No. 17

NOTE ON DE SITTER'S UNIVERSE

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The equations of the element of interval of a four-dimensional universe of constant positive curvature have been given by de Sitter in the form

$$ds^2 = R^2[-d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) + \cos^2\chi d\tau^2], \qquad (1)$$

where R is a constant called the radius of the four-dimensional universe and χ, θ, ϕ, τ are coördinates. When the division of time and space is made as suggested by these coördinates, the space is itself of constant curvature and has the same radius as the universe.

In the four-dimensional universe, every point is equivalent to any other and the choice of any origin is immaterial. In the above separation of space and time, this symmetry is broken. The lines of constant coördinates χ, θ, ϕ, which are considered as the direction of the time, are not geodesics with the exception of the one $\chi = 0$, which passes through the origin. This point has now properties different from any other; it is a *center*. Free points or rays of light describe geodesics of the universe and do not move along geodesics of the space, with the exception of those which pass through the center.

It is clear that such an introduction of an apparent center in a universe which, by definition, has none is objectionable for a study of the properties of this universe. The purpose of this note is to look for a separation of space and time which is free from this objection. We shall be led to a homogenous field, non-statical and euclidean.

A homogenous division of space and time is immediately found by writing

$$ds^2 = R^2\{-\cosh^2\tau'[d\chi'^2 + \sin^2\chi'(d\theta'^2 + \sin^2\theta' d\phi'^2) + d\tau'^2\} \qquad (2)$$

With the division of time and space suggested by these coördinates, the radius of space is constant at any place, but is variable with time. It is proportional to cosh τ'.

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The fact that the field is no more static may be urged as an objection against the significance of de Sitter's solution, but not against the way we have divided space and time to interpret it. There is, however, a serious objection against this procedure: The origin of time becomes a time absolutely distinct from every other. At the instant $\tau'=0$, all the geodesics of the space are geodesics of the universe; at any other instant τ', there is no geodesic of the space which is a geodesic of the universe. The central point in space is removed, but now a central time has been introduced.

We shall see at once that the interval of the field may be written

$$ds^2=R^2\frac{-dx^2-dy^2-dz^2+dt^2}{t^2} \tag{3}$$

When we separate time from space as suggested by these coördinates, *i.e.*, along the lines of constant x, y, z, we see that the geometry is euclidean. The time is now proportional to $T=\pm\int dt/t=\pm ln\ t$, and $t=0$ represents the infinite past or future. For $t=\infty$, the distances of the points of constant dx, dy, dz tend to zero; the geodesics are asymptotic, *i.e.*, parallel, to the future or from the past. The equation of the field may also be written

$$ds^2=R^2[-e^{-2T}(dx^2+dy^2+dz^2)+dT^2] \tag{4}$$

for geodesics parallel to the future, or

$$ds^2=R^2[-e^{2T}(dx^2+dy^2+dz^2)+dT^2] \tag{5}$$

if parallel in the past direction.

This choice of coördinates is free from the objection of introducing a spurious assymmetry in space and time.

Before using this solution to discuss the properties of de Sitter's universe, we shall first establish the validity of Equation (3).

Using the transformation

$$\begin{cases} r=e\tau \tan \chi \\ t=e\tau \sec \chi \end{cases} \tag{6}$$

or

$$\begin{cases} \chi=\text{arc sin } r/t \\ \tau=1/2\ ln(t^2-r^2), \end{cases} \tag{7}$$

the original expression (1) of de Sitter becomes

$$\frac{1}{R^2}ds^2 = -\frac{1}{1-r^2/t^2}\;\frac{(t\,dr-r\,dt)^2}{t^4} - \frac{r^2}{t^2}\,(d\Theta^2 - \sin^2\Theta d\phi^2)$$
$$-\left(1-\frac{r^2}{t^2}\right)\frac{(t\,dt-r\,dr)^2}{(t^2-r^2)^2}$$

or

$$ds^2 = \frac{R^2}{t^2}\left[-dr^2 - r^2\,d\Theta^2 - r^2\sin^2\,\Theta d\phi^2 + dt^2\right]. \tag{8}$$

The expression inside the brackets is a Galilean ds^2 written in polar coördinates. By transforming to Cartesian coördinates, we get (3).

