MATH 317 FINAL

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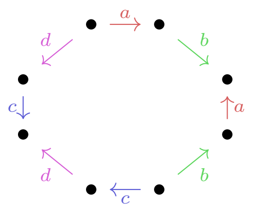
Problem 1: Let F be the closed oriented surface of genus 2.

(a) Give a CW complex structure for F with one 0-cell, four 1-cells, and one 2-cell.
(b) From this CW complex structure, write down a presentation for the fundamental group and compute the homology and cohomology in every dimension.
(c) Compute the ring structure on cohomology with coefficients in $\mathbb{Z}$.

Proof. (a):



Consider the CW structure depicted above. a, b, c, d are the 1-cells, the black dots are all identified and that's the 0-cell, and the octagonal region is the 2-cell, attached via $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.

(b): By Van Kampen's Theorem, the fundamental group is generated by the 1-cells with relations given by the 2-cell's attaching map, giving

$$\pi_1(\Sigma_2) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle.$$

To calculate the homology groups, $H_0(\Sigma_2) = \mathbb{Z}$ because Σ_2 is connected, $H_1(\Sigma_2)$ is the abelianization of $\pi_1(\Sigma_2)$, which is $\mathbb{Z}^4$, and $H_2(\Sigma_2) = \mathbb{Z}$ because Σ_2 is an oriented 2-manifold (by Poincaré duality). In higher dimensions the homology is 0. In summary,

$$H_*(\Sigma_2) = (\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}, 0, 0, \dots).$$

The cohomology can now be computed by universal coefficient theorem:

$$H^j(\Sigma_2; G) = \mathrm{Hom}(H_j(\Sigma_2), G) \oplus \mathrm{Ext}(H_{j-1}(\Sigma_2), G) = \mathrm{Hom}(H_j(\Sigma_2), G)$$

noting that H_* is free in all dimensions so the ext vanishes. $\mathrm{Hom}(\mathbb{Z}, G) = G$ for all abelian groups G, as a unique homomorphism exists bringing 1 to a given group element g. Likewise $\mathrm{Hom}(\mathbb{Z}^4, G) = G^4$. So we have the cohomology

$$H^*(\Sigma_2; G) = (G, G^4, G, 0, 0, \dots).$$

(c): Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be generators of $H^1(\Sigma_2; \mathbb{Z})$ and let γ be a generator of $H^2(\Sigma_2; \mathbb{Z})$. We can associate to each of these cocycles a loop on Σ_2. I really don't have time to make a diagram for this but say α_1 goes "around" hole 1, β_1 goes "through" hole 1, and similarly for α_2, β_2. Cup product is dual to intersection, so the ones which intersect transversely will have nontrivial cup product and the others will have trivial cup product. This is of course invariant under homotopies of these loops, as we showed in class. This gives the following cup product structure (noting that orientation is reversed when swapping the order):

$\smile$	α_1	α_2	β_1	β_2
α_1	0	0	γ	0
α_2	0	0	0	γ
β_1	$-\gamma$	0	0	0
β_2	0	$-\gamma$	0	0

$\square$

Problem 2: Let W be the space obtained from $\mathbb{C}^2$ by removing three complex lines: the line $x = 0$, the line $y = 0$, and the line $x + y = 1$.

(a) Compute the fundamental group of W.
(b) Show that W is not a $K(\pi, 1)$.

Proof. (a): We use Van Kampen's theorem to deduce $\pi_1(W)$ by breaking it into three pieces. Consider the following three open subsets $P, Q, R \subset W$:

$$P := \{(x, y) \in \mathbb{C}^2 \mid x \neq 0, y \neq 0, \Re(x + y) < 1\}$$
$$Q := \{(x, y) \in \mathbb{C}^2 \mid x \neq 0, x + y \neq 1, \Re(y) > 0\}$$
$$R := \{(x, y) \in \mathbb{C}^2 \mid y \neq 0, x + y \neq 1, \Re(x) > 0\}.$$

Note that $P \cup Q \cup R = W$; clearly all three are subsets of W, and if $(x, y) \in W$ but not in Q, R then $\Re(x), \Re(y) < 0$ so $\Re(x + y) < 0 < 1$, thus $(x, y) \in P$.

