SIMPLE GUIDE TO SOLVING PROBLEMS IN ALGEBRAIC TOPOLOGY

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Abstract. This is a consolidation of my algebraic topology notes. My intention is to organize things for practical use. Results are sometimes bizarrely out of logical order because I order them by their *usefulness* rather than how easy they are to prove. I've also omitted a lot of important definitions.

Contents

1. Basic Terminology

The most basic questions one asks in topology are

"Does there exist a map $X \to Y$ with property P?"

To be able to answer these questions positively, we need methods of constructing maps, but this is usually the simple part. To answer them negatively, we need *algebraic invariants*, properties of a space which are preserved by sufficiently nice maps.

Types of maps (in order of increasing strictness):

- Homotopy equivalence: continuous maps with continuous "inverses" *up to homotopy equivalence.*
- Homeomorphism: continuous maps with continuous inverses.

It's a lot easier to work with homotopy equivalence in practice because there's more you can do (contract simply-connected subspaces for example). The invariants of interest are all preserved by homotopy equivalence:

Algebraic Invariants:

- Fundamental group: $\pi_1(X)$
- Homology groups: $H_0(X), H_1(X), \ldots$
- Cohomology groups: $H^0(X;G), H^1(X;G), \ldots$
- Cohomology ring: $H^*(X;G)$ as a graded ring with $\smile$.
- Higher homotopy groups: $\pi_2(X), \pi_3(X), \ldots$

Here are these properties of X partially-ordered from fine to coarse (arrow $A \to B$ indicates that A determines B)

$$\begin{tikzcd}
 & & & \text{homeomorphism type} \arrow[d] & & \\
 & & & \text{homotopy type} \arrow[dlll] \arrow[dl] \arrow[dd] \arrow[drr] & & \\
\pi_1\text{-module} \arrow[dr] & & \pi_*(X) \text{ ring} \arrow[dl] & & & H^*(X;R) \text{ ring} \arrow[d] \\
 & \pi_*(X) \arrow[d] & & H_*(X) \arrow[dr] & & H^*(X;G) \arrow[dl] \\
 & \pi_1(X) & & & \chi(X) &
\end{tikzcd}$$

First, we'll discuss how to compute all of these algebraic invariants, or at least glean information about them. We'll assume X is a connected CW complex.

2. Fundamental Group

The fundamental group of X is a group $\pi_1(X)$ whose elements are homotopy equivalence classes of loops in X, and whose multiplication is the concatenation of loops (at some base point, but if X is connected then the basepoint doesn't matter).

Quick Facts:

- If $\pi_1(X) = 0$ then we say X is "simply connected," i.e. all loops can be contracted.
- $\pi_1(X)$ is in general non-Abelian.

How to Calculate:

(I) **Van Kampen's Theorem**:
- Write X as a CW complex.
- If X has more than one 0-cell, quotient by a maximal spanning tree in X^1 so that X has a single 0-cell (this is a homotopy equivalence so it preserves $\pi_1(X)$).
- Let the 1-cells (loops) at this 0-cell be $\alpha_1, \alpha_2, \ldots, \alpha_k$. The boundaries of the 2-cells are loops in X^1, so they can each be written as products of the α_j. Let these boundaries be $\beta_1, \beta_2, \ldots, \beta_\ell \in \langle \alpha_1, \ldots, \alpha_k \rangle$.
- Now $\pi_1(X)$ is the free group on $\alpha_1, \ldots, \alpha_k$ mod the boundaries of the 2-cells:

$$\pi_1(X) := \langle \alpha_1, \ldots, \alpha_k \mid \beta_1 = \beta_2 = \cdots = \beta_\ell = 1 \rangle.$$

(II) **Covering Space**:
- bla
- bla

3. Computing with Exact Sequences

A sequence of Abelian groups with group homomorphisms between them

$$\cdots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} A_0 \to 0$$

is called *exact* if $\ker(f_n) = \operatorname{im}(f_{n+1})$ for each n. This is a condition which actually comes up a lot. When computing the homology or homotopy groups of a space, it is common to have an exact sequence where some groups are known and some are not. If you have enough information, you can deduce the identities of the missing groups by using exactness.[1]

Computing Missing Groups:

- *Zero*: If
$$A \xrightarrow{f} B \xrightarrow{g} C$$
is an exact sequence and $f = g = 0$ then $B = 0$. So if $A = C = 0$ then $B = 0$.
- *Isomorphisms*: If
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$
is exact, then g is an isomorphism iff $f = h = 0$. So if A, D are 0 then you can conclude $B \cong C$, though the converse is not true.
- *Rank*: If
$$0 \to A \to B \to C \to 0$$
is exact, then $\operatorname{rk}(A) + \operatorname{rk}(C) = \operatorname{rk}(B)$. That is, the free part of B is the direct sum of the free parts of A and C. This is a generalization of the rank-nullity theorem from linear algebra. More generally, if
$$0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$$
is exact, then
$$\sum_{k=1}^{n} (-1)^k \operatorname{rk}(A_k) = 0.$$
- *Splitting*: A short exact sequence
$$0 \to A \to B \to C \to 0$$
is called *split* if $B \cong A \oplus C$. This is guaranteed to be true if A, C are free, but not in general; for example,
$$0 \to \mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$
is exact, yet $\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}_2$.

  The *splitting lemma* says that a short exact sequence being split is equivalent to the existence of bla bla

  A special case of this is when C is a free group, in which case a retract automatically exists.

[1]A lot of this is very specific to the category of Abelian groups and wouldn't hold in more general settings.

4. Homology

The Homology Groups of X are a sequence $H_0(X), H_1(X), \ldots$ of *Abelian groups*, so they are of the form $\mathbb{Z}^a \oplus \mathbb{Z}_{q_1}^{b_{q_1}} \oplus \mathbb{Z}_{q_2}^{b_{q_2}} \oplus \cdots$ where q_i are prime powers.

Terminology:

- The *reduced homology* $\tilde{H}_j(X)$ is equal to $H_j(X)$ for $j > 0$, and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.
- The *relative homology* of a CW complex X and its subcomplex Y, denoted $H_j(X, Y)$, can oftentimes be computed by the long exact sequence

  $$\cdots \to H_j(Y) \to H_j(X) \to H_j(X, Y) \to H_{j-1}(Y) \to H_{j-1}(X) \to H_{j-1}(X, Y) \to \cdots$$

  I won't define it in more detail than that.

Quick Facts:

- $H_0(X) = \mathbb{Z}^a$ where a is the number of connected components of X.
- $H_1(X)$ is the Abelianization of $\pi_1(X)$.
- $H_j(X) = 0$ if X has no j-cells, in particular if $j > \dim(X)$.
- $H_j(X \vee Y) \cong H_j(X) \oplus H_j(Y)$.
- *Excision*: If Y is a subcomplex of X, $\tilde{H}_j(X/Y) \cong H_j(X, Y)$.
- If X is a manifold of dimension n:
  - $H_n(X) = \mathbb{Z}$ if X is oriented and $H_n(X) = 0$ otherwise.
  - $H_{n-1}(X)$ has torsion subgroup 0 if X is oriented and $\mathbb{Z}_2$ otherwise.

How to Calculate:

(I) **Quotients**:
- Express X as a quotient Y/A where Y is a CW complex and A is a subcomplex. Try to choose Y and A such that $H_*(Y)$ and $H_*(A)$ are both simpler than $H_*(X)$.
- By excision $\tilde{H}_j(X) \cong H_j(Y, A)$, yielding the long exact sequence:

  $$\cdots \to H_j(A) \to H_j(Y) \to \tilde{H}_j(X) \to H_{j-1}(A) \to H_{j-1}(Y) \to \tilde{H}_{j-1}(X) \to \cdots$$

- Now use $H_*(Y)$ and $H_*(A)$, along with exactness of the sequence, to deduce $H_*(X)$.
- *Special Cases / Notable Examples*:
  (i) If A is contractible then $H_*(A) = 0$, so $H_*(X) = H_*(Y)$.
  (ii) When $X = S^n$, we can express it as $X = D^n/S^{n-1}$. Because $H_*(D^n) = 0$, the exact sequence yields $H_j(S^n) \cong H_{j-1}(S^{n-1})$, so by induction we can see that $H_j(S^n) = \mathbb{Z}$ iff $j = n$, otherwise it is 0.

