Linear algebra

introduction to matrices

Part → 1

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 0 & -5 \end{bmatrix} = A\ (2\times 3)\ \text{matrix} \qquad A[2,2] = 0$$

$a_{2,2} = 0$

$a_{1,3} = 2$

$$B_{5\times 2} = \begin{bmatrix} 1 & 10 \\ 2 & 3 \\ 0 & 7 \\ -5 & 2 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = A,\ B = \begin{bmatrix} -7 & 2 \\ 3 & 5 \end{bmatrix},\ A+B = \begin{bmatrix} (3+-7) & (-1+2) \\ (2+3) & (0+5) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 5 \end{bmatrix}$$

$$A - B \in A + -1\cdot\begin{bmatrix} 7 & 2 \\ 3 & 5 \end{bmatrix} \in A + \begin{bmatrix} 7 & -2 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 5 \end{bmatrix}$$

Video 2. matrix multiplication

$$\begin{bmatrix} 1 & -3 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 10 & -8 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} (2\cdot 10 + -3\cdot 12) & (-16+6) \\ (-10+60) & (-56-10) \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 130 & -66 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 11 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} \qquad B\cdot A = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$A\cdot B = \begin{bmatrix} 14 & 22 \\ 43 & 50 \end{bmatrix}$$

Video 3 matrix multiplication (Part 2)

$$\underset{2\times 3\ \text{matrix}}{\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 5 \end{bmatrix}} \times \underset{3\times 2\ \text{matrix}}{\begin{bmatrix} -1 & 3 \\ 0 & 5 \\ 2 & 5 \end{bmatrix}} = \begin{bmatrix} (3\cdot -1 + 1\cdot 0 + 2\cdot 2) & (3\cdot 3 + 1\cdot 5 + 2\cdot 5) \\ (-2\cdot -1 + 0\cdot 0 + 5\cdot 2) & (-2\cdot 3 + 0\cdot 5 + 5\cdot 5) \end{bmatrix} = \underset{2\times 2\ \text{matrix}}{\begin{bmatrix} 1 & 24 \\ 12 & 4 \end{bmatrix}}$$

$$\underset{3\times 2}{\begin{bmatrix} -1 & 3 \\ 0 & 5 \\ 2 & 5 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 5 \end{bmatrix}} = \underset{3\times 3.}{\begin{bmatrix} -4 & -1 & 13 \\ -10 & 0 & 25 \\ -4 & 2 & 14 \end{bmatrix}}$$

Video-4

idea behid inverting a 2x2 matrix/matrices | freecodecamp!

$I \cdot A = A$
$AI = A$ $\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$1 \cdot a = a$

$\frac{1}{a} a = 1$

$A^{-1} A = I$

$A A^{-1} = I$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = ad - bc$

$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|B| = -7$

$B = \begin{bmatrix} 3 & -4 \\ 2 & -5 \end{bmatrix}$ or, $B^{-1} = \frac{1}{-15+8} \begin{bmatrix} -5 & 4 \\ -2 & 3 \end{bmatrix}$ or, $B^{-1} = \frac{1}{7} \begin{bmatrix} -5 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{2}{7} & \frac{-3}{7} \end{bmatrix}$

So,

$$\begin{bmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{2}{7} & \frac{-3}{7} \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} \frac{15}{7} - \frac{8}{7} & \frac{-20}{7} + \frac{20}{7} \\ \frac{6}{7} - \frac{6}{7} & \frac{8}{7} + \frac{15}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Video5: classic video on inverting a 3x3 matrix part 1

matrix of minor

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = D \begin{bmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

* matrix of cofactors

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} = Adj(A) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ or, $A^{-1} = \frac{1}{|A|} \times adj(A)$ $|A| = -1$

$|A| = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot -2$

$= 1 + 0 - 2$

$= -1$ $= 1 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ so, $|A| = -1$

Page-3

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

Video→6: classic video on inventing a 3x3 matrix part 2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right]$$

or, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right]$ $A = I$

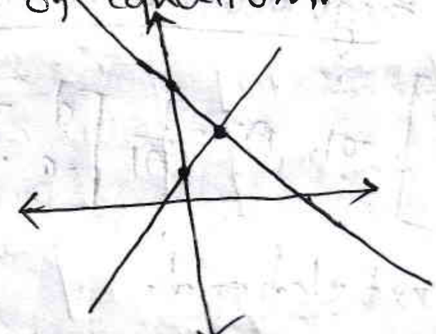
$A^{-1}A = I$

$\downarrow$ I $\downarrow$ A^{-1}

Video 7: Matrices to solve a system of equations.

$3x + 2y = 7$ or, $y = \frac{-3}{2}x + \frac{7}{2}$

$-6x + 6y = 6$ $y = x + 1$



$$\underset{A}{\begin{bmatrix} 3 & 2 \\ -6 & 6 \end{bmatrix}} \cdot \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 7 \\ 6 \end{bmatrix}}$$

$ax = b$

$\cancel{\frac{1}{a}} \cancel{a} x = \frac{1}{a} \cdot b$

$x = \frac{b}{a}$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

$x = A^{-1}b$

$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} 6 & -2 \\ 6 & 3 \end{bmatrix}$

$|A| = 18 + 12 = 30$

$A^{-1} = \frac{1}{30}\begin{bmatrix} 6 & -2 \\ 6 & 3 \end{bmatrix}$ or, $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 6 & -2 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 30 \\ 60 \end{bmatrix}$

video-4

idea behind inverting a 2x2 matrix | matrices | Precalculus

$I \cdot A = A$
$AI = A$ $\rightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $1\cdot a = a$, $\frac{1}{a}a = 1$

$A^{-1}A = I$
$AA^{-1} = I$

$A=\begin{bmatrix}a&b\\c&d\end{bmatrix} = A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$|A| = ad-bc$
$A^{-1} = \frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$
$|B| = -7$

$B = \begin{bmatrix}3&-4\\2&-5\end{bmatrix}$ or, $B^{-1} = \frac{1}{-15+8}\begin{bmatrix}-5&4\\-2&3\end{bmatrix}$ or, $B^{-1} = \frac{1}{7}\begin{bmatrix}-5&4\\-2&3\end{bmatrix} = \begin{bmatrix}\frac{5}{7}&\frac{-4}{7}\\\frac{2}{7}&\frac{-3}{7}\end{bmatrix}$

So,
$\begin{bmatrix}\frac{5}{7}&\frac{-4}{7}\\\frac{2}{7}&\frac{-3}{7}\end{bmatrix}\begin{bmatrix}3&-4\\2&-5\end{bmatrix} = \begin{bmatrix}\frac{15}{7}-\frac{8}{7} & \frac{-20}{7}+\frac{20}{7}\\\frac{6}{7}-\frac{6}{7} & -\frac{8}{7}+\frac{15}{7}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

Video5: classic video on inverting a 3x3 matrix part 1

matrix of minor

$A = \begin{bmatrix}1&0&1\\0&2&1\\1&1&1\end{bmatrix} = \begin{bmatrix}\begin{vmatrix}2&1\\1&1\end{vmatrix}&\begin{vmatrix}0&1\\1&1\end{vmatrix}&\begin{vmatrix}0&2\\1&1\end{vmatrix}\\\begin{vmatrix}0&1\\1&1\end{vmatrix}&\begin{vmatrix}1&1\\1&1\end{vmatrix}&\begin{vmatrix}1&1\\0&1\end{vmatrix}\\\begin{vmatrix}0&1\\2&1\end{vmatrix}&\begin{vmatrix}1&1\\0&1\end{vmatrix}&\begin{vmatrix}1&0\\0&2\end{vmatrix}\end{bmatrix} = D\begin{bmatrix}1&-1&-2\\-1&0&1\\-2&1&2\end{bmatrix}$

matrix of cofactors

$\begin{bmatrix}1&1&-2\\1&0&-1\\-1&-1&1\end{bmatrix} = Adj(A) = \begin{bmatrix}1&1&-2\\1&0&-1\\2&-1&2\end{bmatrix}$

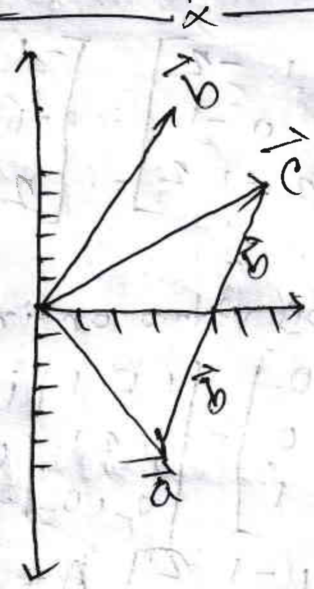
$A = \begin{bmatrix}1&0&1\\0&2&1\\1&1&1\end{bmatrix}$ or, $A^{-1} = \frac{1}{|A|} \times adj(A)$ $|A| = -1$

$|A| = 1\cdot1 + 0\cdot1 + 1\cdot-2$
$= 1+0-2$
$= -1$ $= 1\cdot\begin{bmatrix}2&1\\1&1\end{bmatrix} - 0\cdot\begin{bmatrix}0&1\\1&1\end{bmatrix} + 1\begin{bmatrix}0&2\\1&1\end{bmatrix}$ so, $|A| = -1$

Video-8

Solve a vector combination problem

$\vec{a} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$



$\vec{a}x + \vec{b}y = \vec{c}$

$\begin{bmatrix} 3 \\ -6 \end{bmatrix}x + \begin{bmatrix} 2 \\ 6 \end{bmatrix}y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $3x+2y=7$

$6x+6y=6$

$A^{-1}\begin{bmatrix} 3 & 1 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

$A^{-1} = \frac{1}{30}\begin{bmatrix} 6 & -2 \\ 6 & 3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 6 & -2 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 36 \\ 60 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

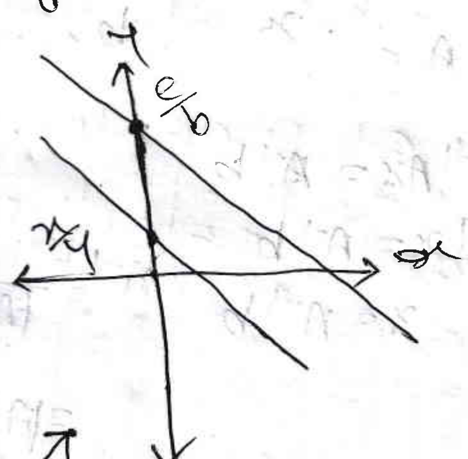
Video-9 Singular matrices:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ বা, $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

A^{-1} = is not defiend.

if f. $|A| = 0$ বা, $|A| = ad - bc = \frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$

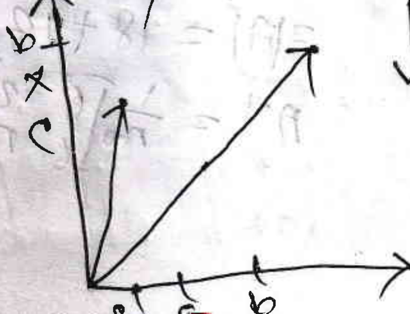
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ x \end{bmatrix}$



$ax + by = e$ বা, $y = -\frac{a}{b}x + \frac{e}{b}$

$cx + b7 = f$ বা, $y = \frac{f}{d}x + \frac{f}{y}$

$\begin{bmatrix} a \\ c \end{bmatrix}x + \begin{bmatrix} b \\ d \end{bmatrix}y = \begin{bmatrix} e \\ f \end{bmatrix}$



Video → 10

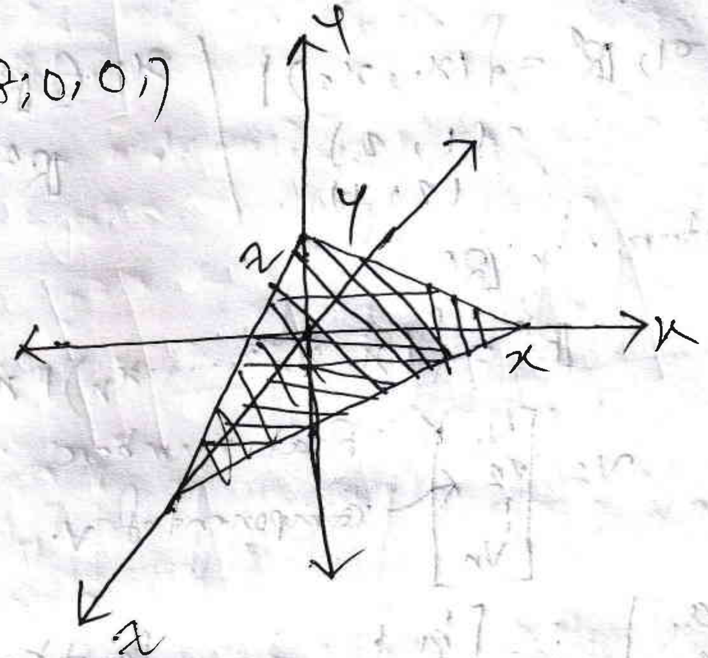
3 Variable linear equation (part 1)

Page → 2

$x + 4y + z = 8$

$(0,0,8) = (0,2,0)$ or, $(8,0,0)$

$x + y + 3z = 3$



Video 11 : Solving 3 equations with Unknowns:

$2x - y = 10 \xrightarrow{\times 3} 6x - 3y = 30$

$x + 3y = 5 \longrightarrow x + 3y = 5$

$7x = 35$

$x = 5$

$2(5) - y = 10$

$10 - y = 10$

$y = 0$

$-x + 2y - z = 4 \rightarrow -3x + 6y - 3z = 27$

$3x - 7y - 2z = -20$

$2x + 2y + z = 2$

$-x + 2y - z = 4$

$-y - 5z = 7$

$z = 7$

$-y - 5(2) = 7$

$-y + 10 = 7$

$-y + 10 = 7 \quad y = -3$

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 4 \\ 3 & -7 & -2 & -20 \\ 2 & 2 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1 & 2 & -1 & 4 \\ 0 & -1 & -5 & 7 \\ 0 & 6 & -1 & 20 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 2 & -1 & 4 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & -3 & 62 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & 2 & -1 & 4 \\ 0 & -1 & -5 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$-x + 2(3) - (2) = 4$

$-x + 6 + 4 = 4$

$-x + 10 = 4$

$-x = -1$

$x = 1$

Video - 12
linear algebra

$\mathbb{R}$ OR R ← set of all real numbers.

