TOPOLOGY NOTES

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ABSTRACT. These are my notes for Topology I-II-III (Math 317-319) at UChicago, 2025-2026.

CONTENTS

Topology I with Danny Calegari

This is an Algebraic Topology course.

Housekeeping:

- HW due Thursday midnight.
- Take-home midterm and final will replace HW.
- Textbook: Hatcher.
- Collaboration is encouraged on homework (but give credit where it is due).
- Grades will be roughly 50% homework 50% exams, with some generous weighting.
- Office Hours: Thursday 5-6 p.m. in Eckhart E7 (basement).

0.1. **Homotopy.**

Rather than equivalence by homeomorphism, which is "too fine to be useful," we'll use the coarser equivalence of homotopy.

We'll also be looking at a lot of computable information about topological spaces.

Suppose $f_0, f_1 : X \to Y$ are two (continuous) maps between topological spaces X and Y. We say f_0, f_1 are *homotopic* if one can be continuously turned into the other, i.e. if there is a continuous map

$$F : [0,1] \to \mathrm{Hom}(X, Y)$$

for which $F(0) = f_0, F(1) = f_1$. Such an F is a homotopy. We write $f_0 \simeq f_1$.

Two spaces X and Y are *homotopy-equivalent* if there is a map $f : X \to Y$ that is an isomorphism "up to homotopy," i.e. there is a map $g : Y \to X$ for which $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Homotopy equivalence is indeed an equivalence relation (not too hard to show). Equivalence of maps is also stable under composition, which makes homotopy classes of spaces and maps a category.

If $f_0, f_1 : X \to Y$ and $A \subseteq X$ is a subset on which f_0 and f_1 agree, and additionally there is a homotopy F which transforms f_0 into f_1 while remaining constant on A, then we say $f_0 \simeq f_1$ relative to A.

We say that a space X is *contractible* if it is homotopy-equivalent to a single point. For example, $\mathbb{R}^n$ is contractible, as constant maps on $\mathbb{R}^n$ are homotopic with the identity map by straight-line contraction.

Another example: given $f : X \to Y$, there is a *mapping cylinder* M_f which is $X \times [0,1] \coprod Y$ under the gluing equivalence $(x, 1) \sim f(x)$. Then $M_f \simeq Y$ via the maps

$$h_0 : (x, t) \mapsto f(x), \quad h_1 : y \mapsto y$$

The thing that must be checked is that $h_1 \circ h_0 : M_f \to M_f$ is homotopy-equivalent to the identity on M_f. This is an example of *deformation retraction*, which means that it is a homotopy relative to Y.

0.2. **CW Complexes.**

General topology is difficult to say much about because of all the pathological cases. So we'll focus mainly on *nice* topological spaces, and in particular *CW-complexes*.

A *CW-complex* is built from cells of different dimensions and attaching maps. Each cell is a pair (D^n, S^{n-1}) consisting of a ball and its surface. We build up the complex by a "skeleton" $X_0 \subseteq X_1 \subseteq \ldots$ where X_n consists of all the cells of dimension at most n and their gluing instructions. The attaching map φ for a cell maps its boundary S^{n-1} into X^{n-1}.

The topology on a CW-complex is the *weak topology* (no relation to functional analysis) which says that A is open iff $A \cap X^n$ is open for all n.

Examples:

- A 0-dimensional CW-complex is just a collection of discrete points.
- A 1-dimensional CW-complex is essentially a graph (with possibly loops and multiple edges).
- Klein bottle, torus, two-holed torus etc. all have presentations as 2-dim CW complexes.

- One can write $\mathbb{CP}^n$ as the union of a 0-cell, a 2-cell, a 4-cell, ..., and a $2n$-cell, where gluing takes the boundary of each to the infinite line of the previous.

Some operations on CW complexes:

- *Product*: $X \times Y$ is given by the union of all products of a cell in X and a cell in Y. Its topology as a CW-complex (i.e. the weak topology) is the same as the product topology in cases where there are only a *countable* number of cells in each or if one is locally compact, but in general the topology is actually finer.
- *Quotient*: X/A, where A is a *subcomplex* of X (i.e. a closed union of cells in X) that is also *contractible*, is given by the union of cells in $X - A$ plus an additional 0-cell representing the image of all cells in A. Such a pair (X, A) is called a CW pair.
- *Suspension*: SX is $X \times [0, 1]$ where $(X, 0)$ is identified and $(X, 1)$ is identified.
- *Cone*: CX is $X \times [0, 1]$ where $(X, 1)$ is identified.
- *Join*: $X * Y$ is the space $X \times I \times Y$ quotiented such that all $(x, 0, Y)$ are identified and all $(X, 1, y)$ are identified. In the case $X = Y = [0, 1]$, the resulting $X * Y$ looks like a tetrahedron. One can think of the points of $X * Y$ as pairs $(x, y) \in X \times Y$ along with a weight $t \in [0, 1]$, such that $(x, y, 0) = x$ and $(x, y, 1) = y$.
- *Wedge*: $X \vee Y$ is $X \coprod Y$ with two specific points x and y identified.
- *Smash*: $X \wedge Y$ is $X \times Y$ with $X \vee Y$ all identified.

An important example of a CW complex obtained this way is the n-simplex, which is the join of n discrete points.

