CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Rao — HW 2

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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2 Sum of Inverses

Prove that for every positive integer k, the following is true:

> For every real number $r > 0$, there are only finitely many solutions in positive integers to
>
> $$\frac{1}{n_1} + \cdots + \frac{1}{n_k} = r.$$

In other words, there exists some number m (that depends on k and r) such that there are at most m ways of choosing a positive integer n_1, and a (possibly different) positive integer n_2, etc., that satisfy the equation.

Solution:

We will first transfer the problem to mathematical notation:

Claim: $\forall k \in \mathbb{Z}\ \forall r \in \mathbb{R}\ \big((k > 0 \wedge r > 0) \Rightarrow (\text{there are finitely many solutions to } n_1^{-1} + \cdots + n_k^{-1} = r, n_i \in \mathbb{Z}, n_i > 0)\big)$

Proof: We will prove this by induction on k. For our base case, $k = 1$. In the base case, iff r can be written as $1/n_1$ when n_1 is a positive integer, then there is exactly one solution, $n_1 = 1/r$. If r cannot be written in that form, then there are exactly zero solutions. In all cases, there is a finite number of solutions. For the inductive hypothesis, assume that there are finitely many solutions for some $k \geq 1$ for all r. Each real number r_1 either can or cannot be written as the sum of $k+1$ integers' inverses. If r_1 cannot be written in that form, then there are exactly zero solutions. If r_1

can be written in that form, then the integers' inverses can be ordered. Since r_1 is the sum of $k+1$ integers' inverses, the largest $1/n_i$ must be at least $r_1/(k+1)$. This means that the smallest n_i must be at most $(k+1)/r_1$, which means that the smallest n_i has finitely many possible values. For each of the possible smallest n_i values, there is a real number $r_1 - 1/n_i$ that can be written as the sum of k integers' inverses in finitely many ways (using the induction hypothesis). This means that there are only finitely many possible solutions for $k+1$ (combining all solutions (finitely many) for each possible smallest n_i values (finitely many)). By the principle of induction, there are finitely many solutions for all k for all r.

3 Stable Marriage

Consider a set of four men and four women with the following preferences:

men	preferences	women	preferences
A	1>2>3>4	1	D>A>B>C
B	1>3>2>4	2	A>B>C>D
C	1>3>2>4	3	A>B>C>D
D	3>1>2>4	4	A>B>D>C

(a) Run on this instance the stable matching algorithm presented in class. Show each stage of the algorithm, and give the resulting matching, expressed as $\{(M,W),\ldots\}$.

(b) We know that there can be no more than n^2 stages of the algorithm, because at least one woman is deleted from at least one list at each stage. Can you construct an instance with n men and n women so that $c \cdot n^2$ stages are required for some respectably large constant c? (We are looking for a *general pattern* here, one that results in $c \cdot n^2$ stages for any n.)

(c) Suppose we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Can the order of the proposals change the resulting pairing? Give an example of such a change or prove that the pairing that results is the same.

Solution:

(a) The situations on the successive days are:
Day 1: Proposals: $\{(A,1),(B,1),(C,1),(D,3)\}$, B and C are rejected.
Day 2: Proposals: $\{(A,1),(B,3),(C,3),(D,3)\}$, C and D are rejected.
Day 3: Proposals: $\{(A,1),(B,3),(C,2),(D,1)\}$, A is rejected.
Day 4: Proposals: $\{(A,2),(B,3),(C,2),(D,1)\}$, C is rejected.
Day 5: Proposals: $\{(A,2),(B,3),(C,4),(D,1)\}$, no one is rejected.
Final matching: $(A,2),(B,3),(C,4),(D,1)$.