$\mathbb{R}^n$ →1, $\mathbb{R}^2 = \{(x_1, x_2) \mid$ / $x_i \in \mathbb{R}$ for $i \leq 2$

(1, 2)
(2, 1)

$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

* Vector in $\mathbb{R}^n$.

$\mathbb{R}^n = \{(x_1, x_2 \ldots\ldots x_n) \mid x_i \in \mathbb{R}$ (ordered)

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

v_1 ← → Real number

v_2 ← Component of v.

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad a+b = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

* scalar multiple.

$$ca = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \qquad -3\begin{bmatrix} \pi \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3\pi \\ 6 \\ 0 \end{bmatrix}$$

Zero vector:

in $\mathbb{R}^n$ $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ n → components.

Video → 15

Page 8

Linear combination and span

Linear combination:

$v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$

* $c_1 v_1 + c_2 v_2 + \ldots c_n v_n$

$c_1 \to c_n \in \mathbb{R}$

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$0\vec{a} + 0\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3\vec{a} + 2\vec{b} = \begin{bmatrix} 3 & -0 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

* $\text{span}(\vec{0}) = 0 \left[c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]$

$\left(\begin{array}{l} \text{span}(v_1 v_2 \ldots v_n) \\ = c_1 v_1 + c_2 v_2 + \ldots c_n v_n \mid c_i \in \mathbb{R} \end{array} \right)$

$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. $\vec{x} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$c_1\vec{a} + c_2\vec{b} = x$

$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$1c_1 + 0c_2 = x_1$

$2c_1 + 3c_2 = x_2$

$-2c_1 + 0 \cdot = 2x_1$

$3c_2 = x_2 - 2x_1$

$c_2 = \frac{1}{3}(x_2 - x_1)$

$c_2 = x_1$

$c_1 = 2$

$c_2 = \frac{1}{3}(0) = 0$

Video → 16
introduction to linear independence

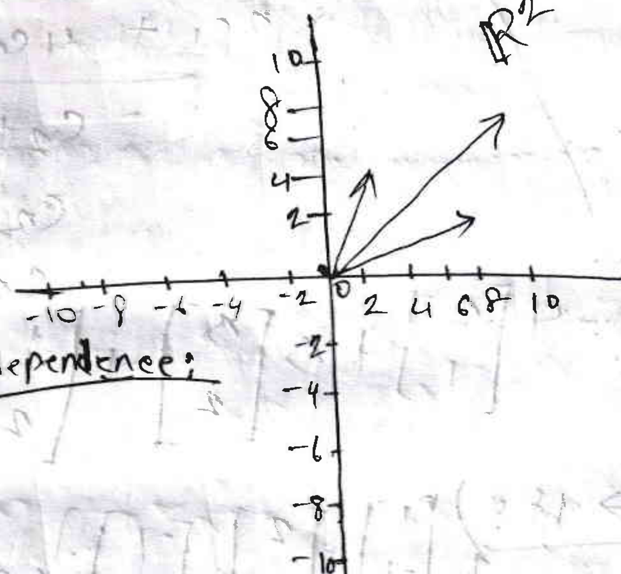
$\Rightarrow \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix} \Rightarrow c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\6\end{bmatrix} \Rightarrow c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\, 2\begin{bmatrix}2\\3\end{bmatrix}$

$\Rightarrow (c_1 + 2c_2)\begin{bmatrix}2\\3\end{bmatrix} \Rightarrow c_3\begin{bmatrix}2\\3\end{bmatrix}$

Ex $\begin{bmatrix}2\\3\end{bmatrix} \begin{bmatrix}7\\2\end{bmatrix} \begin{bmatrix}9\\5\end{bmatrix}$ linear dependent set

$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

Square $(v_1, v_2) = R^2$



Video → 17 : more on linear independence:

* linear Dependance:

$S = \{v_1, v_2, \dots, v_n\}$

linear dependent; iff.

$$c_1v_1 + c_2v_2 + \cdots c_nv_n = 0 = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$$

Ex One vector,

$v_1 = a_2v_2 + a_3v_3 + \cdots a_nv_n.$

$0 = (-1)v_1 + a_2v_2 + a_3v_3 + \cdots a_nv_n$

→ non zero.

Assume

$c_1 \neq 0$

$v_1 + \frac{c_2}{c_1}v_2 + \cdots \frac{c_n}{c_1} = 0$

$\Rightarrow \frac{c_2}{c_1}v_2 + \cdots \frac{c_n}{c_1}v_n = -v_1$

Ex $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right\}$

$c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\end{bmatrix} = 0 \begin{bmatrix}0\\0\end{bmatrix}$

c_1 or c_2 non zero ⇒ dependent

$c_1 + c_2$ both zero ⇒ independent

$2c_1 + 3c_2 = 0$

$c_1 + 2c_2 = 0$

Page → 8

$c_1 + \frac{3}{2} c_2 = 0$

$\frac{1}{2} c_2 = 0$

$c_2 = 0$

$c_1 = 0$

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ → Linearly dependent

$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2c_1 + 3c_2 + c_3 = 0$

$c_1 + 2c_2 + 2c_3 = 0$

$2c_1 + 3c_2 - 1 = 0$

$2c_1 + 4c_2 - 4 = 0$

$-c_2 + 3 = 0$

$c_2 = -3$

$c_2 = 3$

$c_1 + 6 - 2 = 0$

$c_1 + 4 = 0$

$c_1 = -4$

$c_3 = -1$

$c_2 = 3$

$c_1 = -4$

So, $-4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → linearly dependent.

video → 18 : $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\} = S$

Span $(S) = \mathbb{R}^3$ → Linearly indepen

$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$c_1 + 2c_2 - c_3 = a$ or, $c_1 + 2c_2 - c_3 = a$

$\Rightarrow -c_1 + c_2 = b$ $\Rightarrow 3c_2 - c_3 = b + a$

$2c_1 + 3c_2 + 2c_3 = c$ or, $-c_2 + 4c_3 = c - 2a$.

$c_3 = \frac{1}{11}(3c - 5a)$

$c_2 = \frac{1}{3}(b + a + c_3)$

$c_1 = a - 2c_2 + c_3$

$\Rightarrow c_1 + 2c_2 - c_3 = a$

$\Rightarrow 3c_2 - c_3 = b + a$

$\Rightarrow 11c_3 = 3c - 6a + b + a$

$\Rightarrow 3c_2 = b + a + c_3$

Video → 10

Page → 9: Linear subspaces:

$\vec{a}$ in V $\quad \vec{a}+\vec{b}$ = in V

$\vec{b}$ in V

$V=\{0\} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ | V a subspace of $\mathbb{R}^n$, zero vector V

$c\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ closed under addition

Ex: $S = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} \in R^2 \;\middle|\; x_1 \geq 0 \right\}$

is S subspace of $\mathbb{R}^2$

Column $\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}a+c\\b+d\end{bmatrix}$

closed under addition

closed under scalar multiplication

$-1\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}-a\\-b\end{bmatrix}$ (not closed under

$U = \text{span}(v_1, v_2, v_n)$ = Valid subspace of $\mathbb{R}^n$

$0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v} + c_3\vec{v}$

$a\vec{x} = ac_1\vec{v}_1 + ac_2\vec{v} + ac_3\vec{v}$

$= c_4\vec{v}_1 + c_5 v_2 + c_6\vec{v}_3$

$\vec{y} = d_1\vec{v} + d_2 v_2 + d_3\vec{v}_3$

$\vec{x}+\vec{y} = (c_1 d_1)\vec{v}_1 + (c_2+d_2)v_2 + (c_3+d_3)\vec{v}_3$

Basis of a subspace.

* $V = \text{span}(\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n)$ linearly independent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_n\vec{v}_n = \vec{0}$

$c_1 = c_2 = \cdots c_n = 0$ / $T = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n, \vec{v}_s\}$

$S = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$ (S is a basis for V)

span(T) = V (T is linearly dependant)

(T is not a basis for V)

* Basis minimum set of vectors

$S = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right\}$. Span (S) = ?

$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$2c_1 + 7c_2 = x_1$ or, $\frac{2}{3}x_2 + 7c_2 = x_1$

$3c_1 + 0 = x_2$ or, $7c_2 = 1 - \frac{2}{3}x_2$

$\Rightarrow c_1 = \frac{x_2}{3}$

$c_2 = \frac{x_1}{7} - \frac{2}{21}x_2$

$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = 0$ / $c_1 = 0$

$x_2 = 0$ / $c_2 = 0$

S is a basis for $\mathbb{R}^2$

$T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ linearly indipependent.

$\{v_1 v_2 \cdots\cdots v_n\}$ = Basis for U subspace.

Video → 20

Page → 11

vector dot product and vector length

addition

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

Scalar multiplication

$$c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

⊗ dot product: $\vec{a}\cdot\vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \dots a_nb_n$

⊗ $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2\cdot 7 + 5\cdot 1$
$= 14 + 5 = 14$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = 13$

$\Rightarrow 1\cdot(-2) + 2\cdot 0 + 3\cdot 5$

$= -2 + 0 + 15$

Lenth:

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

$$\vec{a}\cdot\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \dots a_n^2$$

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$$

$$\|\vec{a}\| = \vec{a}\cdot\vec{a}$$

Video → 21

Proving vector dot product properties

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots v_n w_n$

$\vec{w} \cdot \vec{v} = w_1 v_1 + w_2 v_2 + \cdots w_n v_n$

$v_1 w_1 = w_1 v_1$ (commutative property)

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ or, $(\vec{v} + \vec{w}) \cdot \vec{x} = (\vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x})$

$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ or, $(v_1 + w_1) x_1 + (v_2 + w_2) x_2 + (v_n + w_n) x_n$

or, $\vec{v} \cdot \vec{x} = v_1 x_1 + v_2 x_2 + \cdots v_n x_n$

or, $\vec{w} \cdot \vec{x} = w_1 x_1 + w_2 x_2 + \cdots w_n x_n$

→ or, $\vec{v} \cdot \vec{x} + \vec{w} \cdot x = (v_1 x_1 + w_1 x_1) + (v_2 x_2 + w_2 x_2) \cdots + (v_n x_n + w_n x_n)$

or, $(v_1 + w_1) x_1 + (v_2 + w_2) x_2 \cdots (v_n + w_n) x_n$

$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

$\begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} \cdot \vec{w} = cv_1 w_1 + cv_2 w_2 + \cdots cv_n w_n$

or, $c(\vec{v} \cdot \vec{w}) = c(v_1 w_1 + v_2 w_2 + \cdots v_n w_n)$

$= cv_1 w_1 + cv_2 w_2 + \cdots cv_n w_n)$

Video → 22: Proof of the cauchy-schwarz inequality.

$\vec{x}, \vec{y} \in \mathbb{R}^n$ non zero

or, $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$

or, $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\| \Rightarrow \vec{x} = c\vec{y}$ (cauchy-schwarz)

$P(t) = \|t\vec{y} - \vec{x}\|^2 \geq 0$

$= (t\vec{y} - \vec{x}) \cdot (t\vec{y} \cdot \vec{x})$

$= t\vec{y} \cdot t\vec{y} - \vec{x} \cdot t\vec{y} - t\vec{y} \cdot x + -1\vec{x} \cdot -1\vec{x}$

$\Rightarrow \underbrace{(\vec{y}\cdot\vec{y})}_{a} t^2 - 2\underbrace{(\vec{x}\cdot\vec{y})}_{b} t + \underbrace{\vec{x}\cdot\vec{x}}_{c} \geq 0$

$\Rightarrow P(t) = at^2 - bt + c \geq 0$

$\Rightarrow P(\frac{b}{2a}) = a\frac{b^2}{4a} - b\frac{b}{2a} + c \geq 0$ বা, $\frac{b^2}{4a} - \frac{2b^2}{4a} + c \geq 0$

বা, $\frac{-b^2}{4a} + c \geq 0$ বা, $c \geq \frac{b^2}{4a}$ বা, $4ac \geq b^2$ বা, $4\|\vec{y}\|^2\|\vec{x}\|^2 \geq 4(\vec{x}\cdot\vec{y})^2$

বা, $4(\|\vec{y}\|\cdot\|\vec{y}\|^2) \geq (2(\vec{x}\cdot\vec{y})^2)$

বা, $\|\vec{y}\|\,\|\vec{x}\| \geq |\vec{x}\cdot\vec{y}| \rightarrow$ chuchy-chwraz.

