TOPOLOGY NOTES

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Abstract. These are my notes for Topology I-II-III (Math 317-319) at UChicago, 2025-2026.

Contents

Topology I with Danny Calegari

This is an Algebraic Topology course.

Housekeeping:

- HW due Thursday midnight.
- Take-home midterm and final will replace HW.
- Textbook: Hatcher.
- Collaboration is encouraged on homework (but give credit where it is due).
- Grades will be roughly 50% homework 50% exams, with some generous weighting.
- Office Hours: Thursday 5-6 p.m. in Eckhart E7 (basement).

0.1. **Homotopy.**

Rather than equivalence by homeomorphism, which is "too fine to be useful," we'll use the coarser equivalence of homotopy.

We'll also be looking at a lot of computable information about topological spaces.

Suppose $f_0, f_1 : X \to Y$ are two (continuous) maps between topological spaces X and Y. We say f_0, f_1 are *homotopic* if one can be continuously turned into the other, i.e. if there is a continuous map

$$F : [0,1] \to \mathrm{Hom}(X, Y)$$

for which $F(0) = f_0, F(1) = f_1$. Such an F is a homotopy. We write $f_0 \simeq f_1$.

Two spaces X and Y are *homotopy-equivalent* if there is a map $f : X \to Y$ that is an isomorphism "up to homotopy," i.e. there is a map $g : Y \to X$ for which $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Homotopy equivalence is indeed an equivalence relation (not too hard to show). Equivalence of maps is also stable under composition, which makes homotopy classes of spaces and maps a category.

If $f_0, f_1 : X \to Y$ and $A \subseteq X$ is a subset on which f_0 and f_1 agree, and additionally there is a homotopy F which transforms f_0 into f_1 while remaining constant on A, then we say $f_0 \simeq f_1$ relative to A.

We say that a space X is *contractible* if it is homotopy-equivalent to a single point. For example, $\mathbb{R}^n$ is contractible, as constant maps on $\mathbb{R}^n$ are homotopic with the identity map by straight-line contraction.

Another example: given $f : X \to Y$, there is a *mapping cylinder* M_f which is $X \times [0,1] \coprod Y$ under the gluing equivalence $(x, 1) \sim f(x)$. Then $M_f \simeq Y$ via the maps

$$h_0 : (x,t) \mapsto f(x), \quad h_1 : y \mapsto y$$

The thing that must be checked is that $h_1 \circ h_0 : M_f \to M_f$ is homotopy-equivalent to the identity on M_f. This is an example of *deformation retraction*, which means that it is a homotopy relative to Y.

0.2. **CW Complexes.**

General topology is difficult to say much about because of all the pathological cases. So we'll focus mainly on *nice* topological spaces, and in particular *CW-complexes*.

A *CW-complex* is built from cells of different dimensions and attaching maps. Each cell is a pair (D^n, S^{n-1}) consisting of a ball and its surface. We build up the complex by a "skeleton" $X_0 \subseteq X_1 \subseteq \dots$ where X_n consists of all the cells of dimension at most n and their gluing instructions. The attaching map φ for a cell maps its boundary S^{n-1} into X^{n-1}.

The topology on a CW-complex is the *weak topology* (no relation to functional analysis) which says that A is open iff $A \cap X^n$ is open for all n.

Examples:

- A 0-dimensional CW-complex is just a collection of discrete points.
- A 1-dimensional CW-complex is essentially a graph (with possibly loops and multiple edges).
- Klein bottle, torus, two-holed torus etc. all have presentations as 2-dim CW complexes.

- One can write $\mathbb{CP}^n$ as the union of a 0-cell, a 2-cell, a 4-cell, ..., and a $2n$-cell, where gluing takes the boundary of each to the infinite line of the previous.

Some operations on CW complexes:

- *Product*: $X \times Y$ is given by the union of all products of a cell in X and a cell in Y. Its topology as a CW-complex (i.e. the weak topology) is the same as the product topology in cases where there are only a *countable* number of cells in each or if one is locally compact, but in general the topology is actually finer.
- *Quotient*: X/A, where A is a *subcomplex* of X (i.e. a closed union of cells in X) that is also *contractible*, is given by the union of cells in $X - A$ plus an additional 0-cell representing the image of all cells in A. Such a pair (X, A) is called a CW pair.
- *Suspension*: SX is $X \times [0, 1]$ where $(X, 0)$ is identified and $(X, 1)$ is identified.
- *Cone*: CX is $X \times [0, 1]$ where $(X, 1)$ is identified.
- *Join*: $X * Y$ is the space $X \times I \times Y$ quotiented such that all $(x, 0, Y)$ are identified and all $(X, 1, y)$ are identified. In the case $X = Y = [0, 1]$, the resulting $X * Y$ looks like a tetrahedron. One can think of the points of $X * Y$ as pairs $(x, y) \in X \times Y$ along with a weight $t \in [0, 1]$, such that $(x, y, 0) = x$ and $(x, y, 1) = y$.
- *Wedge*: $X \vee Y$ is $X \coprod Y$ with two specific points x and y identified.
- *Smash*: $X \wedge Y$ is $X \times Y$ with $X \vee Y$ all identified.

An important example of a CW complex obtained this way is the n-simplex, which is the join of n discrete points.

One thing to note about the quotient is that $X/A \simeq X$.

