MATH 317 HW 3

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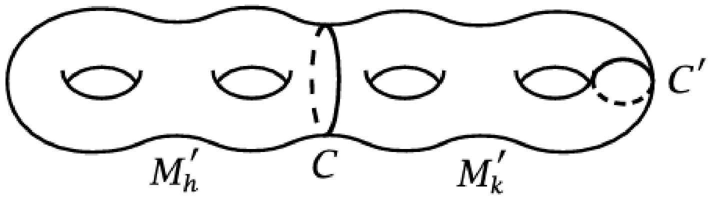
Problem 1 (Hatcher 1.2:9): In the surface M_g of genus g, let C be a circle that separates M_g into two compact surfaces M_h' and M_k' obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M_h' does not retract onto C, and hence M_g does not retract onto C. But show that M_g does retract onto the non-separating circle C' in the figure.



Proof. If there is any retraction $r : M_h' \to C$, then this r induces a map of the first homology groups r_*. Composing r with the inclusion $\iota : C \to M_h'$, induces the identity on $H_1(C) = \mathbb{Z}$:

$$H_1(C) \xrightarrow{\iota_*} H_1(M_h') \xrightarrow{r_*} H_1(C)$$

Now we will show that $\mathrm{id}_{\mathbb{Z}}$ cannot factor through $H_1(M_h')$:

We know that $H_1(M_h)$ can be presented as the Abelianization of

$$\pi_1(M_h) = \langle a_1, b_1, a_2, b_2, \ldots, a_h, b_h \mid [a_1, b_1][a_2, b_2] \cdots [a_h, b_h] \rangle,$$

which is $\mathbb{Z}^{2h}$. For M_h', things are different because there is a puncture, which we can represent as an additional loop c at one of the vertices, so rather than the product of commutators being a loop, it is just equal to c; that is,

$$\pi_1(M_h') = \langle a_1, b_1, \ldots, a_h, b_h, c \mid [a_1, b_1] \cdots [a_h, b_h] = c \rangle$$

so $H_1(M_h')$ is the Abelianization of this. The image $\iota_*(1) \in \pi_1(M_h')$ is exactly the loop c, but this c is 0 in $H_1(M_h')$, so ι_* is trivial on the homology groups, and thus $r_* \circ \iota_*$ cannot be the identity. So r cannot exist in the first place.

On the other hand, C' corresponds to one of the sides of the $2g$-gon, not an additional loop, so the argument does not work for C'. And it is possible to retract to C' by multiple methods. For one, you can fold up the $2g$-gon along the long diagonals repeatedly until left with four edges corresponding to one commutator (i.e. a single torus). Then with this torus, just project down onto one of the sides of the square. $\square$

Problem 2 (Hatcher 1.2:11): The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f preserving the basepoint, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. T_f has a CW-complex construction with a single 0-cell x, two 1-cells α and β, and two 2-cells which attach via f, giving the relations $\alpha = f_*(\alpha)$ and $\beta = f_*(\beta)$. So the group presentation is

$$\pi_1(T_f) = \langle \alpha, \beta \mid \alpha = f_*(\alpha),\ \beta = f_*(\beta) \rangle.$$

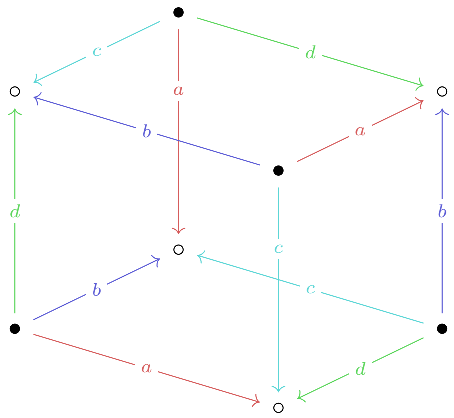
When $X = S^1 \times S^1$, we start with the free abelian group on two generators $\mathbb{Z}^2$ and again identify $\alpha = f_*(\alpha)$ and $\beta = f_*(\beta)$, giving

$$\pi_1(T_f) = \langle \alpha, \beta \mid \alpha = f_*(\alpha),\ \beta = f_*(\beta),\ \alpha\beta = \beta\alpha \rangle.$$

$\square$

Problem 3 (Hatcher 1.2:14): Consider the quotient space X of a cube I^3 obtained by identifying each square face with the opposite square face via a clockwise quarter-rotation and translation by one unit. Show that X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group.

