MATH 317 HW 1

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Problem 1 (Hatcher 0.4): A deformation retraction "in the weak sense" of $X \to A$ is a homotopy $f_t : X \to X$ such that f_0 is the identity map on X, f_1 is a map $X \to A$, and $f_t(A) \subseteq A$ for all $t \in [0, 1]$ (this is weaker because it does not require f_t be *constant* on A). Show that if such a map exists, then the inclusion $\iota : A \to X$ is a homotopy equivalence.

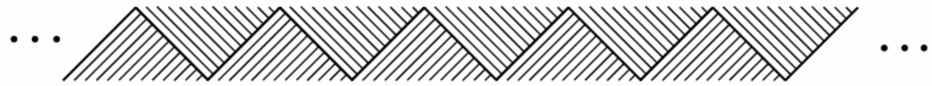
Proof. To show that ι is a homotopy equivalence it is sufficient and necessary to produce an inverse up to homotopy. This inverse will be f_1. The composition $\iota \circ f_1 : X \to X$ is homotopic to id_X via f_t. $f_1 \circ \iota : A \to A$ is also not necessarily id_A (as it would be if f_t were a deformation retraction), but is homotopic to id_A via $f_t|_A$ (here we need the fact that f_t is a weak deformation retraction, which makes $f_t|_A$ a map $A \to A$). $\square$

Problem 2 (Hatcher 0.6):

(a) Let X be the subspace of $\mathbb{R}^2$ consisting defined by
$$X := \Big([0,1]\times\{0\}\Big) \cup \bigcup_{q\in\mathbb{Q}[0,1]} \{q\}\times[0,1-q].$$
Show that X deformation retracts to any point in $[0,1]\times\{0\}$, but not to other points in X.

(b) Let Y be the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show that there is a deformation retraction $Y \to Z$ in the weak sense, but not in the regular sense.



Proof. (a): For any point $r \in [0,1]$, we can construct a deformation retraction from X to $\{r\}\times\{0\}$ as follows: let $d_t : X \to X$ be given by
$$d_t(q,h) = \begin{cases} (q, h(1-2t)) & t\in[0,\frac{1}{2}] \\ ((2t-1)r + (2-2t)q, 0) & t \in [\frac{1}{2},1] \end{cases}$$
It is easy to check that d_t is continuous and that $d_t((r,0)) = (r,0)$ for all $t\in[0,1]$.

On the other hand, every point $x := (q,h)$ with $h \neq 0$ has the property that there is an ε (namely $\varepsilon = h/2$) such that within every neighborhood of x there are points y such that there are no paths between y and x within $B_x(\varepsilon)$. Any point with this property cannot be the target of a deformation retraction:

If there was such a deformation retraction, then for every point y there is a closed time interval T_y for which $d_t(y) \notin B_x(\varepsilon)$ for $t \in T_y$. Let $y_1, y_2, \dots$ be a sequence in X approaching x. By compactness of $[0,1]$, there is a time t such that $t \in T_{y_j}$ for infinitely many y_j. This shows d_t is not continuous in t at x, since $|d_t(x) - d_t(y)| > \varepsilon$ for arbitrarily small $|x-y|$. Thus such a deformation retraction cannot exist.

(b): The difference between being contractible to a point x and being *deformation retractable* to x is that in the former case, the homotopy need not keep x fixed. Y is contractible because it is a 1-dimensional CW complex with no cycles (i.e. a tree). Yet it is not deformation retractable to any point because every point of Y has the property described in the second part of (a).

(c): Here is a somewhat vague description of a weak deformation retraction to Z: for every point $y \in Y$, there is a unique always-rightward path in Y beginning at y. We can let every point simultaneously walk along its path at the same constant rate. As a result, the "bristles" move together with the parallel section of Z, making the map continuous. At time $t = 1$, the bristles will have all rejoined Z, making this a weak deformation retraction. But all points in Z also moved, so it is not a true deformation retraction.

And moreover, we know that there can be no true deformation retraction because if there was, then we could compose it with a deformation retraction of Z to a point, and get that Y deformation retracts to a point, which we showed was impossible. ∎

Problem 3 (Hatcher 0.16): Show that S^∞ is contractible.

