

# ALGEBRA I NOTES

JALEN CHRYSOS

ABSTRACT. These are my notes from Victor Ginzburg's Representation Theory (Math 325) class at UChicago, Autumn 2025.

## CONTENTS

1. Introduction	2
1.1. Differential Algebra	3
1.2. Representation Theory Basics	4
1.3. Spectral Theorem	5
1.4. Modules	6
1.5. Representations of $S_n$	6
1.6. Hilbert's Nullstellensatz	8
1.7. Invariant Theory	9
1.8. Invariant Part	10
1.9. Topological Groups and Integration	11
1.10. Continuous Representations	12
1.11. Calculus on Banach Algebras	13
1.12. Lie Groups	14
1.13. Representations of Lie Groups and Lie Algebras	16
1.14. Complex Lie Groups	17
1.15. Holomorphic Irreps of $GL_n(\mathbb{C})$	19
1.16. Tensors of representations	20

## 1. INTRODUCTION

In this class we'll be interested in the representations of matrix groups. Something like  $\mathrm{GL}(V)$  or  $\mathrm{SO}(V)$  clearly acts on  $V$ , but it can also act on other interesting spaces. One relevant case of this for us will be when  $G$  acts on polynomials in  $x_1, \dots, x_n$ . Let

$$P_d \subseteq \mathbb{C}[x_1, \dots, x_n]$$

be the subspace of homogeneous degree- $d$  polynomials in  $n$  variables. This space has a basis given by the monomials

$$\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\}$$

and hence is finite-dimensional.  $P_d$  is stable under action by  $\mathrm{GL}_n$ . This is because linear transformation does not affect the degree of monomials (every  $x_j$  is sent to a linear combination of  $x_1, x_2, \dots, x_n$ ).

Consider the case of  $G = \mathrm{O}_n$ , the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by  $R$  is an intertwining map  $P_d \rightarrow P_{d+2}$ , meaning  $R \circ g^* f = g^*(R \circ f)$ .

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This  $\Delta$  is an  $\mathrm{O}_n$ -intertwining operator.

We call a function  $f$  *harmonic* if it has  $\Delta(f) = 0$ . The space of harmonic polynomials in  $n$  variables of degree  $d$  is denoted  $H_d \subseteq P_d$ . For  $d \in \{0, 1\}$ ,  $H_d = P_d$ , but for  $d \geq 2$   $H_d$  is strictly smaller. Note that  $H_d$  is stable under orthogonal transformations.

We will now work toward showing that  $H_d$  is an irreducible  $\mathrm{SO}_n$ -representation for  $n \geq 3$ .

A representation  $\rho : G \mapsto \mathrm{GL}(V)$  is *unitary* if  $G$  always acts as a unitary operator (i.e. preserves Hermitian inner product) on  $V$ . We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} e^{-|x|^2} dx$$

where  $dx$  is the Lebesgue measure. Action of  $\mathrm{SO}_n$  on  $P_d$  preserves this inner product.

Alternatively, we could put an inner product on  $P_d$  (or on all functions) from integration over  $S^{n-1}$  (the sphere). And polynomials in  $P_d$  are determined by their behavior on  $S^{n-1}$ .

**Proposition:** If  $V$  is a finite-dimensional vector space with an inner product, then any *unitary* action of  $G$  on  $V$  is completely reducible. Specifically, if  $W \subseteq V$  is a  $G$ -stable subspace, then one can decompose the action into  $V = W \oplus W^\perp$ .

*Proof.* The thing that we need to prove is that if  $W$  is  $G$ -stable then  $W^\perp$  is as well. Let  $x \in W^\perp$  and  $w \in W$ . Because  $g$  acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since  $g^{-1} \cdot w \in W$  by  $G$ -stability of  $W$ . □

**Key Lemma:** If  $F \subseteq C(S^{n-1})$  is any subspace stable under  $\mathrm{SO}_n$ , then it has an element fixed by  $\mathrm{SO}_{n-1}$ .

*Proof.* Let  $N := (0, 0, \dots, 0, 1) \in S^{n-1}$ . We have the evaluation map  $\alpha : C(S^{n-1}) \rightarrow \mathbb{C}$  given by evaluating functions at  $N$ . We have an inner product on  $F$  given by

$$\langle f, g \rangle := \int_{S^{n-1}} f \bar{g}$$

which is clearly fixed by  $\mathrm{SO}_n$ , thus  $F$  is a unitary representation of  $\mathrm{SO}_n$ . By Riesz representation theorem,  $\alpha(f) \equiv \langle f, \varphi \rangle$  for some  $\varphi \in F$ . For any  $g \in \mathrm{SO}_{n-1}$ ,  $g$  fixes  $N$ , thus

$$\langle f, \varphi \rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi \rangle = \langle f, g\varphi \rangle$$

and because this is true for arbitrary  $f \in F$  and  $g \in \mathrm{SO}_{n-1}$ ,  $\varphi$  is fixed by  $\mathrm{SO}_n$ . Now it remains to show that  $\varphi \neq 0$ . We can get this by assuming that some function in  $F$  takes a nonzero value on  $N$  (we can move  $N$  to some point where this is true, since  $F$  contains a nonzero function).  $\square$

We can apply this key lemma to  $P_d$  or  $H_d$  as  $F$ .

Consider  $P_d^{\mathrm{SO}_{n-1}}$ , the homogeneous polynomials fixed by  $\mathrm{SO}_{n-1}$ . On homework we showed that this is a subspace of  $\mathbb{C}\langle x_n, R \rangle$  (where  $R := x_1^2 + \cdots + x_n^2$ ). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \dots \rangle$$

thus  $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$ .

A very important fact about  $P_d$  is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2H_{d-4} \oplus R^3H_{d-6} \oplus \cdots \end{aligned}$$

(we will show this later). This allows us to deduce the dimension of  $H_d$  from  $P_d$ :

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1.$$

Likewise, we can decompose  $P_d^{\mathrm{SO}_{n-1}}$  the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of  $H_d^{\mathrm{SO}_{n-1}}$  as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each  $H_d$  is an *irreducible* representation of  $\mathrm{SO}_n$ ; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an  $\mathrm{SO}_n$ -representation,  $P_d$  decomposes exactly into the sequence  $H_{d-2j}$  for  $2j \leq d$ .

**Theorem:** If  $n \geq 3$ , then for each  $d \geq 0$ , the representation of  $\mathrm{SO}_n$  in  $H_d$  is irreducible, and moreover the representations are all distinct for different  $d$ .<sup>1</sup>

*Proof.* To show that the representations are distinct, we can use a homework problem which shows that the dimension of  $H_d$  is always increasing in  $d$  for any  $n \geq 3$ .  $\square$

**1.1. Differential Algebra.** Let  $W$  be a vector space over  $k$  with basis  $w_1, \dots, w_n$ , and let  $x_1, \dots, x_n$  be a dual basis for  $W^*$ . We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over  $W$ , where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \rightarrow k[W]_{j-1}$$

which acts on  $k[W]$  in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \dots, \partial_{w_n}]$$

<sup>1</sup>In the case  $n = 3$  this gives *all* the irreps. In general you miss  $\Lambda^2(\mathbb{C}^n)$ , but when  $n = 3$  this is just  $\mathbb{C}^3$ , which you get from  $H_1$ .

acting on  $k[W]$ . There is a natural correspondence between  $k[W]$  and  $\mathcal{D}(W)$ , if one assumes that  $k$  is characteristic 0. We have a  $k$ -bilinear pairing

$$\mathcal{D}(W) \times k[W] \rightarrow k$$

by  $\langle u, f \rangle \mapsto u(f)(0)$ . This is a *perfect pairing*. And in general we can do the same thing with

$$\text{Sym}^j(W) \times \text{Sym}^j(W^*) \rightarrow k.$$

**Lemma:** Let  $\xi \in W$  and  $f \in k[W]$ . Then

$$\langle \xi^m, f \rangle = m!f(\xi).$$

In particular, if  $f = \varphi \in W^*$ ,  $\langle \xi^m, \varphi^m \rangle = m!\varphi^m(\xi)$ .

*Proof.* We will show this for homogeneous  $f$  first, and the general result will follow from expressing  $f$  as a sum of homogeneous polynomials. Let the degree of  $f$  be  $d$ . Then by Taylor expansion,

$$f(\xi) = \sum_{k \geq 0} \frac{1}{k!} (\partial_\xi^k f)(0).$$

But note that only the  $d$ th term of this is nonzero, since  $\partial_\xi^j f = 0$  unless  $j = d$  (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other  $j$  both sides are 0. □

We can use this pairing to get another inner product on polynomials in  $k[W]$  given by

$$\langle p, q \rangle := p(\partial)(q)(0)$$

where  $p(\partial)$  is the corresponding element to  $p$  in  $\mathcal{D}(W)$ .<sup>2</sup> For this inner product, we have that multiplication by  $p$  is *adjoint* to  $p(\partial)$ , i.e.

$$\langle r, p(\partial)q \rangle = \langle pr, q \rangle.$$

With this fact, we can finally show why  $P_d = H_d \oplus RP_{d-2}$ :

$$W = \ker(\Delta) \oplus \text{im}(\Delta^*) = \ker(\Delta) \oplus \text{im}(R) = H_d \oplus RP_{d-2}.$$

Another application of this pairing: Let  $V$  be a finite dimensional vector space and  $A \subseteq V$  a subset of  $V$  (not necessarily subspace). Let  $\text{span}^d(A) \subseteq \text{Sym}^d(V)$  be generated over  $\mathbb{C}$  by  $a^d$  for  $a \in A$ . If  $A$  is dense in  $V$  then  $\text{span}^d(A) = \text{Sym}^d(V)$ . We will show this by using the pairing.

Assume for contradiction that  $\text{span}^d(A) \neq \text{Sym}^d(V)$ . Then there is some nonzero linear functional  $F : \text{Sym}^d(V) \rightarrow \mathbb{C}$  which vanishes on  $\text{span}^d(A)$ . Then  $F$  corresponds to some differential polynomial  $f$ , and  $\partial_a^d f(0) = 0$  for all  $a \in A$ . But  $\partial_a^d f(0) = d!f(a)$ , so  $f(a) = 0$ . But then  $A$  is dense, so  $f = 0$ .