Proof. The cells are as in the diagram. The two 0-cells are denoted by black and white circles, which are identified. The four 1-cells are the arrows a, b, c, d. The three 2-cells are the faces of the cube (with opposite faces identified by quarter turn). The single 3-cell is the cube itself.



In the 1-skeleton, X^1 has 4 edges a, b, c, d. The fundamental group $\pi_1(X^1)$ is generated by

$$i := ab^{-1},\ j := ac^{-1},\ k = ad^{-1}.$$

The attaching maps of the three 2-cells yield the relations

$$ad^{-1}bc^{-1} = 1,\ \ ac^{-1}db^{-1} = 1,\ \ ab^{-1}cd^{-1} = 1.$$

Translating this to i, j, k, we can say $j = ac^{-1} = bd^{-1} = ba^{-1}ad^{-1} = i^{-1}k \Rightarrow ij = k$. Likewise,

$$i = jk,\ \ j = ki,\ \ k = ij.$$

And we also have

$$\begin{aligned} kji &= (ad^{-1})ac^{-1}ab^{-1} \\ &= c(b^{-1}ac^{-1})ab^{-1} \\ &= cd^{-1}ab^{-1} \\ &= 1 \end{aligned}$$

(this can be seen as a deformation of the loop in the diagram). Thus, i, j, k obey the group presentation for Q: note that

$$i^2 = ijk = k^2 = kij = j^2$$

and

$$(ijk)^2 = (ij)(ki)(jk) = kji = 1.$$

$\square$

Problem 4 (Hatcher 1.2:16): Show that the fundamental group of the surface of infinite genus is free on an infinite number of generators.

Proof. Similarly to the surfaces of finite genus, we can give a CW complex structure to the surface M_ω^+ with infinite genus *in one direction*: one 0-cell x, infinitely many 1-cells $a_1, b_1, a_2, b_2, \dots$, and a single 2-cell which attaches via

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots$$

This makes $\pi_1(M_\omega^+) = \mathbb{Z}^\omega$, as this infinite product of commutators is not actually a group element.

Now to get the fundamental group of M_ω, which has infinitely many holes extending in both directions: we can write $M_\omega \simeq M_\omega^+ \vee M_\omega^+$, so by Van Kampen's Theorem,

$$\pi_1(M_\omega) = \mathbb{Z}^\omega \oplus \mathbb{Z}^\omega = \mathbb{Z}^{\omega+\omega} = \mathbb{Z}^\omega.$$

□

Problem 5 (Hatcher 1.2:22): In this exercise, we describe an algorithm for computing the *Wirtinger presentation* of the fundamental group of the complement of a knot in $\mathbb{R}^3$. We begin with the knot lying almost flat on a table T so that K consists of finitely-many disjoint arcs α_i contained within T and finitely-many β_ℓ which leave T to cross over another part of K. We build a 2-dimensional complex X that is a deformation retract of $\mathbb{R}^3 - K$ by taking X to essentially be a tube that surrounds the string and contacts itself when there is a crossing:

For each α_i, let R_i be a curved rectangle so that its long edges are on T and it has α_i underneath it, and let and β_ℓ crossing over α_i lie along the curve of R_i. For each β_ℓ, let S_ℓ be the square-shaped piece which covers the crossing, so that β_ℓ lies under S_ℓ but inside R_i. Let X be the union of T, R_i, and S_ℓ for all i, ℓ. Lift K off the table slightly so that it does not intersect X. Then we can deformation retract $\mathbb{R}^3 - K$ to X.

(a) Assuming that this retraction is possible, show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each R_i and one relation $x_i x_j x_i^{-1} = x_k$ whenever α_j, α_k are two pieces which cross over α_i via some β_ℓ.
(b) Use this presentation to show that the Abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$.

Proof. (a): By the deformation retraction, $\pi_1(\mathbb{R}^3 - K) = \pi_1(X)$. X can be constructed as a CW complex with 1-cells x_i corresponding to the arcs where α_i meets the square S_ℓ, and 2-cells forming the tubes (note that this makes the arcs at the beginning and end of tube α_i *both* equivalent to x_i) and the identifications given by each crossing square S_ℓ. Traversing around a crossing counterclockwise (or clockwise, as long as it's consistent), the attaching map gives $x_i x_j x_i^{-1} x_k^{-1}$, so by Van Kampen's theorem we quotient by this relation to get the correct group presentation.

