

Problem 1: Let $M_n = M_n(\mathbb{C})$ and let

$$\det(\lambda I - a) = \lambda^n + s_1(a)\lambda^{n-1} + \cdots + s_{n-1}(a)\lambda + s_n(a)$$

where $a \in M_n$. Note that $s_i \in \mathbb{C}[M_n]$. Let $G = \mathrm{GL}_n(\mathbb{C})$ act on M_n by conjugation. For $g \in G$,

$$\det(\lambda I - gag^{-1}) = \det(\lambda I - a)$$

so $s_i \in \mathbb{C}[M_n]^G$ for all i . Show that $\mathbb{C}[M_n]^G$ is a free polynomial algebra with generators s_1, \dots, s_n .

Proof. G -invariant polynomials are determined by their values on G -orbits, i.e. the conjugacy classes of matrices. If $p \in \mathbb{C}[M_n]^G$, p is determined by its values on inputs a_{ij} that are diagonalizable as matrices (since this is a dense subset of \mathbb{C}^{n^2}). And on such inputs, by G -invariance, p is determined by its value on the conjugate diagonal matrix. Thus, p is determined by the values it takes on all diagonal matrices.

Now, if a is a diagonal matrix, we get

$$\det(\lambda I - a) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

so s_i take the values of the elementary symmetric polynomials on a_{11}, \dots, a_{nn} . And we know that the space of all symmetric polynomials on n variables is generated as a free algebra by the elementary symmetric polynomials. Thus, there is some algebraic combination of s_1, \dots, s_n which matches p on the diagonal matrices, and thus (because of G -invariance and the density of the diagonalizable matrices) on all inputs in \mathbb{C}^{n^2} . That is, all of $\mathbb{C}[M_n]^G$ is generated as an algebra by s_1, \dots, s_n . \square

Problem 2: Let $M = \bigoplus_{i \geq 0} M_i$ be a graded A -module (A is itself a graded k -algebra), let $\{m_s \in M, s \in S\}$ be a collection of homogeneous elements and $\overline{m_s}$ the image of m_s under the projection $M \rightarrow M/A_{>0}M$.

- (a) Show that m_s generate M as an A -module iff the elements $\overline{m_s}$ span $M/A_{>0}M$ as a k -vector space.
- (b) Deduce that M is finitely generated iff the k -vector space $M/A_{>0}M$ has finite dimension.

Proof. (a): In one direction, if m_s span M then clearly $\overline{m_s}$ span $M/A_{>0}M$, as the projection is surjective. Everything in $M/A_{>0}M$ exists in M as a k -linear combination in M .

In the other direction, suppose $\overline{m_s}$ span $M/A_{>0}M$. Let M' be the A -submodule of M generated by m_s . We will show that $M_i \subseteq M'$ for each i , inductively. For M_0 this is clear, as M_0 is fixed by the projection. Now assume that $M_i \subseteq M'$ for a given i and we will show that the same is true for M_{i+1} . Each $m \in M_{i+1}$ can be written as

$$a_{i+1}m_0 + a_i m_1 + a_{i-1}m_2 + \cdots + a_0 m_{i+1}$$

where $a_j \in A_j$ and $m_j \in M_j/A_{>0}M$. Then we have $\overline{m} \equiv a_0 m_{i+1}$. We assumed that this was in the span of $\overline{m_s}$, and by inductive hypothesis all of the other components are in M' , so $m \in M'$ as well. Thus the induction is complete and $M' = M$.

- (b): By part (a), if M is finitely-generated by m_s , then $M/A_{>0}M$ is spanned over k by $\overline{m_s}$.

Conversely, if $M/A_{>0}M$ is finite-dimensional with basis $m_0, m_1, \dots, m_n \subset M_N$, then M is generated over A by these same elements by (a), as they project down to themselves. \square

Problem 3: Let $R = \{a_s \in A_{>0}, s \in S\}$ be a collection of homogeneous elements.

- (a) Prove that $A_{>0}$ is generated by R as an ideal iff A is generated by R as an algebra.
- (b) Deduce that $A_{>0}$ is finitely-generated as an ideal iff A is finitely-generated as an algebra.