A CW-complex X is connected (and path-connected) iff X^1 is a connected graph. Thus, if X is connected then we can give a spanning tree T of its 1-skeleton X^1. Every tree is contractible, thus one can take the quotient $X/T \simeq X$.

Moreover, the quotient has a very simple structure in its low-dimension cells: $Y := X/T$ has Y^0 a single point and Y^1 a wedge of some circles. So we've shown that one can always put a connected CW-complex into this nice form while preserving its homotopy class.

If (X, A) is a CW pair and $f : A \to Y$ is some map into another CW complex (or any topological space), then one can form the space

$$X \cup_f Y := X \times Y/(a \sim f(a)).$$

And if $f, g : A \to Y$ are two homotopy-equivalent maps, then $X \cup_f Y \simeq X \cup_g Y$. This shows in particular that in the construction of CW complexes, the homotopy-type of the complex only depends on the homotopy-classes of the attaching maps.

Both of these facts can be deduced from the *Homotopy Extension Property* for CW-pairs (try this!). (X, A) has the HEP if for all spaces Y, every map $f : X \times 0 \cup A \times I \to Y$ factors through the inclusion into $X \times I$:

$$\begin{tikzcd}
X \times I \arrow[drr, dashed, "\exists g"] & & \\
X \times 0 \cup A \times I \arrow[u, hook] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

That is, a partial homotopy $f : A \to Y$ can always be extended to a homotopy $g : X \to Y$, hence the name. The HEP is equivalent to the specific case for f the identity map on $X \times 0 \cup A \times I$. Thus, to prove the HEP for CW-pairs, it suffices to show the following:

Proposition: If (X, A) is a CW pair then there is a retraction from $X \times I$ to $X \times 0 \cup A \times I$.

Proof. If X has dimension n, then $X = X^n$. We will produce by a series of retractions:

$$X \times I = X^n \times I \cup A \times I \to (X \cup 0) \times (X^{n-1} \times I \cup A \times I) \to (X \cup 0) \times (X^{n-2} \times I \cup A \times I) \to \dots$$

In each step we only need to retract every j-cell onto its boundary. We can do this because it has an *open side*. (check Hatcher to get the details straight later). □

0.3. **The Fundamental Group.** Let X be a space. A path f in X is a map $I \to X$. A homotopy between paths f, g is a homotopy (in the sense defined before) which fixes the endpoints of the paths (so it must be that $f(0) = g(0)$ and $f(1) = g(1)$ for this to be possible). We say that f, g are homotopy-equivalent if one exists.

Two paths can be composed (concatenated) if the end point of one is the start point of the other. This is denoted $f * g$, and corresponds to a path which does f from $[0, \frac{1}{2}]$ and then does g from $[\frac{1}{2}, 1]$. If f and g are both *loops* with $f \simeq f'$ and $g \simeq g'$, then

$$f * g \simeq f' * g'.$$

This can be proven by drawing a picture. Basically the homotopies $f \to f'$ and $g \to g'$ can be concatenated.

The *fundamental group* of X, denoted $\pi(X, x)$, is made up of homotopy-classes of loops beginning and ending at $x \in X$. The operation is concatenation. The identity is given by the constant map and the inverse is given by $f^{-1}(t) := f(1-t)$. We can check that this is a genuine inverse by drawing a picture.

We also have to check that $*$ is associative, i.e. $f * (g * h) \simeq (f * g) * h$. This can also be shown by a simple picture (we're essentially just changing the rate of movement along the image of the path in different segments).

If $\pi(X, x)$ is trivial, we say X is *simply connected* (note that this does not depend on x). In general, the fundamental group only depends on the path-connected component of X in which x lies. If there is a path $\beta : x \to y$ in X then $\pi(X, x)$ is just $\beta^{-1}\pi(X, y)\beta$. This gives a group isomorphism between $\pi(X, x)$ and $\pi(X, y)$.

Any map $f : X \to Y$ induces a group homomorphism between the fundamental groups:

$$f_* : \pi(X, x) \mapsto \pi(Y, f(x))$$

given by $f_* : \alpha \mapsto f \circ \alpha$. If $f \simeq g : X \to Y$, then f_* and g_* differ by an inner automorphism. Suppose f, g are homotopic via $F : X \times I \to Y$, and let $\beta(t) = F(x, t)$. Then $f_* = \beta^{-1} g_* \beta$. And in particular, if $f(x) = g(x)$, β is the constant path at x, so $f_* = g_*$.

If X, Y are homotopy-equivalent and path-connected, then their fundamental groups are isomorphic: the composition

$$(X, x) \xrightarrow{f} (Y, f(x)) \xrightarrow{g} (X, g \circ f(x))$$

is an isomorphism up to homotopy equivalence, therefore (X, x) and $(Y, f(x))$ have isomorphic fundamental group.

An example: let $T^n := (S^1)^n$. Or equivalently, $T^n = \mathbb{R}^n/\mathbb{Z}^n$. $\mathbb{R}^n$ is a covering space of T^n. The fundamental group of T^n is $\mathbb{Z}^n$. $\mathrm{GL}_n(\mathbb{Z})$ acts on T^n in a natural way (these are outer automorphisms).

