

# ALGEBRA I NOTES

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ABSTRACT. These are my notes from Victor Ginzburg's Representation Theory (Math 325) class at UChicago, Autumn 2025.

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## 1. INTRODUCTION

In this class we'll be interested in the representations of matrix groups. Something like  $\mathrm{GL}(V)$  or  $\mathrm{SO}(V)$  clearly acts on  $V$ , but it can also act on other interesting spaces. One relevant case of this for us will be when  $G$  acts on polynomials in  $x_1, \dots, x_n$ . Let

$$P_d \subseteq \mathbb{C}[x_1, \dots, x_n]$$

be the subspace of homogeneous degree- $d$  polynomials in  $n$  variables. This space has a basis given by the monomials

$$\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\}$$

and hence is finite-dimensional.  $P_d$  is stable under action by  $\mathrm{GL}_n$ . This is because linear transformation does not affect the degree of monomials (every  $x_j$  is sent to a linear combination of  $x_1, x_2, \dots, x_n$ ).

Consider the case of  $G = \mathrm{O}_n$ , the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by  $R$  is an intertwining map  $P_d \rightarrow P_{d+2}$ , meaning  $R \circ g^* f = g^*(R \circ f)$ .

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This  $\Delta$  is an  $\mathrm{O}_n$ -intertwining operator.

We call a function  $f$  *harmonic* if it has  $\Delta(f) = 0$ . The space of harmonic polynomials in  $n$  variables of degree  $d$  is denoted  $H_d \subseteq P_d$ . For  $d \in \{0, 1\}$ ,  $H_d = P_d$ , but for  $d \geq 2$   $H_d$  is strictly smaller. Note that  $H_d$  is stable under orthogonal transformations.

We will now work toward showing that  $H_d$  is an irreducible  $\mathrm{SO}_n$ -representation for  $n \geq 3$ .

A representation  $\rho : G \mapsto \mathrm{GL}(V)$  is *unitary* if  $G$  always acts as a unitary operator (i.e. preserves Hermitian inner product) on  $V$ . We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} e^{-|x|^2} dx$$

where  $dx$  is the Lebesgue measure. Action of  $\mathrm{SO}_n$  on  $P_d$  preserves this inner product.

Alternatively, we could put an inner product on  $P_d$  (or on all functions) from integration over  $S^{n-1}$  (the sphere). And polynomials in  $P_d$  are determined by their behavior on  $S^{n-1}$ .

**Proposition:** If  $V$  is a finite-dimensional vector space with an inner product, then any *unitary* action of  $G$  on  $V$  is completely reducible. Specifically, if  $W \subseteq V$  is a  $G$ -stable subspace, then one can decompose the action into  $V = W \oplus W^\perp$ .

*Proof.* The thing that we need to prove is that if  $W$  is  $G$ -stable then  $W^\perp$  is as well. Let  $x \in W^\perp$  and  $w \in W$ . Because  $g$  acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since  $g^{-1} \cdot w \in W$  by  $G$ -stability of  $W$ . □

**Key Lemma:** If  $F \subseteq C(S^{n-1})$  is any subspace stable under  $\mathrm{SO}_n$ , then it has an element fixed by  $\mathrm{SO}_{n-1}$ .

*Proof.* Let  $N := (0, 0, \dots, 0, 1) \in S^{n-1}$ . We have the evaluation map  $\alpha : C(S^{n-1}) \rightarrow \mathbb{C}$  given by evaluating functions at  $N$ . We have an inner product on  $F$  given by

$$\langle f, g \rangle := \int_{S^{n-1}} f \bar{g}$$

which is clearly fixed by  $\mathrm{SO}_n$ , thus  $F$  is a unitary representation of  $\mathrm{SO}_n$ . By Riesz representation theorem,  $\alpha(f) \equiv \langle f, \varphi \rangle$  for some  $\varphi \in F$ . For any  $g \in \mathrm{SO}_{n-1}$ ,  $g$  fixes  $N$ , thus

$$\langle f, \varphi \rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi \rangle = \langle f, g\varphi \rangle$$

and because this is true for arbitrary  $f \in F$  and  $g \in \mathrm{SO}_{n-1}$ ,  $\varphi$  is fixed by  $\mathrm{SO}_n$ . Now it remains to show that  $\varphi \neq 0$ . We can get this by assuming that some function in  $F$  takes a nonzero value on  $N$  (we can move  $N$  to some point where this is true, since  $F$  contains a nonzero function).  $\square$

We can apply this key lemma to  $P_d$  or  $H_d$  as  $F$ .

