MATH 317 HW 1

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Problem 1 (Hatcher 0.4): A deformation retraction "in the weak sense" of $X \to A$ is a homotopy $f_t : X \to X$ such that f_0 is the identity map on X, f_1 is a map $X \to A$, and $f_t(A) \subseteq A$ for all $t \in [0,1]$ (this is weaker because it does not require f_t be *constant* on A). Show that if such a map exists, then the inclusion $\iota : A \to X$ is a homotopy equivalence.

Proof. To show that ι is a homotopy equivalence it is sufficient and necessary to produce an inverse up to homotopy. This inverse will be f_1. The composition $\iota \circ f_1 : X \to X$ is homotopic to id_X via f_t. $f_1 \circ \iota : A \to A$ is also not necessarily id_A, but is homotopic to id_A via f_t (here we need the fact that f_t is always a map $A \to A$). $\square$

Problem 2 (Hatcher 0.6):

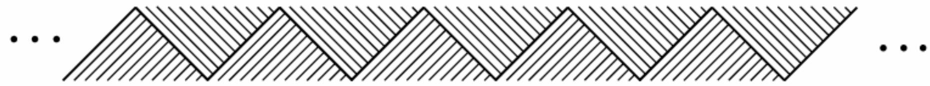
(a) Let X be the subspace of $\mathbb{R}^2$ consisting defined by

$$X := \Big([0,1] \times \{0\}\Big) \cup \bigcup_{q \in \mathbb{Q}[0,1]} \{q\} \times [0, 1-q].$$

Show that X deformation retracts to any point in $[0,1] \times \{0\}$, but not to other points in X.

(b) Let Y be the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show that there is a deformation retraction $Y \to Z$ in the weak sense, but not in the regular sense.



Proof. (a): For any point $r \in [0,1]$, we can construct a deformation retraction from X to $\{r\} \times \{0\}$ as follows: let $d_t : X \to X$ be given by

$$d_t(q,h) = \begin{cases} (q, h(1-2t)) & t \in [0, \frac{1}{2}] \\ ((2t-1)r + (2-2t)q, 0) & t \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to check that d_t is continuous and that $d_t((r,0)) = (r,0)$ for all $t \in [0,1]$.

On the other hand, if there were a deformation retraction to a point (q,h) where $h \neq 0$, then in every neighborhood of (q,h) there are points which must travel down to $[0,1] \times \{0\}$ at some t, a nonzero distance of h from $d_t(q,h)$. But if d_t is continuous in q,h and t, then there must be **i'm confused**

(b): The difference between being contractible to a point x and being *deformation retractable* to x is that in the former case, the homotopy need not keep x fixed. I □

Problem 3 (Hatcher 0.16):

Problem 4 (Hatcher 0.20):

Problem 5 (Hatcher 0.23):

Problem 6 (Hatcher 1.1.6):

Problem 7 (Hatcher 1.1.13):

Problem 8 (Hatcher 1.1.20):