

**Problem 1:** Let  $M_n = M_n(\mathbb{C})$  and let

$$\det(\lambda I - a) = \lambda^n + s_1(a)\lambda^{n-1} + \cdots + s_{n-1}(a)\lambda + s_n(a)$$

where  $a \in M_n$ . Note that  $s_i \in \mathbb{C}[M_n]$ . Let  $G = \mathrm{GL}_n(\mathbb{C})$  act on  $M_n$  by conjugation. For  $g \in G$ ,

$$\det(\lambda I - gag^{-1}) = \det(\lambda I - a)$$

so  $s_i \in \mathbb{C}[M_n]^G$  for all  $i$ . Show that  $\mathbb{C}[M_n]^G$  is a free polynomial algebra with generators  $s_1, \dots, s_n$ .

*Proof.*  $G$ -invariant polynomials are determined by their values on  $G$ -orbits, i.e. the conjugacy classes of matrices. If  $p \in \mathbb{C}[M_n]^G$ ,  $p$  is determined by its values on inputs  $a_{ij}$  that are diagonalizable as matrices (since this is a dense subset of  $\mathbb{C}^{n^2}$ ). And on such inputs, by  $G$ -invariance,  $p$  is determined by its value on the conjugate diagonal matrix. Thus,  $p$  is determined by the values it takes on all diagonal matrices.

Now, if  $a$  is a diagonal matrix, we get

$$\det(\lambda I - a) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

so  $s_i$  take the values of the elementary symmetric polynomials on  $a_{11}, \dots, a_{nn}$ . And we know that the space of all symmetric polynomials on  $n$  variables is generated as a free algebra by the elementary symmetric polynomials. Thus, there is some algebraic combination of  $s_1, \dots, s_n$  which matches  $p$  on the diagonal matrices, and thus (because of  $G$ -invariance and the density of the diagonalizable matrices) on all inputs in  $\mathbb{C}^{n^2}$ . That is, all of  $\mathbb{C}[M_n]^G$  is generated as an algebra by  $s_1, \dots, s_n$ .  $\square$

**Problem 2:** Let  $M = \bigoplus_{i \geq 0} M_i$  be a graded  $A$ -module ( $A$  is itself a graded  $k$ -algebra), let  $\{m_s \in M, s \in S\}$  be a collection of homogeneous elements and  $\overline{m_s}$  the image of  $m_s$  under the projection  $M \rightarrow M/A_{>0}M$ .

- (a) Show that  $m_s$  generate  $M$  as an  $A$ -module iff the elements  $\overline{m_s}$  span  $M/A_{>0}M$  as a  $k$ -vector space.

- (b) Deduce that  $M$  is finitely generated iff the  $k$ -vector space  $M/A_{>0}M$  has finite dimension.

*Proof.* (a): In one direction, if  $m_s$  span  $M$  then clearly  $\overline{m_s}$  span  $M/A_{>0}M$ , as the projection is surjective. Everything in  $M/A_{>0}M$  exists in  $M$  as a  $k$ -linear combination in  $M$ .

In the other direction, suppose  $\overline{m_s}$  span  $M/A_{>0}M$ . Let  $M'$  be the  $A$ -submodule of  $M$  generated by  $m_s$ . We will show that  $M_i \subseteq M'$  for each  $i$ , inductively. For  $M_0$  this is clear, as  $M_0$  is fixed by the projection. Now assume that  $M_i \subseteq M'$  for a given  $i$  and we will show that the same is true for  $M_{i+1}$ . Each  $m \in M_{i+1}$  can be written as

$$a_{i+1}m_0 + a_im_1 + a_{i-1}m_2 + \cdots + a_0m_{i+1}$$

where  $a_j \in A_j$  and  $m_j \in M_j/A_{>0}M$ . Then we have  $\overline{m} \equiv a_0m_{i+1}$ . We assumed that this was in the span of  $\overline{m_s}$ , and by inductive hypothesis all of the other components are in  $M'$ , so  $m \in M'$  as well. Thus the induction is complete and  $M' = M$ .

(b): By part (a), if  $M$  is finitely-generated by  $m_s$ , then  $M/A_{>0}M$  is spanned over  $k$  by  $\overline{m_s}$ .

Conversely, if  $M/A_{>0}M$  is finite-dimensional with basis  $m_0, m_1, \dots, m_n \subset M_N$ , then  $M$  is generated over  $A$  by these same elements by (a), as they project down to themselves.  $\square$

**Problem 3:** Let  $R = \{a_s \in A_{>0}, s \in S\}$  be a collection of homogeneous elements.

- (a) Prove that  $A_{>0}$  is generated by  $R$  as an ideal iff  $A$  is generated by  $R$  as an algebra.
- (b) Deduce that  $A_{>0}$  is finitely-generated as an ideal iff  $A$  is finitely-generated as an algebra.

