MATH 317 HW 7

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Problem 1 (Hatcher 4.1.5): For a pair (X, A) of path-connected spaces, show that $\pi_1(X, A, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, x_0)$ represented by loops in A at x_0.

Proof. By definition, $\pi_1(X, A, x_0)$ is the group of (homotopy classes of) maps $\gamma : I \to X$ beginning at x_0 and ending in A, where the homotopies are rel x_0, A. Since A is connected, such γ can be homotopically extended to end at x_0 and thus form a loop. All possible extensions within A are homotopic to γ, and thus are all equivalent in $\pi_1(X, A, x_0)$.

The correspondence is as follows: each class $[\gamma] \in \pi_1(X, A, x_0)$ contains loops in $\pi_1(X)$. If α is such a loop in $[\gamma]$, then all of αH is in $[\gamma]$ because concatenating loops in A (elements of H) can be done homotopically rel A. So this αH is the class we associate to $[\gamma]$. It is well-defined because if $\beta \simeq \alpha$ in $\pi_1(X)$, then the homotopy between them fixes both endpoints at x_0 which is strictly stronger than fixing one and keeping the other in A, so $\beta \simeq \alpha$ in $\pi_1(X, A, x_0)$ as well.

The inverse is just given by inclusion of αH into $\pi_1(X, A, x_0)$, as they are all in the same class (as shown above). It is easy to check that the composition of these two is the identity.

$\square$

Problem 2 (Hatcher 4.1.8): Show that the sequence

$$\pi_1(X, x_0) \xrightarrow{f} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{g} \pi_0(X, x_0)$$

is exact.

Proof. $\ker(\partial)$ is exactly the set of paths in $\pi_1(X, A, x_0)$ which are loops, equivalently the paths whose endpoint is x_0, equivalently the image of $\pi_1(X, x_0)$.

Next, $\mathrm{im}(\partial)$ is the points in A that are the other endpoint of a path in X from x_0, i.e. the points in the same connected component of X as x_0. And this is also the kernel of g, so indeed the sequence is exact here too. $\square$

Problem 3 (Hatcher 4.1.15): Show that every map $f : S^n \to S^n$ is homotopic to a multiple of the identity map by the following steps:

(a) Use Lemma 4.10 to reduce to the case that there is a point $q \in S^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible linear map near each p_i.

(b) For f as in (a), consider the composition gf where $g : S^n \to S^n$ collapses the complement of a small ball about q to the basepoint. Use this to reduce (a) further to the case $k = 1$.

(c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection.

Proof. (a): Applying lemma 4.10 in the case $Z = S^n$ and $W = \emptyset$ says that f is homotopic to a map $f_1 : S^n \to S^n$ which is piecewise linear (viewing S^n as $\mathbb{R}^n$ locally) on some polyhedron $K \subset S^n$ such that K contains $f_1^{-1}(U)$ for some $U \subset \mathbb{R}^n$. Therefore, it suffices to show the case where f is of this form.

Since f is piecewise linear on K (with finitely many pieces), the points p defining the corners of linear pieces can have $f(p)$ moved slightly while keeping f linear on the adjoining pieces, and this is clearly continuous. We can homotope f by adjusting its value on these points p so that all the piecewise linear parts become invertible. Then any given point $q \in U$ can have at most one preimage per linear piece and none outside of K, so the preimage is finite. And one can slightly move q such that the preimages are all strictly inside the pieces, achieving the desired conditions.

(b): Let $|f^{-1}(q)| = k$. Outside k small neighborhoods of preimages of q, the image of f is contained in $S^n \setminus \{q\} \simeq D^n$, which is contractible, so we can homotope f so that it is the basepoint outside of these k neighborhoods. But then f is literally a product (in the π_n sense) of k maps, each of which has a single preimage of q. Thus if we show that any f' with $k = 1$ is of the form $\pm\,\mathrm{id} \in \pi_n$, then this f will be $\pm k \cdot \mathrm{id}$, which proves the theorem for general k. So it suffices to address the case $k = 1$.

(c): Suppose we have a map f of this type with $k = 1$. In the same way as part (b), we can homotope f so that it is constant outside a neighborhood of $f^{-1}(q)$, and we already know it is invertible inside such a neighborhood from part (a). If we can deform f to $\pm\,\mathrm{id}$ in this neighborhood, then that will prove the theorem.

The space of $n \times n$ matrices with *positive* determinant is path-connected, as a subspace of $\mathbb{R}^{n^2}$. One way to see this is that every matrix $a \in M_n(\mathbb{R})$ has a well defined logarithm

$$\log(1 + a) = a + a^2/2 + a^3/3 + a^4/4 + \cdots$$

and exponential

$$e^a = I + a + a^2/2 + a^3/6 + a^4/24 + \cdots$$

so one can let $x = \log(a)$ and take the path $\gamma(t) = e^{tx}$. Then $\gamma(0) = I$, $\gamma(1) = e^x = a$, and $e^{tx} \in \mathrm{GL}_n(\mathbb{R})$ because it has inverse e^{-tx}.

