Problem 1:

(a) Let A be a finite-dimensional $\mathbb{C}$-algebra such that the rank 1 free A-module A_{mod} is completely reducible. For an irreducible V, write $[A_{mod} : V]$ for the multiplicity of V in A_{mod} and let $Z(A)$ be the center of A. Prove that $[A_{mod} : V] = \dim(V)$ for all irreducible V, and

$$|\operatorname{Irr}(A)| = \dim Z(A), \quad \dim(A) = \sum_{V \in \operatorname{Irr}(A)} \dim(V)^2.$$

(b) Let G be a finite group. Prove that $[(\mathbb{C}G)_{mod} : V] = \dim(V)$ for all $V \in \operatorname{Irr}(G)$, and

$$|\operatorname{Irr}(G)| = |\text{Conjugacy classes of } G|, \quad |G| = \sum_{V \in \operatorname{Irr}(G)} \dim(V)^2.$$

Proof. (a): If A decomposes as

$$A \cong V_1^{\ell_1} \oplus \cdots \oplus V_r^{\ell_r}$$

where $V_1, \ldots, V_r$ are the simple submodules of A_{mod}, then by Wedderburn's Theorem A can also be written as a direct sum of matrix rings

$$M_{\ell_1}(\mathbb{C}) \oplus \cdots \oplus M_{\ell_r}(\mathbb{C}).$$

The dimension of each isotypic component is thus expressed in two ways, $\dim(V_j^{\ell_j}) = \dim(V_j) \cdot \ell_j$ and $\dim(M_{\ell_j}(\mathbb{C})) = \ell_j^2$. Since these must be equal and $\ell_j \neq 0$, we get $\ell_j = \dim(V_j)$.

Using the matrix decomposition, the center of A is

$$Z(A) = Z(M_{\ell_1}) \oplus \cdots \oplus Z(M_{\ell_r}) = (\mathbb{C}I_{\ell_1}) \oplus \cdots \oplus (\mathbb{C}I_{\ell_r}) \cong \mathbb{C}^r$$

thus $\dim Z(A) = r$, the number of simple submodules. Furthermore,

$$\dim(A) = \sum_{j=1}^{r} \dim(M_{\ell_j}) = \sum_{j=1}^{r} \ell_j^2 = \sum_{j=1}^{r} \dim(V_j)^2.$$

(b): By Maschke's Theorem, $\mathbb{C}G$ is semisimple, so we can apply (a) in the case $A = \mathbb{C}G$. This yields $[(\mathbb{C}G)_{mod} : V] = \dim(V)$, and

$$|\operatorname{Irr}(\mathbb{C}G)| = \dim Z(\mathbb{C}G), \quad \dim(\mathbb{C}G) = \sum_{V \in \operatorname{Irr}(\mathbb{C}G)} \dim(V)^2.$$

Now, $\operatorname{Irr}(\mathbb{C}G) = \operatorname{Irr}(G)$ by the usual correspondence, and $\dim(\mathbb{C}G) = |G|$ as $\mathbb{C}G$ is spanned by the elements of G. Moreover, the center of $\mathbb{C}G$ can be spanned by the sums of each conjugacy class; conjugation by h transitively permutes elements within a conjugacy class, so anything in $Z(\mathbb{C}G)$ must have the same coefficient on all elements of a conjugacy class. Thus, $Z(\mathbb{C}G)$ is exactly spanned by sums of each conjugacy class, and its dimension is the number of conjugacy classes.

With these facts, the equations become

$$|\operatorname{Irr}(G)| = |\text{Conjugacy classes of } G|, \quad |G| = \sum_{V \in \operatorname{Irr}(G)} \dim(V)^2$$

as desired.

$\square$

Problem 2:

(a) Show that U_n, $\mathrm{SO}_n(\mathbb{R})$ are compact and path-connected, and the group $\mathrm{O}_n(\mathbb{R})$ is not connected.

(b) Show that there is no compact subgroup $K \subset \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{U}_n \subsetneq K$.

Proof. (a): In $\mathrm{SO}_n(\mathbb{R})$, it suffices to show that every $M \in \mathrm{SO}_n(\mathbb{R})$ has a path within $\mathrm{SO}_n(\mathbb{R})$ to I_n. Let $\gamma : [0,1] \to \mathrm{SO}_n(\mathbb{R})$ be the path

$$\gamma(t) = M^t = e^{\log(M) \cdot tI_n}$$

(note that $\log(M)$ and tI_n commute) so that $\gamma(1) = M, \gamma(0) = I_n$. γ is clearly continuous in t, so it remains to show that M^t is actually in $\mathrm{SO}_n(\mathbb{R})$. This follows from two identities about matrix exponentiation:

$$(M^t)^\top = (M^\top)^t, \quad (AB)^t = A^tB^t \text{ if } A, B \text{ commute.}$$

From these we can show

$$M^t(M^t)^\top = M^t(M^\top)^t = M^t(M^{-1})^t = (MM^{-1})^t = I_n$$

Noting that $M^{-1} = M^\top$ because $M \in \mathrm{SO}_n(\mathbb{R})$. Thus $M^t \in \mathrm{SO}_n(\mathbb{R})$ as well.