(b) Consider the case where the preference lists have the following structure:

men	preferences	women	preferences
1	$1>2>\cdots>n-1>n$	1	$2>3>\cdots>n>1$
2	$2>3>\cdots>1>n$	2	$3>4>\cdots>1>2$
3	$3>4>\cdots>2>n$	3	$4>5>\cdots>2>3$
	$\cdots$		$\cdots$
$n-1$	$n-1>1>\cdots>n-2>n$	$n-1$	$n>1>\cdots>n-2>n-1$
n	$1>2>\cdots>n-1>n$	n	$1>2>\cdots>n-1>n$

In this case, man 1 and n go to woman 1 on the first day (while any other man i goes to woman i), and woman 1 rejects man 1. He then goes to woman 2 the next day, who rejects man 2, and so on. It can be shown that there is exactly one man rejected every day, and on the i-th day, the $((i-1) \mod n)+1$ man is rejected by woman $((i-1) \mod (n-1))+1$. This continues until the man 1 proposes to woman n. The number of days required for this to terminate is $n^2-2n+2 \geq n^2/2$.

(c) Assume, that when a proposal is made and an answer is received, we write down it on a list L and enumerate them. Now the proof is similar to the proof in class that the algorithms finds male optimal pairing.

We should prove that the pairing P' that results is the same as pairing P found by the regular algorithm. Assume the opposite, so either there is a woman that rejects a man who was her pair in P or there is a woman who do not receive a proposal from a man who was her pair in P. By Well Ordering Principle, there is the first entry in the list L for one of that happens, let W denote the woman for whom it happens and let M denote her pair in P.

First, if W rejected M then it is because she get proposal from other man M' who is higher in her preference list. But M' proposed her only because he was rejected by other woman W' who is higher in his preference list and did not rejects him in P. In other words, even before W rejected M, W' rejected her previous pair M'. Contradiction.

Second, if W did not get a proposal from M then it is because M was not rejected by some other woman W' who is higher in his preference list and who rejects him in P. Contradiction.

Therefore, the resulting pairing P' is the same as P.

4 Bieber Fever

In this world, there are only two kinds of people: people who love Justin Bieber, and people who hate him. We are searching for a stable matching for everyone. The situation is as follows:

- For some $n \geq 5$, there are n men, n women, and one Justin Bieber[1].
- Men can be matched with women; or anyone can be matched with Justin Bieber.

[1]For the purposes of this problem, Justin Bieber is neither male nor female.

- Everyone is either a Hater or a Belieber. Haters want to be matched with anyone but Justin Bieber. Beliebers really want to be matched with Justin Bieber but don't mind being matched with other people.
- Men and women still have preference lists, as usual, but if they are a Belieber, Justin Bieber is always in the first position. If they are a hater, Justin Bieber is always in the last position.
- Justin Bieber desires to have 10 individuals matched with him (to party forever). As Justin Bieber is a kind person and wishes to be inclusive, he wishes to have exactly 5 women and 5 men in his elite club.
- Justin Bieber also has a preference list containing all $2n$ men and women.

A stable matching is defined as follows:

- Justin Bieber has 10 partners, of which 5 are men and 5 are women.
- All men and women not matched up with Justin Bieber are married to someone of the opposite gender.
- No rogue couples exist; i.e., there is no man M and woman W such that M prefers W to his current wife, and W prefers M to her current husband.
- No Hater is matched with Justin Bieber.
- There is no man who (1) is not matched with Justin Bieber; and (2) who is preferred by Justin Bieber over one of his current male partners; and (3) who prefers Justin Bieber over his wife. And similarly for women vis-a-vis Justin Bieber relative to their husbands and Justin Bieber's female partners.

(a) Show that there does not necessarily exist a stable matching.

(b) Provide an "if-and-only-if" condition for whether a stable matching exists. (*No need to prove anything in this part. That comes in later parts of this question.*)

(c) Is Justin Bieber guaranteed to always get his Bieber-optimal group if a stable matching exists? (Bieber-optimal means that he gets the best possible group that could be matched to him in any stable matching.)

(d) Give an algorithm which finds a stable matching if the condition you gave in (b) holds. Argue why this algorithm works.

(e) Prove that a stable matching cannot exist if the condition you gave in (b) does not hold.

Solution:

(a) Suppose that there are 5 men and 5 women, all haters. Then Justin Bieber certainly cannot be matched with 5 men and 5 women without being matched with a hater. Therefore a stable matching does not exist.

