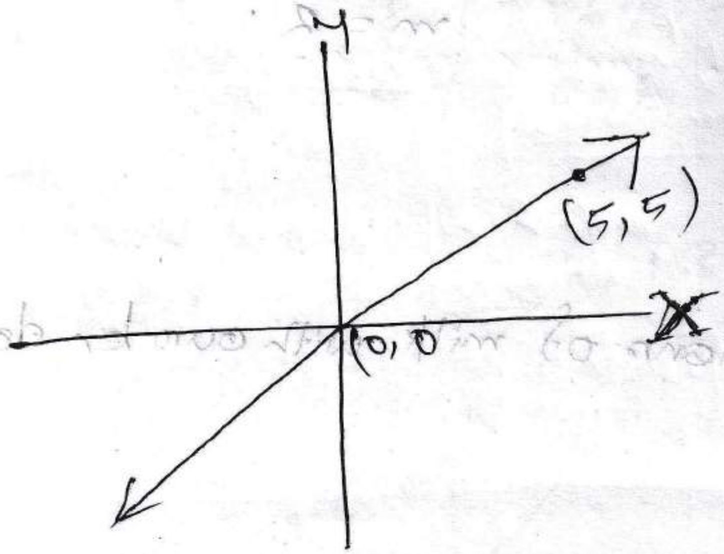
Video 1

Page: 1

Linear Equation

$$y = mx + b$$

Understanding slopes and intercepts



$y = x$

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{slope} \rightarrow \frac{\text{Change in } y}{\text{Change in } x}$$

Why are Linear equations so common?

Ans: $y = mx + b$

intermediate

Simplicity

$f(x) = c$

(line at $y = c$)

Complexity

$f(x) = ax^2 + bx + c$

$f(x) = ax^3 + bx^2 + cx + d$

Note:

So, Linear equations are right in the middle, not too simple, not too complicated, which actually end up modeling a lot of real-world relationships.

Page: 2 (video-1) System of Linear Equations.

* we have y dollars.
milk costs m dollars / gallon.

dollar = $
gallon = gal.

$$y = mx$$
$$\$ = \frac{\$}{gal}\, gal$$

$y = 10\$$
$m = 2$.

$$x = \frac{y}{m}$$
$$x = \frac{10}{2}$$
$$= 5$$

So, we can get five gallons of milk with our ten dollar

example 2:

$m_1 = \$/gal$ milk
$x_1 =$ gal milk we get.
and $m_2 = \$/gal$ OJ (OJ mine orange juice)
$x_2 =$ gal OJ we get
$y_1 =$ Our $
$W_1 =$ lbs/gal milk and $w_2 =$ lbs/gal. O.J.

$$y_1 = m_1x_1 + m_2x_2$$
and $$y_2 = w_1x_1 + w_2x_2$$

Those all all linear equations, and this is a system of linear equations.

* linear equation,
$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

or example, if we have three equations in ... three variables, as is shown here,

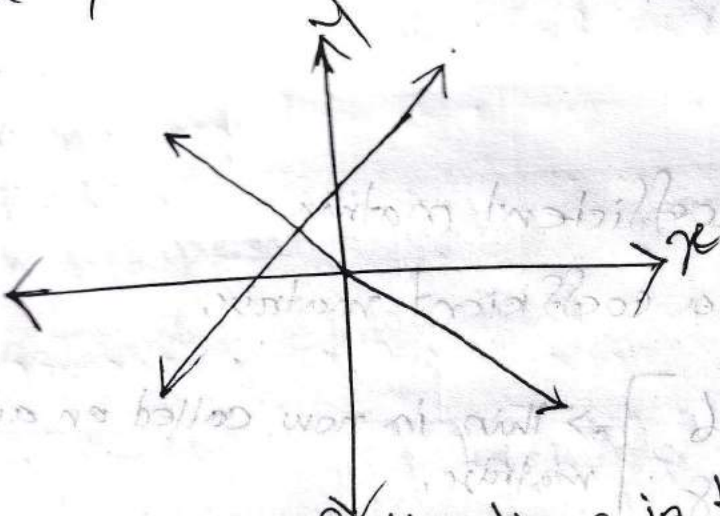
Linear equation

$$a_1x + a_2y + a_3z = b_1$$
$$a_4x + a_5y + a_6z = b_2$$
$$a_7x + a_8y + a_9z = b_3$$

here,

$3x - y = -6 \rightarrow y = 3x + 6$

$-2x - y = 0 \qquad y = -2x$



The intersection of the lines is the solution to the system.

$\rightarrow x_1 = -6/5$

$x_2 = 12/5$

Possible solutions:

- → a point
- → a line
- → a plane

} infinitely many solutions.

Video - 02
Solving systems of linear equations.

- $a_1 x + a_2 y + a_3 z = b_1$
  $a_4 x + a_5 y + a_6 z = b_2$
  $a_7 x + a_8 y + a_9 z = b_3$ → matrix nation.

- Understanding matrices and matrix nation.

$7x + 5y - 3z = 16$
$3x + 5y + 2z = -8$
$5x + 3y + 7z = 0$

$$\begin{bmatrix} 7 & 5 & -3 \\ 3 & -5 & 2 \\ 5 & 3 & -7 \end{bmatrix}$$

matrix - m * n coefficient matrix
we have just contructed a coefficient matrix.

$$\begin{bmatrix} 7 & 5 & -3 \\ 3 & -5 & 2 \\ 5 & 3 & -7 \end{bmatrix} + \begin{bmatrix} 16 \\ -8 \\ 0 \end{bmatrix}$$ → This is now called an augmente matrix,

3×4 augmente matrix = $\begin{bmatrix} 7 & 5 & -3 & 16 \\ 3 & -5 & 2 & -8 \\ 5 & 3 & -7 & 0 \end{bmatrix}$

- $m \times (n+1)$ augmnted matrix

Solving Systems of Linear Equations (Video → 3)

- divide by 2 $(4x + 6y = -2)$

$$5x - 3y = 8$$

divide by 2 → $\begin{bmatrix} 4 & 6 & -2 \\ 5 & -3 & 8 \end{bmatrix}$

- we can multiply or divide any equations by some number we want except for zero.
- we can add or subtract any two equations to get a new one.
- when we perform this on a matrix alone technique is Gauss-Jordan elimination.

