MATH 325 HW 7

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Problem 1: Let $\mathcal{U}(e,h,f)$ be an associative $\mathbb{C}$-algebra with generators e,h,f with relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h.$$

Let V be a $\mathcal{U}(e,h,f)$-module and $v \in V$ a nonzero element such that $h(v) = \lambda v$ where $\lambda \in \mathbb{C}$.

(a) Find an explicit formula for $h(f^i(v))$ as a function of λ.
(b) Assume that $e(v) = 0$. Find explicit formulas for $e(f^i(v))$.
(c) Show that if V is finite-dimensional then there is some nonzero $v \in V$ and nonnegative $d \in \mathbb{Z}$ such that $e(v) = 0$ and $h(v) = dv$.
(d) Classify all simple finite-dimensional $\mathcal{U}(e,h,f)$-modules up to isomorphism.

Proof. (a): We know $hf - fh = -2f$, so

$$hf(v) = (fh - 2f)(v) = (\lambda - 2)fv.$$

I claim that in general $h(f^i v) = (\lambda - 2i)f^i v$. This can be seen inductively:

$$\begin{aligned} hf^i v &= hf(f^{i-1}v) \\ &= (fh - 2f)(f^{i-1}v) \\ &= f(hf^{i-1}v) - 2f^i(v) \\ &= f(\lambda - 2(i-1))f^{i-1}v - 2f^i v \\ &= (\lambda - 2i)f^i v. \end{aligned}$$

Similarly $h(e^i v) = (\lambda + 2i)e^i v$.

(b): Because $ef - fe = h$,

$$ef(v) = (fe + h)(v) = f(0) + \lambda v = \lambda v.$$

For general i, we have $ef^i(v) = \lambda f^i v + ((i-1)\lambda - i(i+1))f^{i-1}v$ by induction:

$$\begin{aligned} ef^i(v) &= ef(f^{i-1}v) \\ &= (fe + h)(f^{i-1}v) \\ &= f(ef^{i-1}v) + hf^{i-1}v \\ &= (\lambda f^i v + ((i-2)\lambda - i(i-1))f^{i-1}v) + (\lambda - 2(i-1))f^{i-1}v \\ &= \lambda f^i v + ((i-1)\lambda - i(i+1))f^{i-1}(v) \end{aligned}$$

(c): Let v be any nonzero vector in V. By part (a), we see that $e^i v$ is an eigenvector of h for all i, and they all have different eigenvalues, so $e^i v = 0$ for all but finitely-many $i \in \mathbb{N}$. Thus let d be maximal such that $e^d v \neq 0$. Then $e(e^d v) = 0$ and $h(e^d v) = (\lambda + 2d)(e^d v)$.

We can view h with respect to the eigenbasis $v, ev, e^2 v, \ldots, e^d v$ as the matrix

$$h = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda+2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda+4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda+2d \end{bmatrix}$$

and thus h has trace

$$\mathrm{tr}(h) = \sum_{i=0}^{d} \lambda + 2i = (d+1)\lambda + (d)(d+1) = (d+1)(\lambda + d).$$

But since $\mathrm{tr}(h) = \mathrm{tr}(ef - fe) = \mathrm{tr}(ef) - \mathrm{tr}(fe) = 0$, this implies $\lambda = -d$, and hence $h(e^d v) = de^d v$.

(d): Let V be a simple $\mathcal{U}(e, h, f)$-module. By (c), there is some v with $e(v) = 0$ and $h(v) = dv$. By (a) and (b), the subspace $V' = \langle v, f(v), f^2(v), \ldots \rangle$ is closed under multiplication by h and e, and clearly f as well, so V' is a submodule. Since V is simple and $V' \neq 0$ (as $v \neq 0$), $V' = V$.

As before, $f^i(v)$ is an h-eigenvector for all i, and they all have different eigenvalues, which implies that the nonzero $f^i(v)$ are all distinct and linearly independent. So because V is finite-dimensional, there must be some minimal n for which $f^n(v) = 0$. Then V has basis

$$V := \langle v, fv, f^2 v, \ldots, f^n v \rangle$$

By (a) and (b), it's already determined how h, e, f act on this basis. So there is exactly one simple $\mathcal{U}(e, f, h)$-module of dimension n up to isomorphism, for each n.