First, we show that $\pi_1(P) = \pi_1(Q) = \pi_1(R) = \mathbb{Z}^2$ by deformation-retracting each of P, Q, R to a torus. But note that P, Q, R are all symmetric; taking a linear change of variables $z = 1 - x - y$, we have

$$Q = \{(x, z) \in \mathbb{C}^2 \mid x \neq 0, z \neq 0, \Re(x + z) < 1\} = P$$

so in fact it suffices to just show this for P.

- P: Project (x, y) through $(0, 0)$ to $(\varepsilon x/|x|, \varepsilon y/|y|) \in S^1 \times S^1$. Note that as $x + y$ begins with real part < 1, and its real part linearly shifts to something at most 2ε, it cannot go above 1 during this transformation. Also x, y never become 0. So we stay in P.

Now we can show that each of the pairwise intersections has fundamental group $\mathbb{Z}$ by showing that each can be deformation retracted to a circle. Again, they are symmetric so I didn't really need to do all three, but I'll leave them in anyways:

- $P \cap Q$: Send (x, y) linearly to $(\varepsilon x/|x|, \varepsilon) \in S^1 \times \{\varepsilon\}$ where $\varepsilon \in \mathbb{R}^+$ (this relies on the fact that $x \neq 0$ initially). Note that $\Re(x + y)$ is never 1 during this, nor is $\Re(y)$ ever 0.
- $P \cap R$: Send (x, y) linearly to $(\varepsilon, \varepsilon y/|y|) \in \{\varepsilon\} \times S^1$.
- $Q \cap R$: Let $S_{1/2}(\varepsilon)$ be the circle of radius ε centered at $\frac{1}{2} \in \mathbb{R}$. We are given that x, y both have positive real part and that $m := \frac{1}{2}(x + y)$ is not $\frac{1}{2}$. Thus project m from $\frac{1}{2}$ onto $S_{1/2}(\varepsilon)$ to get m^* and send (x, y) linearly to (m^*, m^*). One can easily see that m is going away from $\frac{1}{2}$ during this retraction and thus cannot become $\frac{1}{2}$, and similarly x, y retain positive real part.

We can also easily check that $P \cap Q \cap R$ is path-connected by linearly mapping (x, y) to $(\frac{1}{4}, \frac{1}{4})$.

Let a, b, c be generators of $\pi_1(Q \cap R), \pi_1(P \cap R), \pi_1(P \cap Q)$. $\pi_1(P) = \mathbb{Z}^2$ is generated by the inclusions of b and c, which we can call b_P, c_P, and likewise for Q and R. By Van Kampen's theorem, we calculate $\pi_1(W)$ as

$$\pi_1(W) = \frac{\pi_1(P) * \pi_1(Q) * \pi_1(R)}{\langle a_Q = a_R, b_P = b_R, c_P = c_Q \rangle} = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle = \mathbb{Z}^3.$$

I think it makes sense to think of this visually as: if you have a composition of loops ab going around the lines $x + y = 1$ and $y = 0$, you can homotope this loop and bring it near the point $(1, 0)$ (in R) where locally the fundamental group is like a torus, so the loops a and b commute, and you can swap their order to get ba.

(b): The homotopy type of a $K(G, 1)$ is determined by G, and we already know that the space $(S^1)^3$ is a $K(\mathbb{Z}^3, 1)$, so it suffices to show that W differs from $(S^1)^3$ in any homotopy-invariant. We can calculate $H_*(W)$ using Alexander duality: considering W as a subspace of S^4, a one-point compactification of $\mathbb{C}^2$ (which has the same homology by excision), let $K = S^4 \setminus W$. K is the union of three complex lines, which are 2-cells all attached at the infinite point, giving K a CW structure with three 2-cells and one 0-cell. Thus K has homology

