ALGEBRA I NOTES

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Abstract. These are my notes from Victor Ginzburg's Representation Theory (Math 325) class at UChicago, Autumn 2025.

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1. Introduction

In this class we'll be interested in the representations of matrix groups. Something like $\mathrm{GL}(V)$ or $\mathrm{SO}(V)$ clearly acts on V, but it can also act on other interesting spaces. One relevant case of this for us will be when G acts on polynomials in $x_1, \ldots, x_n$. Let

$$P_d \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

be the subspace of homogeneous degree-d polynomials in n variables. This space has a basis given by the monomials

$$\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\}$$

and hence is finite-dimensional. P_d is stable under action by GL_n. This is because linear transformation does not affect the degree of monomials (every x_j is sent to a linear combination of $x_1, x_2, \ldots, x_n$).

Consider the case of $G = \mathrm{O}_n$, the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by R is an intertwining map $P_d \to P_{d+2}$, meaning $R \circ g^* f = g^*(R \circ f)$.

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This Δ is an O_n-intertwining operator.

We call a function f *harmonic* if it has $\Delta(f) = 0$. The space of harmonic polynomials in n variables of degree d is denoted $H_d \subseteq P_d$. For $d \in \{0, 1\}$, $H_d = P_d$, but for $d \geq 2$ H_d is strictly smaller. Note that H_d is stable under orthogonal transformations.

We will now work toward showing that H_d is an irreducible SO_n-representation for $n \geq 3$.

A representation $\rho : G \mapsto \mathrm{GL}(V)$ is *unitary* if G always acts as a unitary operator (i.e. preserves Hermitian inner product) on V. We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} e^{-|x|^2} \, \mathrm{d}x$$

where $\mathrm{d}x$ is the Lebesgue measure. Action of SO_n on P_d preserves this inner product.

Alternatively, we could put an inner product on P_d (or on all functions) from integration over S^{n-1} (the sphere). And polynomials in P_d are determined by their behavior on S^{n-1}.

Proposition: If V is a finite-dimensional vector space with an inner product, then any *unitary* action of G on V is completely reducible. Specifically, if $W \subseteq V$ is a G-stable subspace, then one can decompose the action into $V = W \oplus W^\perp$.

Proof. The thing that we need to prove is that if W is G-stable then $W^\perp$ is as well. Let $x \in W^\perp$ and $w \in W$. Because g acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since $g^{-1} \cdot w \in W$ by G-stability of W. □

Key Lemma: If $F \subseteq C(S^{n-1})$ is any subspace stable under SO_n, then it has an element fixed by SO_{n-1}.

Proof. Let $N := (0, 0, \ldots, 0, 1) \in S^{n-1}$. We have the evaluation map $\alpha : C(S^{n-1}) \to \mathbb{C}$ given by evaluating functions at N. We have an inner product on F given by

$$\langle f, g \rangle := \int_{S^{n-1}} f \overline{g}$$

which is clearly fixed by SO_n, thus F is a unitary representation of SO_n. By Riesz representation theorem, $\alpha(f) \equiv \langle f, \varphi\rangle$ for some $\varphi \in F$. For any $g \in \mathrm{SO}_{n-1}$, g fixes N, thus

$$\langle f, \varphi\rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi\rangle = \langle f, g\varphi\rangle$$

and because this is true for arbitrary $f \in F$ and $g \in SO_{n-1}$, φ is fixed by SO_n. Now it remains to show that $\varphi \neq 0$. We can get this by assuming that some function in F takes a nonzero value on N (we can move N to some point where this is true, since F contains a nonzero function). □

We can apply this key lemma to P_d or H_d as F.

Consider $P_d^{\mathrm{SO}_{n-1}}$, the homogeneous polynomials fixed by SO_{n-1}. On homework we showed that this is a subspace of $\mathbb{C}\langle x_n, R\rangle$ (where $R := x_1^2 + \cdots + x_n^2$). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \ldots\rangle$$

thus $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$.

A very important fact about P_d is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2 H_{d-4} \oplus R^3 H_{d-6} \oplus \cdots \end{aligned}$$

(we will show this later). This allows us to deduce the dimension of H_d from P_d:

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1.$$

Likewise, we can decompose $P_d^{\mathrm{SO}_{n-1}}$ the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2 H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3 H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of $H_d^{\mathrm{SO}_{n-1}}$ as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each H_d is an *irreducible* representation of SO_n; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an SO_n-representation, P_d decomposes exactly into the sequence H_{d-2j} for $2j \leq d$.