Video 29

vectoro triangle inequality:

যদি, $\vec{x}, \vec{y} \in \mathbb{R}^n$ বা, $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$ বা, $\vec{x} = c\vec{y}$ (non-zero)

বা, বা, $|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\,\|\vec{y}\|$ বা, $\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

বা, $\|\vec{x}+\vec{y}\| = \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$

বা, $\|\vec{x}+\vec{y}\|^2 = \vec{x}\cdot(\vec{x}+\vec{y}) + \vec{y}\cdot(\vec{x}+\vec{y})$

বা, $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2$

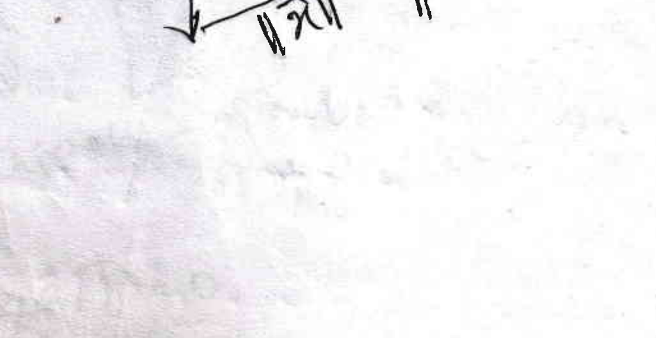
বা, $\|\vec{x}+\vec{y}\|^2 \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$

বা, $\|\vec{x}+\vec{y}\|^2 \leq (\|\vec{x}\| + \|\vec{y}\|)^2$

বা, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \rightarrow$ Triangle inequality.

বা, $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$.

বা, $\vec{x} = c\vec{y}$ $c > 0$

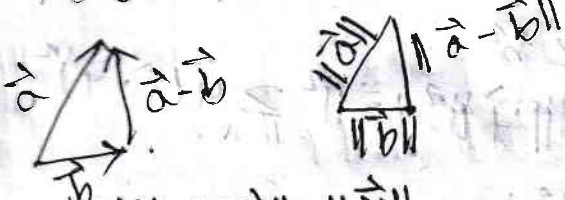


Video 25

Page-15 : Defining the angle between vectors.

$\|\vec{x}\|$ = length of scalars. | angle between vectors.

$\vec{a}, \vec{b} \in R^n$; nonzero.



$\|\vec{b}\| > \|\vec{a}\| + \|\vec{a}-\vec{b}\|$

$\|\vec{a}\| > \|\vec{a}-\vec{b}\| + \|\vec{b}\|$

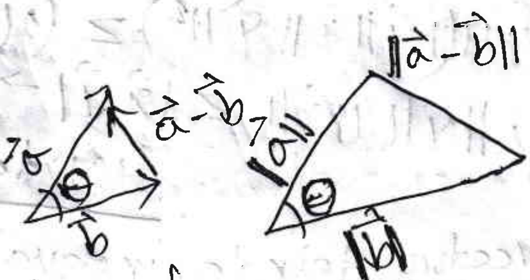
$\|\vec{a}-\vec{b}\| > \|\vec{a}\| + \|\vec{b}\|$

$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

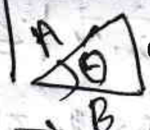
$\|\vec{a}\| = \|\vec{b} + (\vec{a}-\vec{b})\| \le \|\vec{b}\| + \|\vec{a}-\vec{b}\|$

$\|\vec{b}\| = \|\vec{a} + (\vec{b}-\vec{a})\| \le \|\vec{a}\| + \|\vec{b}-\vec{a}\|$ $\|\vec{b}\|$

$\|\vec{a}-\vec{b}\| = \|\vec{a} + (-\vec{b})\| \le \|\vec{a}\| + \cancel{\|-\vec{b}\|}$



low of cosine



$C^2 = A^2 + B^2 - 2AB\cos\theta$

$\|\vec{a}-\vec{b}\|^2 = \|\vec{b}\| + \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$\cancel{\|\vec{a}\|^2} - \cancel{2}(\vec{a}\cdot\vec{b}) + \cancel{\|\vec{b}\|^2} = \cancel{\|\vec{b}\|^2} + \cancel{\|\vec{a}\|^2} - \cancel{2}\|\vec{a}\|\|\vec{b}\|\cdot\cos\theta$

$(\vec{a}\cdot\vec{b}) = \|\vec{a}\|\|\vec{b}\|\cos\theta \rightarrow$ angol between the vectors

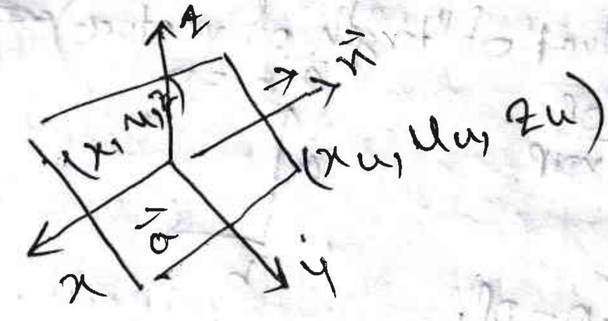
$$\boxed{\begin{aligned}\vec{a} = c\vec{b} \cdot \; & c>0 \Rightarrow \theta = 0 \\ & c<0 \quad \theta = 180\end{aligned}}$$

$0 = 0 \cdot \cos\theta, \quad \cos\theta = \frac{0}{0}$

Video - 26

Defining a plane in $\mathbb{R}^3$ with point an normal vector.

- equality of a plane in $\mathbb{R}^3$



$Ax + By + Cz = D$

$\vec{n} = B$ "notormal"

everything on the plane

$\vec{n} + (x_0, y_0, z_0) \rightarrow Ax+By+Cz=D$

$\vec{n} \cdot \vec{a} = 0$ (perpendicular)

$\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \cdot \vec{x} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (\vec{x} - \vec{x}_0)$

$\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \rightarrow n \cdot (\vec{x} - \vec{x}_0) = 0$

$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$

$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \rightarrow Ax+By+Cz=D$

$\vec{n} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on, $\vec{x} - \vec{x}_0 = \begin{bmatrix} x - 1 \\ y - 2 \\ z - 3 \end{bmatrix}$ on the plane

Video 27: cross product introduction:

- Dot product:

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -7.4 - 1.2 \\ 1.5 - 1.4 \\ 1.2 - (-7).5 \end{bmatrix} = \begin{bmatrix} -30 \\ 1 \\ 37 \end{bmatrix}$

$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1$

$\Rightarrow \vec{a} \cdot \vec{b} = 0$

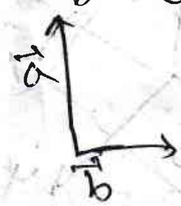
Video → 29

Dot and cross product comparison.

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ / $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

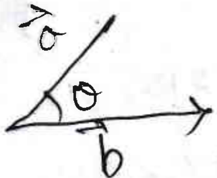
$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ (dot => product of length of vector)

$\vec{a} \cdot \vec{b} = 0$



$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$ $\quad \sin\theta = \frac{op}{hyp}$

$= \|\vec{b}\| \|\vec{a}\| \sin\theta$



$\sin\theta = \frac{a}{\|\vec{a}\|}$ $\therefore \|\vec{a}\| \sin\theta = op.$

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\|$ or, $\|\vec{a} \times \vec{b}\| = 0$ or, $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$

Video → 30:

$x_1 + 2x_2 + x_3 + x_4 = 7$

$x_1 + 2x_2 + 2x_3 - x_4 = 12$

$2x_1 + 4x_2 \quad + 6x_4 = 4$

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \text{rref}(A)$$

$x_1 + 2x_2 + 3x_4 = 2$

$x_3 - 2x_4 = 5$

$x_1 = 2 - 2x_2 - 3x_4$

$x_3 = 5 + 2x_4$

or, $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

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Video - 31

$x+y+z=3$
$x+2y+3z=0$
$x+3y+4z=-2$

$$=\left[\begin{array}{ccc|c}1&1&1&3\\1&2&3&0\\1&3&4&-2\end{array}\right]=\left[\begin{array}{ccc|c}1&1&1&3\\0&1&2&-3\\0&2&3&-5\end{array}\right]=\left[\begin{array}{ccc|c}1&0&-1&6\\0&1&2&-3\\0&0&-1&-1\end{array}\right]$$

$= -5-2(-3) = -5-(6) = -5+6 = 1$

or, $\left[\begin{array}{ccc|c}1&0&0&5\\0&1&0&-1\\0&0&1&-1\end{array}\right]$ or, $-3-2(-1) = -3-(-2)$
$= -3+2 = -1$

$x=5$
$y=-1$
$z=-1$

Video 32

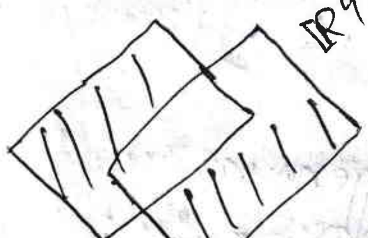
$x_1+2x_2+x_3+x_4=8$
$x_1+2x_2+2x_3-x_4=12$
$2x_1+4x_2 \quad +6x_4=4$

$$\left[\begin{array}{cccc|c}1&2&1&1&8\\1&2&2&-1&12\\2&4&0&6&4\end{array}\right]=\left[\begin{array}{cccc|c}1&2&1&1&8\\0&0&1&-2&4\\0&0&-2&4&12\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1&2&0&3&4\\0&0&1&-2&4\\0&0&0&0&-4\end{array}\right]$$

$1x_1+2x_2+0x_3+3x_4=4$
$1x_3-2x_4=4$

$\boxed{0=-4}$ → impossible.
or, $0=3$ (no solution)



$\mathbb{R}^4$ $\quad 3x+6y+4z=5$
$\mathbb{R}^3$ $\quad 3x+6y+4z=2$

Video: 33 matrix vector product: defining $A\vec{x}$

matrix: $m \times n$ ← coloumns

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&&&\vdots\\a_{m1}&&&a_{mn}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix}a_{11}x_1+a_{12}x_2+\cdots a_{1n}x_n\\a_{21}x_1+a_{22}x_2+\cdots a_{2n}x_n\\\vdots\\a_{m1}x_1+a_{m2}x_2+\cdots a_{mn}x_n\end{bmatrix}=\vec{b}=\overset{m\times 1}{\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}}$$

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$$\begin{bmatrix} -3 & 0 & 3 & 2 \\ 1 & 7 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \cdot 2 + 0 \cdot -3 + 3 \cdot 4 + 2 \cdot -1 \\ 1 \cdot 2 + 7 \cdot -3 + -1 \cdot 4 + 4 \cdot -1 \end{bmatrix}$$

$$\begin{bmatrix} -6 + 0 + 12 - 2 \\ 2 - 21 - 4 - 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \end{bmatrix}, \quad \vec{a}_1 = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 7 \\ -1 \\ 4 \end{bmatrix}$$

$$a_1^T = \begin{bmatrix} -3 & 0 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix}$$

$$a_2^T = \begin{bmatrix} 1 & 7 & -1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 0 & 3 \\ 2 & 4 & 7 & 0 \\ -1 & 2 & 3 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad A\vec{x} = \begin{bmatrix} 3x_1 + 1x_2 + 0x_3 + 3x_4 \\ 2x_1 + 4x_2 \cdots \\ -1x_1 + 2x_2 \cdots \end{bmatrix}$$

$$A \cdot \vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4$$

video: 34: introduction to the null space of a matrix:

Subspace - S

$\vec{0} \in S$

$\vec{v}_1, \vec{v}_2 \in S \Rightarrow \vec{v}_1 + \vec{v}_2 \in S$

$c \in R \quad \vec{v}_1 \in S . \quad c\vec{v}_1 \in S$

$A\vec{0} = \vec{0}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & + a_{22} & + \cdots & a_{2n} \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{v}_1 \in N$

$c\vec{v}_1 \in N$?

$A(c\vec{v}_1) = cA\vec{v}_1 = c\vec{0} = \vec{0}$

A: $m \times n$

$A\vec{x} = \vec{0} \rightarrow$ homogenous

$\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

$\vec{0} \in N$

$\vec{v}_1, \vec{v}_2 \in N$

$\rightarrow A\vec{v}_1 = \vec{0}$

$A\vec{v}_2 = \vec{0}$

$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$

$\vec{v}_1 + \vec{v}_2 \in N$.