**1.2. Representation Theory Basics.** If  $G$  acts on sets  $X$  and  $Y$ , then  $G$  can also act on the space of maps  $X \rightarrow Y$  via conjugation:

$$g : f \mapsto g \circ f \circ g^{-1}.$$

We can ask about the space of maps which commute with this  $G$ -action. Or, equivalently, the maps which are fixed by the  $G$ -action. We call these *intertwining operators*. The set of such operators is denoted  $\text{Hom}_G(X, Y)$ .

We are usually interested in the case where  $X, Y$  are vector spaces and  $\text{Hom}(X, Y)$  is the space of linear maps.

**Schur-Weyl Duality:** Let  $W$  be a finite-dimensional vector space over  $\mathbb{C}$ .  $\text{GL}(W)$  can act on  $W^{\otimes d}$  with  $g$  acting as  $g^{\otimes d}$ .  $S_d$  also acts on  $W^{\otimes d}$  by permutation. It is not too hard to see that these two actions commute. But moreover, action by  $\text{GL}(W)$  *spans* the space of  $S_d$ -intertwiners on  $W^{\otimes d}$ .

---

<sup>2</sup>In the homework, we establish that on  $H_d$ , this is actually *equivalent* to the inner product from integrating over  $S^{n-1}$ !

*Proof.* Let  $\Phi : (\text{End}(W))^{\otimes d} \rightarrow \text{End}(W^{\otimes d})$  be given by

$$\Phi : a_1 \otimes \cdots \otimes a_d \mapsto (w_1 \otimes \cdots \otimes w_d \mapsto a_1(w_1) \otimes \cdots \otimes a_d(w_d)).$$

$\Phi$  is an invertible linear map with inverse

$$\Phi^{-1}f \mapsto f|_{W_1} \otimes \cdots \otimes f|_{W_d}$$

where  $W_j$  is  $0 \otimes \cdots \otimes W \otimes \cdots \otimes 0$  with the  $W$  in the  $j$ th spot. Note also that  $\Phi$  commutes with the action of  $S_d$ . By using  $\Phi$ , we see that

$$\text{Sym}^d(\text{End } W) = ((\text{End } W)^{\otimes d})^{S_d} \xrightarrow{\Phi^{-1}} \text{End}_{S_d}(W^{\otimes d}).$$

So we only need to understand  $\text{Sym}^d(\text{End } W)$ . But  $\text{GL}(W)$  is dense in  $\text{End}(W)$ , so by a previous lemma, we see that  $\text{span}^d(\text{GL}(W)) = \text{Sym}^d(\text{End } W)$ .  $\square$

**1.3. Spectral Theorem.** Let  $A$  be a  $k$ -algebra with  $a \in A$ . We have an evaluation map

$$\text{ev}_a : k[t] \rightarrow A \quad \text{ev}_a : p \mapsto p(a).$$

Let  $A_a := \text{im}(\text{ev}_a)$ , i.e. the subalgebra of  $A$  generated by  $a$ . The kernel  $\ker(\text{ev}_a)$  is an ideal of  $k[t]$ , and it is a principal ideal since  $k[t]$  is a PID. Thus, in the case that  $\text{ev}_a$  is non-injective, there is a unique *minimal polynomial* of  $a$ ,  $p_a$ , which divides every polynomial which vanishes at  $a$ .

**Lemma:**  $a$  is algebraic iff  $A_a$  is finite-dimensional.

*Proof.* If  $A_a$  is finite-dimensional then there is a relation between  $1, a, a^2, \dots, a^n$  for some  $n$ , i.e. a polynomial that  $a$  solves. Conversely if  $a$  solves a polynomial of degree  $n$  then every linear combination of powers of  $a$  can be expressed by the first  $n$  powers of  $a$ .  $\square$

We define the *spectrum* of  $a$ , denoted  $\text{Spec}(a)$ , as

$$\text{Spec}(a) := \{\lambda \in k : (a - \lambda) \text{ is not invertible}\}.$$

So for example, if  $A$  is a function algebra,  $\text{Spec}(a)$  denotes the values that  $a$  can take. In the case that  $A$  is the matrix algebra  $M_n(k)$ ,  $\text{Spec}(a)$  is the set of eigenvalues of  $a$ .

**The Spectral Theorem:** Let  $A$  be a  $k$ -algebra of one of the following types:

- $A$  is finite-dimensional over  $k$  and  $k$  is algebraically closed.
- $A$  is countable-dimension and  $k$  is uncountable.

Then,

- (i)  $\text{Spec}(a)$  is nonempty.
- (ii)  $a$  is nilpotent iff  $\text{Spec}(a) = \{0\}$ .
- (iii) If  $A$  is a division algebra then  $A = k$ .<sup>3</sup>

*Proof.* Lemma: If  $\lambda_1, \dots, \lambda_n \notin \text{Spec}(a)$ , i.e.  $(a - \lambda_j)$  is invertible for each  $j$ , then if

$$\sum_j c_j (a - \lambda_j)^{-1} = 0$$

for some  $c_j \in k$  then  $a$  is algebraic (proof is by clearing denominators). We will use this fact.

(i): We will split into two cases: if  $a$  is algebraic then  $\text{Spec}(a)$  is finite but nonempty and if  $a$  is not algebraic then  $\text{Spec}(a)$  is uncountable (and the converses to both of these are true).

If  $a$  is algebraic, then  $\text{Spec}(a)$  is the roots of the minimal polynomial (HW), and particular this means  $\text{Spec}(a)$  is finite and nonempty because  $k$  is algebraically closed.

If  $a$  is not algebraic, then by the Lemma, there is no linear relation between any finitely-many  $(a - \lambda)^{-1}$  for  $\lambda \notin \text{Spec}(a)$ . We assumed that  $\dim(A)$  is at most countable, and it has an independent set of size  $|k \setminus \text{Spec}(a)|$ , so  $\text{Spec}(a)$  must be uncountable (because  $k$  is).

<sup>3</sup>For a counterexample of this when  $k$  is not algebraically closed, take the Quaternions over  $\mathbb{R}$ .

(ii): If  $a^n = 0$ , then  $0 \in \text{Spec}(a)$  because  $a$  is not invertible, but all other  $(a - \lambda)$  are invertible:

$$(a - \lambda)(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}) = a^n - \lambda^n = -\lambda^n.$$

so

$$(a - \lambda)^{-1} = -\lambda^{-n}(a^{n-1} + a^{n-2}\lambda + a^{n-3}\lambda^2 + \cdots + \lambda^{n-1}).$$

Conversely, suppose that  $\text{Spec}(a) = \{0\}$ .  $\{0\}$  is a finite set, so by part (i),  $a$  is algebraic, but its minimal polynomial only has root  $a = 0$ , so  $a^n = 0$  for some  $n$ .

(iii): Assume for contradiction that  $A$  is a division algebra yet  $\exists a \in A \setminus k$ . Then  $(a - \lambda)$  is invertible for all  $\lambda \in k$ , but then  $\text{Spec}(a)$  would be empty, contradicting (i).  $\square$

**1.4. Modules.** A module  $M$  over ring  $A$  is called *simple* if it is nonzero and has no proper non-trivial submodules (i.e. its only submodules are 0 and  $M$ ).

**Schur's Lemma:** If  $f : M \rightarrow N$  is an  $A$ -linear map between *simple*  $A$ -modules  $M$  and  $N$ , then  $f$  is either 0 or an isomorphism.

*Proof.*  $\ker(f)$  is a submodule of  $M$  and  $\text{im}(f)$  is a submodule of  $N$ . By simplicity, both must be either trivial or the full module. This implies that  $f$  is either injective or 0, and either surjective or 0.  $\square$

As a corollary, we see that  $\text{End}_A M$  is a division ring.

**Schur's Lemma for Algebras:** If  $A$  is a  $k$ -algebra and  $M$  a simple  $A$ -module either

- $k$  is algebraically closed and either  $A$  or  $M$  is finite-dimensional over  $k$ .
- $k = \mathbb{C}$  and either  $A$  or  $M$  is countable-dimension over  $k$ .

Then,  $\text{End}_A M = k \text{id}_M$ .

*Proof.* On HW we showed that  $\dim(\text{End}_A M) \leq \dim_A M$ . The lemma can be proven by applying the spectral theorem to the algebra  $\text{End}_A M$ .  $\square$

If  $A$  satisfies the hypotheses of the Spectral Theorem and  $M$  is a simple  $A$ -module, then the center  $Z$  of  $A$  acts in  $M$  by scalars, as  $z \cdot am = az \cdot m$  for  $z \in Z, a \in A, m \in M$ . And in particular if  $A$  is commutative then  $\dim_k M = 1$  because every subspace of  $M$  is  $A$ -stable.

**Schur's Lemma for Group Representations:** If  $V, W$  are representations of a group  $G$  over a field  $k$ ,

- (i) If  $V, W$  are irreducible then all intertwiners are either 0 or isomorphisms.
- (ii) If  $\dim_k(V)$  is finite and  $k$  is algebraically closed or  $k = \mathbb{C}$  and  $|G| = \aleph_0$ , then

$$\text{End}_G V = k \cdot \text{id}_V.$$

*Proof.* This follows from applying Schur's Lemma for Algebras. A representation of  $G$  corresponds to a module over the group algebra  $A := kG$ . Note that  $\dim_k(A) = |G|$ .  $\square$

**1.5. Representations of  $S_n$ .**  $S_n$  acts on  $\mathbb{R}^n$  by permuting the coordinates. We have the sign representation given by taking the determinant.

$S_n$  also acts on  $P_d$  by permuting the variables:

$$\sigma(f)(x_1, \dots, x_n) := f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}).$$

Motivated by this action, we can consider the symmetric polynomials  $P^{S_n}$ .

A *partition* of  $n$  is a finite non-increasing sequence of positive integers  $\lambda_1 \geq \cdots \geq \lambda_k$  whose sum is  $n$ . Let the set of partitions of  $n$  be  $\mathcal{P}_n$ . Corresponding to a partition, we have a decomposition of  $[1, n]$  into  $I_1, I_2, \dots, I_k$  of length  $|I_j| = \lambda_j$ .

The *Vandermonde Determinant* is the polynomial

$$\Delta_n := \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

which can also be written as the determinant

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix}$$

Corresponding to a given partition  $\lambda$ , define

$$\Delta(I_m) := \prod_{i < j \in I_m} (x_j - x_i)$$

and

$$\Delta_\lambda := \prod_m \Delta(I_m).$$

$\Delta_\lambda$  is a homogeneous polynomial, and its degree is

$$d := \sum_m \frac{\lambda_m(\lambda_m - 1)}{2}$$

so  $\Delta_\lambda \in P_d$ .

