Solving Differential Equations

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1 Power Series Solution of $y'' + p(t)y' + q(t)y = 0$

This is a generalized ODE where $y = y(t)$. The solution is a power series function.

Ordinary Points

A function $f(t)$ is called analytic at t_0 if it has a Taylor Series expansion i.e. all its derivatives from 1 to ∞ exist. For our DE $y'' + p(t)y' + q(t) = 0$, t_0 is an ordinary point if both $p(t)$ and $q(t)$ are analytic. If this is the case, the solution below solves the DE.

Ordinary Point Solution

1. Use the **trial function**:

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

   t_0 is an arbitrary center point. If not given, it is assumed to be 0. Our goal is to find a_n.

2. The **index shifted** derivatives of $y(t)$ are:

$$y'(t) = \sum_{n=1}^{\infty} a_n n (t - t_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(t - t_0)^n$$
$$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1)(t - t_0)^{n-2} = \sum_{n=2}^{\infty} a_{n+2}(n+1)(n+2)(t - t_0)^n$$

3. Substitute back into the DE.

4. Collect the $(t - t_0)^n$ terms to obtain a **recurrence relation**. It should look like:

$$a_{n+2} = (...)a_n$$

5. If possible, generalize the recurrence relation into even and odd cases. Write out a_0 through a_3 and simplify to look for patterns. The generalized even case will be a_{2n} and the odd case will be a_{2n+1}.

6. The final solution is:

$$y(t) = \sum_{n=0}^{\infty} a_{2n}(t-t_0)^{2n} + \sum_{n=0}^{\infty} a_{2n+1}(t-t_0)^{2n+1}$$

Singular Points

If t_0 is not ordinary, it is singular.

Regular Singular Point if $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are both analytic. Alternatively, you can check if $\lim_{t\to t_0}(t-t_0)p(t)$ and $\lim_{t\to t_0}(t-t_0)^2q(t)$ exist.

Irregular Singular Point otherwise.

Regular Singular Point Solution aka Method of Frobenius

Pretty much the same as the Ordinary Point Solution except you use a slightly different trial function.

1. Use the trial function:
$$y(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^{n+\lambda}$$
The goal is to find λ and a_n.

2. The derivatives of $y(t)$ are:
$$y'(t) = \sum_{n=1}^{\infty} a_n(n+\lambda)(t-t_0)^{n+\lambda-1}$$
$$y''(t) = \sum_{n=2}^{\infty} a_n(n+\lambda)(n+\lambda-1)(t-t_0)^{n+\lambda-2}$$

3. Index shift the derivatives so that the exponents are the same.

4. You will find some of your series start at $n=-1$ while others start at $n=0$. Unroll the $n=-1$ term so that all the series start at 0 and you can combine them.

5. Extract a quadratic equation for λ. This comes from the unrolling of the $n=-1$ term from the previous step.

6. Solve for λ_1 and λ_2.

7. For each λ, find the recurrence relation.

8. Write out a_1, a_2, a_3 for each lambda.

9. The linearly independent solution for each λ is:

$$y_{1,2}(t) = (t - t_0)^{\lambda_{1,2}}[a_0 + a_1 t + a_2 t^2 + ...]$$

10. The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

Note If $\lambda_1 - \lambda_2$ is an integer, you will only get 1 linearly independent solution. The other one is found using the Extended Method of Frobenius.

2 Cauchy-Euler Equation $t^2 y'' + aty' + by = 0$

This is a special case of the DE in section 1.