Video → 35

Null space: 2

Calculating the null space of a matrix.

$$A=\begin{bmatrix}1&1&1&1\\1&2&3&4\\4&3&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad N(A)=\{\vec{x}\in\mathbb{R}^4 \mid A\vec{x}=\vec{0}\}$$

$\vec{x}\in\mathbb{R}^4$

$x_1+x_2+x_3+x_4=0$

$x_1+2x_2+3x_3+4x_4=0$

$4x_1+3x_2+2x_3+x_4=0$

$$\Rightarrow \left[\begin{array}{cccc|c}1&1&1&1&0\\1&2&3&4&0\\4&3&2&1&0\end{array}\right] \Rightarrow \left[\begin{array}{cccc|c}1&1&1&1&0\\0&1&2&3&0\\0&1&2&3&0\end{array}\right]$$

(over the augmented matrix: A, $\vec{0}$)

$$\left[\begin{array}{cccc|c}1&0&-1&-2&0\\0&1&2&3&0\\0&0&0&0&0\end{array}\right] \Rightarrow \begin{array}{l} x_1-x_3-2x_4=0\\ x_2+2x_3+3x_4=0\end{array} \;\Big|\; \begin{array}{l} x_1=x_3+2x_4\\ x_2=-2x_3-3x_4\end{array}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}1\\2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}2\\-3\\0\\1\end{bmatrix} \quad N(A)=\text{span}\left(\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\right)$$

Video = 36:

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&&&\\\vdots&&&\\a_{mn}&&&a_{mn}\end{bmatrix} \quad A\vec{x}=\vec{0} \quad \vec{0}\in\mathbb{R}^m$$

$N(A)=\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=0\}$

$$A=[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=x_1\vec{v}_1+x_2\vec{v}_2+\cdots x_n\vec{v}_n=\vec{0}$$

$\vec{v}_1,\vec{v}_2\cdots\vec{v}_n \rightarrow$ Linear independent

$A\vec{x}=\vec{0}$. $\Rightarrow x_1 x_2 \cdots = 0$ | $N(A)=N(\text{ref}(A))$, $\text{ref}(A)=\{0\}$

$\text{ref}(A)\cdot\vec{x}=\vec{0}$

$\vec{x}=\vec{0}$

$A\vec{x}=0$ only solution. $\vec{x}=0$

$N(A)=\{0\}$

$x_1 x_2 = \cdots = 0$

$$\text{ref}(A)=n\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$$

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Video → 37

Column space of a matrix

* Column space:

$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$

$\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n \in \mathbb{R}^m$

$C(A) = \text{span}(\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n)$

$\vec{a} \in C(A)$

$\vec{a} = c_1\vec{v}_1, c_2\vec{v} \ldots c_n\vec{v}_n$

$s\vec{a} = sc_1\vec{v}_1 + sc_2\vec{v}_2 \ldots sc_n\vec{v}_n$

$= s\vec{a} \in C(A)$

$C(A) \Rightarrow$ valid subspace

$\vec{b} \in C(A)$

$\{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

$\vec{b} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots b_n\vec{v}_n$

$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$

$\vec{a} + \vec{b} \in C(A)$

$= (c_1+b_1)\vec{v}_1 + (c_2+b_2)\vec{v}_2 + (b_n c_n)\vec{v}_n$

$\{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots x_n\vec{v}_n \mid x_1, x_2, \ldots x_n \in \mathbb{R}\} = \text{space}\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$

$= C(A)$

$\vec{b}_1$ not $\in C(A) \Rightarrow A\vec{x} = \vec{b}_1$ has no solution

$A\vec{x}_1 = \vec{b}$ has at least 1 solution $\Rightarrow \vec{b}_2 \in C(A)$

Video 39:

$A \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ rref(A) $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right)$

not $C(A)$

$x_1 + 0x_2 + 3x_3 + 2x_4 = 0$

$0x_1 + x_2 - 2x_3 - x_4 = 0$

$N(A) = N(\text{rref}(A))$

$x_1 = -3x_3 - 2x_4$

$x_2 = 2x_3 + x_4$

$N(A) = N(\text{rref}(A)) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$= \text{span}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$

$A\vec{x} = 0$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \rightarrow$ Linear independent.

L.I. $\Rightarrow$ one solution $\Rightarrow N(A) \cdot \{\vec{0}\}$

$A\vec{x} = 0, \ \vec{x} = \vec{0}$

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$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 0$ $\quad x_1 = -3x_3 - 2x_4$

$x_2 = 2x_3 + x_4$

$3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + -2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ | $c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1c \\ 2c \\ 3c \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ is a basis for $C(A)$

$C(A) = \text{span}\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\in R^3}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right)$ — R^3

$n \cdot (\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = 0$

Video 40 proof: Any subspace basis has same number of elements:

$A = \{a_1, a_2 \dots a_n\}$ basis of V

$B = \{b_1, b_2 \dots b_m\}$ $m < n$ span V

$B_1 = \{a_1, b_1, b_2 \dots b_m\}$ linearly dependent.

$a_1 = d_1 b_1 + d_2 b_2 \dots d_m b_m \quad (d_j \neq 0)$

$b_j = -\frac{1}{d_j}(-a_1 + d_1 b_1 + \dots (\;) + d_m b_m$

$B_1 = \{a_1, b_2, b_3 \dots b_m\}$ span V

$B_2' = \{a_1, a_2, b_2, b_3 \dots b_m\}$ L.D

$a_2 = c_1 a_1 + c_2 b_2 + c_3 b_3 \dots c_m b_m \quad (c_i \neq 0)$

Video 41

Dimension of the null space or nullity.

$$B = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix} \quad N(B) = \{\vec{x} \in R^n \mid B\vec{x} = 0\}$$

$N(ref(B)) = N(B) \rightarrow ref(B)$

$$= \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 7 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$x_1 + x_2 + 7x_4 - 2x_5 = 0$

$x_3 - 2x_4 + 2x_5 = 0$

$x_1 = -x_2 - 7x_4 + 2x_5$

$x_3 = 2x_4 - 2x_5$

Soluation set is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$N(A) = N(ref(A)) = span(\vec{V_1}, \vec{V_2}, \vec{V_3})$

linearly independent

$\{v_1, v_2, v_3\}$ basis fo $n(B)$

Dimension of subspace

Dimension of subspace = # of elements in a basis for subspace

dim $(n(B)) = 3$.

nullity $(B) = 3$

column space

$C(A) = span(\vec{a}, \vec{a_2}, \vec{a_3}, \vec{a_4}, a_5)$ linearly independent

basis for $C(A)$

Video 42:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 4 \\ -1 & -2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & -1 & -2 & -2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ basis for $c(A)$

$\dim(c(A)) = 3$, rank $(A) = 3$

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Video → 43

$$A = \begin{bmatrix} 0 & 0 & -1 & 4 \\ 2 & 1 & 0 & 9 \\ -1 & 2 & 1 & -5 \\ 1 & -1 & -2 & 9 \end{bmatrix}$$

$\{\vec{r}_1, \vec{r}_2, \vec{r}_n\} \Rightarrow L.I$

$c_1 \vec{r}_1 + c_2 \vec{r}_2 + c_n \vec{r}_n = 0$

$c_1 c_2 c_4 = 0$

$$R \begin{bmatrix} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ c_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 \vec{r}_1 + c_2 \vec{r}_2 + c_n \vec{r}_n = 0$

$R\vec{x} = 0$ $\quad A\vec{x} = 0$

Video 44 : Candidate basis does span $C(A)$

$\{a_1, a_2 \cdots a_n\}$ basis for $C(A)$

Linearly independent

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & 2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix}$$

$R = \text{rref}(A)$

$\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, a_5\} = C(A)$

$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} : R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + x_4 \vec{a}_4 + x_5 \vec{a}_5 = 0$

$x_3, x_5 \in R$ ~~→~~ 0

$x_1 = Ax_3 + Bx_3$, $\quad x_2 = Cx_3 + Dx_5$, $\quad x_4 = Ex_3 + Ex_5$

$-x_3 \vec{a}_3 = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_4 \vec{a}_4 + x_5 \vec{a}_5$

$x_3 = -1 \quad x_5 = 0$

Video 46

~~understanding of functions~~

Vector transformation

* vectors

$\vec{x} \in \mathbb{R}^n$

$\mathbb{R}^n = \{ n\text{-tuple } (x_1, x_2 \dots x_n) \mid x_1, x_2 \dots x_n \in R \}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $x_1, x_2 \dots x_n \in R$

subset $(\mathbb{R}^n \cdot m > 1)$ $(f: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x_1, x_2, x_3) = (x_1, 2x_2, 3x_3)$

$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_1 \\ 3x_3 \end{bmatrix}$

$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $f\left(\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

Video 47: Linear transformation:

Linear Transformation: $T: \mathbb{R}^n \to \mathbb{R}^m$

Linear independent $\to$ $\vec{a}, \vec{b} \in \mathbb{R}^n$

$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(c\vec{a}) = cT(\vec{a})$

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$

$T: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}$

$T(x_1, x_2) = (x_1 + x_2, 3x_1)$

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\vec{a} + \vec{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

$T(\vec{a} + \vec{b}) = T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$

$c\vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$

$T(a) + T(\vec{b}) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$

Video : 48

matrix vector products as linear transformation

$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$

$\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n \in \mathbb{R}^m$

$T: \mathbb{R}^n \to \mathbb{R}^m$. $T(\vec{x}) = A\vec{x}$

$B = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$ $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = B\vec{x} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$ $T(x_1, x_2) = (2x_1 - x_2, 3x_1 + 4x_2)$

$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$. $T(c\vec{a}) = cT(\vec{a})$

$Ax = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 \cdots x_n\vec{v}_n$

$\underset{m\times n}{A}(\vec{a} + \vec{b}) = A \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = (a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \cdots + (a_n + b_n)\vec{v}_n$

$A(c\vec{a}) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = ca_1\vec{v}_1 + ca_2\vec{v}_2 + ca_n\vec{v}_n$
$= c(a_1\vec{v}_1 + a_2\vec{v} + \cdots a_n\vec{v}_n$

Video → 49: linear transformation as matrix vector product:

$\begin{array}{c} m\times n \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix} \\ \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \quad \cdot \quad \vec{e}_n \end{array} \begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ \vec{x} \end{array}$ $I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

standard basis for $\mathbb{R}^n$
$(e_1\ e_2 \cdots e_n)$ standard basis

$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n e_n)$
$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots x_n T(\vec{e}_n)$

$T(\vec{x}) = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(x_1, x_2) = (x_1 + 3x_2, 5x_2 - x_1, 4x_1 + x_2)$

$\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$ $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

video → 50

* subset under a transformation:

$\vec{x}_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $\vec{x}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$L_0 = \{ x_0 + t(\vec{x}_1 - x_0) \mid 0 \leq t \leq 1 \}$

$L_1 = \{ x_1 + t(\vec{x}_2 - \vec{x}_1) \mid 0 \leq t \leq 1 \}$

$L_2 = \{ x_2 + t(\vec{x}_0 - \vec{x}_2) \mid 0 \leq t \leq 1 \}$ $S = \{ L_0, L_1, L_2 \}$.

$T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$T(L_0) = \{ T(\vec{x}_0 - t(\vec{x}_1 - \vec{x}_0) \mid 0 \leq t \leq 1 \}$

$= \{ T(\vec{x}_0) - T(t(\vec{x}_1 - \vec{x}_0)) \mid 0 \leq t \leq 1 \}$

$= \{ T(\vec{x}_0) - tT(\vec{x}_1 - \vec{x}_0) \mid 0 \leq t \leq 1 \}$

$T(L_0) = \{ T(\vec{x}_0) - t(T(\vec{x}_1) \cdot T(\vec{x}_0) \mid 0 \leq t \leq 1 \}$

Video → 51 , image of a transformation:

V subspace R^n

$\vec{a}, \vec{b} \in V \rightarrow \vec{a} + \vec{b} \in V$ / $c\bar{a} \in V$

$\vec{a}, \vec{b} \in V \rightarrow$ $T: R^n \rightarrow R^m$. $T(V)$: imag of V

$\vec{0} \in V$

$T(a), T(b) \in T(V)$.

$T(\vec{a}) + T(\vec{b}) = T(\vec{a} + \vec{b}) \in T(V)$.

$cT(\vec{a}) = T(c\vec{a}) \in T(V)$

$T(R^n)$: image of R^n . $T = \{ T(\vec{x}) \mid \vec{x} \in R^n \}$.

$T: R^n \rightarrow R^m$ / $T(\vec{x})$ $\underset{m \times n}{A} \vec{x}$

Video → 52

Preimage of a set

$T: X \to Y$

$A \subseteq T \to$ subspace

$T(A)$: image of a mom $T = \{T(\vec{x}) \in Y \mid \vec{x} \in A\}$

$T(T^{-1}(S)) \subseteq S$

Video 53: Sums and scalar multiplication

$S: R^n \to R^m \cdot T: R^n \to R^m$

Def: $(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x})$.

