Homework 1

ORF522: Linear and Nonlinear Optimization

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Problem 1

Consider the problem of minimizing a cost function of the form

$$c^T x + f(d^T x),$$

subject to the linear constraints $Ax \leq b$. Here, d is a given vector and the function $f : \mathbf{R} \to \mathbf{R}$ is as specified in Figure 1. Provide a linear programming formulation of this problem.

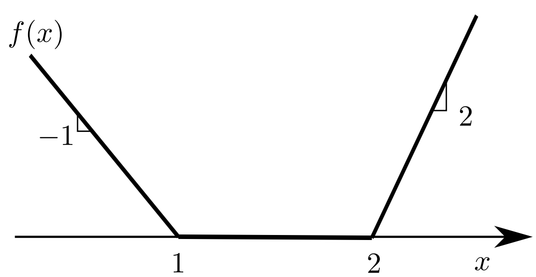

Figure 1: The function f in Problem 1.

Problem 2

Consider a set P described by linear inequality constraints, that is,

$$P = \{x \in \mathbf{R}^n \mid a_i^T x \leq b_i, \quad i = 1, \cdots, m\}.$$

A ball with center y and radius r is defined as the set of all points within (Euclidean) distance r from y. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P. Provide a linear programming formulation of this problem.

Problem 3

Consider the LP whose feasible region is the standard form polyhedron $P = \{x | Ax = b, x \geq 0\}$. Suppose that the matrix A has dimension $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.

1. if $n = m + 1$, then P has at most two basic feasible solutions.
2. The set of all optimal solutions is bounded.
3. At every optimal solution, no more than m variables can be positive.
4. If there is more than one optimal solution, then there are uncountably many optimal solutions.
5. Consider the problem of minimizing $\max\{c^T x, d^T x\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

Justify your answer in 1-2 lines rather than providing a formal proof.

Problem 4

We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n,\ t = 0, \cdots, N$, and actuator or input signal $u(t) \in \mathbf{R}$, for $t = 0, \cdots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \cdots, N-1,$$

where $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$. The *minimum fuel optimal control problem* is to choose the input $u(0), \cdots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbf{R}^n$ is the (given) final or target state. The function $f : \mathbf{R} \to \mathbf{R}$ is the *fuel use map* for the actuator, which gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

1. Formulate the minimum fuel optimal control problem as an LP.

2. Solve the following instance of the problem:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad x(0) = (0,0), \quad x_{\text{des}} = (10,0), \quad N = 20.$$

We can interpret the system as a simple model of a vehicle moving in one dimension. The state dimension is $n = 2$, with $x_1(t)$ denoting the position of the vehicle at time t and $x_2(t)$ giving its velocity. The initial state is $(0,0)$, which corresponds to the vehicle at rest at position 0; the final state is $x_{\text{des}} = (10,0)$, which corresponds to the vehicle being at rest at position 10. Roughly speaking, this means that the actuator input affects the velocity, which in turn affects the position. The coefficient $A_{22} = 0.95$ means that velocity decays by 5 in one sample period, if no actuator signal is applied.

- Solve the problem using CVXPY **in two ways** and check that the solutions match:
  - By keeping operations $|\cdot|$, max, min implicit and making CVXPY convert it to LP for you.
  - By applying directly the LP formulation you obtained in point 1.
- Plot the input signal $u(t)$ for $t = 0, \cdots, 19$, and the position and velocity (*i.e.*, $x_1(t)$ and $x_2(t)$) for $t = 0, \cdots, 20$.