SIMPLE GUIDE TO SOLVING PROBLEMS IN ALGEBRAIC TOPOLOGY

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Abstract. This is a consolidation of my algebraic topology notes. I don't intend to replicate any proofs of important statements unless they're immediate. I'm just aiming to make a dummyproof flowchart-style guide to solving Hatcher questions for my own use.

1. Basic Terminology

The most basic questions one asks in topology are

"Does there exist a map $X \to Y$ with property P?"

To be able to answer these questions positively, we need methods of constructing maps, but this is usually the simple part. To answer them negatively, we need *algebraic invariants*, properties of a space which are preserved by sufficiently nice maps.

Types of maps (in order of increasing strictness):

- Homotopy equivalence: continuous maps with continuous "inverses" *up to homotopy equivalence.*
- Homeomorphism: continuous maps with continuous inverses.

It's a lot easier to work with homotopy equivalence in practice because there's more you can do (contract simply-connected subspaces for example). The invariants of interest are all preserved by homotopy equivalence:

Algebraic Invariants:

- Fundamental group: $\pi_1(X)$
- Homology groups: $H_0(X), H_1(X), \dots$
- Cohomology groups: $H^0(X;G), H^1(X;G), \dots$
- Cohomology ring: $H^*(X;G)$ as a graded ring with $\smile$.
- Higher homotopy groups: $\pi_2(X), \pi_3(X), \dots$

First, we'll discuss how to compute all of these algebraic invariants, or at least glean information about them. We'll assume X is a connected CW complex.

2. Fundamental Group

The fundamental group of X is a group $\pi_1(X)$ whose elements are homotopy equivalence classes of loops in X, and whose multiplication is the concatenation of loops (at some base point, but if X is connected then the basepoint doesn't matter).

Quick Facts:
- If $\pi_1(X) = 0$ then we say X is "simply connected," i.e. all loops can be contracted.
- $\pi_1(X)$ is in general non-Abelian.

How to Calculate:

(I) **Van Kampen's Theorem**:
- Write X as a CW complex.
- If X has more than one 0-cell, quotient by a maximal spanning tree in X^1 so that X has a single 0-cell (this is a homotopy equivalence so it preserves $\pi_1(X)$).
- Let the 1-cells (loops) at this 0-cell be $\alpha_1, \alpha_2, \ldots, \alpha_k$. The boundaries of the 2-cells are loops in X^1, so they can each be written as products of the α_j. Let these boundaries be $\beta_1, \beta_2, \ldots, \beta_\ell \in \langle \alpha_1, \ldots, \alpha_k \rangle$.
- Now $\pi_1(X)$ is the free group on $\alpha_1, \ldots, \alpha_k$ mod the boundaries of the 2-cells:

$$\pi_1(X) := \langle \alpha_1, \ldots, \alpha_k \mid \beta_1 = \beta_2 = \cdots = \beta_\ell = 1 \rangle.$$

(II) **Covering Space**:
- bla
- bla

3. Homology

The Homology Groups of X are a sequence $H_0(X), H_1(X), \ldots$ of *Abelian groups*, so they are of the form $\mathbb{Z}^a \oplus \mathbb{Z}_{q_1}^{b_{q_1}} \oplus \mathbb{Z}_{q_2}^{b_{q_2}} \oplus \cdots$ where q_i are prime powers.

Quick Facts:

- $H_0(X) = \mathbb{Z}^a$ where a is the number of connected components of X.
- $H_1(X)$ is the Abelianization of $\pi_1(X)$.
- $H_j(X) = 0$ if X has no j-cells, in particular if $j > \dim(X)$.
- If X is a manifold of dimension n:
  - $H_n(X) = \mathbb{Z}$ if X is oriented and $H_n(X) = 0$ otherwise.
  - $H_{n-1}(X)$ has torsion subgroup 0 if X is oriented and $\mathbb{Z}_2$ otherwise.

