MATH 325 HW 6

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Problem 1: Let A be Banach algebra. Prove that for sufficiently small $a \in A$, one has

$$\log(\exp(a)) = a, \quad \exp(\log(1-a)) = 1-a.$$

Proof. Recall the power series definitions of exp and log:

$$\exp(a) = 1 + a + \tfrac{1}{2}a^2 + \cdots \qquad \log(1+b) = b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \cdots$$

We also know by Cauchy-Hadamard that $\log(1+b)$ converges for $|b| < 1$, and $\exp(a)$ converges everywhere. Using these we can check that $\log'(1+b) = (1+b)^{-1}$ and $\exp'(a) = \exp(a)$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(\exp(ta)) = \frac{1}{\exp(ta)} \cdot a\exp(ta) = a \quad \Longrightarrow \log(\exp(ta)) = ta + c$$

for some $c \in A$, but when $t = 0$ we clearly get $\log(\exp(0)) = 0$, implying $c = 0$. The other identity follows similarly.

Or alternatively one could expand out the composed series and compute all of the coefficients.
□

Problem 2: Let $A^\times$ be the group of invertible elements of A, where A is a Banach algebra. Show that any continuous group homomorphism $f : (\mathbb{R}, +) \to A^\times$ has the form $t \mapsto e^{ta}$ for some particular $a \in A$.

Proof. By Problem 9 on the previous homework, any map $g : \mathbb{R} \to \mathbb{R}^r$ is $t \mapsto tv$ for some fixed $v \in \mathbb{R}^r$. The proof naturally extends to any algebra in place of $\mathbb{R}^r$. To get such a map, compose f with log, giving

$$(\mathbb{R}, +) \xrightarrow{f} A^\times \xrightarrow{\log} (A, +)$$

The composite map g has the form $g : t \mapsto ta$ where $a \in A$. Since log is locally invertible near 1, we have $f(t) = \exp(g(t)) = e^{ta}$ for small t. And f is a continuous group homomorphism, so if it agrees with e^{ta} on an open neighborhood of 0 it agrees everywhere. $\square$

Problem 3: Let A be a Banach algebra. Prove that for $a, b \in A$, for sufficiently large $n \in \mathbb{N}$ one has

$$e^{a/n} \cdot e^{b/n} = e^{\frac{1}{n}(a+b+\alpha_n)} \quad \text{and} \quad e^{a/n} \cdot e^{b/n} \cdot e^{-\frac{1}{n}(a+b)} = e^{\frac{1}{n^2}(\frac{1}{2}(ab-ba)+\beta_n)}$$

where $\alpha_n, \beta_n \in A$ are sequences converging to 0 in A.

Proof. Let

$$f_n(x) := e^{a/n}e^{b/n} - e^{(a+b+x)/n} = -x/n + o(n^2)$$

As $n \to \infty$, the $-x/n$ term dominates all others. Let B_ε be the ball pf radius ε in A. For sufficiently large n, $f_n(B_\varepsilon)$ contains 0, and thus there is some solution $f_n(x) = 0$ with B_ε. Let α_n be the smallest solution to f_n for each n sufficiently large. We've shown that $|\alpha_n|$ will eventually be below ε, so $\alpha_n \to 0$.

Similarly, we have

$$g_n(x) := e^{a/n}e^{b/n}e^{-\frac{1}{n}(a+b)} - e^{\frac{1}{n^2}(\frac{1}{2}(ab-ba)+x)} = -x/n^2 + o(n^3)$$

and the same argument applies to produce β_n. ∎

Problem 4: Find the Lie algebras of $\mathrm{O}_n(\mathbb{R})$ and U_n.

Proof. If $a \in \mathrm{Lie}(\mathrm{O}_n(\mathbb{R}))$ then $e^{ta} \in \mathrm{O}_n(\mathbb{R})$, or equivalently $e^{ta}(e^{ta})^\top = 1$. So

$$e^{ta^\top} = (e^{ta})^{-1} = e^{-ta}.$$

Now, since exp is invertible near 1, this gives

$$ta^\top = -ta \text{ for small } t \iff a^\top + a = 0.$$

That is, a is a *skew-symmetric* matrix, i.e. its cross-diagonal terms sum to 0 and its diagonal is all 0. So $\mathrm{Lie}(\mathrm{O}_n(\mathbb{R}))$ is the algebra of such matrices.

