MATH 325 HW 1

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Problem 1: Let $V = k^n$ and $P_d \subseteq k[x_1, \ldots, x_n]$ be the space of degree-d polynomials in n variables over k. Let $a : V \to V$ be some linear map with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show the following generating-function identities:

(a)
$$\sum_{d\geq 0} \mathrm{tr}_{V^{\otimes d}}(a^{\otimes d}) \cdot t^d = \frac{1}{1 - \mathrm{tr}_V(a) \cdot t}.$$

(b)
$$\sum_{d\geq 0} \mathrm{tr}_{\mathrm{Sym}^d V}(\mathrm{Sym}^d(a)) \cdot t^d = \prod_{i=1}^n \frac{1}{1 - \lambda_i t}.$$

(c)
$$\sum_{d\geq 0} \mathrm{tr}_{\wedge^d V}(\wedge^d a) \cdot t^d = \prod_{i=1}^n (1 + \lambda_i t).$$

Proof. (a): The trace of $a^{\otimes d}$ is $\mathrm{tr}(a)^d$. It suffices to take some basis where a is in Jordan normal form, as trace is invariant under conjugation. In this case, $a^{\otimes d}$ acts by

$$a^{\otimes d}(e_{k_1} \otimes \cdots \otimes e_{k_d}) = \Big(\prod_j \lambda_{k_j}\Big)(e_{k_1} \otimes \cdots \otimes e_{k_d}) + \text{off-diagonal terms}$$

and the trace is thus

$$\sum_{(k_j)\in[n]^d} \prod_j \lambda_{k_j} = \Big(\sum_k \lambda_k\Big)^d = \mathrm{tr}(a)^d$$

as desired. The generating function immediately follows from the geometric sum formula.

(b): The trace of $\mathrm{Sym}^d(a)$ is the sum of all degree-d monomials in $\lambda_1, \ldots, \lambda_n$. Again we can assume that a is in Jordan normal form wrt some basis. Then $\mathrm{Sym}^d(a)$ acts by

$$e_1^{d_1} e_2^{d_2} \cdots e_n^{d_n} \mapsto \lambda_1^{d_1} \lambda_2^{d_2} \cdots \lambda_n^{d_n} (e_1^{d_1} e_2^{d_2} \cdots e_n^{d_n}) + \text{off-diagonal terms}$$

giving the trace as desired. From there, the generating function is

$$\sum_{d\geq 0} \mathrm{tr}(\mathrm{Sym}^d(a)) \cdot t^d = \prod_i \sum_{d_i \geq 0} \lambda_i^{d_i} t^{d_i} = \prod_i \frac{1}{1 - \lambda_i t}.$$

(c): The trace of $\wedge^d a$ is the sum of all degree d monomials in $\lambda_1, \ldots, \lambda_n$ without repeating terms. Again we assume that a is in Jordan normal form wrt some basis. $\wedge^d a$ acts by

$$e_{k_1} \wedge \cdots \wedge e_{k_d} \mapsto \lambda_{k_1} \cdots \lambda_{k_d} (e_1 \wedge \cdots \wedge e_d)$$

giving the desired trace. The generating function follows. It is like in (b) except that terms cannot repeat, thus we have $1 + \lambda_i t$ instead of the full series $1 + \lambda_i t + \lambda_i^2 t^2 + \ldots$ $\square$

Problem 2: Prove the following claims made in class:
(a) The algebra $\mathbb{C}[x_1, \ldots, x_n]^{\mathrm{SO}_n}$ is the free algebra generated by $R := x_1^2 + \cdots + x_n^2$.
(b) The algebra $\mathbb{C}[x_1, \ldots, x_n]^{\mathrm{SO}_{n-1}}$ is the free algebra generated by R and x_n.

