MATH 325 FINAL

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Problem 1: Fix $F \in \mathrm{GL}_n(\mathbb{R})$ and consider the following set:

$$G_F = \{g \in M_n(\mathbb{R}) \mid F^{-1}g^\top F g = \mathrm{id}\}.$$

Check that G_F is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ and find an explicit description of $\mathrm{Lie}(G_F)$ in terms of F.

Proof. G_F is closed because it is the preimage of the closed set $\{I_n\} \in M_n(\mathbb{R})$ under the continuous function $g \mapsto F^{-1}g^\top F g$ (it is a degree-2 polynomial in the matrix entries of g hence continuous).

To determine $\mathrm{Lie}(G_F)$, $x \in \mathrm{Lie}(G_F)$ if the following holds for all $t \in \mathbb{R}$:

$$\begin{aligned} F^{-1}(e^{tx})^\top F e^{tx} &= I_n \\ F^{-1}e^{tx^\top} F e^{tx} &= I_n \\ e^{tF^{-1}x^\top F} e^{tx} &= I_n \\ e^t F^{-1}x^\top F + x &= I_n \end{aligned}$$

the last step follows because if $e^a e^b = \mathrm{id}$ then $e^b e^a = \mathrm{id}$ so $e^{a+b} = \mathrm{id}$. The only matrix a for which $e^{ta} = \mathrm{id}$ for all t is $a = 0$, so

$$\mathrm{Lie}(G_F) = \{x \in M_n(\mathbb{C}) \mid F^{-1}x^\top F + x = 0\}$$

which is just a system of n^2 linear equations in the matrix entries of x. $\square$

Problem 2: The action of the unitary group U_n in $M_n(\mathbb{C})$ by conjugation $g : x \mapsto gxg^{-1}$ gives a representation $\rho : U_n \to \mathrm{GL}(M_n(\mathbb{C}))$.

(i) Show that ρ is not irreducible.

(ii) Decompose ρ into a direct sum of irreps of U_n (in complex vector spaces).

Proof. (i): The identity matrix commutes with all other matrices, so $\rho(g)$ fixes every matrix λI. Thus, the subspace $\mathbb{C}I$ is a subrepresentation of ρ, so ρ cannot be irreducible.

(ii): As we've shown in class, $\mathrm{d}\rho(a) : x \mapsto [a, x]$. Any irreducible subrepresentation of $\mathrm{d}\rho$ corresponds to an irreducible subrepresentation of ρ. So it suffices to decompose $\mathrm{d}\rho$.

A potentially useful fact is that $\mathfrak{gl}_n = \mathfrak{u}_n \oplus i\,\mathfrak{u}_n$. We know that the action of $\mathrm{GL}_n(\mathbb{C})$ on $M_n(\mathbb{C})$ by conjugation has its irreps corresponding exactly to the eigenvalues of the matrices. So the corresponding irreps of $\mathrm{d}\rho$ as a representation of $GL_n(\mathbb{C})$ are the same. The decomposition gives that any matrix in $\mathfrak{gl}_n$ can be given as a sum $a + bi$ where $a, b \in \mathfrak{u}_n$, so that

$$[a + bi, x] = [a, x] + i[b, x].$$

As a complex vector space, any subprepresentation V of $\mathrm{d}\rho$ (as a representation of $\mathfrak{u}_n$) is also a subrepresentation of $\mathfrak{gl}_n$, since

$$[a, x], [b, x] \in V \implies [a + bi, x] \in V.$$

Thus, the irreps are actually the same, and ρ can be decomposed as irreps corresponding to matrices with given eigenvalues. $\square$

Problem 3: Let V be a complex vector space of dimension $n > 0$. The group $\mathrm{GL}(V) = \mathrm{GL}_{\mathbb{C}}(V)$ acts in V and in V^*. Thus, there is a natural $\mathrm{GL}(V)$-action in $\mathrm{Sym}^p(V)$ and $\mathrm{Sym}^p(V^*)$. Show that for each $p \geq 1$, the resulting representations of $\mathrm{GL}(V)$ in $\mathrm{Sym}^p(V)$ and $\mathrm{Sym}^p(V^*)$ are irreducible and non-isomorphic.

Proof. Let $v_1, \ldots, v_n$ be a basis for V. The polynomial v_1^p is a highest-weight for $\mathrm{Sym}^p(V)$, since every $\hat{1}_{ij}$ for $i < j$ kills it. Its weight is $\lambda = (p, 0, 0, \ldots, 0)$, as

$$h \cdot v_1^p = (h_1 v_1)^p = h_1^p v_1^p.$$

This is the unique vector of weight λ, since any polynomial with a factor of v_j for $j \neq 1$ will be preserved by $\hat{1}_{jj}$. Also, λ is the unique highest-weight; for v to be a common eigenvector of all diagonal matrices, it must be a monomial, and v_1^p is the only monomial which is killed by all the upper-triangular matrices. We know that $\mathrm{GL}_n(\mathbb{C})$ is reductive, so this is a completely-reducible representation. So if it were not $M(\lambda, v_1^p)$, then we could take an irreducible subrepresentation outside of $M(\lambda, v_1^p)$ which would have no highest-weight, which by the classification theorem is impossible. Thus, $\mathrm{Sym}^p(V) = M(\lambda, v_1^p)$.