□

Problem 2:

(a) Check that the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form an $\mathbb{R}$-basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and that these matrices satisfy the relations in problem 1.

Proof. $\mathfrak{sl}_2(\mathbb{R})$ is exactly the matrices x such that $\det(e^{tx}) = e^{t\cdot\mathrm{tr}(x)} = 1$ for all $t \in \mathbb{R}$, or equivalently $\mathrm{tr}(x) = 0$. Thus, $\mathfrak{sl}_2(\mathbb{R})$ consists of the matrices

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH + bE + cF$$

for $a, b, c \in \mathbb{R}$. Thus E, F, H are a basis for $\mathfrak{sl}_2(\mathbb{R})$.

It is easy to check the three relations by just doing the matrix multiplications. In fact, we can check that $HE = E, EH = -E$, and and $HF = -F, FH = F$, and

$$EF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad FE = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

The relations follow. □

Problem 3: Let $1_{ij} \in M_n(\mathbb{R})$ denote the matrices with 1 in the (i, j) place and 0 elsewhere.

(a) Check that for any $i < j$, the matrices $e = 1_{ij}, h = 1_{ii} - 1_{jj}, f = 1_{ji}$ satisfy the relations in problem 1.

(b) Let $\phi : M_n(\mathbb{R}) \to \mathrm{End}_{\mathbb{C}}(V)$ be a Lie algebra representation, where $M_n(\mathbb{R}) = \mathrm{Lie}(\mathrm{GL}_n(\mathbb{R}))$ is viewed as a Lie algebra wrt the commutator and V is a finite-dimensional complex vector space. Prove (without using arguments from class) that there is a nonzero $v \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

- $\phi(1_{ii})(v) = \lambda_i v$ for all i.
- $\phi(1_{ij})(v) = 0$ for all $i < j$.
- $\lambda_i - \lambda_{i+1}$ is a nonnegative integer for all $1 \le i \le n - 1$.

Proof. (a) Just as in problem 2, we can check that $he = e, eh = -e, hf = -f, fh = f$. And $ef = 1_{ii}, fe = 1_{jj}$, which shows $ef - fe = h$.

(b): Since $[1_{ii}, 1_{jj}] = 0$ for all i, j, the actions of $\phi(1_{ii})$ all pairwise commute as well. Thus if v is any eigenvector of $\phi(1_{ii})$ with eigenvalue λ_i, then

$$\phi(1_{jj})v = \lambda_i^{-1}\phi(1_{jj})\phi(1_{ii})v = \lambda_i^{-1}\phi(1_{ii})\phi(1_{jj})v$$

so $\phi(1_{jj}v)$ is also an eigenvector of $\phi(1_{ii})$ with eigenvalue λ_i. Thus the $\phi(1_{ii})$-eigenspace W corresponding to λ_i is both $\phi(1_{jj})$-invariant.

Since $\phi(1_{jj})$ preserves W, there is at least some eigenvector $w \in W$ of $\phi(1_{jj})$ because $\mathbb{C}$ is algebraically closed. Thus, the space of common eigenvectors between the two operators is nontrivial. In this way, one can continue inductively and show that the space of common eigenvectors between all the $\phi(1_{ii})$ is nontrivial, and thus contains some v with corresponding eigenvalues λ_i for $\phi(1_{ii})$.

Now for any $i < j$, $w := \phi(1_{ij})v$ is also an eigenvector for all ϕ_{kk}, though with different eigenvalues;

$$\phi(1_{kk})w = \phi(1_{kk})\phi(1_{ij}v) = \big(\phi(1_{ij})\phi(1_{kk}) + [\phi(1_{kk}), \phi(1_{ij})]\big)v$$

in the case $k \notin \{i, j\}$, the commutator is 0, yielding

$$\phi(1_{kk})w = \big(\phi(1_{ij})\lambda_k\big)v = \lambda_k w$$

so the eigenvalue does not change. In the case $k = i$, the commutator is λ_{ij}, resulting in

$$\phi(1_{ii})w = \big(\phi(1_{ij})\lambda_i + \phi(1_{ij})\big)v = (\lambda_i + 1)w$$