for example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

we want the end result to look like this

- what if we subtract both the first and second rows from the third row, to ge a new third row.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 3 & 7 & 11 & 8 \end{bmatrix} \quad R_3 - R_2 - R_2 = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 0 & 1 & 1 & 5 \end{bmatrix}$$

Second row coeffients, so what we can do is subtract twice the first row from the second to get a new second row.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 2 \\ 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 \end{bmatrix} \quad z = 0$$

Let's subtract the second raw from the third row to get a new third row,

$$z = 0. \quad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix} \rightarrow y = 5$$

And now it's easy to finish things off, because we have tow variables isolated.

$$y = 5, \; z = 0, \quad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix} \xrightarrow{R_1 - 2R_3} \begin{bmatrix} 1 & 0 & 3 & -9 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ hance, } x = -9, \; y = 5, \; z = 0$$

Video - 3

$$E \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

There is a way to tell how many solution there are Once Gauss-Jordan elimination is complete, which will be the case once the matrix is in reduced row-echelon form, can be abbreviated this way

$E = rref(m) \rightarrow$ origial matrix.

Here, Gauss - Jordan elimination is completed.

Video → 4

Let's look at matrics on theair own, and describe some terminology, we can apply to them, and also some simple operations we can do with multiple matrices, starting with simple matrix addition.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ m = number of rows. n = number of columns.

* whe $m = n$ the matrics is square matrix.

and, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This is a diagonal matrix. and also called an identity matrix.

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video - 4

* upper triangular matrix $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

* Lower triangular matrix $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$

* matrix consisting of just one column it's called a Vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and a matrix with a single row called row vector $\begin{bmatrix} a & b & c \end{bmatrix}$

* Systed of linear equations can be represented with vectors. $3x + y = 7$

$x + 2y = 4$

#

$x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ This is the vector form of the linear system.

* when multipling a matrix by a scalar we multiply each entry by the scalar.

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$$

Video-4

- Two matrices to be added together they must have identical dimensions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix}$$

each entry in the new matrix is the sum of the corresponding entries

$$= \begin{bmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{bmatrix}$$

- matrix subtraction is very similar to matrix addition.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$

matrix addition is commutative

$A + B = B + A$

but matrix subtraction is not commutative.

$A - B \neq B - A$

Checking comprehension

$$A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 6 & -5 \\ 7 & 1 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 9 & 4 \\ 0 & 11 & 5 \\ -3 & 2 & -7 \end{bmatrix}$$

Video - 4

Find A + B:
$$\begin{bmatrix} 4 & 7 & 3 \\ 2 & 17 & 0 \\ 4 & 3 & -15 \end{bmatrix}$$

find A - B:
$$\begin{bmatrix} 2 & -11 & -5 \\ 2 & -5 & -10 \\ 10 & -1 & -1 \end{bmatrix}$$

Video → 5

Performing matrix multiplication

- for this multiplication to work these matrices must have specific dimensions.

for example: for this multiplication to work A must have the same number of columns as B has rows

$A = m \times n$ $B = q \times p$ (n must equal q)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

~~a = [(1)(5) + (2)(7)] = 5 + 14 = 19~~

AB = [

Video-5

$a = [(1)(5) + (2)(7)] = 5 + 14 = 19$

$b = [(1)(6) + (2)(8)] = 6 + 16 = 22$

$c = [(3)(5) + (4)(7)] = 15 + 28 = 43$

$d = [(3)(6) + (4)(8)] = 18 + 32 = 50$

So, $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

This is the algorithm for matrix multiplication

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$AB = \begin{bmatrix} (ae + bg) & (af + bh) \\ (ce + dg) & (cf + dh) \end{bmatrix}$$

note: columns in A must equal rows in B,

2 by 3 matrix defenation for multiplication.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$AB = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$ the product is a 2×2 matrix

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* If we switch their orders, we will get a cometely different result,

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$BA = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix}$ This is 3×3 matrix.

* matrix multiplication is not commutative $AB \neq BA$.
however, matrix multiplication is associative $(AB)C = A(BC)$
matrix multiplication can be distributive, if A and B are m×n matrices and C and D are n×p matrices → $A(C+D) = AC + AD$ and $(A+B)C = AC+BC$

* CHecking COMPRENSION

$A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 6 & -5 \\ 7 & 1 & -8 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 9 & 4 \\ 0 & 11 & 5 \\ -3 & 2 & 7 \end{bmatrix}$

find $AB = \begin{bmatrix} 6 & 3 & 9 \\ 17 & 74 & 73 \\ 31 & 58 & 89 \end{bmatrix}$ find BA?

$\begin{bmatrix} 49 & 56 & -78 \\ 57 & 71 & -95 \\ -54 & 11 & 49 \end{bmatrix}$

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Page-13 Evaluating the determinant of a matrix

* finding the determinant of a square matrix.

$\det(A)$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

For example → $A = \begin{bmatrix} 2 & 1 \\ -6 & 4 \end{bmatrix}$ 2×2 matrix

$\det(A = [(2)(4)] - [(1)(-6)]$

$= 8 - (-6)$

$= 14$

* loking 3×3 matrix: $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$\det(A) = a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$

$= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$

for example → $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{bmatrix}$

$\det(A) = 1 \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} - 2 \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 & 0 \\ -5 & 4 \end{bmatrix}$

$= 1(-4) - 2(11) + (-1)[(3)(4) - (0)(-5)]$

$= (-4) - (22) + (-12)$

$= -38$

Video-6

an matrices get bigger finding the determinant requires many more steps.