The *Specht Module* associated with  $\lambda$ , denoted  $V(\lambda)$  is the  $k$ -span of  $\Delta_\lambda$  under the action of  $S_n$ . It is clearly stable under the action of  $S_n$ .<sup>4</sup>

Examples:

- Let  $\lambda = (1, 1, 1, \dots, 1)$ . Then  $\Delta_\lambda = 1$ , and  $V(\lambda) = P_0$ , the constant polynomials. The action of  $S_n$  on  $V(\lambda)$  is trivial. Thus, this  $\lambda$  represents the trivial representation.
- Let  $\lambda = (n)$ . Then  $\Delta_\lambda = \Delta_n$ , the entire Vandermonde determinant. Since this is just a determinant whose columns are permuted by the action of  $S_n$ , the action scales by the sign of the permutation. This makes  $V(\lambda) = k\Delta_n$ , which is one-dimensional. It is the sign representation of  $S_n$ .

Note that in all of these cases  $V(\lambda)$  is irreducible. This is actually true in general:

**Theorem:** Assuming that the underlying field  $k$  is characteristic 0, the Specht module is always an irreducible representation. Moreover, all irreps of  $S_n$  can be expressed as  $V(\lambda)$  for some partition  $\lambda$ .<sup>5</sup>

*Proof.* The proof has three steps. The first step will be to show that  $V(\lambda)$  is irreducible, which we do on homework. Step 2 is that the number of irreducible representations of  $S_n$  is equal to the number of partitions of  $n$ . Step 3 will show that the modules  $V(\lambda)$  are pairwise non-isomorphic for different  $\lambda$ , and hence we have a bijection.

Step 3: In homework (it is fairly clear I think) we showed that if  $d_\mu \neq d_\lambda$  then  $V(\mu) \not\cong V(\lambda)$ , so it remains to show this for  $\mu, \lambda$  that have equal degree.

Notation: for  $\nu \in \mathbb{Z}_{\geq 0}^n$  (note: may have repeats!),  $S_n$  acts on  $\nu$  in the natural way. Let  $m_j(\nu)$  denote the number of elements of  $\nu$  that are equal to  $j$  (this is invariant under  $S_n$ ). Let  $\nu(\lambda)$  be

$$\nu(\lambda) := (1, 2, \dots, \lambda_1, 1, 2, \dots, \lambda_2, \dots, 1, 2, \dots, \lambda_n).$$

Then  $m_j(\nu(\lambda))$  is the length of the  $j$ th column in the Young diagram of  $\lambda$ ,  $D(\lambda)$ . Or equivalently, the  $m_j$  form another partition corresponding to the transposed Young diagram  $D^T(\lambda)$ .

<sup>4</sup>This is not the most common way to construct the Specht module of  $\lambda$ .

<sup>5</sup>It is also true that  $V(\lambda)$  is a subspace of the  $S_n$ -harmonic polynomials (as defined on HW) and the index is  $\dim(V(\lambda))$ .

Similarly, we can apply all this to polynomials in  $n$  variables. The monomial  $x^\nu$  is

$$x^\nu := \prod_{j=1}^n x_j^{\nu_j}$$

so that for a partition  $\lambda$ ,

$$x^{\nu(\lambda)} := \prod_{i=1}^k \prod_{j=1}^{\lambda_i} x_{\lambda_i+j}^j.$$

Recall that  $\Delta_\lambda$  is defined in terms of Vandermonde determinants, which are expressed as

$$\Delta_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1}.$$

And for  $\Delta_\lambda$ , we have a similar thing but with Young subgroups:

$$\Delta_\lambda = \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1} = \frac{1}{x_1 x_2 \cdots x_n} \sum_{\sigma \in S_\lambda} \text{sgn}(\sigma) x^{s(\nu(\lambda))}.$$

Now, if  $V(\mu) = V(\lambda)$ , then  $\Delta_\mu \in V(\lambda)$ , i.e. it is a linear combination of permutations of  $\Delta_\lambda$ . So for example, the monomial  $x^{\nu(\mu)}$  appears as  $x^{\sigma(\nu(\lambda))}$  for some  $\sigma$ . But we can show that this fails (*fill in later*).  $\square$

The *Young Subgroup* of  $S_n$  corresponding to  $\lambda$  consists of all permutations which preserve all the pieces  $I_m$ . It is denoted  $S_\lambda$  and is isomorphic to  $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$ .

Define  $P^{S_\lambda}$  to be the polynomials fixed by  $S_\lambda$ , and  $P^{\text{sgn}(\lambda)}$  the polynomials on which  $S_\lambda$  acts in an anti-symmetric way. We can see that  $P^{\text{sgn}(\lambda)}$  is stable under scaling by  $P^{S_\lambda}$ , i.e. it is a  $P^{S_\lambda}$ -submodule.

**Lemma:** Let  $F : V(\lambda) \rightarrow P_d$  be an  $S_n$ -intertwiner. Then

- (1) If  $d = d_\lambda$ , then  $F$  acts by scaling.
- (2) If  $d < d_\lambda$ , then  $F$  is trivial.

*Proof.* For  $s \in S_\lambda$ ,  $s(F(\Delta_\lambda)) = F(s(\Delta_\lambda))$ , and  $s(\Delta_\lambda) = \text{sgn}_\lambda(s) \cdot \Delta_\lambda$ , so  $F$  acts by scaling. *fill in later*  $\square$

In general, we can see that  $V(\lambda^t) = V(\lambda) \otimes \text{sgn}$ .

**1.6. Hilbert's Nullstellensatz.** Let  $k$  be an algebraically closed field and let  $P = k[x_1, \dots, x_n]$ . An *algebraic* subset of  $k^n$  is the vanishing set of an ideal  $I \subset P$ , denoted  $V(I)$ . In the other direction, we have an ideal  $I_V$  corresponding to polynomials vanishing on a given algebraic set  $V$ .

For any ideal  $I \subset P$ , there is a *radical* of  $I$ , denoted  $\sqrt{I}$ , which consists of all elements of  $P$  for which some power lies in  $I$ . Note that  $V(I) = V(\sqrt{I})$ . Moreover, ideals of the form  $I_V$  are already radical. The interesting thing is that the correspondence goes both ways:

**Nullstellensatz:** Algebraic sets are in bijection with *radical* ideals of  $P$ , via  $V \mapsto \sqrt{I_V}$  and  $V(I) \mapsto I$ . This bijection restricts to one between single points of  $k^n$  and maximal ideals of  $P$ .

*Proof.* We will show that  $I_{V(I)} = \sqrt{I}$ , the content of which is that every polynomial  $f$  vanishing on  $V(I)$  has  $f^n \in I$  for some  $n$ .

Lemma: for  $z \in k^n$ , let  $\text{ev}_z : P \rightarrow k$  be the algebra homomorphism given by evaluation at  $z$ , i.e.  $\text{ev}_z : f \mapsto f(z)$ . In fact, *every* algebra homomorphism  $\chi : P \rightarrow k$  is  $\text{ev}_z$  for some  $z$ . In particular, take  $z = (\chi(x_1), \chi(x_2), \dots, \chi(x_n))$  and note that  $\chi(f) = \text{ev}_z(f)$ .

Now, let  $A$  be a commutative  $k$ -algebra as in the setting of the Spectral theorem (i.e. finite-dimensional or countable dimension with uncountable  $k$ ). We claim that all maximal ideals of  $A$



have the form  $\ker(\chi)$  for some algebra homomorphism  $\chi : A \rightarrow k$ , and moreover  $\text{Spec}(a)$  is exactly  $\{\chi(a) : \chi \in \text{Hom}(A, k)\}$ .

To prove this, let  $I \subset A$  be a maximal ideal. This implies that  $A/I$  is a field, and in particular a division algebra, so the spectral theorem implies that  $A/I = k$ . Thus the projection onto  $I$  gives a character  $A \rightarrow A/I = k$ .

Now for the general case. Suppose  $f$  vanishes on  $V(I)$  but none of its powers is in  $I$ . Let  $\bar{f}$  be  $f \bmod I$ . The assumption that  $f \notin \sqrt{I}$  is equivalent to saying that  $\bar{f}$  is not nilpotent, so by the spectral theorem,  $\text{Spec}(\bar{f}) \neq \{0\}$ . Thus, there is some  $\lambda \neq 0$  so that  $\bar{f} - \lambda$  is non-invertible in  $A$ . Now consider the ideal

$$I + P \cdot (f - \lambda) \subset P.$$

Because  $(\bar{f} - \lambda)$  is non-invertible, there can be no  $b \in P$  such that  $b(\bar{f} - \lambda) = 1 \bmod I$ , thus  $I + P \cdot (f - \lambda)$  is not all of  $P$ . By the maximal case, this shows that

$$V(I) \cap V(f - \lambda) = V(I + P \cdot (f - \lambda)) \neq \emptyset$$

but this implies that  $f = \lambda$  on at least one point of  $V(I)$ , which contradicts the assumption that  $f$  vanishes on  $V(I)$ .  $\square$

As a corollary, we see that every proper ideal  $I \subsetneq P$  there is some  $z$  such that  $f(z) = 0$  for  $z \in I$ . This is because otherwise  $V(I) = \emptyset$ , so  $\sqrt{I} = P$ .

Another corollary: if  $f_1, \dots, f_r \in P$  have no common solution, then there is some  $P$ -linear combination of these equal to 1, i.e.  $1 \in (f_1, \dots, f_r)$ .

A third corollary: if  $V(I_1) \cap V(I_2) = \emptyset$ , then there is some polynomial  $f$  such that  $f = 0$  on  $V(I_1)$  and  $f = 1$  on  $V(I_2)$ . To get such an  $f$ , we use the fact that  $I_1 + I_2 = P$ , so there is  $f \in I_1$  and  $g \in I_2$  such that  $f + g = 1$ , so we can take this  $f$ .

**1.7. Invariant Theory.** Let  $G$  act on  $X$ . If  $f$  is a  $G$ -invariant function on  $X$ , another way to think of this is that  $f$  takes a value on each  $G$ -orbit.

Consider the case where  $G \subset \text{GL}_n(k)$  and  $X = k^n$ . What does the space of  $G$ -invariant polynomials on  $X$  look like? Hilbert showed that it is finitely-generated for sufficiently well-behaved  $G$ .

If  $G = S_n$  for example, we get the symmetric polynomials, and this space is generated by the elementary symmetric functions, i.e. the symmetric sums of square-free monomials in each degree from 1 to  $n$ ; we denote these  $\sigma_j(x)$  for  $1 \leq j \leq n$ .

