# ALGEBRA I NOTES

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ABSTRACT. These are my notes from Victor Ginzburg's Representation Theory (Math 325) class at UChicago, Autumn 2025.

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## 1. Introduction

In this class we'll be interested in the representations of matrix groups. Something like $\mathrm{GL}(V)$ or $\mathrm{SO}(V)$ clearly acts on $V$, but it can also act on other interesting spaces. One relevant case of this for us will be when $G$ acts on polynomials in $x_1, \ldots, x_n$. Let

$$P_d \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

be the subspace of homogeneous degree-$d$ polynomials in $n$ variables. This space has a basis given by the monomials

$$\left\{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \mid \sum_i d_i = d\right\}$$

and hence is finite-dimensional. $P_d$ is stable under action by $\mathrm{GL}_n$. This is because linear transformation does not affect the degree of monomials (every $x_j$ is sent to a linear combination of $x_1, x_2, \ldots, x_n$).

Consider the case of $G = \mathrm{O}_n$, the orthogonal group. This group fixes the polynomial

$$R := x_1^2 + \cdots + x_n^2$$

so as a result, multiplication by $R$ is an intertwining map $P_d \to P_{d+2}$, meaning $R \circ g^* f = g^*(R \circ f)$.

Likewise, let

$$\Delta := \sum_j \frac{\partial^2}{\partial x_j^2}$$

be the Laplacian. This $\Delta$ is an $\mathrm{O}_n$-intertwining operator.

We call a function $f$ *harmonic* if it has $\Delta(f) = 0$. The space of harmonic polynomials in $n$ variables of degree $d$ is denoted $H_d \subseteq P_d$. For $d \in \{0, 1\}$, $H_d = P_d$, but for $d \geq 2$ $H_d$ is strictly smaller. Note that $H_d$ is stable under orthogonal transformations.

We will now work toward showing that $H_d$ is an irreducible $\mathrm{SO}_n$-representation for $n \geq 3$.

A representation $\rho : G \mapsto \mathrm{GL}(V)$ is *unitary* if $G$ always acts as a unitary operator (i.e. preserves Hermitian inner product) on $V$. We can give an inner product between polynomials (or any functions) by

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} e^{-|x|^2} \, \mathrm{d}x$$

where $\mathrm{d}x$ is the Lebesgue measure. Action of $\mathrm{SO}_n$ on $P_d$ preserves this inner product.

Alternatively, we could put an inner product on $P_d$ (or on all functions) from integration over $S^{n-1}$ (the sphere). And polynomials in $P_d$ are determined by their behavior on $S^{n-1}$.

**Proposition**: If $V$ is a finite-dimensional vector space with an inner product, then any *unitary* action of $G$ on $V$ is completely reducible. Specifically, if $W \subseteq V$ is a $G$-stable subspace, then one can decompose the action into $V = W \oplus W^\perp$.

*Proof.* The thing that we need to prove is that if $W$ is $G$-stable then $W^\perp$ is as well. Let $x \in W^\perp$ and $w \in W$. Because $g$ acts as a *unitary* operator, we have

$$\langle g \cdot x, w \rangle = \langle x, g^{-1} \cdot w \rangle = 0$$

since $g^{-1} \cdot w \in W$ by $G$-stability of $W$. □

**Key Lemma**: If $F \subseteq C(S^{n-1})$ is any subspace stable under $\mathrm{SO}_n$, then it has an element fixed by $\mathrm{SO}_{n-1}$.

*Proof.* Let $N := (0, 0, \ldots, 0, 1) \in S^{n-1}$. We have the evaluation map $\alpha : C(S^{n-1}) \to \mathbb{C}$ given by evaluating functions at $N$. We have an inner product on $F$ given by

$$\langle f, g \rangle := \int_{S^{n-1}} f \overline{g}$$

which is clearly fixed by $\mathrm{SO}_n$, thus $F$ is a unitary representation of $\mathrm{SO}_n$. By Riesz representation theorem, $\alpha(f) \equiv \langle f, \varphi \rangle$ for some $\varphi \in F$. For any $g \in \mathrm{SO}_{n-1}$, $g$ fixes $N$, thus

$$\langle f, \varphi \rangle = f(N) = f(gN) = (g^{-1}f)(N) = \langle g^{-1}(f), \varphi \rangle = \langle f, g\varphi \rangle$$

and because this is true for arbitrary $f \in F$ and $g \in SO_{n-1}$, $\varphi$ is fixed by $\mathrm{SO}_n$. Now it remains to show that $\varphi \neq 0$. We can get this by assuming that some function in $F$ takes a nonzero value on $N$ (we can move $N$ to some point where this is true, since $F$ contains a nonzero function). □

We can apply this key lemma to $P_d$ or $H_d$ as $F$.

Consider $P_d^{\mathrm{SO}_{n-1}}$, the homogeneous polynomials fixed by $\mathrm{SO}_{n-1}$. On homework we showed that this is a subspace of $\mathbb{C}\langle x_n, R\rangle$ (where $R := x_1^2 + \cdots + x_n^2$). One basis of this space is

$$P_d^{\mathrm{SO}_{n-1}} := \mathbb{C}\langle x_n^d, x_n^{d-2}R, x_n^{d-4}R^2, \ldots\rangle$$

thus $\dim(P_d^{\mathrm{SO}_{n-1}}) = \lfloor \frac{d}{2} \rfloor + 1$.