One thing to note about the quotient is that $X/A \simeq X$.

A CW-complex X is connected (and path-connected) iff X^1 is a connected graph. Thus, if X is connected then we can give a spanning tree T of its 1-skeleton X^1. Every tree is contractible, thus one can take the quotient $X/T \simeq X$.

Moreover, the quotient has a very simple structure in its low-dimension cells: $Y := X/T$ has Y^0 a single point and Y^1 a wedge of some circles. So we've shown that one can always put a connected CW-complex into this nice form while preserving its homotopy class.

If (X, A) is a CW pair and $f : A \to Y$ is some map into another CW complex (or any topological space), then one can form the space

$$X \cup_f Y := X \times Y/(a \sim f(a)).$$

And if $f, g : A \to Y$ are two homotopy-equivalent maps, then $X \cup_f Y \simeq X \cup_g Y$. This shows in particular that in the construction of CW complexes, the homotopy-type of the complex only depends on the homotopy-classes of the attaching maps.

Both of these facts can be deduced from the *Homotopy Extension Property* for CW-pairs (try this!). (X, A) has the HEP if for all spaces Y, every map $f : X \times 0 \cup A \times I \to Y$ factors through the inclusion into $X \times I$:

$$\begin{tikzcd}
X \times I \arrow[dr, dashed, "\exists g"] & \\
X \times 0 \cup A \times I \arrow[u, hook] \arrow[r, "f"'] & Y
\end{tikzcd}$$

That is, a partial homotopy $f : A \to Y$ can always be extended to a homotopy $g : X \to Y$, hence the name. The HEP is equivalent to the specific case for f the identity map on $X \times 0 \cup A \times I$. Thus, to prove the HEP, it suffices to show the following:

Proposition: If (X, A) is a CW pair then there is a retraction from $X \times I$ to $X \times 0 \cup A \times I$.

Proof. If X has dimension n, then $X = X^n$. We will produce by a series of retractions:

$$X \times I = X^n \times I \cup A \times I \to (X \cup 0) \times (X^{n-1} \times I \cup A \times I) \to (X \cup 0) \times (X^{n-2} \times I \cup A \times I) \to \dots$$

In each step we only need to retract every j-cell onto its boundary. We can do this because it has an *open side*. (check Hatcher to get the details straight later). $\square$

0.3. **The Fundamental Group.** Let X be a space. A path f in X is a map $I \to X$. A homotopy between paths f, g is a homotopy (in the sense defined before) which fixes the endpoints of the paths (so it must be that $f(0) = g(0)$ and $f(1) = g(1)$ for this to be possible). We say that f, g are homotopy-equivalent if one exists.

Two paths can be composed (concatenated) if the end point of one is the start point of the other. This is denoted $f * g$, and corresponds to a path which does f from $[0, \frac{1}{2}]$ and then does g from $[\frac{1}{2}, 1]$. If f and g are both *loops* with $f \simeq f'$ and $g \simeq g'$, then

$$f * g \simeq f' * g'.$$

This can be proven by drawing a picture. Basically the homotopies $f \to f'$ and $g \to g'$ can be concatenated.

The *fundamental group* of X, denoted $\pi(X, x)$, is made up of homotopy-classes of loops beginning and ending at $x \in X$. The operation is concatenation. The identity is given by the constant map and the inverse is given by $f^{-1}(t) := f(1-t)$. We can check that this is a genuine inverse by drawing a picture.

We also have to check that $*$ is associative, i.e. $f * (g * h) \simeq (f * g) * h$. This can also be shown by a simple picture (we're essentially just changing the rate of movement along the image of the path in different segments).

If $\pi(X, x)$ is trivial, we say X is *simply connected* (note that this does not depend on x). In general, the fundamental group only depends on the path-connected component of X in which x lies. If there is a path $\beta : x \to y$ in X then $\pi(X, x)$ is just $\beta^{-1}\pi(X, y)\beta$. This gives a group isomorphism between $\pi(X, x)$ and $\pi(X, y)$.

Any map $f : X \to Y$ induces a group homomorphism between the fundamental groups:

$$f_* : \pi(X, x) \mapsto \pi(Y, f(x))$$

given by $f_* : \alpha \mapsto f \circ \alpha$. If $f \simeq g : X \to Y$, then f_* and g_* differ by an inner automorphism. Suppose f, g are homotopic via $F : X \times I \to Y$, and let $\beta(t) = F(x, t)$. Then $f_* = \beta^{-1} g_* \beta$. And in particular, if $f(x) = g(x)$, β is the constant path at x, so $f_* = g_*$.

If X, Y are homotopy-equivalent and path-connected, then their fundamental groups are isomorphic: the composition

$$(X, x) \xrightarrow{f} (Y, f(x)) \xrightarrow{g} (X, g \circ f(x))$$

is an isomorphism up to homotopy equivalence, therefore (X, x) and $(Y, f(x))$ have isomorphic fundamental group.

An example: let $T^n := (S^1)^n$. Or equivalently, $T^n = \mathbb{R}^n / \mathbb{Z}^n$. $\mathbb{R}^n$ is a covering space of T^n. The fundamental group of T^n is $\mathbb{Z}^n$. $\mathrm{GL}_n(\mathbb{Z})$ acts on T^n in a natural way (these are outer automorphisms).