**Problem 1 (Hatcher 3.2:1)**: Assuming the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for $M_g$ the closed orientable surface of genus $g$, by using the quotient map from $M_g$ to the wedge of $g$ tori.

*Proof.* Let $T = S^1 \times S^1$. By the quotient map, we have

$$H^*(M_g) = H^*(\vee_g T) = \prod_g H^*(T).$$

Thus, using the fact that $H^*(T; R)$ is the exterior algebra over $R$ generated by two elements,

$$H^*(M_g; R) = \prod_{i=1}^{g} \Lambda_R[\alpha_i, \beta_i].$$

□

**Problem 2 (Hatcher 3.2:2)**: Using the cup product

$$H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R),$$

show that if $X$ is the union of contractible open subsets $A$ and $B$, then all cup products of positive-dimensional classes in $H^*(X; R)$ are 0. This applies in particular if $X$ is a suspension. Generalize to the situation that $X$ is the union of $n$ contractible open subsets, to show that all $n$-fold cup products of positive-dimensional classes are 0.

*Proof.* If $A, B$ are contractible then $H^k(A; R) = H^\ell(B; R) = 0$, so in the long exact sequence defining $H^k(X, A; R)$ we have

$$0 \cong H^{k+1}(A; R) \to H^k(X, A; R) \to H^k(X; R) \to H^k(A; R) \cong 0$$

so $H^k(X, A; R) \cong H^k(X; R)$ and similarly $H^\ell(X, B; R) = H^\ell(X; R)$. If $X = A \cup B$ then

$$H^{k+\ell}(X, A \cup B; R) = H^{k+\ell}(X, X; R) = 0.$$

So the cup product yields a map

$$H^k(X; R) \times H^\ell(X; R) \to 0$$

i.e. it is always 0.

Similarly, if $X = A_1 \cup A_2 \cup \cdots \cup A_n$, a finite union of contractible open sets, then

$$H^{k_1 + \cdots + k_n}(X, A_1 \cup \cdots \cup A_n; R) = 0$$

and $H^{k_j}(X, A_j; R) = H^{k_j}(X; R)$, so the $n$-fold cup product is

$$H^{k_1}(X; R) \times \cdots \times H^{k_n}(X; R) \to 0$$

and thus it is always 0. $\square$

**Problem 3 (Hatcher 3.2:3)**:

(a) Using the cup product structure, show that there is no map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ which induces a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$?

(b) Prove the Borsuk-Ulam theorem by the following argument: suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all $x$. Then define $g : S^n \to S^{n-1}$ by

$$g(x) := \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

so that $g(-x) = -g(x)$ and $g$ induces a map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n-1}$. Show that part (a) applies to this map.

*Proof.* (a): We know that $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$. A map from $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ induces a map on the cohomology, and thus a ring homomorphism

$$\mathbb{Z}_2[\beta]/(\beta^{m+1}) \to \mathbb{Z}_2[\alpha]/(\alpha^{n+1}).$$

This map is determined by where it sends $\beta$. To induce a nontrivial map on $H^1$, it would have to send $\beta$ to $\alpha$ (the only degree-1 term in $\mathbb{Z}_2[\alpha]$), but then the map would be an isomorphism which is impossible since $n \neq m$.

For $\mathbb{C}\mathrm{P}$, we can take coefficients in $\mathbb{Z}$ to get the cohomology ring $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$. In this case a map $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$ would induce a ring map

$$\mathbb{Z}[\beta]/(\beta^{m+1}) \to \mathbb{Z}[\alpha]/(\alpha^{n+1}).$$

If this is nontrivial on $H^1$, then $\beta$ must be mapped to $t\alpha$ for some $t \in \mathbb{Z}$. But $\beta^{m+1}$ must be mapped to 0, so $(t\alpha)^{m+1} = t^{m+1}\alpha^{m+1} = 0$, which implies $\alpha^{m+1} = 0$, but $m < n$ so this is false in $\mathbb{Z}[\alpha]/(\alpha^{n+1})$.

