MANIFOLDS

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These are my notes for Topology II taught by Eduard Looijenga at UChicago in Winter 2026.

1. Basic Definitions

A *topological manifold* of dimension m is a topological space M that is Hausdorff and locally homeomorphic to $\mathbb{R}^m$. Such an M has an open covering $\mathcal{A} = \{U_\alpha\}$ called an *atlas* with associated homeomorphisms (*charts*) $\kappa_\alpha : U_\alpha \to \mathbb{R}^m$ which are compatible, meaning that in each intersection $U_\alpha \cap U_\beta$, we have a homeomorphic coordinate change map:

$$\mathbb{R}^m \supset \kappa_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\kappa_\beta \kappa_\alpha^{-1}} \kappa_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^m$$

The atlas is C^k if all the coordinate change maps are C^k.

$\mathcal{A}$ must be C^k in order to define the notion of a C^k function $M \to \mathbb{R}$ (relative to $\mathcal{A}$); otherwise, we could have $f : M \to \mathbb{R}$ that is C^k through one chart but not another. Naturally, which functions $M \to \mathbb{R}$ are C^k depends on $\mathcal{A}$. And in fact, **atlases define the same notion of C^k iff they are compatible**. That is, all possible notions of a C^k function on M correspond to maximal atlases, or "C^k *structures*."

The presence of a C^k structure enriches M and allows one to say more about it, so it is natural to ask whether a given M has a C^k structure. Whitney showed that all manifolds with a C^k structure also have a C^∞ structure that can be obtained by restricting the corresponding atlas (and hence a C^j structure for $j > 0$). So the C^k structures come together. However, there are topological manifolds with no C^1 structure, and hence no C^k structure for any $k > 0$. Thus the only distinction is between smooth manifolds and non-differentiable manifolds. We will be concerned only with the former.

A bijection $f : M \to N$ between smooth manifolds is called a *diffeomorphism* if it and its inverse are both C^1. This is more strict than a *homeomorphism*, which is only required to be continuous in both directions.

While in Algebraic topology we are concerned with the homotopy types of spaces, which is a coarser characterization than homeomorphism type, when studying smooth manifolds we can also ask about the diffeomorphism type, which is finer[1].

1.1. Submanifolds, Immersions, Submersions, and Embeddings.

All m-manifolds M, because they are locally homeomorphic to $\mathbb{R}^m$, have an associated *tangent space* T_pM at each point p. This is literally the set of tangent vectors to M at p. It is in fact a vector space of dimension m. For smooth maps $f : M \to N$, we can think of the derivative D_pf as literally a linear map between tangent spaces $D_pf : T_pM \to T_{f(p)}N$.

The quality of D_pf locally tells us a lot about its overall properties.

- f is called an *immersion* if D_pf is injective at all points p.
- f is called a *submersion* if D_pf is surjective (i.e. full rank) at all points p.
- f is called a *local diffeomorphism* if D_pf is invertible at all points p.

Note that a local diffeomorphism need not be a diffeomorphism because though it is locally invertible it might not be globally (consider e.g. the map $\mathbb{R} \to S^1$ given by $x \mapsto e^{ix}$).

The image of a local diffeomorphism need not even be a manifold. Consider the map taking S^1 to a figure-eight. Locally, every section of the circle is sent to a segment of the figure-eight, yet near the crossing point there is no homeomorphism to $\mathbb{R}$, so the figure-eight isn't a manifold.

[1] For example, the homeomorphism type of S^7 splits into 28 diffeomorphism types, as shown by Milnor.

Another way of looking at this is that the tangent space to the figure-eight at the crossing point is *not* a vector-space. For a higher-dimensional example, take the Klein bottle. There is a local diffeomorphism to a "fake Klein bottle" in $\mathbb{R}^3$ which self-intersects in a circle.

If an immersion is injective (so its image has no self-intersections), it is called an *embedding*. The image of an embedding $f : M \to N$ is always a manifold, and is called a *submanifold* of N. Submanifolds can be equivalently characterized in another way: a manifold $A \subset N$ is a submanifold of N if the charts $\kappa : N \to \mathbb{R}^n$ send A to a linear subspace $\mathbb{R}^k \subset \mathbb{R}^n$.

Question: Can a given N be realized as a submanifold of a given M (especially when $M = \mathbb{R}^m$)? Or in another way, can we classify all submanifolds of a given M up to diffeomorphism type?

- Any smooth manifold (as long as its topology has a countable basis) can be embedded in $\mathbb{R}^m$ for sufficiently large m, though exactly what m is the minimum is not always easy to determine.
  - **Whitney's Embedding Theorem**: a smooth m-manifold can always be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$ (smaller dimensions may be possible).
  - Cohen proved more generally that a smooth m-manifold could be immersed in $\mathbb{R}^{2m-a(m)}$, where $a(m)$ is the number of 1's in the binary expansion of m.
  - For example, the Klein bottle K can be described without reference to an underlying space via gluing instructions, and the lowest-dimensional space it can be embedded in is $\mathbb{R}^4$. In $\mathbb{R}^3$ it can be *immersed*, but not embedded.
- **Inverse/Implicit Function Theorem**:
  - If $f : M \to N$ is an immersion at p, $f(U)$ is a submanifold of N for some open $U \ni p$.
  - If $f : M \to N$ is a submersion at $p \mapsto q$, then the fiber $f^{-1}(q)$ is a submanifold of M.
- **Transverse Intersections**:
  - We say $f : M \to N$ is *transverse* to a submanifold $Q \subseteq N$ if for $q = f(p) \in Q$, $D_q f(T_p(M))$ and $T_q(Q)$ span $T_q(N)$. This is denoted $f \pitchfork Q$.
  - In this case, $f^{-1}(Q)$ is a submanifold of M with the same codimension as Q in N.

1.2. Tangent Bundles and Vector Fields.

Every smooth manifold M has an associated *tangent bundle* TM whose elements are pairs (p, v) where $v \in T_pM$. It is a $2m$-dimensional manifold.

A *vector field* over a manifold is a smooth map $V : M \to TM$ with $V(p) \in T_p(M)$. For some manifolds M, it is possible to give a basis for TM by vector fields; that is, to give vector fields $V_1, V_2, \ldots, V_m$ such that their values at $p \in M$ are always a basis of T_pM.

Question: For which M is there a basis of vector spaces?

- Examples:
  - For $M = S^1$ there is a basis, given by a 90-degree rotation at every p. Thinking of S^1 as $\mathbb{C}^\times$, this corresponds algebraically to a multiplication by i.
  - For $M = S^2$, there is not. There isn't even a nonzero vector field $V : M \to TM$. Suppose there were such $V : M \to TM$. Such V induces by projection a map $V' : S^2 \to S^2$ with $V'(p) \perp p$ for all $p \in S^2$. But we know from Algebraic Topology that every map $S^n \to S^n$ either maps at least a point to its antipode or has a fixed point (if $p, V'(p)$ are not antipodes then there is a unique shortest path between them, so we can homotope every $V'(p)$ continuously along this shortest path to p, giving a homotopy to the identity map, and hence there is a fixed point).
  - For $M = S^3$, there is a basis. It is given by analogy to the Quaternions: thinking of S^3 as $\mathbb{H}^\times$, then we have a basis via the three vector fields $p \mapsto ip, p \mapsto jp, p \mapsto kp$.