$$H_*(K) = (\mathbb{Z}, 0, \mathbb{Z}^3, 0, \dots)$$

and the exact same integral cohomology because all of the homology groups are free abelian. By Alexander duality,

$$\tilde{H}_i(W) \cong \tilde{H}^{3-i}(K) \cong \tilde{H}_{3-i}(K)$$

which gives

$$H_*(W) = (\mathbb{Z}, \mathbb{Z}^3, 0, 0, \dots)$$

so in particular $\chi(W) = -2$, but this differs from $\chi((S^1)^3) = \chi(S^1)^3 = 0^3 = 0$, so the spaces must be different, and W is therefore not a $K(\mathbb{Z}^3, 1)$. $\square$

Problem 3: Give an example of a connected CW complex X with fundamental group $\mathbb{Z}$ and vanishing homology in dimensions above 1 but for which X is not homotopy-equivalent to S^1.

Proof. Consider the following 3-dimensional CW complex X: X has one 0-cell, one 1-cell attached trivially, one 2-cell attached trivially, and one 3-cell attached via a map $\varphi : S^2 \to S^1 \vee S^2$ corresponding to the polynomial $2 - t \in \mathbb{Z}[t, t^{-1}] \cong \pi_2(S^1 \vee S^2)$.

To be more concrete, this attaching map (viewed as a loop of loops $S^1 \times I \to S^1 \vee S^2$) goes over the S^2 twice, then cycles around the S^1 once, then goes back over the S^2 in the opposite direction once, and then cycles back around the S^1. Note that the degree of φ restricted to S^2 is $(2) + (-1) = 1$.

We can verify the homology groups of X using cellular homology: the cellular chain of X is

$$\underbrace{H_3(X^3, X^2)}_{\mathbb{Z}} \xrightarrow{\mathrm{d}_3} \underbrace{H_2(X^2, X^1)}_{\mathbb{Z}} \xrightarrow{0} \underbrace{H_1(X^1, X^0)}_{\mathbb{Z}} \xrightarrow{0} \underbrace{H_0(X^0)}_{\mathbb{Z}} \to 0.$$

Because φ has degree 1 restricted to the 2-cell, the first map is surjective, so $H_3(X) = H_2(X) = 0$. And we also see that $H_1(X) = H_0(X) = \mathbb{Z}$.

But $\pi_2(X) = \mathbb{Z}[t, t^{-1}]/(2 - t) = \mathbb{Z}[\frac{1}{2}]$. To see this, consider the universal cover $\overline{X}$, which is a line with 2-cells e^2_α attached at integer points and 3-cells e^3_β attached to each neighboring pair of 2-cells. $\overline{X}$ is simply connected, as is its 2-skeleton, so by limited-range excision we can say

$$\pi_j(\overline{X}, \overline{X}^2) \cong \pi_j(\overline{X}/\overline{X}^2)$$

for $j \leq 3$. Applied to the exact sequence of the pair $(\overline{X}, \overline{X}^2)$, this yields

$$\pi_3(\overline{X}/\overline{X}^2) \xrightarrow{\partial} \pi_2(\overline{X}^2) \to \pi_2(\overline{X}) \to \pi_2(\overline{X}/\overline{X}^2) \cong 0$$

noting that $\overline{X}/\overline{X}^2$ has no 2-cells and thus by cellular approximation it has no π_2. This gives

$$\pi_2(\overline{X}) = \pi_2(\overline{X}^2)/\operatorname{im}(\partial) = \mathbb{Z}[t, t^{-1}]/(2 - t) = \mathbb{Z}[\tfrac{1}{2}].$$

And π_n for $n \geq 2$ is shared by spaces and their covering spaces, so in particular $\pi_2(X) = \pi_2(\overline{X}) = \mathbb{Z}[\frac{1}{2}]$. Thus, $\pi_2(X) \neq \pi_2(S^1) = 0$, so X is not homotopy-equivalent to S^1.

□

Problem 4: Give an example of two connected CW complexes X and Y with isomorphic homotopy groups in every dimension but with different homology groups in some dimensions.