Proof. (a): Let p be a polynomial fixed by SO_n. Let $q(t)$ be the polynomial

$$q(t) := p(t, 0, 0, \ldots, 0).$$

Because p is fixed by SO_n, we have $p(x_1, \ldots, x_n) = q(\sqrt{R})$ for all $x_1, \ldots, x_n$. Moreover, q has no odd-degree terms, since $q(t)$ is invariant under switching the sign of t (as this corresponds to a π-rotation about the x_2 axis, which is in SO_n). Thus, $q(\sqrt{R})$ is a polynomial in R, and hence p is as well.

(b) If p is fixed by SO_{n-1}, then we can think of p as a polynomial in $\mathbb{C}[x_n][x_1, \ldots, x_{n-1}]$, from which the same method from part (a) shows that p is generated by $x_1^2 + \cdots + x_{n-1}^2$ over $\mathbb{C}[x_n]$, and thus that p is generated by $x_1^2 + \cdots + x_{n-1}^2$ and x_n over $\mathbb{C}$, or equivalently R and x_n. $\square$

Problem 3: The case $n = 2$: Let $H_d \subseteq P_d$ be the space of degree-d harmonic homogeneous polynomials in x, y over $\mathbb{C}$.

(a) Find an explicit $\mathbb{C}$-basis for H_d.

(b) Show that the representation of SO_2 in H_d is not irreducible.

Proof. (a): For all d, $\dim(H_d) = 2$. Let A be the homogeneous polynomial

$$A(x, y) := \sum_{k=0}^{d} a_k x^k y^{d-k}.$$

If $A \in H_d$, then

$$\begin{aligned} 0 &= \Delta(A) \\ &= \sum_{k=0}^{d} a_k \cdot (k)(k-1)x^{k-2}y^{d-k} + a_k \cdot (d-k)(d-k-1)x^k y^{d-k-2} \\ &= \sum_{k=0}^{d-2} \Big(a_k(d-k)(d-k-1) + a_{k+2}(k+2)(k+1)\Big)x^k y^{d-k-2} \end{aligned}$$

and thus

$$a_k(d-k)(d-k-1) + a_{k+2}(k+2)(k+1) = 0$$

for all $0 \le k \le d-2$. Thus, given that A is harmonic, the ratios between all even coefficients are fixed, and the ratios between all odd coefficients are fixed, i.e. A is a linear combination of the two harmonic polynomials

$$y^d - \Big(\frac{d(d-1)}{2}\Big)x^2y^{d-2} + \Big(\frac{d(d-1)(d-2)(d-3)}{(4)(3)(2)}\Big)x^4y^{d-4} - \cdots$$

and

$$xy^{d-1} - \Big(\frac{(d-1)(d-2)}{(3)(2)}\Big)x^3y^{d-3} + \Big(\frac{(d-1)(d-2)(d-3)(d-4)}{(5)(4)(3)(2)}\Big)x^5y^{d-5} - \cdots$$

which are also proportional to the real and imaginary parts of $(ix + y)^d$.

(b): The action of SO_2 on H_d preserves the 1-dimensional subspace $\mathbb{C}\langle(ix + y)^d\rangle \subset H_d$. All elements of SO_2 are rotations by some θ, corresponding to matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

which maps $(ix + y)^d$ to

$$(i(\cos\theta x + \sin\theta y) + \cos\theta y - \sin\theta x)^d = (e^{i\theta}(ix + y))^d = e^{i\theta d}(ix + y)^d$$

and thus preserves the subspace. Thus the action of SO_2 is reducible.

$\square$

Problem 4: In the case $n = 3$:
(a) Check that $\dim(H_d) = 2d + 1$ and find a nonzero element of H_2 which is fixed by SO_2.
(b) For each $d \geq 0$, find the trace of the operator g_θ, which is the rotation by θ about the z axis.

