ALGEBRA I NOTES

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Abstract. These are my notes from Victor Ginzburg's Representation Theory (Math 325) class at UChicago, Autumn 2025.

Contents

1. Introduction

In this class we'll be interested in the representations of matrix groups. Something like $\mathrm{GL}(V)$ or $\mathrm{SO}(V)$ clearly acts on V, but it can also act on other interesting spaces. One relevant case of this for us will be when G acts on polynomials in $x_1, \ldots, x_n$. Let

$$P_d \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

be the subspace of homogeneous degree-d polynomials in n variables. This space has a basis given by the monomials

$$\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\}$$

and hence is finite-dimensional. P_d is stable under action by GL_n. This is because linear transformation does not affect the degree of monomials (every x_j is sent to a linear combination of $x_1, x_2, \ldots, x_n$).

Consider the case of $G = \mathrm{O}_n$, the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by R is an intertwining map $P_d \to P_{d+2}$, meaning $R \circ g^* f = g^*(R \circ f)$.

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This Δ is an O_n-intertwining operator.

We call a function f *harmonic* if it has $\Delta(f) = 0$. The space of harmonic polynomials in n variables of degree d is denoted $H_d \subseteq P_d$. For $d \in \{0, 1\}$, $H_d = P_d$, but for $d \geq 2$ H_d is strictly smaller. Note that H_d is stable under orthogonal transformations.

We will now work toward showing that H_d is an irreducible SO_n-representation for $n \geq 3$.

A representation $\rho : G \mapsto \mathrm{GL}(V)$ is *unitary* if G always acts as a unitary operator (i.e. preserves Hermitian inner product) on V. We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x)\overline{f_2(x)} e^{-|x|^2} \, \mathrm{d}x$$

where $\mathrm{d}x$ is the Lebesgue measure. Action of SO_n on P_d preserves this inner product.

Alternatively, we could put an inner product on P_d (or on all functions) from integration over S^{n-1} (the sphere). And polynomials in P_d are determined by their behavior on S^{n-1}.

Proposition: If V is a finite-dimensional vector space with an inner product, then any *unitary* action of G on V is completely reducible. Specifically, if $W \subseteq V$ is a G-stable subspace, then one can decompose the action into $V = W \oplus W^\perp$.

Proof. The thing that we need to prove is that if W is G-stable then $W^\perp$ is as well. Let $x \in W^\perp$ and $w \in W$. Because g acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since $g^{-1} \cdot w \in W$ by G-stability of W. □

Key Lemma: If $F \subseteq C(S^{n-1})$ is any subspace stable under SO_n, then it has an element fixed by SO_{n-1}.

Proof. Let $N := (0, 0, \ldots, 0, 1) \in S^{n-1}$. We have the evaluation map $\alpha : C(S^{n-1}) \to \mathbb{C}$ given by evaluating functions at N. We have an inner product on F given by

$$\langle f, g \rangle := \int_{S^{n-1}} f\overline{g}$$

which is clearly fixed by SO_n, thus F is a unitary representation of SO_n. By Riesz representation theorem, $\alpha(f) \equiv \langle f, \varphi\rangle$ for some $\varphi \in F$. For any $g \in \mathrm{SO}_{n-1}$, g fixes N, thus

$$\langle f, \varphi\rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi\rangle = \langle f, g\varphi\rangle$$

and because this is true for arbitrary $f \in F$ and $g \in SO_{n-1}$, φ is fixed by SO_n. Now it remains to show that $\varphi \neq 0$. We can get this by assuming that some function in F takes a nonzero value on N (we can move N to some point where this is true, since F contains a nonzero function). □

We can apply this key lemma to P_d or H_d as F.

Consider $P_d^{\mathrm{SO}_{n-1}}$, the homogeneous polynomials fixed by SO_{n-1}. On homework we showed that this is a subspace of $\mathbb{C}\langle x_n, R\rangle$ (where $R := x_1^2 + \cdots + x_n^2$). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \ldots\rangle$$

thus $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$.

A very important fact about P_d is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2 H_{d-4} \oplus R^3 H_{d-6} \oplus \cdots \end{aligned}$$

(we will show this later). This allows us to deduce the dimension of H_d from P_d:

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d+1.$$

Likewise, we can decompose $P_d^{\mathrm{SO}_{n-1}}$ the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2 H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3 H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of $H_d^{\mathrm{SO}_{n-1}}$ as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each H_d is an *irreducible* representation of SO_n; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an SO_n-representation, P_d decomposes exactly into the sequence H_{d-2j} for $2j \leq d$.