Proof. Similarly to the previous problem, let X, Y be CW complexes defined in the following way: for X, start with $S^1 \vee S^2$ and attach a 3-cell via an attaching map $S^2 \to S^1 \vee S^2$ corresponding to the polynomial $2 - t \in \mathbb{Z}[t, t^{-1}] \cong \pi_2(S^1 \vee S^2)$, just as in problem 3. For Y, do the same but with the polynomial $4 - t$. As established in problem 3,

$$\pi_*(X) = (\mathbb{Z}, \mathbb{Z}[\tfrac{1}{2}], 0, \dots), \quad H_*(X) = (\mathbb{Z}, 0, 0, \dots).$$

For Y, $\pi_2(Y) = \mathbb{Z}[t, t^{-1}]/(4 - t) = \mathbb{Z}[\frac{1}{4}] = \mathbb{Z}[\frac{1}{2}]$, so all homotopy groups are the same. However, the degree of the attaching map of the 3-cell in Y, restricted to S^2, is $(4) + (-1) = 3$, so the map

$$\mathrm{d}_3 : H_3(Y^3, Y^2) \to H_2(Y^2, Y^1)$$

is multiplication by 3. Thus, $H_2(Y) = \ker(\mathrm{d}_2)/\operatorname{im}(\mathrm{d}_3) = \mathbb{Z}/3\mathbb{Z}$. So although X, Y have the same homotopy groups, they differ in H_2! □

Problem 5: Show that the real projective plane $\mathbb{R}\mathrm{P}^2$ is not the boundary of any compact 3-manifold M.

Proof. In the case where n is odd, any n-dimensional manifold N without boundary (oriented or not) has Euler characteristic 0. In the oriented case this follows from Poincaré duality and the universal coefficient theorem:

$$\begin{aligned}
\chi(N) &= \sum_{j=0}^{n} (-1)^j \operatorname{rk} H_j(N) \\
&= \sum_{j=0}^{n} (-1)^j \operatorname{rk} H^{n-j}(N;\mathbb{Z}) \\
&= \sum_{j=0}^{n} (-1)^j \Big(\operatorname{rk} \operatorname{Hom}(H_{n-j}(N),\mathbb{Z}) + \operatorname{rk} \operatorname{Ext}(H_{n-j-1}(N),\mathbb{Z}) \Big) \\
&= \sum_{j=0}^{n} (-1)^j \operatorname{rk} H_{n-j}(N) \\
&= (-1)^n \sum_{j=0}^{n} (-1)^{n-j} \operatorname{rk} H_{n-j}(N) \\
&= (-1)^n \chi(N).
\end{aligned}$$

So $\chi(N) = -\chi(N)$, implying $\chi(N) = 0$. And in the general case we can use the fact that N has a 2-sheeted orientable cover $\tilde{N}$, so $\chi(N) = \frac{1}{2}\chi(\tilde{N}) = 0$.

Now let D be the 3-manifold obtained by gluing together two copies of M along their shared boundary. D has two cells for every cell of M, except those in the boundary which are only included in D once, giving

$$\chi(D) = 2\chi(M) - \chi(\partial M).$$

Since D is a 3-manifold without boundary, $\chi(D) = 0$, which yields

$$\chi(\partial M) = 2\chi(M).$$

Yet $\chi(\mathbb{R}\mathrm{P}^2) = 1$, as $\chi(\mathbb{R}\mathrm{P}^2) = \frac{1}{2}\chi(S^2) = 1$ due to the 2-sheeted covering $S^2 \to \mathbb{R}\mathrm{P}^2$, so it cannot be that $\partial M = \mathbb{R}\mathrm{P}^2$.

$\square$

Problem 6: Let W be a closed oriented simply-connected 4-manifold and let $f : W \to W$ be a self-map which is homotopic to the identity. Show that f has a fixed point.

Proof. We will show that the Lefschetz trace $\tau(f)$ is nonzero, from which it follows that f has a fixed point by the Lefschetz fixed point theorem.