Proof. (a): To show that $\dim(H_d) = 2d + 1$, we use the decomposition of P_d as

$$P_d = H_d \oplus R \cdot P_{d-2}$$

which gives

$$\begin{aligned}
\dim(P_d) &= \dim(H_d) + \dim(P_{d-2}) \\
\binom{d+2}{2} &= \dim(H_2) + \binom{d}{2} \\
\dim(H_2) &= \frac{(d+2)(d+1) - d(d-1)}{2} \\
&= \frac{4d+2}{2} = 2d + 1
\end{aligned}$$

as expected.

To give a nonzero element of H_2 fixed by SO_2, take $x^2 + y^2 - 2z^2$. Clearly $x^2 + y^2$ and z^2 are each fixed by SO_2, so $x^2 + y^2 - 2z^2$ is also fixed, and it is harmonic.

(b): The eigenvalues of g_θ are $e^{i\theta}, e^{-i\theta}, 1$, so g_θ can be represented by the matrix

$$g_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its action on the basis elements of P_d is

$$g_\theta : x^a y^b z^c \mapsto (e^{i\theta} x)^a (e^{-i\theta} y)^b z^c = e^{(a-b)i\theta} \cdot x^a y^b z^c.$$

This gives the trace

$$\mathrm{tr}_{P_d}(g_\theta) = \sum_{a+b \leq d} e^{(a-b)i\theta}.$$

Now to determine $\mathrm{tr}_{H_d}(g_\theta)$ from this. We can decompose P_d into subspaces $H_d \oplus R \cdot P_{d-2}$. Thus, the trace of g_θ is the sum of its trace on the subspaces H_d and $R \cdot P_{d-2}$. In the latter subspace, the trace is not affected by R, since g_θ preserves R. Thus,

$$\begin{aligned}
\mathrm{tr}_{H_d}(g_\theta) &= \mathrm{tr}_{P_d}(g_\theta) - \mathrm{tr}_{P_{d-2}}(g_\theta) \\
&= \sum_{a+b \in \{d-1, d\}} e^{i\theta(a-b)} \\
&= \sum_{k=-d}^{d} e^{ki\theta} \\
&= \frac{e^{(d+1)i\theta} - e^{-di\theta}}{e^{i\theta} - 1}.
\end{aligned}$$

The last line only follows if $e^{i\theta} \neq 1$, of course. In the case $\theta = 0$, the geometric sum yields $\mathrm{tr}_{H_d}(\mathrm{id}) = \dim(H_d) = 2d + 1$ as expected.

$\square$

Problem 5: For $n \geq 3$, prove the following generating function identity:

$$\sum_{d\geq 0} \dim(H_d) \cdot t^d = \frac{1+t}{(1-t)^{n-1}}.$$

Proof. For general n, $\dim(H_d)$ is

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1}$$

which yields the generating function

$$\begin{aligned}\sum_{d\geq 0} \dim(H_d) t^d &= \sum_{d\geq 0} \left(\binom{d+n-1}{n-1} - \binom{d+n-3}{n-1} \right) t^d \\ &= \frac{1}{(1-t)^n} - \frac{t^2}{(1-t)^n} \\ &= \frac{1+t}{(1-t)^{n-1}}\end{aligned}$$

as desired. Here we used the generating function identity

$$\frac{1}{(1-t)^m} = \sum_{d\geq 0} \binom{d+m}{m} t^d$$

which can be proven by the generalized binomial theorem:

$$(1-t)^{-m} = \sum_{d\geq 0} \binom{-m}{d} (-t)^d = \sum_{d\geq 0} \binom{m+d}{d} t^d = \sum_{d\geq 0} \binom{d+m}{m} t^d.$$

$\square$

Problem 6: Let res : $\mathbb{C}[x_1, \dots, x_n] \to C(S^{n-1})$ be the restriction map to S^{n-1}. If $\overline{P_d} := \text{res}(P_d)$, let M_d be the orthogonal complement of $\overline{P_{d-2}}$ in $\overline{P_d}$. Show that res induces an isomorphism $H_d \to M_d$ of SO_n-representations.