Checking comprehension:

$$\begin{bmatrix} 1 & 2 & 9 & 6 \\ 2 & 3 & -4 & 2 \\ 5 & 0 & 1 & 2 \\ 5 & 1 & 0 & 8 \end{bmatrix} = -338$$

video 7

The vector cross product

Vector multiplication: $a = \langle a_1, a_2, a_3 \rangle$
$b = \langle b_1, b_2, b_3 \rangle$

vector dot product =
$a \cdot b = (a_1, b_1) + (a_2\ b_2) + (a_3, b_3)$
Vector cross product of $a \times b$

$a = \langle 1, 3, 4 \rangle \quad b = \langle 2, 7, -5 \rangle$

$a = 1 + 3j + 4k \qquad b = 2i + 7j - 5k$

$$a \times b = \begin{bmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & -5 \end{bmatrix} i - \begin{bmatrix} 1 & 4 \\ 2 & -5 \end{bmatrix} j + \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} k$$

$a \times b = (-15 - 28)i - (-5 - 8)j + (7 - 6)k$

$a \times b = -43i + 13j + k$

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The cross product will always be orthogonal to the plane containing the original two vectors.

$$a \times a = 0$$

$$a \times a = \begin{bmatrix} i & j & k \\ a & b & c \\ a & b & c \end{bmatrix} = \begin{bmatrix} b & c \\ b & c \end{bmatrix} i - \begin{bmatrix} a & c \\ a & c \end{bmatrix} j + \begin{bmatrix} a & b \\ a & b \end{bmatrix} k$$

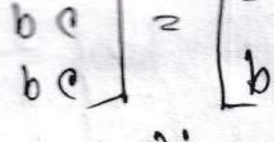
$$a \times a = (bc - bc)i - (ac - ac)j + (ab - ab)k$$

$$a \times a = 0i + 0j + 0k$$

$$|a \times b| = |a||b| \sin\theta$$

we can also find the cross product using this method.

1) find the magnitude of the cross product.
2) find its direction using the right-hand rule



any two parallel vectors must have a cross product of zero.

$$a \times b = 0$$

$$|a \times b| = |a||b| \sin\theta \rightarrow (\sin 0 = 0)$$

the cross product is equal to the area of the parallelogram formed by these two vectors.

properties of the cross product.

$a \times b \neq b \times a$. $a \times b = -b \times a$ (not commutative)

$a \times (b + c) = (a \times b) + (a \times c)$ (distributive)

$(a \times b) \times c \neq a \times (b \times c)$ (not associative)

Checking comprehension

$a = \langle 4, -2, 5 \rangle$, $b = \langle -1, 3, -6 \rangle$, $c = \langle 7, -5, 1 \rangle$

find $a \times b$: $-3i + 19j + 10k$

find $b \times c$: $-27i - 41j - 16k$

find $a \times c$: $23i + 31j - 6k$

Video - 8

Inverse matrices and their properties.

inverse functions:

$f(x) = 2x + 3$

$y = 2x + 3$

$x = 2y + 3$

solve for y

$y = (x-3)/2$

$f^{-1}(x) = (x-3)/2$

$\underset{\text{matrix}}{A} \rightarrow \underset{\text{inverse matrix}}{A^{-1}}$

$A^{-1} \neq \frac{1}{A}$ this does not mean reciprocal

numbers

$x \cdot \frac{1}{x} = 1$

matrices

$AA^{-1} = I$

$A^{-1}A = I$

$\boxed{\text{identity matrix}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Video 8

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & ad-bc \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1. a and d swap positions.
2. b and c switch their sign.
3. divide by the determinant.

$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$. or, $A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$ or, $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$

or, $AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ or, $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the identity matrix confirms this inverse relationship.

Solving equations with matrices

$\frac{XA}{A} = \frac{B}{A}$ (this is not possible

$X = BA^{-1}$ (this is only possible if the two matrices have appropriate dimensions.)

 ~~$XAA^{-1} = BA^{-1}$~~ Video-8

$XAA^{-1} = BA^{-1}$

matrix multiplication is not commutative

$X = A^{-1}B$

not every matrix will have an inverse.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if det(A) = 0 then:

$A^{-1} = \frac{1}{0}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ = undefined. here, A is a singular matrix.

Stape 1 — Generate the matrix of minors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -24 & -20 & -5 \\ -18 & -15 & -4 \\ 5 & 4 & 1 \end{bmatrix}$$ → matrix of minors.

Stape 2 → Generate the matrix of cofactors.

$$\begin{bmatrix} -24 & -20 & -5 \\ -18 & -15 & -4 \\ 5 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$$

minors matrix — cofactors matrix.

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Stape 3: find the Adjugate (Adjoint

$$\begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Cofactors matrix — adjugate/adjoint

Just reflect entries across the main diagonal.

Stape 4: Devide Adjugate by Determinant.

$$\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \div \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix}$$

adjugate/adjoint — original matrix

$$= 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$$= -24 + 40 - 15$$

$$= 1$$

So, $$\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \div 1 = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & 4 \\ -5 & 4 & 1 \end{bmatrix}$$

This is our final answer for the inverse matrix.

- checking comprehension:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -6 \\ -6 & 3 & -7 \\ -2 & 1 & -2 \end{bmatrix}$$

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Page-19 Solving systems Using Cramer's Rule.

- Defining Cramer's Rule: $a_1x_1 + a_2x_2 + a_3x_3 = b_1$

$$a_4x_1 + a_5x_2 + a_6x_3 = b_2$$
$$a_7x_1 + a_8x_2 + a_9x_3 = b_3$$

- the i^{th} component of th solution:

$$x_1 = \frac{|A_i|}{|A|}$$

- Practice using Cramer's Rule: $x_1 + 3x_2 = 5$

$$2x_1 + 2x_2 = 6$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$|A| = (1)(2) - (3)(2)$

$|A| = 2 - 6$

$|A| = -4$

Now = $A_1 = \begin{bmatrix} 5 & 3 \\ 6 & 2 \end{bmatrix}$

$|A_1| = (5)(2) - (6)(3)$

$|A_1| = 10 - 18$

$|A_1| = -8$

$A_2 = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$

$|A_2| = (1)(6) - (2)(5)$

$|A_2| = -4$

$x_1 = \frac{|A_1|}{|A|}$

$= \frac{-8}{-4}$

$x_1 = 2$

now, $x_2 = \frac{|A_2|}{|A|}$

$= \frac{-4}{-4}$

$x_2 = 1$

Video

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Solution: $x_1 = 2, x_2 = 1$

So, $x_1 + 3x_2 = 5$

$2x_1 + 2x_2 = 6$

$= (2) + 3(1) = 5$

$2(2) + 2(1) = 6$

$x_1 = \frac{|A_i|}{|A|}$ (This algorithm never changes.)

another example: $2x_1 + x_2 + x_3 = 1$

$3x_2 - x_3 = 7$

$x_1 + 2x_2 + 2x_3 = -4$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 1 & 2 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 7 & -1 \\ 1 & -4 & 2 \end{bmatrix}$ $A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 7 \\ 1 & 2 & -4 \end{bmatrix}$

