MATH 325 MIDTERM

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Problem 1:

Proof. (i): Suppose for the sake of contradiction that there is some $W \subsetneq V$ stable under $\mathrm{SL}_6(\mathbb{C})$. Then I claim that it is also stable under the action of $\mathrm{GL}_6(\mathbb{C})$, contradicting the irreducibility of V. Indeed, every $g \in \mathrm{GL}_6(\mathbb{C})$ is of the form ag' where $g' \in \mathrm{SL}_6(\mathbb{C})$ and $a = \det(g)^{1/6} \cdot \mathrm{id} \in \mathrm{GL}_6(\mathbb{C})$. So

$$g(W) = ag'(W) = aW = W.$$

$\square$

Problem 2:

Proof. A conjugacy class of $\mathrm{GL}_n(\mathbb{C})$ is determined by its eigenspaces and eigenvalues. In this set $x^2 = I$ so all eigenvalues must be ± 1. The eigenspaces themselves are indistinguishable, as they can be permuted by change of basis. Thus, the conjugacy class is determined uniquely by the number $N(d, \lambda)$ of eigenspaces of dimension d and eigenvalue λ for each $d \in \{1, 2, \ldots, n\}$ and $\lambda \in \{-1, 1\}$. The dimensions must all add to n.

To calculate this, we can go by partitions of n. For each partition of n, let m_j be the number of pieces of size j. Then choosing a conjugacy class of $\mathrm{GL}_n(\mathbb{C})$ is equivalent to choosing, for each j, the number of the m_j that have eigenvalue 1 and the number that have -1 (of which there are $m_j + 1$ ways). Thus, we get

$$\sum_{\lambda \in P_n} \prod_{j=1}^{n} m_j + 1.$$

The first couple of values are $2, 5, 10, 20, 36$ for $n = 1, 2, 3, 4, 5$. I'm sure there's a generating function for this as well. $\square$

Problem 3:

Proof. If $(x, y) \in (\mathbb{C}^2)^G$ then

$$(x, y) = (x + ay, by)$$

for all a, b with $b \neq 0$, which in the case $a \neq 0$ implies $y = 0$. Thus, this subspace is 1-dimensional and spanned by $(1, 0)$.

To show that there is no G-stable complement, note that the G-orbit of $(0, 1)$ is (a, b) for all $b \neq 0$, i.e. the entire space except for the G-fixed subspace. Thus, the smallest G-stable subspace outside of $(\mathbb{C}^2)^G$ must include all of $\mathbb{C}^2$.

If $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ is a G-intertwiner, then we can use the fact that it commutes with the action of g in particular on the input $(0, 1)$ to get

$$\varphi_1(a, b) = \varphi_1(0, 1) + a\varphi_2(0, 1), \quad \varphi_2(a, b) = b\varphi_2(0, 1)$$

for all $a, b \in \mathbb{C}$ with $b \neq 0$. The second of these equations shows that $\varphi_2(a, b)$ depends only on b and scales it by $c := \varphi_2(0, 1)$. Then the first equation shows that $\varphi_1(a, b)$ depends only on a. In the case $(a, b) = (0, 1)$, it gives

$$\varphi_1(0, 1) = \varphi_1(0, 1) + ac$$

so $\varphi_1(0, 1) = 0$, and thus in general $\varphi_1(a, b) = ac$, so φ scales both coefficients by c. It remains to show that this is also true when $b = 0$, but this follows from linearity of φ, as

$$\varphi(a, 0) = \varphi(a/2, b) + \varphi(a/2, -b) = (ac/2, bc) + (ac/2, -bc) = (ac, 0).$$

$\square$

Problem 4:

Proof. Recalling the definition of Specht modules, we have

$$V(\lambda) = \mathbb{C}\langle x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n\rangle, \quad V(\lambda^t) = \mathbb{C}\langle \Delta_{\hat{1}}, \Delta_{\hat{2}}, \dots, \Delta_{\hat{n}}\rangle$$

where by $\Delta_{\hat{m}}$ I mean

$$\Delta_{\hat{m}} := \prod_{i<j\in[n]\setminus\{m\}} (x_j - x_i) = \Delta_n \cdot \prod_{i\neq m} (x_m - x_i)^{-1} \cdot (-1)^{n-m}.$$