so the eigenvalue increases by 1. In the case $k = j$, the commutator is $-\lambda_{ij}$, giving

$$\phi(1_{jj})w = \big(\phi(1_{ij})\lambda_j - \phi(1_{ij})\big)v = (\lambda_j - 1)w$$

so the eigenvalue decreases by 1. If we begin by maximizing the metric

$$n\lambda_1 + (n-1)\lambda_2 + \cdots + \lambda_n$$

then replacing v by w would result in increasing this metric by

$$(n - i + 1) - (n - j + i) = j - i$$

which would contradict maximality. Thus, w cannot exist, so $\phi(1_{ij})v = 0$.

To get the third condition, note that $\lambda_i - \lambda_{i+1}$ is the eigenvalue of v for $h = 1_{ii} - 1_{jj}$. By the same trace argument as in problem 1(c), we see that this eigenvalue must be a positive integer. $\square$

The group $\mathrm{GL}_2(k)$ acts on k^2 in the usual way. For $k = \mathbb{R}$ this action induces an action on $C^\infty(\mathbb{R}^2)$ by $g : p \mapsto g^* p$. For any $p \in C^\infty(\mathbb{R}^2)$ and $\chi \in \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R})) = M_2(\mathbb{R})$, the Lie derivative $L_\xi(p)$ is a function on $\mathbb{R}^2$ defined by

$$(L_\xi(p))(x,y) = \frac{\mathrm{d}(e^{t\xi})^*(p)\cdot(x,y)}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}(p(e^{-t\xi}(x,y)))}{\mathrm{d}t}\Big|_{t=0}.$$

Problem 4: For $\chi = E, H, F$ as in problem 2, find an explicit formula for $L_\xi(p)$ in terms of the partials of the function p and check that the operators L_H, L_E, L_F satisfy relations in problem 1.

Proof. Let

$$e^{-t\xi} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}.$$

Note that

$$\xi = \begin{bmatrix} -a'(0) & -b'(0) \\ -c'(0) & -d'(0) \end{bmatrix}.$$

We can calculate $L_\xi(p)$ as

$$\begin{aligned} L_\xi(p) &= \frac{\mathrm{d}(p(e^{-t\xi}(x,y)))}{\mathrm{d}t}\Big|_{t=0} \\ &= \frac{\mathrm{d}p(a(t)x + b(t)y, c(t)x + d(t)y)}{\mathrm{d}t}\Big|_{t=0} \\ &= (\partial_1 p)(a'(0)x + b'(0)y) + (\partial_2 p)(c'(0)x + d'(0)y) \\ &= (\partial_1 p)\cdot -\xi_1(x,y) + (\partial_2 p)\cdot -\xi_2(x,y). \end{aligned}$$

For $\xi \in \{H, E, F\}$ from the previous problem, this yields

$$L_H(p) = \partial_2 p \cdot y - \partial_1 p \cdot x, \quad L_E(p) = -\partial_1 p \cdot y, \quad L_F(p) = -\partial_2 p \cdot x.$$

Checking the relations from problem 1,

$$\begin{aligned} (L_H L_E - L_E L_H)(p) &= (\partial_2 L_E)y - (\partial_1 L_E)x - (\partial_1 L_H)y \\ &= (-\partial_1 p)y - 0 + (-\partial_1 p)y \\ &= -2(\partial_1 p)y = 2L_E(p) \end{aligned}$$

and similarly

$$\begin{aligned} (L_H L_F - L_F L_H)(p) &= (\partial_2 L_F)y - (\partial_1 L_F)x - (\partial_2 L_H)x \\ &= 0 - (-\partial_2 L_F)x - (-\partial_2 p)x \\ &= 2(\partial_2 p)x = -2L_F(p) \end{aligned}$$

and

$$\begin{aligned} (L_E L_F - L_F L_E)(p) &= (-\partial_1 L_F)y - (-\partial_2 L_E)x \\ &= (\partial_2 p)y - (\partial_1 p)x \\ &= L_H(p). \end{aligned}$$

□

Problem 5: Use problems 1 and 4 to prove that the representations $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(P_d)$ are precisely the irreducible finite dimensional continuous representations of $\mathrm{SL}_2(\mathbb{R})$.