Similarly, the space of negative determinant matrices is path connected. Whichever one of these we are in initially, we can deform to get id or $-\,\mathrm{id}$. $\square$

Problem 4 (Hatcher 4.1.17): Show that if X and Y are CW complexes with X m-connected and Y n-connected, then $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected, as is the smash product $X \wedge Y$.

Proof. We have the long exact sequence

$$\cdots \to \pi_j(X \vee Y) \to \pi_j(X \times Y) \to \pi_j(X \times Y, X \vee Y) \to \pi_{j-1}(X \vee Y) \to \cdots$$

X is homotopy equivalent to a CW complex X' with one 0-cell and no other cells of dimension $\leq m$, and likewise Y has a corresponding Y'. $(X \times Y, X \vee Y) \simeq (X' \times Y', X' \vee Y')$ so it suffices to show this for the latter.

The cells of $X' \times Y'$ are the same as $X' \vee Y'$ in small dimensions. The smallest-dimension cells in $X' \times Y'$ not in $X' \vee Y'$ are of dimension $m+n$, so $\pi_j(X' \vee Y') \to \pi_j(X' \times Y')$ is automatically an isomorphism for $j \leq m+n$, and thus in the long exact sequence

$$\cdots \to \underbrace{\pi_{m+n+1}(X' \times Y', X' \vee Y')}_{0} \to \pi_{m+n}(X' \vee Y') \cong \pi_{m+n}(X' \times Y') \to \underbrace{\pi_{m+n}(X' \times Y', X' \vee Y')}_{0} \to \cdots$$

implies $\pi_{m+n+1}(X' \times Y', X' \vee Y') = 0$ and likewise for all smaller dimensions.

The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$. Excision for homotopy groups implies that

$$\pi_j(X \times Y / X \vee Y) \cong \pi_j(X \times Y, X \vee Y)$$

for $j \leq m+n$. For $j = m+n+1$, the inclusion induces a surjection, and the only thing 0 can surject onto is 0, so $\pi_{m+n+1}(X \wedge Y) = 0$ as well. $\square$

Problem 5 (Hatcher 4.2.1): Use homotopy groups to show that there is no retraction $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ for $n > k > 0$.

Proof. $\mathbb{R}\mathrm{P}^n$ has a 2-sheeted cover from S^n, as we've seen before. So $\pi_k(\mathbb{R}\mathrm{P}^n) = \pi_k(S^n) = 0$, but $\pi_k(\mathbb{R}\mathrm{P}^k) = \pi_k(S^k) = \mathbb{Z}$ (assuming $k \geq 2$). And in the case $k = 1$, $\mathbb{R}\mathrm{P}^1 = S^1$, so it is still true that $\pi_1(\mathbb{R}\mathrm{P}^1) = \mathbb{Z}$. A retraction would induce an isomorphism between $\pi_k(S^n)$ and $\pi_k(S^k)$, but they are not isomorphic, so there is no retraction. $\square$

Problem 6 (Hatcher 4.2.5): Let $f : S^2_\alpha \vee S^2_\beta \to S^2_\alpha \vee S^2_\beta$ be the map which is the identity on S^2_α, and the sum of the identity and a map $S^2_\beta \to S^2_\alpha$ on S^2_β. Let X be the mapping torus of f, i.e.

$$X := \frac{(S^2_\alpha \vee S^2_\beta) \times I}{(x, 0) \sim (f(x), 1)}$$

The mapping torus of the restriction of f to S^2_α forms a subspace $A = S^1 \times S^2_\alpha \subset X$. Show that the maps $\pi_2(A) \to \pi_2(X) \to \pi_2(X, A)$ form a short exact sequence

$$0 \to \underbrace{\pi_2(A)}_{\mathbb{Z}} \to \underbrace{\pi_2(X)}_{\mathbb{Z}^2} \to \underbrace{\pi_2(X, A)}_{\mathbb{Z}} \to 0$$

and compute the action of $\pi_1(A)$ on these three groups. In particular show that the action is trivial on $\pi_2(A)$ and $\pi_2(X, A)$ but nontrivial on $\pi_2(X)$.

Proof. First, we already know that the sequence

$$\pi_3(X, A) \to \pi_2(A) \to \pi_2(X) \to \pi_2(X, A) \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A)$$

is exact. To show the short exactness of this piece, it suffices to show that $\pi_1(A) \to \pi_1(X)$ is an isomorphism, which is clear, and that $\pi_3(X, A) = 0$. Note that this incidentally shows $\pi_1(X, A) = 0$, which we will use later.

Now let's calculate each of these groups π_2. We can immediately see that

$$\pi_2(A) = \pi_2(S^1 \times S^2) = \pi_2(S^1) \times \pi_2(S^2) = \mathbb{Z}.$$

Similarly $\pi_2(X, A)$ must be $\pi_2(X/A) = \mathbb{Z}$ as well by Hurewicz, because $\pi_1(X, A) = 0$ (we will show next that the action of $\pi_1(A)$ is trivial). And it follows by exactness of the sequence above that $\pi_2(X)$ must be $\mathbb{Z}^2$.