W is simply-connected, which implies that $H_1(W) = 0$ (since H_1 is the abelianization of π_1). Because W is an oriented manifold, Poincaré duality also gives some information about the homology groups: $H_4(W) = \mathbb{Z}$,

$$H_3(W) = H^1(W, \mathbb{Z}) = \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$$

and

$$H_2(W) = H^2(W, \mathbb{Z}) = \mathrm{Hom}(H_2(W), \mathbb{Z}) \oplus \mathrm{Ext}(0, \mathbb{Z}) = \mathrm{Hom}(H_2(W), \mathbb{Z})$$

which shows that $H_2(W)$ is free. Thus W has homology only in even dimensions, and in particular its Euler characteristic is at least 2. And since f is homotopic to the identity, $\tau(f) = \chi(W)$, as the trace of the identity map is the rank of the matrix. Thus $\tau(f) > 0$, so f has a fixed point. $\square$

Problem 7: Exhibit a nontrivial element of $\pi_5(S^3 \vee S^3)$.

Proof. In the long exact sequence of the homotopy groups of the pair $(S^3 \times S^3, S^3 \vee S^3)$, we can see that the generators of $\pi_n(S^3 \times S^3) = \pi_n(S^3) \oplus \pi_n(S^3)$, restricted to one or the other S^3, are also contained in $\pi_n(S^3 \vee S^3)$, so the map between them is surjective, which implies that the map $\pi_n(S^3 \times S^3) \to \pi_{n-1}(S^3 \times S^3, S^3 \vee S^3)$ is trivial. Moreover, there is section taking these generators back. So in particular at $n = 5$, the sequence

$$\cdots \xrightarrow{0} \pi_6(S^3 \times S^3, S^3 \vee S^3) \overset{\partial}{\hookrightarrow} \pi_5(S^3 \vee S^3) \twoheadrightarrow \pi_5(S^3 \times S^3) \xrightarrow{0} \cdots$$

is short exact and *splits* (because of the section, by the splitting lemma) to give

$$\pi_5(S^3 \vee S^3) \cong \pi_6(S^3 \times S^3, S^3 \vee S^3) \oplus \pi_5(S^3) \oplus \pi_5(S^3).$$

All three of these component parts are nontrivial. $\pi_5(S^3) = \mathbb{Z}_2$ via the table in Hatcher, but I don't really know what this element is. Instead, let's consider $\pi_6(S^3 \times S^3, S^3 \vee S^3)$. $(S^3 \times S^3, S^3 \vee S^3)$ is 5-connected and $S^3 \vee S^3$ is 2-connected, so we can apply limited-range excision to get

$$\pi_6(S^3 \times S^3, S^3 \vee S^3) \cong \pi_6(S^3 \times S^3/(S^3 \vee S^3)) = \pi_6(S^6) = \mathbb{Z}.$$

A generator of this group is a map including the 6-cell into $S^3 \times S^3$, and its image in $\pi_5(S^3 \vee S^3)$ is the 5-cell boundary of the 6-cell, i.e. the attaching map where that 5-cell is glued onto $S^3 \vee S^3$.

What *is* this map $S^5 \to S^3 \vee S^3$? If we view $S^3 \times S^3$ as a quotient of the 6-cube I^6,

$$S^3 \times S^3 = \frac{(a,b,c,d,e,f) \in I^6}{(a,b,c,d,e,f) = (a',b',c',d,e,f) \text{ for } (a,b,c),(a',b',c') \in \partial I^3, \text{ etc.}}$$

then the boundary of the 6-cell is the quotient of ∂I^6, which is the union of 12 5-cell faces. Each of these cells corresponds to a locus where one of the six indices is fixed at 0 or 1. Each of the twelve cells maps entirely inside either S^3_α or S^3_β. A cell mapping to S^3_α has ten neighbors, six of which map to S^3_β, and the other four to S^3_α. This is a pretty difficult-to-describe map. □

Problem 8 (Hatcher 4.2.31): For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a contant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ this yields isomorphisms

$$\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3), \quad \pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$$

Thus $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain $\mathbb{Z}$ summands.

