SIMPLE GUIDE TO SOLVING PROBLEMS IN ALGEBRAIC TOPOLOGY

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ABSTRACT. This is a consolidation of my algebraic topology notes. I don't intend to replicate any proofs of important statements unless they're immediate. I'm just aiming to make a dummyproof flowchart-style guide to solving Hatcher questions for my own use.

1. Basic Terminology

The most basic questions one asks in topology are

"Does there exist a map $X \to Y$ with property P?"

To be able to answer these questions positively, we need methods of constructing maps, but this is usually the simple part. To answer them negatively, we need *algebraic invariants*, properties of a space which are preserved by sufficiently nice maps.

Types of maps (in order of increasing strictness):

- Homotopy equivalence: continuous maps with continuous "inverses" *up to homotopy equivalence.*
- Homeomorphism: continuous maps with continuous inverses.

It's a lot easier to work with homotopy equivalence in practice because there's more you can do (contract simply-connected subspaces for example). The invariants of interest are all preserved by homotopy equivalence:

Algebraic Invariants:

- Fundamental group: $\pi_1(X)$
- Homology groups: $H_0(X), H_1(X), \ldots$
- Cohomology groups: $H^0(X;G), H^1(X;G), \ldots$
- Cohomology ring: $H^*(X;G)$ as a graded ring with $\smile$.
- Higher homotopy groups: $\pi_2(X), \pi_3(X), \ldots$

First, we'll discuss how to compute all of these algebraic invariants, or at least glean information about them. We'll assume X is a connected CW complex.

2. Fundamental Group

The fundamental group of X is a group $\pi_1(X)$ whose elements are homotopy equivalence classes of loops in X, and whose multiplication is the concatenation of loops (at some base point, but if X is connected then the basepoint doesn't matter).

Quick Facts:

- If $\pi_1(X) = 0$ then we say X is "simply connected," i.e. all loops can be contracted.
- $\pi_1(X)$ is in general non-Abelian.

How to Calculate: Use **Van Kampen's Theorem**:

- Write X as a CW complex.
- If X has more than one 0-cell, quotient by a maximal spanning tree in X^1 so that X has a single 0-cell (this is a homotopy equivalence so it preserves $\pi_1(X)$).
- Let the 1-cells (loops) at this 0-cell be $\alpha_1, \alpha_2, \ldots, \alpha_k$. The boundaries of the 2-cells are loops in X^1, so they can each be written as products of the α_j. Let these boundaries be $\beta_1, \beta_2, \ldots, \beta_\ell \in \langle \alpha_1, \ldots, \alpha_k \rangle$.

- Now $\pi_1(X)$ is the free group on $\alpha_1, \dots, \alpha_k$ mod the boundaries of the 2-cells:

$$\pi_1(X) := \langle \alpha_1, \dots, \alpha_k \mid \beta_1 = \beta_2 = \dots = \beta_\ell = 1 \rangle.$$

3. Homology

The Homology Groups of X are a sequence $H_0(X), H_1(X), \dots$ of *Abelian groups*, so they are of the form $\mathbb{Z}^a \oplus \mathbb{Z}_2^{b_2} \oplus \mathbb{Z}_3^{b_3} \oplus \mathbb{Z}_5^{b_5} \oplus \cdots$.

Quick Facts:

- $H_0(X) = \mathbb{Z}^a$ where a is the number of connected components of X.
- $H_1(X)$ is the Abelianization of $\pi_1(X)$.
- $H_n(X) = 0$ for $n > \dim(X)$.
- If X is a manifold of dimension n:
  - $H_n(X) = \mathbb{Z}$ if X is oriented and $H_n(X) = 0$ otherwise.
  - $H_{n-1}(X)$ has torsion subgroup 0 if X is oriented and $\mathbb{Z}_2$ otherwise.

How to Calculate: There are a couple of useful methods.