Showing that U_n is path-connected is similar, as

$$\gamma(t) = M^t$$

for unitary M is a path between M and I_n for the same reasoning.

To show compactness, because we are in an ambient Euclidean space it suffices to show that both $U_n, \mathrm{SO}_n(\mathbb{R})$ are closed, since they are bounded (each column has norm 1 so any matrix in either group has norm at most n). And the property of being in U_n or $\mathrm{SO}_n(\mathbb{R})$ is the finite intersection of polynomial conditions saying "columns are orthogonal" and "determinant is 1." These are closed sets because they are continuous preimages of $\{0\}$ and $\{1\}$, which are closed. Thus their finite intersection is closed.

The reason $\mathrm{O}_n(\mathbb{R})$ is not connected is that the pieces with determinant 1 and -1 are disconnected. Note that $\det(\mathrm{O}_n(\mathbb{R})) = \{-1, 1\}$, a disconnected set, but det is continuous so it preserves the property of connectedness. $\square$

Problem 3: Show that $\mathrm{SL}_n(\mathbb{R})$ is path-connected and not compact.

Proof. To show $\mathrm{SL}_n(\mathbb{R})$ is path-connected, it suffices to exhibit a path between any $M \in \mathrm{SL}_n(\mathbb{R})$ and some element of $\mathrm{SO}_n(\mathbb{R})$, which we already know is path-connected from Problem 2. Let M be composed of columns $m_1, m_2, \ldots, m_n \in \mathbb{R}^n$. Fixing all but the first column, we have the linear $\mathbb{R}^n \to \mathbb{R}$ function

$$v \mapsto \det(v, m_1, \ldots, m_n).$$

Since this mapping is linear and nontrivial (as m_1 produces the output 1) it must be equivalent to an inner product with some fixed nonzero vector w (in fact this is the cross product of $m_1, \ldots, m_n$). Thus m_1 can vary freely within the hyperplane $\{v : \langle v, w\rangle = 1\}$ without leaving $\mathrm{SL}_n(\mathbb{R})$.

Now, we'd like to continuously move m_1 to some scaled basis vector λe_j while staying inside $\mathrm{SL}_n(\mathbb{R})$. There is such a path within $\{v : \langle v, w\rangle = 1\}$ unless $w \perp e_j$, and w cannot be orthogonal to the entire basis, so it is possible for some j. Repeating for each m_j, we get a matrix in $\mathrm{SL}_n(\mathbb{R})$ each of whose columns is a basis vector. None of the columns can be repeated as the determinant remains 1, so the result is a permutation matrix, and hence in $\mathrm{SO}_n(\mathbb{R})$ as desired.

Within $\mathrm{SL}_n(\mathbb{R})$ we have the sequence

$$M_k = \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

for $k \in \mathbb{N}_+$. This sequence has no convergent subsequence. That shows that $\mathrm{SL}_n(\mathbb{R})$ cannot be compact. $\square$

Problem 4: Let $\text{Aff}(\mathbb{R})$ be the group of affine linear transformations of the form $g_{a,b} : x \mapsto ax + b$ with $a \neq 0$. Find a pair ϕ, ψ of continuous functions

$$\phi, \psi : \{(a, b) \in \mathbb{R}^2 | a \neq 0\} \to \mathbb{R}_{>0}$$

such that $\phi(a, b)\mathrm{d}a\mathrm{d}b$ is a left-invariant measure on $\text{Aff}(\mathbb{R})$ and $\psi(a, b)\mathrm{d}a\mathrm{d}b$ is a right-invariant measure on $\text{Aff}(\mathbb{R})$.

Proof. For this to be left-invariant means that for all functions $f : \text{Aff}(\mathbb{R}) \to \mathbb{R}$, and all affine transformations $g_{c,d} \in \text{Aff}(\mathbb{R})$,

$$\int_{\text{Aff}(\mathbb{R})} f(g_{a,b})\phi(a, b)\ \mathrm{d}a\mathrm{d}b = \int_{\text{Aff}(\mathbb{R})} f(g_{c,d} \cdot g_{a,b})\phi(a, b)\ \mathrm{d}a\mathrm{d}b = \int_{\text{Aff}(\mathbb{R})} f(g_{ac,bc+d})\phi(a, b)\ \mathrm{d}a\mathrm{d}b.$$