To compute the action of $\pi_1(A)$, consider the universal cover $p : X' \to X$. X' is the mapping torus rolled out into a line. $\pi_1(A)$ acts on this space by translating the line.

For $\pi_2(A)$, the action is trivial because the 2-sphere is just being pushed down the $S^2_\alpha \times \mathbb{R}$ tube, and can be retracted back again homotopically. Similarly for $\pi_2(X, A)$ with the $S^2_\beta \times \mathbb{R}$ tube. But with $\pi_2(X)$, the action of $\pi_1(A)$ can split a sphere in S^2_β between the two tubes, so it is nontrivial. $\square$

Problem 7 (Hatcher 4.2.7): Construct a CW complex with prescribed homotopy groups and prescribed actions of π_1 on those homotopy groups.

Proof. One can construct, by taking a product of $K(G_n, n)$ for $n \geq 2$, a space X' which has homotopy groups $\pi_n(X) = G_n$ for $n \geq 2$. Now we want X' to be a covering space of some X, $p : X' \to X$ (which necessarily has the same homotopy groups for $n \geq 2$) while making $\pi_1(X) = G_1$ *and* ensuring the desired actions of G_1 on each G_n. It suffices to do this for one n at a time.

We just need the deck group of this covering to match the desired action of π_1. We can attach a contractible tree of 1-cells representing the free group, and then in X we can identify some of the branches where there are relations that want to impose. □

Problem 8 (Hatcher 4.2.10): Let X be the CW complex obtained from $S^1 \vee S^n$ (where $n \geq 2$) by attaching e^{n+1} by a map representing the polynomial $p(t) \in \mathbb{Z}[t, t^{-1}] \cong \pi_n(S^1 \vee S^n)$, so that $\pi_n(X) \cong \mathbb{Z}[t, t^{-1}]/(p(t))$. Show that $\pi'_n(X)$ is cyclic and compute its order in terms of $p(t)$. Give examples showing that the group $\pi_n(X)$ can be finitely-generated or not, independently of whether $\pi'_n(X)$ is finite or infinite.

Proof. $S^1 \vee S^n$ has a universal cover Y that is a line with $\mathbb{Z}$ copies of S^n attached along it. π_1 acts on Y by shifting this line.

$\pi'_n(X)$ is $\pi_n(X)$ but quotienting out the action of $\pi_1(X)$. Since $\pi_1(X)$ acts as multiplication by t, this is the same as setting t to 1, or quotienting by $(t-1)$. That is,

$$\pi'_n(X) = \mathbb{Z}[t, t^{-1}]/(p(t), t-1) = \mathbb{Z}/(p(1))$$

So $\pi'_n(X)$ is cyclic of order $p(1)$ (or infinite if $p(1) = 0$).

$\pi'_n(X)$ finite, $\pi_n(X)$ finitely-generated: take $p(t) = 1 - t - t^2$, for which $\pi_n(X)$ can be generated by just 1 and t, and $\pi'_n(X) = 0$.

$\pi'_n(X)$ finite, $\pi_n(X)$ infinitely-generated: take $p(t) = 2 - t$, for which $\pi_n(X) = \mathbb{Z}[t, t^{-1}]/(2-t) = \mathbb{Z}[\frac{1}{2}]$ which is infinitely generated, but $\pi'_n(X) = 0$.

$\pi'_n(X) = \mathbb{Z}$, $\pi_n(X)$ finitely-generated: take $p(t) = t - 1$, for which $\pi_n(X) = \pi'_n(X) = \mathbb{Z}$.

$\pi'_n(X) = \mathbb{Z}$, $\pi_n(X)$ infinitely-generated: take $p(t) = 2t - 2$, for which $\pi'_n(X) = \mathbb{Z}$ and $\pi_n(X)$ has nonzero and unrelated t^n for all $n \in \mathbb{Z}$, so it is infinitely-generated. $\square$

Problem 9 (Hatcher 4.2.13): Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leq n$.

Proof. If $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leq n$ then $\pi_i(Y, X) = 0$ for $1 < i \leq n$. The case $i = 1$ may not be true. But if we pass to universal covers Y' and X', then $\pi_i(Y', X') = 0$ for $1 \leq i \leq n$, so $H_i(Y', X') = 0$ by Hurewicz Theorem. And for $i > n$, $H_i(Y', X') = H_i(Y'/X') = 0$ automatically because Y'/X' has no i-cells, as Y', X' are n-dimensional. Thus, by Hurewicz again, $\pi_i(Y', X') = \pi_i(Y, X) = 0$ for all i, and so f_* induces an isomorphism in all dimensions, and by Whitehead's theorem f is therefore a homotopy equivalence. $\square$