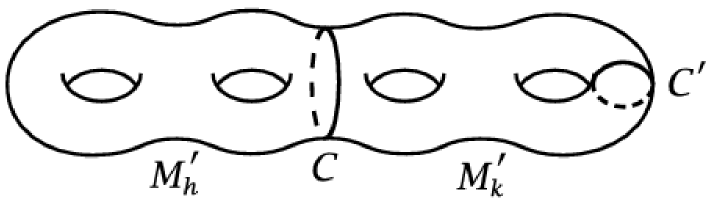
MATH 317 HW 3

JALEN CHRYSOS

Problem 1 (Hatcher 1.2:9): In the surface M_g of genus g, let C be a circle that separates M_g into two compact surfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto C, and hence M_g does not retract onto C. But show that M_g does retract onto the non-separating circle C' in the figure.



Proof. We know that $\pi_1(M_g)$ can be presented as

$$\pi_1(M_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle.$$

The Abelianization of this is $\mathbb{Z}^{2g}$. If M'_h retracts onto C, then

On the other hand, for C', there is a retraction given by $\square$

Problem 2 (Hatcher 1.2:11): The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f preserving the basepoint, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. T_f has a CW-complex construction with a single 0-cell x, two 1-cells α and β, and two 2-cells which attach via f, giving the relations $\alpha = f_*(\alpha)$ and $\beta = f_*(\beta)$. So the group presentation is

$$\pi_1(T_f) = \langle \alpha, \beta \mid \alpha = f_*(\alpha),\ \beta = f_*(\beta) \rangle.$$

When $X = S^1 \times S^1$, we start with the free abelian group on two generators $\mathbb{Z}^2$ and again identify $\alpha = f_*(\alpha)$ and $\beta = f_*(\beta)$, giving

$$\pi_1(T_f) = \langle \alpha, \beta \mid \alpha = f_*(\alpha),\ \beta = f_*(\beta),\ \alpha\beta = \beta\alpha \rangle.$$

$\square$

Problem 3 (Hatcher 1.2:14): Consider the quotient space X of a cube I^3 obtained by identifying each square face with the opposite square face via a clockwise quarter-rotation and translation by one unit. Show that X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group.

Proof. The cells are as in the diagram:

Diagram goes here

In the 1-skeleton, X^1 has 4 edges a, b, c, d. The fundamental group $\pi_1(X^1)$ is generated by

$$i := ab^{-1},\ j := ac^{-1},\ k = ad^{-1}.$$

The attaching maps of the three 2-cells yield the relations

$$ad^{-1}bc^{-1} = 1,\ \ ac^{-1}db^{-1} = 1,\ \ ab^{-1}cd^{-1} = 1.$$

Translating this to i, j, k, we can say $j = ac^{-1} = bd^{-1} = ba^{-1}ad^{-1} = i^{-1}k \Rightarrow ij = k$. Likewise,

$$i = jk,\ \ j = ki,\ \ k = ij.$$

And we also have

$$\begin{aligned} kji &= (ad^{-1})ac^{-1}ab^{-1} \\ &= c(b^{-1}ac^{-1})ab^{-1} \\ &= cd^{-1}ab^{-1} \\ &= 1 \end{aligned}$$

(this can be seen as a deformation of the loop in the diagram). Thus, i, j, k obey the group presentation for Q: note that

$$i^2 = ijk = k^2 = kij = j^2$$

and

$$(ijk)^2 = (ij)(ki)(jk) = kji = 1.$$

□

Problem 4 (Hatcher 1.2:16): Show that the fundamental group of the surface of infinite genus is free on an infinite number of generators.

Proof. Similarly to the surfaces of finite genus, we can give a CW complex structure to the surface M_ω^+ with infinite genus *in one direction*: one 0-cell x, infinitely many 1-cells $a_1, b_1, a_2, b_2, \ldots$, and a single 2-cell which attaches via

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots$$

This makes $\pi_1(M_\omega^+) = \mathbb{Z}^\omega$, as this infinite product of commutators is not actually a group element.

Now to get the fundamental group of M_ω, which has infinitely many holes extending in both directions: we can write $M_\omega \simeq M_\omega^+ \vee M_\omega^+$, so by Van Kampen's Theorem,

$$\pi_1(M_\omega) = \mathbb{Z}^\omega \oplus \mathbb{Z}^\omega = \mathbb{Z}^{\omega+\omega} = \mathbb{Z}^\omega.$$

$\square$

Problem 5 (Hatcher 1.2:22): In this exercise, we describe an algorithm for computing the *Wirtinger presentation* of the fundamental group of the complement of a knot in $\mathbb{R}^3$. We begin with the knot lying almost flat on a table T so that K consists of finitely-many disjoint arcs α_i contained within T and finitely-many β_ℓ which leave T to cross over another part of K. We build a 2-dimensional complex X that is a deformation retract of $\mathbb{R}^3 - K$ in the following steps:

For each α_i, let R_i be a curved rectangle so that its long edges are on T and it has α_i underneath it, and let and β_ℓ crossing over α_i lie along the curve of R_i. For each β_ℓ, let S_ℓ be the square-shaped piece which covers the crossing. Let X be the union of T, R_i, and S_ℓ for all i, ℓ. Lift K off the table slightly so that it does not intersect X. Then we can retract $R^3 - K$ to X.

(a) Assuming that this retraction is possible, show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each R_i and one relation $x_i x_j x_i^{-1} = x_k$ whenever α_j, α_k are two pieces which cross over α_i via some β_ℓ.

(b) Use this presentation to show that the Abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$.

Problem 6 (Hatcher 1.B:5): Consider the graph of groups Γ having one vertex $\mathbb{Z}$ and one edge $n \mapsto 2n$. Show that $\pi_1(K\Gamma)$ has presentation $\langle a, b | bab^{-1}a^{-2} \rangle$ and describe the universal cover of $K\Gamma$ explicitly as a product $T \times \mathbb{R}$ with T a tree.

Problem 7 (Hatcher 2.1:5): Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure.

Problem 8 (Hatcher 2.1:8): Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$, with all sharing a common edge and each sharing a face with its two neighbors. Show that the simplicial homology groups of X in dimensions 0,1,2,3 are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$ respectively.

Problem 9 (Hatcher 2.1:11): Show that if X is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.