Proof. (a): Suppose $A_{>0}$ is generated by R as an ideal, and let A' be the graded algebra generated over k by R . We will show $A_i \subseteq A'$ for each i inductively. For A_0 , it is automatic because $k = A_0$. Now assume that $A_i \subseteq A'$ and we will show the same for A_{i+1} . For each $a \in A_{i+1}$, since $A_{>0}$ is generated by R as an ideal, we have

$$a = a_1 r_1 + a_2 r_2 + \cdots + a_n r_n$$

for some $r_j \in R$ and $a_j \in A$. And since everything in R has degree at least 1, all of these a_j are strictly lower in degree than a , so they're in A_i . By the inductive hypothesis, they are all in A' . Thus $a \in A'$ as well. This concludes the induction, showing that $A = A'$.

Conversely if A is generated by R as an algebra, then $A = k[R]$, so for any $a \in A_j$ with $j \geq 1$, a can be written as a k -linear combination of polynomials of positive degree in R , which is also an A -linear combination of elements in R (just take all but one of the R terms in each monomial to be the coefficient in A), thus $A_{>0}$ is generated by R as an ideal.

(b): If A is finitely-generated as an algebra, then there is some finite algebra generator set R (we can assume it is positive-degree because everything in k is already generated in any algebra over k). So by (a), $A_{>0}$ is finitely generated as an ideal (by R).

Conversely if $A_{>0}$ is finitely-generated as an ideal by some R , then we can assume R is positive-degree, so part (a) applies. \square

Problem 4: Let A be a finite-dimensional algebra over an algebraically closed field k and A_{mod} the algebra A viewed as a rank 1 free A -module. Thus, a left ideal of A is the same thing as a left A -submodule of A_{mod} . Show that if the A -module A_{mod} is completely reducible then the following holds:

- (a) Any left ideal of A is of the form Ae where $e = e^2 \in A$.
- (b) (Wedderburn theorem) A is isomorphic to $M_{\ell_1}(k) \oplus \cdots \oplus M_{\ell_r}(k)$ for some $\ell_1, \dots, \ell_r \geq 0$.

Proof. (a): Let I be a left ideal of A . Then I is also a left A -submodule of A_{mod} . By complete reducibility we can express A as a direct sum with I , $A = I \oplus I'$. Then $1 \in A$ can be expressed uniquely as $1 = e + e'$ where $e \in I$ and $e' \in I'$.

First, note that

$$e + e' = 1 = \langle e + e', e + e' \rangle = e^2 + e'^2 \implies e = e^2, e' = e'^2,$$

using the fact that the decomposition is unique, i.e. this is the only way to express 1 as an element of $I \oplus I'$. Moreover, $I = Ae$ and $I' = Ae'$; if $b \in I$ then

$$b = b(e + e') = be + be' = be$$

as b has no I' component. Thus $I \subseteq Ae$ and the other direction is trivial, so $I = Ae$.

(b): First, because A_{mod} is rank 1 as an A -module, $A^{op} \cong \text{End}_A(A_{mod})$ via the map

$$a \mapsto [b \mapsto ab].$$

Note that aa' acts as scaling by $a'a$ so the order is reversed, hence A^{op} . The map is invertible because A_{mod} is 1-dimensional as an A -module, hence every automorphism is scaling by something in A (Schur's Lemma).

Now we can show that $\text{End}_A(A_{mod})$ has the desired form; this follows from the fact that A is completely reducible, using problem 8 on the previous homework. Hence A^{op} has the desired form, and also A does as well, using the fact that $M_\ell(k) \cong M_\ell(k)^{op}$ (such a map can be given by transposition).

□

Problem 5: Let W be a vector subspace of a k -vector space V .

- (a) Let $f \in \text{End}_k(V)$ with $\text{im}(f) = W$. Show that f is idempotent (i.e. $f^2 = f$) iff f projects onto W , i.e. $V = W \oplus W'$ and $f(w, w') = (w, 0)$.
 (b) Define

$$I_W := \{f \in \text{End}_k(V) \mid \ker(f) \subseteq W\}, \quad J_W := \{f \in \text{End}_k(V) \mid \text{im}(f) \subseteq W\}$$

I_W and J_W are left and right ideals of $\text{End}_k(V)$ respectively. Show that if V is finite-dimensional then all left ideals I are of the form I_W for

$$W := \bigcap_{f \in I} \ker(f)$$

and similarly every right ideal J is of the form J_W where

$$W := \text{span}\{\text{im}(f)\}_{f \in J}$$

Proof. (a): If f projects onto W then it is clearly idempotent, as $f^2(w, w') = f(w, 0) = (w, 0)$.

