MATH 325 HW 3

JALEN CHRYSOS

Problem 1: Prove that $P^{\mathrm{sign}_\lambda}$ is a rank 1 free P^{S_λ}-module with generator Δ_λ.

Proof. $\Delta_\lambda \in P^{\mathrm{sign}_\lambda}$ because each $\sigma \in S_\lambda$ is the product of permutations σ_m on each index set I_m, each of which acts on $\Delta(I_m)$ as multiplication by $\mathrm{sign}(\sigma_m)$ (because the determinant is alternating), so their product also acts on Δ_λ as multiplication by $\mathrm{sign}(\sigma_1)\cdot\mathrm{sign}(\sigma_2)\cdots\mathrm{sign}(\sigma_k) = \mathrm{sign}(\sigma)$.

Moreover, I claim that *every* polynomial in $P^{\mathrm{sign}_\lambda}$ is a multiple of Δ_λ. For any $p \in P^{\mathrm{sign}_\lambda}$, $(x_j - x_i)|p$ for all pairs $i < j \in I_m$, as swapping x_i, x_j inverts p. To show this, express p as a polynomial in x_i, x_j with coefficients in $k[x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_n]$, and note that the coefficients of terms $x_i^{e_1}x_j^{e_2}$ and $x_i^{e_2}x_j^{e_1}$ must add to 0 (since swapping x_i, x_j inverts p). Thus, p is a linear combination of terms $(x_i^{e_1}x_j^{e_2} - x_i^{e_2}x_j^{e_1})$, each of which is a multiple of $(x_i - x_j)$. Now, since p must be a multiple of $(x_i - x_j)$ for all such i, j, and these polynomials are all irreducible, their product Δ_λ must also divide p, which was the desired result. $\square$

Problem 2:

(a) Prove that all polynomials $f \in V(\lambda)$ are S_n-harmonic.
(b) (Optional) Show that there is no nonzero homogeneous S_n-harmonic polynomial f of degree greater than $\deg(\Delta_n)$ (which is $\binom{n}{2}$).

Proof. (a): To be S_n-harmonic means that for any *symmetric* polynomial p, $\langle p(\partial), f\rangle = 0$.

For $p \in P_d^{S_n}$, p induces a linear map $p(\partial) : V(\lambda) \to P_{d_\lambda - d}$ which is an S_n-intertwining map because p is symmetric. Thus, by Lemma 4.2.7, this map must be the constant zero map. $\square$

Problem 3: Using Lemmas 4.2.7 and 4.2.9, deduce Corollary 4.2.11:

(a) The representation $V(\lambda)$ is irreducible.
(b) If $d_\lambda = d_\mu$ and $V(\lambda) \cong V(\mu)$ then $V(\lambda) = V(\mu)$ as subspaces of $P_{d_\lambda} = P_{d\mu}$.
(c) If $d_\lambda \neq d_\mu$ then $V(\lambda) \ncong V(\mu)$.

Proof. (a): Every permutation is a unitary operator, so every S_n-representation is unitary and hence completely reducible. Thus $V(\lambda)$ is completely reducible. So we can use Lemma 4.2.9 to say that $V(\lambda)$ is irreducible iff $\dim_k(\operatorname{End}_{S_n} V(\lambda)) = 1$. And Lemma 4.2.7 showed that the only S_n-intertwining maps $V(\lambda) \to P_{d_\lambda}$ are scaling by some constant in $\mathbb{C}$, which implies in particular that maps in $\operatorname{End}_{S_n} V(\lambda)$ are also scaling by constants, and hence it is dimension 1. Thus, $V(\lambda)$ is irreducible.

(b): By Lemma 4.2.7, if $d_\mu = d_\lambda$, any S_n-intertwiner $V(\lambda) \to V(\mu)$ must be scaling by a constant. Thus, if $V(\mu) \cong V(\lambda)$ then the intertwiner between them is a constant, so the subspaces $V(\mu)$ and $V(\lambda)$ are actually the same.

(c): Likewise, if $d_\mu \neq d_\lambda$ then any S_n-intertwiner $V(\lambda) \to P_{d_\mu}$ must be trivial, so in particular there can be nonzero map $V(\lambda) \to V(\mu)$. $\square$

Problem 4: Prove the generating function identity

$$\sum_{d\geq 0}\dim(P_d^{S_n})t^d = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