Theorem: If $n \geq 3$, then for each $d \geq 0$, the representation of SO_n in H_d is irreducible, and moreover the representations are all distinct for different d.[1]

Proof. To show that the representations are distinct, we can use a homework problem which shows that the dimension of H_d is always increasing in d for any $n \geq 3$. □

1.1. **Differential Algebra.** Let W be a vector space over k with basis $w_1, \ldots, w_n$, and let $x_1, \ldots, x_n$ be a dual basis for W^*. We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over W, where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \to k[W]_{j-1}$$

which acts on $k[W]$ in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \ldots, \partial_{w_n}]$$

[1] In the case $n = 3$ this gives *all* the irreps. In general you miss $\Lambda^2(\mathbb{C}^n)$, but when $n = 3$ this is just $\mathbb{C}^3$, which you get from H_1.

acting on $k[W]$. There is a natural correspondence between $k[W]$ and $\mathcal{D}(W)$, if one assumes that k is characteristic 0. We have a k-bilinear pairing

$$\mathcal{D}(W) \times k[W] \to k$$

by $\langle u, f\rangle \mapsto u(f)(0)$. This is a *perfect pairing.* And in general we can do the same thing with

$$\mathrm{Sym}^j(W) \times \mathrm{Sym}^j(W^*) \to k.$$

Lemma: Let $\xi \in W$ and $f \in k[W]$. Then

$$\langle \xi^m, f\rangle = m!f(\xi).$$

In particular, if $f = \varphi \in W^*$, $\langle \xi^m, \varphi^m\rangle = m!\varphi^m(\xi)$.

Proof. We will show this for homogeneous f first, and the general result will follow from expressing f as a sum of homogeneous polynomials. Let the degree of f be d. Then by Taylor expansion,

$$f(\xi) = \sum_{k\geq 0} \frac{1}{k!}(\partial_\xi^k f)(0).$$

But note that only the dth term of this is nonzero, since $\partial_\xi^j f = 0$ unless $j = d$ (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other j both sides are 0. $\square$

We can use this pairing to get another inner product on polynomials in $k[W]$ given by

$$\langle p, q\rangle := p(\partial)(q)(0)$$

where $p(\partial)$ is the corresponding element to p in $\mathcal{D}(W)$.[2] For this inner product, we have that multiplication by p is *adjoint* to $p(\partial)$, i.e.

$$\langle r, p(\partial)q\rangle = \langle pr, q\rangle.$$

With this fact, we can finally show why $P_d = H_d \oplus RP_{d-2}$:

$$W = \ker(\Delta) \oplus \mathrm{im}(\Delta^*) = \ker(\Delta) \oplus \mathrm{im}(R) = H_d \oplus RP_{d-2}.$$

Another application of this pairing: Let V be a finite dimensional vector space and $A \subseteq V$ a subset of V (not necessarily subspace). Let $\mathrm{span}^d(A) \subseteq \mathrm{Sym}^d(V)$ be generated over $\mathbb{C}$ by a^d for $a \in A$. If A is dense in V then $\mathrm{span}^d(A) = \mathrm{Sym}^d(A)$. We will show this by using the pairing.

Assume for contradiction that $\mathrm{span}^d(A) \neq \mathrm{Sym}^d(V)$. Then there is some nonzero linear functional $F : \mathrm{Sym}^d(V) \to \mathbb{C}$ which vanishes on $\mathrm{span}^d(A)$. Then F corresponds to some differential polynomial f, and $\partial_a^d f(0) = 0$ for all $a \in A$. But $\partial_a^d f(0) = d!f(a)$, so $f(a) = 0$. But then A is dense, so $f = 0$.

1.2. Representation Theory Basics.

If G acts on sets X and Y, then G can also act on the space of maps $X \to Y$ via conjugation:

$$g : f \mapsto g \circ f \circ g^{-1}.$$

We can ask about the space of maps which commute with this G-action. Or, equivalently, the maps which are fixed by the G-action. We call these *intertwining operators.* The set of such operators is denoted $\mathrm{Hom}_G(X, Y)$.

We are usually interested in the case where X, Y are vector spaces and $\mathrm{Hom}(X, Y)$ is the space of linear maps.

Schur-Weyl Duality: Let W be a finite-dimensional vector space over $\mathbb{C}$. $\mathrm{GL}(W)$ can act on $W^{\otimes d}$ with g acting as $g^{\otimes d}$. S_d also acts on $W^{\otimes d}$ by permutation. It is not too hard to see that these two actions commute. But moreover, action by $\mathrm{GL}(W)$ *spans* the space of S_d-intertwiners on $W^{\otimes d}$.

[2] In the homework, we establish that on H_d, this is actually *equivalent* to the inner product from integrating over S^{n-1}!

Proof. Let $\Phi : (\mathrm{End}(W))^{\otimes d} \to \mathrm{End}(W^{\otimes d})$ be given by

$$\Phi : a_1 \otimes \cdots \otimes a_d \mapsto \big(w_1 \otimes \cdots \otimes w_d \mapsto a_1(w_1) \otimes \cdots \otimes a_d(w_d)\big).$$

Φ is an invertible linear map with inverse

$$\Phi^{-1} f \mapsto f|_{W_1} \otimes \cdots \otimes f|_{W_d}$$

where W_j is $0 \otimes \cdots \otimes W \otimes \cdots \otimes 0$ with the W in the jth spot. Note also that Φ commutes with the action of S_d. By using Φ, we see that

$$\mathrm{Sym}^d(\mathrm{End}\, W) = ((\mathrm{End}\, W)^{\otimes d})^{S_d} \xrightarrow{\Phi^{-1}} \mathrm{End}_{S_d}(W^{\otimes d}).$$

So we only need to understand $\mathrm{Sym}^d(\mathrm{End}\, W)$. But $\mathrm{GL}(W)$ is dense in $\mathrm{End}(W)$, so by a previous lemma, we see that $\mathrm{span}^d(\mathrm{GL}(W)) = \mathrm{Sym}^d(\mathrm{End}\, W)$. □