Proof. Let $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(P_d)$ be the representation

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^i y^j = (ax+by)^i (cx+dy)^j.$$

This is $d+1$ dimensional. ρ is irreducible because $\mathrm{d}\rho$ is, as we will show:

Note that

$$e^{tH} = I + tH + t^2 I/2 + t^3 H/6 + \cdots = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

and

$$e^{tE} = I + tE = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad e^{tF} = I + tF = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

So can calculate $\mathrm{d}\rho$ on the basis H, E, F of $\mathfrak{sl}_2(\mathbb{R})$ as

$$\begin{aligned} \mathrm{d}\rho(H)(x^i y^j) &= \partial_t \rho(e^{tH}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t ((e^t x)^i (e^{-t} y)^j)\big|_{t=0} \\ &= \partial_t e^{t(i-j)} x^i y^j\big|_{t=0} \\ &= (i-j) x^i y^j. \end{aligned}$$

For E, we have

$$\begin{aligned} \mathrm{d}\rho(E)(x^i y^j) &= \partial_t \rho(e^{tE}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t (x+ty)^i (y)^j\big|_{t=0} \\ &= y \cdot i(x+ty)^{i-1} \cdot y^j\big|_{t=0} \\ &= i x^{i-1} y^{j+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{d}\rho(F)(x^i y^j) &= \partial_t \rho(e^{tF}) \cdot (x^i y^j)\big|_{t=0} \\ &= \partial_t (x)^i (y+tx)^j\big|_{t=0} \\ &= x^i \cdot j(y+tx)^{j-1} \cdot x\big|_{t=0} \\ &= j x^{i+1} y^{j-1}. \end{aligned}$$

The action of L_H, L_E, L_F on $C^\infty(\mathbb{R}^2)$ restricts to an action on P_d. It is easy to check, using the calculations from problem 4, that $\mathrm{d}\rho(H)$ acts the same as L_H on P_d (up to sign), and similarly for $\mathrm{d}\rho(E)$ and $\mathrm{d}\rho(F)$. Thus, P_d is equivalent as a representation of $\mathbb{R}\langle L_H, L_E, L_F\rangle$ and as a representation of $\mathfrak{sl}_2(\mathbb{R})$, so the conclusion of problem 1 implies that there is exactly one n-dimensional irrep of $\mathfrak{sl}_2(\mathbb{R})$, which is P_{n-1} (one can see that the action on the basis $\{x^i y^j\}$ is the same).

Conversely, all irreps of $\mathrm{SL}_2(\mathbb{R})$ are of this form. This follows from the classification theorem which says that there is exactly one irrep with a given highest weight, which in this case is $(d, d-2, d-4, \ldots, -d)$. ∎

Problem 6:

(a) Show that the group SU_2 of unitary 2×2 matrices with determinant 1 is formed by the matrices

$$\left\{ g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

(b) Check that the following matrices form an $\mathbb{R}$-basis of the Lie algebra $\mathfrak{su}_2 = \mathrm{Lie}(SU_2) \subset M_2(\mathbb{C})$:

$$I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ I_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(c) Find $[I_i, I_j]$ for $i, j \in \{1, 2, 3\}$ and express I_i in terms of H, F, E.

Proof. (a): If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2$$

then being unitary implies that $|a|^2 + |b|^2 = 1$ and $a\overline{c} + b\overline{d} = 0$, so $\overline{d}/a = -\overline{c}/b$. If this ratio is $x \in \mathbb{C}$, then we have $d = \overline{ax}$ and $c = \overline{-bx}$. Using the fact that g has determinant 1,

$$1 = ad - bc = a(\overline{ax}) - b(\overline{-bx}) = (|a|^2 + |b|^2)\overline{x} \implies x = 1$$

so

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

as desired. Moreover, for any $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, this g is clearly in SU_2.

(b): We showed in a previous homework that $\mathfrak{u}_2$ consists of the skew-hermitian matrices, and that $\mathfrak{sl}_2$ is the trace 0 matrices. Thus, $\mathfrak{su}_2$ is the intersection of these:

$$\begin{pmatrix} ir & s + it \\ -s + it & -ir \end{pmatrix}$$

where $r, s, t \in \mathbb{R}$. So over $\mathbb{R}$, it is clearly generated by I_1, I_2, I_3.