The map  $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  given by

$$\pi : x \mapsto (\sigma_1(x), \dots, \sigma_n(x))$$

has fibers exactly equal to the orbits of  $S_n$ .

Let  $\zeta$  be a primitive  $n$ th root of unity. Consider the subgroup  $\Gamma \subset \text{SL}_2(\mathbb{C})$  generated by

$$\gamma = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$$

Now we can ask about  $\mathbb{C}[x, y]^\Gamma$ . We immediately see that  $x^n, y^n, xy \in \mathbb{C}[x, y]^\Gamma$ . Taking  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^3$  as before, defined by

$$(x, y) \mapsto (x^n, y^n, xy)$$

and we again see that  $\mathbb{C}^2/\Gamma \cong \text{im}(\pi)$ .

In more generality, suppose  $\Gamma \subset \text{SL}_2(\mathbb{C})$  is a finite subgroup. If  $f(x, y) \in \mathbb{C}[x, y]^\Gamma$ , supposing  $f$  is homogeneous of degree  $d$ , we can write  $f$  as the product of  $d$  lines through 0. Then any  $g \in \Gamma$  must permute these linear factors.

Let  $G = \mathbb{R}^\times$ , acting on  $\mathbb{R}$ . The orbits of the action are  $(-\infty, 0), \{0\}, (0, \infty)$ . Thus the only  $G$ -invariant polynomials on  $\mathbb{R}$  are constants.

But if we let this  $G$  act on  $\mathbb{R}^2$  via  $t : (x, y) \mapsto (tx, t^{-1}y)$ , the orbits are halves of hyperbolas and the four pieces of the coordinate axes and the origin.

Let  $G = \mathrm{GL}_n(k)$  act on  $M_n(k)$ . The  $G$ -orbits are represented by matrices in Jordan Canonical Form. Let's look at the case  $n = 2$ . The  $G$ -invariant polynomials in the four matrix entries are generated by  $ad - bc$ , the determinant. The orbits can be made to look like hyperboloids or cones.

**1.8. Invariant Part.** For any completely-reducible representation  $V$  of  $G$ , there is a canonical projection from  $\mathrm{Inv}_V : V \rightarrow V^G$  which commutes with action of  $G$  (hence fixes  $V^G$ ) and also commutes with any other  $G$ -intertwiner  $f : V \rightarrow W$ . Also  $\mathrm{Inv}_V$  agrees with  $\mathrm{Inv}_W$  on  $W$  for any subrepresentation  $W$ .

*Proof.* Now let's construct this  $\mathrm{Inv}_V$ . Let  $A$  be an algebra over  $k$  and  $E$  a simple finite-dimensional  $A$ -module. If  $M$  is another finite-dimensional  $A$ -module that is completely reducible, of the form

$$M = M^{(E)} \oplus M'$$

where  $M^{(E)}$  is the isotypic component of  $E$  and  $M'$  is the direct sum of other isotypic components (since  $M$  is completely reducible). We want the projection  $M \rightarrow M^{(E)}$ . If  $N = N^{(E)} \oplus N'$  is another such module, then we claim that the projections commute with any  $A$ -linear map  $f : M \rightarrow N$ . This is not so hard to check.

To convert this proof about modules into one about representations, we take  $A = kG$  as usual and say that  $A$ -modules are  $G$ -representations. We take  $E$  to be the trivial representation of  $G$  as a module, so that  $V^{(E)}$  is the  $G$ -fixed points of  $G$ .  $\square$

Let  $G \subset \mathrm{GL}_n(\mathbb{C})$ , acting on  $P = \mathbb{C}[x_1, \dots, x_n]$ . We say that  $G$  is *reductive* if the representation given by the action of  $G$  on  $P_d$  is completely reducible.

Examples:

- Any finite group  $G$  is reductive.
- “Any group with a name” is reductive ( $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n, \mathrm{SO}_n$ , etc)

A non-example is given by the group

$$G := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C} \right\}.$$

**Theorem:** If  $G$  is reductive then there is some  $r \geq 0$  and a map

$$\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^r$$

where each component  $\varphi_i$  is a homogeneous  $G$ -invariant polynomial of positive degree, and

- (1)  $\mathrm{im}(\varphi)$  is an algebraic subset of  $\mathbb{C}^r$ .
- (2) The fiber  $\varphi^{-1}(y)$  of any  $y \in \mathrm{im}(\varphi)$  contains a *unique* closed  $G$ -orbit  $O_y$ , and moreover any  $G$ -orbit in  $\varphi^{-1}(y)$  contains  $O_y$  in its closure.
- (3)  $\varphi^{-1}(0)$  consists of the points  $z \in \mathbb{C}^n$  for which  $0 \in \overline{G \cdot z}$  (i.e. the closure of the  $G$ -orbit of  $z$  contains 0).

*Proof.*  $\square$

Examples in some cases we looked at before:

- (Non-example) If  $G = \mathbb{R}^{>0}$  acting on  $\mathbb{R}^2$  by  $t : (x, y) \mapsto (tx, t^{-1}y)$ , then we have  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  (so  $r = 1$ ) defined

$$\varphi : (x, y) \mapsto xy.$$

The orbits look like half-hyperbolas or half-axes or the origin. The theorem tells us that  $\varphi^{-1}(y)$  should contain a unique closed orbit, but there are two when  $y \neq 0$ . So this fails because in  $\mathbb{R}$  the orbits are not connected, whereas in  $\mathbb{C}$  they would be.

If  $V_1, V_2 \subseteq \mathbb{C}^n$  are two  $G$ -stable algebraic subsets, then there is a  $G$ -invariant polynomial  $f$  such that  $f|_{V_1} = 0$  and  $f|_{V_2} = 1$ .

*Proof.* Let  $I_1 = I_{V_1}$  and  $I_2 = I_{V_2}$ , since  $V_1, V_2$  are algebraic. For all  $p \in I_1$ ,  $\text{Inv}_P(p) = \text{Inv}_{I_1}(p)$  (why?)

By the Nullstellensatz, take  $p_1 \in I_1$ ,  $p_2 \in I_2$  such that  $p_1 + p_2 = 1$ , and then take the invariant parts of each over  $P$ .  $\square$

**Theorem (Hilbert):** If  $G \subseteq \text{GL}_n(\mathbb{C})$  is reductive then the algebra  $\mathbb{C}[x_1, \dots, x_n]^G$  is finitely generated.

*Proof.* Let  $J := P_{>0}^G$ . It suffices to show that  $J$  is finitely-generated in  $P^G$ . We know by Hilbert basis theorem that  $JP$  is a finitely-generated ideal of  $P$ . (proof omitted)  $\square$

So let  $\varphi_1, \varphi_2, \dots, \varphi_r$  generate  $P^G$  and take a map  $F : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[y_1, \dots, y_r]$  via

$$F : p(x_1, \dots, x_n) \mapsto p(\varphi_1(x_1, \dots, x_n), \varphi_2(x_1, \dots, x_n), \dots, \varphi_r(x_1, \dots, x_n)).$$

**1.9. Topological Groups and Integration.** Almost all of our results rest on the complete reducibility of the representations we're interested in. The invariant part depends on this, for example. We initially got this from unitary representations, but in recent examples (e.g.  $t : (x, y) \mapsto (tx, t^{-1}y)$  etc) we don't have an inner product yet we want to get the same results. We can get this another way: from compact topological groups. In fact, when working with these groups one can get an explicit formula for the invariant part.

Hermann Weyl showed reductivity for some non-compact groups by relating them to compact groups, using the *Unitary Trick*.

A *topological group* is a group  $G$  which is also a Hausdorff topological space, with the additional property that multiplication and inversion are continuous. We also say that  $G$  acts topologically on a set  $X$  if the action is continuous. We will be most interested in *compact* topological groups.

- Any finite group can be made into a compact topological group by giving it the discrete topology.
- $M_n(\mathbb{R})$  can be put in bijection with  $\mathbb{R}^{n^2}$  and thus given a metric. So the group  $\text{GL}_n(\mathbb{R})$  is a topological group (one has to check that multiplication and inversion are continuous). Similarly for any subgroup of  $\text{GL}_n(\mathbb{R})$ .
- Any finite-dimensional vector space under  $+$  is a topological group.
- The circle  $S^1$  is a compact topological group.

Let  $X$  be a locally compact topological space,  $C(X)$  be the space of continuous functions  $X \rightarrow \mathbb{R}$ , and  $C_c(X)$  the space of such functions with compact support. Let  $G$  act on  $X$ . A Borel measure  $\mu$  on  $X$  is  $G$ -invariant if action of  $G$  preserves measure of all Borel sets. It is equivalent to make the stronger statement that  $G$  preserves integrals:

$$\int_X g^* f(x) \, d\mu = \int_X f(x) \, d\mu.$$

where  $g^* f(x) = f(g^{-1}(x))$ .

Let  $L_g$  and  $R_g$  be the actions of  $G$  on itself by

$$L_g : x \mapsto xg, \quad R_g : x \mapsto gx.$$

These actions commute.

**Theorem (Haar):** Any locally compact topological group  $G$  has a unique (up to scaling) left-invariant measure  $\mu$ , and also a unique right-invariant measure. They may be different in general, but if  $G$  is compact then they are the same.

*Proof.* The following construction is due to Von Neumann. Fix some continuous  $f : G \rightarrow \mathbb{R}$ . For any finite subset  $F \subseteq G$ , let the left average of  $f$  at  $x$  be

$$A_F(f)(x) = \frac{1}{|F|} \sum_{g \in F} f(gx)$$

and likewise for the right average. Let  $\delta(f)$  be

$$\delta(f) := \max_{x \in G} f(x) - \min_{x \in G} f(x)$$

which exists because of the local compactness of  $G$ . We will show that for non-constant  $f$ , there is a finite  $F \subseteq G$  such that

$$\delta(A_F(f)) < \delta(f).$$

And in fact  $\delta(A_F(f))$  is arbitrarily small as  $F$  ranges over subsets of  $G$ . Thus  $A_F(f)$  approaches a limit as  $F$  gets larger, and we let this limit be the integral of  $f$ .  $\square$

Properties of the Haar measure:

- $f \rightarrow \int f$  is linear.
- $f \geq 0$  for all  $x$  implies  $\int f \geq 0$ .
- $\int f = 0$  iff  $f = 0$  everywhere (assuming  $f$  is continuous).
- $\int 1 = 1$ .
- $\int$  is both left- and right-  $G$ -invariant.
- $\int f(x) = \int f(x^{-1})$ .