$(S+T): R^n \to R^m$

Def: $(cS)(\vec{x}) = c(S(\vec{x}))$

$cS: R^n \to R^m$

$S(\vec{x}) = \underset{m\times n}{A}\vec{x}$ $\quad T(\vec{x}) = \underset{m\times n}{B}\vec{x}$

$A = [\vec{a}_1 \ \vec{a}_2 - \vec{a}_n]$ $\quad B = [b_1 \ b_2 -- b_n]$

$(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x})$

$= A\vec{x} + B\vec{x}$

$= x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n$

$= x_1\vec{b}_1 + x_2\vec{b}_2 \cdots x_n\vec{b}_n$

$= x_1(\vec{a}_1 + \vec{b}_1) + x_2(\vec{a}_2 + \vec{b}_2) + \cdots + x_n(\vec{a}_n + \vec{b}_n)$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$\begin{bmatrix} \vec{a}_1 + \vec{b}_1 & \vec{a}_2 + \vec{b}_2 & \cdots & \vec{a}_n + \vec{b}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$(S+T) = S(\vec{x}) + T(\vec{x})$

$= A\vec{x} + B\vec{x}$

$= (A+B)\vec{x}$

Def $= A+B = [\vec{a}_1 + \vec{b}_1, \cdots a_n + b_n]$

$(cS)(\vec{x}) = c(\underset{Ax}{S(\vec{x})}) = cAx$

$= c(x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n)$

$= x_1 c\vec{a}_1 + x_2 c\vec{a}_2 + \cdots x_n c\vec{a}_n$

$= [c\vec{a}_1 \cdot c\vec{a}_2 \cdots c\vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

Def: $cA = [c\vec{a}_1 \cdot c\vec{a}_2 -- c\vec{a}_n]$

video - 55

matrix addition and scalar multiplication

$S: R^n \to R^m \qquad T: R^n \to R^m$

$(S+T)(\vec{x}) = S(x) + T(x) \cdot \; S+T: R^n \to R^m$

$S(\vec{x}) = \underset{m\times n}{A}\vec{x} \cdot \quad T(\vec{x}) = \underset{m\times n}{B}x$

$A+B = [\vec{a}_1+\vec{b}_1, \; a_2+b_2, \; \cdots a_n+b_n] = \begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix} + \begin{bmatrix}2 & 7\\ -3 & -1\end{bmatrix} = \begin{bmatrix}1+2 & 3+7\\ -2-3 & 4-1\end{bmatrix}$

$= \begin{bmatrix}3 & 10\\ -5 & 3\end{bmatrix}$

$A = \begin{bmatrix}a_{11}, & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{m1} & \vdots & & a_{mn}\end{bmatrix} \quad B = \begin{bmatrix}b_{11}, & b_{12} & \cdots & b_{1n}\\ b_{21} & \ddots & & \\ b_{m1} & b_{m2} & \ddots & b_{mn}\end{bmatrix}$

$(cT)(\vec{x}) = c(T(x)) = c(B\vec{x}) = (cB)\vec{x}$

$cA = [c\vec{a}_1, c\vec{a}_2 \cdots c a_n]$

video 56 : scaling and reflection:

$T: R^n \to R^m \cdot \quad T(\vec{x}) = \underset{m\times n}{A}\vec{x} \qquad I_n \begin{bmatrix}1 & 0 & 0\\ 0 & \ddots & \\ 0 & & 1\end{bmatrix} \qquad A = [T(e_1)\, T(e_2) \cdots T(e_n)]$

$R^2 \begin{bmatrix}2\\3\end{bmatrix}; \begin{bmatrix}-3\\2\end{bmatrix}, \begin{bmatrix}-3\\-2\end{bmatrix}$

$-1 \times \cdot x_1$

$2x \; y$

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$I_2 \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

$\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-3\\4\end{bmatrix}$

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video → 60

Unit vectors

vector length:

$u \in R^n \begin{bmatrix} u_1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\|\vec{u}\| = \sqrt{u_n^2 + u_2^2 - - + u_n^2}$

unit vector $= \|\vec{u}\| = 1$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \to \vec{u}$.

$\|\vec{u}\| = 1$

$\|\vec{v}\| \cdot \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} \cdot \|u\| = \|\frac{1}{\|\vec{v}\|} \cdot \vec{v}\| - \frac{1}{\|\vec{v}\|} \|\vec{v}\|$

$\|c\vec{v}\| = c\|\vec{v}\|$ $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\|v\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

Video 61 ; Expressing a projection;

$\vec{x} \cdot \vec{x} = \|x\|^2$

$L = \{c\vec{v} \mid c \in R\}$

$Proj_L : R^n \to R^n$. $Proj_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v}$

$= \frac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v}$

$L = \{c\vec{u} \mid c \in R\}$

$Proj_L(\vec{x}) = (\vec{x} \cdot \vec{u}) \cdot \vec{u}$

$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\|\vec{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\vec{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Linear transformation; $Proj_L(\vec{a} + \vec{b}) = ((\vec{a} + \vec{b}) \cdot \vec{u})\vec{u}$

$= (\vec{a} \cdot \vec{u} + \vec{b} \cdot \vec{u})\vec{u}$

$Proj_L(\vec{x}) = (\vec{x} \cdot \vec{u}) \cdot \vec{u} = A\vec{x}$

$Proj(\vec{a}) + Proj_L(\vec{b})$

$Proj_L : R^2 \to R^2$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$Proj_L(c\vec{a}) = (c\vec{a} \cdot \vec{u})\vec{u}$

$= c(\vec{a} \cdot \vec{u}) \cdot \vec{u}$

$= c\, Proj_L\, \vec{a}$

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$S: X \to Y$

Video → 63
Composition of linear transformation

$X \subseteq R^n \quad Y \subseteq R^n \qquad S(\vec{x}) = \underset{m\times n}{A}\vec{x}$

$T \circ S : X \to Z$ The composition of T with S

Def: $T \circ S : T(S(\vec{x}))$ Linear transformation ?

$T \circ S(\vec{x}+\vec{y}) = T(S(\vec{x}+\vec{y})) = T(S(\vec{x}) + S(\vec{y})) = T(S(\vec{x})) + T(S(\vec{y}))$
$= T\circ S(\vec{x}) + T\circ S(\vec{y})$

$T \circ S(c\vec{x}) = T(S(c\vec{x})) = T(c S(\vec{x}))$

$= cT(S(\vec{x})) = c(T\circ S)(\vec{x})$

Video - 64: $T(\vec{x}) = \underset{l\times m}{B}\vec{x}$ $\qquad S(\vec{x}) = \underset{m\times n}{A}\vec{x}$ $\qquad T \circ S(\vec{x}) = T(S(\vec{x})) = T(A\vec{x})$

$T\circ S(\vec{x}) = B(A\vec{x}) = C\vec{x}$

$I_n \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & , & 1 \\ 0 & \vdots & & 0 \\ & 0 & & 1 \end{bmatrix}$ $A = [\vec{a}_1 \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n$

$C = \left[B\left(A\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}\right) \; B\left(A\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right) \cdots B\left(A\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\right) \right]$

$S: X \to Y \subseteq R^m$ $\quad (\subseteq R^n)$

$C = [B(\vec{a}_1) \; B(\vec{a}_2) \cdots B(\vec{a}_n)]$ $\qquad S(\vec{x}) = \underset{m\times n}{A}(\vec{x})$

$T\circ S(\vec{x}) = B(A\vec{x}) = C\vec{x}$
$= [B\vec{a}_1 \; B\vec{a}_2 \cdots B\vec{a}_n]\vec{x}$

$T: Y \to Z \subseteq R^l$
$T(x) = \underset{l\times m}{B}\vec{x}$

Video 6 III

matrix product examples.

$A_{m\times n}$ $B_{n\times k} = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_k]$

$AB = [A\vec{b}_1\ A\vec{b}_2\ A\vec{b}_3 \cdots A\vec{b}_k]$

$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ $\cdot$ $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$

2×3 $\quad$ 3×4

$$AB = \left[A\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot A\begin{bmatrix}0\\0\\1\end{bmatrix}\ A\begin{bmatrix}1\\1\\0\end{bmatrix}\ A\begin{bmatrix}1\\-1\\2\end{bmatrix}\right]$$

$$AB = \begin{bmatrix} \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} & \begin{bmatrix}1\\-1\\2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\-1\\2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} & \begin{bmatrix}1\\-1\\2\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix} \\ \begin{bmatrix}0\\-2\\1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} & \begin{bmatrix}0\\-2\\1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\-2\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} & \begin{bmatrix}0\\-2\\1\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1+-2+6 & 0+0+2 & 1-1+0 & 1+1+4 \\ 0-4+3 & 0+0+1 & 0-2+0 & 0+2+2 \end{bmatrix}$$

$$\underset{2\times 4}{AB} \begin{bmatrix} 5 & 2 & 0 & 6 \\ -1 & 1 & -2 & 4 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

$$BA = \left[B\begin{bmatrix}1\\0\end{bmatrix} \cdot B\begin{bmatrix}-1\\-2\end{bmatrix}\ B\begin{bmatrix}2\\1\end{bmatrix}\right]$$

$S: R^3 \to R^2 \cdot S(\vec{x}) = A\vec{x}$

$T: R^4 \to R^3 \quad T(\vec{x}) = B(\vec{x})$

$S\circ T(\vec{x}) = S(T(\vec{x})) = A(B\vec{x})$

$S\circ T: R^4 \to R^2 \cdot S\circ T(\vec{x}) = AB\vec{x}$

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Video 66

Distribute property of matrix products

$B = [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n]$ $C = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]$

$A(B+C) = A[\vec{b}_1+\vec{c}_1 \cdot \vec{b}_2+\vec{c}_2 \cdots \vec{b}_n+\vec{c}_n] = [A(\vec{b}_1+\vec{c}_1)\ A(\vec{b}_2+\vec{c}_2) \cdots A(\vec{b}_n+\vec{c}_n)]$

$= [A\vec{b}_1 + A\vec{c}_1\ \ A\vec{b}_2+A\vec{c}_2 \cdots A\vec{b}_n + A\vec{c}_n]$

$= [A\vec{b}_1\ A\vec{b}_2 \cdots A\vec{b}_n] + [A\vec{c}_1\ A\vec{c}_2 \cdots A\vec{c}_n]$ $(B+C)A = ?$

$(B+C)[a_1, a_2 \cdots a_m] = [(B+C)\vec{a}_1\ (B+C)\vec{a}_2 \cdots (B+C)\vec{a}_n]$

$= [B\vec{a}_1\ B\vec{a}_2 \cdots B\vec{a}_n] + [C\vec{a}_1\ C a_2 \cdots C\vec{a}_n]$

$= (B+C)A = BA + CA$

Video → 68 : introduction to the inverse of a function.

$f: x \to y$

$a \in X$ $b \in Y$

$f(a) = b$

f. is invertible.

exists a function $f^{-1}: Y \to X$

Such that

$f^{-1} \circ f = I_x$

$x \to x$ f^{-1}

f $\to x$

$x \to y$

Identity function

$I_x : X \to X$

$a \in X$

$I_x(a) = a$

$b \in Y$

$I_y(b) = b$

and $f \circ f^{-1} = I_y$

$(f^{-1} \circ f)(a) = I_x(a) = a$

$f^{-1}(f(a)) = a$

$f(f^{-1}(y))$

$y \in Y$

video 69

surgective (onto) and ingective (one to one)

$f: x \to Y$

surgective "onto"

any $y \in Y$ $\exists$ at least one $x \in Y$ such that $f(x) = y$

f surgective (or only)

int(f) = Y

range (f) = Y

ingective function one to one

for $y \in Y \Rightarrow$ at most x such that $f(x) = y$

Video → 70 : Relating inverotibility to being onto and one to one.

invertible if for any $y \in Y$

there exists unique $x \in X$ such that $f(x) = y$

f is surgective (onto)

f is ingective (one to one)

$f: x \to y$ is invertible if and only

f is both surgective and ingective

(onto) → (one-to-one)

Page → 36:

Video → 71

$T: R^n \to R^m$. $T(x) = \underset{m\times n}{A}\vec{x}$

$A\vec{x} = \vec{b}$ $[\vec{a}_1 \vec{a}_2 \dots \vec{a}_n] = A$

when, $\vec{x} \in R^n$

$\underset{\vec{x}}{A}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n$

for T to be onto → span $(a_1\, a_2 \dots \vec{a}_n) = R^m$

$C(A) = R^m$

$A\vec{x} = \vec{b}$. $[A|\vec{b}]$

Reff (A) $[R|\vec{c}]$?

basis for $C(A)$.

$A \xrightarrow{\text{reff}(A)} R$

$[\vec{a}_1\, \vec{a}_1 \dots \vec{a}_n]$

$\downarrow$

$\begin{bmatrix} 1 & 2 & 5 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$

$\text{Rank}(A) = \dim(C(A))$
= # of basis vector for $C(A)$

if Rank (A) = m

$S: R^2 \to R^3$. $S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\vec{x}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\text{Rank}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right) = 2$

S is not onto

S is not invertible.

Video → 73

Exploring the soluation set of $Ax = b$

Ex: $T: R^2 \to R^2$

$T(x) = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \vec{x}$ or, $\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

assume → $b_1 + b_2 = 0$

$x_1 - 3x_2 = b_1$

$x_1 = b_1 + 3x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

for a partical $\vec{b}$ that has a soluation $A\vec{x} = \vec{b}$

Soluation set $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

assume $A\vec{x} = \vec{b}$ has a soluation.

the soluation set $= \{\vec{x}_p\} \cup N(A)$

Video → 74: Simpliying conditions for invertibility.

$\text{Rank}(A) = m = n$

$A_{m \times n} = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$

$\text{ref}(A) = m \times n$ matrix when every column L.I. column

$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

Video → 75

Showing that inverses are linear

Ex — $T(x) = \underset{n \times n}{A}\vec{x}$. $T : R^n \to R^n$

$Rref(A) = I_n$. → T is invertible.

$T^{-1} \circ T = I_{R^n}$. $T \circ T^{-1} = I_{R^n}$

$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$.