Proof. $P_d^{S_n}$ has a basis consisting of the S_n-orbits of monomials in P_d, and these correspond to partitions of d with at most n parts (for the degrees of the n variables $x_1, \dots, x_n$). To express the number of such partitions as a generating function, it is easier to count the transposed partitions, i.e. those which have parts of size no larger than n (the count will be the same, naturally). To do this, take the infinite power series

$$Q(t) := (1+t+t^2+\cdots)(1+t^2+t^4+\cdots)\cdots(1+t^n+t^{2n}+\cdots).$$

One can see by expanding the products that the t^d term of $Q(t)$ is equal to the number of ways to partition d into parts of size no larger than n. And Q can be expressed using the geometric series identity as

$$Q(t) = \left(\frac{1}{1-t}\right)\left(\frac{1}{1-t^2}\right)\cdots\left(\frac{1}{1-t^n}\right)$$

as desired. $\square$

Problem 5: (Optional) Show that P is a free P^{S_n} module with basis

$$\{x_2^{m_2}x_3^{m_3}\cdots x_n^{m_n}|m_j \in [0, j-1]\forall j\}.$$

Problem 6: Let $g \in \mathrm{GL}_n(\mathbb{C})$ be a diagonal matrix with diagonal entries $z_1, \ldots, z_n$, and $\sigma \in S_d$ a permutation. Consider the linear operator $(\mathbb{C}^n)^{\otimes d} \to (\mathbb{C}^n)^{\otimes d}$ given by composing the action of σ with that of g, i.e.

$$v_1 \otimes \cdots \otimes v_d \mapsto g(v_{\sigma^{-1}(1)}) \otimes \cdots \otimes g(v_{\sigma^{-1}(d)}).$$

Show that its trace is

$$\prod_{j \geq 1} ((z_1)^j + \cdots + (z_n)^j)^{m_j}$$

where m_j is the number of cycles of length j in the cycle type of σ.

Proof. For any basis element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$, this operator Φ acts as

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \mapsto g(e_{i_{\sigma^{-1}(1)}}) \otimes \cdots \otimes g(e_{i_{\sigma^{-1}(d)}}) = (z_{i_{\sigma^{-1}(1)}} \cdots z_{i_{\sigma^{-1}(d)}})(e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(d)}}).$$

To get the trace, we only need to consider these products for basis elements which are scaled by Φ, i.e. those for which $i_t = i_{\sigma^{-1}(t)}$ for all $1 \leq t \leq d$. That is, for each cycle $(t_1\ t_2\ t_3\ \ldots\ t_j)$ in σ,

$$i_{t_1} = i_{t_2} = \cdots = i_{t_j} \in \{1, \ldots, n\}.$$

In choosing such a basis element, then, there is one choice to be made for each cycle in σ. If i is chosen as the index of a cycle, that contributes a multiplication by z_i^j to the corresponding diagonal element (where j is the length of the cycle). Let $\mathrm{cyc}(\sigma)$ denote the set of disjoint cycles whose product is σ. The sum of all possible diagonal elements, i.e. the trace, is the sum over all choice functions $f : \mathrm{cyc}(\sigma) \to \{1, \ldots, \mathrm{n}\}$ of

$$\prod_{c \in \mathrm{cyc}(\sigma)} z_{f(c)}^{|c|}$$

which is the expanded form of the product

$$\prod_{c \in \mathrm{cyc}(\sigma)} (z_1^{|c|} + z_2^{|c|} + \cdots + z_n^{|c|}) = \prod_{j \geq 1} ((z_1)^j + \cdots + (z_n)^j)^{m_j}$$

as desired. $\square$

Problem 8: Let A be an algebra over an ACF k. Let $V_1, \ldots, V_r$ be pairwise non-isomorphic simple finite-dimensional A-modules, and

$$N := (V_1)^{\ell_1} \oplus \cdots \oplus (V_r)^{\ell_r}$$

for some positive integers ℓ_j. Use Schur's Lemma to prove:

(a) Any simple A-submodule $V \subseteq N$ is isomorphic to V_i for some i, and in this case V is contained in $(V_i)^{\ell_i}$.

(b) The algebra $\mathrm{End}_A(N)$ is isomorphic to $M_{\ell_1}(k) \oplus \cdots \oplus M_{\ell_r}$ (where M_n is the matrix algebra).

Proof. (a): Let $V \subseteq N$ be a simple A-module. Then V has a projection map onto each copy of V_j for $1 \leq j \leq r$, and by Schur's Lemma each of these maps must be either trivial or an isomorphism. They cannot all be trivial because they span the entire space of N, and each V_j is pairwise non-isomorphic to the others, so V is isomorphic with exactly one of them.

(b): Every A-algebra homomorphism $(V_j)^{\ell_j} \to (V_i)^{\ell_i}$ where $i \neq j$ is trivial by Schur's Lemma, so any such endomorphism on N preserves $(V_j)^{\ell_j}$, i.e. is a direct sum of matrices in $M_{\ell_j}(k)$ for each j.

And conversely, any linear map $V_j^{\ell_j} \to V_j^{\ell_j}$ is an A-algebra endomorphism, since the action of A is scaling in V_j (again by Schur's Lemma). Thus, these direct sums of matrices are *exactly* the endomorphisms on N. $\square$