MATH 317 HW 7

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Problem 1 (Hatcher 4.1.5): For a pair (X, A) of path-connected spaces, show that $\pi_1(X, A, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, x_0)$ represented by loops in A at x_0.

Proof. By definition, $\pi_1(X, A, x_0)$ is the group of (homotopy classes of) maps $I \to X$ beginning at x_0 and ending in A. In each of those classes there is a loop at x_0, since A is path connected and paths can be extended by homotopy.

If α is a loop at x_0 in X, then everything in αH is homotopic to α (do later) $\square$

Problem 2 (Hatcher 4.1.8): Show that the sequence

$$\pi_1(X, x_0) \xrightarrow{f} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{g} \pi_0(X, x_0)$$

is exact.

Proof. $\ker(\partial)$ is exactly the set of paths in $\pi_1(X, A, x_0)$ which are loops, equivalently the paths whose endpoint is x_0, equivalently the image of $\pi_1(X, x_0)$.

Next, $\mathrm{im}(\partial)$ is the points in A that are the other endpoint of a path in X from x_0, i.e. the points in the same connected component of X as x_0. And this is also the kernel of g, so indeed the sequence is exact here too. $\square$

Problem 3 (Hatcher 4.1.15): Show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity map by the following steps:

(a) Use Lemma 4.10 to reduce to the case that there is a point $q \in S^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible linear map near each p_i.

(b) For f as in (a), consider the composition gf where $g : S^n \to S^n$ collapses the complement of a small ball about q to the basepoint. Use this to reduce (a) further to the case $k = 1$.

(c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection.

Proof. (a): Lemma 4.10 says that f is homotopic to a map $f_1 : S^n \to S^n$ which is piecewise linear on some polyhedron $K \subset S^n$.

(b): Supposing the result is true for $k = 1$,

(c): The space of $n \times n$ matrices with positive determinant is path-connected, as a subspace of $\mathbb{R}^{n^2}$. One way to see this is that every matrix $a \in M_n(\mathbb{R})$ has a well defined logarithm

$$\log(1 + a) = a + a^2/2 + a^3/3 + a^4/4 + \cdots$$

and exponential

$$e^a = I + a + a^2/2 + a^3/6 + a^4/24 + \cdots$$

so one can let $x = \log(a)$ and take the path $\gamma(t) = e^{tx}$. Then $\gamma(0) = I$, $\gamma(1) = e^x = a$, and $e^{tx} \in \mathrm{GL}_n(\mathbb{R})$ because it has inverse e^{-tx}. $\square$

Problem 4 (Hatcher 4.1.17): Show that if X and Y are CW complexes with X m-connected and Y n-connected, then $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected, as is the smash product $X \wedge Y$.

Proof. We have the long exact sequence

$$\cdots \to \pi_j(X \vee Y) \to \pi_j(X \times Y) \to \pi_j(X \times Y, X \vee Y) \to \pi_{j-1}(X \vee Y) \to \cdots$$

X is homotopy equivalent to a CW complex X' with one 0-cell and no other cells of dimension $\leq m$, and likewise Y has a corresponding Y'. $(X \times Y, X \vee Y) \simeq (X' \times Y', X' \vee Y')$ so it suffices to show this for the latter.

The cells of $X' \times Y'$ are the same as $X' \vee Y'$ in small dimensions. The smallest-dimension cells in $X' \times Y'$ not in $X' \vee Y'$ are of dimension $m+n$, so $\pi_j(X' \vee Y') \to \pi_j(X' \times Y')$ is automatically an isomorphism for $j \leq m+n$, and thus in the long exact sequence

$$\cdots \to \underbrace{\pi_{m+n+1}(X' \times Y', X' \vee Y')}_{0} \to \pi_{m+n}(X' \vee Y') \cong \pi_{m+n}(X' \times Y') \to \underbrace{\pi_{m+n}(X' \times Y', X' \vee Y')}_{0} \to \cdots$$

implies $\pi_{m+n+1}(X' \times Y', X' \vee Y') = 0$ and likewise for all smaller dimensions.