0.4. **Covering Spaces.** $\hat{X}$ is a *covering space* of X when there is a map $p : \hat{X} \to X$ such that for every $x \in X$ there is a neighborhood $U \subset X$ such that $p^{-1}(U)$ is homeomorphic to a union of disjoint copies of U in $\hat{X}$. We say that such a U is *evenly covered* by $\hat{X}$.

Examples:

- The classic example is that the infinite helix covers S^1 by projection.
- One could also cover S^1 by S^1 with a map $z \mapsto z^n$. The order of the covering is n.
- $\mathbb{R}^n/\mathbb{Z}^n$ (the n-torus) is covered by $\mathbb{R}^n$ in the natural way.
- Graphs can be covered by other graphs. The only thing that must be obeyed by the covering space is the local behavior near vertices.

Homotopy Lifting Property: Let $p : \hat{X} \to X$ be a covering of X. If f_t is a homotopy from Y to X, then there is a lifting of f_t to a homotopy $\hat{f}_t$ from Y to $\hat{X}$, and $\hat{f}_t$ is *uniquely determined* by $\hat{f}_0$.

Proof. Let $y \in Y$. $f_t(y)$ for $t \in [0,1]$ is a path in X. This path is covered by finitely many neighborhoods which are evenly covered by $\hat{X}$. Given $\hat{f}_0$, we have a single preimage set that we must choose for $\hat{f}_0(y)$. Now, to choose $\hat{f}_t(y)$ for the next $t \in [0,1]$, we choose the preimage set which intersects with the previous, etc. The fact that $p : \hat{X} \to X$ is continuous guarantees that we can do this and result in a continuous path in $\hat{X}$. □

In the case that Y is a single point, this shows that individual paths can always be lifted through coverings. It is important to note that loops may not remain loops when they are lifted, since the first and last points may both map to x while not being the same.

In the case $Y = [0,1]$, this is saying that entire homotopies of paths can be lifted. In contrast to the case of paths in general, two paths that are *homotopic* will maintain their homotopy (and as a result their endpoints will stay together) when lifted. This implies that loops that are the identity element in $\pi(X, x)$ stay as the identity element in $\pi(\hat{X}, p^{-1}(x))$.

As the previous example implies, we have a correspondence between the fundamental groups

$$p_* : \pi_1(\hat{X}, \hat{x}) \to \pi_1(X, x).$$

The important property to know here is that p_* is *injective*. That is, we can think of $\pi_1(\hat{X}, \hat{x})$ as a *subgroup* of $\pi_1(X, x)$.

To show this, we'll first show that if $p_*(\alpha) = 1$ then $\alpha = 1$. This follows from the preceding discussion about homotopy lifting: the homotopy $p(\alpha) \simeq 1_X$ lifts to a homotopy between α and $1_{\hat{X}}$.

Lifting Criterion: Let Y be path connected and locally-path connected (CW complex suffices). Given $f : (Y, y) \to (X, x)$ with a covering $p : (\hat{X}, \hat{x}) \to (X, x)$, when is there a lift $\hat{f} : (Y, y) \to (\hat{X}, \hat{x})$? The answer is that it exists iff the image of

$$f_* : \pi_1(Y, y) \to \pi_1(X, x)$$

lands inside the subgroup $\pi_1(\hat{X}, \hat{x})$. It is clearly necessary, but the interesting thing is that it's sufficient.

Proof. We construct $\hat{f}(z)$ by lifting $f(z)$ through p. The thing that needs to be checked is that it preserves homotopy classes of paths. **fill in later** □

Classification of Covering Spaces: If X satisfies some basic properties (which connected CW complexes satisfy) then covering spaces $\hat{X}$ correspond exactly with subgroups of $\pi_1(X, x)$.

To show this we will construct a *Universal Cover* of X, which we denote $\tilde{X}$, whose fundamental group is trivial. All other coverings will appear as quotients of this universal cover.

Proof. **Step 1**: We define $\tilde{X}$ as the set of homotopy classes of paths starting at x. The covering map comes from

$$p : \gamma \mapsto \gamma(1).$$

The topology of $\tilde{X}$ will have open sets

$$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta \text{ is a path in } U \text{ extending } \gamma\}.$$

for all *simply-connected* open sets $U \subset X$.

$\tilde{X}$ is path-connected because every γ (i.e. every point in $\tilde{X}$) can be contracted to the identity.

Now to show $\tilde{X}$ is simply-connected. (*omitted*).

Step 2: A *Deck transformation* is a specific case of lifting in which the space Y is *also* $\hat{X}$, but with a different basepoint. The Deck transformations form a group under composition, called the *Deck group* of $\hat{X}$, which we denote $G(\hat{X})$.

We can show that that

$$G(\hat{X}) = N(\pi_1(\hat{X}, \hat{x}))/\pi_1(\hat{X}, \hat{x}).$$

This implies, in the case of the universal cover we just constructed, that the $G(\tilde{X}) = \pi_1(X, x)$. Now for any subgroup H of $\pi_1(X, x)$, we can look at H as a subgroup of $G(\tilde{X})$, and thus take the quotient

$$X_H := \tilde{X}/H$$

for which $\pi_1(X_H, x) = \pi_1(X, x)/H$. It remains to show that this X_H is actually a covering space of X. (*omitted*). □

Examples:

- In the covering of S^1 by $\mathbb{R}/\mathbb{Z}$, the deck group is $\mathbb{Z}$ and thus $\pi_1(S^1) = \mathbb{Z}$.
- For spaces X, Y with universal covers $\tilde{X}$ and $\tilde{Y}$, $\tilde{X} \times \tilde{Y}$ is also the universal cover for $X \times Y$. Thus
  $$\pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y).$$
  This implies that $\pi_a(T^n) = \mathbb{Z}^n$.
- Let Γ be a 1-dimensional connected CW complex (a connected graph). Up to homotopy we can assume that Γ is a wedge of n circles for some n. To calculate the fundamental group of Γ then, we get the free group on n generators. The universal covering is an infinite fractal tree. Ping Pong Lemma.