(II) **Mayer-Vietoris**:
- Express X as the union of two *open* subcomplexes $A \cup B$. Try to choose A and B so that A,B, and $A \cap B$ are all simpler than X.
- There is a long exact sequence relating $H_*(A), H_*(B), H_*(A \cap B)$, and $H_*(X)$:

  $$\cdots \to H_j(A \cap B) \to H_j(A) \oplus H_j(B) \to H_j(X) \to H_{j-1}(A \cap B) \to \cdots$$

- Now use $H_*(A), H_*(B), H_*(A \cap B)$, and the exactness of the sequence to deduce $H_*(X)$.
- *Special Cases / Notable Examples*:
  (i) If $A \cap B$ is contractible, then $H_j(X) \cong H_j(A) \oplus H_j(B)$. In particular, we see that $H_j(A \vee B) = H_j(A) \oplus H_j(B)$.

(III) **Cellular Homology**:
- Use this if X is a CW complex with relatively few cells.
- For each j, let c_j be the number of j-cells in X, and let $C_j(X) \cong \mathbb{Z}^{c_j}$ be the Abelian group with generators corresponding to the j-cells of X.
- We have the long exact sequence

  $$C_n(X) \xrightarrow{\mathrm{d}_n} C_{n-1}(X) \xrightarrow{\mathrm{d}_{n-1}} \cdots \xrightarrow{\mathrm{d}_1} C_0(X) \to 0$$

  in which d_n acts by taking cells to their boundaries, where each cell is counted *with degree.* d_n is a linear map $\mathbb{Z}^{c_n} \to \mathbb{Z}^{c_{n-1}}$, so we can see d_n as a $c_n \times c_{n-1}$ matrix whose (i, j)th entry is the degree of the restriction of d_n to a boundary map from the ith n-cell to the jth $(n-1)$-cell. (see examples to get a sense of how this works)

- $H_j(X)$ is the quotient
$$H_j(X) \cong \frac{\ker(\mathrm{d}_j)}{\mathrm{im}(\mathrm{d}_{j+1})}.$$

(IV) **Poincaré Duality**:
- Use this if X is an oriented n-manifold (recall that X is oriented iff $H_n(X) \cong \mathbb{Z}$).
- If X is a closed, oriented n-manifold, then
$$H_j(X) \cong H^{n-j}(X; Z) \cong \mathrm{Hom}(H_{n-j}(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-j-1}(X), \mathbb{Z}).$$
For instructions on how to easily compute Hom and Ext, see the Cohomology section.
- If X is an oriented n-manifold *with boundary* ∂X, then
$$H_j(X, \partial X) \cong H^{n-j}(X; \mathbb{Z}) \quad \text{and} \quad H_{n-j}(X) \cong H^j(X, \partial X; \mathbb{Z}).$$
- Also note that every manifold X has a 2-sheeted cover $\tilde{X} \to X$ where $\tilde{X}$ is oriented.

<u>Related Ideas</u>:
- **Euler Characteristic**: This is a coarser invariant than Homology, and often easier to compute, especially when working with manifolds.
- The Euler characteristic of a space X, denoted $\chi(X)$, is the integer
$$\chi(X) := \sum_{j=0}^{n} (-1)^j \, \mathrm{rk}(H_j(X))$$
where $\mathrm{rk}(A)$ is the number of copies of $\mathbb{Z}$ in the group A.
- It is also equal to the alternating sum
$$\chi(X) = \sum_{j=0}^{n} (-1)^j \#(n\text{-cells of } X).$$
- If X is a closed n-manifold (not necessarily oriented!) then $\chi(X) = 0$.
- If X is a manifold with boundary, then $\chi(\partial X) = 2\chi(X)$. So the Euler characteristic of a boundary is always even.

5. Cohomology

The Cohomology of X (wrt G) is a sequence of Abelian groups $H^0(X;G), H^1(X;G), \ldots$.

How to Calculate $H^*(X;G)$: Use **Universal Coefficient Theorem**:

- Use this if you know $H_*(X)$.
- The Cohomology can be determined from the Homology via the isomorphism

$$H^j(X;G) = \mathrm{Hom}(H_j(X),G) \oplus \mathrm{Ext}(H_{j-1}(X),G).$$

- To calculate Ext: each $H_k(X)$ is an Abelian group, so it has some free parts and some torsion parts. If $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_{q_1})^{b_{q_1}} \oplus (\mathbb{Z}_{q_2})^{b_{q_2}} \oplus \cdots$ then

$$\mathrm{Ext}(H_k(X),G) = (G/(q_1G))^{b_{q_1}} \oplus (G/(q_2G))^{b_{q_2}} \oplus \cdots$$