Moreover, we can extend this measure to integrate functions  $G \rightarrow V$  where  $V$  is any finite-dim vector space, e.g.  $\mathbb{C}$ .

Examples:

- For  $\mathbb{R}^n$ , we get the Lebesgue measure.
- For  $S^1$ , we have  $\mu = d\theta$ .

**1.10. Continuous Representations.** A representation  $\rho : G \rightarrow \mathrm{GL}_N(k)$  of a topological group  $G$  is *continuous* if each entry of the matrix  $\rho(g)$  is continuous wrt  $g$ .

**Theorem:** Continuous irreps of  $\mathrm{GL}_N^+$  (the matrices with positive determinant) are in bijection with descending  $n$ -tuples in  $\mathbb{Z}$ , which we call “collections.”

*Proof.* Let  $H$  be the group of diagonal matrices in  $\mathrm{GL}_N^+$ ,  $B$  be the group of upper triangular matrices in  $\mathrm{GL}_N^+$ , and  $U$  be the group of upper triangular matrices with 1s on the diagonal.

Let  $\rho$  be a continuous irrep of  $\mathrm{GL}_N^+$ . Then we claim that there is a nonzero eigenvector  $v \in k^N$  with eigenvalue 1 for all matrices in  $\rho(U)$ , and eigenvalue

$$h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_N^{\lambda_N}$$

for  $\rho(h)$ , where  $h$  is the diagonal matrix with entries  $h_1, \dots, h_N$ .

It follows by Borel fixed point theorem applied to  $\mathbb{P}^{N-1}$ . (?)  $\square$

Let  $G$  be a compact topological group acting continuously on a topological space  $X$ . For  $f \in C(X)$ , we can define

$$\mathrm{Av}(f) : X \rightarrow \mathbb{C}$$

by

$$\mathrm{Av}(f)(x) = \int_G f(g^{-1}x) \, dg.$$

Note that  $\mathrm{Av}(f)$  is continuous and  $G$ -invariant:

$$h^* \mathrm{Av}(f) = \mathrm{Av}(h^{-1}f) = \int_G f(g^{-1}h^{-1}x) \, dg = \int_G f((gh)^{-1}x) \, dg = \mathrm{Av}(f)$$

using the fact that  $Gh = G$ .

**Theorem:** Any finite-dimensional continuous representation  $V$  of  $G$  (a compact topological group) can be made unitary.

*Proof.* Since  $V$  is finite-dimensional assume wlog that  $V = \mathbb{C}^n$  for some  $n$ . We have a hermitian inner product  $V \times V \rightarrow \mathbb{C}$ . Let  $G$  act on  $V \times V$  by

$$g : (v, w) \mapsto (g \cdot v, g \cdot w).$$

We can now define a  $G$ -invariant inner product on  $V \times V$  given by

$$\langle v, w \rangle = \text{Av}((v, w)) = \int_G (g^{-1} \cdot v, g^{-1} \cdot w) dg.$$

To show that it is an inner product we must check that it is linear, which follows because  $(\bullet, \bullet)$  is linear and the integral is linear and  $G$  acts linearly on  $V$ . We also check that it is positive definite, which follows from  $(\bullet, \bullet)$  being positive definite.  $\square$

As a corollary, we see that any finite-dimensional continuous representation of a compact topological group is completely reducible. In particular, this is true of finite-dimensional representations of finite topological groups, which are always compact under the discrete topology, and all of their representations are continuous. This is essentially Maschke's Theorem.

Fix a partition  $\lambda$  of  $n$ . Let  $O_\lambda$  be the conjugacy class of all nilpotent matrices in  $M_n(\mathbb{C})$  whose Jordan blocks are the partition  $\lambda$ . The closure  $\overline{O_\lambda}$  is an algebraic subset of  $M_n(\mathbb{C})$ .

Let  $i$  be the inclusion of diagonal matrices into  $M_n(\mathbb{C})$  and  $i^* : \mathbb{C}[M_n(\mathbb{C})] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  be the corresponding restriction map of algebras. Take the ideal

$$J_\lambda = i^*(I_{\overline{O_\lambda}})$$

and note that  $V(J_\lambda) = 0$  because  $O_\lambda$  is entirely nilpotent. Thus by the nullstellensatz, we have

$$\sqrt{J_\lambda} = (x_1, \dots, x_n)$$

so  $A := \mathbb{C}[x_1, \dots, x_n]/J_\lambda$  is a finite-dimensional  $\mathbb{C}$ -algebra, since for each  $j$ ,  $x_j^k \in J_\lambda$  for some  $k$ . Thus we can apply Maschke's Theorem to see that this algebra is completely reducible, so it can be written as

$$A = \bigoplus \text{something}$$

a graded algebra. Note also that  $S_n$  acts on  $A$  by graded algebra automorphisms. Let  $d_\lambda$  be the maximum  $d$  such that  $A_d \neq 0$  (the grade of  $A$ ). Then the amazing thing<sup>6</sup> is

$$(P/J_\lambda)_{d_\lambda} \cong V(\lambda).$$

This is pretty new stuff. It's a special case of the Hikita conjecture.

**1.11. Calculus on Banach Algebras.** A *Banach Algebra* is an algebra over  $\mathbb{R}$  or  $\mathbb{C}$  equipped with a complete multiplicative norm. So essentially it is a Banach space that is also an algebra. A key example is any matrix algebra.

Let  $D_r(A)$  be the open disk of radius  $r$  in  $A$ , i.e.

$$D_r(A) := \{x \in A : |x| < r\}.$$

If  $a_0, a_1, \dots$  is any infinite sequence in  $A$ , and  $f(x)$  is the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

then  $f$  converges within the radius

$$r(f) := \limsup_{k \rightarrow \infty} |a_k|^{-1/k}$$

---

<sup>6</sup>Ginzburg: "there is no explanation."

which is known as the Cauchy-Hadamard radius of convergence. It is important to note that this  $r(f)$  is the same for equivalent norms, as any constant factor goes to 1 in the long tail of  $|a_n|^{-1/n}$ .

**Proposition:** For all  $r < r(f)$ ,  $f(x)$  converges within  $D_r(A)$  to a continuous function. Moreover, the map  $D_r(\mathbb{R}) \rightarrow A$  given by  $t \mapsto f(t \cdot 1_A)$  is smooth.

*Proof.* Mostly identical to the normal calculus version.  $\square$

Examples:

- For  $f(x) = e^x = \sum_{k \geq 0} x^k/k!$ , we have  $\infty = \limsup_k |1/k!|^{-1/k}$  so  $e^x$  is defined everywhere.
- For  $f(x) = \log(1+x)$ ,  $\sum_{i=1}^{\infty} (-1)^{i-1} x^i/i$ , then  $r(f) = 1$ .
- For  $f(x) = 1+x+x^2+\dots = (1-x)^{-1}$ ,  $r(f) = 1$ . This gives that inversion is defined and continuous on the open disk of radius  $1-\varepsilon$  around  $1_A$ .
- If we conjugate  $f(x)$  by  $g$ , yielding  $f(gxg^{-1})$ , then the radius of convergence becomes

$$\min \left\{ r(f), \frac{r(f)}{|g| \cdot |g^{-1}|} \right\}$$

note that we do not necessarily have  $|g^{-1}| = |g|^{-1}$ .

Properties of the exponential map:

- If  $a, b$  commute (they may not in the matrix case!) then  $e^{a+b} = e^a \cdot e^b$ . The proof is the same as it would be in a field.
- $e^a$  is always invertible, and its inverse is  $e^{-a}$  (note that  $a, -a$  commute).
- $t \mapsto e^{ta}$  is a continuous group homomorphism from  $\mathbb{R}$  to  $A^\times$ .
- For sufficiently small  $r > 0$ ,  $\exp(D_r(A))$  is an open neighborhood of  $1_A$ , and there are mutually inverse homeomorphisms  $\exp$  and  $\log$  between  $D_r(A)$  and  $\exp(D_r(A))$ .

**1.12. Lie Groups.** Let  $V, W$  be normed vector spaces and let  $U$  be an open subset of  $V$ . For maps  $f : U \rightarrow W$ , we say that  $f$  is *differentiable* at  $x \in U$  if there is a linear map  $d_x f : V \rightarrow W$  such that

$$f(x+v) = f(x) + (d_x f)(v) + o(v)$$

within a sufficiently small neighborhood of  $x$ .

For example, taking the function  $\exp : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ , we have  $d_0 \exp = \text{id}$ .

**Lemma:** Let  $M_n = V_1 \oplus V_2$  be a decomposition of  $M_n(\mathbb{R})$ . Define the map  $\Phi : M_n \rightarrow M_n$  by

$$\Phi : (v_1, v_2) \mapsto e^{v_1} e^{v_2}.$$

Then there exist sufficiently small disks  $D_1 \subset V_1$  and  $D_2 \subset V_2$  centered at 0 such that  $\Phi(D_1 \times D_2)$  is an open neighborhood of  $\text{id} \in \text{GL}_n(\mathbb{R})$ , and  $\Phi$  is a diffeomorphism on  $D_1 \times D_2$ .

*Proof.* It suffices to show that  $d_0 \Phi$  exists and is invertible. In fact, it is equal to the identity. By linearity,

$$d_0 \Phi(v_1, v_2) = d_0 \Phi(v_1) + d_0 \Phi(v_2) = \text{id}_{V_1} + \text{id}_{V_2} = \text{id}.$$

$\square$

**Lemma:** For matrices  $x, y \in M_n$ , we have the approximation

$$e^{x/N} e^{y/N} = e^{(x+y+\alpha_N)/N}$$

where  $\alpha_N \in M_n$  such that  $\lim_N \alpha_N = 0$ .

The significance of this is that while  $e^{x+y} \neq e^x e^y$  in general, they are close for small-norm matrices.

**Limit Lemma:** Let  $G \subset \text{GL}_n$  be a closed subgroup. If  $x_i \in M_n$  is a sequence such that

(1)  $e^{x_i} \in G$

- (2)  $|x_i| \rightarrow 0$  but  $x_i \neq 0$   
 (3)  $x_i/|x_i| \rightarrow x$  where  $|x| = 1$   
 then  $e^{tx} \in G$  for all  $t \in \mathbb{R}$ .