Conversely, suppose f is idempotent. f is surjective onto W because W is its image, so for each $w \in W$ let v be such that $f(v) = w$. Then

$$f(v) = w \implies w = f(v) = f^2(v) = f(w)$$

by idempotence of f , i.e. f fixes W .

Now let $W' = \ker(f)$. I claim $V = W \oplus W'$. First, the two subspaces have intersection 0, as $0 = f(x) = x$ for all $x \in W \cap W'$. And their span is all of V : for $v \in V$, suppose $f(v) = w \in W$, and let $w' = v - w$ so that $v = w + w'$. Then $w' \in W'$, as

$$f(w') = f(v) - f(w) = w - w = 0.$$

Thus f is a projection onto W .

(b): $\text{End}_k(V)$ is completely reducible as a module over itself (in fact, it is a simple module), so we can apply problem 4(a). Since I is a left-ideal, by problem 4(a) it is of the form $\text{End}_k(A)e$ where e is an idempotent matrix, i.e. a projection by 5(a). For this e ,

$$W = \bigcap_{f \in I} \ker(f) = \bigcap_{f \in \text{End}_k(V)} \ker(fe) = \ker(e)$$

as $\ker(fe) \supseteq \ker(e)$ with equality when $f = \text{id}_V$. So I_W is the set of f which vanish on $\ker(e)$, i.e.

$$I_W = \{f \in \text{End}_k(V) : \ker(f) \subseteq \ker(e)\} = \text{End}_k(V)e = I.$$

For right ideals, we have $J = e \text{End}_k(V)$ for some projection e . Thus

$$W = \text{span}\{\text{im}(f)\}_{f \in J} = \text{span}\{\text{im}(ef)\}_{f \in \text{End}_k(V)} = \text{im}(e)$$

because $\text{im}(ef) \subseteq \text{im}(e)$ with equality when $f = \text{id}_V$. Now,

$$J_W = \{f \in \text{End}_k(V) : \text{im}(f) \subseteq \text{im}(e)\} = e \text{End}_k(V) = J.$$

□

Problem 6 (Proposition 6.1.2 in notes):

- (a) Any completely reducible finite-dimensional A -module is isomorphic to a finite direct sum of simple modules.
 (b) Let V_1, \dots, V_r be distinct irreducible representations over A and

$$M := (V_1)^{\ell_1} \oplus \dots \oplus (V_r)^{\ell_r}.$$

Then $M^{(V_i)} = (V_i)^{\ell_i}$ (the isotypic component), and $\ell_i = \dim_k(\text{Hom}_A(V_i, M))$.

- (c) Let $f : M \rightarrow N$ be a morphism of finite-dimensional A -modules. Then for any irreducible V , $f(M^{(V)}) \subseteq N^{(V)}$.

Proof. (a): We show this by induction on dimension. A 1-dimensional A -module is simple. Let M be a completely-reducible finite-dimensional A -module. If M is simple then we are done. Otherwise, take some submodule M_1 . By complete-reducibility $M = M_1 \oplus M_2$ for some M_2 . But now M_1, M_2 are lower dimension than M , so by inductive hypothesis we can express both M_1 and M_2 as finite direct sums of simple modules, and hence M can be expressed this way as well.

(b): $M^{(V_i)}$ is by definition the direct sum of all submodules of M isomorphic to V_i . There are ℓ_i copies of V_i in the decomposition given, and this is all of the isomorphic copies of V_i ; if V_i maps into M then restricting to any V_j with $j \neq i$ we get by Schur's Lemma that the map must be trivial. Thus $\text{Hom}_A(V_i, M)$ consists exactly of the direct sum of identity maps into the ℓ_i copies of V_i , and thus has dimension ℓ_i .

(c): For every submodule V' of M isomorphic to V (irreducible), $f(V')$ is an irreducible submodule of N . By Schur's Lemma it is either isomorphic to V or 0. Thus, every such $f(V')$ is contained in $N^{(V)}$. It follows that $f(M^{(V)}) \subseteq N^{(V)}$. \square