How to Calculate:

(I) **Quotients**:

- Express X as a quotient Y/A where Y is a CW complex and A is a subcomplex. Try to choose Y and A such that $H_*(Y)$ and $H_*(A)$ are both simpler than $H_*(X)$.
- There is a long exact sequence relating $\tilde{H}_*(A), \tilde{H}_*(Y)$, and $\tilde{H}_*(X)$:

$$\cdots \to \tilde{H}_j(A) \to \tilde{H}_j(Y) \to \tilde{H}_j(Y/A) \to \tilde{H}_{j-1}(A) \to \tilde{H}_{j-1}(Y) \to \tilde{H}_{j-1}(Y/A) \to \cdots$$

- Now use $\tilde{H}_*(Y)$ and $\tilde{H}_*(A)$, along with exactness of the sequence, to deduce $\tilde{H}_*(X)$.
- *Special Cases / Notable Examples*:
  - (i) If A is contractible then $H_*(A) = 0$, so $H_*(X) = H_*(Y)$.
  - (ii) When $X = S^n$, we can express it as $X = D^n/S^{n-1}$. Because $H_*(D^n) = 0$, the exact sequence yields $\tilde{H}_j(S^n) \cong \tilde{H}_{j-1}(S^{n-1})$, so by induction we can see that $\tilde{H}_j(S^n) = \mathbb{Z}$ iff $j = n$, otherwise it is 0.

(II) **Mayer-Vietoris**:

- Express X as the union of two *open* subcomplexes $A \cup B$. Try to choose A and B so that A,B, and $A \cap B$ are all simpler than X.
- There is a long exact sequence relating $H_*(A), H_*(B), H_*(A \cap B)$, and $H_*(X)$:

$$\cdots \to H_j(A \cap B) \to H_j(A) \oplus H_j(B) \to H_j(X) \to H_{j-1}(A \cap B) \to \cdots$$

- Now use $H_*(A), H_*(B), H_*(A \cap B)$, and the exactness of the sequence to deduce $H_*(X)$.
- *Special Cases / Notable Examples*:
  - (i) If $A \cap B$ is contractible, then $H_j(X) \cong H_j(A) \oplus H_j(B)$. In particular, we see that $H_j(A \vee B) = H_j(A) \oplus H_j(B)$.
  - (ii)

(III) **Cellular Homology**:

- Use this if X is a CW complex with relatively few cells.
- For each j, let c_j be the number of j-cells in X, and let $C_j(X) \cong \mathbb{Z}^{c_j}$ be the Abelian group with generators corresponding to the j-cells of X.
- We have the long exact sequence

$$C_n(X) \xrightarrow{\mathrm{d}_n} C_{n-1}(X) \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0(X) \to 0$$

  in which d_n acts by taking cells to their boundaries, where each cell is counted *with degree*. d_n is a linear map $\mathbb{Z}^{c_n} \to \mathbb{Z}^{c_{n-1}}$, so we can see d_n as a $c_n \times c_{n-1}$ matrix.
- $H_j(X)$ is the quotient

$$H_j(X) \cong \frac{\ker(\mathrm{d}_j)}{\mathrm{im}(\mathrm{d}_{j+1})}.$$

4. Cohomology

The Cohomology of X (wrt G) is a sequence of Abelian groups $H^0(X;G), H^1(X;G), \dots$.

How to Calculate $H^*(X;G)$: Use **Universal Coefficient Theorem**:

- Use this if you know $H_*(X)$.
- The Cohomology can be determined from the Homology via the isomorphism
$$H^j(X;G) = \mathrm{Hom}(H_j(X), G) \oplus \mathrm{Ext}(H_{j-1}(X), G).$$
- To calculate Ext: each $H_k(X)$ is an Abelian group, so it has some free parts and some torsion parts. If $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_{q_1})^{b_{q_1}} \oplus (\mathbb{Z}_{q_2})^{b_{q_2}} \oplus \cdots$ then
$$\mathrm{Ext}(H_k(X), G) = (G/(q_1G))^{b_{q_1}} \oplus (G/(q_2G))^{b_{q_2}} \oplus \cdots$$
(note that the free part $\mathbb{Z}^a$ is dropped entirely).
- To calculate Hom: Hom is additive in its first argument, so if we have an Abelian homology group $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_{q_1})^{b_{q_1}} \oplus (\mathbb{Z}_{q_2})^{b_{q_2}} \oplus \cdots$ then
$$\begin{aligned}\mathrm{Hom}(H_k(X), G) &= \mathrm{Hom}(\mathbb{Z}, G)^a \oplus \mathrm{Hom}(\mathbb{Z}_{q_1}, G)^{q_1} \oplus \mathrm{Hom}(\mathbb{Z}_{q_2}, G)^{q_2} \oplus \cdots \\ &= G^a \oplus G[q_1]^{b_{q_1}} \oplus G[q_2]^{b_{q_2}} \oplus \cdots\end{aligned}$$
where $G[n]$ denotes the n-torsion subgroup
$$G[n] := \{g \in G : \underbrace{g + g + \cdots + g}_{n} = 0\}.$$

If $G = R$, a ring, then the cohomology groups $H^*(X;R)$ together form a graded ring under the cup product operation. With the additional structure of the cup product, the cohomology ring makes a strictly finer invariant than the groups alone. Two spaces might have the same cohomology groups in every dimension but different ring structures, e.g. with $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$.

How to Calculate $H^*(X;R)$ (as a ring):

(I) **Künneth Formula**:
- Use this if $X = A \times B$, and $H^*(A;R), H^*(B;R)$ are both known, and finally $H^*(B;R)$ is free in every dimension.
- There is a ring isomorphism
$$H^*(A \times B; R) \cong H^*(A;R) \otimes_R H^*(B;R).$$
- In particular this also implies that
$$H^n(A \times B; R) \cong \bigoplus_{j=0}^{n} H^j(A;R) \otimes_R H^{n-j}(B;R)$$
as groups.

(II) **Poincaré Duality**:
- Use this if X is a closed, connected R-oriented n-manifold.
- In this case $H^j(X;R) = H^{n-j}(X;R)$ for all j. We can sometimes guarantee that the generators of $H^n(X;R) = \mathbb{Z}$ factor as cup products of elements in these two groups.
- Suppose $R = \mathbb{Z}$. If $H^j(X;\mathbb{Z}) = H^{n-j}(X;\mathbb{Z})$ has positive rank, then for α a $\mathbb{Z}$-generator in $H^j(X;\mathbb{Z})$ and β a $\mathbb{Z}$-generator in $H^{n-j}(X;\mathbb{Z})$,
$$\alpha \smile \beta \text{ generates } H^n(X;\mathbb{Z}).$$
- Suppose R is a field. If $H^j(X;R) = H^{n-j}(X;R)$ are nontrivial, then for any nonzero $\alpha \in H^j(X;R)$ and $\beta \in H^j(X;R)$,
$$\alpha \smile \beta \text{ generates } H^n(X;R).$$

5. Higher Homotopy Groups

The nth homotopy group of X is denoted $\pi_n(X)$. It is like a higher-dimensional analogue of $\pi_1(X)$, where the elements are equivalence classes of maps $S^n \to X$.

Terminology:

- X is *n-connected* if $\pi_i(X) = 0$ for $i \leq n$. So, in particular, 0-connected means connected, and 1-connected means simply connected.

Quick Facts:

- $\pi_n(X)$ is Abelian for $n \geq 2$.
- *Cellular Approximation*: $\pi_j(X) = 0$ if X has no j-cells.
- *Hurewicz*: If $\pi_1(X) = 0$, the first nonzero $\pi_i(X)$ and $H_j(X)$ occur at the same dimension $i = j$, and $\pi_j(X) = H_j(X)$.
- For any collection of path-connected spaces $\{X_\alpha\}$, $\pi_n\big(\prod_\alpha X_\alpha\big) \cong \prod_\alpha \pi_n(X_\alpha)$.