The map $F : V(\lambda) \to V(\lambda^t)$ can be defined on the basis of $V(\lambda)$ by

$$F : (x_m - x_{m+1}) \mapsto \Delta_{\hat{m}} - \Delta_{\widehat{m+1}}$$

Now, if $s \in S_n$ acts on $V(\lambda^t)$, consider how s affects the sign of $\Delta_{\hat{m}}$. It will flip the sign for each two indices (neither of which is m) whose order is inverted by s. The sign of s is the number of *all* pairs of indices whose order is inverted by s. Thus, $s(\Delta_{\hat{m}}) = \operatorname{sign}(s) \cdot \Delta_{\hat{m}} \cdot (-1)^{R_m}$, where R_m is the number of pairs which include m that are inverted by s.

$$s(\Delta_{\hat{m}} - \Delta_{\widehat{m+1}}) = \operatorname{sign}(s) \cdot ((-1)^{R_m} \Delta_{\widehat{s(m)}} - (-1)^{R_{m+1}} \Delta_{\widehat{s(m+1)}})$$

and

$$F(x_{s(m)} - x_{s(m+1)}) = \Delta_{\widehat{s(m)}} - \Delta_{\widehat{s(m+1)}}.$$

Some combinatorics about R_m has to be done to show that this works. $\square$

Problem 5:

Proof. For λ to be an eigenvalue of M_I means that

$$fg \equiv \lambda g \pmod{I} \iff (f-\lambda)g \in I$$

for some $g \notin I$, and similarly for λ to be an eigenvalue of $M_{\sqrt{I}}$ it is equivalent that $(f-\lambda)h \in \sqrt{I}$ for some $h \notin \sqrt{I}$.

If $f - \lambda \in \sqrt{I}$ then immediately λ must be an eigenvalue in both $\sqrt{I}$ and I (if $(f-\lambda)^n \in I$, take $h = (f-\lambda)^{n-1}$). And likewise if $f - \lambda \in I$ then λ is an eigenvalue of $M_{\sqrt{I}}$ and M_I, taking $h = 1$. So assume neither of these is the case.

Assume λ is not an eigenvalue of $M_{\sqrt{I}}$. Since $\mathbb{C}[x,y]/I$ is finite-dimensional, this implies that there are $\mathbb{C}$-linear relations between $1, x, x^2, x^3, \ldots$, so I contains a polynomial $p(x)$ and similarly I contains a polynomial $q(y)$. $V(I)$ is thus finite; for any $(a,b) \in V(I)$, a is among the finitely-many roots of p and b is a root of q, so there are only finitely-many such pairs. Now by the Nullstellensatz, $(f-\lambda)h \in \sqrt{I}$ for some $h \notin \sqrt{I}$ is equivalent to $V(f-\lambda) \cup V(h) \supseteq V(I)$ and $V(h) \not\supset V(I)$. So if such an h exists then $f = \lambda$ at some point in $V(I)$, and conversely because $V(I)$ is finite such an h always exists if $f = \lambda$ somewhere in $V(I)$, since it is possible to construct a polynomial passing through any finite collection of points and avoiding a given point (it is easy to do this with a product of lines). Assuming λ is not an eigenvalue of $M_{\sqrt{I}}$, then $f - \lambda \neq 0$ on $V(I)$, so for any $r \in I$, one can take a linear combination

$$a(f-\lambda) + br = 1$$

with $a, b \in \mathbb{C}[x,y]$, by the Nullstellensatz. But now if λ is an eigenvalue of M_I, so $(f-\lambda)g \in I$ for $g \notin I$, then take $r = (f-\lambda)g$ to get

$$a(f-\lambda) + b(f-\lambda)g = (a+bg)(f-\lambda) = 1$$

which implies $f - \lambda$ and $a + bg$ are nonzero constant polynomials, and thus that either $g = 0$, contradicting that $g \notin I$, or $(f-\lambda)g$ is a nonzero constant in I, so $I = \mathbb{C}[x,y]$, a contradiction. That is, if λ is not an eigenvalue of $M_{\sqrt{I}}$ then λ cannot be an eigenvalue of M_I.

Conversely, if λ is an eigenvalue of $M_{\sqrt{I}}$, then let $(f-\lambda)h \in \sqrt{I}$ with $h \notin \sqrt{I}$. Then $(f-\lambda)^n h^n \in I$ for some n, so

$$(f-\lambda) \cdot (f-\lambda)^{n-1} h^n \in I$$

which shows that $(f-\lambda)^{n-1}h^n$ is an eigenvector for M_I with eigenvalue λ unless it is in I. If it is in I, then $(f-\lambda)^{n-2}h^n$ is an eigenvector unless *it* is in I, and so on. If they all fail then $h^n \in I$ but we assumed $h \notin \sqrt{I}$, so $(f-\lambda)^{n-j}h^n$ must be a nontrivial eigenvector with eigenvalue λ for some j. $\square$