(b): In the Abelianization of $\pi_1(X)$, $x_i x_j x_i^{-1} = x_j$, so this gives $x_k = x_j$ for all k, j that are separated by a crossing. But since K is a knot in the first place, all of the segments are related this way. So $\pi_1(X)$ has a single generator with no relations, i.e. it is $\mathbb{Z}$. □

Problem 6 (Hatcher 1.B:5): Consider the graph of groups Γ having one vertex $\mathbb{Z}$ and one edge $n \mapsto 2n$. Show that $\pi_1(K\Gamma)$ has presentation $\langle a, b \mid bab^{-1}a^{-2}\rangle$ and describe the universal cover of $K\Gamma$ explicitly as a product $T \times \mathbb{R}$ with T a tree.

Proof. $K\Gamma$ consists of S^1 along with a mapping cylinder which attaches x to $2x$. Thus we have the CW complex presentation below:

$$X$$

$$\begin{tikzcd}
& \bullet \arrow[rr, "a"] \arrow[d, two heads, "b"'] & & \bullet \arrow[d, two heads, "b"] \\
{[0,1]} & \bullet \arrow[r, "a"'] & \bullet \arrow[r, "a"'] & \bullet
\end{tikzcd}$$

All the black dots are identified and the similarly labeled arrows are identified. a represents a single loop in S^1 and b the loop around the mapping cylinder (it's really more of a mapping torus in this case). We can see that the single relation given by the 2-cell is $aba^{-2}b^{-1} = 1$, giving the desired group presentation.

In the universal cover, we want to tile some 2-dimensional surface with these squares so that every point is the corner of two squares of sidelength x and two more of sidelength $2x$ (because they are identified with the vertices on the bottom). This forces the two top vertices to be connected to *separate* squares of sidelength $2x$, since they cannot both fit. Thus at the top of each square we have to split into two, and so on. This creates an infinite 3-valent tree T in one direction, and just a non-branching $\mathbb{R}$ in the other. So the universal cover is $T \times \mathbb{R}$.

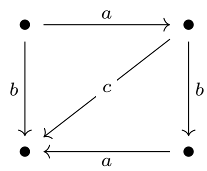
Below is a depiction of the splitting at one of the nodes of T:

$$\begin{tikzcd}
& \bullet \arrow[rr] \arrow[d, two heads] & & \bullet \arrow[rr] \arrow[d, two heads] & & \bullet \arrow[rr] \arrow[d, two heads] & & \bullet \arrow[d, two heads] & & \\
\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \\
& & \bullet \arrow[rr] \arrow[u, two heads] & & \bullet \arrow[rr] \arrow[u, two heads] & & \bullet \arrow[rr] \arrow[u, two heads] & & \bullet \arrow[u, two heads] &
\end{tikzcd}$$

at the top and bottom lines of this figure, there are further branchings to larger squares, and the shorter arrows in the middle are themselves the tops of smaller squares. $\square$

Problem 7 (Hatcher 2.1:5): Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure.

Proof. The Klein Bottle K can be constructed as two 2-simplices:



There is one 0-cell, three 1-cells, and two 2-cells, giving the chain complex

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$$

On $\partial(C_1(K)) = 0$ since there is only one vertex, yielding $H_0(K) = \mathbb{Z}$. On $C_2(K)$, the boundary operator is injective, with image generated by the boundaries of the triangles,