(I) **Quotients**:
- Express X as a quotient Y/A where Y is a CW complex and A is a subcomplex. Try to choose Y and A such that $H_*(Y)$ and $H_*(A)$ are both simpler than $H_*(X)$.
- There is a long exact sequence relating $\tilde{H}_*(A), \tilde{H}_*(Y)$, and $\tilde{H}_*(X)$:

$$\cdots \to \tilde{H}_j(A) \to \tilde{H}_j(Y) \to \tilde{H}_j(Y/A) \to \tilde{H}_{j-1}(A) \to \tilde{H}_{j-1}(Y) \to \tilde{H}_{j-1}(Y/A) \to \cdots$$

- Now use $\tilde{H}_*(Y)$ and $\tilde{H}_*(A)$, along with exactness of the sequence, to deduce $\tilde{H}_*(X)$.
- *Special Cases / Notable Examples*:
  (i) If A is contractible then $H_*(A) = 0$, so $H_*(X) = H_*(Y)$.
  (ii) When $X = S^n$, we can express it as $X = D^n/S^{n-1}$. Because $H_*(D^n) = 0$, the exact sequence yields $\tilde{H}_j(S^n) \cong \tilde{H}_{j-1}(S^{n-1})$, so by induction we can see that $\tilde{H}_j(S^n) = \mathbb{Z}$ iff $j = n$, otherwise it is 0.

(II) **Mayer-Vietoris**:
- Express X as the union of two *open* subcomplexes $A \cup B$. Try to choose A and B so that A,B, and $A \cap B$ are all simpler than X.
- There is a long exact sequence relating $H_*(A), H_*(B), H_*(A \cap B)$, and $H_*(X)$:

$$\cdots \to H_j(A \cap B) \to H_j(A) \oplus H_j(B) \to H_j(X) \to H_{j-1}(A \cap B) \to \cdots$$

- Now use $H_*(A), H_*(B), H_*(A \cap B)$, and the exactness of the sequence to deduce $H_*(X)$.
- *Special Cases / Notable Examples*:
  (i) If $A \cap B$ is contractible, then $H_j(X) \cong H_j(A) \oplus H_j(B)$. In particular, we see that $H_j(A \vee B) = H_j(A) \oplus H_j(B)$.
  (ii)

(III) **Cellular Homology**: Not done.
- Write X as a CW complex.
- For each j,

(IV) **Künneth Formula**:
- Use this if X happens to be a product $A \times B$ and $H_*(B)$ is free in every dimension.
- There is an isomorphism

$$H^*(A \times B; \mathbb{Z}) \cong H^*(A; \mathbb{Z}) \otimes H^*(B; \mathbb{Z})$$

of the cohomology rings, and in particular

$$H^n(A \times B; \mathbb{Z}) \cong \bigoplus_{j=0}^{n} H^j(A; \mathbb{Z}) \oplus H^{n-j}(B; \mathbb{Z})$$

- Putting this through the universal coefficient theorem,

$$\mathrm{Hom}(H_n(A \times B; \mathbb{Z}); \mathbb{Z}) \cong \bigoplus_{j=0}^{n} \mathrm{Hom}(H_j(A), \mathbb{Z}) \oplus \mathrm{Ext}(H_{j-1}(A), \mathbb{Z}) \oplus \mathrm{Hom}(H_{n-j}(B), \mathbb{Z})$$

4. Cohomology

The Cohomology of X (wrt G) is a sequence of Abelian groups $H^0(X; G), H^1(X; G), \ldots$.

<u>How to Calculate</u>: If you know $H_*(X)$, then use **Universal Coefficient Theorem**:

$$H^j(X; G) = \mathrm{Hom}(H_j(X), G) \oplus \mathrm{Ext}(H_{j-1}(X), G).$$

How to calculate Ext? Each $H_k(X)$ is an Abelian group, so it has some free parts and some torsion parts. If $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_2)^{b_2} \oplus (\mathbb{Z}_3)^{b_3} \oplus (\mathbb{Z}_5)^{b_5} \oplus \cdots$ then

$$\mathrm{Ext}(H_k(X); G) = (G/(2G))^{b_2} \oplus (G/(3G))^{b_3} \oplus (G/(5G))^{b_5} \cdots$$

(note that the free part $\mathbb{Z}^a$ is dropped entirely).

If it's not easy to calculate $H_*(X)$ then there's basically only one other thing you can do. If you're in the specific case that $G = R$ a ring, and $X = A \times B$, and $H^*(A; R), H^*(B; R)$ are both known, and finally $H^*(B; R)$ is free in every dimension, then you can apply the **Künneth formula**:

$$H^*(X; R) = H^*(A \times B; R) = H^*(A; R) \otimes_R H^*(B; R).$$

The cohomology ring is a strictly finer invariant. Two spaces might have the same cohomology groups in every dimension but different ring structures

5. Homotopy Groups

6. Invariants of Common Spaces