*Proof.* (a): Suppose  $A_{>0}$  is generated by  $R$  as an ideal, and let  $A'$  be the graded algebra generated over  $k$  by  $R$ . We will show  $A_i \subseteq A'$  for each  $i$  inductively. For  $A_0$ , it is automatic because  $k = A_0$ . Now assume that  $A_i \subseteq A'$  and we will show the same for  $A_{i+1}$ . For each  $a \in A_{i+1}$ , since  $A_{>0}$  is generated by  $R$  as an ideal, we have

$$a = a_1 r_1 + a_2 r_2 + \cdots + a_n r_n$$

for some  $r_j \in R$  and  $a_j \in A$ . And since everything in  $R$  has degree at least 1, all of these  $a_j$  are strictly lower in degree than  $a$ , so they're in  $A_i$ . By the inductive hypothesis, they are all in  $A'$ . Thus  $a \in A'$  as well. This concludes the induction, showing that  $A = A'$ .

Conversely if  $A$  is generated by  $R$  as an algebra, then  $A = k[R]$ , so for any  $a \in A_j$  with  $j \geq 1$ ,  $a$  can be written as a  $k$ -linear combination of polynomials of positive degree in  $R$ , which is also an  $A$ -linear combination of elements in  $R$  (just take all but one of the  $R$  terms in each monomial to be the coefficient in  $A$ ), thus  $A_{>0}$  is generated by  $R$  as an ideal.

(b): If  $A$  is finitely-generated as an algebra, then there is some finite algebra generator set  $R$  (we can assume it is positive-degree because everything in  $k$  is already generated in any algebra over  $k$ ). So by (a),  $A_{>0}$  is finitely generated as an ideal (by  $R$ ).  $\square$

Conversely if  $A_{>0}$  is finitely-generated as an ideal by some  $R$ , then we can assume  $R$  is positive-degree, so part (a) applies.  $\square$

**Problem 4:** Let  $A$  be a finite-dimensional algebra over an algebraically closed field  $k$  and  $A_{\text{mod}}$  the algebra  $A$  viewed as a rank 1 free  $A$ -module. Thus, a left ideal of  $A$  is the same thing as a left  $A$ -submodule of  $A_{\text{mod}}$ . Show that if the  $A$ -module  $A_{\text{mod}}$  is completely reducible then the following holds:

- (a) Any left ideal of  $A$  is of the form  $Ae$  where  $e = e^2 \in A$ .
- (b) (Wedderburn theorem)  $A$  is isomorphic to  $M_{\ell_1}(k) \oplus \cdots \oplus M_{\ell_r}(k)$  for some  $\ell_1, \dots, \ell_r \geq 0$ .

*Proof.* (a): Let  $I$  be a left ideal of  $A$ . Then  $I$  is also a left  $A$ -submodule of  $A_{\text{mod}}$ . By complete reducibility we can express  $A$  as a direct sum with  $I$ ,  $A = I \oplus I'$ . Then  $1 \in A$  can be expressed uniquely as  $1 = e + e'$  where  $e \in I$  and  $e' \in I'$ .

First, note that

$$e + e' = 1 = \langle e + e', e + e' \rangle = e^2 + e'^2 \implies e = e^2, e' = e'^2,$$

using the fact that the decomposition is unique, i.e. this is the only way to express 1 as an element of  $I \oplus I'$ . Moreover,  $I = Ae$  and  $I' = Ae'$ ; if  $b \in I$  then

$$b = b(e + e') = be + be' = be$$

as  $b$  has no  $I'$  component. Thus  $I \subseteq Ae$  and the other direction is trivial, so  $I = Ae$ .

(b): First, because  $A_{\text{mod}}$  is rank 1 as an  $A$ -module,  $A^{op} \cong \text{End}_A(A_{\text{mod}})$  via the map

$$a \mapsto [b \mapsto ab].$$

Note that  $aa'$  acts as scaling by  $a'a$  so the order is reversed, hence  $A^{op}$ . The map is invertible because  $A_{\text{mod}}$  is 1-dimensional as an  $A$ -module, hence every automorphism is scaling by something in  $A$  (Schur's Lemma).

Now we can show that  $\text{End}_A(A_{\text{mod}})$  has the desired form; this follows from the fact that  $A$  is completely reducible, using problem 8 on the previous homework. Hence  $A^{op}$  has the desired form, and also  $A$  does as well, using the fact that  $M_\ell(k) \cong M_\ell(k)^{op}$  (such a map can be given by transposition). □

**Problem 5:** Let  $W$  be a vector subspace of a  $k$ -vector space  $V$ .

- (a) Let  $f \in \text{End}_k(V)$  with  $\text{im}(f) = W$ . Show that  $f$  is idempotent (i.e.  $f^2 = f$ ) iff  $f$  projects onto  $W$ , i.e.  $V = W \oplus W'$  and  $f(w, w') = (w, 0)$ .  
(b) Define

$$I_W := \{f \in \text{End}_k(V) \mid \ker(f) \subseteq W\}, \quad J_W := \{f \in \text{End}_k(V) \mid \text{im}(f) \subseteq W\}$$

$I_W$  and  $J_W$  are left and right ideals of  $\text{End}_k(V)$  respectively. Show that if  $V$  is finite-dimensional then all left ideals  $I$  are of the form  $I_W$  for

$$W := \bigcap_{f \in I} \ker(f)$$

and similarly every right ideal  $J$  is of the form  $J_W$  where

$$W := \text{span}\{\text{im}(f)\}_{f \in J}$$

*Proof.* (a): If  $f$  projects onto  $W$  then it is clearly idempotent, as  $f^2(w, w') = f(w, 0) = (w, 0)$ .

