SIMPLE GUIDE TO SOLVING PROBLEMS IN ALGEBRAIC TOPOLOGY

JALEN CHRYSOS

ABSTRACT. This is a consolidation of my algebraic topology notes. I don't intend to replicate any proofs of important statements unless they're immediate. I'm just aiming to make a dummyproof flowchart-style guide to solving Hatcher questions for my own use.

1. BASIC TERMINOLOGY

The most basic questions one asks in topology are

"Does there exist a map $X \to Y$ with property P?"

To be able to answer these questions positively, we need methods of constructing maps, but this is usually the simple part. To answer them negatively, we need *algebraic invariants*, properties of a space which are preserved by sufficiently nice maps.

Types of maps (in order of increasing strictness):
- Homotopy equivalence: continuous maps with continuous "inverses" *up to homotopy equivalence.*
- Homeomorphism: continuous maps with continuous inverses.

It's a lot easier to work with homotopy equivalence in practice because there's more you can do (contract simply-connected subspaces for example). The invariants of interest are all preserved by homotopy equivalence:

Algebraic Invariants:
- Fundamental group: $\pi_1(X)$
- Homology groups: $H_0(X), H_1(X), \ldots$
- Cohomology groups: $H^0(X; G), H^1(X; G), \ldots$
- Cohomology ring: $H^*(X; G)$ as a graded ring with $\smile$.
- Higher homotopy groups: $\pi_2(X), \pi_3(X), \ldots$

First, we'll discuss how to compute all of these algebraic invariants, or at least glean information about them. We'll assume X is a connected CW complex.

2. FUNDAMENTAL GROUP

The fundamental group of X is a group $\pi_1(X)$ whose elements are homotopy equivalence classes of loops in X, and whose multiplication is the concatenation of loops (at some base point, but if X is connected then the basepoint doesn't matter).

Quick Facts:
- If $\pi_1(X) = 0$ then we say X is "simply connected," i.e. all loops can be contracted.
- $\pi_1(X)$ is in general non-Abelian.

How to Calculate: Use **Van Kampen's Theorem**:
- Write X as a CW complex.
- If X has more than one 0-cell, quotient by a maximal spanning tree in X^1 so that X has a single 0-cell (this is a homotopy equivalence so it preserves $\pi_1(X)$).
- Let the 1-cells (loops) at this 0-cell be $\alpha_1, \alpha_2, \ldots, \alpha_k$. The boundaries of the 2-cells are loops in X^1, so they can each be written as products of the α_j. Let these boundaries be $\beta_1, \beta_2, \ldots, \beta_\ell \in \langle \alpha_1, \ldots, \alpha_k \rangle$.

- Now $\pi_1(X)$ is the free group on $\alpha_1, \dots, \alpha_k$ mod the boundaries of the 2-cells:

$$\pi_1(X) := \langle \alpha_1, \dots, \alpha_k \mid \beta_1 = \beta_2 = \cdots = \beta_\ell = 1 \rangle.$$

3. Homology

The Homology Groups of X are a sequence $H_0(X), H_1(X), \dots$ of *Abelian groups*, so they are of the form $\mathbb{Z}^a \oplus \mathbb{Z}_2^{b_2} \oplus \mathbb{Z}_3^{b_3} \oplus \mathbb{Z}_5^{b_5} \oplus \cdots$.

<u>Quick Facts</u>:

- $H_0(X) = \mathbb{Z}^a$ where a is the number of connected components of X.
- $H_1(X)$ is the Abelianization of $\pi_1(X)$.
- $H_n(X) = 0$ for $n > \dim(X)$.
- If X is a manifold of dimension n:
  - $H_n(X) = \mathbb{Z}$ if X is oriented and $H_n(X) = 0$ otherwise.
  - $H_{n-1}(X)$ has torsion subgroup 0 if X is oriented and $\mathbb{Z}_2$ otherwise.

<u>How to Calculate</u>: There are a couple of useful methods.