(b): We can show that this map $g$ (if it exists) induces a nontrivial map on the first cohomology groups $H^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$. That is, $g$ sends a nontrivial cocycle (which corresponds dually to a nontrivial loop in $\mathbb{R}\mathrm{P}^n$, i.e. a path in $S^n$ whose endpoints are antipodes) to a nontrivial cocycle. And this is indeed true, as $g(x) = -g(-x)$, so $g$ preserves antipodes. $\square$

**Problem 4 (Hatcher 3.2:4)**: Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ has a fixed point if $n$ is even, using the fact that $f^* : H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is a ring homomorphism. When $n$ is odd, show that there is a fixed point unless $f^*(\alpha) = -\alpha$ for some $\alpha$ that is a generator of $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$.

*Proof.* Recall that $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ as a ring with the cup product, where $\alpha$ is a generator of $H^2$. Since $f$ induces a ring map on $\mathbb{Z}[\alpha]/(\alpha^{n+1})$, it must send $\alpha$ to some $t\alpha$ with nonzero $t \in \mathbb{Z}$. Then $f^*(\alpha^k) = t^k \alpha^k$ etc. So the Lefschetz trace of $f$ is

$$\tau(f) = \sum_{k=0}^{n} t^k = \frac{1 - t^{n+1}}{1 - t}.$$

If there are no fixed-points then $\tau(f) = 0$ by Lefschetz fixed point theorem. This can only be 0 if $t \neq 1$ and $t^{n+1} = 1$, which for $t \in \mathbb{Z}$ is only possible if $t = -1$ and $n$ is odd. In the case that $n$ is odd and $t = -1$, it follows that $f^*(\alpha) = -\alpha$. □

**Problem 5 (Hatcher 3.2:8)**: Let $X$ be $\mathbb{C}\mathrm{P}^2$ with a cell $e^3$ attached by a map $S^2 \to \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$ of degree $p$, and let $Y = M(\mathbb{Z}_p, 2) \vee S^4$ (where $M(\mathbb{Z}_p, 2)$ is the *Moore Space* given by $S^2$ with an $S^3$ attached via a degree-$p$ attaching map). Thus $X$ and $Y$ have the same 3-skeleton but differ in the way their 4-cells are attached. Show that $X$ and $Y$ have isomorphic cohomology rings with $\mathbb{Z}$ coefficients but not with $\mathbb{Z}_p$ coefficients.

*Proof.* Recall that the homology of a Moore space $M(G, n)$ is

$$H_j(M(G, n)) = (\mathbb{Z}, 0, 0, \ldots, 0, n)$$

so in particular

$$H_j(M(\mathbb{Z}_p, 2)) = (\mathbb{Z}, 0, \mathbb{Z}_p).$$

Using the universal coefficient theorem we can calculate the cohomology of $M(\mathbb{Z}_p, 2)$ over $\mathbb{Z}$ as

$$H^j(M(Z_p, 2); \mathbb{Z}) = \mathrm{Hom}(H_j(M(\mathbb{Z}_p, 2)), \mathbb{Z}) \oplus \mathrm{Ext}(H_{j-1}(M(\mathbb{Z}_p, 2)), \mathbb{Z}).$$

This Ext is 0 when $H_{j-1}$ is free, i.e. for $j = 1, 2$. For $j = 3$, we have $\mathrm{Ext}(H_2, \mathbb{Z}) = \mathrm{Ext}(\mathbb{Z}_p, \mathbb{Z}) = \mathbb{Z}_p$ and $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}) = 0$, resulting in $H^3 = \mathbb{Z}_p$. Thus,

$$H^j(M(Z_p, 2); \mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}_p).$$

Now we can calculate the cohomology of $Y = M(\mathbb{Z}_p, 2) \vee S^4$:

$$H^*(Y; \mathbb{Z}) = H^*(M(\mathbb{Z}_p, 2); \mathbb{Z}) \oplus H^*(S^4; \mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}_p, \mathbb{Z}).$$

It is easy to check via cellular homology that $H^*(X; \mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}_p, \mathbb{Z})$ as well. We immediately see that the square (cup product with itself) of every ring-element in $H^*(X)$ or $H^*(Y)$ is 0, since there is no nonempty pair $H^n, H^{2n}$. Thus the rings $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$ are automatically isomorphic; one can just send generators in one to same-grade generators in the other.