Incidentally, our discussion so far proves that any subgroup of a free group is free, a highly nontrivial fact in group theory. Every subgroup of a free group $\langle a, b\rangle$ has some generators in terms of a, b, and one can make a graph with these generators as loops. The resulting CW complex is homotopy-equivalent to a wedge of circles, and thus a free group.[1]

Van Kampen Theorem: Let $X = A \cup B$ where A, B are both open in X, with basepoint $x \in A \cap B$. Assume $A, B, A \cup B$ are all path-connected. Then $\pi_1(X, x)$ is freely generated by $\pi_1(A, x), \pi_1(B, x)$ under the identification of $\pi_1(A \cap B, x)$. That is, it is the group such that the following diagram commutes for all groups G such that the outer square commutes.

$$\begin{tikzcd}
& \pi_1(A,x) \arrow[dr] \arrow[drr] & & \\
\pi_1(A\cap B,x) \arrow[ur] \arrow[dr] & & \pi_1(X,x) \arrow[r, "\exists!"'] & G \\
& \pi_1(B,x) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Another way to say this is that it's the simplest group that $\pi_1(A, x), \pi_1(B, x)$ can both map into while commuting with the inclusions from $A \cap B$. We can explicitly determine $\pi_1(X, x)$ as

$$\pi_1(X, x) \cong \pi_1(A, x) * \pi_1(B, x)/\langle\langle \iota_A(w) = \iota_B(w)\rangle\rangle$$

where $\langle\langle \bullet \rangle\rangle$ denotes the normal subgroup generated by the enclosed relations.

The same will hold for more than two (even an arbitrary infinite family) open spaces, as long that any *three* of them have path-connected intersection.

Proof. There is a map $\pi_1(A, x) * \pi_1(B, x) \to \pi_1(X, x)$ by inclusion. We will show that it is surjective and that its kernel is exactly the words w in $\pi_1(A \cap B, x)$ for which $\iota_A(w) = \iota_B(w)$.

Step 1: Let $\alpha : [0, 1] \to X$ be a loop at x in X. We want to show that α is homotopy equivalent (rel endpoints) to a path which is a concatenation of paths entirely in A and entirely in B.

Let $I_A \subset [0, 1]$ be $\alpha^{-1}(A)$ and $I_B = \alpha^{-1}(B)$. Since A, B are open, these sets are open, and they cover $[0, 1]$. Thus by compactness, they are each expressible as finite collections of open intervals.

[1]Hatcher p. 58 has an excellent table of diagrams like this.

So we can split α into finitely-many parts, each of which is in A or B, and whose endpoints are in $A \cap B$. Because $A \cap B$ is path-connected, each of these are homotopic to a path which begins at x.

Step 2: Suppose $\alpha, \beta \in \pi_1(A, x) * \pi_1(B, x)$ are identified by this map. There is a nice diagram that looks like a brick wall. Basically, if α, β are identified, there is a homotopy between their images, and one can factor this homotopy into individual steps ("bricks") which correspond with identifying a path in $A \cap B$ in $\pi(A, x)$ with its other inclusion in $\pi(B, x)$. □

Examples:

- Let $X = S^n$ decomposed into D^n_- and D^n_+, with intersection S^{n-1}. Then Van Kampen implies that $\pi(S^n)$ is trivial.
- In any connected CW complex with $X^1 = \vee_\alpha S^1_\alpha$, we have $\pi_1(X^2)$ is the free group on generators α quotiented by the attaching maps.
- Similarly, $\pi(X) = \pi(X^n) = \pi(X^2)$ for all connected CW complexes X. This is because attaching 3-cells yields 2-cell intersections, which have trivial fundamental group.
- Let Σ_g be a g-holed torus in $\mathbb{R}^3$. This can be given a CW structure with $2g$ 1-cells and one 2-cell. The fundamental group $\pi_1(\Sigma_g)$ has a presentation

$$\pi_1(\Sigma_g) := \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g | \prod_{i=1}^{g} [\alpha_i, \beta_i] \rangle.$$

Given a group π, A $K(\pi, 1)$ is a path-connected space X for which $\pi_1(X) \cong \pi$ and whose universal cover is contractible (universal covers are always simply-connected, but not necessarily contractible).

Let X be a connected CW complex and Y a $K(\pi, 1)$ where $\pi = \pi_1(Y, y)$. Then maps $X \to Y$ induce homomorphisms $\rho : \pi_1(X, x) \to \pi_1(Y, y)$. But moreover, *every* such homomorphism is induced by a map $X \to Y$.

