CS 70	Discrete Mathematics and Probability Theory	
Spring 2017	Satish Rao	DIS 1a

1 Key Facts

Key Facts are included to jog your memory. Your TA will explain these in detail.

We have covered several types of proofs so far, the most recent of which was *induction*.

1. **Direct Proof**: Prove a statement using a collection of other facts you know.
2. **Contraposition**: Consider $P \implies Q$. Then, prove $\neg Q \implies \neg P$.
3. **Contradiction**: Consider P. Assume otherwise and show it leads to a contradiction.
4. **Induction**: Use a base case, inductive hypothesis, and inductive step.

What is **strong induction**? As it turns out, strong induction is simply changing our inductive hypothesis: assume the *first* k are true, instead of assuming only the *previous* $(k-1)$*th* is true.

How do know when to **strengthen our inductive hypothesis**? In general, strengthen when you're stuck. (e.g., instead of proving $a_{i+1} = a_i^2 - a_i + 1 \geq 0$, where $a_0 = 1$, prove $0 \leq a_i^2 - 2a_i + 1$. Hint: How to simplify $a_i^2 - 2a_i + 1$?)

2 Fibonacci Proof

Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2} = F_{i+1} + F_i$ and $F_0 = 0$, $F_1 = 1$. Prove that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

Solution:

We proceed by induction on n.

Base case: $\sum_{i=0}^{0} F_i^2 = F_0^2 = 0 = F_0 F_1$.

Inductive hypothesis: Assume $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.

Inductive step: We have

$$\begin{aligned}
\sum_{i=0}^{n+1} F_i^2 &= F_{n+1}^2 + \sum_{i=0}^{n} F_i^2 \\
&= F_{n+1}^2 + F_n F_{n+1} \\
&= F_{n+1}(F_n + F_{n+1}) \\
&= F_{n+1} F_{n+2}
\end{aligned}$$

where the second equality is the inductive hypothesis and the last equality is the definition of the Fibonacci numbers.

3 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

(b) For all positive integers n, $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$.

(c) For all natural numbers n, $(5/4)8^n + 3^{3n-1}$ is divisible by 19.

Solution:

(a) The inequality is true for $n = 3$ because $8 > 7$. Let the inequality be true for $n = m$. Then,

$$2^{m+1} = 2 \cdot 2^m > 2 \cdot (2m+1) = 4m+2.$$

Since $2m > 1$, we have $4m+2 > 2m+3$, which completes the inductive step.

(b) For $n = 1$, the statement is $1 = 1$, which is true. Assume that it holds for $n = m$. Then,

$$\begin{aligned}\sum_{k=1}^{m+1} (2k-1)^3 &= \sum_{k=1}^{m} (2k-1)^3 + (2m+1)^3 = m^2(2m^2-1) + (2m+1)^3 \\ &= 2m^4 + 8m^3 + 11m^2 + 6m + 1 = (m+1)^2(2(m+1)^2-1).\end{aligned}$$

(c) For $n = 1$, the statement is "$10+9$ is divisible by 19", which is true. Assume that it holds for $n = m$. Then,

$$8 \cdot \frac{5}{4} \cdot 8^m + 27 \cdot 3^{3m-1} = 8\left(\frac{5}{4} \cdot 8^m + 3^{3m-1}\right) + 19 \cdot 3^{3m-1}.$$

The first term is divisible by the inductive hypothesis and the second term is clearly divisible by 19.

4 Seating Arrangement

N people have come to watch a play and were given a row with exactly N consecutive seats. They have decided on the following seating arrangement. After the first person sits down, the next person has to sit next to the first. The third sits next to one of the first two and so on until all N are seated. In other words, no person can take a seat that separates him/her from at least one other person. How many different ways can this be accomplished? Note that the first person can choose any of the N seats. (Hint: Use induction.)

Solution:

This is to introduce students to induction and counting. If there is just a one person and one seat, that person has only one option. If there are two persons and two seats, it can be accomplished in 2 different ways. If there are three persons and three seats, it can be accomplished in 4 different ways. Remember that no person can take a seat that separates him/her from at least one other person. Similarly, four persons and four seats produce 8 different ways. And five persons with five seats produce 16 different ways. It can be seen that with each additional person and seat, the different ways increase by the power of two. For any number N, the different possible ways are $2^{(N-1)}$.

5 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every natural number n.

1. Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.
2. Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

Solution:

1. Try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

   Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.

   Inductive Step: Assuming the statement is true for an n, we have

   $$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

   However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

2. This time the induction works.

   Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.

   Inductive Step: Assuming the hypothesis holds for n, we get

   $$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n - 1})^2 = 3 \times 3^{2 \times (2^n - 1)} = 3 \times 3^{2^{n+1} - 2} = 3^{2^{n+1} - 1}.$$

   This is exactly the induction hypothesis for $n+1$. Note that for every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n - 1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved here does indeed imply what we wanted to prove in the previous part.

6 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

Solution:

Prove by strong induction on n. Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.

- *Base Case:* $n = 1$ can be written with 1×2^0.
- *Inductive Hypothesis:* Assume that the statement is true for all $1 \leq k \leq n$.
- *Inductive Step:* If $n+1$ is divisible by 2, then it can use the representation of $(n+1)/2$.

  $$\begin{aligned} (n+1)/2 &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0 \\ n+1 = 2 \cdot (n+1)/2 &= c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \ldots + c_1 \cdot 2^2 + c_0 \cdot 2^1. \end{aligned}$$

  Otherwise, n must be divisible by 2 and have $c_0 = 0$. We can obtain the representation of $n+1$ from n.

  $$\begin{aligned} n &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 0 \cdot 2^0 \\ n+1 &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 1 \cdot 2^0 \end{aligned}$$

  Therefore, the statement is true.