Theorem: If $n \geq 3$, then for each $d \geq 0$, the representation of SO_n in H_d is irreducible, and moreover the representations are all distinct for different d.[1]

Proof. To show that the representations are distinct, we can use a homework problem which shows that the dimension of H_d is always increasing in d for any $n \geq 3$. □

1.1. **Differential Algebra.** Let W be a vector space over k with basis $w_1, \ldots, w_n$, and let $x_1, \ldots, x_n$ be a dual basis for W^*. We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over W, where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \to k[W]_{j-1}$$

which acts on $k[W]$ in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \ldots, \partial_{w_n}]$$

[1]In the case $n = 3$ this gives *all* the irreps. In general you miss $\Lambda^2(\mathbb{C}^n)$, but when $n = 3$ this is just $\mathbb{C}^3$, which you get from H_1.

acting on $k[W]$. There is a natural correspondence between $k[W]$ and $\mathcal{D}(W)$, if one assumes that k is characteristic 0. We have a k-bilinear pairing

$$\mathcal{D}(W) \times k[W] \to k$$

by $\langle u, f\rangle \mapsto u(f)(0)$. This is a *perfect pairing.* And in general we can do the same thing with

$$\mathrm{Sym}^j(W) \times \mathrm{Sym}^j(W^*) \to k.$$

Lemma: Let $\xi \in W$ and $f \in k[W]$. Then

$$\langle \xi^m, f\rangle = m! f(\xi).$$

In particular, if $f = \varphi \in W^*$, $\langle \xi^m, \varphi^m\rangle = m!\varphi^m(\xi)$.

Proof. We will show this for homogeneous f first, and the general result will follow from expressing f as a sum of homogeneous polynomials. Let the degree of f be d. Then by Taylor expansion,

$$f(\xi) = \sum_{k\geq 0} \frac{1}{k!}(\partial_\xi^k f)(0).$$

But note that only the dth term of this is nonzero, since $\partial_\xi^j f = 0$ unless $j = d$ (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other j both sides are 0. □

We can use this pairing to get another inner product on polynomials in $k[W]$ given by

$$\langle p, q\rangle := p(\partial)(q)(0)$$

where $p(\partial)$ is the corresponding element to p in $\mathcal{D}(W)$.[2] For this inner product, we have that multiplication by p is *adjoint* to $p(\partial)$, i.e.

$$\langle r, p(\partial)q\rangle = \langle pr, q\rangle.$$

With this fact, we can finally show why $P_d = H_d \oplus RP_{d-2}$:

$$W = \ker(\Delta) \oplus \mathrm{im}(\Delta^*) = \ker(\Delta) \oplus \mathrm{im}(R) = H_d \oplus RP_{d-2}.$$

Another application of this pairing: Let V be a finite dimensional vector space and $A \subseteq V$ a subset of V (not necessarily subspace). Let $\mathrm{span}^d(A) \subseteq \mathrm{Sym}^d(V)$ be generated over $\mathbb{C}$ by a^d for $a \in A$. If A is dense in V then $\mathrm{span}^d(A) = \mathrm{Sym}^d(A)$. We will show this by using the pairing.

Assume for contradiction that $\mathrm{span}^d(A) \neq \mathrm{Sym}^d(V)$. Then there is some nonzero linear functional $F : \mathrm{Sym}^d(V) \to \mathbb{C}$ which vanishes on $\mathrm{span}^d(A)$. Then F corresponds to some differential polynomial f, and $\partial_a^d f(0) = 0$ for all $a \in A$. But $\partial_a^d f(0) = d!f(a)$, so $f(a) = 0$. But then A is dense, so $f = 0$.

1.2. Representation Theory Basics.

If G acts on sets X and Y, then G can also act on the space of maps $X \to Y$ via conjugation:

$$g : f \mapsto g \circ f \circ g^{-1}.$$

We can ask about the space of maps which commute with this G-action. Or, equivalently, the maps which are fixed by the G-action. We call these *intertwining operators.* The set of such operators is denoted $\mathrm{Hom}_G(X, Y)$.

We are usually interested in the case where X, Y are vector spaces and $\mathrm{Hom}(X, Y)$ is the space of linear maps.

Schur-Weyl Duality: Let W be a finite-dimensional vector space over $\mathbb{C}$. $\mathrm{GL}(W)$ can act on $W^{\otimes d}$ with g acting as $g^{\otimes d}$. S_d also acts on $W^{\otimes d}$ by permutation. It is not too hard to see that these two actions commute. But moreover, action by $\mathrm{GL}(W)$ *spans* the space of S_d-intertwiners on $W^{\otimes d}$.