*Proof.* Fix  $t$ . Choose integers  $m_i$  so that  $m_i|x_i|$  is as close as possible to  $t$ . Since  $|x_i| \rightarrow 0$ ,  $m_i|x_i| \rightarrow t$ , and  $m_i x_i \rightarrow tx$ . But now since  $m_i$  is an integer we can say

$$e^{m_i x_i} = (e^{x_i})^{m_i} \in G$$

and  $e^{m_i x_i} \rightarrow e^{tx}$  so because  $G$  is closed,  $e^{tx} \in G$ .  $\square$

A *Lie Group* is a closed subgroup  $G \subseteq \mathrm{GL}_n(\mathbb{R})$ . Corresponding to a Lie group we have a Lie algebra of  $G$ ,

$$\mathrm{Lie}(G) = \{x \in M_n : e^{tx} \in G \forall t \in \mathbb{R}\}$$

To show that this is actually an algebra requires some work. Note that  $\mathrm{Lie}(G)$  is automatically closed under scaling and multiplication. To show that it is also closed under addition, we use the lemmas: if  $a, b \in \mathrm{Lie}(G)$ , then

$$e^{a/N} e^{b/N} \in G \implies e^{(a+b+\alpha_N)/N} \in G$$

where  $\alpha_N \rightarrow 0$ . Take  $x_N = (a+b+\alpha_N)/N$ . Now by the limit lemma,  $x_N$  satisfies all the conditions and  $a+b = x$ , so indeed  $e^x = e^{a+b} \in G$  as desired.

Examples:

- For  $G = \mathrm{GL}_n(\mathbb{R})$ ,  $\mathrm{Lie}(G) = G$ .
- For  $G = \mathrm{SL}_n(\mathbb{R})$ ,

$$\mathrm{Lie}(G) = \{x \in M_n : \det(e^{tx}) = 1 \forall t\} = \{x \in M_n : e^{t \mathrm{tr}(x)} = 1\} = \{x \in M_n : \mathrm{tr}(x) = 0\}.$$

**Cartan's Theorem:** Let  $G \subseteq \mathrm{GL}_n(\mathbb{R})$  be a Lie group with unit  $e$ . Then there is an open neighborhood of  $\mathbf{0}_n \in M_n(\mathbb{R})$  such that

$$\exp(U \cap \mathrm{Lie}(G)) = \exp(U) \cap G.$$

*Proof.* It is clear that

$$\exp(U \cap \mathrm{Lie}(G)) \subset \exp(U) \cap \exp(\mathrm{Lie}(G)) \subset \exp(U) \cap G.$$

The other direction is the interesting one. How can we choose  $U$  so that if  $g = e^u$  then  $u$  must be in  $\mathrm{Lie}(G)$ ?

Pick a complementary subspace  $V \subset M_n$  of  $\mathrm{Lie}(G)$ , so that

$$\mathrm{Lie}(G) \oplus V = M_n.$$

Recall the map

$$\Phi : (u, v) \mapsto e^u e^v.$$

Let  $U_m$  be the disk of radius  $\frac{1}{m}$  in  $\mathrm{Lie}(G)$ , and similarly  $V_m$  in  $V$ . We claim that for all sufficiently large  $m$ ,

$$G \cap \Phi(U_m \times V_m) \subset \Phi(U_m \times \{0\}).$$

If this is not true, then there is a sequence  $m_i \rightarrow \infty$  and  $g_i = e^{u_i} e^{v_i} \in G$  for some  $|u_i|, |v_i| < \frac{1}{m_i}$  with  $v_i \neq 0$ . By compactness, we can take a subsequence where  $v_i/|v_i|$  converges to some limit  $|v| = 1$ , since the sphere is compact. Now by the Limit Lemma,  $v \in \mathrm{Lie}(G)$ . But  $V \cap \mathrm{Lie}(G) = \{0\}$  so  $v = 0$ , and this contradicts  $|v| = 1$ . Thus we have this claim.

Let  $m$  be sufficiently large so that the claim holds. Since  $\exp$  is a local diffeomorphism, we can find an open neighborhood of 0 in  $U \subset U_m \times V_m$  so that  $\exp$  is invertible on  $U$ . For this  $U$ , we claim that  $\exp(U \cap \mathrm{Lie}(G)) = \exp(U) \cap G$ .

Suppose that  $g \in \exp(U) \cap G \subseteq \Phi(U_m \times V_m) \cap G$ . Then because  $\exp$  is a bijection, there is a *unique* way to write  $g$  as  $e^u$ , so (revisit!)  $\square$

We can also think of the Lie algebra  $\text{Lie}(G)$  as the tangent space at 1 to  $G$ . For example the Lie algebra of  $S^1 \subset \mathbb{C}$  is the complex numbers  $x$  so that  $e^{tx} \in S^1$  for all  $t \in \mathbb{R}$ , i.e.  $x \in i\mathbb{R}$ . Note that this is the tangent space to  $S^1$  at 1.

It is an important fact that  $\text{Lie}(G)$  is closed under conjugation by elements in  $G$ .

The *Lie Bracket* is  $[x, y] := xy - yx$ . The Lie bracket of two elements in  $\text{Lie}(G)$  is in  $\text{Lie}(G)$ .

*Proof.* If  $x, y \in \text{Lie}(G)$ , then  $e^{tx}$  and  $e^{ty}$  are both in  $G$  for all  $t \in \mathbb{R}$ . Thus,  $e^{tx}ye^{-tx} \in \text{Lie}(G)$  as well. And when we expand this, we get

$$(1 + tx + \frac{t^2}{2}x^2 + \cdots)y(1 - tx + \frac{t^2}{2}x^2 - \cdots) = y + t(xy - yx) + \cdots$$

and in particular the derivative in  $t$  at  $t = 0$  is  $xy - yx$ . So  $xy - yx$  is in the tangent space, i.e. it is in  $\text{Lie}(G)$ .  $\square$

A morphism of Lie groups is a continuous group homomorphism. A morphism of Lie algebras is a linear map  $\varphi : \text{Lie}(G) \rightarrow \text{Lie}(H)$  which preserves the Lie bracket.

**Proposition:** If  $f : G \rightarrow H$  is a morphism of Lie groups, then there is a unique algebra morphism  $df : \text{Lie}(G) \rightarrow \text{Lie}(H)$  such that  $f(e^x) = e^{df(x)}$  for all  $x \in \text{Lie}(G)$ . Moreover,  $\text{Lie}(\ker(f)) = \ker(df)$ .

*Proof.* Fixing  $x \in \text{Lie}(G)$ , we have the following sequence of maps:

$$(\mathbb{R}, +) \xrightarrow{e^{\bullet x}} G \xrightarrow{f} H \hookrightarrow \text{GL}_n$$

We showed on homework that any continuous group homomorphism  $\mathbb{R} \rightarrow M_n^\times$  must be of the form  $t \mapsto e^{t \cdot v}$  for some  $v \in M_n^\times$ . Call this  $v$   $df(x)$ . Thus we have  $f(e^{tx}) = e^{t \cdot df(x)}$ . This satisfies the desired property that  $f(e^x) = e^{df(x)}$  (take  $t = 1$ ). If  $f$  were differentiable then we could say

$$df(x) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tx})$$

but we do not know this.

Now we will show that this constructed map  $df$  is actually a Lie algebra morphism. First, it is linear (I didn't get what he did to show this, maybe revisit later).

Next, we show that  $df$  preserves the Lie bracket. Let  $g \in G$  and  $y \in \text{Lie}(G)$ . Then

$$e^{t df(gyg^{-1})} = f(e^{t(gyg^{-1})}) = f(ge^{ty}g^{-1}) = f(g)f(e^{ty})f(g)^{-1} = f(g)e^{t df(y)}f(g)^{-1} = e^{t(f(g)df(y)f(g)^{-1})}$$

for all  $t \in \mathbb{R}$ , which implies

$$df(gyg^{-1}) = f(g)df(y)f(g)^{-1}.$$

Let  $g = e^{tx}$ . Expanding the LHS gives  $df([x, y])$  and expanding the RHS gives  $[df(x), df(y)]$ .  $\square$

**1.13. Representations of Lie Groups and Lie Algebras.** Let  $V$  be a finite-dimensional complex vector space  $V \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$  and consider  $\text{GL}(V) \subset \text{GL}_{2n}(\mathbb{R})$  as a Lie group.

Let  $G$  be a Lie group in  $\text{GL}(V)$ . A *continuous* representation of  $G$  in  $V$  is a morphism of Lie groups  $\rho : G \rightarrow \text{GL}(V)$  (so it must be continuous). For each representation  $\rho$  there is a corresponding  $d\rho : \text{Lie}(G) \rightarrow \text{End}(V)$ .

**Proposition:** Let  $\rho : G \rightarrow \text{GL}(V)$  be a continuous representation. Then

- (i) If  $W \subseteq V$  is  $G$ -stable then  $W$  is  $\text{Lie}(G)$ -stable as well.
- (ii) If  $f : V \rightarrow V$  is a  $G$ -intertwiner, then  $f$  is a  $\text{Lie}(G)$ -intertwiner as well.

Moreover, if  $G$  is connected, then the converses of (i) and (ii) hold.



*Proof.* (i): Suppose  $W$  is  $G$ -stable. We want to show that if  $x \in \text{Lie}(G)$  then  $d\rho(x)(w) \in W$  for  $w \in W$ .  $x \in \text{Lie}(G)$  implies  $e^{tx} \in G$ , so we know that

$$e^{td\rho(x)}(w) = \rho(e^{tx})(w) \in W$$

and taking the derivative of this at  $t = 0$ , we get  $d\rho(x)(w) \in W$ .

(ii): Let  $f : V \rightarrow V'$  be a  $G$ -intertwiner. Then for  $x \in \text{Lie}(G)$  we have

$$f \circ e^{td\rho(x)} = f \circ \rho(e^{tx}) = \rho(f(e^{tx})) = e^{td\rho'(x)} \circ f.$$

so again taking the derivative in  $t$  we see that  $f \circ d\rho(x) = d\rho'(x) \circ f$ .

To show the converses, we use the following fact: if  $G$  is connected then every element of  $G$  can be written as a product  $g = e^{x_1}e^{x_2} \cdots e^{x_n}$  for some  $x_1, \dots, x_n \in \text{Lie}(G)$ . On homework we showed that for connected  $G$ , every element can be expressed as a product of elements in  $U$ , if  $U$  is any open neighborhood of 1. And we know that there is some  $\exp(U)$  that is an open neighborhood of 1 in  $G$  by Cartan's Theorem, so this claim follows.

