Homework 2

ORF522: Linear and Nonlinear Optimization

Instructor: Bartolomeo Stellato
AI: Irina Wang

Due on October 6

Problem 1 (Conditions for a unique optimum)

Let x be a basic feasible solution associated with some basis matrix A_B. Prove that:

1. If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.
2. If x is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Problem 2 (Sherman-Morrison formula)

Suppose we have a matrix M and we have computed its inverse M^{-1}. Now consider the matrix

$$\tilde{M} := M + uv^T,$$

where uv^T corresponds to a rank-1 matrix.

1. Prove the following. $\tilde{M}$ is invertible if and only if $v^\top M^{-1}u \neq -1$, in which case:

$$\tilde{M}^{-1} = \left[I - \frac{M^{-1}uv^T}{1 + v^T M^{-1}u}\right] M^{-1}.$$

2. Suppose we have already computed an LU factorization of M (therefore we have a convenient way to solve $Mx = b$), but that we want to solve the linear system

$$\tilde{M}x = b.$$

   Explain how we can do it with the previous formula. What is the computational complexity of your approach?
3. Explain how we can apply this approach in simplex method. What is the potential bottleneck?

Problem 3

Consider an LP problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq -c \\ & x \geq 0 \end{array}$$

where A satisfies $A^T = -A$. Prove that this problem has an optimal solution if it is feasible.

Problem 4

Solve the following linear program on your computer using the simplex algorithm. Implement each one of the two pivoting rules:

1. Bland's pivoting rule (smallest subscript).
2. Most negative reduced cost index (if there is a tie, pick the index that comes last).

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}. \end{array}$$

Start the algorithm at the extreme point $x = (2, 2, 0)$, with active set $\mathcal{I}(x) = \{3, 4, 5\}$. Code up your own simplex method for above problem without using CXVPY. The maximum number of iterations should be set to 20. Please print out the values of

$$x, \quad c^T x, \quad A_B, \quad p, \quad \bar{c}, \quad j, \quad d_B, \quad d, \quad i, \quad \theta^*,$$

in each iteration (see lecture notes 6, page 22-23). Compare your result with the solution using an LP solver (e.g., GLPK) in CXVPY.