Proof. I'll first give a very analytic proof, and then a short algebraic one:

First, we'll show that $\text{res}(H_d) \subseteq M_d$ by showing that for $h \in H_d, p \in P_{d-2}$,

$$\int_{S^{n-1}} h\overline{p} = 0.$$

Because of the decomposition of P_{d-2}, and since $R = 1$ on S^{n-1}, it suffices to show this for $p \in H_{d-2j}$ for all $j \geq 1$. We will prove this be induction on d. For $d = 1, 2$ the result is clear. For the inductive step, assume that this holds for $d - 1$, and we'll show that it does for d as well.

To do this we will use the divergence theorem

$$\int_{S^{n-1}} \nabla f \cdot \vec{x} = \int_{B^n} \Delta f,$$

Euler's identity for homogeneous functions,

$$\nabla f \cdot \vec{x} = \deg(f) \cdot f,$$

and the Laplacian identity

$$\Delta(ab) = a\Delta(b) + b\Delta(a) + 2\nabla a \cdot \nabla b.$$

Using these, we get

$$\begin{aligned}
\int_{S^{n-1}} h\overline{p} &= \frac{1}{2d-2} \int_{S^{n-1}} \nabla(h\overline{p}) \cdot \vec{x} \\
&= \frac{1}{2d-2} \int_{B^n} \Delta(h\overline{p}) \\
&= \frac{1}{2d-2} \int_{B^n} \overline{p}\Delta(h) + h\Delta(\overline{p}) + 2\nabla(h) \cdot \nabla(\overline{p}) \\
&= \frac{1}{d-1} \int_{B^n} \nabla(h) \cdot \nabla(\overline{p}).
\end{aligned}$$

Now, since $h, \overline{p}$ are harmonic, $\partial_j h$ and $\partial_j \overline{p}$ are also harmonic for each $j \in [n]$, thus $\nabla(h) \cdot \nabla(\overline{p})$ is the sum of products in $H_{d-1} \cdot H_{d-3}$. By the inductive hypothesis, we know these integrate to 0 over S^{n-1} and thus over B^n as well. So $\langle h, \overline{p} \rangle = 0$, as desired.

Conversely, if $q \in P_d$, and $\langle q, p \rangle = 0$ for all $p \in P_{d-2}$, then q must be harmonic: otherwise, $q = Rq_{d-2}$ where $q_{d-2} \in P_{d-2}$, so setting $p = q_{d-2}$ yields

$$\langle q, p \rangle = \langle Rq_{d-2}, q_{d-2} \rangle = \int_{S^{n-1}} |q_{d-2}|^2 > 0.$$

So we see that res induces an isomorphism on the vector spaces H_d and M_d. And noting that res clearly commutes with the action of SO_n on H_d and M_d, this map is also an isomorphism of representations.

Algebraic proof: To show that H_d is orthogonal to $H_{d'}$ for $d' \neq d$, we first note that the projection map $\pi : H_d \to H_{d'}$ is an SO_n-intertwining operator. Then by Schur's Lemma, because H_d and $H_{d'}$ are both irreducible representations of different dimensions (hence non-isomorphic), the map π must be trivial, and thus $H_d \perp H_{d'}$. $\square$

Problem 7: Prove that for $d \geq 0$, there is a constant $c_d > 0$ such that for all $p, q \in H_d$,

$$p(\partial)(\overline{q})(0) = c_d \cdot \langle \operatorname{res}(p), \operatorname{res}(q)\rangle_{L^2}$$

where the inner product $\langle \bullet, \bullet \rangle_{L^2}$ is given by

$$\langle f, g\rangle := \int_{S^{n-1}} f(x)\overline{g}(x)\, \mathrm{d}x.$$

Proof. We will show this by induction on d. In the case $d = 1$, p and q are each linear combinations of $x_1, \ldots, x_n$, so

$$p = p_1x_1 + p_2x_2 + \cdots + p_nx_n, \quad q = q_1x_1 + q_2x_2 + \cdots + q_nx_n.$$