$|A| = 12$, $|A_1| = 24$, $|A_2| = 12$ $|A_3| = -48$

$x_1 = 2$ $x_2 = 1$ $x_3 = -4$

checking comprehension: $2x + y + z = 3$

$x - y - z = 0$

$x + 2y - z = 0$

~~$|A| = 4$~~ $|A| = 3$ $|A|_1 = 3$ $|A_2| = -6$ $|A_3| = 9$

$x = 3/3 = 1$

$y = -6/3 = -2$

$z = 9/3 = 3$

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Video – 10

Understanding Vector spaces

V

$\vec{a} \in V$

- $\vec{a}$ is an element of V
- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property of addition)
- $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ (associative property of addition)

Sol

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$, $\quad \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

$\vec{a} + \vec{0} = a$, $\quad \vec{a} + (-\vec{a}) = \vec{0}$, $\quad c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$

$(c + d)\vec{a} = c\vec{a} + d\vec{a}$, $\quad cd(\vec{a}) = c(d\vec{a})$; $\quad 1\vec{a} = \vec{a}$

- V is a collection of elements that can be:

① added together in any combination

② multiplication by scalars in any combination

Closure:

① given $\vec{a} \in V$ and scalar c, than $c\vec{a} \in V$

② given $\vec{a} \in V$ and $\vec{b} \in V$, then $\vec{a} + \vec{b} \in V$

- set of real vectors of length n

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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- set of real $m \times n$ matrices:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- set of linear polynomials: $ax+b$

$c(ax+b) = (ca)x + (bc)$

- $(a_1x + b_1) + (a_2x + b_2)$

$(a_1 + a_1)x + (b_1 + b_2)$

- this set is not a vector space $\begin{bmatrix} a_1 \\ 2 \end{bmatrix}$

$\vec{a} = \begin{bmatrix} a_1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ 2 \end{bmatrix}$ $\vec{a} + \vec{b} = \begin{bmatrix} a_1 + b_1 \\ 4 \end{bmatrix}$

- checking comprehension:

label the following statements as true or false:

(1) A vector space can be made of vectors.

Ans: False.

(2) $\begin{bmatrix} 2 & -1 \\ 0 & 7 \end{bmatrix} \in R^{2\times 2}$

Ans: True

(3) A space must satisfy closure to be a vector space.

Ans: True.

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Examples of subspaces

$\vec{x} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$, $c\vec{x} = c\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \begin{bmatrix} cx \\ 0 \\ -cx \end{bmatrix}$

(1) given $\vec{x} \in S$ and scalar c, then $c\vec{x} \in S$

(2) " " " $\vec{y} \in S$, then $\vec{x} + \vec{y} \in S$

$\vec{x} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y \\ 0 \\ -y \end{bmatrix}$, so, $\vec{x} + \vec{y} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} y \\ 0 \\ -y \end{bmatrix}$

so, $\vec{x} + \vec{y} = \begin{bmatrix} x+y \\ 0 \\ -(x+y) \end{bmatrix}$

V

$\vec{v}_1, \vec{v}_2 \dots \vec{v}_n \in V$

$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots a_n\vec{v}_n$

set all linear combinations is the span

$\mathbb{R}^3$ $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$= \begin{bmatrix} 2a \\ a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -2b \end{bmatrix} + \begin{bmatrix} -c \\ -c \\ -c \end{bmatrix} = \begin{bmatrix} 2a-c \\ a+2b-c \\ a+2b+c \end{bmatrix}$

is important for describing vector spaces.

[an,]

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Linear independence

* $\vec{a}$ $\vec{b}$ $\vec{c}$ $\rightarrow$ $\vec{c} = \vec{a} + 2\vec{b}$

Linear combination of these vectors:

$2\vec{a} + \vec{b} - (\vec{a} + 2\vec{b})$

* $\vec{v}_1; \vec{v}_2 \ldots \vec{v}_n$ in vector space V.

for linear independence this equation will require that all scallars ~~equal~~ equal zero.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_n\vec{v}_n = \vec{0}$$

* Lets check it now: $\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{bmatrix}c_1\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\-c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \text{add these together}$$

$$\begin{matrix}c_1 + c_2 = 0\\ c_1 + c_2 = 0\end{matrix} \rightarrow \left.\begin{matrix}c_1 = 0\\ c_2 = 0\end{matrix}\right\}$$ these vectors are linearly independent.

note: all zero $\rightarrow$ linearly independent

some nonzero $\rightarrow$ linearly dependent.

$$\begin{bmatrix}1\\1\\3\\3\end{bmatrix} \begin{bmatrix}-1\\1\\-1\\3\end{bmatrix} \begin{bmatrix}2\\5\\9\\6\end{bmatrix} \xrightarrow{\text{linear combination}} c_1\begin{bmatrix}1\\1\\3\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\-1\\-3\end{bmatrix} + c_3\begin{bmatrix}2\\5\\9\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 2\\1 & 1 & 5\\3 & -1 & 9\\3 & -3 & 6\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$ some thing ~~this~~ but as a matrix equation

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- we can solve this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & 5 & 0 \\ 3 & -1 & 9 & 0 \\ 3 & -3 & 6 & 0 \end{array}\right] = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 5 & 0 \\ 3 & -1 & 9 & 0 \\ 3 & -3 & 6 & 0 \end{bmatrix}$$

- we can solve coefficient matrix.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 5 \\ 3 & -1 & 9 \\ 3 & -3 & 6 \end{bmatrix}$$

- $R_2 - R_1$ to get a new R_2

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 5 \\ 3 & -1 & 9 \\ 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- we need a 1 here and 0 below

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

divide R_2 by two to get a new R_2

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we must have a free variable.

the set of vectors is linearly dependent $= c_1 - c_2 + 2c_3 = 0$

$c_2 + (3/2)c_3 = 0$

$c_3 = 0$

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$$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\-1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$ → this should have a nonzero determinat.

