}t}\Big|_{t=0} \\ &= \frac{\mathrm{d}p(a(t)x + b(t)y, c(t)x + d(t)y)}{\mathrm{d}t}\Big|_{t=0} \\ &= (\partial_1 p)(a'(0)x + b'(0)y) + (\partial_2 p)(c'(0)x + d'(0)y) \\ &= (\partial_1 p) \cdot \xi_1(x, y) + (\partial_2 p) \cdot \xi_2(x, y). \end{aligned}$$

For $\xi \in \{H, E, F\}$ from the previous problem, this yields

$$L_H(p) = \partial_1 p \cdot x - \partial_2 p \cdot y, \;\; L_E(p) = \partial_2 p \cdot x, \;\; L_F(p) = \partial_1 p \cdot y.$$

Checking the relations from problem 1,

$$\begin{aligned} (L_H L_E - L_E L_H)(p) &= (\partial_1 L_E)x - (\partial_2 L_E)y - (\partial_2 L_H)x \\ &= (\partial_2 p)x - 0 + (\partial_2 p)x \\ &= 2(\partial_2 p)x = 2L_E(p) \end{aligned}$$

and similarly

$$\begin{aligned} (L_H L_F - L_F L_H)(p) &= (\partial_1 L_F)x - (\partial_2 L_F)y - (\partial_1 L_H)y \\ &= 0 - (\partial_1 L_F)y - (\partial_1 p)y \\ &= -2(\partial_1 p)y = -2L_F(p) \end{aligned}$$

and

$$\begin{aligned} (L_E L_F - L_F L_E)(p) &= (\partial_2 L_F)x - (\partial_1 L_E)y \\ &= (\partial_1 p)x - (\partial_2 p)y \\ &= L_H(p). \end{aligned}$$

$\square$

Problem 5: Use problems 1 and 4 to prove that the representations $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(P_d)$ are precisely the irreducible finite dimensional continuous representations of $\mathrm{SL}_2(\mathbb{R})$.

Proof. In problem 4 we showed that L_H, L_E, L_F satisfy the relations of problem 1, so the conclusion of problem 1 follows: there is exactly one simple n-dimensional (L_H, L_E, L_F)-module up to isomorphism.

Let $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(P_d)$ be the representation

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^i y^j = (ax+by)^i (cx+dy)^j.$$

We showed in class that if ρ is irreducible then $\mathrm{d}\rho : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{End}(P_d)$ is irreducible, since $\mathrm{SL}_2(\mathbb{R})$ is connected.

Note that

$$e^{tH} = I + tH + t^2 I/2 + t^3 H/6 + \cdots = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

and

$$e^{tE} = I + tE = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad e^{tF} = I + tF = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}.$$

So can calculate $\mathrm{d}\rho$ on the basis H, E, F of $\mathfrak{sl}_2(\mathbb{R})$ as

$$\begin{aligned} \mathrm{d}\rho(H)(x^i y^j) &= \partial_t \rho(e^{tH}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t (e^{ti} e^{-tj})\big|_{t=0} \\ &= i - j. \end{aligned}$$

For E, we have

$$\begin{aligned} \mathrm{d}\rho(E)(x^i y^j) &= \partial_t \rho(e^{tE}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t (x+ty)^i (y)^j\big|_{t=0} \\ &= y \cdot i(x+ty)^{i-1} \cdot y^j\big|_{t=0} \\ &= i x^{i-1} y^{j+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{d}\rho(F)(x^i y^j) &= \partial_t \rho(e^{tF}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t (x)^i (y-tx)^j\big|_{t=0} \\ &= x^i \cdot j(y-tx)^{j-1} \cdot -x\big|_{t=0} \\ &= -j x^{i+1} y^{j-1}. \end{aligned}$$

In particular for y^d, $\mathrm{d}\rho(H)(y^d) = -d$, $\mathrm{d}\rho(E)(y^d) = 0$, and $\mathrm{d}\rho(F)(y^d) = -dxy^{d-1}$.

□

Problem 6:

(a) Show that the group SU_2 of unitary 2×2 matrices with determinant 1 is formed by the matrices

$$\left\{ g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

(b) Check that the following matrices form an $\mathbb{R}$-basis of the Lie algebra $\mathfrak{su}_2 = \mathrm{Lie}(SU_2) \subset M_2(\mathbb{C})$:

$$I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(c) Find $[I_i, I_j]$ for $i, j \in \{1, 2, 3\}$ and express I_i in terms of H, F, E.

Problem 7:

(a) Classify irreducible representations of $\mathfrak{su}_2$ in finite-dimensional complex vector spaces up to isomorphism.

(b) Let $g_d : SU_2 \to \mathrm{GL}(P_d)$ be the representation of SU_2 in the vector space P_d. Show that the representations g_d are precisely the irreducible finite-dimensional continuous representations of SU_2.

Problem 8: Let $Ad : SU_2 \to \mathrm{GL}(P_d)$ be the representation of SU_2 that sends $g \in SU_2$ to $Adg : \mathfrak{su}_2 \to \mathfrak{su}_2$, where $Adg : x \mapsto gxg^{-1}$. Let $\mathrm{d}(Ad) : \mathfrak{su}_2 \to \mathrm{End}_{\mathbb{R}}(\mathfrak{su}_2) = |lie(\mathrm{GL}(\mathfrak{su}_2))$ be the differential of the representation Ad.

(a) Show that $\ker(Ad) = \pm\,\mathrm{id}$ and that $\mathrm{d}(Ad)$ is injective.

(b) Construct a surjective morphism of Lie groups $SU_2 \to \mathrm{SO}_3(\mathbb{R})$ with kernel $\pm\,\mathrm{id}$. That is, construct an isomorphism $\mathrm{SO}_3(\mathbb{R}) \cong SU_2/\{\pm\,\mathrm{id}\}$.

Problem 9:

(a) Prove that any continuous irreducible representation of the group $\mathrm{SO}_3(\mathbb{R})$ has odd dimension. Moreover, for every odd integer $2m+1$, there is exactly one continuous irreducible representation of dimension $2m+1$.

(b) Prove theorem 2.1.2(2), which says that any continuous finite-dimensional representation of $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to the representation H_d for some $d \geq 0$.