I claim $\phi(a, b) = a^{-2}$ works. To see this, it is equivalent to show that the measure of rectangles is unaffected by left action. That is, if $[a_0, a_1] \times [b_0, b_1]$ is a rectangle in $\text{Aff}(\mathbb{R})$,

$$\int_{a_0}^{a_1} \int_{b_0}^{b_1} a^{-2}\ \mathrm{d}a\mathrm{d}b = -2(b_1 - b_0)\Big(\frac{1}{a_1} - \frac{1}{a_0}\Big)$$

and

$$\int_{ca_0}^{ca_1} \int_{cb_0+d}^{cb_1+d} a^{-2}\ \mathrm{d}a\mathrm{d}b = -2c(b_1 - b_0)\Big(\frac{1}{ca_1} - \frac{1}{ca_0}\Big) = -2(b_1 - b_0)\Big(\frac{1}{a_1} - \frac{1}{a_0}\Big)$$

thus $a^{-2}\ \mathrm{d}a\mathrm{d}b$ is left-invariant.

Right action is $g_{a,b} \cdot g_{c,d} = g_{ca,da+b}$. For this, $\psi(a, b) = a^{-1}$ works. Again we can show that the measure is right-invariant on rectangles, which implies it generally:

$$\int_{a_0}^{a_1} \int_{b_0}^{b_1} a^{-1}\ \mathrm{d}a\mathrm{d}b = (b_1 - b_0)(\log(a_1) - \log(a_0))$$

and

$$\int_{ca_0}^{ca_1} \int_{da+b_0}^{da+b_1} a^{-1}\ \mathrm{d}a\mathrm{d}b = (b_1 - b_0)(\log(ca_1) - \log(ca_0)) = (b_1 - b_0)(\log(a_1) - \log(a_0))$$

thus $a^{-1}\ \mathrm{d}a\mathrm{d}b$ is right-invariant. $\square$

Problem 5: Let $\mathrm{d}x$ be the standard Lebesgue measure on $M_n(\mathbb{R})$, and view $\mathrm{GL}_n(\mathbb{R})$ as an open subset of $M_n(\mathbb{R})$. Find a function $f : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_{>0}$ such that $f(x)\mathrm{d}x$ is a bi-invariant measure on $\mathrm{GL}_n(\mathbb{R})$.

Proof. Let $f(x) = |\det(x)|^{-n}$. This is bi-invariant, and it follows as a special case of the Jacobian change-of-variables formula, which says in general that for a region S and a linear transformation M,

$$\int_{M(S)} g(x) = \int_S |\det(M)| \cdot g(M(x)).$$

In this case, M is actually acting on the space of matrices. If e_{ij} is a basis for this space (where e_{ij} is the matrix with 1 in the ijth entry and 0 elsewhere), then M left-acts on the basis by sending e_{ij} to $\sum_{i=1}^n m_{ji}e_{ij}$. Thus, as a matrix acting on $\mathrm{GL}_n(\mathbb{R})$, M looks like n copies of M (as a matrix acting on $\mathbb{R}^n$) along the diagonal, so its determinant is $\det(M)^n$. Similarly for right-action. Hence, by the change-of-variables formula, $|\det(x)|^{-n}$ is bi-invariant. $\square$

Problem 6: (Optional) Give an example of a discrete subgroup H of the additive group $(\mathbb{R}^2, +)$ such that the image of H under the first projection $\mathbb{R}^2 \to \mathbb{R}$ is not a discrete subgroup of $(\mathbb{R}, +)$.

Proof. Take $H = \langle (1, \pi), (-\pi, 1) \rangle$. The elements of H are of the form

$$(a - b\pi, a\pi + b) \;\; a, b \in \mathbb{Z}.$$

This is a square lattice within $\mathbb{R}^2$, hence discrete. But in the projection, $\{a - b\pi : a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, so it is not a discrete subgroup. $\square$

Problem 7: View a finite-dimensional $\mathbb{R}$-vector space V as a topological group wrt addition and let H be a discrete (wrt to the topology) subgroup of V. Prove that one can find an $\mathbb{R}$-basis of V, $e_1, \dots, e_n$ (with $n = \dim(V)$) such that

$$H = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_d$$

for some $d \leq n$. To prove this, choose a Euclidean inner product on V and use the following strategy:

(1) Let e_1 be a nonzero element H of minimal length (why does one exist?). Check that in the case $\dim(V) = 1$ and $H \neq \{0\}$, $H = \mathbb{Z}e_1$.
(2) Let $V' := e_1^{\perp}$, and let $p : V \to V'$ be an orthogonal projection along the line $\mathbb{R}e_1$. Prove that $p(H)$ is a discrete subgroup of V'.
(3) Complete the proof by induction on $\dim(V)$.