There are linearly independent

$= (1)(-1) - (1)(1) = -2$

$x_2 + x - 2 \quad x^2 - 3x + 5 \quad 2x^2 + 6x - 11$

$c(x^2 + x - 2) + c_2(x^2 - 3x + 5) + c_3(2x^2 + 6x - 11) = 0$

$(c_1 + c_2 + 2c_3)x^2 + (c_1 - 3c_1 + 6c_3)x + (-2c_1 + 5c_2 - 11c_3) = 0$

$$\begin{bmatrix}1 & 1 & 2\\1 & -3 & 6\\-2 & 5 & -11\end{bmatrix} = 1[(-3)(-11) - (6)(5)] - 1[(1)(-11) - (6)(-2)] + 2[(1)(5) - (-3)(-2)] = 3 - 1 - 2 = 0$$

the polynomials are linearly dependent.

Checking comprension

1. Use the determinat method if the following vectors are linearlyindependent: $\langle 1, 2, 1\rangle \langle -1, -1, 2\rangle \langle -1, 0, 5\rangle$

Ans: Linearly independent.

2. $\langle 1, -3, 2, 1\rangle \langle -1, 2, 2, 5\rangle \langle 2, -2, -3, 3\rangle$ Use elements row operations to check if the following vectors are linearly independent.

Ans: Linearly indepentent.

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Basis and dimension

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$

this from vectors form a basis for v if:

1. They are linearly independent.
2. They span the vector space v.

finding a basis.

$\mathbb{R}$ $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix}$

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

these terms are our variables and regular numbers.

$$\begin{bmatrix}c_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\c_3\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} \Rightarrow \begin{matrix}c_1 = a\\ c_2 = b\\ c_3 = c\end{matrix}$$

These three vectors span $\mathbb{R}^3$

$\mathbb{R}^{2\times2}$ $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$

$$\Rightarrow c_1\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + c_2\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} + c_3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c_4\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 & c_1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\c_2 & c_2\end{bmatrix} + \begin{bmatrix}c_3 & 0\\0 & c_3\end{bmatrix} + \begin{bmatrix}0 & c_4\\c_4 & c_4\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1 + c_3 & c_1 + c_4\\c_2 + c_4 & c_2 + c_3 + c_4\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

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now find the determinant:

$$c_1 + 0c_2 + c_3 + 0c_4 = a$$
$$c_1 + 0c_2 + 0c_3 + c_4 = b$$
$$0c_1 + c_2 + 0c_3 + c_4 = c$$
$$0c_1 + c_2 + c_3 + c_4 = d$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

now find the determinant.

now let's check for linear independence

$R^{2\times 2}$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_3 & c_1 + c_4 \\ c_2 + c_4 & c_2 + c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_2 + c_3 + c_4 &= 0 \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

let's solve this system using elementary row operations.

$= R_2 - R_1$ to get new $R_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

now $R_4 - R_2$ to get a new $R_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

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① these matrices span the vector space
② these matrices are linearly independent.

Understanding Dimension

$\mathbb{R}^3$ $\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$ this vector space has dimension 3

$\mathbb{R}^{2\times 2}$ $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ this ~~v~~ vector space has dimension 4.

a vector space can be infinitely dimentionl

V

the dimension of a vector space is fixed

Checking comprehension.

① are the folowing vectors a basis of $\mathbb{R}^2$?
$\langle 1, 0\rangle$ $\langle 1, 1\rangle$ $\langle 3, 2\rangle$
(no, they are not linearly independent)

② are the following vectors a basis of $\mathbb{R}^3$?
$\langle 0, 1, 1\rangle$ $\langle -2, 1, 0\rangle$ $\langle -1, 0, -1\rangle$

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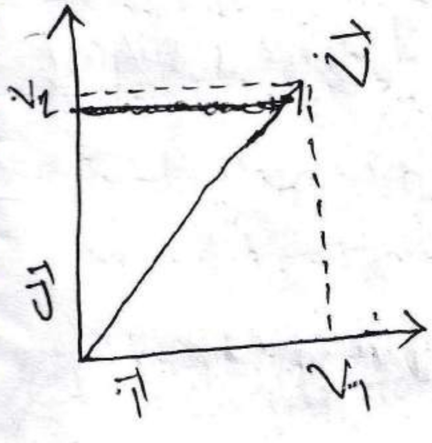
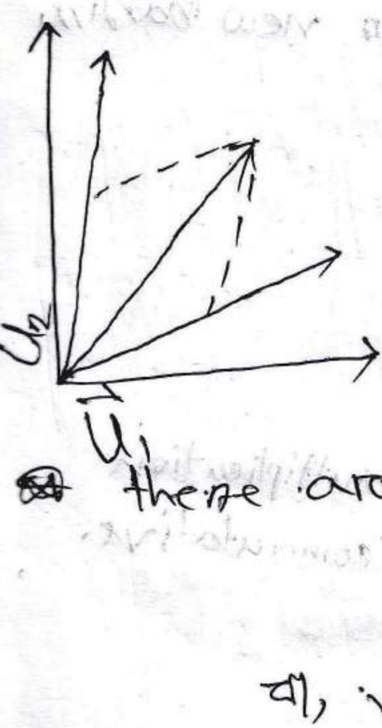
Change of Basis



$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= v_1 \hat{i} + v_2 \vec{j}$$

$\mathbb{R}^2$



$$\vec{U}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{U}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\vec{v} = v_1' \vec{U}_1 + v_2' \vec{U}_2$$

these are the new coordinates for this basis.

$$\vec{U}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{U}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$\therefore$, $v_1' \vec{U}_1 + v_2' \vec{U} = v_1 \vec{i} + v_2 \vec{j}$

$$\vec{U}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{i} + 2\vec{j}$$

$$U_2 = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\vec{i} + 3\vec{j}$$

$$v_1'(\vec{i} + 2\vec{j}) + v_2'(3\vec{j} + 3\vec{j}) = v_1 \vec{i} + v_2 \vec{j},$$

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$$(v_2' + 3v_3')\vec{i} + (2v_1' + 3v_2')\vec{j} = v_1\vec{i} + v_2\vec{j}$$

$$\begin{bmatrix} v_1' + 3v_2' \\ 2v_1' + 3v_2' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

let's find the coefficients in this new basis.

$$\vec{v} = U\vec{v}'$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = U \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

$$AA^{-1} = I$$

$$U^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} = U \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \right)$$

matrix multiplication is not commutative.

$$U = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \quad U^{-1} = \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix} \frac{1}{-3}$$

$$\Rightarrow -\frac{1}{3} \begin{bmatrix} 3 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

$$\Rightarrow -\frac{1}{3} \begin{bmatrix} (3)(2) + (-3)(3) \\ (-2)(2) + (1)(3) \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

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$\ominus_1, -\frac{1}{3}\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}V_1'\\V_2'\end{bmatrix}$

Coordinaten in the new basin.