  (note that the free part $\mathbb{Z}^a$ is dropped entirely).
- To calculate Hom: Hom is additive in its first argument, so if we have an Abelian homology group $H_k(X) = \mathbb{Z}^a \oplus (\mathbb{Z}_{q_1})^{b_{q_1}} \oplus (\mathbb{Z}_{q_2})^{b_{q_2}} \oplus \cdots$ then

$$\begin{aligned}\mathrm{Hom}(H_k(X),G) &= \mathrm{Hom}(\mathbb{Z},G)^a \oplus \mathrm{Hom}(\mathbb{Z}_{q_1},G)^{q_1} \oplus \mathrm{Hom}(\mathbb{Z}_{q_2},G)^{q_2} \oplus \cdots \\ &= G^a \oplus G[q_1]^{b_{q_1}} \oplus G[q_2]^{b_{q_2}} \oplus \cdots\end{aligned}$$

  where $G[n]$ denotes the n-torsion subgroup

$$G[n] := \{g \in G : \underbrace{g + g + \cdots + g}_{n} = 0\}.$$

If $G = R$, a ring, then the cohomology groups $H^*(X;R)$ together form a graded ring under the cup product operation. With the additional structure of the cup product, the cohomology ring makes a strictly finer invariant than the groups alone. Two spaces might have the same cohomology groups in every dimension but different ring structures, e.g. with $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$.

How to Calculate $H^*(X;R)$ (as a ring):

(I) **Geometric Intuition**:
- Use this if you're in a *very* simple or easy-to-visualize space X.
- Elements of the cohomology group $H^j(X;R)$ can be identified with homotopy classes of $(n-j)$-dimensional surfaces in X.
- In this setting the cup product of two elements is the intersection of the corresponding surfaces (with orientation). It's important to note that this is invariant under homotopies of the two elements. This is a pretty bad explanation.

(II) **Künneth Formula**:
- Use this if $X = A \times B$, and $H^*(A;R), H^*(B;R)$ are both known, and finally $H^*(B;R)$ is free in every dimension.
- There is a ring isomorphism

$$H^*(A \times B;R) \cong H^*(A;R) \otimes_R H^*(B;R).$$

- In particular this also implies that

$$H^n(A \times B;R) \cong \bigoplus_{j=0}^{n} H^j(A;R) \otimes_R H^{n-j}(B;R)$$

  as groups.

(III) **Poincaré Duality**:
- Use this if X is a closed, connected R-oriented n-manifold.
- In this case $H^j(X;R) = H^{n-j}(X;R)$ for all j. We can sometimes guarantee that the generators of $H^n(X;R) = \mathbb{Z}$ factor as cup products of elements in these two groups.
- Suppose $R = \mathbb{Z}$. If $H^j(X;\mathbb{Z}) = H^{n-j}(X;\mathbb{Z})$ has positive rank, then for α a $\mathbb{Z}$-generator in $H^j(X;\mathbb{Z})$ and β a $\mathbb{Z}$-generator in $H^{n-j}(X;\mathbb{Z})$,

$$\alpha \smile \beta \text{ generates } H^n(X;\mathbb{Z}).$$

- Suppose R is a field. If $H^j(X;R) = H^{n-j}(X;R)$ are nontrivial, then for any nonzero $\alpha \in H^j(X;R)$ and $\beta \in H^j(X;R)$,
$$\alpha \smile \beta \text{ generates } H^n(X;R).$$

6. Higher Homotopy Groups

The nth homotopy group of X is denoted $\pi_n(X)$. It is like a higher-dimensional analogue of $\pi_1(X)$, where the elements are equivalence classes of maps $S^n \to X$.

Terminology:

- X is *n-connected* if $\pi_i(X) = 0$ for $i \leq n$. So, in particular, 0-connected means connected, and 1-connected means simply connected.
- The *relative homotopy* of Y wrt A, denoted $\pi_i(Y, A)$, can oftentimes be computed from $\pi_*(Y)$ and $\pi_*(A)$ via the long exact sequence
$$\cdots \to \pi_n(A) \to \pi_n(Y) \to \pi_n(Y, A) \to \pi_{n-1}(A) \to \cdots$$
I won't define it.