(c): Note that $I_1 = iH$, $I_2 = E - F$, and $I_3 = i(E + F)$. This gives

$[I_1, I_2] = iH(E - F) - (E - F)iH = i(HE - HF - EH + FH) = i(2E + 2F) = 2I_3.$

$[I_2, I_3] = (E - F)i(E + F) - i(E + F)(E - F) = 2i(EF - FE) = 2iH = 2I_1.$

$[I_3, I_1] = i(E + F)iH - iHi(E + F) = HE + HF - EH - FH = 2E - 2F = 2I_2.$

$\square$

Problem 7:

(a) Classify irreducible representations of $\mathfrak{su}_2$ in finite-dimensional complex vector spaces up to isomorphism.

(b) Let $g_d : SU_2 \to \mathrm{GL}(P_d)$ be the representation of SU_2 in the vector space P_d. Show that the representations g_d are precisely the irreducible finite-dimensional continuous representations of SU_2.

Proof. (a): As shown in problem 6, $I_1 = iH, I_2 = E - F, I_3 = i(E + F)$. It follows that $H, E, F \in \mathfrak{su}_2$ as linear combinations of I_1, I_2, I_3:

$$H = -iI_1, \quad E = \frac{1}{2}(I_2 - iI_3), \quad F = \frac{1}{2}(-I_2 - iI_3)$$

In fact, $\mathfrak{su}_2$ is a $\mathbb{C}$-algebra generated by H, E, F, since they generate I_1, I_2, I_3 which in turn generate $\mathfrak{su}_2$. Thus, the conclusion in problem 1 (d) applies, showing that there is exactly one n-dimensional irrep for each n, up to isomorphism.

(b): First, g_d is irreducible because $\mathrm{d}g_d$ is, which follows from problem 1 if we check that $\mathrm{d}g_d(H)$ (and E, F) have the same action on the basis elements of $x^i y^i$ as dictated in problem 1. And unlike in problem 5, the converse also holds automatically because SU_2 is simply connected, so because $\mathrm{d}g_d$ is the unique irrep of $\mathfrak{su}_2$, g_d must be the unique irrep of SU_2 as well. □

Problem 8: Let $\mathrm{Ad} : SU_2 \to \mathrm{GL}(\mathfrak{su}_2)$ be the representation of SU_2 that sends $g \in SU_2$ to $\mathrm{Ad}\, g : \mathfrak{su}_2 \to \mathfrak{su}_2$, where $\mathrm{Ad}\, g : x \mapsto gxg^{-1}$. Let $\mathrm{d}(\mathrm{Ad}) : \mathfrak{su}_2 \to \mathrm{End}_{\mathbb{R}}(\mathfrak{su}_2) = \mathrm{Lie}(\mathrm{GL}(\mathfrak{su}_2))$ be the differential of the representation Ad.

(a) Show that $\ker(\mathrm{Ad}) = \pm\,\mathrm{id}$ and that $\mathrm{d}(\mathrm{Ad})$ is injective.
(b) Construct a surjective morphism of Lie groups $SU_2 \to \mathrm{SO}_3(\mathbb{R})$ with kernel $\pm\,\mathrm{id}$. That is, construct an isomorphism $\mathrm{SO}_3(\mathbb{R}) \cong SU_2/\{\pm\,\mathrm{id}\}$.

Proof. (a): If $\mathrm{Ad}(g) = \mathrm{id}$, then $gxg^{-1} = x$ for all $x \in \mathfrak{su}_2$, or equivalently for each of the three basis matrices. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If g commutes with I_1 then it follows that $b = c = 0$. If g commutes with I_2 then it follows that $a = d$. Finally, because $g \in SU_2$ to begin with, $ad - bc = 1$, hence $a^2 = 1$. Thus g is either id or $-\,\mathrm{id}$, as expected.

To show that $\mathrm{d}(\mathrm{Ad})$ is injective, we have by proposition 9.6.2(2) that

$$\ker(\mathrm{d}\,\mathrm{Ad}) = \mathrm{Lie}(\ker(\mathrm{Ad})) = \mathrm{Lie}(\pm\,\mathrm{id}) = 0$$

noting that $e^{tx} = \pm\,\mathrm{id}$ for all $t \in \mathbb{R}$ implies $x = 0$, as exp is invertible near 0.