Solution

1. Use the trial function:
$$y(t) = |t|^\lambda$$
2. Find its derivatives:
$$y'(t) = \lambda t^{\lambda-1}$$
$$y''(t) = \lambda(\lambda - 1)t^{\lambda-2}$$
3. Substitute into the DE.
4. Collect terms by exponent.
5. Extract quadratic equation for λ (indicial equation) and solve for λ_1 and λ_2.
6. Only 2 cases we care about:

   Distinct real roots $y_1(t) = t^{\lambda_1}$ and $y_2(t) = t^{\lambda_2}$

   Repeated real roots $y_1(t) = t^{\lambda_1}$ and $y_2(t) = t^{\lambda_2} \ln(t)$
7. The general solution in both cases is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

3 Bessel's DE $t^2 y'' + ty' + (t^2 - \nu^2)y = 0$

This DE is usually arrived at from other DEs through change of variables, which is most of the work. Actually solving the DE is just a matter of stating the general solution.

Solution

You only get one linearly independent solution:

$$y(t) = c_1 J_0(t) + c_2 Y_0(t)$$

$J_0(t)$ is called the Bessel's function of the 1st kind of order $\nu = 0$. $Y_0(t)$ is called the Bessel's function of the 2nd kind of order 0.

4 Heat Equation $u_t = \kappa u_{xx}$

4.1 Homogenous Boundary Conditions

This means that all the BCs equal 0. There are two types of BCs.

Frozen ends $u(0,t) = u(l,t) = 0$

Insulated ends $u_x(0,t) = u_x(l,t) = 0$

Solution

1. Separation of Variables: assume the solution $u(x,t) = X(x)T(t)$
2. Substitute into the PDE to obtain:
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = \sigma$$
3. The solution $T(t)$ is:
$$T(t) = Ce^{\kappa\sigma t}$$
4. Assuming $\sigma < 0$, the solution $X(x)$ is:
$$X(x) = A\cos(-\sqrt{\sigma}x) + B\sin(-\sqrt{\sigma}x)$$
5. Plug the BCs into $X(x)$ to find σ:
$$\sigma = -(\frac{n\pi}{l})^2$$
6. Substitute $X_n(x)$ and $T_n(t)$ back into $u(x,t)$ to obtain:
$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} B_n \sin(\frac{n\pi}{l}x)e^{-(\frac{n\pi}{l})^2\kappa t}$$
For frozen ends, you end up with just sin. For insulated ends, you end up with just cos.
7. To find B_n, sub in the IC $u(x,0) = f(x)$ to find that it resembles the Fourier series, and so:
$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi}{l}x)dx$$
8. If the integral is straightforward, solve it, otherwise get it into one of the following forms through trig identities:

$$
\begin{aligned}
&\frac{1}{l}\int_{-l}^{l} cos(\frac{n\pi}{l}x)cos(\frac{m\pi}{l}x) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \\
&\frac{1}{l}\int_{-l}^{l} cos(\frac{n\pi}{l}x)sin(\frac{m\pi}{l}x) = 0 \\
&\frac{1}{l}\int_{-l}^{l} sin(\frac{n\pi}{l}x)sin(\frac{m\pi}{l}x) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}
\end{aligned}
$$

9. Plug in B_n back into $u(x,t)$ to complete the solution.

4.2 Non-Homogenous Boundary Conditions

Separation of variables cannot be used because not all the BCs are 0. The plan is to change variables until all the BCs are 0 and then the solution is the same as the homogenous case.

Solution

1. Find the steady-state temperature, i.e. the temperature as $t \to \infty$:

$$
\begin{aligned}
&\implies u_t = 0 \\
&\implies u_{xx} = 0 \text{ (from PDE)} \\
&\implies u''_{ss}(x) = 0 \\
&\implies u'_{ss}(x) = a \\
&\implies u_{ss}(x) = ax + b
\end{aligned}
$$

2. Sub in the BCs to find u_{ss}.

3. Define $w(x,t)$ to be the transient temperature and:

$$u(x,t) = w(x,t) + u_{ss}(x)$$

4. Rewrite the I/BVP in terms of w. The PDE remains unchanged but the BCs should now be 0.

5. Solve the new I/BVP for $w(x,t)$ using the same method as the previous section on homogenous BCs.

6. Write the final solution in terms of $u(x,t)$.

4.3 Non-Homogenous Heat Problem $u_t = \kappa u_{xx} + g(x,t)$

Solution

1. Assume g can be expressed as a Fourier sine series (if frozen ends) or cosine series (if insulated ends).

$$g(x,t) = \sum_{n=0}^{\infty} g_n(t)\cos(\frac{n\pi}{l}x)$$
$$g_n(t) = \frac{1}{l}\int_{-l}^{l} g(x,t)\cos(\frac{n\pi}{l}x)dx$$

2. The solution is of the form:

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(\frac{n\pi}{l}x)$$

   The goal is to find $T_n(t)$.