Quick Facts:

- $\pi_n(X)$ is Abelian for $n \geq 2$.
- X is *contractible* iff $\pi_j(X) = 0$ for all j. The groups $\pi_*(X)$ do not generally determine the homotopy-type of X but in this special case they do.
- *Cellular Approximation*: $\pi_j(X) = 0$ if X has no cells of dimension $\leq j$.
- *Hurewicz*: If $\pi_1(X) = 0$, the first nonzero $\pi_i(X)$ and $H_j(X)$ occur at the same dimension $i = j$, and $\pi_j(X) = H_j(X)$.
  - *Wedge of n-cells*: It follows from cellular approximation that any wedge of n-spheres is $(n-1)$-connected. So by Hurewicz,
$$\pi_n\Big(\bigvee_\alpha S^n_\alpha\Big) = H_n\Big(\bigvee_\alpha S^n_\alpha\Big) = \mathbb{Z}^\alpha$$
  - *Wedge of n-cells with attached $(n+1)$-cells*: Similarly, if X has only cells of dimensions n and $n+1$, with n-cells $\vee_\alpha S^n_\alpha$ and $(n+1)$-cells S^{n+1}_β attached via some attaching maps $\varphi_\beta : S^n \to \vee_\alpha S^n_\alpha$, then
$$\pi_n(X) = H_n(X) = \mathbb{Z}^\alpha / \langle \varphi_\beta \rangle_\beta$$
by Cellular approximation.[2]
- For any collection of path-connected spaces $\{X_\alpha\}$, $\pi_n\big(\prod_\alpha X_\alpha\big) \cong \prod_\alpha \pi_n(X_\alpha)$.

How To Calculate $\pi_n(X)$:

(I) **Fiber Bundles**:

- A map $p : E \to X$ is called a *fiber bundle* if there is an open cover $\{U\}$ of X such that $p^{-1}(U)$ is homeomorphic to $U \times F$ for all U in the open cover (F is the *fiber*).
- If there is such a fiber bundle p, then we have the long exact sequence
$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(X) \to \pi_{n-1}(F) \to \cdots \to \pi_0(F) \to \pi_0(E) \to 0$$
and one can use this sequence to gain information about $\pi_n(X)$.
- *Special Cases*:
  - *Covering space*: If $p : \tilde{X} \to X$ is a covering space of X, then it is a fiber bundle with F some discrete set, so $\pi_n(F) = 0$ for $n \geq 2$, hence
$$p_* : \pi_n(\tilde{X}) \to \pi_n(X)$$
is an isomorphism for all $n \geq 2$.
  - *Universal cover*: If X has a universal cover $\tilde{X}$ (so that $\pi_1(\tilde{X}) = 0$) then $\pi_2(X) = \pi_2(\tilde{X}) = H_2(\tilde{X})$.
  - *Group action*: If X is acted upon by a topological group G, then one can obtain a fiber bundle by taking the quotient of this action:
$$G \to X \to X/G$$

[2] These two examples are typically used to *prove* Hurewicz's theorem, so this is a bit backwards. But it's easy to remember them this way.

where X/G is the space of G-orbits of X.
   - *Hopf Fibration*: *check later to get definition right*

(II) **Quotients / Excision**:
- Express X as a quotient Y/A, where Y is a CW complex and A a subcomplex.
- If (Y, A) is r-connected (a common way to tell this is if Y, A have the same r-skeleton) and A is s-connected, then the map
$$\pi_j(Y, A) \to \pi_j(X)$$
induced by the quotient map $Y \to X$ is an isomorphism for $j \le r + s$ and a surjection for $j = r + s + 1$.
- This is in contrast to the situation with homology groups, in which this holds for all j.
- *Important Cases*:
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The group $\pi_n(X)$ has additional structure: it is acted upon by $\pi_1(X)$. A loop $\gamma \in \pi_1(X)$ acts on a map $f : S^n \to X$ in $\pi_n(X)$ by sending the basepoint of S^n around γ. This additional structure can sometimes help to calculate $\pi_n(X)$.