Now, suppose  $W \subseteq V$  is  $\text{Lie}(G)$ -stable. Then for any  $x \in \text{Lie}(G)$ ,  $x$  preserves  $W$ , but this also implies that  $e^x$  preserves  $W$  (since it is a sum of powers of  $x$ ). Now, every  $g \in G$  can be written as a product of exponentials  $e^x$ , each of which preserves  $W$ , so  $g$  also preserves  $W$ .

For the converse of (ii), suppose  $\rho, \rho'$  are representations and  $f : V \rightarrow V'$  is a  $\text{Lie}(G)$ -intertwining operator. For any  $g \in G$ , we will show that  $f(\rho(g) \cdot v) = \rho'(g) \cdot f(v)$ . Again, this follows because  $g$  can be written as a product of  $e^x$ , which we know can be commuted with  $f$ . Commute each of them one-by-one and the result follows.  $\square$

The significance of this proposition is that we can now study the representation theory of connected Lie groups through their (much simpler) Lie algebras. The irreps of  $G$  correspond to irreps of  $\text{Lie}(G)$ , and if  $G$  is *simply connected* then the correspondence is bijective

**Puzzle:** What is the complexification of  $S_n$ ? (informal question)

**1.14. Complex Lie Groups.** A  $\mathbb{C}$ -vector space is the same thing as an  $\mathbb{R}$ -vector space  $M$  along with an  $\mathbb{R}$ -linear operator  $I : M \rightarrow M$  such that  $I^2 = -\text{id}$  (which corresponds to multiplication by  $i$ ). In this way, we can say that a map  $a : V \rightarrow V$  is  $\mathbb{C}$ -linear if it commutes with addition, scaling by  $\mathbb{R}$ , and the operator  $I$ .

If  $V$  is a  $\mathbb{C}$ -vector space, we define  $V_{\mathbb{R}}$  to be the  $\mathbb{R}$ -vector space given by forgetting the operator  $I$ . We say that

$$\text{GL}(V) := \{g \in \text{GL}(V_{\mathbb{R}}) \mid gI = Ig\}.$$

We can compute the Lie algebra of this  $\text{GL}(V)$  as

$$\text{Lie}(\text{GL}(V)) = \{x \in \text{End}_{\mathbb{R}}(V_{\mathbb{R}}) \mid e^{tx} \in \text{GL}(V) \forall t \in \mathbb{R}\} = \text{Lie}(\text{GL}(V_{\mathbb{R}})) \cap \{e^{tx} I e^{-tx} = 1\} = \text{End}_{\mathbb{C}}(V).$$

If  $V$  is a  $\mathbb{C}$  vector space, a closed subgroup  $G \subset \text{GL}(V)$  is called a *complex Lie group* if  $\text{Lie}(G)$  is a complex vector space. For example if  $G = S^1 \subset \text{GL}_1(\mathbb{C})$  then  $\text{Lie}(G) = i\mathbb{R}$ , but this is *not* a complex vector space so  $G$  is not a complex Lie group.

A representation  $\rho : G \rightarrow \text{GL}(V)$  of a complex Lie group  $G$  is called *holomorphic* if  $d\rho : \text{Lie}(G) \rightarrow \text{End}(V)$  is  $\mathbb{C}$ -linear. For example, the representation  $\det : \text{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}^*$  has  $d(\det) = \text{tr}$ , which is  $\mathbb{C}$ -linear hence  $\det$  is a holomorphic representation. But  $\overline{\det}$  and  $|\det|$  are not holomorphic.

A complex Lie group  $G$  is *reductive* if it has a compact subgroup  $K \subset G$  such that  $\text{Lie}(G) = \text{Lie}(K) + i \text{Lie}(K)$ . For example,  $G = O_n(\mathbb{C})$  has  $\text{Lie}(G)$  as the complex skew-symmetric matrices. These can be decomposed into real and imaginary skew-symmetric parts, so  $G$  is reductive.

**Unitary Trick:** Any holomorphic representation of a reductive, connected group  $G$  is completely reducible.

*Proof.* Let  $\rho : G \rightarrow \mathrm{GL}(V)$  be a holomorphic representation. Let  $W \subseteq V$  be a  $G$ -stable subspace. Because  $G$  is reductive it has such a compact subgroup  $K$ . Representations of  $K$  are completely reducible because  $K$  is compact. Thus if  $W'$  is a complement of  $W$  in  $V$ , then  $V = W \oplus W'$ , so  $W'$  is  $K$ -stable as well.

Now,  $d\rho(x)$  preserves  $W'$  for  $x \in \mathrm{Lie}(K)$ . For general  $a = x + iy$  where  $x, y \in \mathrm{Lie}(K)$ ,

$$d\rho(a)(W') = d\rho(x)(W') + id\rho(y)(W') \subset W'$$

(note that we use the fact that  $\rho$  is holomorphic to take  $i$  out of  $d\rho(iy)$ ) so  $W'$  is  $\mathrm{Lie}(G)$ -stable as well. Since  $G$  is connected it follows that  $W'$  is  $G$ -stable, thus  $\rho$  is completely reducible.  $\square$

Consider the  $\mathbb{C}$ -bilinear form  $M_n(\mathbb{C}) \times M_n(\mathbb{C}) \rightarrow \mathbb{C}$  given by  $(a, b) \mapsto \mathrm{tr}(ab)$ . If  $a, b$  are skew-hermitian (i.e. in  $\mathrm{Lie}(U_n)$ ), then

$$\begin{aligned} \mathrm{tr}(ab) &= \sum_i a_{ii}b_{ii} + \sum_{i < j} a_{ij}b_{ji} + \sum_{i > j} a_{ij}b_{ij} \\ &= \sum_i a_{ii}b_{ii} + 2\Re\left(\sum_{i < j} a_{ij}b_{ji}\right) \end{aligned}$$

Noting that  $a_{ii}, b_{ii} \in i\mathbb{R}$ , we see that  $\mathrm{tr}(ab) \in \mathbb{R}$  in this case. Moreover  $\mathrm{tr}(a^2) \leq 0$ . Thus  $-\mathrm{tr}(a, b)$  is a positive-definite inner product.

**Proposition:**  $\mathrm{GL}_n(\mathbb{C})$  is reductive.

*Proof.* In the case  $n = 1$ ,  $\mathrm{GL}_n(\mathbb{C}) = \mathbb{C}^\times$ .  $S^1$  is a compact subgroup with Lie algebra  $i\mathbb{R}$ , so

$$\mathrm{Lie}(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C} = i\mathbb{R} \oplus \mathbb{R} = \mathrm{Lie}(S^1) + i\mathrm{Lie}(S^1)$$

thus  $\mathrm{GL}_n(\mathbb{C})$  is reductive.

For general  $n$ ,  $\mathrm{Lie}(\mathrm{GL}_n(\mathbb{C})) = M_n(\mathbb{C})$ . We want to split this into a real and imaginary part. If we take  $U_n$ , this is compact and as we showed on homework  $\mathrm{Lie}(U_n)$  is the set of skew-hermitian matrices. So

$$\mathrm{Lie}(\mathrm{GL}_n(\mathbb{C})) = M_n(\mathbb{C}) = \mathrm{Lie}(U_n) \oplus i\mathrm{Lie}(U_n)$$

because for any  $a \in M_n(\mathbb{C})$ ,

$$a = \left(\frac{a - a^*}{2}\right) + i\left(-i\frac{a + a^*}{2}\right).$$

$\square$

Now back to polynomials. Let  $P_d \subset \mathbb{C}[x_1, \dots, x_n]$  be the space of degree  $d$  homogeneous polynomials over  $\mathbb{C}$ .  $\mathrm{GL}_n(\mathbb{C})$  acts on  $P_d$ , thus giving a representation  $\rho_d$  of  $\mathrm{GL}_n(\mathbb{C})$ . We will show that this representation is holomorphic.

*Proof.* We need to check that  $d\rho_d : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathrm{End}_{\mathbb{C}}(P_d)$  is  $\mathbb{C}$ -linear.

First consider  $n = 2$ .  $P_d$  has basis  $x^d, x^{d-1}y, \dots, y^d$ , so let  $f(x, y) = x^i y^j$ . For  $u \in M_2(\mathbb{C})$  with entries

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we can calculate

$$\begin{aligned} (d\rho(u))(f) &= \frac{d}{dt} \rho(e^{tu})f|_{t=0} \\ &= \frac{d}{dt} f\left(e^{-tu} \begin{pmatrix} x \\ y \end{pmatrix}\right)|_{t=0} \\ &= \frac{d}{dt} f\left(\begin{pmatrix} x \\ y \end{pmatrix} - t \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} + o(t)\right)|_{t=0} \end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt} (x - t(ax + by) + o(t))^i (y - t(cxy + dy) + o(t))^j \Big|_{t=0} \\
&= (x - t(ax + by) + o(t))^i (y - t(cxy + dy) + o(t))^j \Big|_{t=0} \\
&=
\end{aligned}$$

and we can see that this is linear in  $a, b, c, d$  and thus linear in  $u$  as desired.  $\square$

As a corollary, for any reductive subgroup of  $\mathrm{GL}_n(\mathbb{C})$ , the representation by  $P_d$  is holomorphic and hence completely reducible by the unitary trick.

**1.15. Holomorphic Irreps of  $\mathrm{GL}_n(\mathbb{C})$ .** Because we have the unitary trick, we know that holomorphic representations of  $\mathrm{GL}_n(\mathbb{C})$  are completely reducible. But what are these representations?

Let  $T$  be the torus  $(\mathbb{C}^*)^n$ .

**Lemma:** Any holomorphic homomorphism  $T \rightarrow \mathbb{C}^*$  has the form  $\chi : t \mapsto t^\lambda$  for  $\lambda \in \mathbb{Z}^n$ .

*Proof.* Holomorphic means that  $d\chi : \mathbb{C}^n \rightarrow \mathbb{C}$  is linear, thus  $d\chi$  is an inner product with some  $(\lambda_1, \dots, \lambda_n)$ .  $\chi$  is thus an exponential of this, but it doesn't work unless  $\lambda_j$  are integers (details omitted).  $\square$

Let  $\rho : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}(M)$  be a holomorphic representation. Let  $T$  be the matrices in  $\mathrm{GL}_n$  with diagonal  $t_1, \dots, t_n$ . We can restrict  $\rho$  to  $T$  to give a representation of  $T$ . We decompose  $M$  into pieces

$$M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_\lambda$$

where

$$M_\lambda = \{m \in M : \rho(t)m = t^{\lambda m}\}.$$

We say  $\lambda$  is a “weight” of  $M$  if  $M_\lambda \neq 0$ . This decomposition is called the *weight space* of  $M$ .