3. Plug in the $u(x,0)$ BC to find $T_n(0)$, which serves as initial conditions.
4. Sub everything into the PDE to end up with:

$$T_n'(t) + \kappa(\frac{n\pi}{l})^2 T_n(t) = g_n(t)$$

5. Solve the n DEs from the previous equation to solve the problem.

5 Wave Problem $u_{tt} = c^2 u_{xx}$

Here, $0 < x < l, t > 0$.

BCs $u(0,t) = u(l,t) = 0$

ICs $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Solution

1. Separation of Variables: assume the solution $u(x,t) = X(x)T(t)$
2. Substitute into the PDE to obtain:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \sigma$$

3. The solution $X(x)$ is:
$$X(x) = c_1\cos(-\sqrt{\sigma}x) + c_2\sin(-\sqrt{\sigma}x)$$

4. Plug the BCs into $X(x)$ to find σ:

$$\sigma = -(\frac{n\pi}{l})^2$$

5. The solution $T(t)$ is:

$$T_n(t) = a_n \cos(\frac{n\pi}{l}ct) + b_n \sin(\frac{n\pi}{l}ct)$$

6. Substitute $X_n(x)$ and $T_n(t)$ back into $u(x,t)$ to obtain:

$$u(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos(\frac{n\pi}{l}ct) + b_n \sin(\frac{n\pi}{l}ct)\right) \sin(\frac{n\pi}{l}x)$$

7. Plug in the ICs to get a_n, b_n.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(\frac{n\pi}{l}x)dx$$
$$b_n = \frac{1}{n\pi c}\int_{-l}^{l} g(x)\sin(\frac{n\pi}{l}x)dx$$

6 Traveling Wave $u_{tt} = c^2 u_{xx}$

The difference between the Wave Problem and the Traveling Wave is that here, $-\infty < x < \infty$. There are no boundary conditions.

ICs $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Solution

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

7 Dirichlet Problem $u_{xx} + u_{yy} + y_{zz} = 0$

7.1 All corners 0

Solution

1. Separation of Variables: assume the solution $u(x,t) = X(x)Y(y)$

2. Substitute into the PDE to obtain:

$$-\frac{X''(t)}{X(x)} = \frac{Y''(y)}{Y(y)} = \sigma$$

3. Assuming $\sigma > 0$, the solution $X(x)$ is:

$$X(x) = a\cos(\sqrt{\sigma}x) + b\sin(\sqrt{\sigma}x)$$

4. Plug in the corners into $X(x)$ to find σ:

$$\sigma = (\frac{n\pi}{l})^2$$

5. The solution $Y(y)$ is:

$$Y_n(y) = c_1 \cosh(\frac{n\pi}{l}y) + c_2 \sinh(\frac{n\pi}{l}y)$$

6. Plug in the corners to find the constants.

7. Substitute $X_n(x)$ and $Y_n(y)$ back into $u(x, y)$ to obtain:

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin(\frac{n\pi}{l}x) \sinh(\frac{n\pi}{l}y)$$

8. Plug in the BCs to get b_n.

7.2 Corners non-zero

Solution

1. Make the corners 0 by using the change of variables:

$$U(x,y) = u(x,y) - \nu(x,y)$$

   where

$$\nu(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

2. Setting $u = \nu$ at the corners results in a 4x4 system.

3. Once ν is known, write the new problem in terms of $U(x, y)$ and solve using the same method as above.