Proof. Normally, the long exact sequence is

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \cdots$$

If the inclusion $F \hookrightarrow E$ is homotopic to a constant map, then it induces the trivial map on $\pi_i(F) \to \pi_i(E)$ for all i. So the sequence becomes

$$\pi_n(F) \xrightarrow{0} \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \xrightarrow{0} \pi_{n-1}(E) \to \cdots$$

making the middle part exact.

To show that it's split, by the splitting lemma it suffices to produce a group homomorphism $r : \pi_{n-1}(F) \to \pi_n(B)$ such that the composition

$$\pi_{n-1}(F) \xrightarrow{r} \pi_n(B) \to \pi_{n-1}(F)$$

is the identity on $\pi_{n-1}(F)$. Let r be the map which takes $\varphi : S^{n-1} \to F$ to ψ which attaches an n-cell to E via φ on the fiber F_b (where b is the basepoint of B), and then project this down via the fiber bundle to get an n-cell in B attached at b, i.e. an element of $\pi_n(B)$. It is clear to see that this is a retract of the boundary map.

The map $S^3 \to S^7$ in the Hopf fibration is homotopic to a constant map because $\pi_3(S^7) = 0$ via cellular approximation, and likewise for $S^7 \to S^{15}$. Setting n to be 7 and 15 respectively gives

$$\pi_7(S^4) \cong \mathbb{Z} \oplus \pi_6(S^3), \quad \pi_{15}(S^8) \cong \mathbb{Z} \oplus \pi_{14}(S^7).$$

$\square$

Problem 9 (Hatcher 4.3.3): Suppose that a CW complex X contains a subcomplex S^1 such that the inclusion $S^1 \hookrightarrow X$ induces an injection $H_1(S^1) \to H_1(X)$ with image a direct summand of $H_1(X)$. Show that S^1 is a retract of X.

Proof. It suffices to prove this for connected CW complexes X, as every connected component not containing the S^1 can be crushed to any point of S^1 without affecting the continuity of the map from other components.

Let $s : S^1 \hookrightarrow X$ be the inclusion of the S^1 into X and let $A = \mathrm{im}(s)$. We are given that s induces an inclusion of homology groups

$$H_1(S^1) = \mathbb{Z} \to \mathbb{Z} \oplus G = H_1(X)$$

where G is some Abelian group. Let $p : \mathbb{Z} \oplus G \to \mathbb{Z}$ be the projection in the first coordinate, forming the sequence

$$H_1(S^1) \xrightarrow{s*} H_1(X) \xrightarrow{p} H_1(S^1).$$

where the composition $p \circ s*$ is the identity on $\mathbb{Z}$.

Because S^1 is a $K(\mathbb{Z}, 1)$, by proposition 1B.9 in Hatcher (which requires that X is connected), every map $\pi_1(X) \to \pi_1(S^1) = \mathbb{Z}$ is induced by a map $f : X \to S^1$. Using this fact, we can show that there is a map f which induces p on $H_1(X)$.

Let $\mathrm{ab} : \pi_1(X) \to H_1(X)$ be the surjective group homomorphism given by abelianization. Then there is a group homomorphism $p \circ \mathrm{ab} : \pi_1(X) \to H_1(S^1) = \pi_1(S^1)$, and by 1B.9 it must be induced by some $f : X \to S^1$. This f induces the map p on $H_1(X) \to H_1(S^1)$.

It follows that the composition $s \circ f|_A : A \to A$ induces the identity map $p \circ s^*$ on $H_1(A) = \mathbb{Z}$ and thus also on $\pi_1(A) = \mathbb{Z}$, so it is homotopic to the identity on A via some homotopy $h_t : A \to A$. By the homotopy extension property for CW pairs (using the fact that A is assumed to be a subcomplex of X), this homotopy can be extended to a homotopy $\tilde{h}_t : X \to A$. Then $\tilde{h}_1 : X \to A$ is a continuous map which restricts to the identity on A, i.e. a retract of X onto A. $\square$