[2] In the homework, we establish that on H_d, this is actually *equivalent* to the inner product from integrating over S^{n-1}!

Proof. Let $\Phi : (\mathrm{End}(W))^{\otimes d} \to \mathrm{End}(W^{\otimes d})$ be given by

$$\Phi : a_1 \otimes \cdots \otimes a_d \mapsto \big(w_1 \otimes \cdots \otimes w_d \mapsto a_1(w_1) \otimes \cdots \otimes a_d(w_d)\big).$$

Φ is an invertible linear map with inverse

$$\Phi^{-1} f \mapsto f|_{W_1} \otimes \cdots \otimes f|_{W_d}$$

where W_j is $0 \otimes \cdots \otimes W \otimes \cdots \otimes 0$ with the W in the jth spot. Note also that Φ commutes with the action of S_d. By using Φ, we see that

$$\mathrm{Sym}^d(\mathrm{End}\, W) = ((\mathrm{End}\, W)^{\otimes d})^{S_d} \xrightarrow{\Phi^{-1}} \mathrm{End}_{S_d}(W^{\otimes d}).$$

So we only need to understand $\mathrm{Sym}^d(\mathrm{End}\, W)$. But $\mathrm{GL}(W)$ is dense in $\mathrm{End}(W)$, so by a previous lemma, we see that $\mathrm{span}^d(\mathrm{GL}(W)) = \mathrm{Sym}^d(\mathrm{End}\, W)$. □

1.3. **Spectral Theorem.** Let A be a k-algebra with $a \in A$. We have an evaluation map

$$\mathrm{ev}_a : k[t] \to A \quad \mathrm{ev}_a : p \mapsto p(a).$$

Let $A_a := \mathrm{im}(\mathrm{ev}_a)$, i.e. the subalgebra of A generated by a. The kernel $\ker(\mathrm{ev}_a)$ is an ideal of $k[t]$, and it is a principal ideal since $k[t]$ is a PID. Thus, in the case that ev_a is non-injective, there is a unique *minimal polynomial* of a, p_a, which divides every polynomial which vanishes at a.

Lemma: a is algebraic iff A_a is finite-dimensional.

Proof. If A_a is finite-dimensional then there is a relation between $1, a, a^2, \ldots, a^n$ for some n, i.e. a polynomial that a solves. Conversely if a solves a polynomial of degree n then every linear combination of powers of a can be expressed by the first n powers of a. □

We define the *spectrum* of a, denoted $\mathrm{Spec}(a)$, as

$$\mathrm{Spec}(a) := \{\lambda \in k : (a - \lambda) \text{ is not invertible}\}.$$

So for example, if A is a function algebra, $\mathrm{Spec}(a)$ denotes the values that a can take. In the case that A is the matrix algebra $M_n(k)$, $\mathrm{Spec}(a)$ is the set of eigenvalues of a.

The Spectral Theorem: Let A be a k-algebra of one of the following types:

- A is finite-dimensional over k and k is algebraically closed.
- A is countable-dimension and k is uncountable.

Then,

(i) $\mathrm{Spec}(a)$ is nonempty.
(ii) a is nilpotent iff $\mathrm{Spec}(a) = \{0\}$.
(iii) If A is a division algebra then $A = k$.[3]

Proof. Lemma: If $\lambda_1, \ldots \lambda_n \notin \mathrm{Spec}(a)$, i.e. $(a - \lambda_j)$ is invertible for each j, then if

$$\sum_j c_j (a - \lambda_j)^{-1} = 0$$

for some $c_j \in k$ then a is algebraic (proof is by clearing denominators). We will use this fact.

(i): We will split into two cases: if a is algebraic then $\mathrm{Spec}(a)$ is finite but nonempty and if a is not algebraic then $\mathrm{Spec}(a)$ is uncountable (and the converses to both of these are true).

If a is algebraic, then $\mathrm{Spec}(a)$ is the roots of the minimal polynomial (HW), and particular this means $\mathrm{Spec}(a)$ is finite and nonempty because k is algebraically closed.

If a is not algebraic, then by the Lemma, there is no linear relation between any finitely-many $(a - \lambda)^{-1}$ for $\lambda \notin \mathrm{Spec}(a)$. We assumed that $\dim(A)$ is at most countable, and it has an independent set of size $|k \setminus \mathrm{Spec}(a)|$, so $\mathrm{Spec}(a)$ must be uncountable (because k is).

(iii): Assume for contradiction that A is a division algebra yet $\exists a \in A \setminus k$. □

[3]For a counterexample of this when k is not algebraically closed, take the Quaternions over $\mathbb{R}$.