Problem 1: Let G be a subgroup of $\mathrm{GL}_n(k)$ and

$$I := \bigoplus_{d>0} P_d^G, \quad I_D := \bigoplus_{d>0} (D_d(\mathbb{R}^n))^G$$

(note that we exclude constants). A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called *G-harmonic* if $u(f) = 0$ for all $u \in I_D$.

(a) Show that f is G-harmonic iff $\langle u_1 u_2, f \rangle = 0$ for all $u_1 \in D(\mathbb{R}^n)$ and $u_2 \in I_D$.

(b) (optional) Let H be the vector space of G-harmonic polynomials. Prove that if G is a subgroup of O_n, then

$$\mathbb{C}[x_1, \dots, x_n] = H \oplus \mathbb{C}[x_1, \dots, x_n] I.$$

(c) Show that f is SO_n-harmonic iff $\Delta(f) = 0$ and interpret the statement of (ii) in this case.

Proof. (a): If $\langle u_1 u_2, f \rangle = 0$ for all $u_1 \in D(\mathbb{R}^n)$ then in particular for $u_1 = 1$, $\langle u_2, f \rangle = 0$.

Conversely, suppose $u(f) = 0$ for $u \in I_D$. f can be split into $f_0 + f_1 + \cdots + f_d$ where $f_k \in P_k$. *to do*

(c): $\square$

Problem 2: Let A be a k-algebra and a be algebraic over A with minimal polynomial p_a. Show that $\mathrm{Spec}(a)$ is the set of roots of p_a in k.

Proof. $\mathrm{Spec}(a)$ is defined as the set of $\lambda \in k$ for which $(a-\lambda)$ is invertible. We use the fact that

$$p_a(t) - p_a(\lambda) = (t-\lambda)q(t)$$

for some polynomial $q(t)$ with degree $\deg(p_a) - 1$. In the case $t = a$, this gives

$$\begin{aligned} p_a(a) - p_a(\lambda) &= (a-\lambda)q(a) \\ -p_a(\lambda) &= (a-\lambda)q(a) \\ (a-\lambda)^{-1} &= -p_a(\lambda)^{-1}q(a) \end{aligned}$$

(note that $q(a) \neq 0$ since p_a is minimal and $\deg(q) < \deg(p_a)$). So as long as $p_a(\lambda) \neq 0$, $\lambda \notin \mathrm{Spec}(a)$. However, if $p_a(\lambda) = 0$, then $(a-\lambda)$ is not invertible because it divides $p_a(\lambda)$ i.e. it is a zero-divisor. $\square$

Problem 3: Let A be a k-algebra and M a *simple* A-module. Prove that

$$\dim_k(\operatorname{End}_A M) \leq \dim_k M \leq \dim_k A.$$

(here $\leq$ is the order on cardinals).

Proof. Pick some nonzero $m \in M$ and let $f : \operatorname{End}_A M \to M$ be defined by $f : \varphi \mapsto \varphi(m)$. f is injective as follows: suppose $\varphi, \psi \in \operatorname{End}_A M$ and $\varphi(m) = \psi(m)$. Then define $M' \subseteq M$ by

$$M' := \{m' \in M : \varphi(m') = \psi(m')\}.$$

Because φ and ψ are linear, M' is a submodule of M. But M is simple, so M' is either 0 or M. If it is 0 then no such φ, ψ exist, thus f is injective. If $M' = M$, then $\operatorname{End}_A M$ is trivial. In either case, $\dim_k(\operatorname{End}_A M) \leq \dim_k M$.

For the other inequality, let $g : A \to M$ be the map $g : a \mapsto a \cdot m$. The image of g is a submodule of M, but M is simple so $\operatorname{im}(g)$ must be 0 or M and it is clearly not 0 because $g(1) = m \neq 0$. Thus g is surjective, establishing that $\dim_k M \leq \dim_k A$. $\square$

Problem 4: Let A be a ring with nonzero idempotent $e = e^2$. Note that the set eAe forms a ring with unit e. Prove the following:

(a) For any left A-module M, there is a natural eAe-module structure on eM. Furthermore, for any A-modules M, N, there is a natural morphism of additive groups $f : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{eAe}(eM, eN)$.
(b) Multiplication on the right gives an algebra isomorphism $(eAe)^{\mathrm{op}} \to \mathrm{Hom}_A(Ae, Ae)$.
(c) Assuming that $AeA = A$, the map f in (a) is a bijection.
(d) Assuming $AeA = A$, an A-module M is simple iff the eAe-module eM is simple (and similarly for direct sums of simple modules).
(e) (optional) Assuming $AeA = A$, there is an algebra isomorphism $A^{\mathrm{op}} \to \mathrm{Hom}_{eAe}(eA, eA)$.

Proof. (a): To give a module structure we have to specify how scaling and addition work. They are as follows:

$$eae \cdot em = e(aem) \in eM, \quad em_1 + em_2 = e(m_1 + m_2) \in eM.$$

For the morphism $f : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{eAe}(eM, eN)$, take

$$f : \varphi \mapsto \big(em \mapsto e\varphi(m) \in eN\big).$$

To verify that $f(\varphi)$ is indeed eAe-linear,

$$f(\varphi)(eae \cdot em) = e\varphi(aem) = eae\varphi(m) = eae \cdot f(\varphi)(em)$$
$$f(\varphi)(em_1 + em_2) = e\varphi(m_1 + m_2) = e\varphi(m_1) + e\varphi(m_2) = f(\varphi)(em_1) + f(\varphi)(em_2).$$

(b): For $ebe \in eAe$, we have the associated linear map $Ae \to Ae$ given by

$$\Phi(ebe) = ae \mapsto ae \cdot ebe = (aeb)e \in Ae.$$

This map $\Phi : (eAe)^{\mathrm{op}} \to \mathrm{Hom}_A(Ae, Ae)$ is an algebra homomorphism; it is clearly linear, and to show that it is multiplicative,

$$\Phi(ebe) \circ \Phi(ece) = \big(ae \mapsto ae \cdot ece \cdot ebe = aecebe\big) = \Phi(ece \cdot ebe)$$

(note that the order is switched, reflecting the oppositeness of $(eAe)^{\mathrm{op}}$). To show that it is an isomorphism, we have the inverse

$$\Phi^{-1} : \varphi \mapsto e\varphi(e)$$

and one can check that Φ, Φ^{-1} are indeed inverses:

$$(\Phi^{-1} \circ \Phi)(ebe) = \Phi^{-1}\big(ae \mapsto (aeb)e\big) = eeebe = ebe$$

and

$$(\Phi \circ \Phi^{-1})(\varphi) = \Phi(e\varphi(e)) = \big(ae \mapsto aee\varphi(e) = \varphi(ae)\big) = \varphi.$$

(c): With the assumption that $AeA = A$, there is some linear combination $\sum_{i=1}^n a_i e b_i = 1$ where $a_i, b_i \in A$. Thus, any m can be expressed as a A-linear combination of elements in eM:

$$m = \sum_{i=1}^n a_i \cdot e(b_i m).$$

Using this idea, we can define the inverse of f as

$$f^{-1} : \varphi \mapsto \big(m \mapsto \sum_{i=1}^n a_i \cdot \varphi(eb_i m)\big).$$

It is simple to check that these are inverses.

□