(b) A stable matching exists if and only if there are at least 5 male Beliebers and at least 5 female Beliebers.

(c) Yes. Consider the group which consists of Justin Bieber's favorite 5 male Beliebers, and favorite 5 female Beliebers. If a stable matching exists, Justin Bieber must be matched with this group. Suppose towards a contradiction that he is matched with some other set of 5 male Beliebers. Then he is matched with some M who is not in his top 5 (i.e. M is ranked 6th or worse in Justin Bieber's preferences among male Beliebers), and he is not matched with some M^* who is in his top 5. But Justin prefers M^* to M, and M^* is a Belieber, violating condition 5. Thus this matching is not stable. The same reasoning follows symmetrically for females; therefore, there is only one group that Justin Bieber can possibly be matched with, and thus any stable matching is Justin Bieber-optimal.

(d) First, take Justin Bieber's 5 favorite male Beliebers and his 5 favorite female Beliebers, and match them with him. For example, suppose that Justin Bieber's preference list is A B 1 2 C D E F 3 4 5 G 6 H 7 I 8 9, and that A, B, D, E, G, H, I, 1, 3, 4, 7, 8, and 9 are Beliebers. Then we would match Justin Bieber with A, B, D, E, G and 1, 3, 4, 7, 8. Remove Justin Bieber and his crew from all remaining preference lists, and run the propose and reject algorithm on everyone else ($n-5$ men and $n-5$ women). Note that Justin Bieber's "favorite 5 male Beliebers" must exist, because there are at least 5 male Beliebers, and similarly for females. Now let us check the conditions. Certainly Justin Bieber is matched with 5 men and 5 women, none of whom are Haters, by construction (conditions 1 and 4). Because we matched Justin Bieber with his 5 favorite male Beliebers, there is no Belieber who is not matched with Justin Bieber who Justin Bieber prefers over one of his current partners (condition 5), and similarly for women. Finally, because we ran the propose-and-reject algorithm on the remaining people, we are guaranteed to have a matching with no rogue couples (conditions 2 and 3). Note that the definition of rogue couple can only apply to people who are not matched with Justin Bieber, so we don't need to concern ourselves with rogue couples involving anyone in Justin Bieber's crew.

For this problem, we saw several examples of incorrect algorithms. Algorithms that had two stages (first Justin Bieber collecting his crew, and then the rest running the propose-and-reject algorithm) generally worked. The algorithms which tried to run the propose-and-reject algorithm on Justin Bieber and everyone else "in parallel" were often incorrect in subtle ways. We also saw many algorithms which were not precisely stated. For example, students had men proposing to Justin Bieber without specifying how Justin Bieber should process those proposals, or other things like this.

In addition, although many students gave correct algorithms, many students did not give adequate arguments for stability. In particular, trying to prove stability for a single-stage algorithm is quite involved, because the old proofs no longer apply (one would essentially have to prove everything from scratch), and very few students gave correct arguments for such an algorithm. There were also other errors; for example, some students tried to use part c) to prove that Justin Bieber always gets his optimal group, but this is circular reasoning (this is a stable matching, therefore Justin Bieber gets his optimal group, therefore this is a stable matching).

(e) Suppose, without loss of generality, that there are less than 5 male Beliebers. Then, if Justin Bieber is matched with 5 males, at least one must be a hater, so the matching is unstable; and otherwise, Justin Bieber is not matched with 5 males, so the matching is also unstable. Therefore a stable matching does not exist.

5 Combining Stable Marriages

In this problem we examine a simple way to *combine* two different solutions to a stable marriage problem. Let R, R' be two distinct stable matchings. Define the new matching $R \wedge R'$ as follows:

> For every man m, m's date in $R \wedge R'$ is whichever is better (according to m's preference list) of his dates in R and R'.