Let $v_1^*, \ldots, v_n^*$ be a basis for V^*. In this case, v_n^{*p} is the highest weight and its weight is $\mu = (0, 0, \ldots, 0, -p)$, we

$$h \cdot v_n^{*p} = (v_n^* \circ h^{-1})^p = h_n^{-p} v_n^{*p}.$$

Similarly, this is the unique vector of weight μ, and all the same arguments follow to show that $\mathrm{Sym}^p(V^*)$ is irreducible.

Now, each irreducible representation has a *unique* highest weight, and λ, μ are different, and they are different for all p, so all of these are distinct irreps of $\mathrm{GL}_n(\mathbb{C})$. □

Problem 4: Let G be a Lie group and H the connected component of G containing 1_G.

(i) Show that H is a normal subgroup of G.

(ii) Show that if G is commutative and the group $C := G/H$ is a finite cyclic group $\mathbb{Z}_n$, then there is a group homomorphism $f : C \to G$ such that for all $c \in C$ one has $f(c) \equiv c \pmod{H}$.

Proof. (i): We'll use the fact that connected implies path-connected for Euclidean spaces, which a Lie group is.

First, we show that H is a group: if $a, b \in H$ then there is a continuous path $\gamma : [0,1] \to H$ such that $\gamma(0) = 1_G$ and $\gamma(1) = b$. Then $a\gamma(t)$ is a continuous path $[0,1] \to G$ because a acts continuously on G, and it connects a to ab, so $ab \in H$ because H is connected.

Similarly, if $g \in G$ and $h \in H$, there is a path $\gamma : [0,1] \to H$ with $\gamma(0) = 1_G$ and $\gamma(1) = h$, so $g\gamma(t)g^{-1}$ is a continuous path in G connecting 1_G to ghg^{-1}, thus $ghg^{-1} \in H$, making H a normal subgroup.

(ii): Let $\mathrm{Lie}(G)$ be the Lie algebra of G. Since H is connected, everything in H is generated by $e^{\mathrm{Lie}(G)}$ as we've shown on homework, and G is commutative so $e^a e^b = e^{a+b}$, thus everything in H can actually be represented directly as an exponential of something in $\mathrm{Lie}(G)$, i.e. $H = e^{\mathrm{Lie}(G)}$.

Fix some a in the coset of H corresponding to $1 \in \mathbb{Z}_n$. Because $G/H = \mathbb{Z}_n$, every element of G is $a^j e^x$ for some $x \in \mathrm{Lie}(G)$ and $j \in [0, n-1]$. And because $H = e^{\mathrm{Lie}(G)}$ and $a^n \in H$, we have that $a^n = e^y$ for some $y \in \mathrm{Lie}(G)$. To get an element $g \in G$ with order exactly n, we can take $g := ae^{-y/n}$ to get

$$g^n = a^n e^{-y} = 1$$

so that $\langle g \rangle \cong \mathbb{Z}_n$. So f can be defined by $f : j \mapsto g^j$.

$\square$

Problem 5: Let $\mathrm{Aff}^{>0}(\mathbb{R})$ be the group of affine linear transformations $g_{a,b} : \mathbb{R} \to \mathbb{R}$

$$g_{a,b} : x \mapsto ax + b$$

for $a \in \mathbb{R}^{>0}$ and $b \in \mathbb{R}$. Let $\rho : \mathrm{Aff}^{>0}(\mathbb{R}) \to \mathrm{GL}(V)$ be a continuous representation in a finite-dimensional complex vector space V. Prove that if $v \in V$ is such that $\rho(g_{1,b})(v) = e^b v$ for all $b \in \mathbb{R}$ then $v = 0$.

Proof. We have the commutator relation

$$g_{1,b} \circ g_{a,0} = g_{a,0} \circ g_{1,b/a}.$$

If such a v exists, this implies

$$g_{1,b} \cdot (g_{a,0} \cdot v) = g_{a,0} \cdot e^{b/a} v = e^{b/a}(g_{a,0} \cdot v).$$

That is, every $g_{a,0}v$ is an eigenvector for every $g_{1,b}$, with eigenvalue $e^{b/a}$.

This implies that any nonzero vectors $\{g_{a,0}v\}_{a \in \mathbb{R}^+}$ are linearly independent, since they have different eigenvalues wrt $g_{1,b}$ for $b \neq 0$. So because V is finite-dimensional, we can conclude that all but finitely many a have $g_{a,0} \cdot v = 0$. But ρ is a *continuous* representation, so $\rho(g_{a,0})v$ is a continuous function in a that is 0 for all but finitely many a, and thus it must be 0 for all a. But then when $a = 1$,

$$v = g_{1,0} \cdot v = 0.$$

So such a v cannot exist. $\square$