$T(c\vec{x}) = cT(\vec{x})$

is T^{-1} a linear transformation

$(T \circ T^{-1})(\vec{a} + \vec{b}) = \vec{a} + \vec{b}$

$= (T \circ T^{-1})(\vec{a}) + (T \circ T^{-1})(\vec{b})$

$= (T \circ T^{-1})(\vec{a}) + T(T^{-1}(\vec{b}))$

$= T(T^{-1}(\vec{a} + \vec{b})) = T(T^{-1}(\vec{a})) + T(T^{-1}(\vec{b}))$

$= T(T^{-1}(\vec{a}) + T^{-1}(\vec{b}))$

$= T^{-1}(T(T^{-1}(\vec{a} + \vec{b})) = T^{-1}(T(T^{-1}(a) + T^{-1}(\vec{b}))$

$= (T^{-1} \circ T)(T^{-1}(\vec{a} + \vec{b})) = (T^{-1} \circ T)(T^{-1}(\vec{a}) + T^{-1}(\vec{b}))$

$= T^{-1}(\vec{a} + \vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$

Video 76 : Deriving a method for determining inverses :

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$T_2(x) = S_2\vec{x}$ so, $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_3 - 2x \end{bmatrix}$

$T_3\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 - 2x_2 \\ x_3 \end{bmatrix}$

so, $T_0^{-1}(\vec{x}) = A^{-1}\vec{x}$ $\boxed{A^{-1}A = I}$

$[A \mid I] - - - [I \mid A^{-1}]$

Video 77

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & -1 \\ 0 & 1 & 0 & 7 & 5 & -2 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right]$$

$rref(A)$. A^{-1}

Video → 78: formal for 2x2 inverse.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \text{or}; \; T_1\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ ac_1 - cc_1 \end{bmatrix} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$T_2\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} (ad-bc)c_1 - b(c_2) \\ c_2 \end{bmatrix} \quad \begin{bmatrix} (ad-bc)a & (ad-bc)b - b(a) - bc & ad-bc+bc \\ 0 & ad-bc & \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} (ad-bc)a & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$T_3\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{(ad-bc)a} c_1 \\ \frac{1}{(ad-bc)} c_2 \end{bmatrix} \left[\begin{array}{cc|cc} 1 & 0 & \frac{a d}{(ad-bc)\cancel{a}} & \frac{-\cancel{a}b}{ad-bc\cancel{a}} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ not allowed if $ad-bc=0$

$ad-bc \neq 0$ A is invertible

Video → 79 : 3x3 determinant.

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det(B) = |B| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [ad-bc]$

$$B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \operatorname{Det}(A) = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix} \quad \det(c) = 1\begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$$

$$= 1(-1\cdot 1 - 0\cdot 3) - 2(2\cdot 1 - 4\cdot 3) + 4(2\cdot 0 - (-4))$$

$$= -1 + 20 + 16$$

$$= 35$$

Video → 81

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Determinants along other rows/cols

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix} = 7 = 1\begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

sign(i, j) $-1^{(i+j)}$

$$-2\left(2\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}\right) + 3\left(-2\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\right)$$

Video 82: Rule of sarrus of determinants:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\begin{bmatrix} a & f \\ g & i \end{bmatrix} + c\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$= a(ei - fh) - b(di - fg) + c(dh - eg)$

$= aei - afh - bdg + bfg + cdh - ceg$

$= aei + bfg + cdh - afh - bdi - ceg$

Video 83: determinant when row multiplied by scalar:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \Rightarrow |kA| = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k^2ad - k^2bc$$

or, $k^2(ad - bc)$

or, $k^2|A|$

$$A\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad |A| = -d\begin{bmatrix} b & c \\ h & i \end{bmatrix} + e\begin{bmatrix} a & c \\ g & i \end{bmatrix} - f\begin{bmatrix} a & b \\ g & h \end{bmatrix}$$

Page 42: Video = 83

$$A' = \begin{bmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{bmatrix} \quad |A'| = -kd\begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke\begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= k|A|$$

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{nn} & \cdots & a_{nn} \end{bmatrix} \quad \det(A) = (-1)^{i+1} a_{i1} A_{i1} + (-1)^{i+2} \cdot a_{i2} A_{i2} + (-1)^{i+n} a_{in} \cdot A_{in}$$

$$\det(A) = \boxed{\sum_{j=1}^{j=1} (-1)^{i+j} a_{ij} A_{ij}}$$

Video → 84: (correction) scalar multiplication of row:

$$A_{n\times n} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \quad \det(A) = (-1)^{i+1} a_{i1} A_{ii} \times + a_{i2} A_{i2} + \cdots a_{in} A_{in}$$

$$\det(A) = (-1)^{i+1} \cdot a_{i1} \det(A_{i1}) + a_{i2} \det(A_{i2}) + \cdots a_{in} \det(A_{in})$$

Video → 85 | Determinant when row is added.

$$X = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix} \cdot Y = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix} \cdot Z = \begin{bmatrix} a & b \\ x_1+y_1 & x_2+y_2 \end{bmatrix}$$

$$|X| = ax_2 - bx_1 \cdot |Y| = ay_2 - by_1 \cdot |Z| = a(x_2+y_2) - b(x_1+y_1)$$

$$= ax_2 + ay_2 - bx_1 - by_1$$

$$= \underbrace{ax_2 - bx_1}_{|X|} + \underbrace{ay_2 - by_1}_{|Y|}$$

$$X = \begin{bmatrix} a & b & c \\ x_1 & x_2 & x_3 \\ d & e & f \end{bmatrix} \quad \det(X) = -x_1\begin{bmatrix} b & c \\ d & f \end{bmatrix} + x_2\begin{bmatrix} a & c \\ d & f \end{bmatrix} - x_3\begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$Y = \begin{bmatrix} a & b & c \\ y_1 & y_2 & y_3 \\ d & e & f \end{bmatrix} \quad \det(Y) = -y_1\begin{bmatrix} b & c \\ d & f \end{bmatrix} + y_2\begin{bmatrix} a & c \\ d & f \end{bmatrix} + y_3\begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

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$$Z = \begin{bmatrix} a & b & c \\ x_1+y_1 & x_2+y_2 & x_n+y_n \\ d & e & f \end{bmatrix}$$

$\det(Z) = -(x_1+y_1)\begin{vmatrix} b & c \\ e & f \end{vmatrix} + (x_2+y_2)\begin{vmatrix} a & c \\ d & f \end{vmatrix} - (x_n+y_n)\begin{vmatrix} a & c \\ d & e \end{vmatrix}$

$$\det(x) = \sum_{j=1}^{n} (-1)^{i+j} x_j |A_{ij}|$$

$$\det(y) \sum_{j=1}^{n} (-1)^{i+j} y_i |A_{ij}|$$

$$\det(z) = \sum_{j=1}^{n} (-1)^{i+j} (x_j + y_j) |A_{ij}|$$

Video 86 : Duplicate row determinant !

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ a_{i1} & \cdots & \cdots & a_{in} \\ a_{j1} & \cdots & \cdots & a_{jn} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad r_i = [a_{i1} \; a_{i2} \cdots a_{in}]$$

$$A_{n\times n} \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_n \\ \vec{r}_n \end{bmatrix} \qquad S_{ij} = \begin{bmatrix} \vec{r}_n \\ \vec{r}_n \\ \vec{r}_j \\ \vec{r}_n \end{bmatrix}$$

$\det(S_{ij}) = -\det(A)$

if row i = row j $\det(S_{ij}) = \det(A) = -\det(A)$

$S_{ij} = A \Rightarrow \det(S_{ij})$

$x = -x$

$x = 0$

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Video → 87

$$\underset{n\times n}{A} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_i \\ \vec{r}_j \\ \vec{r}_n \end{bmatrix} \qquad \vec{r}_K = [a_{K1} \; a_{K2} \cdots a_{Kn}]$$

$$\underset{n\times n}{B} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_i \\ \vec{r}_j - c\vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix} \quad \det(B) = a_{j1} - ca_{i1} \cdot a_{j2} - ca_2 \cdots a_{jn} - ca_{in}$$

Vide 88

$$A = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \quad \det(A) = ad$$

$$B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \quad \det(B) = a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - 0\begin{vmatrix} b & e \\ 0 & f \end{vmatrix} + 0\begin{vmatrix} b & c \\ d & e \end{vmatrix}$$

$$= adf$$

$$\underset{n\times n}{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & & a_{nn} \end{vmatrix}$$

$$\det(A) = a_{11}\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & a_{23} & \\ \vdots & & \\ 0 & \cdots 0 & a_{nn} \end{bmatrix}$$

$$a_{11}a_{22}\begin{bmatrix} a_{33} & \cdots & a_{n4} \\ 0 & & \vdots \\ 0 & \cdots & 0 \; a_n \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

$$\begin{bmatrix} 7 & 3 & 4 & 2 \\ 0 & -2 & 3 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 7 \times -2 \times 1 \times 3 = -42$$

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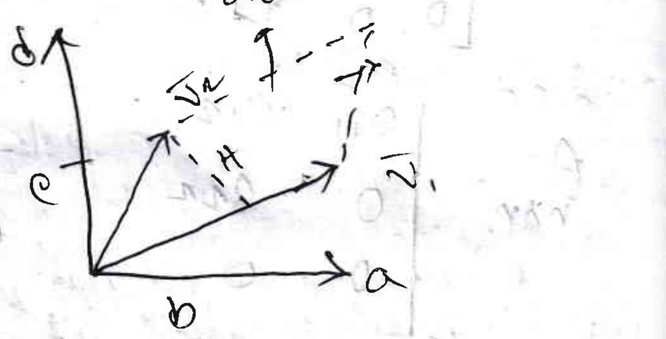
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{bmatrix} \quad \det(A)$$

Replace row j with row j − c × row 1 → not change det(A)

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 7 & 5 & 2 \\ -1 & 4 & -6 & 3 \end{vmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 6 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 6 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -6 & 4 \end{bmatrix}$$

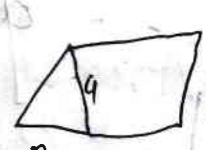
$$= \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} = -(1 \times 3 \times 2 \times 7)$$
$$= 42$$

Video → 91: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



$\vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$

$A = BH$
$= \|\vec{v}_1\|$



$A = B \cdot H$

$H^2 + \|Proj_L \vec{v}_2\|^2 = \|\vec{v}_2\|^2$ L = is a line spanned by $\vec{v}_1$

→ pythagolean Theory

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$$\operatorname{Proj}_{\vec{v}_1}\vec{v}_2 = \left(\frac{\vec{v}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)$$

$$H^2 = \|\vec{v}_2\|^2 - \|\operatorname{Proj}_{\vec{v}_1}\vec{v}_2\|^2$$

$$= \vec{v}_2\cdot\vec{v}_2 - \left\|\frac{\vec{v}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\cdot\vec{v}_1\right\|^2$$

$$H^2 = \vec{v}_2\cdot\vec{v}_2 - \left(\frac{\vec{v}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\cdot\vec{v}_1\cdot\frac{\vec{v}_2\cdot\vec{v}}{\vec{v}_1\cdot\vec{v}_1}\cdot\vec{v}_1\right)$$

$$\left(\frac{(\vec{v}_2\cdot\vec{v}_1)(\vec{v}_2\cdot\vec{v}_1)}{(\vec{v}_1\cdot\vec{v})\cancel{(\vec{v}_1\cdot\vec{v}_1)}}\cdot\cancel{\vec{v}_1\cdot\vec{v}}\right)$$

$$H^2 = \vec{v}_2\cdot\vec{v}_2 - \frac{(\vec{v}_2\cdot\vec{v}_1)^2}{\vec{v}_1\cdot\vec{v}_1}$$

$$A = BH \rightarrow A^2 = B^2H^2$$

$$B^2 = \|\vec{v}_1\|^2 = \vec{v}_1\cdot\vec{v}_1$$

$$A^2 = \vec{v}_1\cdot\vec{v}_1\left(\vec{v}_2\cdot\vec{v}_2 - \frac{(\vec{v}_2\cdot\vec{v}_1)^2}{\vec{v}_1\cdot\vec{v}_1}\right) = (\vec{v}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\vec{v}_2) - (\vec{v}_2\cdot\vec{v}_1)^2$$

$$= (a^2+c^2)(b^2+d^2) - (ab+cd)^2$$

$$A^2 = a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - (a^2b^2 + 2abcd + c^2d^2)$$

$$A^2 = \cancel{a^2b^2} + a^2d^2 + c^2b^2 + \cancel{c^2d^2} - \cancel{a^2b^2} - 2abcd - \cancel{c^2d^2}$$

$$A^2 = a^2d^2 - 2abcd + c^2b^2$$

$$= x^2 - 2xy + y^2 = (x-y)^2$$

$$A^2 = (ad - bc)^2$$

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Video 92

Transpose of a matrix

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transpose A^T

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\underset{n\times m}{A^T} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{mn} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\underset{4\times 3}{C} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 7 & -5 \\ 4 & -3 & 2 \\ -1 & 3 & 0 \end{bmatrix}$$

$\to$ 2nd row 3rd column

3rd row 2nd column

$$\underset{3\times 4}{C^T} = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 7 & -3 & 3 \\ -1 & -5 & 2 & 0 \end{bmatrix}$$

$\to$ 3rd column 2nd row

$\to$ 2nd column 3rd row

Video: 93: Determinant of transpose:

2×2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

2×2: $Det(A) = Det(A^T)$

$$\underset{n+1\times n+1}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\underset{(n+1)\times(n+1)}{A^T} \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$\det(A) = a_{11}\det(A_1) - a_{12}\cdot\det(A_{12}) + (-1)^{1+m}\det(A_{1m})$$

$$\det(A^T) = a_{11}\det((A_{11})^T) - a_{12}\det((A_{12})^T) + (-1)^{1+m}\cdot\det((A_{1m})^T)$$