Also, we will say that a man/woman *prefers* a matching R to a matching R' if he/she prefers his/her date in R to his/her date in R'. We will use the following example:

men	preferences	women	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

(a) $R = \{(A,4),(B,3),(C,1),(D,2)\}$ and $R' = \{(A,3),(B,4),(C,2),(D,1)\}$ are stable matchings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

(b) Prove that, for any matchings R, R', no man prefers R or R' to $R \wedge R'$.

(c) Prove that, for any stable matchings R, R' where m and w are dates in R but not in R', one of the following holds:

- m prefers R to R' and w prefers R' to R; or
- m prefers R' to R and w prefers R to R'.

[*Hint*: Let M and W denote the sets of mens and women respectively that prefer R to R', and M' and W' the sets of men and women that prefer R' to R. Note that $|M|+|M'| = |W|+|W'|$. (Why is this?) Show that $|M| \leq |W'|$ and that $|M'| \leq |W|$. Deduce that $|M'| = |W|$ and $|M| = |W'|$. The claim should now follow quite easily.]

(You may assume this result in subsequent parts even if you don't prove it here.)

(d) Prove an interesting result: for any stable matchings R, R', (i) $R \wedge R'$ is a matching [*Hint*: use the results from (c)], and (ii) it is also stable.

Solution:

(a) $R \wedge R' = \{(A,3),(B,4),(C,1),(D,2)\}$.

(b) Let m be a man, and let his dates in R and R' be w and w' respectively, and without loss of generality, let $w > w'$ in m's list. Then his date in $R \wedge R'$ is w, whom he prefers over w'. However, for m to prefer R or R' over $R \wedge R'$, he must prefer w or w' over w, which is not possible (since $w > w'$ in his list).

(c) Let M and W denote the sets of men and women respectively that prefer R to R', and M' and W' the sets of men and women that prefer R' to R. Note that $|M| + |M'| = |W| + |W'|$, since the left-hand side is the number of men who have different partners in the two matchings, and the right-hand side is the number of women who have different partners.

Now, in R there cannot be a pair (m,w) such that $m \in M$ and $w \in W$, since this will be a rogue couple in R'. Hence the partner in R of every man in M must lie in W', and hence $|M| \leq |W'|$. A similar argument shows that every man in M' must have a partner in R' who lies in W, and hence $|M'| \leq |W|$.

Since $|M| + |M'| = |W| + |W'|$, both these inequalities must actually be tight, and hence we have $|M'| = |W|$ and $|M| = |W'|$. The result is now immediate: if the man m does not date the woman w in one but not both matchings, then

- either $m \in M$ and $w \in W'$, i.e., m prefers R to R' and w prefers R' to R,
- or $m \in M'$ and $w \in W$, i.e., m prefers R' to R and w prefers R to R'.

(d) (i) If $R \wedge R'$ is not a matching, then it is because two men get the same woman, or two women get the same man. Without loss of generality, assume it is the former case, with $(m,w) \in R$ and $(m',w) \in R'$ causing the problem. Hence m prefers R to R', and m' prefers R' to R. Using the results of the previous part would imply that w would prefer R' over R, and R over R' respectively, which is a contradiction.

(ii) Now suppose $R \wedge R'$ has a rogue couple (m,w). Then m strictly prefers w to his partners in both R and R'. Further, w prefers m to her partner in $R \wedge R'$. But w is matched to the better of her partners in R and R'. Let w's partners in R and R' be m_1 and m_2. If she is finally matched to m_1, then (m,w) is a rogue couple in R; on the other hand, if she is matched to m_2, then (m,w) is a rogue couple in R'. Since these are the only two choices for w's partner, we have a contradiction in either case.

6 Better Off Alone

In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the preference orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing could ultimately have to be partial, i.e., some individuals would remain single.

The notion of stability here should be adjusted a little bit. A pairing is stable if

- there is no paired individual who prefers being single over being with his/her current partner,

- there is no paired man and single woman (or paired woman and single man) that would both prefer to be with each other over being single or with his/her current partner,
- there is no paired man and paired woman that would both prefer to be with each other over their current partners, and
- there is no single man and single woman that would both prefer to be with each other over being single.

(a) Prove that a stable pairing still exists in the case where we allow single individuals. You can approach this by introducing imaginary mates that people "marry" if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?