(I) **Quotients**:
  - Express X as a quotient Y/A where Y is a CW complex and A is a subcomplex. Try to choose Y and A such that $H_*(Y)$ and $H_*(A)$ are both simpler than $H_*(X)$.
  - There is a long exact sequence relating $\tilde{H}_*(A), \tilde{H}_*(Y)$, and $\tilde{H}_*(X)$:

$$\cdots \to \tilde{H}_j(A) \to \tilde{H}_j(Y) \to \tilde{H}_j(Y/A) \to \tilde{H}_{j-1}(A) \to \tilde{H}_{j-1}(Y) \to \tilde{H}_{j-1}(Y/A) \to \cdots$$

  - Now use $\tilde{H}_*(Y)$ and $\tilde{H}_*(A)$, along with exactness of the sequence, to deduce $\tilde{H}_*(X)$.
  - *Special Cases / Notable Examples*:
    (i) If A is contractible then $H_*(A) = 0$, so $H_*(X) = H_*(Y)$.
    (ii) When $X = S^n$, we can express it as $X = D^n/S^{n-1}$. Because $H_*(D^n) = 0$, the exact sequence yields $\tilde{H}_j(S^n) \cong \tilde{H}_{j-1}(S^{n-1})$, so by induction we can see that $\tilde{H}_j(S^n) = \mathbb{Z}$ iff $j = n$, otherwise it is 0.

(II) **Mayer-Vietoris**:
  - Express X as the union of two *open* subcomplexes $A \cup B$. Try to choose A and B so that A, B, and $A \cap B$ are all simpler than X.
  - There is a long exact sequence relating $H_*(A), H_*(B), H_*(A \cap B)$, and $H_*(X)$:

$$\cdots \to H_j(A \cap B) \to H_j(A) \oplus H_j(B) \to H_j(X) \to H_{j-1}(A \cap B) \to \cdots$$

  - Now use $H_*(A), H_*(B), H_*(A \cap B)$, and the exactness of the sequence to deduce $H_*(X)$.
  - *Special Cases / Notable Examples*:
    (i) If $A \cap B$ is contractible, then $H_j(X) \cong H_j(A) \oplus H_j(B)$. In particular, we see that $H_j(A \vee B) = H_j(A) \oplus H_j(B)$.
    (ii)

(III) **Cellular Homology**:
  - Use this if X is a CW complex with relatively few cells.
  - For each j, let c_j be the number of j-cells in X, and let $C_j(X) \cong \mathbb{Z}^{c_j}$ be the Abelian group with generators corresponding to the j-cells of X.
  - We have the long exact sequence

$$C_n(X) \xrightarrow{\mathrm{d}_n} C_{n-1}(X) \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0(X) \to 0$$

  in which d_n acts by taking cells to their boundaries. d_n is a linear map $\mathbb{Z}^{c_n} \to \mathbb{Z}^{c_{n-1}}$, so we can see d_n as a $c_n \times c_{n-1}$ matrix.
  - $H_j(X)$ is the quotient

$$H_j(X) \cong \frac{\ker(\mathrm{d}_j)}{\mathrm{im}(\mathrm{d}_{j+1})}$$

4. Cohomology

The Cohomology of X (wrt G) is a sequence of Abelian groups $H^0(X; G), H^1(X; G), \ldots$.

How to Calculate $H^*(X; G)$: Use **Universal Coefficient Theorem**:
- Use this if you know $H_*(X)$.
- The Cohomology can be determined from the Homology via the isomorphism

$$H^j(X; G) = \mathrm{Hom}(H_j(X), G) \oplus \mathrm{Ext}(H_{j-1}(X), G).$$

- To calculate Ext: each $H_k(X)$ is an Abelian group, so it has some free parts and some torsion parts. If $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_2)^{b_2} \oplus (\mathbb{Z}_3)^{b_3} \oplus (\mathbb{Z}_5)^{b_5} \oplus \cdots$ then

$$\mathrm{Ext}(H_k(X); G) = (G/(2G))^{b_2} \oplus (G/(3G))^{b_3} \oplus (G/(5G))^{b_5} \cdots$$

(note that the free part $\mathbb{Z}^a$ is dropped entirely).

If $G = R$, a ring, then the cohomology groups $H^*(X; R)$ together form a graded ring under the cup product operation. With the additional structure of the cup product, the cohomology ring makes a strictly finer invariant than the groups alone. Two spaces might have the same cohomology groups in every dimension but different ring structures, e.g. with $S^2 \vee S^4$ and $\mathbb{CP}^2$.