Video 94

Transpose of matrix product

$$\underset{m\times n}{A}=\begin{bmatrix} a_{11} & a_{12} & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{2j} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & a_{ij} & \cdots & a_{in} \\ a_{m1} & a & \cdots & a_{mj} & a_{mn} \end{bmatrix} \quad \underset{m\times n}{B}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2m} \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{im} \\ b_{n1} & \cdots & & b_{ij} & \cdots & b_{nm} \end{bmatrix} \quad \underset{m\times m}{C=AB}$$

$$\underset{m\times m}{B^T}=\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{j1} & \cdots & b_{m1} \\ b_{12} & b_{22} & & b_{ij} & \cdots & b_{m2} \\ b_{ij} & b_{22} & & b_{ij} & & b_{m2} \\ b_{im} & b_{2m} & & b_{im} & & b_{nm} \end{bmatrix} \quad \underset{n\times m}{A^T}=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{i1} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{i2} & \cdots & \cancel{a_{m2}} \\ a_{ij} & a_{2j} & \cdots & a_{ij} & & \\ a_{in} & \cdots & a_{2n} & \cdots & a_{in} & \cdots a_{mn} \end{bmatrix} \quad \underset{m\times m}{D=B^TA^T}$$

$$d_{ji} = a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$D=\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ & \ddots & & \\ & d_{i2} & \cdots & d_{mn} \end{bmatrix}$$

$$C\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ & c_{22} & & \\ & & c_{ij} & \\ & & & c_{mn} \end{bmatrix} \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$c_{ij} + d_{ij}$$

$$(AB)^T = B^TA^T$$

$$C = AB. \quad D = B^TA^T$$

$$C^T = (AB)^T = B^TA^T$$

$$(XYZ)^T - Z^TY^TX^T$$

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Transposes of sums and inverses:

$C = A+B$

$c_{ij} = a_{ij} + b_{ij}$. $A \begin{vmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{vmatrix}$

$c'_{ij} = c_{ji} = a_{ij} + b_{ij} + a_{ij} + b_{ij}$

$C^T = (A+B)^T$.
$= A^T + B^T$

A^{-1} is inverses of $A \Rightarrow AA^{-1} = I_n$ and $A^{-1}A = I_n$

$(AA^{-1})^T = I_n^T = I_n$. $(A^{-1}A)^T = I_n^T = I_n$

$(A^{-1})^T A^T = I_n$

Video 96: Transpose of a vector:

$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{V}^T = [v_1\ v_2 \cdots v_n]$

$n \times 1$ $1 \times n$

$A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$

$\vec{W} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ $\vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + \cdots v_n w_n$

$= [v_1\ v_2 \cdots v_n]^T [w_1\ w_2 \cdots w_n] = [w_1 v_1 + w_2 v_2 + \cdots v_n w_n]$
1×1

$A\vec{x} = z$
$(m \times n)(n \times 1)$

$(A\vec{x}) \in R^n$ $Y \in R^m$

$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ $(A\vec{x}) \cdot Y = (A\vec{x})^T Y = (\vec{x}^T A^T) Y \in R$
$1 \times m$

$= \vec{x}^T (A^T Y)$

Video 98

* Visualization of left ~~null~~ nullspace and rowspace.

$$A_{2\times 3} = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}$$

$N(A)\ \{\vec{x} \in R^3 \,/\, A\vec{x} = \vec{0}\}$

$C(A) = span\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix}\right) = span\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right)$

$Rank(A) = 1$

$$\begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -4 & 2 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & \frac{1}{2} & \frac{3}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3 = 0$$

$$x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \quad N(A) = span\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right)$$

$$A^T_{3\times 2} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ -3 & 6 \end{bmatrix} \quad N(A^T) = \{\vec{x} \in R^2 \,/\, A^T\vec{x} = \vec{0}\}$$

$C(A) = span\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right)$ $\quad N(A^T) = span\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$N(A) = span\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right)$

Orthogonal complements

V some subspace of R^n orthogonal complement of V

$V^\perp = \{\vec{x} \in R^n \mid \vec{x} \cdot \vec{v} = 0$ for every $\vec{v} \in V\}$

$V^\perp$ subspace?

$\vec{a}, \vec{b} \in V^\perp$, $\vec{a}+\vec{b} \in V^\perp$?, $c\vec{a} \in V^\perp$?

$\vec{a} \cdot \vec{v} = 0$ for any $\vec{v} \in V$

$\vec{b} \cdot \vec{v} = 0$ for any $\vec{v} \in V$

$(\vec{a}+\vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0+0 = 0$

$c\vec{a} \cdot \vec{v} = c(\vec{a} \cdot \vec{v}) = c(0) = 0$

$A_{m \times n}$. $N(A)$ is the orthogonal complement of the rowspace of (A)

$C(A^T)$.

$N(A) = (C(A^T))^\perp$

$$A = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{matrix} \vec{r}_1^T \cdot \vec{x} \\ \vec{r}_2^T \cdot \vec{x} \end{matrix}$$

$\vec{w} = c_1 \vec{r}_1 + \vec{c}_2 \vec{r}$

$N(A) = \{\vec{x} \in R^n \mid A\vec{x} = 0\}$

$\vec{r} \cdot \vec{w} = c_1(\vec{v} \cdot \vec{r}_1) + c(v \cdot \vec{r}_2) + c_n(\vec{v}_n \cdot \vec{r}_m)$

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video → 100

Rank(a) = rank (transpose of a)

Rank (A) = Rank(A^T).

Rowspace of A

Rank (A^T) = dim(C(A^T)) → of basis vectors for $\boxed{C(A^T)}$

$$\underset{m\times n}{A} = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vec{r}_m^{\,T} \end{bmatrix} \qquad \underset{n\times m}{A^T} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \cdots & \vec{r}_m \end{bmatrix}$$

Rank (A) = of pivot entries in ref(A)

Rank(A) = dim(C(A)) = of vectors in the basis for C(A)

video – 101: dim(v) + dim(orthogonal complement of v)

V subspace of R^n

$\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$ basis for V

dim(V) = k

$N(A^T) = C(A)^\perp = V^\perp$

$V = span(\vec{v}_1 \vec{v}_2 \cdots \vec{v}_k) = C(A)$

$$\underset{n\times k}{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}$$

$\dim(V^\perp) = \dim(N(A^T))$

$$\underset{k\times n}{A^T} \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vdots \\ \vec{v}_k^{\,T} \end{bmatrix}$$

Rank (A^T) + Nullity(A^T) = n

Rank (A) + Nullity (A^T) = n

dim (C(A)) + dim (N(A^T)) = n

dim (V) + dim ($V^\perp$) = n

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Orthogonal complement of the orthogonal complement

$V^{\perp} = \{ x \in \mathbb{R}^n \mid x \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V \}$

$(V^{\perp})^{\perp} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{w} = 0 \text{ for every } \vec{w} \in V^{\perp} \}$

$\vec{x} \in (V^{\perp})^{\perp}$

$\vec{x} = (\vec{v} + \vec{w})$ where $\vec{v} \in V$ and $\vec{w} \in V^{\perp}$

$\vec{x} \cdot \vec{w} = 0 = (\vec{v} + \vec{w}) \cdot \vec{w} = \vec{v} \cdot \vec{w}^{\,0} + \vec{w} \cdot \vec{w} = \|\vec{w}\|^2$

$\|\vec{w}\|^2 = 0 \Rightarrow \vec{w} = \vec{0} \Rightarrow \vec{x} = \vec{v} \Rightarrow \vec{x} \in V$

Video → 104 | Orthogonal complement of the nullspace

$C(A^T)^{\perp} = N(A)$; $C(A)^{\perp} = N(A^T) \rightarrow$ left null space

$N(A)^{\perp} = (C(A^T)^{\perp})^{\perp} = C(A^T)$

$(V^{\perp})^{\perp} = V$; $N(A^T)^{\perp} = (C(A)^{\perp})^{\perp} = C(A)$

Video 105 | Unique rowspace solution to $Ax = b$

$\underset{m \times n}{A} = [a_1, a_2 \cdots a_n]$ $\quad b \in C(A)$

$\Rightarrow b = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$

$\Rightarrow [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = b$

$\vec{r} = \vec{x} - \vec{n}_0$ $\quad A\vec{r}_0 = (A(\vec{x} - \vec{n}_0)) = Ax - A\vec{n}_0 = \vec{b}$

$r_1 \in C(A^T) \cdot \exists$ a solution $Ax = \vec{b}$

$(\vec{r}_1 - \vec{r}_0) \in C(A^T) \Rightarrow A(\vec{r}_1 - r_0) = A\vec{r}_1 - A\vec{r}_0 = \vec{b} - \vec{b} = \vec{0})$ $\vec{x} = \vec{r}_0 + \vec{n}_0$

Any solution $\vec{x}$ to $Ax = \vec{r}$ can be write as $\vec{x} = \vec{r}_0 + \vec{n}_0$

$\|x\|^2 = x \cdot x = (\vec{r}_0 + \vec{n}_0) \cdot (\vec{r}_0 + \vec{n}_0) = \vec{r}_0 \cdot \vec{r}_0 + \cancel{\vec{n}_0 \cdot \vec{r}_0} + \cancel{\vec{n}_0 \cdot \vec{r}_0} + \vec{n}_0 \cdot \vec{n}_0$

$\|x\|^2 = \|\vec{r}_0\| + \|\vec{n}_0\| \geq \|\vec{r}_0\|^2$

Video 107

Rowspace solution to $Ax = b$ example

$$A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 18 \end{bmatrix}$$

$N(A) = N(\text{rref}(A))$

$$\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad N(A) \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

$t = 3c \quad = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

$A\vec{x} = \vec{b}$

$$\left[\begin{array}{cc|c} 3 & -2 & 9 \\ 6 & -4 & 18 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & -2 & 9 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 3 \\ 0 & 0 & 0 \end{array}\right]$$

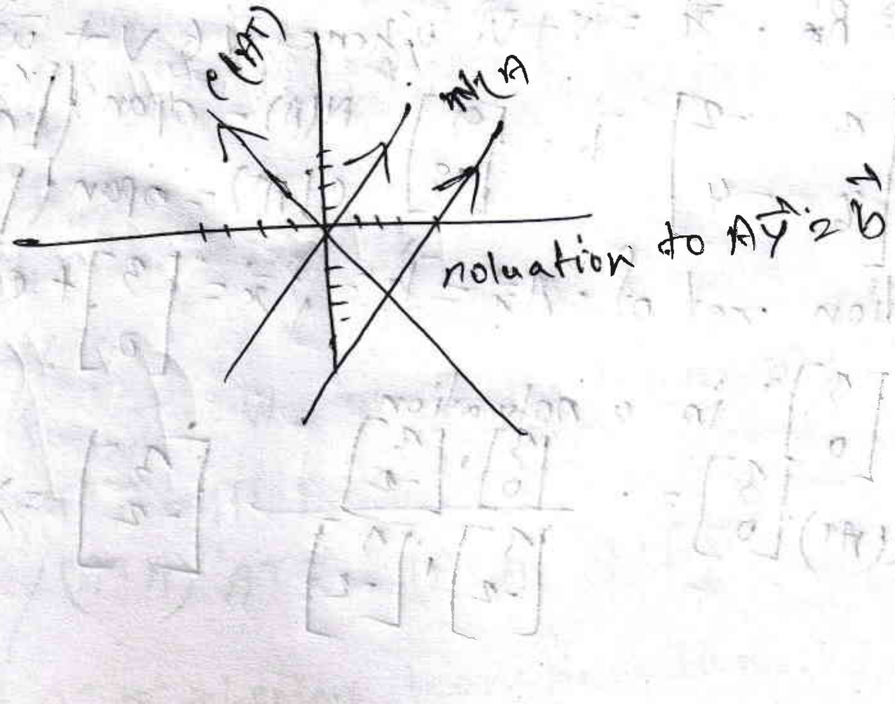
$$\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad x_1 - \frac{2}{3}x_2 = 3 \Rightarrow x_1 = 3 + \frac{2}{3}x_2$$

$x_2 = t$

$x_1 = 3 + \frac{2}{3}t$

$$S.S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \overset{3c}{t} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

$$C(A^T) = \text{span}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -4 \end{bmatrix}\right) = \text{span}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



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Showing that A-transpose   Video → 108

$$\underset{n\times k}{A} = \begin{bmatrix} \vec a_1 & \vec a_2 & \cdots & \vec a_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \vec 0$$

Only soluation is $A\vec x = 0$ is $\vec x = \vec 0$

$\vec a_1, \vec a_2 \dots \vec a_k \Rightarrow x_1\vec a_1 + x_2\vec a_2 \dots x_k\vec a_k = 0$

$\vec v \in N(A^T A) \Rightarrow \vec v A^T A \vec v = \vec v\,\vec 0 = \vec 0$

$\vec v A^T = (A\vec v)^T$. so, $(A\vec v)^T A\vec v = 0$

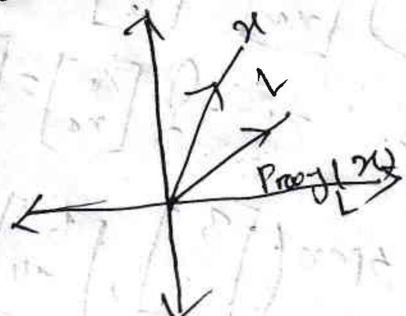
$v \in N(A^T A)$ then $\vec v \in N(A)$. $N(A^T A) = N(A) = \{\vec 0\}$

$\Rightarrow \vec v = \vec 0$

Video 109: Projection onto subspace.