$\vec{v} = \vec{U}_1 - \frac{1}{3}\bar{U}_2$

Relating two non-standard basen.

$\vec{u}_1 = \begin{bmatrix}1\\2\end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix}3\\3\end{bmatrix}$

$U = \begin{bmatrix}1 & 3\\2 & 3\end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix}1\\2\end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix}3\\3\end{bmatrix}$

$\vec{w}_1 = \begin{bmatrix}-1\\-1\end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix}3\\0\end{bmatrix}$

$U = \begin{bmatrix}1 & 3\\2 & 3\end{bmatrix} \qquad W = \begin{bmatrix}-1 & 3\\-1 & 0\end{bmatrix}$

$\vec{v} = U\cdot\vec{v}_u \qquad \vec{v} = W\vec{v}_w$

$U\vec{v}_u = W\vec{v}_w$

$\hookrightarrow \vec{v}_w = W^{-1}U\vec{v}_u$

$\vec{v}_u = U^{-1}W\cdot\vec{v}_w$

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Checking Comprehension

① Give the transition matrix for the following basis:

$\langle 0,1,1\rangle \ \langle -2,1,0\rangle \ \langle -1,0,-1\rangle$

$$\begin{bmatrix} 0 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

② Given the vector $\langle 5,4\rangle$ write this in terms of the basis

$u_1 = \langle 2,-1\rangle \quad u_2 = \langle 1,4\rangle$

$(16/9)u_1 + (13/9)u_2$

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Linear Transformations on vector spaces

$$V \xrightarrow{L} W$$

* a line transformation will map one vector space on to another.

$L: V \to W$

taken in vectors of length two and then given back scallars. linear transformations:

$L: \mathbb{R}^2 \to \mathbb{R}$

$\Rightarrow L: V \to V \Rightarrow L(c\vec{v}) = cL(\vec{v})$

$\Rightarrow L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$.

Let's verify that this is a linear transformation.

$L(c\vec{v}) = cL(\vec{v})$

$c\vec{v} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$

$L(c\vec{v}) = \begin{bmatrix} cv_2 \\ cv_1 + cv_2 \\ cv_1 - cv_2 \end{bmatrix} = \begin{bmatrix} c(v_2) \\ c(v_1 + v_2) \\ c(v_1 - v_2) \end{bmatrix} = c\begin{bmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{bmatrix}$.

this is same thing as $cL(\vec{v})$

and: $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

$\vec{v}_i = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

$L(\vec{v} + \vec{w}) = \begin{bmatrix} (v_2 + w_2) \\ (v_1 + w_1) + (v_2 + w_2) \\ (v_1 + w_1) - (v_2 + w_2) \end{bmatrix}$

$= \begin{bmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{bmatrix} + \begin{bmatrix} w_2 \\ w_1 + w_2 \\ w_1 - w_2 \end{bmatrix}$

this is the same as $L(\vec{v}) + L(\vec{w})$

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* $L(c\vec{v}) = cL(\vec{v})$

the processes of scalar multiplication and applying the function commute.

* $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

the processes of vector addition and applying the function commute

* $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$L(\vec{v}) = A\vec{v}$

A is an $m \times n$ matrix.

* $L: R^2 \to R^3$ $\quad L(\vec{v}) = \begin{bmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{bmatrix}$

$L(\vec{v}) = A\vec{v}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0v_1 + 1v_2 \\ 1v_1 + 1v_2 \\ 1v_1 - 1v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{bmatrix}$$

$L: R^3 \to R^2$ $\quad L(v) = \begin{bmatrix} v_1 + v_3 \\ 2v_2 - v_3 \end{bmatrix}$

(1) what is the matrix representation of $L(v)$?

$$L(v) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot v$$

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Imag and Kernel

$L: \mathbb{R}^3 \to \mathbb{R}^2 \qquad L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad L\begin{bmatrix} c \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c - 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix}$

$S = \begin{bmatrix} c \\ 2c \\ 0 \end{bmatrix}$ this is the image of the subspace S

$L: \mathbb{R}^3 \to \mathbb{R}^2 \qquad L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$

$L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_1 = 0, \; v_2 = v_3 \quad \begin{bmatrix} 0 \\ c \\ c \end{bmatrix}$

this set of vectors in the kernel of L.

checking comprehension

$L: \mathbb{R}^2 \to \mathbb{R}^1 \cdot L(v) = \begin{bmatrix} v_2 \\ -3v_1 \\ 2v_1 + 3v_2 \end{bmatrix}$

1) if is the subspace of $\mathbb{R}^2$ with vectors of the form $\langle c, 0 \rangle$ then find the image $L(S)$:
$L(S)$ includes vectors of the form $\langle 0, -3c, 2c \rangle$

2) find Ker(L
Ker(L) includes only the vector $\langle 0, 0 \rangle$

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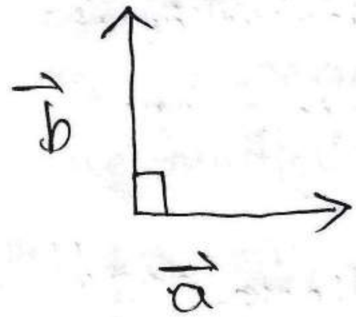
Orthogonality and orthonormality



$\theta = \pi/2$ radians or $90°$

vector dot product

$\vec{a} \cdot \vec{b} = 0$

* defining orthogonality:

$\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$\vec{a} \cdot \vec{b} = (a_1)(b_1) + (a_2)(b_2) + (a_3)(b_3)$

$\vec{a} \cdot \vec{b} = (4)(1) + (2)(-3) + (-1)(-2)$

$= 4 - 6 + 2$

$= 0$ | these vectors are orthogonal.

