MATH 317 HW 1

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Problem 1 (Hatcher 0.4): A deformation retraction "in the weak sense" of $X \to A$ is a homotopy $f_t : X \to X$ such that f_0 is the identity map on X, f_1 is a map $X \to A$, and $f_t(A) \subseteq A$ for all $t \in [0, 1]$ (this is weaker because it does not require f_t be *constant* on A). Show that if such a map exists, then the inclusion $\iota : A \to X$ is a homotopy equivalence.

Proof. To show that ι is a homotopy equivalence it is sufficient and necessary to produce an inverse up to homotopy. This inverse will be f_1. The composition $\iota \circ f_1 : X \to X$ is homotopic to id_X via f_t. $f_1 \circ \iota : A \to A$ is also not necessarily id_A, but is homotopic to id_A via f_t (here we need the fact that f_t is always a map $A \to A$). $\square$

Problem 2 (Hatcher 0.6):

(a) Let X be the subspace of $\mathbb{R}^2$ consisting defined by

$$X := \Big([0,1] \times \{0\}\Big) \cup \bigcup_{q \in \mathbb{Q}[0,1]} \{q\} \times [0, 1-q].$$

Show that X deformation retracts to any point in $[0,1] \times \{0\}$, but not to other points in X.

(b) Let Y be the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show that there is a deformation retraction $Y \to Z$ in the weak sense, but not in the regular sense.

Proof. (a): For any point $r \in [0,1]$, we can construct a deformation retraction from X to $\{r\} \times \{0\}$ as follows: let $d_t : X \to X$ be given by

$$d_t(q,h) = \begin{cases} (q, h(1-2t)) & t \in [0, \frac{1}{2}] \\ ((2t-1)r + (2-2t)q, 0) & t \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to check that d_t is continuous and that $d_t((r,0)) = (r,0)$ for all $t \in [0,1]$.

On the other hand, every point $x := (q,h)$ with $h \neq 0$ has the property that there is an ε (namely $\varepsilon = h/2$) such that within every neighborhood of x there are points y such that there are no paths between y and x within $B_x(\varepsilon)$. Any point with this property cannot be the target of a deformation retraction:

If there was such a deformation retraction, then for every point y there is a closed time interval T_y for which $d_t(y) \notin B_x(\varepsilon)$ for $t \in T_y$. Let $y_1, y_2, \ldots$ be a sequence in X approaching x. By compactness of $[0,1]$, there is a time t such that $t \in T_{y_j}$ for infinitely many y_j. This shows d_t is not continuous in t at x, since $|d_t(x) - d_t(y)| > \varepsilon$ for arbitrarily small $|x - y|$. Thus such a deformation retraction cannot exist.

(b): The difference between being contractible to a point x and being *deformation retractable* to x is that in the former case, the homotopy need not keep x fixed. Y is contractible because it is a 1-dimensional CW complex with no cycles (i.e. a tree). Yet it is not deformation retractable to any point because every point of Y has the property described in the second part of (a).

(c): Here is a somewhat vague description of a weak deformation retraction to Z: every point $y \in Y$ has a single always-rightward path which extends to infinity through Y, so we let every point simultaneously walk along this path at the same constant rate. As a result, the "bristles" move together with the parallel section of Z, making the map continuous. At time $t = 1$, the bristles will have all rejoined Z, making this a weak deformation retraction. But all points in Z also moved, so it is not a true deformation retraction. □

Problem 3 (Hatcher 0.16): Show that S^∞ is contractible.

Proof. Let x be the single point in the 0-skeleton of S^∞. Let A_k^+, A_k^- be the upper and lower open half-k-sphere, so that $A_k^+ \cap A_k^- = S^{k-1}$ and all $A_k^\pm$ contain x. Every loop at x in S^∞ is entirely contained in A_k^+ or A_k^- for some finite k, and thus contractible to x. Thus S^∞ is also contractible. $\square$

Problem 4 (Hatcher 0.20): Show that the space $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Proof. We can think of this Klein bottle embedded in $\mathbb{R}^3$ as a square with the usual Klein bottle edge-gluing and an additional identification of two disjoint circles. $\square$

Problem 5 (Hatcher 0.23): Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Proof. Let X, Y be two contractible CW-complexes with intersection Z, and let $x \in Z$. If ℓ is a loop at x in $X \cup Y$, then □

Problem 6 (Hatcher 1.1.6):

Problem 7 (Hatcher 1.1.13):

Problem 8 (Hatcher 1.1.20):