Consider  $P_d^{\mathrm{SO}_{n-1}}$ , the homogeneous polynomials fixed by  $\mathrm{SO}_{n-1}$ . On homework we showed that this is a subspace of  $\mathbb{C}\langle x_n, R \rangle$  (where  $R := x_1^2 + \cdots + x_n^2$ ). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \dots \rangle$$

thus  $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$ .

A very important fact about  $P_d$  is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2H_{d-4} \oplus R^3H_{d-6} \oplus \cdots \end{aligned}$$

(we will show this later). This allows us to deduce the dimension of  $H_d$  from  $P_d$ :

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1.$$

Likewise, we can decompose  $P_d^{\mathrm{SO}_{n-1}}$  the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of  $H_d^{\mathrm{SO}_{n-1}}$  as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each  $H_d$  is an *irreducible* representation of  $\mathrm{SO}_n$ ; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an  $\mathrm{SO}_n$ -representation,  $P_d$  decomposes exactly into the sequence  $H_{d-2j}$  for  $2j \leq d$ .

**Theorem:** If  $n \geq 3$ , then for each  $d \geq 0$ , the representation of  $\mathrm{SO}_n$  in  $H_d$  is irreducible, and moreover the representations are all distinct for different  $d$ .<sup>1</sup>

*Proof.* To show that the representations are distinct, we can use a homework problem which shows that the dimension of  $H_d$  is always increasing in  $d$  for any  $n \geq 3$ .  $\square$

**1.1. Differential Algebra.** Let  $W$  be a vector space over  $k$  with basis  $w_1, \dots, w_n$ , and let  $x_1, \dots, x_n$  be a dual basis for  $W^*$ . We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over  $W$ , where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \rightarrow k[W]_{j-1}$$

which acts on  $k[W]$  in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \dots, \partial_{w_n}]$$

<sup>1</sup>In the case  $n = 3$  this gives *all* the irreps. In general you miss  $\Lambda^2(\mathbb{C}^n)$ , but when  $n = 3$  this is just  $\mathbb{C}^3$ , which you get from  $H_1$ .

acting on  $k[W]$ . There is a natural correspondence between  $k[W]$  and  $\mathcal{D}(W)$ , if one assumes that  $k$  is characteristic 0. We have a  $k$ -bilinear pairing

$$\mathcal{D}(W) \times k[W] \rightarrow k$$

by  $\langle u, f \rangle \mapsto u(f)(0)$ . This is a *perfect pairing*. And in general we can do the same thing with

$$\text{Sym}^j(W) \times \text{Sym}^j(W^*) \rightarrow k.$$

**Lemma:** Let  $\xi \in W$  and  $f \in k[W]$ . Then

$$\langle \xi^m, f \rangle = m!f(\xi).$$

In particular, if  $f = \varphi \in W^*$ ,  $\langle \xi^m, \varphi^m \rangle = m!\varphi^m(\xi)$ .

*Proof.* We will show this for homogeneous  $f$  first, and the general result will follow from expressing  $f$  as a sum of homogeneous polynomials. Let the degree of  $f$  be  $d$ . Then by Taylor expansion,

$$f(\xi) = \sum_{k \geq 0} \frac{1}{k!} (\partial_\xi^k f)(0).$$

But note that only the  $d$ th term of this is nonzero, since  $\partial_\xi^j f = 0$  unless  $j = d$  (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other  $j$  both sides are 0. □

We can use this pairing to get another inner product on polynomials in  $k[W]$  given by

$$\langle p, q \rangle := p(\partial)(q)(0)$$

where  $p(\partial)$  is the corresponding element to  $p$  in  $\mathcal{D}(W)$ .<sup>2</sup> For this inner product, we have that multiplication by  $p$  is *adjoint* to  $p(\partial)$ , i.e.