Over $\mathbb{Z}_p$, things are different because $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p$ whereas before $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}) = 0$. This makes $H^2(X; \mathbb{Z}_p) = \mathbb{Z}_p$ rather than 0, and similarly for $Y$. So now there is (potentially) a nontrivial cup product $H^2 \times H^2 \to H^4$. In $Y$, the cup product of any two 2-cocycles is 0, as it is 0 in $M(\mathbb{Z}_p, 2)$ (which has no $H^4$) and the ring splits:

$$H^*(M(\mathbb{Z}_p, 2) \vee S^4; \mathbb{Z}_p) = H^*(M(\mathbb{Z}_p, 2); \mathbb{Z}_p) \oplus H^*(S^4; \mathbb{Z}_p).$$

However, in $\mathbb{C}\mathrm{P}^2$, the ring structure is $\mathbb{Z}_p[\alpha]/(\alpha^3)$, so there is a 2-cocycle class $\alpha$ with a nontrivial square, and this remains true in $X$. Thus, $H^*(X; \mathbb{Z}_p)$ and $H^*(Y; \mathbb{Z}_p)$ are non-isomorphic rings. $\square$

**Problem 6 (Hatcher 3.2:11)**: Using cup products, show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism $H_{k+\ell}(S^{k+\ell}) \to H_{k+\ell}(S^k \times S^\ell)$ if $k, \ell > 0$.

*Proof.* First we'll show the corresponding result on cohomology: Let $f : S^{k+\ell} \to S^k \times S^\ell$. Then $f$ induces maps on cohomology:

$$H^j(S^k \times S^\ell) \to H^j(S^{k+\ell}).$$

All of these maps are necessarily trivial for $j \notin \{0, k+\ell\}$. In particular we can break it into $j = k, \ell$ via the cup product. $f^*$ maps

$$H^k(S^k \times S^\ell) \times H^\ell(S^k \times S^\ell) \to H^{k+\ell}(S^k \times S^\ell)$$

to

$$H^k(S^{k+\ell}) \times H^\ell(S^{k+\ell}) \to H^{k+\ell}(S^{k+\ell})$$

and since the maps on $H^k, H^\ell$ are trivial, the map on $H^{k+\ell}$ must also be trivial.

Now we will transfer this result to homology. By the Universal Coefficient theorem,

$$H^j(S^k \times S^\ell; \mathbb{Z}) = \mathrm{Hom}(H_j(S^k \times S^\ell), \mathbb{Z}) \oplus \mathrm{Ext}(H_j(S^k \times S^\ell), \mathbb{Z}) = \mathrm{Hom}(H_j(S^k \times S^\ell), \mathbb{Z})$$

noting that $H_j(S^k \times S^\ell)$ is free, whatever $j$ is, so the Ext has to be 0. So the Hom functor takes $f^*$ to $f_*$, showing that the latter is also trivial on $H_{k+\ell}$. $\square$

**Problem 7 (Hatcher 3.2:14)**: Let $q : \mathbb{R}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ be the natural quotient map obtained by regarding both spaces as quotients of $S^\infty$. Show that the induced map

$$q^* : H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \to H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z})$$

is surjective in even dimensions by showing first by a geometric argument that the restriction $q : \mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^1$ induces a surjection on $H^2$, and then appealing to the cup product structures. Next, form a quotient space $X$ of $\mathbb{R}\mathrm{P}^\infty \coprod \mathbb{C}\mathrm{P}^n$ by identifying each point $x \in \mathbb{R}\mathrm{P}^{2n}$ with $q(x) \in \mathbb{C}\mathrm{P}^n$. Show that there are ring isomorphisms

$$H^*(X; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha^{n+1}) \quad \text{and} \quad H^*(X; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta]/(\beta^2 - \alpha^{2n+1})$$

where $|\alpha| = 2$ and $|\beta| = 2n + 1$. Make a similar construction and analysis for the quotient map $q : \mathbb{C}\mathrm{P}^\infty \to \mathbb{H}\mathrm{P}^\infty$.