How To Calculate $\pi_n(X)$:

(I) **Covering Space**:
- If $p : \tilde{X} \to X$ is a covering space of X, then
$$p_* : \pi_n(\tilde{X}) \to \pi_n(X)$$
is an isomorphism for all $n \geq 2$.
- *Special Cases*:
  - If X has a contractible covering space then $\pi_n(X) = 0$ for $n \geq 2$.
  - If X has a universal cover $\tilde{X}$ (so that $\pi_1(\tilde{X}) = 0$) then $\pi_2(X) = \pi_2(\tilde{X}) = H_2(\tilde{X})$.

(II) **Quotients / Excision**:
- Express X as a quotient Y/A, where Y is a CW complex and A a subcomplex.
- If (Y, A) is r-connected and A is s-connected, then the map
$$\pi_j(Y, A) \to \pi_j(X)$$
induced by the quotient map $Y \to X$ is an isomorphism for $j \leq r + s$ and a surjection for $j = r + s + 1$.
- The relative homology groups $\pi_i(Y, A)$ are potentially easier to compute if you have information about $\pi_*(A)$ and $\pi_*(Y)$, as we have the long exact sequence
$$\cdots \to \pi_n(A) \to \pi_n(Y) \to \pi_n(Y, A) \to \pi_{n-1}(A) \to \cdots$$
- This is in contrast to the situation with homology groups, in which this holds for all j.
- *Special Cases*:
  - If Y is r-connected and A is s-connected where $s \geq r$, then it follows from the long exact sequence that (Y, A) is r-connected so the theorem holds. If $r > s$, in particular if Y is contractible, then (Y, A) is s-connected.

6. Existence Results

How to Build a Space with Given Algebraic Properties:

- *Moore Spaces*: For any Abelian group G, there is a Moore space $M(G,n)$ for which
$$\tilde{H}_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$
- By taking wedge products of Moore spaces $M(G_1,1), M(G_2,2), \dots$ one can construct a space with any arbitrary homology sequence.
- *Eilenberg-MacLane*: For any group G, there is a space X of type $K(G,1)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = 1 \\ 0 & j > 1 \end{cases}$$
Moreover, if G is Abelian, then for any n there is a space X of type $K(G,n)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$
All $K(G,n)$ spaces are weak-homotopy equivalent, and all CW complexes of type $K(G,n)$ are fully homotopy equivalent by Whitehead's Theorem.
- *Postnikov Tower*: For any connected CW complex X, there is decreasing sequence of CW complexes $X_1, X_2, \dots$ such that
  - X is a subcomplex of each X_j,
  - The inclusion $X \hookrightarrow X_j$ induces an isomorphism on π_i for $i \leq j$,
  - $\pi_i(X_j) = 0$ for $i > j$.

How to Construct Homotopy Equivalences:

- *Whitehead's Theorem*: To show that a map $f : X \to Y$ is a homotopy equivalence, it suffices to show that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism in all dimensions.
- It also suffices to show that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism in all dimensions.

7. Worked Examples

Example 1: The Klein Bottle.

Fundamental Group: The Klein bottle can be represented as a CW complex with one 0-cell, two 1-cells, and one 2-cell as depicted below:

DIAGRAM

Letting the 1-cells be α, β, the attaching map of the 2-cell has boundary $\alpha\beta\alpha^{-1}\beta$, so by Van Kampen's Theorem we have

$$\pi_1(X) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle$$

In other words $\alpha\beta = \beta^{-1}\alpha$. Every string of α and β can be rearranged using this rule into a canonical representative $\alpha^n\beta^m$ with $n, m \in \mathbb{Z}$. These multiply by

$$\alpha^{n_1}\beta^{m_1} \cdot \alpha^{n_2}\beta^{m_2} = \alpha^{n_1+n_2}\beta^{(-1)^{n_2}m_1+m_2}$$