$$\langle r, p(\partial)q \rangle = \langle pr, q \rangle.$$

With this fact, we can finally show why  $P_d = H_d \oplus RP_{d-2}$ :

$$W = \ker(\Delta) \oplus \text{im}(\Delta^*) = \ker(\Delta) \oplus \text{im}(R) = H_d \oplus RP_{d-2}.$$

Another application of this pairing: Let  $V$  be a finite dimensional vector space and  $A \subseteq V$  a subset of  $V$  (not necessarily subspace). Let  $\text{span}^d(A) \subseteq \text{Sym}^d(V)$  be generated over  $\mathbb{C}$  by  $a^d$  for  $a \in A$ . If  $A$  is dense in  $V$  then  $\text{span}^d(A) = \text{Sym}^d(V)$ . We will show this by using the pairing.

Assume for contradiction that  $\text{span}^d(A) \neq \text{Sym}^d(V)$ . Then there is some nonzero linear functional  $F : \text{Sym}^d(V) \rightarrow \mathbb{C}$  which vanishes on  $\text{span}^d(A)$ . Then  $F$  corresponds to some differential polynomial  $f$ , and  $\partial_a^d f(0) = 0$  for all  $a \in A$ . But  $\partial_a^d f(0) = d!f(a)$ , so  $f(a) = 0$ . But then  $A$  is dense, so  $f = 0$ .

**1.2. Representation Theory Basics.** If  $G$  acts on sets  $X$  and  $Y$ , then  $G$  can also act on the space of maps  $X \rightarrow Y$  via conjugation:

$$g : f \mapsto g \circ f \circ g^{-1}.$$

We can ask about the space of maps which commute with this  $G$ -action. Or, equivalently, the maps which are fixed by the  $G$ -action. We call these *intertwining operators*. The set of such operators is denoted  $\text{Hom}_G(X, Y)$ .

We are usually interested in the case where  $X, Y$  are vector spaces and  $\text{Hom}(X, Y)$  is the space of linear maps.

**Schur-Weyl Duality:** Let  $W$  be a finite-dimensional vector space over  $\mathbb{C}$ .  $\text{GL}(W)$  can act on  $W^{\otimes d}$  with  $g$  acting as  $g^{\otimes d}$ .  $S_d$  also acts on  $W^{\otimes d}$  by permutation. It is not too hard to see that these two actions commute. But moreover, action by  $\text{GL}(W)$  *spans* the space of  $S_d$ -intertwiners on  $W^{\otimes d}$ .

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<sup>2</sup>In the homework, we establish that on  $H_d$ , this is actually *equivalent* to the inner product from integrating over  $S^{n-1}$ !

*Proof.* Let  $\Phi : (\text{End}(W))^{\otimes d} \rightarrow \text{End}(W^{\otimes d})$  be given by

$$\Phi : a_1 \otimes \cdots \otimes a_d \mapsto (w_1 \otimes \cdots \otimes w_d \mapsto a_1(w_1) \otimes \cdots \otimes a_d(w_d)).$$

$\Phi$  is an invertible linear map with inverse

$$\Phi^{-1}f \mapsto f|_{W_1} \otimes \cdots \otimes f|_{W_d}$$

where  $W_j$  is  $0 \otimes \cdots \otimes W \otimes \cdots \otimes 0$  with the  $W$  in the  $j$ th spot. Note also that  $\Phi$  commutes with the action of  $S_d$ . By using  $\Phi$ , we see that

$$\text{Sym}^d(\text{End } W) = ((\text{End } W)^{\otimes d})^{S_d} \xrightarrow{\Phi^{-1}} \text{End}_{S_d}(W^{\otimes d}).$$

So we only need to understand  $\text{Sym}^d(\text{End } W)$ . But  $\text{GL}(W)$  is dense in  $\text{End}(W)$ , so by a previous lemma, we see that  $\text{span}^d(\text{GL}(W)) = \text{Sym}^d(\text{End } W)$ .  $\square$

**1.3. Spectral Theorem.** Let  $A$  be a  $k$ -algebra with  $a \in A$ . We have an evaluation map

$$\text{ev}_a : k[t] \rightarrow A \quad \text{ev}_a : p \mapsto p(a).$$

Let  $A_a := \text{im}(\text{ev}_a)$ , i.e. the subalgebra of  $A$  generated by  $a$ . The kernel  $\ker(\text{ev}_a)$  is an ideal of  $k[t]$ , and it is a principal ideal since  $k[t]$  is a PID. Thus, in the case that  $\text{ev}_a$  is non-injective, there is a unique *minimal polynomial* of  $a$ ,  $p_a$ , which divides every polynomial which vanishes at  $a$ .

