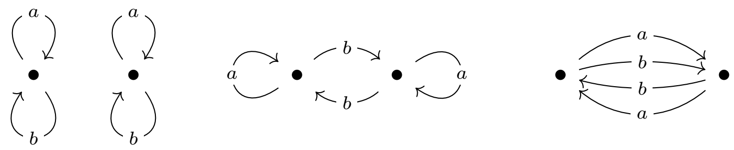
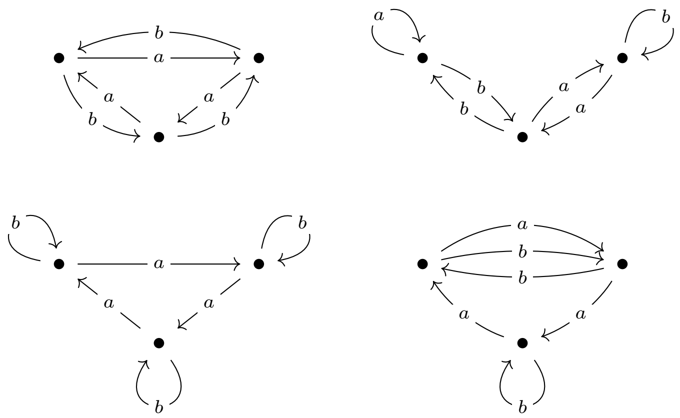
Problem 1 (Hatcher 1.3:10): Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces (without basepoints).

Proof. There are three 2-sheeted coverings



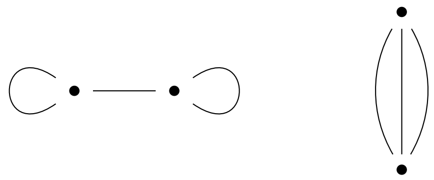
and seven 3-sheeted coverings; three of them are the 2-sheeted coverings with another copy of $S^1 \vee S^1$, and the remaining four are below:



□

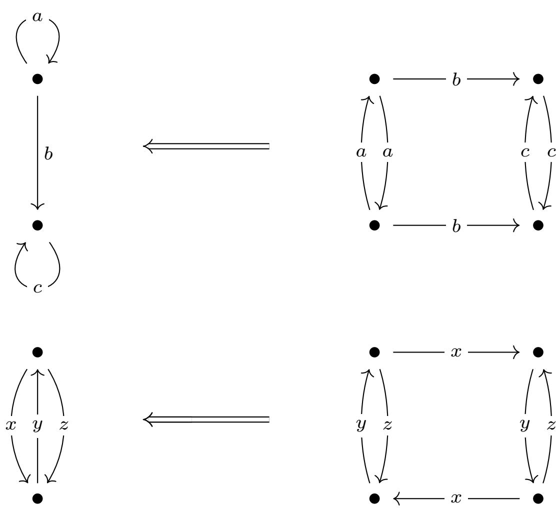
Problem 2 (Hatcher 1.3:11): Construct finite graphs X_1 and X_2 having a common finite-sheeted covering space $\tilde{X_1} = \tilde{X_2}$, but such that there is no space which X_1 and X_2 both cover.

Proof. Consider the two graphs A and B pictured below:



Both have valence 3, but they cannot both cover any other graph: suppose that both cover some graph G. Because A covers G, G has a loop. But because B covers G, there can be no loop unless G has only one vertex. If G is a single vertex, then all its edges are loops, but then it cannot have valence 3 (since 3 is odd), so this is impossible.

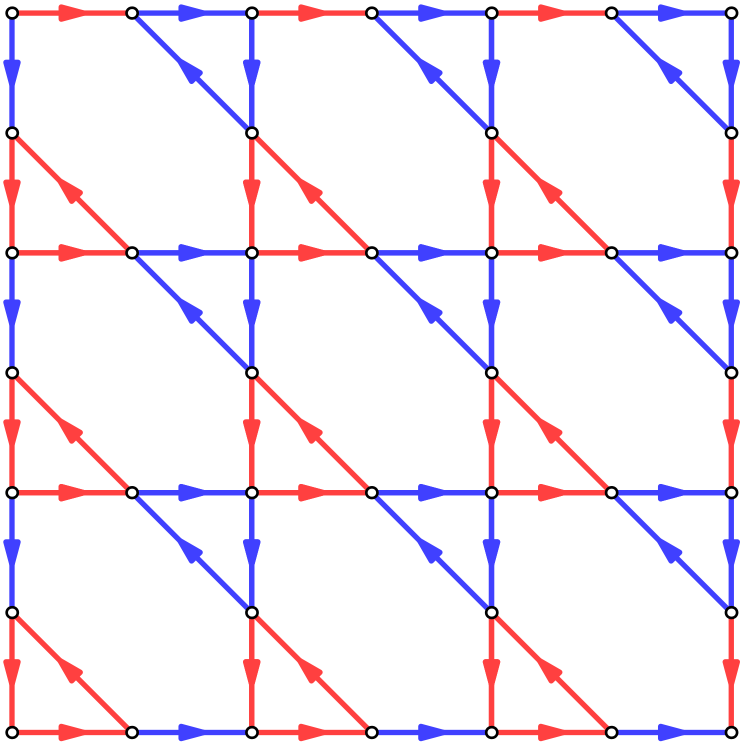
On the other hand, they can both be covered by the same graph. To see how, consider the two coverings below:



$\square$

Problem 3 (Hatcher 1.3:13): Determine the covering space of $S^1 \vee S^1$ corresponding to the subgroup of $\pi_1(S^1 \vee S^1)$ generated by the cubes of all elements. The covering space is 27-sheeted and can be drawn on a torus so that the complementary regions are nine triangles with edges labeled aaa, nine triangles with edges labeled bbb, and nine hexagons with edges $ababab$.

Proof. The answer is given below. Red denotes a, blue denotes b, and opposite edges of the square are identified (making it a torus). Note that the covering is 27-sheeted because there are 27 vertices and they all have the same valence:



The red and blue triangles indicate that a^3, b^3 are in this fundamental group, the hexagons give loops for $(ab^{-1})^3$, and the torus structure gives loops for $(ab)^3$.

In fact for any word w, w^3 is a loop: let w be some word in a, b, a^{-1}, b^{-1}. Since a^2, a^{-1} have the same start and endpoints, we can consider the case where w is just a string $a^{\pm}b^{\pm}a^{\pm}\dots b^{\pm}$. Geometrically, each ab^{-1} or $b^{-1}a$ denotes a right turn while each ba^{-1} or $a^{-1}b$ denotes a left turn, and $ab, a^{-1}b^{-1}, ba, b^{-1}a^{-1}$ are straight. Thus each word w has some "net rotation" that is either two right turns, two left turns, or straight (this is not hard to show).

In the cases where w is not straight, w^3 forms a path with threefold rotational symmetry, and is therefore a loop. When w is straight, w must have an even number of letters, so w^3 has a multiple of 6 letters, and thus it cycles around the torus to form a loop. $\square$

Problem 4 (Hatcher 1.3:18): For a space X that is path-connected, locally path-connected, and semilocally simply-connected, call a covering space $\hat{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has a 'universal' abelian covering space (i.e. one that covers every other abelian covering space of X) and it is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof. The universal abelian covering space is the one whose fundamental group is the Abelianization of $\pi_1(X, x)$. The Deck transformation group is $\pi_1(X, x)/p_*(\pi_1(\hat{X}, \hat{x}))$, and if this is Abelian then $p_*(\pi_1(\hat{X}, \hat{x}))$ must include the commutator subgroup of $\pi_1(X, x)$, and in the maximal case it must be exactly the commutator subgroup.

For $S^1 \vee S^1$, the covering space is an infinite square grid, and the deck group is the translation group $\mathbb{Z} \times \mathbb{Z}$; for $S^1 \vee S^1 \vee S^1$ it is an infinite triangle grid.

$\square$

Problem 5 (Hatcher 1.3:23): Show that if a group G acts freely (no fixed points) and properly discontinuously (i.e. every $x \in X$ has a neighborhood U with only finitely-many g s.t. $U \cap g(U) \neq \emptyset$) on a Hausdorff space X, then the action is a covering space action.

Proof. First, I claim that if G acts on X freely, then for every $g \in G$ and $x \in X$ there is some neighborhood U_g of x such that $x \notin g(U_g)$. To get this neighborhood, let V_1, V_2 be neighborhoods of $x, g(x)$ which are disjoint (X is Hausdorff) and take $U_g = V_1 \cap g^{-1}(V_2)$.

Using this fact, if U is such that only finitely many a finite subset $G' \subset G$ has $g(U) \cap U \neq \emptyset$, then take the intersection of U_g for $g \in G'$ to get a neighborhood U_G of x for which $x \notin g(U_G)$ for any $g \in G$. Then by Hausdorff again, take a neighborhood $U' \subset U_G$ of x disjoint from $g(U_G)$ for $g \in G'$, and thus for all $g \in G$. The fact that this U' exists for all x shows that G is a covering space action. $\square$

Problem 6 (Hatcher 1.3:25): Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\varphi(x, y) = (2x, y/2)$. Let $X := \mathbb{R}^2 - \{0\}$. $\mathbb{Z}$ acts on X by $n : (x, y) \mapsto \varphi^n(x, y)$. Show that this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show that the orbit space $X/\mathbb{Z}$ is non-Hausdorff and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and y-axis.

Proof. First, this is a covering space action. One can check that the open square $(\frac{1}{2}x, 2x)\times(\frac{1}{2}y, 2y)$ is disjoint from its image under φ^n for $n \neq 0$.