Similarly in the case of U_n we have $a \in \mathrm{Lie}(U_n)$ iff $e^{ta}e^{t\overline{a}^\top} = 1$, which implies $a + \overline{a}^\top = 0$. So $\mathrm{Lie}(U_n)$ is the space of conjugate-skew-symmetric matrices. $\square$

Problem 5: Let $T = (S^1)^r$. Show that

(a) Any finite-dimensional *complex* representation of T is a direct sum of 1-dimensional representations.

(b) Any 1-dimensional continuous representation $\rho : T \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ has the form

$$\rho(e^{i\theta_1}, \dots, e^{i\theta_r}) = e^{i(m_1\theta_1 + \cdots + m_r\theta_r)}$$

for some $m_1, \dots, m_r \in \mathbb{Z}$.

Proof. (a): The Torus has commutative multiplication, so multiplication by any group element is an intertwining operator and thus by Schur's lemma, within any irreducible subrepresentation (using the fact that this is a *complex* representation) $\rho(g)$ must be scaling by some complex number. This implies that every irrep is one-dimensional, since scaling preserves all subspaces. Moreover, T is a compact group so it is completely reducible, thus any representation is the sum of irreps, each of which is finite-dimensional.

(b): As ρ is multiplicative, it is determined by its behavior on points of T with all but one θ_j equal to 0. Also $(e^{i\theta})^{2\pi/\theta} = 1$ so $\rho(e^{i\theta})^{2\pi/\theta} = 1$ as well. Thus $\rho(e^{i\theta}) = e^{i\varphi}$ for some φ, and in particular

$$(e^{i\varphi})^{2\pi/\theta} = 1 \implies \varphi \cdot 2\pi/\theta = 2\pi \cdot m \implies \varphi = m\theta$$

for some integer m. The claim follows. $\square$

Problem 6: Let G be a connected Lie group. Prove the following are equivalent:

(1) G is commutative.
(2) Lie(G) is Abelian in the sense that $[a, b] = 0$.
(3) For some $m, n \geq 0$ such that $m+n = \dim(\mathrm{Lie}(G))$, there is an isomorphism $G \cong \mathbb{R}^m \times (S^1)^n$ of topological groups.

Proof. (2) $\implies$ (1): Since G is connected, each $g \in G$ can be written $g = e^{x_1} e^{x_2} \cdots e^{x_n}$ for $x_1, \ldots, x_n \in \mathrm{Lie}(G)$. Let $h \in G$ be written similarly as $y = e^{y_1} \cdots e^{y_m}$. Then using (2), every pair x_i, y_j commute, so powers of x_i and y_j commute, and thus e^{x_i}, e^{y_j} also commute. Thus, we can commute all of the exponentials past each other to yield $gh = hg$.

(1) $\implies$ (2): We can define $[x, y]$ in terms of a function $\mathbb{R} \to M_n(\mathbb{R})$:

$$[x, y] = xy - yx = \partial_t\big|_{t=0} e^{tx} e^y - e^y e^{tx}.$$

It is easy to confirm this by expanding the power series and multiplying the first few terms. Since G is commutative and $e^{tx}, e^y \in G$, $e^{tx} e^y - e^y e^{tx} = 0$ for all t. Thus, $[x, y] = 0$ as well.

(2) $\implies$ (3): Every $g \in G$ can be written as a product of exponentials

$$g = e^{a_1} e^{a_2} \cdots e^{a_k}$$

for $a_i \in \mathrm{Lie}(G)$. By (2), all of these a_i commute, so

$$g = e^{a_1 + \cdots + a_k}$$

That is to say that $\exp : \mathrm{Lie}(G) \to G$ is surjective. It may not be injective though. Let Lie(G) have basis $x_1, \ldots, x_m, y_1, \ldots, y_n$ where x_j have e^{tx_j} injective in t, and e^{ty_j} is non-injective with t_j minimal such that $e^{t_j y_j} = 1$. In fact, e^{ty_j} is periodic with period t_j, as

$$e^{(t+t_j)y_j} = e^{ty_j + t_j y_j} = e^{ty_j} e^{t_j y_j} = e^{ty_j}.$$

So exp puts G in bijection with the quotient

$$\mathbb{R}\langle x_1, \ldots, x_m \rangle \times \mathbb{R}\langle y_1, \ldots, y_n \rangle / (t_1 y_1, t_2 y_2, \ldots, t_n y_n) \cong \mathbb{R}^m \times (S^1)^n.$$

And this is a group isomorphism, as exp is continuous and preserves the group operations.

(3) $\implies$ (1): If $G \cong \mathbb{R}^m \times (S^1)^n$, then since multiplication is commutative in $\mathbb{R}^m$ and $(S^1)^n$, it is also commutative in $\mathbb{R}^m \times (S^1)^n$, and hence in G.

$\square$