**Lemma:**  $a$  is algebraic iff  $A_a$  is finite-dimensional.

*Proof.* If  $A_a$  is finite-dimensional then there is a relation between  $1, a, a^2, \dots, a^n$  for some  $n$ , i.e. a polynomial that  $a$  solves. Conversely if  $a$  solves a polynomial of degree  $n$  then every linear combination of powers of  $a$  can be expressed by the first  $n$  powers of  $a$ .  $\square$

We define the *spectrum* of  $a$ , denoted  $\text{Spec}(a)$ , as

$$\text{Spec}(a) := \{\lambda \in k : (a - \lambda) \text{ is not invertible}\}.$$

So for example, if  $A$  is a function algebra,  $\text{Spec}(a)$  denotes the values that  $a$  can take. In the case that  $A$  is the matrix algebra  $M_n(k)$ ,  $\text{Spec}(a)$  is the set of eigenvalues of  $a$ .

**The Spectral Theorem:** Let  $A$  be a  $k$ -algebra of one of the following types:

- $A$  is finite-dimensional over  $k$  and  $k$  is algebraically closed.
- $A$  is countable-dimension and  $k$  is uncountable.

Then,

- (i)  $\text{Spec}(a)$  is nonempty.
- (ii)  $a$  is nilpotent iff  $\text{Spec}(a) = \{0\}$ .
- (iii) If  $A$  is a division algebra then  $A = k$ .<sup>3</sup>

*Proof.* Lemma: If  $\lambda_1, \dots, \lambda_n \notin \text{Spec}(a)$ , i.e.  $(a - \lambda_j)$  is invertible for each  $j$ , then if

$$\sum_j c_j (a - \lambda_j)^{-1} = 0$$

for some  $c_j \in k$  then  $a$  is algebraic (proof is by clearing denominators). We will use this fact.

(i): We will split into two cases: if  $a$  is algebraic then  $\text{Spec}(a)$  is finite but nonempty and if  $a$  is not algebraic then  $\text{Spec}(a)$  is uncountable (and the converses to both of these are true).

If  $a$  is algebraic, then  $\text{Spec}(a)$  is the roots of the minimal polynomial (HW), and particular this means  $\text{Spec}(a)$  is finite and nonempty because  $k$  is algebraically closed.

If  $a$  is not algebraic, then by the Lemma, there is no linear relation between any finitely-many  $(a - \lambda)^{-1}$  for  $\lambda \notin \text{Spec}(a)$ . We assumed that  $\dim(A)$  is at most countable, and it has an independent set of size  $|k \setminus \text{Spec}(a)|$ , so  $\text{Spec}(a)$  must be uncountable (because  $k$  is).

<sup>3</sup>For a counterexample of this when  $k$  is not algebraically closed, take the Quaternions over  $\mathbb{R}$ .

(ii): If  $a^n = 0$ , then  $0 \in \text{Spec}(a)$  because  $a$  is not invertible, but all other  $(a - \lambda)$  are invertible:

$$(a - \lambda)(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}) = a^n - \lambda^n = -\lambda^n.$$

so

$$(a - \lambda)^{-1} = -\lambda^{-n}(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}).$$

Conversely, suppose that  $\text{Spec}(a) = \{0\}$ .  $\{0\}$  is a finite set, so by part (i),  $a$  is algebraic, but its minimal polynomial only has root  $a = 0$ , so  $a^n = 0$  for some  $n$ .

