Problem 1: Let $M_n = M_n(\mathbb{C})$ and let
$$\det(\lambda I - a) = \lambda^n + s_1(a)\lambda^{n-1} + \cdots + s_{n-1}(a)\lambda + s_n(a)$$
where $a \in M_n$. Note that $s_i \in \mathbb{C}[M_n]$. Let $G = \mathrm{GL}_n(\mathbb{C})$ act on M_n by conjugation. For $g \in G$,
$$\det(\lambda I - gag^{-1}) = \det(\lambda I - a)$$
so $s_i \in \mathbb{C}[M_n]^G$ for all i. Show that $\mathbb{C}[M_n]^G$ is a free polynomial algebra with generators $s_1, \ldots, s_n$.

Proof. Let $p \in \mathbb{C}[M_n]^G$. If p is linear, then I claim p is diagonal. $\square$

Problem 2: Let $M = \bigoplus_{i\geq 0} M_i$ be a graded A-module (A is itself a graded k-algebra), let $\{m_s \in M, s \in S\}$ be a collection of homogeneous elements and $\overline{m_s}$ the image of m_s under the projection $M \to M/A_{>0}M$.

(a) Show that m_s generate M as an A-module iff the elements $\overline{m_s}$ span $M/A_{>0}M$ as a k-vector space.

(b) Deduce that M is finitely generated iff the k-vector space $M/A_{>0}M$ has finite dimension.

Proof. (i): Let M' be the A-submodule generated by m_s. We will show that $M_i \subseteq M'$. For M_0 this is clear (why) $\square$

Problem 3: Let $R = \{a_s \in A_{>0}, s \in S\}$ be a collection of homogeneous elements.

(a) Prove that $A_{>0}$ is generated by R as an ideal iff A is generated be R as an algebra.

(b) Deduce that $A_{>0}$ is finitely-generated as an ideal iff A is finitely-generated as an algebra.

Problem 4: Let A be a finite-dimensional algebra over an algebraically closed field k and A_{mod} the algebra A viewed as a rank 1 free A-module. Thus, a left ideal of A is the same thing as a left A-submodule of A_{mod}. Show that if the A-module A_{mod} is completely reducible then the following holds:

(a) Any left ideal of A is of the form Ae where $e = e^2$.
(b) (Wedderburn theorem) The algebra A is isomorphic to $M_{\ell_1}(k) \oplus \cdots \oplus M_{\ell_r}$ for some $\ell_1, \ldots, \ell_r \geq 0$.

Problem 5: Let W be a vector subspace of a k-vector space V.

(a) Let $f \in \mathrm{End}_k(V)$ with $\mathrm{im}(f) = W$. Show that f is idempotent (i.e. $f^2 = f$) iff f projects onto W, i.e. $V = W \oplus W'$ and $f(w, w') = (w, 0)$.

(b) Define

$$I_W := \{f \in \mathrm{End}_k(V) \mid f|_W = 0\}, \quad J_W := \{f \in \mathrm{End}_k(V) \mid \mathrm{im}(f) \subseteq W\}$$

I_W and J_W are left and right ideals of $\mathrm{End}_k(V)$ respectively. Show that if V is finite-dimensional then all left ideals I are of the form I_W for

$$W := \bigcap_{f \in I} \ker(f)$$

and similarly every right ideal J is of the form J_W where

$$W := \mathrm{span}\{\mathrm{im}(f)\}_{f \in J}$$

Problem 6 (Proposition 6.1.2 in notes):

(a) Any completely reducible finite-dimensional A-module is isomorphic to a finite direct sum of simple modules.

(b) Let $V_1, \dots, V_r$ be distinct irreducible representations over A and

$$M := (V_1)^{\ell_1} \oplus \cdots \oplus (V_r)^{\ell_r}$$

then $M^{(V_i)} = (V_i)^{\ell_i}$ (the isotypic component), and $\ell_i = \dim_k(\mathrm{Hom}_A(V_i, M))$.

(c) Let $f : M \to N$ be a morphism of finite-dimensional A-modules. Then for any irreducible V, $f(M^{(V)}) \subseteq N^{(V)}$.