MANIFOLDS

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These are my notes for Topology II taught by Eduard Looijenga at UChicago in Winter 2026.

1. Basic Definitions

A *topological manifold* of dimension m is a topological space M that is Hausdorff and locally homeomorphic to $\mathbb{R}^m$. Such an M has an open covering $\mathcal{A} = \{U_\alpha\}$ called an *atlas* with associated homeomorphisms (*charts*) $\kappa_\alpha : U_\alpha \to \mathbb{R}^m$ which are compatible, meaning that in each intersection $U_\alpha \cap U_\beta$, we have a homeomorphic coordinate change map:

$$\mathbb{R}^m \supset \kappa_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\kappa_\beta \kappa_\alpha^{-1}} \kappa_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^m$$

The atlas is C^k if all the coordinate change maps are C^k.

$\mathcal{A}$ must be C^k in order to define the notion of a C^k function $M \to \mathbb{R}$ (relative to $\mathcal{A}$); otherwise, we could have $f : M \to \mathbb{R}$ that is C^k through one chart but not another. Naturally, which functions $M \to \mathbb{R}$ are C^k depends on $\mathcal{A}$. And in fact, **atlases define the same notion of C^k iff they are compatible**. That is, all possible notions of a C^k function on M correspond to maximal atlases, or "*C^k structures*."

The presence of a C^k structure enriches M and allows one to say more about it, so the natural question is which M have such structures. Whitney showed that all manifolds with a C^k structure also have a C^∞ structure that can be obtained by restricting the corresponding atlas (and hence a C^j structure for $j > 0$). So the C^k structures come together. However, there are topological manifolds with no C^1 structure, and hence no C^k structure for any $k > 0$. Thus the only distinction is between smooth manifolds and non-differentiable manifolds. We will be concerned only with the former.

Smooth manifolds are always (given some cardinality restrictions) diffeomorphic to smooth submanifolds of $\mathbb{R}^n$ for some n.

A *submanifold* $N \subset M$ is a subset of M for which the charts $\kappa : M \to \mathbb{R}^m$ send N to a linear subspace $\mathbb{R}^k \subset \mathbb{R}^m$. These charts naturally give N the structure of a k-manifold.

1.1. Tangent Space.

If manifold M is smooth, it has the additional structure of *tangent spaces* T_pM at each point $p \in M$. These are vector spaces of the same dimension as M. Specifically, T_pM is the set of vectors that are $\gamma'(0)$ for curves $\gamma : I \to M$ with $\gamma(0) = p$.

For each map $f : M \to N$ where M, N are smooth manifolds, we can see the derivative of f at $p \in M$ as a linear map between the tangent spaces $D_pf : T_pM \to T_{f(p)}N$.

1.2. Basic Results.