Problem 1: Let G be a subgroup of $\mathrm{GL}_n(k)$ and

$$I := \bigoplus_{d>0} P_d^G, \quad I_D := \bigoplus_{d>0} (D_d(\mathbb{R}^n))^G$$

(note that we exclude constants). A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called *G-harmonic* if $u(f) = 0$ for all $u \in I_D$.

(a) Show that f is G-harmonic iff $\langle u_1 u_2, f \rangle = 0$ for all $u_1 \in D(\mathbb{R}^n)$ and $u_2 \in I_D$.

(b) (optional) Let H be the vector space of G-harmonic polynomials. Prove that if G is a subgroup of O_n, then

$$\mathbb{C}[x_1, \dots, x_n] = H \oplus \mathbb{C}[x_1, \dots, x_n] I.$$

(c) Show that f is SO_n-harmonic iff $\Delta(f) = 0$ and interpret the statement of (ii) in this case.

Proof. (a): If $\langle u_1 u_2, f \rangle = 0$ for all $u_1 \in D(\mathbb{R}^n)$ then in particular for $u_1 = 1$, $\langle u_2, f \rangle = 0$.

Conversely, suppose $u(f) = 0$ for $u \in I_D$. f can be split into $f_0 + f_1 + \cdots + f_d$ where $f_k \in P_k$. *to do*

(c): □

Problem 2: Let A be a k-algebra and a be algebraic over A with minimal polynomial p_a. Show that $\mathrm{Spec}(a)$ is the set of roots of p_a in k.

Proof. $\mathrm{Spec}(a)$ is defined as the set of $\lambda \in k$ for which $(a - \lambda)$ is invertible. We use the fact that

$$p_a(t) - p_a(\lambda) = (t - \lambda)q(t)$$

for some polynomial $q(t)$ with degree $\deg(p_a) - 1$. In the case $t = a$, this gives

$$\begin{aligned} p_a(a) - p_a(\lambda) &= (a - \lambda)q(a) \\ -p_a(\lambda) &= (a - \lambda)q(a) \\ (a - \lambda)^{-1} &= -p_a(\lambda)^{-1}q(a) \end{aligned}$$

(note that $q(a) \neq 0$ since p_a is minimal and $\deg(q) < \deg(p_a)$). So as long as $p_a(\lambda) \neq 0$, $\lambda \notin \mathrm{Spec}(a)$. However, if $p_a(\lambda) = 0$, then $(a - \lambda)$ is not invertible because it divides $p_a(\lambda)$ i.e. it is a zero-divisor. $\square$