(each β in β^{m_1} has its sign swapped n_2 times when swapping with the α^{n_2} block). Thus we can consider $\pi_1(X)$ as $\mathbb{Z} \ltimes \mathbb{Z}$ with the action $n : m \mapsto (-1)^n m$. (revisit)

Homology: A natural way to calculate the homology of K is via cellular homology. We have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\mathrm{d}_2} \mathbb{Z}^2 \xrightarrow{\mathrm{d}_1} \mathbb{Z} \xrightarrow{\mathrm{d}_0} 0$$

where the boundary maps look like

$$\mathrm{d}_0 : \begin{pmatrix} 0 \end{pmatrix}, \quad \mathrm{d}_1 = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \mathrm{d}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

noting in d_2 that the boundary of the 2-cell is $\alpha + \beta - \alpha + \beta = 0\alpha + 2\beta$. So we have

$$\begin{aligned} H_0(K) &= \frac{\ker(\mathrm{d}_0)}{\mathrm{im}(\mathrm{d}_1)} = \frac{\mathbb{Z}}{0} = \mathbb{Z}, \\ H_1(K) &= \frac{\ker(\mathrm{d}_1)}{\mathrm{im}(\mathrm{d}_2)} = \frac{\mathbb{Z}^2}{0 \times 2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}_2, \\ H_2(K) &= \frac{\ker(\mathrm{d}_2)}{\mathrm{im}(\mathrm{d}_3)} = \frac{0}{0} = 0. \end{aligned}$$

These also could have been calculated faster by using other methods:

- $H_0(K) = \mathbb{Z}$ because K is connected.
- $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ because this is the abelianization of $\pi_1(K)$:
$$\mathrm{Ab}\left(\langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle\right) = \mathbb{Z}\langle \alpha, \beta \rangle / (2\beta) = \mathbb{Z} \oplus \mathbb{Z}_2.$$
- $H_2(K) = 0$ because $H_1(K)$ is not free, which implies K is non-orientable.

Cohomology: Let the coefficient ring be G. Having solved $H_*(K)$ already, we can use Universal Coefficient Theorem:

$$H^j(K; G) = \mathrm{Hom}(H_j(K), G) \oplus \mathrm{Ext}(H_{j-1}(K), G).$$

In the case $G = \mathbb{Z}$, we have

$$\begin{aligned} H^0(K;\mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}(0,\mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z} \\ H^1(K;\mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}_2,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z}, \\ H^2(K;\mathbb{Z}) &= \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2,\mathbb{Z}) = (0) \oplus (\mathbb{Z}_2) = \mathbb{Z}_2. \end{aligned}$$

For the ring structure, beginning with $\mathbb{Z}[x, y]$ where x generates $H^1(K;\mathbb{Z}) = \mathbb{Z}$ and y generates $H^2(K;\mathbb{Z}) = \mathbb{Z}_2$. We clearly have the relation $2y = 0$ because $H^2(K;\mathbb{Z}) = \mathbb{Z}_2$, and $xy = y^2 = 0$ because $H^3 = H^4 = 0$. The only product that could be nontrivial is x^2.

When $G = \mathbb{Z}_2$,

$$H^0(K;\mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) \oplus \mathrm{Ext}(0,\mathbb{Z}_2) = (\mathbb{Z}_2) \oplus (0) = \mathbb{Z}_2$$

$$H^1(K;\mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) = (\mathbb{Z}_2^2)\oplus(0) = \mathbb{Z}_2^2,$$
$$H^2(K;\mathbb{Z}_2) = \mathrm{Hom}(0,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2) = (0)\oplus(\mathbb{Z}_2) = \mathbb{Z}_2.$$

Now for the ring structure.

Homotopy Groups: There is a covering $\mathbb{R}^2 \to K$ as depicted below:

Diagram

Xovering induces isomorphism on homotopy groups above dimension 1, so $\pi_n(K) = 0$ for $n \geq 2$.

8. Table of Computation Results