The equation is convenient for discussion, as we can easily get a representation of de Sitter's universe in a Galilean universe of Cartesian coördinates x, y, z. The lines parallel to the t-axis (x, y, z=Const.) are obviously geodesics. The rays of light are represented by the rays of light of the Galilean field (straight lines at 45° to the plane $t=0$).

The other geodesics are easily found. If $\dot{x}$, $\dot{y}$, $\dot{z}$, are the derivatives with respect to t, we must have

$$\delta\int\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}\;\frac{dt}{t}=0.$$

Euler's equations are

$$\frac{d}{dt}\left(\frac{\dot{x}}{t\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}}\right)=0$$

and similarly in y and z. Integrating,

$$\frac{\dot{x}}{\alpha t}=\frac{\dot{y}}{\beta t}=\frac{}{\gamma t}=\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}=\frac{1}{\sqrt{1+(\alpha^2+\beta^2+\gamma^2)t^2}}$$

where α, β, γ are integration constants, real for time-like lines, imaginary for space-like ones. Writing $C=\alpha^2+\beta^2+\gamma^2$ and integrating

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z-c}{\gamma}=\frac{1}{C}\sqrt{1+Ct^2}$$

where a, b, c are integration constants and therefore,

$$(x-a)^2+(y-b)^2+(z-c)^2-t^2=1/C.$$

The geodesics are equilateral hyperbolae lying in planes perpendicular to the plane $t=0$ and having an axis in this plane.

Their asymptotes are rays of light.

De Sitter separated time from space along the lines of constant χ, θ, ϕ. From (7) these lines are now represented by a pencil of straight lines passing through the point $r=0$, $t=0$. These lines are n t geodesics except the central one $r=0$, $\chi=0$ and the lines $r=t$, $\chi=\pi/2$ which are rays of light.

This explains the paradoxical singularity which occurs for this value of χ.

The Doppler-Fizeau effect is easy to calculate. Let us consider two sources of light of same proper period ds. They are supposed to describe geodesics of constant, x, y, z, then

$$\frac{1}{R}ds=\frac{dt}{t}=\frac{dt_0}{t_0}.$$

The equation of the ray of light from M to M_0 is

$$t_0=t+r;$$

for two rays, we have

$$t_0'-t_0=t'-t$$

so that the periods, as observed by an observer in M_0, are dt_0 and dt. The Doppler effect will be

$$\frac{d\lambda}{\lambda}=\frac{dt}{dt_0}-1=\frac{t}{t_0}-1=\frac{-r}{t_0}.$$

If we introduce de Sitter's coördinates, taking as center the light-source M, the observer M_0 will be at a distance

$$\sin\chi=\frac{r}{t_0};$$

we shall have,

$$\frac{d\lambda}{\lambda}=-\frac{r}{t_0}=-\sin\chi. \tag{9}$$

This expression has been given by Silberstein. He supposes the light-source at the origin. The observer describes a geodesic which neither goes through the origin, nor passes at a minimum distance from it, but is at the intermediate case. Introducing this condition in a general expression of the Doppler effect, he finds the above formula.

In the preceding reasoning, we have supposed $t=0$ in the past so that our lines of universe converge in the future. If we reverse the sign of t and suppose that our lines are parallel in the direction of the past, we find the same value with the opposite sign. These two solutions cannot be compounded without introducing an assymmetry of the kind we are just criticizing.

We may sum up the above discussion in the following way: de Sitter's coördinates introduced a spurious inhomogeneity of the field which is not simply the mathematical appearance of center of an origin of coördinates, but really attributes distinct absolute properties to a center.

We tried to remove the difficulty by introducing other coördinates and were led to a homogeneous field; but first the field is not static and secondly, the space has no curvature. The first point may probably be accepted. Eddington writes on this subject: "It is sometimes urged against de Sitter's world that it becomes non-statical as soon as any matter is inserted in it. But this property is perhaps rather in favor of de Sitter's theory than against it."[2] Our treatment evidences this non-statical character of de Sitter's world which gives a possible interpretation of the mean receding motion of spiral nebulae.

The second point, on the contrary, seems completely inadmissible. We are led back to the euclidean space and to the impossibility of filling up an infinite space with matter which cannot but be finite. De Sitter's solution has to be abandoned, not because it is non-static, but because it does not give a finite space without introducing an impossible boundary.

[2]Eddington. The Mathematical Theory of Relativity, p. 161, Cambridge, 1923.

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