*Proof.* The map $q$ specifically acts on projective points in $\mathbb{R}\mathrm{P}^\infty$ via

$$q : [a_1 : b_1 : a_2 : b_2 : \cdots] \mapsto [a_1 + ib_1 : a_2 + ib_2 : \cdots].$$

The restriction to $\mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^1$ looks like

$$[a_1 : b_1] \mapsto [a_1 + ib_1].$$

Looking at the induced map, we see that $q^* : H^2(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) \to H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z})$ is a map $q^* : \mathbb{Z} \to \mathbb{Z}_2$, thus is surjective simply because it is nontrivial. Likewise, $H^{2j}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}) = \mathbb{Z}_2$ for all $j$, so by a similar argument we can say that $q^*$ is surjective everywhere.

$X$ is the pushout of $q : \mathbb{R}\mathrm{P}^{2n} \to \mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^{2n} \hookrightarrow \mathbb{R}\mathrm{P}^\infty$:

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{2n} \arrow[r, hook] \arrow[d, "q"'] & \mathbb{R}\mathrm{P}^\infty \arrow[d] & & \mathbb{Z}[\alpha]/(2\alpha, \alpha^{n+1}) & \mathbb{Z}[\alpha]/(2\alpha) \arrow[l] \\
\mathbb{C}\mathrm{P}^n \arrow[r] & X \arrow[ur, phantom, "\xRightarrow{H^*(\bullet,\mathbb{Z})}"] & & \mathbb{Z}[\alpha]/(\alpha^{n+1}) \arrow[u, "q^*"'] & H^*(X; \mathbb{Z}) \arrow[l] \arrow[u]
\end{tikzcd}$$

The cohomologies are given by Hatcher (p. 222). Note that because $q^*$ is surjective, the generator of $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ does map to a generator of $H^2(\mathbb{R}\mathrm{P}^{2n}; \mathbb{Z})$. This yields $H^*(X; \mathbb{Z}) = \mathbb{Z}[\alpha]/(2\alpha^{n+1})$.

For $R = \mathbb{Z}_2$, $q^*$ is no longer necessarily surjective, so $H^*(X; \mathbb{Z}_2)$ has two generators now, $\alpha$ and $\beta$. Calculating the cohomology groups $H^*(X; \mathbb{Z}_2)$, we get $\mathbb{Z}_2$ in even dimensions below $2n$ and in all dimensions above $2n$. Thus we can let $\alpha$ generate the even dimensions and $\beta$ generate the odd dimensions above $2n$, so $\alpha \in H^2(X; \mathbb{Z}_2)$ and $\beta \in H^{2n+1}(X; \mathbb{Z}_2)$. But $\alpha^{2n+1}, \beta^2 \in H^{4n+2}(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and as neither is 0 they must be equal. Thus we have $H^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha, \beta]/(\alpha^{2n+1} - \beta^2)$.

Similarly if $X'$ is constructed via the quotient map to the quaternions $\mathbb{H}\mathrm{P}^\infty$, then the cohomology is $\mathbb{Z}$ in dimensions $4j$ for $j \leq n$ and in even dimensions above $4n$. One can show that $H^*(X'; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta]/(\alpha^{2n+1} - \beta^2)$ in the same way, except with $|\alpha| = 4$ and $|\beta| = 4n + 2$. $\square$

**Problem 8 (Hatcher 3.2:15)**: For a fixed coefficient field $F$, define the *Poincaré series* of a space $X$ to be the formal power series

$$p(t) = \sum_k a_k t^k$$

where $a_k$ is the dimension of $H^k(X;F)$ as a vector space over $F$, assuming this dimension is finite for all $k$. Show that $p(X \times Y) = p(X)p(Y)$. Compute the Poincaré series for $S^n$, $\mathbb{R}\mathrm{P}^n$, $\mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^\infty$, and the space in the preceding exercise.