Proof. Let X have one 0-cell $X^0 = \{x\}$. For up to the 2-skeleton X^2, we can construct a map $X \to Y$ inducing ρ without having any $K(\pi, 1)$ property of Y. To get to X^3 and higher, we need this property. (details omitted) □

For any group G there is a CW complex that has G as its fundamental group and has a contractible universal cover (i.e. a $K(G, 1)$).

Proof. Let EG be the simplicial complex which has n-simplices corresponding to all n-tuples of elements of G, with triangles corresponding to the same three elements identified. EG is deformation-retractable to id_G.

G acts on EG by acting on the vertices. Let $BG = EG/G$. This BG is a $K(G, 1)$. □

0.5. **Homology.** Homology groups are somewhat like fundamental groups (they are invariant properties of a topological space) but they are always Abelian. There are two equivalent definitions *on CW complexes*: one will be easy to compute but non-trivially invariant (simplicial/cellular homology), and one will be clearly invariant but difficult to compute (singular homology).

Let Δ^n be the canonical n-simplex, defined as the set of points in $\mathbb{R}^{n+1}$ whose coordinates are all non-negative and sum to 1. We say that the jth face of Δ^n, denoted Δ^n_j, is the $(n-1)$-simplex obtained by taking all the vertices except for j. The orientation (i.e. the ordering on vertices) is preserved if j is even and reversed if j is odd. There may be other n-simplices in space which are images of Δ^n under some continuous map. A Δ-complex is basically a CW complex but with simplices instead of balls and the attaching maps additionally have to obey some orientation stuff.

If X is a Δ-complex, $\Delta_n(X)$ is the free abelian group generated by the n-simplices of X. We have a map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ given by the oriented sum of all the faces:

$$\partial_n : \sigma \mapsto \sum_{i=1}^{n} (-1)^i \sigma_i$$

where σ_i is the ith face of σ. This is the boundary operator. We can get the familiar property

$$\partial_{n-1} \circ \partial_n = 0.$$

To check this,

$$\partial_{n-1}\partial_n\sigma = \partial_{n-1} \sum_{i=1}^{n} (-1)^i \sigma_i = \sum_{i=1}^{n} \sum_{j}^{n-1} (-1)^{i+j} (\sigma_i)_j$$

and note that $(\sigma_i)_j = -(\sigma_j)_i$ because i and j are swapped, which negates the sign of the permutation. The idea is essentially that $(n-2)$-edges all appear twice with opposite orientations (one can check this on the 3-simplex).

An element in $\ker(\partial_n) =: Z_n$ is an n-cycle, and element in $\operatorname{im}(\partial_{n+1}) =: B_n$ is an n-boundary. Note that B_n is a subgroup of Z_n, which is a subgroup of Δ_n. The nth *Homology Group* is

$$H_n = Z_n / B_n.$$

Elements of H_n are equivalence classes of cycles (which are equal mod boundaries). It's not clear a priori that this is well-defined (i.e. does not depend on the parametrization of X) but we will show this later. This is *simplicial homology.*

Alternatively, we could take a given X which is not canonically given by a Δ-complex, and form a simplicial complex $S(X)$ in such a way that it contains *all* continuous maps from Δ_n into X, and define homology of X to be $H_n(S(X))$. This is called the *singular homology* of X.

A sequence of Abelian groups C_n with ∂ maps between them (these maps could actually be anything as long as $\partial^2 = 0$, i.e. it is a long exact sequence) is called a *chain complex*, and a *chain map* is a sequence of maps $f_n : C_n \to C'_n$ from one chain complex to another such that the diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial"] & C'_{n+1} \arrow[r, "\partial"] & C'_n \arrow[r, "\partial"] & C'_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

This chain map induces a homomorphism between the homology groups $f_* : H_n(C_*) \to H_n(C'_*)$ (this is diagram chasing).

Theorem: If $f, g : X \to Y$ are homotopic maps of topological spaces, then the homomorphisms of homology groups induced by f_*, g_* are equal.

Proof. Let $f_\sharp, g_\sharp : C_*(X) \to C_*(Y)$ be the induced chain maps. A *dual homotopy* is a map $P : C_*(X) \to C_{*+1}(Y)$ with the property that

$$\partial P + P\partial = g_\sharp - f_\sharp$$

If such a P exists, then $f_* = g_*$, since the boundary of any loop is 0, so

$$(g_\sharp - f_\sharp)(\alpha) = (\partial P + P\partial)(\alpha) = \partial P(\alpha) + 0$$

and this is a boundary, so in the homology group $f_\sharp = g_\sharp$.

So it suffices to find such a P. To make this P, we begin with the homotopy between f and g. *add more details later.* □

Examples:

- If X is a single point, then there is only a single map from the n-simplex to X, so each $C_n(X)$ is the Abelian group generated by a single generator σ_n, i.e. $C_n(X) = \mathbb{Z}$. For this n-simplex σ_n, we have
$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$
so the maps in this complex are
$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \cdots$$
thus giving $H_n(X) = 0$ for all n (except $H_0(X) = \mathbb{Z}$).

This $X = \{x\}$ example has a sort of annoying property that $H_0(X)$ is an exception. So we have a *reduced homology* which just slips in a $\mathbb{Z}$ between C_0 and 0 at the end of the homology sequence, with the map $\varepsilon : C_0 \to \mathbb{Z}$ which maps every individual point to 1. This allows for $\widetilde{H_0}(X)$ to be 0. Every other dimension is the same as ordinary homology.