Proof. Let x be the single point in the 0-skeleton of S^∞. Let S^n be the n-skeleton of S^∞, i.e. the n-sphere. For $n \geq 2$, S^n is contractible, as we've seen. Now, every loop α at x in S^∞ lies entirely in S^n for some n. This follows because the open sets $\alpha^{-1}(S^n)$ cover $[0,1]$, hence by compactness there is a finite subcover which has some maximum n. For this n, α is contained entirely in S^n, thus it is contractible. This shows that all loops at x in S^∞ are contractible, thus S^∞ is contractible. $\square$

Problem 4 (Hatcher 0.20): Show that the space $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Proof. The disk-shaped piece where the Klein-bottle intersects itself is contractible, thus we can collapse it to a point p without changing the homotopy type of X. We now have a 2-cell with three points identified (those at p), which is homotopy-equivalent to a sphere with a three-pronged fork sticking into it. By collapsing the two paths connecting the three points of this fork, the fork becomes two loops. Thus we are left with $S^2 \vee S^1 \vee S^1$ as desired. □

Problem 5 (Hatcher 0.23): Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Proof. Let X, Y be two contractible CW-complexes with intersection Z. Since Z is contractible and a subcomplex of both X and Y, we can take the quotients $X' = X/Z$ and $Y' = Y/Z$. Note that $X' \simeq X$ and $Y' \simeq Y$, so both are contractible. Now

$$X \cup Y \simeq (X \cup Y)/Z \simeq X' \vee Y' \simeq \{*\} \vee \{*\} \simeq \{*\}$$

so $X \cup Y$ is contractible as well. $\square$

Problem 6 (Hatcher 1.1.6): Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ *without* specifying a basepoint. There is a natural inclusion $\Phi : \pi_1(X, x) \to [S^1, X]$. Assuming X is path-connected, show that Φ is surjective and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x)$.

Proof. Given any loop α in X, let $a = \alpha(0) = \alpha(1)$. Because X is path-connected, we have a path γ connecting x with a. Then the path $\gamma \cdot \alpha \cdot \gamma^{-1}$ is a loop at x. And $\Phi(\gamma\alpha\gamma^{-1}) = [\alpha]$ because the γ part of the path can be smoothly contracted to the constant loop at a, leaving α. Thus Φ is surjective.

If $[f]$ and $[g]$ are conjugate in $\pi_1(X, x)$, i.e. if there is some loop $\gamma \in \pi_1(X, x)$ such that $\gamma f \gamma^{-1} \simeq g$, then $\Phi([f]) = \Phi([\gamma f \gamma^{-1}]) = \Phi([g])$. Conversely, if $\Phi([f]) = \Phi([g])$, then there is a homotopy taking f to g so $[f] = [g]$. □

Problem 7 (Hatcher 1.1.13): Given a space X and a path-connected subspace A containing the basepoint x, show that the map $\pi_1(A,x) \to \pi_1(X,x)$ induced by the inclusion is surjective iff every path in X with endpoints in A is homotopic to a path entirely in A.

Proof. Suppose that every path in X with endpoints in A is homotopic to one in A. Let $\alpha \in \pi_1(X,x)$. Then by the hypothesis, α is homotopic to a loop in A (but not necessarily based at x) so let this loop be β with basepoint b. Then because A is path-connected, we have a path γ connecting x to b, and $\gamma \cdot \beta \cdot \gamma^{-1}$ is a path in $\pi_1(A,x)$ homotopic to α.

Conversely, suppose that the inclusion is surjective and let γ be a path in X with endpoints $p, q \in A$. Because A is path-connected, there are paths α_1 from x to p and α_2 from x to q. Thus we have a loop $\alpha_1 \cdot \gamma \cdot \alpha_2^{-1} \in \pi_1(X,x)$. By hypothesis, there is a loop $\beta \in \pi_1(A,x)$ homotopic to this loop. The section of β corresponding to γ in this homotopy is a path in A homotopic to γ. $\square$

Problem 8 (Hatcher 1.1.20): Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for $x \in X$, the loop $f_t(x)$ is in the center of $\pi_1(X, x)$.

Proof. Let β be the loop $f_t(x)$. Lemma 1.19 says that for any loop α at x, $\beta f_1(\alpha)\beta^{-1} \simeq f_0(\alpha)$. In this case, since $f_1(\alpha) = f_0(\alpha) = 1$, this implies that $\beta\alpha\beta^{-1} = \alpha$, i.e. β commutes with α. Since α was arbitrary, this shows that β commutes with all of $\pi_1(X, x)$, and thus it is in the center. $\square$