(b) As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact by contradiction.

Solution:

(a) Following the hint, we introduce an imaginary mate (let's call it a robot) for each person. Note that we introduce one robot for each individual person, i.e. there are as many robots as there are people. For simplicity let us say each robot is owned by the person we introduce it for.

Each robot is in love with its owner, i.e. it puts its owner at the top of its preference list. The rest of its preference list can be arbitrary. The owner of a robot puts it in his/her preference list exactly after the last person he/she is willing to marry. i.e. owners like their robots more than people they are not willing to marry, but less than people they like to marry. The ordering of people who someone does not like to marry as well as robots he/she does not own is irrelevant as long as they all come after their robot.

To illustrate, consider this simple example: there are three men $1,2,3$ and three women A,B,C. The preference lists for men is given below:

Man	Preference List
1	$A > B$
2	$B > A > C$
3	C

The following depicts the preference lists for women:

Woman	Preference List
A	1
B	$3 > 2 > 1$
C	$2 > 3 > 1$

In this example, 1 is willing to marry A and B and he likes A better than B, but he'd rather be single than to be with C. On the other side B has a low standard and does not like being single at all. She likes 3 first, then 2, then 1 and if there is no option left she is willing to be forced into singleness. On the other hand, A has pretty high standards. She either marries 1 or remains single.

According to our explanation we should introduce a robot for each person. Let's name the robot owned by person X as R_X. So we introduce male robots R_A, R_B, R_C and female robots R_1, R_2, R_3. Now we should modify the existing preference lists and also introduce the preference lists for robots.

According to our method, 1's preference list should begin with his original preference list, i.e. $A > B$. Then comes the robot owned by 1, i.e. R_1. The rest of the ordering, which should include C and R_2, R_3 does not matter, and can be arbitrary.

For B, the preference list should begin with $3 > 2 > 1$ and continue with R_B, but the ordering between the remaining robots (R_A and R_C) does not matter.

What about robots' preference lists? They should begin with their owners and the rest does not matter. So for example R_A's list should begin with A, but the rest of the humans/robots (B, C, R_1, R_2, and R_3) can come in any arbitrary order.

So the following is a list of preference lists that adhere to our method. There are arbitrary choices which are shown in bold (everything in bold can be reordered within the bold elements).

Man	Preference List
1	$A > B > R_1 > \mathbf{3} > \mathbf{R_3} > \mathbf{R_2}$
2	$B > A > C > R_2 > \mathbf{R_1} > \mathbf{R_3}$
3	$C > R_3 > \mathbf{R_1} > \mathbf{R_3} > \mathbf{A} > \mathbf{B}$
R_A	$A > \mathbf{B} > \mathbf{C} > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3}$
R_B	$B > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3} > \mathbf{A} > \mathbf{C}$
R_C	$C > \mathbf{A} > \mathbf{R_2} > \mathbf{B} > \mathbf{R_1} > \mathbf{R_3}$

and the following depicts the preference lists for women and female robots:

Woman	Preference List
A	$1 > R_A > \mathbf{3} > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C}$
B	$3 > 2 > 1 > R_B > \mathbf{R_C} > \mathbf{R_A}$
C	$2 > 3 > 1 > R_C > \mathbf{R_A} > \mathbf{R_B}$
R_1	$1 > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C} > \mathbf{3} > \mathbf{R_A}$
R_2	$2 > \mathbf{R_A} > \mathbf{R_C} > \mathbf{1} > \mathbf{3} > \mathbf{R_B}$
R_3	$3 > \mathbf{2} > \mathbf{1} > \mathbf{R_A} > \mathbf{R_C} > \mathbf{R_B}$

Now let us prove that a stable pairing between robots and owners actually corresponds to a stable pairing (with singleness as an option). This will finish the proof, since we know that in the robots and owners case, the propose and reject algorithm will give us a stable matching.