We can directly relate H_0 and $\widetilde{H_0}$ via the short exact sequence
$$0 \to \widetilde{H_0}(X) \to H_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

If X is path-connected, then every cycle (mod boundary) in $H_1(X)$ is induced by some loop in $\pi_1(X)$. The kernel of the map $\pi_1(X) \to H_1(X)$ is exactly the commutator subgroup of $\pi_1(X)$.

0.6. **Relative Homology.** If $A \subset X$, we have chain complexes $C_*(A)$ and $C_*(X)$. There are inclusions $C_*(A) \to C_*(X)$. So corresponding to this inclusion we have a chain complex $C_*(X, A)$ given by the quotient map. Thus the following short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "\partial"] & C_n(X) \arrow[r] \arrow[d, "\partial"] & C_n(X, A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X, A) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We can think of elements in $C_n(X, A)$ as n-chains in X mod the chains in A (or those that are not entirely contained in A). $C_n(X, A)$ has its own homology. This gives rise to a *long* exact sequence of the corresponding homology groups:
$$\cdots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$
The only thing that really needs to be done is to define the maps $H_{n+1}(X, A) \to H_n(A)$ and show that they fit in with their neighbors. This map is given as follows: if $c \in C_{n+1}(X, A)$ is a cycle, then we pull back to some representative $x \in C_n(X)$, take its boundary ∂x (which need not be zero), then pull this back to a cycle in $C_n(A)$, and this is the result of the map. It needs to be shown that this is well-defined etc. but we omit this. The fact that we're working with homology groups specifically is fundamental to why this works because it allows us to ignore boundaries.

In a special case where A is a single point, we get $\widetilde{H_n}(A) = 0$ for all n, which results in $\widetilde{H_n}(X) = \widetilde{H_n}(X, A)$.

Now we have this nice thing but we don't know how to compute it. To do this, we can show
$$H_n(X, A) \cong \widetilde{H_n}(X/A)$$

We use a technique called *excision*: if $Z \subset A$ such that the closure of Z is in the interior of A, then we can say
$$H_*(X - Z, A - Z) \cong H_*(X, A).$$

Proof. Let $\mathcal{U}$ be an open cover of X, and let $C_*^{\mathcal{U}}(X)$ be the chain subgroup generated by simplices entirely contained in some open set in $\mathcal{U}$. The claim is that this is the same as $C_*(X)$. To show this it suffices to give a map the other way. You can basically represent any simplex as a sum of many arbitrarily small simplices. The simplex is compact so this only needs to be done finitely-many times. This is done by barycentric subdivision (details omitted). But also you have to be careful that the boundary ends up what it should be.

So now we can specialize this to $\mathcal{U} = \{X - Z, Z\}$ to get the desired result. □

Examples:

- Taking the CW-pair (D^n, S^{n-1}), we have the homology sequence
$$\cdots \to \widetilde{H}_i(S^{n-1}) \to \widetilde{H}_i(D^n) \to \widetilde{H}_i(D^n, S^{n-1}) \to \widetilde{H}_{i-1}(S^{n-1}) \to \cdots$$
and we know by using excision that $\widetilde{H}_i(D^n, S^{n-1}) = \widetilde{H}_i(D^n/S^{n-1}) = \widetilde{H}_i(S^n)$ giving
$$\cdots \to \widetilde{H}_i(S^{n-1}) \to 0 \to \widetilde{H}_i(S^n) \to \widetilde{H}_{i-1}(S^{n-1}) \to \cdots$$
which gives $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$, so by induction we see that
$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$
- We can also use excision to show that singular and simplicial homology coincide for Δ-complexes X. There is an inclusion of the simplicial chain group into the singular chain group, $C_i^\Delta(X) \hookrightarrow C_i(X)$. This map between the chain complexes induces a map between homologies:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d] & H_n^\Delta(X^k) \arrow[r] \arrow[d] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_{n-1}^\Delta(X^{k-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

  By the "5-lemma" from category theory, to prove that the middle map here is an isomorphism, it suffices to show this for the outer four maps. Many of them have smaller k and n, so can be assumed by induction (the cases $k = 0$ and $n = 0$ are the base cases and not hard to show). For the leftmost map $H_{n+1}^\Delta(X^k, X^{k-1}) \to H_{n+1}(X^k, X^{k-1})$, we use excision (check Hatcher for details later).

Given a map $f : S^n \to S^n$, we have a an induced map on homology groups. The nth homology group of S^n is the only nonzero one, as we showed, so we look at the map $f_* : H_n(S^n) \to H_n(S^n)$ i.e. $f_* : \mathbb{Z} \to \mathbb{Z}$. So f_* must be a multiplication by some element, and we call this factor $\deg(f)$. Recall that homotopic maps induce the same maps on homology, so $\deg(f)$ only depends on the homotopy class of f.

Some basic facts about degree:

- $\deg(1) = 1$.
- $\deg(\text{reflection}) = -1$.
- $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
- $\deg(-1)$ (antipodal map) is $(-1)^{n+1}$ (it is a composition of $n + 1$ reflections).
- If f is not surjective, then $\deg(f) = 0$, as f factors through a punctured S^n, and this space has trivial homology groups (it is contractible).