(iii): Assume for contradiction that  $A$  is a division algebra yet  $\exists a \in A \setminus k$ . Then  $(a - \lambda)$  is invertible for all  $\lambda \in k$ , but then  $\text{Spec}(a)$  would be empty, contradicting (i).  $\square$

**1.4. Modules.** A module  $M$  over ring  $A$  is called *simple* if it is nonzero and has no proper non-trivial submodules (i.e. its only submodules are 0 and  $M$ ).

**Schur's Lemma:** If  $f : M \rightarrow N$  is an  $A$ -linear map between *simple*  $A$ -modules  $M$  and  $N$ , then  $f$  is either 0 or an isomorphism.

*Proof.*  $\ker(f)$  is a submodule of  $M$  and  $\text{im}(f)$  is a submodule of  $N$ . By simplicity, both must be either trivial or the full module. This implies that  $f$  is either injective or 0, and either surjective or 0.  $\square$

As a corollary, we see that  $\text{End}_A M$  is a division ring.

**Schur's Lemma for Algebras:** If  $A$  is a  $k$ -algebra and  $M$  a simple  $A$ -module either

- $k$  is algebraically closed and either  $A$  or  $M$  is finite-dimensional over  $k$ .
- $k = \mathbb{C}$  and either  $A$  or  $M$  is countable-dimension over  $k$ .

Then,  $\text{End}_A M = k \text{id}_M$ .

*Proof.* On HW we showed that  $\dim(\text{End}_A M) \leq \dim_A M$ . The lemma can be proven by applying the spectral theorem to the algebra  $\text{End}_A M$ .  $\square$

If  $A$  satisfies the hypotheses of the Spectral Theorem and  $M$  is a simple  $A$ -module, then the center  $Z$  of  $A$  acts in  $M$  by scalars, as  $z \cdot am = az \cdot m$  for  $z \in Z, a \in A, m \in M$ . And in particular if  $A$  is commutative then  $\dim_k M = 1$  because every subspace of  $M$  is  $A$ -stable.

**Schur's Lemma for Group Representations:** If  $V, W$  are representations of a group  $G$  over a field  $k$ ,

- (i) If  $V, W$  are irreducible then all intertwiners are either 0 or isomorphisms.
- (ii) If  $\dim_k(V)$  is finite and  $k$  is algebraically closed or  $k = \mathbb{C}$  and  $|G| = \aleph_0$ , then

$$\text{End}_G V = k \cdot \text{id}_V.$$

*Proof.* This follows from applying Schur's Lemma for Algebras. A representation of  $G$  corresponds to a module over the group algebra  $A := kG$ . Note that  $\dim_k(A) = |G|$ .  $\square$

**1.5. Representations of  $S_n$ .**  $S_n$  acts on  $\mathbb{R}^n$  by permuting the coordinates. We have the sign representation given by taking the determinant.

$S_n$  also acts on  $P_d$  by permuting the variables:

$$\sigma(f)(x_1, \dots, x_n) := f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}).$$

Motivated by this action, we can consider the symmetric polynomials  $P^{S_n}$ .

A *partition* of  $n$  is a finite non-increasing sequence of positive integers  $\lambda_1 \geq \cdots \geq \lambda_k$  whose sum is  $n$ . Let the set of partitions of  $n$  be  $\mathcal{P}_n$ . Corresponding to a partition, we have a decomposition of  $[1, n]$  into  $I_1, I_2, \dots, I_k$  of length  $|I_j| = \lambda_j$ .

The *Vandermonde Determinant* is the polynomial

$$\Delta_n := \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

which can also be written as the determinant

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix}$$

Corresponding to a given partition  $\lambda$ , define

$$\Delta(I_m) := \prod_{i < j \in I_m} (x_j - x_i)$$

and

$$\Delta_\lambda := \prod_m \Delta(I_m).$$

$\Delta_\lambda$  is a homogeneous polynomial, and its degree is

$$d := \sum_m \frac{\lambda_m(\lambda_m - 1)}{2}$$

so  $\Delta_\lambda \in P_d$ .