How to Calculate $H^*(X; R)$ (as a ring):
(I) **Künneth Formula**:
- Use this if $X = A \times B$, and $H^*(A; R), H^*(B; R)$ are both known, and finally $H^*(B; R)$ is free in every dimension.
- There is a ring isomorphism

$$H^*(A \times B; R) \cong H^*(A; R) \otimes_R H^*(B; R).$$

- In particular this also implies that

$$H^n(A \times B; R) \cong \bigoplus_{j=0}^{n} H^j(A; R) \otimes_R H^{n-j}(B; R)$$

as groups.
(II) **Poincaré Duality**:
- Use this if X is a closed, connected R-oriented n-manifold.
- In this case $H^j(X; R) = H^{n-j}(X; R)$ for all j. We can sometimes guarantee that the generators $\pm 1 \in H^n(X; R) = \mathbb{Z}$ factor as cup products of elements in these two groups.
- Suppose $R = \mathbb{Z}$. If $H^j(X; \mathbb{Z}) = H^{n-j}(X; \mathbb{Z})$ has positive rank, then for α a $\mathbb{Z}$-generator in $H^j(X; \mathbb{Z})$ and β a $\mathbb{Z}$-generator in $H^{n-j}(X; \mathbb{Z})$,

$$\alpha \smile \beta = \pm 1 \in H^n(X; \mathbb{Z}).$$

- Suppose R is a field. If $H^j(X; R) = H^{n-j}(X; R)$ are nontrivial, then for any nonzero $\alpha \in H^j(X; R)$ and $\beta \in H^j(X; R)$,

$$\alpha \smile \beta = \pm 1 \in H^n(X; R).$$

5. Higher Homotopy Groups

The nth homotopy group of X is denoted $\pi_n(X)$. It is like a higher-dimensional analogue of $\pi_1(X)$, where the elements are equivalence classes of maps $S^n \to X$.

Quick Facts:
- $\pi_n(X)$ is Abelian for $n \geq 2$.
- $\pi_j(S^n) = 0$ for $j < n$, $\pi_n(S^n) = \mathbb{Z}$.
- *Hurewicz*: If $\pi_1(X) = 0$, the first nonzero $\pi_i(X)$ and $H_j(X)$ occur at the same dimension $i = j$, and $\pi_j(X) = H_j(X)$.
- For any collection of path-connected spaces $\{X_\alpha\}$, $\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$.

How To Calculate $\pi_n(X)$:

(I) **Covering Space**:
- If $p : \tilde{X} \to X$ is a covering space of X, then
$$p_* : \pi_n(\tilde{X}) \to \pi_n(X)$$
is an isomorphism for all $n \geq 2$.
- *Special Cases*:
  - If X has a contractible universal cover then $\pi_n(X) = 0$ for $n \geq 2$.

(II) **Excision**:
- Express X as a CW complex that is the union of two subcomplexes A and B whose intersection is subcomplex C.
- If (A, C) is m-connected and (B, C) is n-connected, then the inclusion $(A, C) \hookrightarrow (X, B)$ induces the map
$$\pi_j(A, C) \to \pi_j(X, B)$$
and this map is an isomorphism for $j < m + n$, and a surjection for $j = m + n$.
- *Special Cases*:
  - If C is a single point, i.e. $X = A \vee B$, then the map is $\pi_j(A) \to \pi_j(X, B)$.
  - One can represent X as $(X/A) \vee A$. Applying the theorem in this case gives
$$\pi_j(X/A) \to \pi_j(X, A)$$
  so we can identify quotients with relative homotopy groups in sufficiently good conditions.

6. EXISTENCE RESULTS

How to Build a Space with Given Algebraic Properties:
- *Moore Spaces*: For any Abelian group G, there is a Moore space $M(G, n)$ for which
$$\tilde{H}_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$
- By taking wedge products of Moore spaces $M(G_1, 1), M(G_2, 2), \ldots$ one can construct a space with any arbitrary homology sequence.
- *Eilenberg-MacLane*: For any group G, there is a space X of type $K(G, 1)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = 1 \\ 0 & j > 1 \end{cases}$$

  Moreover, if G is Abelian, then for any n there is a space X of type $K(G, n)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$

  All $K(G, n)$ spaces are weak-homotopy equivalent, and all CW complexes of type $K(G, n)$ are fully homotopy equivalent by Whitehead's Theorem.
- *Postnikov Tower*: For any connected CW complex X, there is decreasing sequence of CW complexes $X_1, X_2, \ldots$ such that
  - X is a subcomplex of each X_j,
  - The inclusion $X \hookrightarrow X_j$ induces an isomorphism on π_i for $i \leq j$,
  - $\pi_i(X_j) = 0$ for $i > j$.

7. STRATEGIES FOR BROAD QUESTION-TYPES

"Are X and Y homotopy equivalent?"
- If the answer is *no*:
  - You can probably determine this from their algebraic invariants.
  - If the fundamental group, homology, cohomology ring, or any higher homotopy groups are different, then X and Y are not homotopy equivalent.
- If the answer is *yes*:
  - Then you should construct the map.
  - If $X \subset Y$, then you may be able to find a deformation retraction.