Conversely, suppose  $f$  is idempotent.  $f$  is surjective onto  $W$  because  $W$  is its image, so for each  $w \in W$  let  $v$  be such that  $f(v) = w$ . Then

$$f(v) = w \implies w = f(v) = f^2(v) = f(w)$$

by idempotence of  $f$ , i.e.  $f$  fixes  $W$ .

Now let  $W' = \ker(f)$ . I claim  $V = W \oplus W'$ . First, the two subspaces have intersection 0, as  $0 = f(x) = x$  for all  $x \in W \cap W'$ . And their span is all of  $V$ : for  $v \in V$ , suppose  $f(v) = w \in W$ , and let  $w' = v - w$  so that  $v = w + w'$ . Then  $w' \in W'$ , as

$$f(w') = f(v) - f(w) = w - w = 0.$$

Thus  $f$  is a projection onto  $W$ .

(b):  $\text{End}_k(V)$  is completely reducible as a module over itself (in fact, it is a simple module), so we can apply problem 4(a). Since  $I$  is a left-ideal, by problem 4(a) it is of the form  $\text{End}_k(A)e$  where  $e$  is an idempotent matrix, i.e. a projection by 5(a). For this  $e$ ,

$$W = \bigcap_{f \in I} \ker(f) = \bigcap_{f \in \text{End}_k(V)} \ker(fe) = \ker(e)$$

as  $\ker(fe) \supseteq \ker(e)$  with equality when  $f = \text{id}_V$ . So  $I_W$  is the set of  $f$  which vanish on  $\ker(e)$ , i.e.

$$I_W = \{f \in \text{End}_k(V) : \ker(f) \subseteq \ker(e)\} = \text{End}_k(V)e = I.$$

For right ideals, we have  $J = e \text{End}_k(V)$  for some projection  $e$ . Thus

$$W = \text{span}\{\text{im}(f)\}_{f \in J} = \text{span}\{\text{im}(ef)\}_{f \in \text{End}_k(V)} = \text{im}(e)$$

because  $\text{im}(ef) \subseteq \text{im}(e)$  with equality when  $f = \text{id}_V$ . Now,

$$J_W = \{f \in \text{End}_k(V) : \text{im}(f) \subseteq \text{im}(e)\} = e \text{End}_k(V) = J.$$

□

**Problem 6 (Proposition 6.1.2 in notes):**

- (a) Any completely reducible finite-dimensional  $A$ -module is isomorphic to a finite direct sum of simple modules.
- (b) Let  $V_1, \dots, V_r$  be distinct irreducible representations over  $A$  and

$$M := (V_1)^{\ell_1} \oplus \cdots \oplus (V_r)^{\ell_r}.$$

Then  $M^{(V_i)} = (V_i)^{\ell_i}$  (the isotypic component), and  $\ell_i = \dim_k(\text{Hom}_A(V_i, M))$ .

- (c) Let  $f : M \rightarrow N$  be a morphism of finite-dimensional  $A$ -modules. Then for any irreducible  $V$ ,  $f(M^{(V)}) \subseteq N^{(V)}$ .

*Proof.* (a): We show this by induction on dimension. A 1-dimensional  $A$ -module is simple. Let  $M$  be a completely-reducible finite-dimensional  $A$ -module. If  $M$  is simple then we are done. Otherwise, take some submodule  $M_1$ . By complete-reducibility  $M = M_1 \oplus M_2$  for some  $M_2$ . But now  $M_1, M_2$  are lower dimension than  $M$ , so by inductive hypothesis we can express both  $M_1$  and  $M_2$  as finite direct sums of simple modules, and hence  $M$  can be expressed this way as well.

(b):  $M^{(V_i)}$  is by definition the direct sum of all submodules of  $M$  isomorphic to  $V_i$ . There are  $\ell_i$  copies of  $V_i$  in the decomposition given, and this is all of the isomorphic copies of  $V_i$ ; if  $V_i$  maps into  $M$  then restricting to any  $V_j$  with  $j \neq i$  we get by Schur's Lemma that the map must be trivial. Thus  $\text{Hom}_A(V_i, M)$  consists exactly of the direct sum of identity maps into the  $\ell_i$  copies of  $V_i$ , and thus has dimension  $\ell_i$ .

(c): For every submodule  $V'$  of  $M$  isomorphic to  $V$  (irreducible),  $f(V')$  is an irreducible submodule of  $N$ . By Schur's Lemma it is either isomorphic to  $V$  or 0. Thus, every such  $f(V')$  is contained in  $N^{(V)}$ . It follows that  $f(M^{(V)}) \subseteq N^{(V)}$ .  $\square$