The LHS is equal to $p \cdot q := \sum_j p_j\overline{q_j}$, naturally. On the RHS, we integrate a quadratic over S^{n-1}, and the cross-terms cancel because they are odd, yielding

$$\begin{aligned}\int_{S^{n-1}} p\overline{q} &= \sum_{i,j=1}^{n} \int_{S^{n-1}} p_i\overline{q_j}x_ix_j \\ &= \sum_j \int_{S^{n-1}} p_j\overline{q_j}x_j^2 \\ &= \left(\int_{S^{n-1}} x_1^2\right) \cdot (p \cdot q).\end{aligned}$$

Indeed, the RHS and LHS differ by a constant factor, as desired. Now, for the inductive step, assume that there is such a c_{d-1}, and we will show that there is a c_d.

As explained in the solution to Problem 6, we can express $\langle \operatorname{res}(p), \operatorname{res}(q)\rangle$ as

$$\begin{aligned}\langle \operatorname{res}(p), \operatorname{res}(q)\rangle &= \int_{S^{n-1}} p\overline{q} \\ &= \frac{1}{d}\int_{B^n} \nabla(p) \cdot \nabla(\overline{q}).\end{aligned}$$

Now, $\nabla(p)$ and $\nabla(\overline{q})$ are both sums of polynomials in H_{d-1}, so by the inductive hypothesis,

$$\begin{aligned}\langle \operatorname{res}(p), \operatorname{res}(q)\rangle &= \frac{1}{d}\int_{B^n} \nabla(p) \cdot \nabla(\overline{q}) \\ &= \frac{1}{d}\int_0^1 \int_{S^{n-1}} \nabla(p(r\vec{x})) \cdot \nabla(\overline{q}(r\vec{x}))\, \mathrm{d}\vec{x}\mathrm{d}r \\ &= \frac{1}{d}\int_0^1 r^{2d-2}\int_{S^{n-1}} \nabla(p(\vec{x})) \cdot \nabla(\overline{q}(\vec{x}))\, \mathrm{d}\vec{x}\mathrm{d}r \\ &= \frac{1}{d}\int_0^1 r^{2d-2}c_{d-1}(\nabla p, \nabla\overline{q})\, \mathrm{d}r \\ &= \frac{c_{d-1}}{d(2d-1)}(\nabla p, \nabla \overline{q}).\end{aligned}$$

Finally, we have

$$(\nabla p, \nabla \overline{q}) = (\nabla p \cdot \vec{x}, \overline{q}) = (dp, \overline{q}) = d(p, \overline{q})$$

giving

$$\langle \operatorname{res}(p), \operatorname{res}(q)\rangle = \frac{c_{d-1}}{(2d-1)} \cdot (p, \overline{q})$$

Thus, c_d exists and is equal to $c_{d-1}/(2d-1)$. $\square$

Problem 8 (Optional): Show that the kernel of res is exactly the principal ideal generated by $R-1$.

Proof. By Hilbert's Nullstellensatz, the ideal of polynomials vanishing on S^{n-1} is exactly $\sqrt{R-1}$. But $R-1$ is irreducible so $\sqrt{R-1}=(R-1)$. One can see this by using Eisenstein's criterion:

We'll show this by induction on n. For the base case $n=2$, x^2+y^2-1 is irreducible as a polynomial in $\mathbb{C}[x][y]$, as $(x-1)$ is prime and divides (only once) the constant term (x^2-1) and the linear term 0, but not the leading term y^2 (Eisenstein's Criterion). Thus it is irreducible as a polynomial in $\mathbb{C}[x,y]$ as well.

For the inductive step, the x_n^0 term $x_1^2+\cdots+x_{n-1}^2-1$ is prime by the inductive hypothesis, and it also divides the x_n^1 term (which is 0), but not the x_n^2 term, so $x_1^2+\cdots+x_n^2-1$ is also irreducible. $\square$