(b): We'll show that Ad is such a morphism by showing that it is surjective onto $\mathrm{SO}_3(\mathbb{R})$.

The basis I_1, I_2, I_3 of $\mathfrak{su}_2$ is orthonormal with inner product $\langle a, b\rangle = -\frac{1}{2}\,\mathrm{tr}(ab)$. Any $\mathrm{Ad}(g)$ preserves this inner product, as

$$-\tfrac{1}{2}\,\mathrm{tr}(\mathrm{Ad}\, g(x)\,\mathrm{Ad}\, g(y)) = -\tfrac{1}{2}\,\mathrm{tr}(gxyg^{-1}) = -\tfrac{1}{2}\,\mathrm{tr}(xy)$$

because trace is invariant under conjugation. Thus, we see that $\mathrm{Ad}\, g$ is an orthogonal transformation on the 3-dimensional basis of $\mathfrak{su}_2$, so the image of Ad lies within $\mathrm{O}_3(\mathbb{R})$. Thus the image of $\mathrm{d}\,\mathrm{Ad}$ lies within $\mathrm{Lie}(\mathrm{O}_3(\mathbb{R}))$, which we know to be the skew-symmetric matrices in $M_3(\mathbb{R})$, a 3-dimensional subspace:

$$\mathfrak{o}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

Then $\mathrm{d}\,\mathrm{Ad} : \mathfrak{su}_2 \to \mathfrak{o}_3$ is a map $\mathbb{R}^3 \to \mathbb{R}^3$. We know from part (a) that it is injective, so it is also surjective. And note that $\mathfrak{o}_3(\mathbb{R}) = \mathfrak{so}_3(\mathbb{R})$, since $\det(e^{tx}) = 1 \iff \mathrm{tr}(x) = 0$ is already true in $\mathfrak{o}_3(\mathbb{R})$. So in fact $\mathrm{d}\,\mathrm{Ad}$ is bijective on $\mathfrak{so}_3(\mathbb{R})$.

Ad cannot be surjective onto $\mathrm{O}_3(\mathbb{R})$, as SU_2 is connected and Ad is continuous, so $\mathrm{im}(\mathrm{Ad})$ must be connected as well. This implies that $\mathrm{im}(\mathrm{Ad})$ is contained in the connected component of 1, which is $\mathrm{SO}_3(\mathbb{R})$ (we showed this to be connected in previous homework). And because $\mathrm{d}\,\mathrm{Ad}$ is surjective on $\mathfrak{so}_3(\mathbb{R})$, Ad must be surjective on $\mathrm{SO}_3(\mathbb{R})$ as well. $\square$

Problem 9:

(a) Prove that any continuous irreducible representation of the group $\mathrm{SO}_3(\mathbb{R})$ has odd dimension. Moreover, for every odd integer $2m+1$, there is exactly one continuous irreducible representation of dimension $2m+1$.

(b) Prove theorem 2.1.2(2), which says that any continuous finite-dimensional representation of $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to the representation H_d for some $d \geq 0$.

Proof. (a): As shown in problem 8, $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to $SU_2/\{\pm \mathrm{id}\}$ via Ad. Thus, representations of $\mathrm{SO}_3(\mathbb{R})$ map to representations of SU_2 by just having $-\mathrm{id}$ act trivially. Because there is a unique representation of SU_2 in each dimension, there can be a representation of SO_3 iff the induced representation of SU_2 is g_p; that is, if $-\mathrm{id}$ acts trivially. In g_d, $-\mathrm{id}$ acts as scaling by $(-1)^d$ on P_d. This is trivial iff d is even, hence when P_d has odd dimension.

(b): Recall that $P_d = RP_{d-2} \oplus H_d$, so
$$\dim(H_d) = \dim(P_d) - \dim(P_{d-2}) = \binom{d}{2} - \binom{d-2}{2} = 2d+1$$
thus H_d is the unique continuous representation of $\mathrm{SO}_3(\mathbb{R})$ in dimension $2d+1$. □