CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao **DIS 12b**

1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains. Keep in mind the following theorems and ideas as you work.

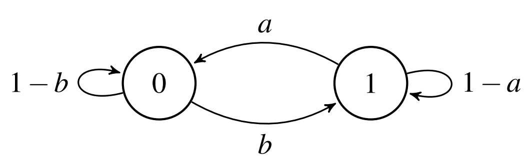
1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.

2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \Pr[X_n = i \mid X_0 = i] > 0\}, i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.

3. (Matrix Representation) Define the transition probability matrix P by filling entry (i,j) with probability $P(i,j)$.

4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.

Use the above theorems and the Markov chain below to answer the following questions.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

(a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one is 0.

(b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2,4,6,...\} = 2.$$

Thus, the chain is periodic.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1,2,3,...\} = 1.$$

Thus, the chain is aperiodic.

(d)

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e)

$$\begin{aligned} \pi(0) &= (1-b)\pi(0) + a\pi(1), \\ \pi(1) &= b\pi(0) + (1-a)\pi(1). \end{aligned}$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

$$\pi = \frac{1}{a+b}\begin{bmatrix} a & b \end{bmatrix}.$$

2 Allen's Umbrellas

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

We will model this as a Markov chain. Let $\mathscr{X} = \{0,1,2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Write down the transition matrix, determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

$$\Pr[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\Pr[X_{n+1}=2 \mid X_n=1]=p, \qquad \Pr[X_{n+1}=1 \mid X_n=1]=1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\Pr[X_{n+1}=1 \mid X_n=2]=p, \qquad \Pr[X_{n+1}=0 \mid X_n=2]=1-p$$

We summarize this with the transition matrix

$$P=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

Observe that the transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P=\pi$, or $\pi(P-I)=0$. This yields the balance equations

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0)+\pi(1)+\pi(2)=1$.

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix}=\frac{1}{3-p}\begin{bmatrix} 1-p & 1 & 1 \end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

3 Continuous Intro

(a) Is

$$f(x)=\begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate $\mathbf{E}[X]$ and $\text{var}(X)$ for X with the density function

$$f(x) = \begin{cases} 1/\ell, & 0 \le x \le \ell, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is their joint distribution?

(d) Calculate $\mathbf{E}[XY]$ for the above X and Y.

Solution:

(a) Yes; it is non-negative and integrates to 1. No; a CDF should go to 1 as x goes to infinity and be non-decreasing.

(b) $\mathbf{E}[X] = \int_{x=0}^{\ell} x \cdot (1/\ell)\,\mathrm{d}x = \ell/2$. $\mathbf{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot (1/\ell)\,\mathrm{d}x = \ell^2/3$.
$\text{var}(X) = \ell^2/3 - \ell^2/4 = \ell^2/12$.

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted $U[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \Pr(X \in [x, x+\mathrm{d}x], Y \in [y, y+\mathrm{d}y]) = \Pr(X \in [x, x+\mathrm{d}x])\Pr(Y \in [y, y+\mathrm{d}y]) \\ &\approx f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint distribution is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) $\mathbf{E}[XY] = \int_{x=0}^{1}\int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = 1/3$.
Alternatively, since X and Y are independent, we can compute $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. Note that

$$\mathbf{E}[X] = \int_0^1 x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\bigg|_0^1 = \frac{2}{3},$$

and $\mathbf{E}[Y] = 1/2$ since the density of Y is symmetric around $1/2$. Hence,

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y] = \frac{1}{3}.$$

4 Uniform Distribution

You have two spinners, each having a circumference of 10, with values in the range $[0,10)$. If you spin both (independently) and let X be the position of the first spinner and Y be the position of the second spinner, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\Pr[X \geq 5 \mid Y \geq X] = \frac{\Pr[Y \geq X \cap X \geq 5]}{\Pr[Y \geq X]}.$$

$\Pr[Y \geq X] = 1/2$ by symmetry. To find $\Pr[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\Pr[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

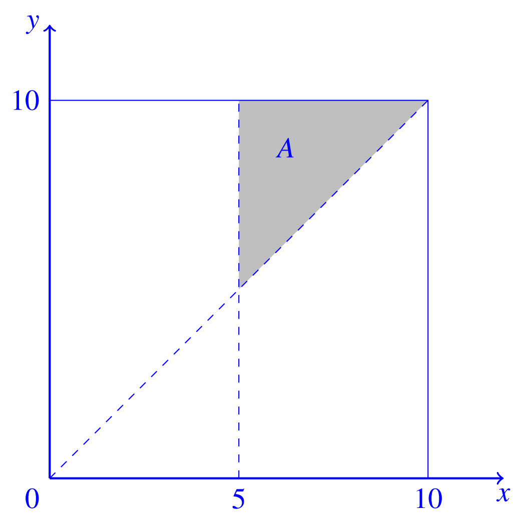


Figure 1: Joint probability density for the spinner.

$$\Pr[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\Pr[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.