It is obvious that to extract a pairing without robots, we should simply remove all pairs in which there is at least one robot (two robots can marry each other, yes). Then each human who is not matched is declared to be single. It remains to check that this is a stable matching (in the new, modified sense). Before we do that, notice that a person will never be matched with another person's robot, because if that were so he/she and his/her robot would form a rogue couple (the robot's love is there, and the owner actually likes his/her robot more than other robots).

(a) No one who is paired would rather break out of his/her pairing and be single. This is because if that were so, that person along with its robot would have formed a rogue couple in the original pairing. Remember, the robot loves its owner more than anything, so if the owner likes it more than his/her mate too, they would be a rogue couple.

(b) There is no rogue couple. If a rogue couple m and w existed, they would also be a rogue couple in the pairing which includes robots. If neither m nor w is single, this is fairly obvious. If one or both of them are single, they prefer the other person over being single, which in the robots scenario means they prefer being with each other over being with their robot(s) which is their actual match.

This shows that each stable pairing in the robots and humans setup gives us a stable pairing in the humans-only setup. It is noteworthy that the reverse direction also works. If there is a stable pairing in the humans-only setup, one can extend it to a pairing for robots and humans setup by first creating pairs of owners who are single and their robots, and then finding an arbitrary stable matching between the unmatched robots (i.e. we exclude everything other than the unmatched robots and find a stable pairing between them). To show why this works, we have to refute the possibility of a rogue pair. There are three cases:

(a) A human-human rogue pair. This would also be rogue pair in the humans-only setup. The humans prefer each other over their current matches. If their matches are robots, that translates to them preferring each other over being single in the humans-only setup.

(b) A human-robot rogue pair. If the human is matched to his/her robot, our pair won't be a rogue pair since a human likes his/her robot more than any other robot. On the other hand if the human is matched to another human, he/she prefers being with that human over being single which places that human higher than any robot. Again this refutes the human-robot pair being rogue.

(c) A robot-robot rogue pair. If both robots are matched to other robots, then by our construction, this won't be a rogue couple (we explicitly selected a stable matching between left-alone robots). On the other hand, if either robot is matched to a human, that human is its owner, and obviously a robot loves its owner more than anything, including other robots. So again this cannot be a rogue pair.

This completes the proof.

(b) We will perform proof by contradiction. Assume that there exists some man m_1 who is paired with a woman w_1 in stable pairing S and unpaired in stable pairing T. Since S is a stable pairing and m_1 is unpaired, w_1 must be paired in T with a man m_2 whom she prefers over m_1. (If w_1 were unpaired or paired with a man she does not prefer over m_1, then (m_1, w_1) would be a rouge couple, which is a contradiction.)

Since m_2 is paired with w_1 in T, he must be paired in S with some woman w_2 whom m_2 prefers over w_1. This process continues (w_2 must be paired with some m_3 in T, m_3 must be paired with some w_3 in S, etc.) until all persons are paired. Since this requires m_1 to be paired in T, where he is known to be unpaired, we have reached a contradiction. Therefore, our assumption must be false, and there cannot exist some man who is paired in a stable pairing S and unpaired in a stable pairing T. A similar argument can be used for women.

Since no man or woman can be paired in one stable pairing and unpaired in another, every man or woman must be either paired in all stable pairings or unpaired in all stable pairings.

Here is another possible proof:

We know that some male-optimal stable pairing exists. Call this pairing M. We first establish two lemmas.

Lemma 1. If a man is single in male-optimal pairing M, then he is single in all other stabling pairings.

Proof. Assume there exists a man that is single in M but not single in some other stable pairing M'. Then M would not be a male-optimal pairing, so this is a contradiction.

Lemma 2. If a woman is paired in male-optimal pairing M, she is paired in all other stable pairings.

Proof. Assume there exists a woman that is paired in M but single in some other stable pairing M'. Then M would not be female-pessimal, so this is a contradiction.

Let there be k single men in M. Let M' be some other stable pairing. Then by Lemma 1, we know single men in M' will be greater than or equal to k. We also know that there are $n-k$ paired men and women in M. Then by Lemma 2, we know that the number of paired women in M' will be greater than or equal to $n-k$.

