MATH 318 HW 1

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Problem 1 (2.6): Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree d. Prove that $f^{-1}(1)$ is a (possibly empty) submanifold of dimension $n - 1$.

Proof. By Example 2.6, it suffices to show that the derivative of a homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$ is surjective on points within $f^{-1}(1)$. In this case because the dimension of the output space is 1, it is equivalent to show that the derivative is nonzero.

For f to be homogeneous of degree d means that $f(\lambda v) = \lambda^d f(v)$ for all $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. In particular, where $f(v) = 1$, $f(\lambda v) = \lambda^d f(v) = \lambda^d$, so the derivative in direction v (which is defined because $f(0) = 0$ so $0 \notin f^{-1}(1)$) at v is nonzero. In particular

$$\partial_t f(v + tv) = \partial_t f((1 + t)v) = \partial_t (1 + t)^d = d(1 + t)^{d-1} = d \text{ (at } t = 0)$$

so f has nonzero derivative at all points in $f^{-1}(1)$, thus making $f^{-1}(1)$ a submanifold of dimension $n - 1$ as desired. $\square$

Problem 2 (2.7): Show that $\mathrm{SL}_n(\mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{n^2}$ and determine its dimension. Prove also that the map $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ via $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is smooth. Do the same for $\mathrm{SO}_n(\mathbb{R})$.

Proof. $\mathrm{SL}_n(\mathbb{R})$ is the preimage $\det^{-1}(1)$ of $\det : \mathbb{R}^{n^2} \to \mathbb{R}$. The determinant is homogeneous of degree n. Thus, by the previous problem, $\mathrm{SL}_n(\mathbb{R})$ is a submanifold of dimension $n^2 - 1$.

$\tau \mapsto \tau^{-1}$ is smooth on $\mathrm{SL}_n(\mathbb{R})$ as it is given by the adjugate matrix, so each coordinate is just the determinant of one of the minors, a polynomial in the matrix entries in τ and thus smooth. Similarly, each entry of $\sigma\tau^{-1}$ is a polynomial in the entries of σ and τ and thus smooth.

$\mathrm{SO}_n(\mathbb{R})$ can be seen as the fiber $f^{-1}(I)$ where $f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ is defined $f : a \mapsto aa^\top$, with the codomain being the symmetric matrices. So by Corollary 4.3, it suffices to show that D_pf is surjective at every special orthogonal matrix p.

To do this, wlog suppose we want to show that the symmetric matrix with 1 in the (i, j) and (j, i) places and 0 elsewhere is in the image of D_pf (the diagonal case is easy). Let c_k be the kth column of p. Assume wlog that c_j is not in the direction $(1, 1, \ldots, 1)$; otherwise swap i, j and the following construction will work (note that they cannot both be in the same direction because p has determinant 1). Consider $D_p(f)$ of the vector v that perturbs the matrix by

$$p_{i1} \mapsto p_{i1} + v_1, \;\; p_{i2} \mapsto p_{i2} + v_2, \;\; \ldots \;\; p_{in} \mapsto p_{in} + v_n.$$

We can choose v to be orthogonal to c_k for $k \neq i, j$ and also orthogonal to $(1, 1, \ldots, 1)$. Due to this choice, the perturbation by v doesn't affect $c_i \cdot c_k$ for any $k \neq i, j$, nor any dot product that doesn't involve c_i. $c_i \cdot c_i$ changes by $2(v_1 + v_2 + \cdots + v_n)$, which is 0 because v to be orthogonal to $(1, 1, \ldots, 1)$. Now, because we assumed that c_j is not in the direction $(1, 1, \ldots, 1)$, and also p is orthogonal so it is not in the direction of any other c_k, $c_i \cdot c_j$ is changed by $v \cdot c_j$, which is nonzero. Thus, this (and its symmetric opposite) is the only entry that has nonzero determinant. So D_pf is surjective, as desired.

Proving that the product and inverse are smooth is the same as for $\mathrm{SL}_n(\mathbb{R})$.

As for the dimension, 4.3 gives that the codimension should be $\dim(\mathrm{Sym}_n(\mathbb{R}))$, which is $n + (n-1) + \cdots + 1 = \frac{1}{2}(n)(n+1)$, so $\dim(\mathrm{SO}_n(\mathbb{R})) = n^2 - \frac{1}{2}(n)(n+1) = \frac{1}{2}(n)(n-1)$.

□

Problem 3 (4.2): Find an embedding of $S^n \times S^m$ in $\mathbb{R}^{n+m+1}$.

Proof. Since S^n and S^m are compact, it suffices to produce an injective immersion. Take S^n and S^m to be the submanifolds of $\mathbb{R}^{n+1}, \mathbb{R}^{m+1}$ given by

$$S^n = \{x_1, \dots, x_{n+1} \ : \ x_1^2 + \cdots + x_{n+1}^2 = 1\}, \quad S^m = \{z_1, \dots, z_{m+1} \ : \ z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

Fixing some $R > 1$, define the map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^{n+m+1}$ by

$$f : (x_1, \dots, x_{n+1}, z_1, \dots, z_{m+1}) \mapsto \Big(x_1(R+z_1), \dots, x_{n+1}(R+z_1), z_2, z_3, \dots, z_{m+1}\Big).$$

f is polynomial in every coordinate and thus smooth. I claim that f is injective as a map restricted to $S^n \times S^m$.

Suppose $(x_1', \dots, x_{n+1}', z_1', \dots, z_{m+1}')$ is another point in $S^n \times S^m$ mapped to the same output by f. The first $n+1$ coordinates give

$$x_j(R+z_1) = x_j'(R+z_1') \implies \frac{x_j}{x_j'} = \frac{R+z_1'}{R+z_1} =: \lambda$$

for all $1 \le j \le n+1$. Because $R > 1$ and $|z_1|, |z_1'| \le 1$, we have $\lambda > 0$. But then

$$1 = x_1^2 + \cdots + x_{n+1}^2 = \lambda^2(x_1'^2 + \cdots + x_{n+1}'^2) = \lambda^2 \implies \lambda = 1.$$

This gives $z_1 = z_1'$ and hence $x_j = x_j'$ for all j. From the remaining m coordinates, it immediately follows that $z_j = z_j'$ for $j \ge 2$, so the two points are indeed equal. That is, f is injective on $S^n \times S^m$, and thus it is an embedding of $S^n \times S^m$ into $\mathbb{R}^{n+m+1}$.

$\square$