Suppose $f : S^n \to S^n$ has the property that some fiber $f^{-1}(y) = \{x_1, \ldots, x_n\}$, a finite set. Then we can calculate $\deg(f)$ as a sum of the degrees *locally* near each point x_i. Let V be a neighborhood of y such that the preimage $f^{-1}(V)$ is a union of disjoint neighborhoods U_i of x_i. We say that the local degree of f near x is the factor by which f multiplies in

$$f_* : H_n(X, X - x) \to H_n(X, X - x).$$

And note that by excision, $H_n(X, X - x_i) = H_n(U_i, U_i - x_i)$ for each i. (rest of proof omitted).

0.7. **Cellular Homology.** We've seen how homology groups can be defined for Δ-complexes. But now we'd like to do the same for CW-complexes; that is, we'd like a way to define $H_n(X)$ based only on the relative homologies $H(X^n, X^{n-1})$. We can do it as follows:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & \ddots \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[r, equal, "\sim"] & H_n(X) & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "\iota"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_n(X^{n-1}) \arrow[ur, "\iota"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Note that the quotient of $\ker(d_n)/\operatorname{im}(d_{n+1})$ is indeed $H_n(X)$ as we wanted.

0.8. **Mayer-Vietoris.** Let $X = A \cup B$ where A, B are both open. We can relate the homologies $H_*(A \cup B), H_*(A), H_*(B), H_*(A \cap B)$ via the Mayer Vietoris sequence:

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A + B) \to 0$$

(these maps commute with the boundary maps going down each column). $C_*(A + B)$ denotes chains that are contained entirely in A or entirely in B. This over-counts chains that are contained in both, hence the short exact sequence. But also $H_*(A + B) = H_*(A \cup B)$ by excision. So this induces a long exact sequence of homology groups:

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B) \to \cdots$$

0.9. **Euler Characteristic and Lefschetz Trace.** If X is a finite CW complex, we define the Euler characteristic as

$$\chi(X) := \sum_d (-1)^d \cdot \#(\text{Cells of dimension } d) = \sum_d (-1)^d \cdot \operatorname{rank}(C_d(X)).$$

This doesn't depend on the CW-complex structure of X, and in fact only depends on the homology groups of X. This is because

$$\operatorname{rank}(C_d) = \operatorname{rank}(Z_d) + \operatorname{rank}(B_{d-1}) = \operatorname{rank}(B_d) + \operatorname{rank}(H_d) + \operatorname{rank}(B_{d-1})$$

so the boundary terms all cancel and we get

$$\chi(X) := \sum_d (-1)^d \operatorname{rank}(H_d)$$

Examples:

- Σ_g, the closed oriented surface of genus g, has $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$.
- If $X = S^1 \times Y$, then $\chi(X) = 0$ because cells in the product come in pairs corresponding to the 0- and 1-cell of S^1 product with cells of Y.
- In general $\chi(A \times B) = \chi(A) \cdot \chi(B)$.
- If $\hat{X} \to X$ is an d-sheeted cover of X, a finite CW-complex, then $\hat{X}$ can be given a CW structure with d cells of dimension i for each i-cell of X. As a result $\chi(\hat{X}) = d \cdot \chi(X)$.

- Suppose $S^{2m} \to M$ is a covering space. Then the degree of the covering is either 1 or 2 because $\chi(S^{2n}) = 1 + 1 = 2$, so $2 = d \cdot \chi(M)$. An example of this is $M = \mathbb{R}P^{2m}$ with $\chi(\mathbb{R}P^{2m}) = 1$, so $d = 2$ (antipodes are identified!).
- $\chi(\mathbb{C}P^n) = n + 1$.

The Lefschetz Trace Formula is a sort of generalization of the Euler characteristic but for maps rather than spaces themselves (functorial).

Let X be a finite CW complex for which the homology groups are all finitely-generated, and $f : X \to X$. Then there is an induced homomorphism $f_* : H_*(X) \to H_*(X)$. If we think of $H_n(X)$ as a vector space over $\mathbb{Z}$, we can take the trace of f_* as a linear map. Specifically, we define the Lefschetz Trace as

$$\tau(f) := \sum_d (-1)^d \operatorname{tr}(f_* : H_d(X) \to H_d(X))$$

These are $\mathbb{Z}$-linear maps, so their traces are all integers, thus $\tau(f)$ is an integer. Note in particular that $\tau(\mathrm{id}) = \chi(X)$.

The Lefschetz Trace can also be defined in terms of the trace on the maps of chain groups, similarly to Euler Characteristic:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

$\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$. Revisit.

Lefschetz Fixed-Point Theorem: If X is a finite simplicial complex (or a retract of one) and there is a map $f : X \to X$ for which $\tau(f) \neq 0$, then f has a fixed point.

Proof. First we show that this also extends to retracts. If $K \to X$ is a retract to X, and $f : K \to K$, then we can factor f through the retract and $f|_X$. Moreover $\tau(f|_X) = \tau(f)$, because the trace of f_* on the retractable loops in $H_*(K)$ is 0. $f|_X$ has a fixed point (assuming that the theorem is true) and this point is also fixed in K by the inclusion and retraction.[2]

Now on to the proof. By Brouwer fixed-point theorem, if f maps any cell to itself, then it has a fixed point. If we knew that $f : X \to X$ were cellular (that is, preserved X^n for each n), then we could say that f permutes the cells of dimension n for each n. But then its trace would be 0 because it takes every element in $H_n(X)$ to a sum not involving itself (it sends it to a different cell). So it suffices to show that we can reduce to the case of cellular maps.