Let  $N$  be the set of upper triangular matrices with 1 on the diagonal. We say that  $\lambda$  is a “highest weight” of  $M$  if  $\rho(g)$  fixes some  $m \in M_\lambda$  for all  $g \in N$ . We say that  $\lambda$  is “dominant” if it is ordered, i.e.  $\lambda_1 \geq \dots \geq \lambda_n$ .

**Classification Theorem:** For every dominant  $\lambda$ , there is a unique irreducible representation  $M(\lambda)$  with highest weight  $\lambda$ . Moreover,  $\lambda$  is the only highest weight of  $M(\lambda)$ , and  $\dim(M(\lambda)) = 1$ .

Let  $h$  be the diagonal matrix with diagonal  $h_1, \dots, h_n$ . Let  $E_i = 1_{i,i+1}$  and  $F_i = 1_{i+1,i}$ . Then

$$[h, E_i] = (h_i - h_{i+1})E_i, \quad [h, F_i] = -(h_i - h_{i+1})F_i$$

Let  $\alpha_i$  be the function on diagonal matrices which sends  $h$  to  $h_i - h_{i+1}$ .

Let  $\phi : \mathfrak{gl}_n(\mathbb{C}) \rightarrow \mathrm{End}(V)$  be a finite-dimensional representation of  $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$ . Denote  $\hat{x} = \phi(x)$ .

$$[\hat{h}, \hat{E}_i] = \alpha_i(h)\hat{E}_i \implies \hat{h}\hat{E}_i = \hat{E}_i(\hat{h} + \alpha_i(h)I).$$

Let  $\mu$  be a linear function on diagonal matrices,  $\mu(h) = \sum \mu_i h_i$ . The  $\mu$ -weight space of  $V$  is

$$V_\mu := \{v \in V \mid \hat{h}v = \mu(h)v \quad \forall h \text{ diagonal}\}$$

So if  $v \in V_\mu$  then

$$\hat{h}\hat{E}_i v = \hat{E}_i(\hat{h} + \alpha_i(h)I)v = (\mu + \alpha_i)(h)\hat{E}_i v.$$

That is,  $\hat{E}_i(V_\mu) \subseteq V_{\mu+\alpha_i}$ , and similarly for  $\hat{F}_i$ .

Let  $g_+$  be the upper triangular matrices with diagonal 1, and similarly  $g_-$ . We say that a nonzero vector  $v \in V_\mu$  is a *highest weight vector* of weight  $\mu$  if  $\hat{X}v = 0$  for all  $X \in g_+$ . Or equivalently, if

$$\widehat{1_{i,j}}v = \begin{cases} 0 & i < j \\ \mu_i v & i = j \end{cases}$$

Let  $v \in V_\lambda$  be a highest weight vector of weight  $\lambda$ . Let  $M$  be the  $\mathbb{C}$ -span of  $v, \hat{F}_i v, \hat{F}_i \hat{F}_j v, \dots$ . We just showed that applying  $\hat{F}_i$  reduces the weight by  $\alpha_i$  for any  $i$ . So

$$M = \bigoplus_{\mu} M_{\alpha - \mu}$$

where  $\mu$  ranges over  $m_1\alpha_1 + m_2\alpha_2 + \dots + m_n\alpha_n$  for  $m_i \in \mathbb{N}$ .

**Proposition:**  $M$  is  $\mathfrak{gl}_n$ -stable.

*Proof.* We just need to show that  $\widehat{1}_{i,j}(M) \subseteq M$  for  $i, j$ .

In the case  $i = j$ ,  $\widehat{1}_{i,i}(M_{\lambda - \mu}) = (\lambda_i - \mu_i)M_{\lambda - \mu} \subset M_{\lambda - \mu}$ .

In the case  $i > j$ , we will show by induction on  $i - j$ . We showed the case  $i - j = 0$  already. If  $i = j + 1$ , then  $1_{i,j} = F_i$  by definition of  $F_i$ , so clearly  $\hat{F}_i$  only reduces weights.

For the inductive step, note that  $1_{ij} = [F_{i-1}, 1_{i-1,j}]$ . Because both  $\widehat{F_{i-1}}$  and  $\widehat{1_{i-1,j}}$  preserve  $M$  (by inductive hypothesis), it follows that their commutator does, so  $\widehat{1_{ij}}$  does.

In the case  $i < j$ , we again use induction. In the case  $j = i + 1$ , then  $1_{i,j} = E_i$ , so  $\hat{E}_i$  adds weight, thus this is less immediate than the  $F_i$  case. We have for some diagonal  $h$ ,

$$\hat{E}_i \hat{F}_j v = \hat{F}_j \hat{E}_i v + \hat{h}v = 0 + (\lambda - \mu)(h)v \in \mathbb{C} \cdot v.$$

using the fact that  $v$  is a highest-weight vector, so it gets killed by  $\hat{E}_i$ . The inductive step is similar.  $\square$

**Proposition:** Let  $\rho : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}(V)$  be a holomorphic representation and  $v \in V_\lambda$  a highest weight vector for  $d\rho : \mathfrak{gl}_n \rightarrow \mathrm{End}(V)$ . Let  $M$  again be the  $\mathbb{C}$ -vectorspace generated by  $v$  and  $\hat{F}_i$  for all  $i$ . Then  $M$  is a  $\mathrm{GL}_n$ -stable subspace of  $V$ . Moreover,  $M$  is an irrep of  $\mathrm{GL}_n(\mathbb{C})$ .

*Proof.* By the previous proposition  $M$  is  $\mathfrak{gl}_n$ -stable, so it follows that  $M$  is  $\mathrm{GL}_n$ -stable. Similarly if  $M$  is a  $\mathfrak{gl}_n$ -irrep then it is a  $\mathrm{GL}_n$ -irrep. So we just need to show that  $M$  is a  $\mathfrak{gl}_n$ -irrep.

Assume that  $M' \subseteq M$  is  $\mathrm{GL}_n$ -stable. Since  $\mathrm{GL}_n(\mathbb{C})$  is reductive, there is a  $\mathrm{GL}_n$ -stable complement of  $M'$  in  $M$ , which we can call  $M''$ , so that  $M = M' \oplus M''$ . Then we also have

$$M_\lambda = M'_\lambda \oplus M''_\lambda$$

for each  $\lambda$ . But now we claim that  $M_\lambda = \mathbb{C}v$  is one-dimensional (why?).

Now for any  $\lambda$ , either  $M'_\lambda$  or  $M''_\lambda$  is trivial. Then we can show that  $M'' = 0$  (details fill in later)  $\square$

**1.16. Tensors of representations.** Let  $G$  be a Lie group and  $\rho, V, \rho', V'$  two representations of  $G$ . Then we can define the tensor product of these two representations as

$$(\rho \otimes \rho')g : v \otimes v' \mapsto \rho(g)v \otimes \rho'(g)v'.$$

What is  $d(\rho \otimes \rho')$ ? It turns out to be

$$d(\rho \otimes \rho') = d\rho \otimes \mathrm{id}_{V'} + d\rho' \otimes \mathrm{id}_V.$$

To show this, we can calculate out how  $e^{t d(\rho \otimes \rho')(x)}$  acts on  $v \otimes v'$  and take the derivative in  $t$ .

As a corollary, if  $v, v'$  are eigenvectors of  $d\rho$  and  $d\rho'$  with eigenvalues  $\lambda$  and  $\mu$ , then  $v \otimes v'$  is an eigenvector of  $d(\rho \otimes \rho')$  with eigenvalue  $\lambda + \mu$ . This will be an important fact.

Returning to the setting of  $\mathrm{GL}_n(\mathbb{C})$ , let  $G = \mathrm{GL}_n(\mathbb{C})$  with two holomorphic representations  $\rho, V$  and  $\rho', V'$ . As before, for each  $\lambda = \lambda_1, \dots, \lambda_n$ , let

$$V_\lambda := \{v \in V \mid d\rho(1_{ii})v = \lambda_i v \ \forall i\}$$

and similarly for  $V'_\mu$ . Using the tensor fact we just proved, we have

$$V_\lambda \otimes V'_\mu \subseteq (V \otimes V')_{\lambda+\mu}.$$

If  $v \in V_\lambda$  and  $v' \in V'_\mu$  are both highest weight vectors, then  $v \otimes v'$  is highest weight in  $(V \otimes V')_{\lambda+\mu}$ .

Examples:

- If  $V = \mathbb{C}^n$  then the highest-weight vector is  $e_1$ , and its weight is  $(1, 0, \dots, 0)$ .
- If  $V = \wedge^2 \mathbb{C}^n$ , then the highest-weight vector is  $e_1 \wedge e_2$ , with weight  $(1, 1, 0, 0, \dots, 0)$ .
- If  $V = \wedge^p \mathbb{C}^n$ , then the highest-weight vector is  $e_1 \wedge \dots \wedge e_p$ , with weight  $(1, 1, 1, \dots, 1, 0, 0, \dots, 0)$ .

In general, let  $\omega_j$  be a tuple of  $j$  1s followed by 0s. These are the “fundamental weights” and are important.

**Theorem:** For all  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_n)$  there is a *unique* holomorphic irrep of  $GL_n$  with highest weight  $\lambda$ .

*Proof.*  $\lambda$  can be written as

$$\lambda_n \omega_n + (\lambda_{n-1} - \lambda_n) \omega_{n-1} + \dots + (\lambda_1 - \lambda_2) \omega_1.$$

We’ve seen that we can achieve  $\omega_1$  as a highest-weight of  $V$  and  $\omega_2$  as a highest-weight of  $\wedge^2 V$  and so on. So we can make a representation

$$M(V, v) := \left( \wedge^1 V \right)^{\otimes (\lambda_1 - \lambda_2)} \otimes \left( \wedge^2 V \right)^{\otimes (\lambda_2 - \lambda_3)} \otimes \dots \otimes \left( \wedge^n V \right)^{\otimes (\lambda_n)}$$

with highest-weight  $\lambda$ . Note that for  $\lambda_n$  it is okay that it may be negative (why?).

Now we show that this representation is unique. Let  $W$  be an irrep with highest weight vector  $w \in W_\lambda$ . Take  $W \oplus M(V, v)$ . The highest weight vector in this direct sum is  $(w, v)$ . Then we can also take

$$M(W \oplus M(V, v), (w, v))$$

which is irreducible. □