Now, we want to prove that if a man is paired in M, then he is paired in every other stable pairing. We prove this by contradiction. Assume that there exists a man m that is paired in M but is single in some other stable pairing M'. Then there must be strictly greater than k single men in M', and thus strictly greater than k single women in M'. Since there are strictly greater than k single women in M', there must be strictly less than $n-k$ paired women in M'. But this contradicts that the number of paired women in M' will be greater than or equal to $n-k$.

We also have to prove that if a woman is single in M, then she must be single every other stable pairing. We again prove this by contradiction. Assume that there exists a woman w that is single in M and paired in some other stable pairing M'. Then there are strictly greater than

$n-k$ paired women in M', which means there are strictly greater than $n-k$ paired men in M'. This means there must be strictly less than k single men in M'. But this contradicts that the number of single men in M' will be greater than or equal to k.

Since we have proved both 1) If a man is single in M then he is single in every other stable pairing and 2) If a man is paired in M then he is paired in every other stable pairing (note that the contrapositive of this is if a man is single in any other stable pairing, then this man is single in M), we know that a man is single in M if and only if he is single in every other stable pairing. Similarly, since we have proved both 1) If a woman is single in M then she is single in every other stable pairing and 2) If a woman is paired in M then she is paired in every other stable pairing, we know that a woman is single in M if and only if she is single in every stable pairing. Thus we have proved that if a person is single in one stable pairing, s/he is single in every stable pairing.

7 Quantitative Stable Marriage Algorithm

Once you have practiced the basic algorithm, let's quantify stable marriage problem a little bit. Here we define the following notation: on day j, let $P_j(M)$ be the rank of the woman that man M proposes to (where the first woman on his list has rank 1 and the last has rank n). Also, let $R_j(W)$ be the total number of men that woman W has rejected up through day $j-1$ (i.e. not including the proposals on day j). Please answer the following questions using the notation above.

(a) Prove or disprove the following claim: $\sum_M P_j(M) - \sum_W R_j(W)$ is independent of j. If it is true, please also give the value of $\sum_M P_j(M) - \sum_W R_j(W)$. The notation, $\sum_M$ and $\sum_W$, simply means that we are summing over all men and all women.

(b) Prove or disprove the following claim: one of the **men or women** must be matched to someone who is ranked in the top half of their preference list. You may assume that n is even.

Solution:

(a) On day $j=1$, each man proposes to the first woman on his list so $\sum_M P_1(M) = n$, and no woman rejected any man through day 0, and therefore $\sum_M P_1(M) - \sum_W R_1(W) = n$. In general, each time a woman rejects a man on day $j-1$, it increases $\sum_W R_j(W)$ by exactly 1. It also increases $\sum_M P_j(M)$ by exactly 1, since the rejected man proposes to the next woman on his list on day j. Therefore $\sum_M P_j(M) - \sum_W R_j(W)$ stays constant and is independent of j. $\square$

More formally, we can prove this by induction on j, with $j=1$ as base case.
Induction Hypothesis: Assume $\sum_M P_j(M) - \sum_W R_j(W) = n$.
Induction Step: The quantity $\sum_W R_{j+1}(W) - \sum_W R_j(W)$ is exactly the number of men rejected by women on day j. But each of the rejected men propose to the next woman on their list on day $j+1$, and so this quantity is also equal to $\sum_M P_{j+1}(M) - \sum_M P_j(M)$. Equating the two, we get

$$\sum_W R_{j+1}(W) - \sum_W R_j(W) = \sum_M P_{j+1}(M) - \sum_M P_j(M).$$

Therefore,

$$\sum_M P_{j+1}(M) - \sum_W R_{j+1}(W) = \sum_M P_j(M) - \sum_W R_j(W)$$

and the right hand side is equal to n by the induction hypothesis. $\square$

(b) Assume that no man is matched with a woman in the top half of his preference list. Each of them must have been rejected at least $n/2$ times, for a total of at least $n^2/2$ rejections. This implies that at least one woman must have rejected at least $n/2$ men (because if not, then the total number of rejections must be less than $(n/2) \cdot n$, contradiction). But now, by the improvement lemma, this woman must be matched with a man she likes more than the $n/2$ men she rejected, meaning that the man she is matched with is in the top half of her preference list. $\square$

Alternative Proof:

Assume towards contradiction that every man and every woman is matched to someone who is ranked in the bottom half of their preference list.