The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$. Excision for homotopy groups implies that

$$\pi_j(X \times Y / X \vee Y) \cong \pi_j(X \times Y, X \vee Y)$$

for $j \leq m+n$. For $j = m+n+1$, the inclusion induces a surjection, and the only thing 0 can surject onto is 0, so $\pi_{m+n+1}(X \wedge Y) = 0$ as well. $\square$

Problem 5 (Hatcher 4.2.1): Use homotopy groups to show that there is no retraction $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ for $n > k > 0$.

Proof. $\mathbb{R}\mathrm{P}^n$ has a 2-sheeted cover from S^n, as we've seen before. So $\pi_k(\mathbb{R}\mathrm{P}^n) = \pi_k(S^n) = 0$, but $\pi_k(\mathbb{R}\mathrm{P}^k) = \pi_k(S^k) = \mathbb{Z}$ (assuming $k \geq 2$). And in the case $k = 1$, $\mathbb{R}\mathrm{P}^1 = S^1$, so it is still true that $\pi_1(\mathbb{R}\mathrm{P}^1) = \mathbb{Z}$. A retraction would induce an isomorphism between $\pi_k(S^n)$ and $\pi_k(S^k)$, but they are not isomorphic, so there is no retraction. $\square$

Problem 6 (Hatcher 4.2.5): Let $f : S^2_\alpha \vee S^2_\beta \to S^2_\alpha \vee S^2_\beta$ be the map which is the identity on S^2_α, and the sum of the identity and a map $S^2_\beta \to S^2_\alpha$ on S^2_β. Let X be the mapping torus of f, i.e.

$$X := \frac{(S^2_\alpha \vee S^2_\beta) \times I}{(x, 0) \sim (f(x), 1)}$$

The mapping torus of the restriction of f to S^2_α forms a subspace $A = S^1 \times S^2_\alpha \subset X$. Show that the maps $\pi_2(A) \to \pi_2(X) \to \pi_2(X, A)$ form a short exact sequence

$$0 \to \underbrace{\pi_2(A)}_{\mathbb{Z}} \to \underbrace{\pi_2(X)}_{\mathbb{Z}^2} \to \underbrace{\pi_2(X, A)}_{\mathbb{Z}} \to 0$$

and compute the action of $\pi_1(A)$ on these three groups. In particular show that the action is trivial on $\pi_2(A)$ and $\pi_2(X, A)$ but nontrivial on $\pi_2(X)$.

Proof. $\pi_3(X, A) = 0$ by cellular approximation, as X has no cells of dimension 3 so one could deform any map $S^3 \to X$ to a point. And we know that

$$\pi_3(X, A) \to \pi_2(A) \to \pi_2(X)$$

is exact in general. This shows that $\pi_2(A) \to \pi_2(X)$ is injective.

Now to show that $\pi_2(X) \to \pi_2(X, A)$ is surjective, it suffices to show that $\pi_1(A) \to \pi_1(X)$ is injective, i.e. that if a loop intersecting A is nullhomotopic in X then it is nullhomotopic in A too. $\square$

Problem 7 (Hatcher 4.2.10): Let X be the CW complex obtained from $S^2 \vee S^n$ (where $n \geq 2$) by attaching e^{n+1} by a map representing the polynomial $p(t) \in \mathbb{Z}[t, t^{-1}] \cong \pi_n(S^1 \vee S^n)$, so that $\pi_n(X) \cong \mathbb{Z}[t, t^{-1}]/(p(t))$. Show that $\pi_n'(X)$ is cyclic and compute its order in terms of $p(t)$. Give examples showing that the group $\pi_n(X)$ can be finitely-generated or not, independently of whether $\pi_n'(X)$ is finite or infinite.

Proof. By the relative Hurewicz theorem, $\square$

Problem 8 (Hatcher 4.2.13): Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leq n$.

Proof. Whitehead's theorem implies that if $f : X \to Y$ induces an isomorphism on π_i for all i then it is a homotopy equivalence. If X, Y are n-dimensional, then $\pi_i(X) = \pi_i(Y) = 0$ for $i > n$, so any such f automatically induces an isomorphism on these groups. Thus it suffices to know that $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leq n$, as desired. $\square$