A very important fact about $P_d$ is that it decomposes into the subspaces

$$\begin{aligned} P_d &= H_d \oplus R \cdot P_{d-2} \\ &= H_d \oplus RH_{d-2} \oplus R^2 H_{d-4} \oplus R^3 H_{d-6} \oplus \cdots \end{aligned}$$

(why? ♠). This allows us to deduce the dimension of $H_d$ from $P_d$:

$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1.$$

Likewise, we can decompose $P_d^{\mathrm{SO}_{n-1}}$ the same way:

$$\begin{aligned} P_d^{\mathrm{SO}_{n-1}} &= H_d^{\mathrm{SO}_{n-1}} \oplus RP_{d-2}^{\mathrm{SO}_{n-1}} \\ &= H_d^{\mathrm{SO}_{n-1}} \oplus RH_{d-2}^{\mathrm{SO}_{n-1}} \oplus R^2 H_{d-4}^{\mathrm{SO}_{n-1}} \oplus R^3 H_{d-6}^{\mathrm{SO}_{n-1}} \oplus \cdots \end{aligned}$$

which gives us the dimension of $H_d^{\mathrm{SO}_{n-1}}$ as

$$\dim(H_d^{\mathrm{SO}_{n-1}}) = \dim(P_d^{\mathrm{SO}_{n-1}}) - \dim(P_{d-2}^{\mathrm{SO}_{n-1}}) = (\lfloor d/2 \rfloor + 1) - (\lfloor (d-2)/2 \rfloor + 1) = 1.$$

As a consequence, we see that each $H_d$ is an *irreducible* representation of $\mathrm{SO}_n$; every stable subspace has a fixed point by the Key Lemma, and the fixed-points are one-dimensional, so there can only be one stable subspace. Thus, as an $\mathrm{SO}_n$-representation, $P_d$ decomposes exactly into the sequence $H_{d-2j}$ for $2j \leq d$.

**Theorem**: If $n \geq 3$, then for each $d \geq 0$, the representation of $\mathrm{SO}_n$ in $H_d$ is irreducible, and moreover the representations are all distinct for different $d$.[1]

Let $W$ be a vector space over $k$ with basis $w_1, \ldots, w_n$, and let $x_1, \ldots, x_n$ be a dual basis for $W^*$. We let

$$k[W] := \bigoplus_{d \geq 0} k[W]_d$$

be the homogeneous polynomials over $W$, where

$$k[W]_j := \mathrm{Sym}^j(W^*).$$

We have the directional derivative operator

$$\partial_\xi : k[W]_j \to k[W]_{j-1}$$

which acts on $k[W]$ in the expected way (product rule). Thus we have the differential algebra

$$\mathcal{D}(W) = k[\partial_{w_1}, \ldots, \partial_{w_n}]$$

acting on $k[W]$. There is a natural correspondence between $k[W]$ and $\mathcal{D}(W)$, if one assumes that $k$ is characteristic 0. We have a $k$-bilinear pairing

$$\mathcal{D}(W) \times k[W] \to k$$

[1] In the case $n = 3$ this gives *all* the irreps. In general you miss $\Lambda^2(\mathbb{C}^n)$, but when $n = 3$ this is just $\mathbb{C}^3$, which you get from $H_1$.

by $\langle u, f\rangle \mapsto u(f)(0)$. This is a *perfect pairing.* And in general we can do the same thing with

$$\mathrm{Sym}^j(W) \times \mathrm{Sym}^j(W^*) \to k.$$

**Lemma**: Let $\xi \in W$ and $f \in k[W]$. Then

$$\langle \xi^m, f\rangle = m! f(\xi).$$

In particular, if $f = \varphi \in W^*$, $\langle \xi^m, \varphi^m\rangle = m!\varphi^m(\xi)$.

*Proof.* We will show this for homogeneous $f$ first, and the general result will follow from expressing $f$ as a sum of homogeneous polynomials. Let the degree of $f$ be $d$. Then by Taylor expansion,

$$f(\xi) = \sum_{k \geq 0} \frac{1}{k!} (\partial_\xi^k f)(0).$$

But note that only the $d$th term of this is nonzero, since $\partial_\xi^j f = 0$ unless $j = d$ (since we are evaluating at 0). Thus,

$$f(\xi) = \frac{(\partial_\xi^d f)(0)}{d!}$$

and for other $j$ both sides are 0. □

We can use this pairing to get another inner product on polynomials in $k[W]$ given by

$$\langle p, q\rangle := p(\partial)(q)(0)$$

where $p(\partial)$ is the corresponding element to $p$ in $\mathcal{D}(W)$.[2] For this inner product, we have that multiplication by $p$ is *adjoint* to $p(\partial)$.

With this fact, we can finally show why $P_d = H_d \oplus RP_{d-2}$:

$$W = \ker(\Delta) \oplus \mathrm{im}(\mu_R) = H_d \oplus RP_{d-2}.$$

[2]In the homework, we establish that on $H_d$, this is actually *equivalent* to the inner product from integrating over $S^{n-1}$!