Observe that a man M is matched to someone in the top half of his preference list if and only if $P_m(M) \leq n/2$, where m is the last day of the algorithm. Therefore, if M is matched to someone in the bottom half of his preference list, then $P_m(M) > n/2$, i.e., $P_m(M) \geq n/2+1$. Summing over the men gives us $\sum_M P_m(M) \geq n^2/2+n$. By part (a), it follows that $\sum_W R_m(W) = \sum_M P_m(M) - n \geq n^2/2$.

Observe also that if $R_m(W) \geq n/2$, then by the improvement lemma, W must be matched to someone in the top half of her preference list. Therefore, from our assumption that W is matched to someone in the bottom half of her preference list, we get $R_m(W) < n/2$. Summing over the women gives us $\sum_W R_m(W) < n^2/2$. But this contradicts our earlier result above! $\square$

8 Short Answer: Graphs

(a) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph? (An expression that may contain n.)

(b) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

(c) Give a gray code for 3-bit strings. (Recall, that a gray code is a sequence of the strings where adjacent elements differ by one. For example, the gray code of 2-bit strings is $00, 01, 11, 10$. Note the last string is considered adjacent to the first and 10 differs in one bit from 00. Answer should be sequence of three-bit strings: 8 in all.)

(d) For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the d-dimensional hypercube for $d = n$. (True or False.)

(e) The complete graph with n vertices where n is an odd prime can have all its edges covered with x Rudrata cycles: a cycle where each vertex appears exactly once. What is the number, x, of such cycles in a cover? (Answer should be an expression that depends on n.)

(f) Give a set of disjoint Rudrata cycles that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5.)

Solution:

(a) **3**.
Each neighbor must be in a different connected component. This follows from a tree having a unique path between each neighbor in the tree as it is acyclic. The removed vertex broke that path, so each neighbor is in a separate component. Moreover, every other node is connected to one of the neighbors as every other vertex has a path to the removed node which must go through a neighbor.

(b) **7**
The problem is asking you to make each component into a tree. The components should have $n_1 - 1$, $n_2 - 1$ and $n_3 - 1$ edges each or a total of $n-3$ edges. The total number of edges after Bob and Alice did their work was $n-1+10-5 = n+4$, thus one needs to remove 7 edges to ensure there are no cycles.

(c) **000, 001, 011, 010, 110, 111, 101, 100.**
The idea is to use the solution to the homework problem that showed that the hypercube has a Rudrata path.

(d) **False**
This is just an exercise in definitions. The complete graph has $n(n-1)/2$ edges where the hypercube has $n2^{n-1}$ edges. For $n \geq 3$, $2^{n-1} \geq (n-1)/2$.

(e) $(n-1)/2$.
Each cycle removes degree 2 from each node. As the degree is $n-1$, we obtain a total of $\frac{n-1}{2}$. This is if it can be done disjointly.

(f) $(0,1),(1,2),(2,3),(3,4),(4,0)$
$(0,2),(2,4),(4,1),(1,3),(3,0)$

The following details a procedure for generating the paths using ideas from modular arithmetic. Note that modular arithmetic is not necessary for the solution, but it provides a clean solution.

The idea is that we can generate disjoint Rudrata cycles by repeatedly adding an element a to the current node. This produces the sequence of edges $(0,a),(a,2a),\ldots,((p-1)a,0)$ which are disjoint for different a, as long as $a \neq -a \pmod p$, as that would simply be subtracting a everytime. (In other words, there exists no integer k such that $-a + pk = a$.)

We use primality to say that inside a sequence the edges are disjoint since the elements $\{0a,\ldots,(p-1)a\}$ are distinct $\pmod p$.