The *Specht Module* associated with  $\lambda$ , denoted  $V(\lambda)$  is the  $k$ -span of  $\Delta_\lambda$  under the action of  $S_n$ . It is clearly stable under the action of  $S_n$ .<sup>4</sup>

Examples:

- Let  $\lambda = (1, 1, 1, \dots, 1)$ . Then  $\Delta_\lambda = 1$ , and  $V(\lambda) = P_0$ , the constant polynomials. The action of  $S_n$  on  $V(\lambda)$  is trivial. Thus, this  $\lambda$  represents the trivial representation.
- Let  $\lambda = (n)$ . Then  $\Delta_\lambda = \Delta_n$ , the entire Vandermonde determinant. Since this is just a determinant whose columns are permuted by the action of  $S_n$ , the action scales by the sign of the permutation. This makes  $V(\lambda) = k\Delta_n$ , which is one-dimensional. It is the sign representation of  $S_n$ .

Note that in all of these cases  $V(\lambda)$  is irreducible. This is actually true in general:

**Theorem:** Assuming that the underlying field  $k$  is characteristic 0, the Specht module is always an irreducible representation. Moreover, all irreps of  $S_n$  can be expressed as  $V(\lambda)$  for some partition  $\lambda$ .<sup>5</sup>

*Proof.* The proof has three steps. The first step will be to show that  $V(\lambda)$  is irreducible, which we do on homework. Step 2 is that the number of irreducible representations of  $S_n$  is equal to the number of partitions of  $n$ . Step 3 will show that the modules  $V(\lambda)$  are pairwise non-isomorphic for different  $\lambda$ , and hence we have a bijection.

Step 3: In homework (it is fairly clear I think) we showed that if  $d_\mu \neq d_\lambda$  then  $V(\mu) \not\cong V(\lambda)$ , so it remains to show this for  $\mu, \lambda$  that have equal degree.

Notation: for  $\nu \in \mathbb{Z}_{\geq 0}^n$  (note: may have repeats!),  $S_n$  acts on  $\nu$  in the natural way. Let  $m_j(\nu)$  denote the number of elements of  $\nu$  that are equal to  $j$  (this is invariant under  $S_n$ ). Let  $\nu(\lambda)$  be

$$\nu(\lambda) := (1, 2, \dots, \lambda_1, 1, 2, \dots, \lambda_2, \dots, 1, 2, \dots, \lambda_n).$$

Then  $m_j(\nu(\lambda))$  is the length of the  $j$ th column in the Young diagram of  $\lambda$ ,  $D(\lambda)$ . Or equivalently, the  $m_j$  form another partition corresponding to the transposed Young diagram  $D^T(\lambda)$ .

<sup>4</sup>This is not the most common way to construct the Specht module of  $\lambda$ .

<sup>5</sup>It is also true that  $V(\lambda)$  is a subspace of the  $S_n$ -harmonic polynomials (as defined on HW) and the index is  $\dim(V(\lambda))$ .

Similarly, we can apply all this to polynomials in  $n$  variables. The monomial  $x^\nu$  is

$$x^\nu := \prod_{j=1}^n x_j^{\nu_j}$$

so that for a partition  $\lambda$ ,

$$x^{\nu(\lambda)} := \prod_{i=1}^k \prod_{j=1}^{\lambda_i} x_{\lambda_i+j}^j.$$

Recall that  $\Delta_\lambda$  is defined in terms of Vandermonde determinants, which are expressed as

$$\Delta_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1}.$$

And for  $\Delta_\lambda$ , we have a similar thing but with Young subgroups:

$$\Delta_\lambda = \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1} = \frac{1}{x_1 x_2 \cdots x_n} \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x^{s(\nu(\lambda))}.$$

Now, if  $V(\mu) = V(\lambda)$ , then  $\Delta_\mu \in V(\lambda)$ , i.e. it is a linear combination of permutations of  $\Delta_\lambda$ . So for example, the monomial  $x^{\nu(\mu)}$  appears as  $x^{\sigma(\nu(\lambda))}$  for some  $\sigma$ . But we can show that this fails (*fill in later*).  $\square$

The *Young Subgroup* of  $S_n$  corresponding to  $\lambda$  consists of all permutations which preserve all the pieces  $I_m$ . It is denoted  $S_\lambda$  and is isomorphic to  $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$ .

