MATH 317 HW 7

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Problem 1 (Hatcher 4.1.5): For a pair (X, A) of path-connected spaces, show that $\pi_1(X, A, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, x_0)$ represented by loops in A at x_0.

Proof. By definition, $\pi_1(X, A, x_0)$ is the group of (homotopy classes of) maps $I \to X$ beginning at x_0 and ending in A. In each of those classes there is a loop at x_0, since A is path connected and paths can be extended by homotopy.

If α is a loop at x_0 in X, then everything in αH is homotopic to α (do later) $\square$

Problem 2 (Hatcher 4.1.8): Show that the sequence

$$\pi_1(X, x_0) \to \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \to \pi_0(X, x_0)$$

is exact.

Problem 3 (Hatcher 4.1.15): Show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity map by the following steps:

(a) Use Lemma 4.10 to reduce to the case that there is a point $q \in S^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible linear map near each p_i.
(b) For f as in (a), consider the composition gf where $g : S^n \to S^n$ collapses the complement of a small ball about q to the basepoint. Use this to reduce (a) further to the case $k = 1$.
(c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection.

Proof. (a): Lemma 4.10 says that f is homotopic to a map $f_1 : S^n \to S^n$ which is piecewise linear on some polyhedron $K \subset S^n$.

(b): Supposing the result is true for $k = 1$,

(c): The space of $n \times n$ matrices with positive determinant is path-connected, as a subspace of $\mathbb{R}^{n^2}$. One way to see this is that every matrix $a \in M_n(\mathbb{R})$ has a well defined logarithm

$$\log(1 + a) = a + a^2/2 + a^3/3 + a^4/4 + \cdots$$

and exponential

$$e^a = I + a + a^2/2 + a^3/6 + a^4/24 + \cdots$$

so one can let $x = \log(a)$ and take the path $\gamma(t) = e^{tx}$. Then $\gamma(0) = I$, $\gamma(1) = e^x = a$, and $e^{tx} \in \mathrm{GL}_n(\mathbb{R})$ because it has inverse e^{-tx}. $\square$

Problem 4 (Hatcher 4.1.17): Show that if X and Y are CW complexes with X m-connected and Y n-connected, then $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected, as is the smash product $X \wedge Y$.

Proof. We have the long exact sequence

$$\cdots \to \pi_j(X \vee Y) \to \pi_j(X \times Y) \to \pi_j(X \times Y, X \vee Y) \to \pi_{j-1}(X \vee Y) \to \cdots$$

By excision for homotopy groups, $\pi_j(X) \cong \pi_j(X \vee Y,)$

$\square$

Problem 5 (Hatcher 4.2.1): Use homotopy groups to show that there is no retraction $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ for $n > k > 0$.

Problem 6 (Hatcher 4.2.5): Let $f : S^2_\alpha \vee S^2_\beta \to S^2_\alpha \to S^2_\beta$ be the map which is the identity on S^2_α, and the sum of the identity and a map $S^2_\beta \to S^2_\alpha$ on S^2_β. Let X be the mapping torus of f, i.e.

$$X := \frac{(S^2_\alpha \vee S^2_\beta) \times I}{(x, 0) \sim (f(x), 1)}$$

The mapping torus of the restriction of f to S^2_α forms a subspace $A = S^1 \times S^2_\alpha \subset X$. Show that the maps $\pi_2(A) \to \pi_2(X) \to \pi_2(X, A)$ form a short exact sequence

$$0 \to \underbrace{\pi_2(A)}_{\mathbb{Z}} \to \underbrace{\pi_2(X)}_{\mathbb{Z}^2} \to \underbrace{\pi_2(X, A)}_{\mathbb{Z}} \to 0$$

and compute the action of $\pi_1(A)$ on these three groups. In particular show that the action is trivial on $\pi_2(A)$ and $\pi_2(X, A)$ but nontrivial on $\pi_2(X)$.

Problem 7 (Hatcher 4.2.10): Let X be the CW complex obtained from $S^2 \vee S^n$ (where $n \geq 2$) by attaching e^{n+1} by a map representing the polynomial $p(t) = \mathbb{Z}[t, t^{-1}] \cong \pi_n(S^1 \vee S^n)$, so that $\pi_n(X) \cong \mathbb{Z}[t, t^{-1}]/(p(t))$. Show that $\pi'_n(X)$ is cyclic and compute its order in terms of $p(t)$. Give examples showing that the group $\pi_n(X)$ can be finitely-generated or not, independently of whether $\pi'_n(X)$ is finite or infinite.

Problem 8 (Hatcher 4.2.13): Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leq n$.