*Proof.* First, to show that $p(X \times Y) = p(X)p(Y)$ will mean that

$$\dim(H^k(X \times Y; F)) = \sum_{j=0}^{k} \dim(H^j(X;F)) \cdot \dim(H^{k-j}(Y;F))$$

This follows from the Künneth formula (which we can always apply for $F$ a field, since $F$-modules are vector spaces and thus free), as

$$H^k(X \times Y; F) \cong \bigoplus_j H^j(X;F) \otimes_F H^{k-j}(Y;F) \implies$$

$$\dim(H^k(X\times Y;F)) = \sum_j \dim(H^j(X;F) \otimes_F H^{k-j}(Y;F)) = \sum_j \dim(H^j(X;F)) \cdot \dim(H^{k-j}(Y;F)).$$

The Poincaré series are as follows:

For $S^n$, $H^k(S^n;F)$ is $F$ for $k = 0, n$ and 0 otherwise, giving $p(t) = 1 + t^n$.

For $\mathbb{R}\mathrm{P}^n$, recall that the homology is

$$H_*(\mathbb{R}\mathrm{P}^n) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \dots)$$

ending in $\mathbb{Z}$ if $n$ is odd and 0 otherwise. Now, extending this to cohomology over $F$, by universal coefficient theorem we get

$$H^j(\mathbb{R}\mathrm{P}^n) = \mathrm{Hom}(H_j(\mathbb{R}\mathrm{P}^n), F) \oplus \mathrm{Ext}(H_{j-1}(\mathbb{R}\mathrm{P}^n), F).$$

In the case that $F = \mathbb{Z}_2$, then $\mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, but otherwise $\mathrm{Ext}(\mathbb{Z}_2, F) = F/2F = 0$ and $\mathrm{Hom}(\mathbb{Z}_2, F) = 0$. So in the $\mathbb{Z}_2$ case we get the cohomology

$$p(t) = 1 + t + t^2 + \cdots + t^n = \frac{1 - t^{n+1}}{1-t}$$

and in the general case $F \neq \mathbb{Z}_2$ we get

$$p(t) = \begin{cases} 1 + t^n & n \text{ odd} \\ 1 & n \text{ even.} \end{cases}$$

These become $1/(1-t)$ for $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ and 1 for $H^*(\mathbb{R}\mathrm{P}^\infty, F)$ with $F \neq \mathbb{Z}_2$.

For $\mathbb{C}\mathrm{P}^n$, the homology is

$$H_*(\mathbb{C}\mathrm{P}^n) = (\mathbb{Z}, 0, \mathbb{Z}, 0, \dots, \mathbb{Z}).$$

Thus by UCT, noting that $\mathrm{Ext}(\mathbb{Z}, F) = 0$ and $\mathrm{Hom}(\mathbb{Z}, F) = \mathbb{Z}$, the cohomology is the same as homology for any field $F$. This gives the Poincaré series

$$p(t) = 1 + t^2 + t^4 + \cdots + t^{2n} = \frac{1 - t^{2n+2}}{1 - t^2}.$$

Finally, let $X$ be the quotient space constructed in the previous problem. In that problem we saw that $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha^{n+1})$ and $H^*(X;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta]/(\beta^2 - \alpha^{2n+1})$. In the first case, the Poincaré series is

$$1 + t^2 + t^4 + \cdots = \frac{1}{1-t^2}$$

and in the second case it is

$$1 + t^2 + t^4 + \cdots + t^{2n} + t^{2n+1} + t^{2n+2} + \cdots = \frac{1}{1-t} - t\Big(\frac{1 - t^{2n}}{1-t^2}\Big).$$

For the $\mathbb{H}\mathrm{P}^\infty$ version, we get the series

$$p(t) = 1 + t^4 + t^8 + \cdots + t^{4n} + t^{4n+2} + \cdots = \frac{1}{1-t^2} - t^2\left(\frac{1-t^{4n}}{1-t^4}\right).$$

$\square$