Define  $P^{S_\lambda}$  to be the polynomials fixed by  $S_\lambda$ , and  $P^{\text{sgn}(\lambda)}$  the polynomials on which  $S_\lambda$  acts in an anti-symmetric way. We can see that  $P^{\text{sgn}(\lambda)}$  is stable under scaling by  $P^{S_\lambda}$ , i.e. it is a  $P^{S_\lambda}$ -submodule.

**Lemma:** Let  $F : V(\lambda) \rightarrow P_d$  be an  $S_n$ -intertwiner. Then

- (1) If  $d = d_\lambda$ , then  $F$  acts by scaling.
- (2) If  $d < d_\lambda$ , then  $F$  is trivial.

*Proof.* For  $s \in S_\lambda$ ,  $s(F(\Delta_\lambda)) = F(s(\Delta_\lambda))$ , and  $s(\Delta_\lambda) = \text{sgn}_\lambda(s) \cdot \Delta_\lambda$ , so  $F$  acts by scaling. *fill in later*  $\square$

In general, we can see that  $V(\lambda^t) = V(\lambda) \otimes \text{sgn}$ .

**1.6. Hilbert's Nullstellensatz.** Let  $k$  be an algebraically closed field and let  $P = k[x_1, \dots, x_n]$ . An *algebraic* subset of  $k^n$  is the vanishing set of an ideal  $I \subset P$ , denoted  $V(I)$ . In the other direction, we have an ideal  $I_V$  corresponding to polynomials vanishing on a given algebraic set  $V$ .

For any ideal  $I \subset P$ , there is a *radical* of  $I$ , denoted  $\sqrt{I}$ , which consists of all elements of  $P$  for which some power lies in  $I$ . Note that  $V(I) = V(\sqrt{I})$ . Moreover, ideals of the form  $I_V$  are already radical. The interesting thing is that the correspondence goes both ways:

**Nullstellensatz:** Algebraic sets are in bijection with *radical* ideals of  $P$ , via  $V \mapsto \sqrt{I_V}$  and  $V(I) \mapsto I$ . This bijection restricts to one between single points of  $k^n$  and maximal ideals of  $P$ .

*Proof.* We will show that  $I_{V(I)} = \sqrt{I}$ , the content of which is that every polynomial  $f$  vanishing on  $V(I)$  has  $f^n \in I$  for some  $n$ .

Lemma: for  $z \in k^n$ , let  $\text{ev}_z : P \rightarrow k$  be the algebra homomorphism given by evaluation at  $z$ , i.e.  $\text{ev}_z : f \mapsto f(z)$ . In fact, *every* algebra homomorphism  $\chi : P \rightarrow k$  is  $\text{ev}_z$  for some  $z$ . In particular, take  $z = (\chi(x_1), \chi(x_2), \dots, \chi(x_n))$  and note that  $\chi(f) = \text{ev}_z(f)$ .

Now, let  $A$  be a commutative  $k$ -algebra as in the setting of the Spectral theorem (i.e. finite-dimensional or countable dimension with uncountable  $k$ ). We claim that all maximal ideals of  $A$



have the form  $\ker(\chi)$  for some algebra homomorphism  $\chi : A \rightarrow k$ , and moreover  $\text{Spec}(a)$  is exactly  $\{\chi(a) : \chi \in \text{Hom}(A, k)\}$ .

To prove this, let  $I \subset A$  be a maximal ideal. This implies that  $A/I$  is a field, and in particular a division algebra, so the spectral theorem implies that  $A/I = k$ . Thus the projection onto  $I$  gives a character  $A \rightarrow A/I = k$ .