7. Existence Results

Building a Space with Given Algebraic Properties:

- *Moore Spaces*: For any Abelian group G, there is a Moore space $M(G, n)$ for which
$$\tilde{H}_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$
- By taking wedge products of Moore spaces $M(G_1, 1), M(G_2, 2), \ldots$ one can construct a space with any arbitrary homology sequence.
- *Eilenberg-MacLane*: For any group G, there is a space X of type $K(G, 1)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = 1 \\ 0 & j > 1 \end{cases}$$
Moreover, if G is Abelian, then for any n there is a space X of type $K(G, n)$, i.e. for which
$$\pi_j(X) = \begin{cases} G & j = n \\ 0 & j \neq n \end{cases}$$
All $K(G, n)$ spaces are weak-homotopy equivalent, and all CW complexes of type $K(G, n)$ are fully homotopy equivalent by Whitehead's Theorem.
- *Postnikov Tower*: For any connected CW complex X, there is decreasing sequence of CW complexes $X_1, X_2, \ldots$ such that
  - X is a subcomplex of each X_j,
  - The inclusion $X \hookrightarrow X_j$ induces an isomorphism on π_i for $i \leq j$,
  - $\pi_i(X_j) = 0$ for $i > j$.

Constructing Homotopy Equivalences:

- *Quotient*: If A is a contractible subspace of X, then $X \simeq X/A$.
- *Deformation Retraction*: A homotopy $f : X \to A$, where A is a subspace of X, is a deformation retraction if it restricts to the identity on A. Any such map is a homotopy equivalence.
- *Whitehead's Theorem*: To show that a map $f : X \to Y$ is a homotopy equivalence, it suffices to show that $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism in all dimensions.
- If X and Y are *simply connected*, it also suffices to show that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism in all dimensions, by Hurewicz's theorem.

Constructing Homotopies and Continuous Maps:

- *Homotopy Extension Property for CW Pairs*: Every CW complex X and subcomplex A has the homotopy extension property, which says that if one has a map $f_0 : X \to Y$ and a homotopy on just A, $h_t : A \to Y$ for which $h_0 = f_0|_A$, then this homotopy can be extended to $\tilde{h}_t : X \to A$ where $\tilde{h}_0 = f_0$.
- As a special case of this, to show that there is a retract $X \to A$ (a continuous map which is the identity on A, a weaker condition than being a deformation retract!) it suffices to find a map $X \to A$ which restricts to a map *homotopic to* id_A. Then HEP will extend this homotopy to X and the resulting map will be a retract onto A.

Ways to show that a map $f : X \to X$ has a fixed point:

- *Brouwer's Fixed Point Theorem*: If X is homeomorphic to a closed ball then f has a fixed point.
- *Lefschetz Fixed Point Theorem*: Assume X is compact and has highest dimension n. The Lefschetz trace of f is the alternating sum
$$\tau(f) := \sum_{i=0}^{n} (-1)^n \operatorname{tr}(f_* : H_i(X) \to H_i(X)).$$
If this sum is nonzero then f has a fixed point.

- *Degree*: If $X = S^n$, then if $f : S^n \to S^n$ has no fixed point, it must be homotopic to the antipodal map and thus have degree $(-1)^{n+1}$; that is, the induced map
$$f_* : \underbrace{H_n(S^n)}_{\mathbb{Z}} \to \underbrace{H_n(S^n)}_{\mathbb{Z}}$$
is a multiplication by $(-1)^{n+1}$. If the degree is anything other than this, then there must be a fixed point.

8. Worked Examples

Example 1: The Klein Bottle.

Fundamental Group: The Klein bottle can be represented as a CW complex with one 0-cell, two 1-cells, and one 2-cell as depicted below:

DIAGRAM

Letting the 1-cells be α, β, the attaching map of the 2-cell has boundary $\alpha\beta\alpha^{-1}\beta$, so by Van Kampen's Theorem we have

$$\pi_1(X) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle$$

In other words $\alpha\beta = \beta^{-1}\alpha$. Every string of α and β can be rearranged using this rule into a canonical representative $\alpha^n\beta^m$ with $n, m \in \mathbb{Z}$. These multiply by

$$\alpha^{n_1}\beta^{m_1} \cdot \alpha^{n_2}\beta^{m_2} = \alpha^{n_1+n_2}\beta^{(-1)^{n_2}m_1+m_2}$$

(each β in β^{m_1} has its sign swapped n_2 times when swapping with the α^{n_2} block). Thus we can consider $\pi_1(X)$ as $\mathbb{Z} \ltimes \mathbb{Z}$ with the action $n : m \mapsto (-1)^n m$. (revisit)