$$\partial(C_2(K)) = \langle a + c - b, b + a - c \rangle.$$

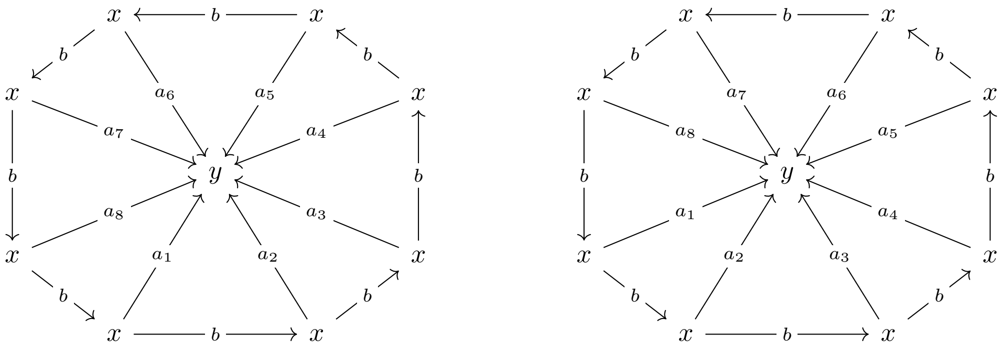
Thus for $H_2(K)$ we get

$$H_2(K) = \langle a, b, c \rangle / \langle a + c - b, b + a - c \rangle = \langle a, b, c \rangle / \langle a, 2b - 2c \rangle = \langle b, c \rangle / \langle 2b - 2c \rangle = \mathbb{Z} \oplus \mathbb{Z}_2$$

So we get $H_0(K) = \mathbb{Z}$, $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_2(K) = 0$ and $H_n(K) = 0$ for $n \geq 3$. $\square$

Problem 8 (Hatcher 2.1:8): Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \dots, T_n$, with all sharing a common edge and each sharing a face with its two neighbors, and such that the bottom face of T_i and the top face of T_{i+1} are identified. Show that the simplicial homology groups of X in dimensions 0,1,2,3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$ respectively.

Proof. The identification collapses all of the equatorial vertices into one (say x) and the top and bottom vertices into one (say y). Label the edges (1-cells) with $a_1, \dots, a_n$ pointing in, b the repeated edge on the circumference, and c the edge $y \to y$, as in the diagram (top-down view on the left, the edges directly below those on the right):



Let C_j be the triangle (2-cell) whose edges are a_j, a_{j+1}, c (indices mod n), and let B_j be the 2-cell whose edges are a_j, a_{j+1}, b.

$Z_0(X) = \langle x, y \rangle$ and $B_0(X) = \langle x - y \rangle$, so $H_0(X) = Z_0/B_0 = \mathbb{Z}$.

$Z_1(X)$ contains b and c as they are loops on their own, and also contains $a_i - a_j$ for any i, j, so $Z_1(X) = \langle b, c, a_1 - a_2, a_2 - a_3, \dots, a_{n-1} - a_n \rangle$. $B_1(X)$ consists of the boundaries of B_j and C_j, so $a_j - a_{j+1} - b$ and $a_j - a_{j+1} - c$. Thus

$$H_1(X) = Z_1/B_1 = \frac{\langle b, c, a_j - a_{j+1} \rangle}{\langle a_j - aj + 1 - b, a_j - a_{j+1} - c \rangle} = \frac{\langle b \rangle}{\langle nb \rangle} = \mathbb{Z}_n.$$

What's happened here is that mod boundaries, $b, c, a_j - a_{j+1}$ are all the same, and

$$nb = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_1) = 0,$$

hence we get $\mathbb{Z}_n$.

Let

$$s := \sum_j p_j B_j + q_j C_j$$

be some element of $Z_2(X)$. What can we say about p_j and q_j? We have $\partial B_j = a_j - a_{j+1} - b$ and $\partial C_j = a_j - a_{j+1} - c$, so

$$\partial s = \sum_j a_j (p_j + q_j - p_{j-1} - q_{j-1}) + b p_j + c q_j.$$

Since $\partial s = 0$, this gives $\sum p_j = \sum q_j = 0$ and $p_j + q_j = p_{j-1} + q_{j-1}$ for each j, from which we can see that $p_j + q_j = 0$ for all j. Thus, $Z_2(X)$ is generated by $(B_j - C_j) - (B_{j+1} - C_{j+1})$. But this is the boundary of the jth 3-cell, so $H_2(X) = 0$.

Finally, $Z_3(X)$ is generated by the sum of all 3-cells, and there are no boundaries because X is only 3-dimensional, so $H_3(X) = \mathbb{Z}$. $\square$

Problem 9 (Hatcher 2.1:11): Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Proof. Let $\iota : A \to X$ be the inclusion and $r : X \to A$ the retraction. $r \circ \iota = \mathrm{id}_A$. These maps induce maps on homology groups

$$H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{r_*} H_n(A)$$

which must compose to the identity. Thus, ι_* must be injective; otherwise $r_* \circ \iota_* = (r \circ \iota)_* = \mathrm{id}_{H_n(A)}$ cannot be injective. $\square$