Now for the general case. Suppose  $f$  vanishes on  $V(I)$  but none of its powers is in  $I$ . Let  $\bar{f}$  be  $f \bmod I$ . The assumption that  $f \notin \sqrt{I}$  is equivalent to saying that  $\bar{f}$  is not nilpotent, so by the spectral theorem,  $\text{Spec}(\bar{f}) \neq \{0\}$ . Thus, there is some  $\lambda \neq 0$  so that  $\bar{f} - \lambda$  is non-invertible in  $A$ . Now consider the ideal

$$I + P \cdot (f - \lambda) \subset P.$$

Because  $(\bar{f} - \lambda)$  is non-invertible, there can be no  $b \in P$  such that  $b(\bar{f} - \lambda) = 1 \bmod I$ , thus  $I + P \cdot (f - \lambda)$  is not all of  $P$ . By the maximal case, this shows that

$$V(I) \cap V(f - \lambda) = V(I + P \cdot (f - \lambda)) \neq \emptyset$$

but this implies that  $f = \lambda$  on at least one point of  $V(I)$ , which contradicts the assumption that  $f$  vanishes on  $V(I)$ .  $\square$

As a corollary, we see that every proper ideal  $I \subsetneq P$  there is some  $z$  such that  $f(z) = 0$  for  $z \in I$ . This is because otherwise  $V(I) = \emptyset$ , so  $\sqrt{I} = P$ .

Another corollary: if  $f_1, \dots, f_r \in P$  have no common solution, then there is some  $P$ -linear combination of these equal to 1, i.e.  $1 \in (f_1, \dots, f_r)$ .

A third corollary: if  $V(I_1) \cap V(I_2) = \emptyset$ , then there is some polynomial  $f$  such that  $f = 0$  on  $V(I_1)$  and  $f = 1$  on  $V(I_2)$ . To get such an  $f$ , we use the fact that  $I_1 + I_2 = P$ , so there is  $f \in I_1$  and  $g \in I_2$  such that  $f + g = 1$ , so we can take this  $f$ .

**1.7. Invariant Theory.** Let  $G$  act on  $X$ . If  $f$  is a  $G$ -invariant function on  $X$ , another way to think of this is that  $f$  takes a value on each  $G$ -orbit.

Consider the case where  $G \subset \text{GL}_n(k)$  and  $X = k^n$ . What does the space of  $G$ -invariant polynomials on  $X$  look like? Hilbert showed that it is finitely-generated for sufficiently well-behaved  $G$ .

If  $G = S_n$  for example, we get the symmetric polynomials, and this space is generated by the elementary symmetric functions, i.e. the symmetric sums of square-free monomials in each degree from 1 to  $n$ ; we denote these  $\sigma_j(x)$  for  $1 \leq j \leq n$ .

The map  $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  given by

$$\pi : x \mapsto (\sigma_1(x), \dots, \sigma_n(x))$$

has fibers exactly equal to the orbits of  $S_n$ .

Let  $\zeta$  be a primitive  $n$ th root of unity. Consider the subgroup  $\Gamma \subset \text{SL}_2(\mathbb{C})$  generated by

$$\gamma = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$$

Now we can ask about  $\mathbb{C}[x, y]^\Gamma$ . We immediately see that  $x^n, y^n, xy \in \mathbb{C}[x, y]^\Gamma$ . Taking  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^3$  as before, defined by

$$(x, y) \mapsto (x^n, y^n, xy)$$

and we again see that  $\mathbb{C}^2/\Gamma \cong \text{im}(\pi)$ .

In more generality, suppose  $\Gamma \subset \text{SL}_2(\mathbb{C})$  is a finite subgroup. If  $f(x, y) \in \mathbb{C}[x, y]^\Gamma$ , supposing  $f$  is homogeneous of degree  $d$ , we can write  $f$  as the product of  $d$  lines through 0. Then any  $g \in \Gamma$  must permute these linear factors.

Let  $G = \mathbb{R}^\times$ , acting on  $\mathbb{R}$ . The orbits of the action are  $(-\infty, 0), \{0\}, (0, \infty)$ . Thus the only  $G$ -invariant polynomials on  $\mathbb{R}$  are constants.

But if we let this  $G$  act on  $\mathbb{R}^2$  via  $t : (x, y) \mapsto (tx, t^{-1}y)$ , the orbits are halves of hyperbolas and the four pieces of the coordinate axes and the origin.

Let  $G = \mathrm{GL}_n(k)$  act on  $M_n(k)$ . The  $G$ -orbits are represented by matrices in Jordan Canonical Form. Let's look at the case  $n = 2$ . The  $G$ -invariant polynomials in the four matrix entries are generated by  $ad - bc$ , the determinant.