$L = span(\vec v)$, $L = \{c\vec v \mid c \in R\}$

$$Proj_L(\vec x) = \frac{\vec x \cdot \vec v}{\vec v \cdot \vec v}\vec v$$



V is a subspace of R^n. $V^\perp$ also a subspace.

$x \in R^n$. $\vec x = \vec v + \vec w$ where $\vec v \in V$ + $\vec w \in V^\perp$

$A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$ $\vec b = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$ $N(A) = span\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$

$C(A^T) = span\left\{\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}$

Soluation set of $A\vec x = \vec b = \left\{\vec x = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 3 \end{bmatrix} \mid c \in R\right\}$

$\vec x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is a solution

$$Proj_{C(A^T)} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}}{\begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \frac{9}{13}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{22}{13} \\ \frac{-18}{13} \end{bmatrix}$$

Video → 110

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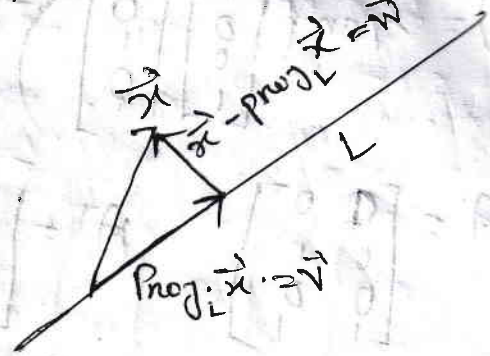
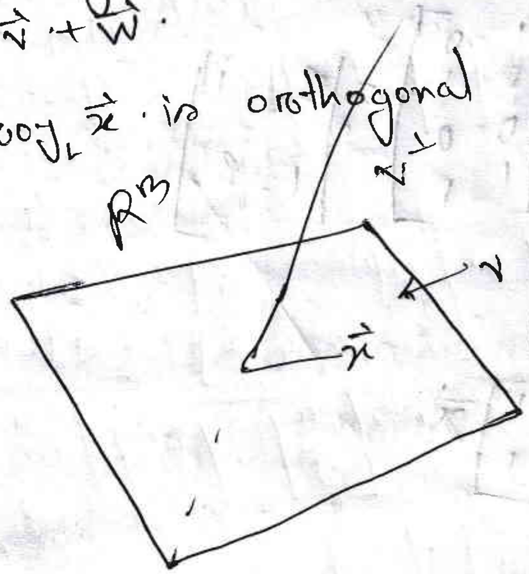
Visualizing a projection onto a plane

$Proj_L \vec{x}$ is the vector $\vec{v}$ in L such that $\vec{x} - \vec{v} = \vec{w}$ which is orthogonal to every things.

$\vec{x} = \vec{v} + \vec{w}$.

$x - Proj_L \vec{x}$ is orthogonal





$Proj_V \vec{x}$ = the unique vector $\vec{v} \in V$ such that $\vec{x} = \vec{v} + \vec{w}$ where $\vec{w}$ is a unic member of $v^\perp$

Video : 111 . A projection onto a subspace is a linear transformation.

V is a subspace of R^n

$\{\vec{b}_1, \vec{b}_2 --- b_K\}$ basis for $V = \vec{a} \in V \Rightarrow \vec{a} = y_1 \vec{b}_1 + y_2 b_2 + \cdots y_K b_K$

$$\underset{n \times k}{A} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & --- & \vec{b}_K \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix} = y_1 \vec{b}_1 + y_2 \vec{b}_2 + y_K \vec{b}_K$$

$\vec{a} \in V \quad A\vec{y} = \vec{a}$ for some $\vec{y} \in R^n$

$\vec{x} \in R^n \quad Proj_V \vec{x} \in V \Rightarrow Proj_V \vec{x} = Ay$ for some vector $\vec{y}$ in R^K

$\vec{x} - Proj_V \vec{x} + w$ or, $\vec{x} - Proj_V \vec{x} = w$ or, $\vec{x} - Proj_V \vec{x} \in N(A^T)$

$A^T(\vec{x} - Proj_V \vec{x}) = \vec{0}$ or, $A^T \vec{x} = A^T \cdot Ay$

$A^T \vec{x} - A^T Ay = \vec{0}$ / $(A^T A)^{-1} A^T \vec{x} = (A^T A)^{-1} A \cdot A\vec{y}$

$Proj_V(\vec{x}) = A(A^T A)^{-1} A^T \vec{x} \rightarrow$ Leniar transformation.

Video → 113

Subspace projection matrix example:

$$V = \text{span}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)$$ — basis for V $\quad \vec{x} \in \mathbb{R}^n$

$$\text{Proj}_V \vec{x} = A(A^T A)^{-1} A^T \vec{x}$$

$$A = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\1 & 1\end{bmatrix} \qquad A^T = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$$

$$\text{Proj}_V(\vec{x}) = \frac{1}{3}\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\end{bmatrix}\vec{x}$$

$$= \frac{1}{3}\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 1\\-1 & 2 & 0 & 1\end{bmatrix}\vec{x}$$

$$\text{Proj}_V \vec{x} = \frac{1}{3}\begin{bmatrix}2 & -1 & 0 & 1\\-1 & 2 & 0 & 1\\0 & 0 & 0 & 0\\1 & 1 & 0 & 2\end{bmatrix}\vec{x}$$

Video → 114

Projection is closest vector in subspace:

$$\|\vec{x} - Proj_V \vec{x}\| \leq \|\vec{x} - \vec{v}\|$$

$$\|\vec{x} - \vec{v}\|^2 = \|\vec{b} + \vec{a}\|^2 = (\vec{b}+\vec{a})(\vec{b}+\vec{a}) = \vec{b}\cdot\vec{b} + 2\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a}$$

$$\|\vec{x} - \vec{v}\|^2 = \|b\|^2 + \|\vec{a}\|^2 \geq \|\vec{a}\|^2 \text{ or, } \|\vec{x} - \vec{v}\|^2 \geq \|\vec{a}\|^2$$

$$\|\vec{x} - \vec{v}\| \geq \|\vec{a}\|$$

So, $\|\vec{x} - \vec{v}\| \geq \|\vec{x} - Proj_V \vec{x}\|$

Video → 115 : Least squares approximation

$A\vec{x} = \vec{b}$. $\vec{x} \in R^k$. $\vec{b} \in R^n$

No soluation to $A\vec{x} = \vec{b}$. (b is not in the $C(A)$)

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_k\vec{a}_k = \vec{b}$$

minimize $\|\vec{b} - Ax\|$

$$\left\| \begin{bmatrix} b_1 - v_1 \\ b_2 - v_2 \\ \vdots \\ b_n - v_n \end{bmatrix} \right\|^2 = (b_1 - v_1)^2 + (b_2 - v_2)^2 + \cdots (b_n - v_n)^2$$

$A\vec{x} = \vec{b}$

no soluation.

x^* that minimize $\|\vec{b} - A\vec{x}^*\|$

$$A^T A \vec{x} = A^T \vec{b}$$

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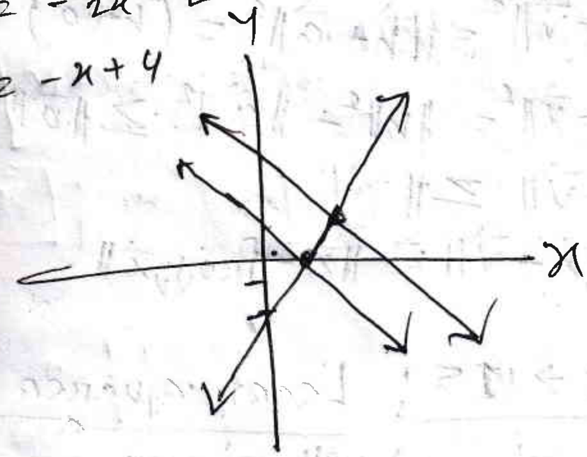
Video → 116

Least squares examples

$2x - y = 2 \qquad -y = -2x + 2 \qquad y = 2x - 2$

$x + 2y = 1 \qquad 2y = -x + 1 \qquad y = -\frac{1}{2}x + \frac{1}{2}$

$x + y = 4 \qquad\qquad\qquad\qquad y = -x + 4$



$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ = no solution

$A^{-1}A\vec{x}^{*} = A^T\vec{b}$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

$2\times 3 \quad \cdot \quad 3\times 2 \qquad A^TA$

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4+1+4 \\ -2+2+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$2\times 3 \quad \cdot \quad 3\times 1$

$$\begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}\vec{x}^{*} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \text{ or, } \left[\begin{array}{cc|c} 6 & 1 & 9 \\ 1 & 6 & 4 \end{array}\right] \text{ or, } \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 6 & 1 & 9 \end{array}\right] 9 - 24$$

$$\text{or, } \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -35 & -15 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & \frac{-15}{-35} = \frac{3}{7} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & \frac{3}{7} \end{array}\right]$$

$$\vec{x}^{*} = \begin{bmatrix} \frac{10}{7} \\ \frac{3}{7} \end{bmatrix}$$

Video → 119

$y = f(x) = mx + b$

$f(-1) = -m + b = 0$

$f(0) = \cdot + b = 1$

$f(1) = m + b = 2$

$f(2) = 2m + b = 1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$A\vec{x} = b$

so let's find x^*

$A^T A x^* = A^T \vec{b}$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} A^T A$$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} A^T b$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} x^* = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} m^* \\ b^* \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$6m^* + 2b^* = 4$

$-2m^* + 4b^* = 4$

$12m^* + 4b^* = 8$

$6\left(\frac{2}{5}\right) + 2b^* = 4$

$\frac{12}{5} = \frac{20}{5} - \frac{12}{5} = \frac{8}{5}$

$b^* = \frac{4}{5}$

Video-120: Invertible change of basis matrix!

$B = \{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_k\}$ $\vec{v}_1 \vec{v}_2, v_k \in R^n$

$C_{n \times m} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\cdot B = \{\vec{v}_1, \vec{v}_2\}$ B is a basis for R^2

$\vec{a} \in R^2$

$\vec{a} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ $[\vec{a}]_B = ?$

$C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $|C| = 1 - 6 = -5$ / $C^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

$[\vec{a}]_B = C^{-1}\vec{a} \Rightarrow [\vec{a}]_B = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 3 \\ -19 \end{bmatrix}$

$\Rightarrow \frac{-3}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{19}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ -\frac{9}{5} \end{bmatrix} + \begin{bmatrix} \frac{38}{5} \\ \frac{19}{5} \end{bmatrix}$

$= \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

Video → 122

Alternate basis transformation matrix

$T: R^n \to R^n$ $\quad T(\vec{x}) = A\vec{x}$. $\quad B = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$ is a basis for R^n

standard coordinates

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, "C^{-1}"] & T(\vec{x}) \arrow[d, "C^{-1}"] \\ {[\vec{x}]_B} \arrow[r, "D"] & {[T(\vec{x})]_B} \end{tikzcd}$$

$C = \begin{bmatrix} \vec{v}_1, \vec{v}_2 \cdots \vec{v}_n \end{bmatrix}$ Change of basis matrix for B

$C[\vec{x}]_B = \vec{x}$ or, $[\vec{x}]_B = C^{-1}\vec{x}$

$D = C^{-1}AC$.

$T: R^2 \to R^2$. $\quad T(\vec{x}) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\vec{x}$. $\quad C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$|C| = -3$

$C^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

Alternate R^2 basis.

$B = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ $\quad$ Looking for D

$T([\vec{x}]_B) = D[\vec{x}]_B$

Video 123 | Alternate basis transformation matrix: Part 2

$C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad T(\vec{x}) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\vec{x}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to T(\vec{x}) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$|C| = -3$ $\quad\quad \downarrow C^{-1}$

$C^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = D$ $\quad [\vec{x}]_B = -\frac{1}{3}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ $\quad [T(\vec{x})]_B$

$[\vec{x}] = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{D} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$B = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$C[\vec{x}]_B = \vec{x}$

$[\vec{x}]_B = C^{-1}\vec{x}$

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Video 129

$B = \{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_k\}$

$\|\vec{v}_i\| = 1$ for $i = 1, 2 \cdots k$

$\|\vec{v}_i\|^2 = 1$

$\vec{v}_i \cdot \vec{v}_i = 1$ for $i = 1, 2 \cdots k$

$\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$

$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$

B is an orthonormal set

$B \rightarrow$ Linearly independent.

$\vec{v}_i, \vec{v}_j \in B \quad i \neq j$

$\vec{v}_i \cdot \vec{v}_j = 0$

Assume that $\vec{v}_i + v_j$ are L. dependent.

$c\,\vec{v}_i \cdot \vec{v}_j = 0 \Rightarrow c(\vec{v}_j \cdot \vec{v}_j) = c\|\vec{v}_j\|^2 = 0$

$\Rightarrow \|v_j\| = 0$

Video → 130: Finding projection onto subspace with orthonormal.

Orthonormal basis for subspace V.

$Proj_V \vec{x} = AA^T\vec{x}$ where $A = [\vec{v}_1 \; \vec{v}_2 \cdots \vec{v}_K]$ basis vector for V

$V = span\left(\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}\right)$ $\{\vec{v}_1\, \vec{v}_2\}$ is an orthonormal basis for V

$Proj_V \vec{x} = AA^T \vec{x}$

$$A = \underset{3\times 2}{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-2}{3} \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \end{bmatrix}} = \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$$