Knowing that this is a covering space action, it implies that $p : X \to X/\mathbb{Z}$ by $(x, y) \mapsto \varphi^{\mathbb{Z}}(x, y)$ is a normal covering. And X is both path-connected and locally path-connected, so we have

$$\mathbb{Z} = \pi_1(X/\mathbb{Z})/p_*(\pi_1(X)) = \pi_1(X/\mathbb{Z})/\mathbb{Z}$$

using the fact that $p_*(\mathbb{Z}) = \mathbb{Z}$, since φ has positive determinant and thus preserves orientation. This produces the short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(X/\mathbb{Z}) \to \mathbb{Z} \to 1$$

which, because $\pi_1(X/\mathbb{Z})$ is Abelian, is a splitting, which gives $\pi_1(X/\mathbb{Z}) = \mathbb{Z}^2$. To show that $\pi_1(X/\mathbb{Z})$ is Abelian, it suffices to give two generators and show that they commute. These generators can be the projections of paths $\alpha, \beta : [0, 1] \to X$ given by

$$\alpha(t) = (\cos(t), \sin(t)) \quad \text{and} \quad \beta(t) = (t + 1, 0)$$

(β is a loop because $(1, 0) \sim (2, 0)$ mod the action of φ). These commute up to homotopy, so the group is Abelian.

$X/\mathbb{Z}$ is not Hausdorff: consider the points $(0, 1)$ and $(1, 0)$. If U_1, U_2 are any open neighborhoods of these points in X, then in $X/\mathbb{Z}$ they correspond to $\varphi^{\mathbb{Z}}(U_1)$ and $\varphi^{\mathbb{Z}}(U_2)$. Because these neighborhoods are open, for some $N \gg 0$, U_1 contains $(2^{-N}, 1)$ and U_2 contains $(1, 2^{-N})$. But then $\varphi^{N/2}(U_1)$ and $\varphi^{-N/2}(U_2)$ both contain $(2^{-N/2}, 2^{-N/2})$. Thus, U_1, U_2 intersect in $X/\mathbb{Z}$.

Finally, for every point $(x, y) \in X$ with $x > 0$, we can find a unique representative of its equivalence class in $[1, 2] \times \mathbb{R}$, with the right and left edges identified so that $(1, y) \sim (2, y/2)$, so that this part of $X/\mathbb{Z}$ is identified with $S^1 \times \mathbb{R}$. Taking the union of these cylinders corresponding to the four half planes $x > 0, y > 0, x < 0, y < 0$ gives us the four components.

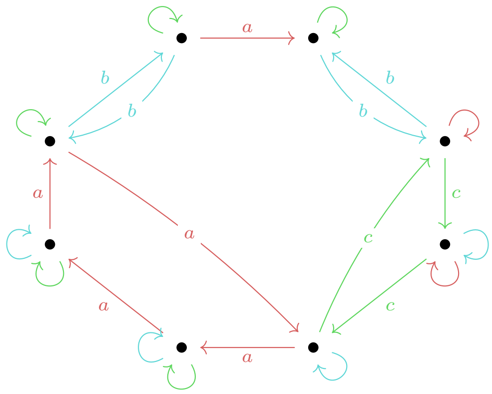
□

Problem 7 (Hatcher 1.A:6): Let F be the free group on two generators and let F' be its commutator subgroup. Find a set of free generators of F' by considering the covering space of the graph corresponding to F'.

Proof. One set of free generators is the commutators of the form $a^n b^m a^{-n} b^{-m}$ for $n, m \in \mathbb{N}$. The commutator subgroup is the fundamental group of the infinite square grid with a on one axis and b on the other. To find its generators, we can quotient by a maximal spanning tree. One such tree is the one consisting of the vertical line $x = 0$ and the horizontal lines $y = n$ for $n \in \mathbb{Z}$. Upon taking the quotient, we get a wedge of infinitely-many loops, each of which is represented by an edge not present in the tree. These are exactly $a^n b^m a^{-n} b^{-m}$ for $n, m \in \mathbb{N}$. $\square$

Problem 8 (Hatcher 1.A:13): Let x be a nontrivial element of a finitely-generated free group F. Show that there is a finite-index subgroup $H \subset F$ in which x is part of a basis.

Proof. Let G be a graph whose fundamental group is F (i.e. a wedge of finitely-many loops). x is a product of these loops in some order. We can make a finite-sheeted covering space $\hat{G}$ which has x as a loop; an example is shown for $x = ab^{-1}ccaaab$, but the same method will work in general:



The fundamental group $p_*(\pi_1(\hat{G})) \subset G$ will be a finite index subgroup (because the covering has finitely-many sheets) and has x as part of a basis because it is a loop in $\hat{G}$ (one can take a maximal spanning tree consisting of all but one edge in the loop representing x). $\square$