* defining orthonormality:

$\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{16 + 4 + 1} = \sqrt{21}$

$|\vec{b}| = \sqrt{\vec{b} \cdot \vec{b}} = \sqrt{1 + 9 + 4} = \sqrt{14}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ -1/\sqrt{21} \end{bmatrix}$ unit vector

$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$ unit vector

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example of orthogonality

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\to = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = \sqrt{(1/2) + (1/2)} = 1$$

$$\to = \sqrt{(1/\sqrt{2})^2 + (-1/\sqrt{2})^2} = \sqrt{(1/2) + (1/2)} = 1$$

Orthogonality in functions:

$f(x) = x$. $\quad g(x) = 1$

$$\langle f, g \rangle = \int_a^b x\,dx$$

$$= \frac{x^2}{2}\Big|_0^1$$

$\frac{(1)^2}{2} - \frac{(-1)^2}{2} = 0 \qquad \frac{(1)^2}{2} - \frac{(0)^2}{2} = \frac{1}{2}$

orthogonal over -1 to 1, not orthogonal over 0 to 1

checking comprihension

1. Check if the vectors $\langle 3, 5, 2 \rangle$ and $\langle -2, 2, -2 \rangle$ are orthogonal

Ans: Orthogonal,

2. Normalize the vectors $\langle -1, 2, 1 \rangle$ and $\langle 3, 0, 4 \rangle$

Ans: $\langle -1/\text{root}(6); 2/\text{root}(6), 1/\text{root}(6) \rangle$ and $\langle 3/5, 0, 4/5 \rangle$

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Perforoming diagnolization

$A = \begin{bmatrix} -3 & -4 \\ -5 & 6 \end{bmatrix}$

find the eigenvalues

$$|A-\lambda I| = 0$$
$$= (-3-1\lambda)(6-\lambda) - (-4)(5) = 0$$
$$= (-18+3\lambda) - 6\lambda + \lambda^2 + 20 = 0$$
$$= \lambda^2 - 3\lambda + 2 = 0$$
$$= (\lambda-1)(\lambda-2) = 0$$

$A-(1)I = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix}$ $x_1 = -1$, $x_2 = 1$ $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A-(2)I = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix}$ $x_1 = -4/5$, $x_2 = 1$ $\vec{x}_2 = \begin{bmatrix} -4/5 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$ $\lambda = 1$ $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -4/5 \\ 1 \end{bmatrix}$

$\lambda - 2$

$$x^{-1} A x = D$$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $x = \begin{bmatrix} -1 & -4/5 \\ 1 & 1 \end{bmatrix}$ $x^{-1} = \begin{bmatrix} 1 & 4/5 \\ -1 & -1 \end{bmatrix} \cdot \frac{1}{|-5|}$

$|x| = (-1)(1) - (-4/5)(1) = -1 + 4/2$

$x^{-1} = \begin{vmatrix} 1 & 4/5 \\ -1 & -1 \end{vmatrix} - 5 = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$

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(3) $\vec{U_3} = \vec{V_3} - \frac{\vec{V_3}\cdot\vec{U_1}}{\vec{U_1}\cdot\vec{U_1}}\vec{U_1} - \frac{\vec{V_3}\cdot\vec{U_2}}{\vec{U_2}\cdot\vec{U_2}}\vec{U_2}$

a, $\vec{V_3}\cdot\vec{U} = (1)(1) + (1)(-1) + (2)(1) = 2$

a, $\vec{U_1}\cdot\vec{U_1} = 3$

a, $\vec{V_3}\cdot\vec{U_2} = (1)(1/3) + (1)(2/3) + (2)(1/3) = \frac{5}{3}$

a, $\vec{U_2}\cdot\vec{U_2} = (1/3)(1/3) + (2/3)(2/3) + (1/3)(1/3)$

$= (1/9) + (4/9) + (1/9) = \frac{6}{9}$

$U_3 = \begin{bmatrix}1\\1\\2\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} - \frac{5/3}{2/3}\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$

$\to \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$

Checking comprehension:

$V_1 = \begin{bmatrix}1\\2\\-2\end{bmatrix}$ $V_2 = \begin{bmatrix}4\\3\\2\end{bmatrix}$ $V_3 \begin{bmatrix}1\\2\\1\end{bmatrix}$

$U_1 = \begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix}$ $U_2 \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$ $U_3 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$

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Finding Eigenvalues and Eigenvectors.

defining eigenvalues and eigenvectors.

square $n\times n$ matrix A

$A\vec{x} = \lambda.\vec{x}$

$\vec{x}$ = eigenvector

λ = eigenvalue.

$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

verify

$A\vec{x} = \lambda\vec{x}$

$A\vec{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-3)(1) + (1)(1) \\ (-2)(1) + (0)(1) \end{bmatrix}$

$= \begin{bmatrix} -3+1 \\ -2+0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ this is an eigenvector and -2 is an eigen value

$(A - \lambda I)\vec{x} = \vec{0}$

this can not be the zero vector.

$(A - \lambda I)^{-1}(A - \lambda I)\vec{x} = (A - \lambda I)^{-1}\vec{0}$

$\vec{x} = (A - \lambda I)^{-1}\vec{0}$

$\vec{x} = \vec{0}$ (the determinant of this matrix must be zero

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$(A - \lambda I)\vec{x} = \vec{0}$

$|A - \lambda I| = 0$ (characteristic equation)

solutions to this equation will be our eigenvalues.

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$((1-\lambda)(1-\lambda)) - (4)(1) = 0$

$(1-\lambda)(1-\lambda) - 4 = 0$

$1 - 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

this is our characteristic equation

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3, \lambda = -1$

$(A - \lambda I)\vec{x} = 0$

the solution of this equation gives us only the form of the eigenvectors.

$A\vec{x} = \lambda\vec{x}$

$A(c\vec{x}) = cA\vec{x} = c\lambda\vec{x} = \lambda(c\vec{x})$

$= \lambda(c\vec{x})$

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$(A-\lambda I)\vec{x} = \vec{0}$

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\lambda = 3$, $\lambda = -1$

$A - 3I = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-2x_1 + x_1 = 0$

$x_2 = 2(1)$

$x_2 = 2$

let's choose $x_1 = 1$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{x} = \begin{bmatrix} c \\ 2c \end{bmatrix}$

$A - (-1)I = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2x + x_2 = 0$ $\vec{x} = \begin{bmatrix} c \\ -2c \end{bmatrix}$

$x_2 = -2$

~~Ex~~ $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

let's find the eigenvalues and the eigenvectors.