Homology: A natural way to calculate the homology of K is via cellular homology. We have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\mathrm{d}_2} \mathbb{Z}^2 \xrightarrow{\mathrm{d}_1} \mathbb{Z} \xrightarrow{\mathrm{d}_0} 0$$

where the boundary maps look like

$$\mathrm{d}_0 : \begin{pmatrix} 0 \end{pmatrix}, \quad \mathrm{d}_1 = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \mathrm{d}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

noting in d_2 that the boundary of the 2-cell is $\alpha + \beta - \alpha + \beta = 0\alpha + 2\beta$. So we have

$$\begin{aligned} H_0(K) &= \frac{\ker(\mathrm{d}_0)}{\mathrm{im}(\mathrm{d}_1)} = \frac{\mathbb{Z}}{0} = \mathbb{Z}, \\ H_1(K) &= \frac{\ker(\mathrm{d}_1)}{\mathrm{im}(\mathrm{d}_2)} = \frac{\mathbb{Z}^2}{0 \times 2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}_2, \\ H_2(K) &= \frac{\ker(\mathrm{d}_2)}{\mathrm{im}(\mathrm{d}_3)} = \frac{0}{0} = 0. \end{aligned}$$

These also could have been calculated faster by using other methods:

- $H_0(K) = \mathbb{Z}$ because K is connected.
- $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ because this is the abelianization of $\pi_1(K)$:
$$\mathrm{Ab}\left(\langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle\right) = \mathbb{Z}\langle \alpha, \beta \rangle / (2\beta) = \mathbb{Z} \oplus \mathbb{Z}_2.$$
- $H_2(K) = 0$ because $H_1(K)$ is not free, which implies K is non-orientable.

Cohomology: Let the coefficient ring be G. Having solved $H_*(K)$ already, we can use Universal Coefficient Theorem:

$$H^j(K; G) = \mathrm{Hom}(H_j(K), G) \oplus \mathrm{Ext}(H_{j-1}(K), G).$$

In the case $G = \mathbb{Z}$, we have

$$\begin{aligned} H^0(K; \mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(0, \mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z} \\ H^1(K; \mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}) \oplus (0) = \mathbb{Z}, \\ H^2(K; \mathbb{Z}) &= \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}) = (0) \oplus (\mathbb{Z}_2) = \mathbb{Z}_2. \end{aligned}$$

For the ring structure, beginning with $\mathbb{Z}[x, y]$ where x generates $H^1(K; \mathbb{Z}) = \mathbb{Z}$ and y generates $H^2(K; \mathbb{Z}) = \mathbb{Z}_2$. We clearly have the relation $2y = 0$ because $H^2(K; \mathbb{Z}) = \mathbb{Z}_2$, and $xy = y^2 = 0$ because $H^3 = H^4 = 0$. The only product that could be nontrivial is x^2.

When $G = \mathbb{Z}_2$,

$$H^0(K; \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) \oplus \mathrm{Ext}(0, \mathbb{Z}_2) = (\mathbb{Z}_2) \oplus (0) = \mathbb{Z}_2$$

$$H^1(K;\mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) = (\mathbb{Z}_2^2)\oplus(0) = \mathbb{Z}_2^2,$$
$$H^2(K;\mathbb{Z}_2) = \mathrm{Hom}(0,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2) = (0)\oplus(\mathbb{Z}_2) = \mathbb{Z}_2.$$

Now for the ring structure.

Homotopy Groups: There is a covering $\mathbb{R}^2 \to K$ as depicted below:

Diagram

Covering induces isomorphism on homotopy groups above dimension 1, so $\pi_n(K) = 0$ for $n \geq 2$.

9. A Couple of Important Results Proved via These Techniques

Brouwer's Fixed Point Theorem: Any continuous map $f : D^2 \to D^2$ has a fixed point.

Proof. Suppose that f has no fixed point. Then we can obtain a deformation retraction $D^2 \to S^1$ by projecting x away from $f(x)$ at a constant rate until it hits ∂D^2 (this is only well-defined because $x \neq f(x)$). This would imply that D^2 and S^1 have the same homotopy type but they clearly don't (D^2 is contractible while S^1 is not). □

This result can be extended to any D^n and hence any contractible space X.

Fundamental Theorem of Algebra: Any polynomial $p(x) = \sum_{j=0}^{n} p_j x^j$ where $p_n \neq 0$ and $n \geq 1$ has n complex roots (counted with multiplicity).

Proof. It suffices to show that there is at least one complex root z; then one can take $p(x)/(x - z)$ and use induction.

If no root exists then $p(z)$ is a nonzero complex number for all $z \in \mathbb{C}$, so we can use it to define a map □

10. Table of Homology and Homotopy for common spaces.