The idea will be that we can re-parameterize our X with a more subdivided CW-complex structure X' so that f is homotopic to a cellular map $g : X' \to X'$. To do that we need a little bit of organizational terminology.

Let $\mathrm{St}(\sigma)$ (pronounced "Star of σ") be the union of all simplices containing σ. We also have the *open* star $\mathrm{st}(\sigma)$ which is the union of the *interiors* of all simplices containing σ. $\overline{\mathrm{st}(\sigma)} = \mathrm{St}(\sigma)$ (this is not so obvious as it seems). Also the intersection of $\mathrm{st}(v_i)$ for vertices v_i is $\mathrm{st}(\tau)$ if they are exactly the vertices of a simplex τ, and empty otherwise.

We just need to choose g so that $g(v) \in \mathrm{St}(f(v))$ for every vertex $v \in X$, and there will be a homotopy equivalence between f and g (many details omitted). This is a case of a more general tool called *simplicial approximation*.

∎

[2] By extending to allow retractions from simplicial complexes, we also get to use this on compact manifolds and $\mathbb{C}$-algebraic varieties. Very broad usage.

0.10. **Cohomology.** We'll initially define Cohomology as an algebraic thing, but then see that it has some important implications for topology and geometry.

Fix some abelian "coefficient group" G. Given any abelian group A, let A^* be the group $\mathrm{Hom}(A, G)$. For any group homomorphism $f : A \to B$ we have the pullback map $f^* : B^* \to A^*$ by precomposition with f. So dualizing is a contravariant functor.

Suppose we have a chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

where C_n are abelian groups. We can dualize this complex to get

$$\cdots \to C^*_{n-1} \xrightarrow{\delta} C^*_n \xrightarrow{\delta} C^*_{n+1} \to \cdots$$

Note that $\delta^2 = 0$ still, as $\delta^2 f = f \circ \partial^2 = 0$. We can thus define the homology of this dual complex,

$$H^n(C_*; G) := \ker(C^*_n \to C^*_{n+1}) / \operatorname{im}(C^*_{n-1} \to C^*_n).$$

This is what we call the *cohomology* relative to G.

$\varphi \in \ker(C^*_n \to C^*n + 1)$ means $\varphi|_{B_n} = 0$; *cocycles vanish on boundaries.* Because of this, cocycles induce maps $H_n \to G$. And $\varphi \in \operatorname{im}(C^*_{n-1} \to C^*_n)$ means φ factors through ∂, i.e. $\varphi|Z_n = 0$; *coboundaries vanish on cycles.* So we can see the cohomology groups as the groups of homomorphisms $H^n(C_*) \to G$. Let

$$h : H^n(C; G) \to \mathrm{Hom}(H_n(C); G).$$

We can show that h is surjective, and in fact

$$H^n(C; G) \cong \mathrm{Hom}(H_n(C), G) \oplus \ker(h).$$

$\ker(h) = \mathrm{Ext}(H_{n-1}(C); G)$ (I don't know what this is).

Take the commutative diagram made up of short exact sequences

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r] \arrow[d, "\partial"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

we can dualize this whole diagram to get

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & Z^*_{n+1} \arrow[l] \arrow[u] & C^*_{n+1} \arrow[l] \arrow[u] & B^*_n \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
0 & Z^*_{n-1} \arrow[l] \arrow[u, "0"] & C^*_{n-1} \arrow[l] \arrow[u, "\delta"] & B^*_{n-2} \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

it is not immediate that these are actually still exact sequences, as dualizing is not an exact functor (it is only right-exact).

$$0 \longrightarrow B_{n-1} \xrightarrow{\iota} Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

$$0 \longleftarrow \operatorname{coker}(\iota) \longleftarrow B_{n-1}^* \xleftarrow{\iota^*} Z_{n-1}^* \longleftarrow H_{n-1}^* \longleftarrow 0$$

Note $\operatorname{coker}(\iota) = \ker(h)$.

A *free resolution* of H is an exact sequence

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to H \to 0$$

where all F_i are free. We can map between free resolutions by some sequence of maps that commute with the maps in the sequence.

Lemma: Given two free resolutions of H, F_i and F_i', any $\alpha : H \to H$ extends to a chain map $\alpha_i : F_i \to F_i'$ which makes the diagram commute. Further, any two chain maps that agree on H are chain-homotopic.

Proof. Because F_i, F_i' are free, to give $\alpha_i : F_i \to F_i'$ it suffices to say where the basis goes, but there are no relations so we have a lot of flexibility.

At every stage, the only thing we need is for there to be some target to map to in F_i' which will commute the square. And we find that by exactness, the thing we want it to map to, $a_{i-1}(f_i(c))$, is in the kernel of the next map, so it's in the image of f_{i-1}', so a target exists. (this is very handwavey in my notes but it makes sense in the diagram).

To prove the second claim it suffices to show that any extension of the 0 map $H \to H$ is also 0. The idea is that you can basically slide the whole bottom sequence over since the bottom copy of H can be replaced by 0. $\square$

We can apply this fact to homology. Namely, we can dualize the lemma to see that cohomology is unique.