$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 2 & 3 & 4-\lambda \end{bmatrix} = (1-\lambda)[(-2-\lambda)(4-\lambda) - (0)(3)] - 0 + 0$

$= (1-\lambda)(-2-\lambda)(4-\lambda) = 0$

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$\lambda = 1$
$\lambda = -2$
$\lambda = 4$

⊕ Solving for eigonvaluen and eigenvectors.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 0 \\ 2 & 3 & 3 \end{bmatrix}$$

$\Rightarrow 3x_1 - 3x_2 = 0$ | $2 + 3 + 3x_3 = 0$ | $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -5/3 \end{bmatrix}$

$x_1 = x_2 = 1$ | $x_3 = -5/3$

$$A - (-2)I = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 3 & 6 \end{bmatrix}$$

$= 3x_1 = 0$ | $2x_1 + 3x_2 + 6x_3 = 0$ | $\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$x_1 = 0$ | $3x_2 + 6x_3 = 0$

$x_1 = 0$ | $x_2 = -2x_3$

$x_2 = -2(1)$

$x_3 = 1$

$$A - (4)I = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

$x_1 = 0$ | $2x_1 + 3x_2 = 0$ | $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$3x_1 - 6x_2 = 0$ | $0 = 0$

$3(0) - 6x_2 = 0$ | this makes x_3 a free varible

$x_2 = 0$

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The gram-schmidt Process.

introducing the gram-schmidt process.

V

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \rightarrow \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$

these vectors serve as a basis for V.

1) $\vec{u}_1 = \vec{v}_1$

2) $\vec{u}_2 = \vec{v}_2 - \dfrac{\langle \vec{v}_2, \vec{u}_1 \rangle}{|\vec{u}_1|^2} \vec{u}_1$

3) $\vec{u}_3 = \vec{v}_3 - \dfrac{\langle \vec{v}_3, \vec{u}_1 \rangle}{|\vec{u}_1|^2} \vec{u}_1 - \dfrac{\langle \vec{v}_3, \vec{u}_2 \rangle}{|\vec{u}_2|^2} \vec{u}_2$

4) $\vec{u}_k = \vec{v}_k - \displaystyle\sum_{i=1}^{k-1} \dfrac{\langle \vec{v}_k, \vec{u}_i \rangle}{|\vec{u}_i|^2} \vec{u}_i$

example

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(1) $\vec{u}_1 = \vec{v}_1$ so; $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(2) $\vec{u}_2 = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \vec{v}_2 - \dfrac{2}{(1)(1) + (-1)(-1) + (1)(1)} \vec{u}_1$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

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Checking comprehension

find the eigenvalues and eigenvectors associated with

$$A = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$$

$\lambda_1 = 2$, with eigenvectors of the form $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$, with eigenvectors of the form $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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Diagonalization

Understanding diagonalization:

$A = XDX^{-1}$ (diagonal matrix)

$x^{-1}(A)x = x^{-1}(xDx^{-1})x$

$x^{-1}(A)x = \cancel{x^{-1}x}D\cancel{x^{-1}x}$

$x^{-1}Ax = D$

the x matrix is said to diagonalize A, this gives us a diagonal matrix.

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ eigenvectors of A.

$$x = [\vec{x}_1 \; \vec{x}_2 \; \ldots \; \vec{x}_n]$$

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$$A = \begin{bmatrix} -1 & -4/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ 10 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 - (4/5)(10) & 4 - (4/5)(10) \\ -5 + 10 & -4 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 - 8 & 4 - 8 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

$A = xDx^{-1}$ (Proved)

Checking comprehension:

"Given matrix A, find x, D, and x^{-1}
Always choose $x_2 = 1$ and order eigenvalues from lowest to highest.

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \quad x^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$$

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Page ; 48 complex, Heromitial, and Unitary matrix

$a + b$ (real part) and imaginary part $= i = \sqrt{-1}$

$A + iB$ (real part) and imaginary matrix $= \begin{bmatrix} 2+3i & 1 & 6-4i \\ 7 & 2-3i & -i \end{bmatrix}$

real part + imaginary part $= \begin{bmatrix} 2 & 0 & 6 \\ 7 & 2 & 0 \end{bmatrix} + i \begin{bmatrix} 3 & 1 & -4 \\ 0 & -3 & -1 \end{bmatrix}$

$a + bi \xrightarrow{\text{complex conjugate}} a - bi$

$A + iB \xrightarrow{\text{complex conjugate}} A - iB$

Orginal matrix: $\begin{bmatrix} 2+3i & 1 & 6-4i \\ 7 & 2-3i & -i \end{bmatrix}$ to complix

Complex conjugate $\begin{bmatrix} 2-3i & -i & 6+4i \\ 7 & 2+3i & i \end{bmatrix}$

Conjugate transpose $\rightarrow \begin{bmatrix} 2-3i & 7 \\ -i & 2+3i \\ 6+4i & i \end{bmatrix}$

$M^H = M$ (Heromitian matrix:

$M = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$

complex conjugate

conjugate transpose (same as M)

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Properties of Unitary matrices

Orthogonal (real)
- columns form an orthonormal basis.

$A^T = A^{-1}$
(Transpose = inverse)

Unitary (complex)
- columns form orthonormal vectors

$U^H = U^{-1}$
(Conjugate transpose = inverse)

$$U = \begin{bmatrix} i/\sqrt{2} & -1\sqrt{2} \\ 1/\sqrt{2} & -i\sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ -1\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

conjugate $\qquad U^H$

$$U^H U = \begin{bmatrix} -1\sqrt{2} & 1/\sqrt{2} \\ -1\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -(-1)/2 + 1/2 & i/2 - i/2 \\ -1/2 + 1/2 & 1/2 - (-1)/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Checking comprehension:

① find the congugate transpose of the following matrix:

$$\begin{bmatrix} i & 2+1 \\ 5 & 5 \\ 9+7i & -3-3i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 5 & 9-7i \\ 2-i & 5 & -3+3i \end{bmatrix}$$

② is the following matrix a hermitian matrix?

$$\begin{bmatrix} 3i & 2+2i \\ 2-2i & 1 \end{bmatrix} \rightarrow \underline{Ans}: \text{No}.$$

③ is the folloing matrix a unitary matrix?

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1+i & -1 \\ 1 & 1-i \end{bmatrix} \quad \underline{Ans}: \text{Yes}.$$