– If you can show that $f : X \to Y$ induces an isomorphism $f_* : H_*(X) \to H_*(Y)$, then this automatically implies that f is a homotopy equivalence by Whitehead's + Hurewicz's Theorems.

"Is there a space X with algebraic property P?"
- If the answer is *no*:
- If the answer is *yes*:

8. Worked Examples

Example 1: The Klein Bottle.

Fundamental Group: The Klein bottle can be represented as a CW complex with one 0-cell, two 1-cells, and one 2-cell as depicted below:

DIAGRAM

Letting the 1-cells be α, β, the attaching map of the 2-cell has boundary $\alpha\beta\alpha^{-1}\beta$, so by Van Kampen's Theorem we have

$$\pi_1(X) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle$$

In other words $\alpha\beta = \beta^{-1}\alpha$. Every string of α and β can be rearranged using this rule into a canonical representative $\alpha^n \beta^m$ with $n, m \in \mathbb{Z}$. These multiply by

$$\alpha^{n_1}\beta^{m_1} \cdot \alpha^{n_2}\beta^{m_2} = \alpha^{n_1+n_2}\beta^{(-1)^{n_2}m_1+m_2}$$

(each β in β^{m_1} has its sign swapped n_2 times when swapping with the α^{n_2} block). Thus we can consider $\pi_1(X)$ as $\mathbb{Z} \ltimes \mathbb{Z}$ with the action $n : m \mapsto (-1)^n m$. (revisit)

Homology: A natural way to calculate the homology of K is via cellular homology. We have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

where the boundary maps look like

$$d_0 : \begin{pmatrix} 0 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

noting in d_2 that the boundary of the 2-cell is $\alpha + \beta - \alpha + \beta = 0\alpha + 2\beta$. So we have

$$H_0(K) = \frac{\ker(d_0)}{\operatorname{im}(d_1)} = \frac{\mathbb{Z}}{0} = \mathbb{Z},$$

$$H_1(K) = \frac{\ker(d_1)}{\operatorname{im}(d_2)} = \frac{\mathbb{Z}^2}{0 \times 2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}_2,$$

$$H_2(K) = \frac{\ker(d_2)}{\operatorname{im}(d_3)} = \frac{0}{0} = 0.$$

These also could have been calculated faster by using other methods:
- $H_0(K) = \mathbb{Z}$ because K is connected.
- $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ because this is the abelianization of $\pi_1(K)$:

$$\operatorname{Ab}\left(\langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle\right) = \mathbb{Z}\langle \alpha, \beta \rangle / (2\beta) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

Also note that $H_2(K) = 0$ implies, by Poincaré Duality, that K is non-orientable.

Cohomology: Let the coefficient ring be G. Having solved $H_*(K)$ already, we can use Universal Coefficient Theorem:

$$H^j(K; G) = \operatorname{Hom}(H_j(K), G) \oplus \operatorname{Ext}(H_{j-1}(K), G).$$

In the case $G = \mathbb{Z}$, we have

$$H^0(K; \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}(0, \mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z}$$

$$H^1(K; \mathbb{Z}) = \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z},$$

$$H^2(K;\mathbb{Z}) = \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}) = (0) \oplus (\mathbb{Z}_2) = \mathbb{Z}_2.$$

For the ring structure, beginning with $\mathbb{Z}[x,y]$ where x generates $H^1(K;\mathbb{Z}) = \mathbb{Z}$ and y generates $H^2(K;\mathbb{Z}) = \mathbb{Z}_2$. We clearly have the relation $2y = 0$ because $H^2(K;\mathbb{Z}) = \mathbb{Z}_2$, and $xy = y^2 = 0$ because $H^3 = H^4 = 0$. The only product that could be nontrivial is x^2.

When $G = \mathbb{Z}_2$,

$$H^0(K;\mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) \oplus \mathrm{Ext}(0,\mathbb{Z}_2) = (\mathbb{Z}_2) \oplus (0) = \mathbb{Z}_2$$

$$H^1(K;\mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) = (\mathbb{Z}_2^2) \oplus (0) = \mathbb{Z}_2^2,$$

$$H^2(K;\mathbb{Z}_2) = \mathrm{Hom}(0,\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2) = (0) \oplus (\mathbb{Z}_2) = \mathbb{Z}_2.$$

Now for the ring structure.

Homotopy Groups: There is a covering $\mathbb{R}^2 \to K$ as depicted below:

Diagram

Xovering induces isomorphism on homotopy groups above dimension 1, so $\pi_n(K) = 0$ for $n \geq 2$.

9. TABLE OF COMPUTATION RESULTS