Proof. (1): Within H, there must be a nonzero element of minimal length; otherwise, 0 is not an isolated point and H is not discrete. Let this minimal element be e_1. If $\dim(V) = 1$, then $V = \mathbb{R}e_1$ and so any other $h \in H$ is a real multiple of e_1. If h is a non-integer multiple of e_1, i.e. $h = (k + \alpha)e_1$ for some $\alpha \in (0, 1)$, then $\alpha e_1 \in H$, but this contradicts the minimality of e_1. Thus $H = \mathbb{Z}e_1$ in this case.

(2): Otherwise suppose V has higher dimension. Let $V' = e_1^{\perp}$, and project H orthographically onto V'. The projection is still discrete in V'; if not, then let $w \in V'$ be some non-isolated point, and let $w_j + \alpha_j e_1$ be a sequence in H where $w_j \in V'$ and $w_j \to w$. We can choose $\alpha_j \in (0, 1)$ because $e_1 \in H$ so it can be added in integer amounts. But now this sequence of elements in H is entirely contained in the compact set $B_r(w) \times [0, |e_1|]$ so by Bolzano-Weierstrass there is a subsequence which does converge to a limit in H, which violates H being discrete.

(3): Induct on the dimension of V. Step (1) showed the base case $\dim(V) = 1$. For the inductive step, take the projection onto V' as in step (2), which is also a discrete subgroup but $\dim(V')$ is smaller by one. Thus by the inductive hypothesis the projection is of the form $\mathbb{Z}e_2 + \cdots + \mathbb{Z}e_d$ for some basis of V'. Now adding in e_1, we see that

$$H = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_d$$

as follows: suppose $h \in H$ is decomposed as $h = r_1 e_1 + a_2 e_2 + \cdots + a_d e_d$ where $r_1 \in \mathbb{R}$ and $a_j \in \mathbb{Z}$. If $r_1 \notin \mathbb{Z}$, then it has a nearest integer r'. Let $h' = r' e_1 + \cdots + a_d e_d$. Clearly $h' \in H$, so $h - h' \in H$, but this is strictly smaller than e_1 which is a contradiction of minimality. $\square$

Problem 8: Let G be a topological group $U \subseteq G$ an open neighborhood of the identity $e \in G$. For $n \geq 1$, define

$$U^n := \{g \in G | \exists g_1, \ldots, g_n \in U \text{ such that } g = g_1 \cdots g_n\}.$$

Prove that if G is connected then we have $G = \cup_{n \geq 1} U^n$.

Proof. First, note that U^n is open for all n. This is because action by both g and g^{-1} is continuous for all $g \in G$, so $g^{-1}U$ and gU are both open. Thus we can write U^n as

$$U^n = \bigcup_{g \in U} gU^{n-1},$$

a union of open sets (by induction on n). Thus,

$$G' := \bigcup_{n \geq 1} U^n$$

is open.

G being connected means that it cannot be written as the union of two disjoint open sets. Let $H \subset G$ be the complement of G'. Assuming H is nonempty, we can show that H is open, as

$$H = \bigcup_{h \in H} hU^{-1}$$

a union of open sets. To see that this is actually an equality, note that if $g \in hU^{-1}$ and $g \in U^n$ then $h \in U^{n+1}$, a contradiction. But now $G = H \cup G'$, a union of disjoint open sets, thus G cannot be connected.

$\square$

Problem 9: Prove that any continuous group homomorphism $\mathbb{R} \to \mathbb{R}^r$ has the form $t \mapsto tv$ for some $v \in \mathbb{R}^r$.

Proof. Suppose $\rho : \mathbb{R} \to \mathbb{R}^r$ is a continuous group homomorphism. Let $\rho(1) = v \in \mathbb{R}^r$. Since this is a homomorphism, we automatically get that $\rho(n) = nv$ for $n \in \mathbb{Z}$. Moreover, $n\rho(1/n) = \rho(1) = v$ implies that $\rho(1/n) = v/n$. Thus, for general $p, q \in \mathbb{Z}$ with $q \neq 0$, we have

$$\rho\left(\frac{p}{q}\right) = p\rho\left(\frac{1}{q}\right) = \frac{p}{q}v$$

so we get that $\rho(t) = tv$ for $t \in \mathbb{Q}$. Without the hypothesis of continuity we could not extend this fact to all of $\mathbb{R}$, and in fact one could construct a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and give a discontinuous solution. But due to continuity, the behavior of ρ on all of $\mathbb{R}$ can be determined from its behavior on $\mathbb{Q}$; if $\rho(r) \neq rv$ for some $r \in \mathbb{R}$, then $\rho' : t \mapsto \rho(t) - tv$ is a continuous function that is 0 on $\mathbb{Q}$ but nonzero on r, which is impossible. $\square$