CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand HW 8

Due Thursday March 17th at 10PM

1. **Independence**

   2 points per sub problem. 16 points total.

   (a) **Independence (due to H.W. Lenstra)**

   Suppose we pick a random card from a standard deck of 52 playing cards. Let A represent the event that the card is an queen, B the event that the card is a spade, and C the event that a red card (a heart or a diamond) is drawn.

   i. Which two of A, B, and C are independent? Justify your answer carefully. (In other words: For each pair of events (AB, AC, and BC), state and prove whether they are independent or not.)

   Answer: A and B are independent, since $\Pr[A \cap B] = 1/52 = 1/13 \times 1/4 = \Pr[A]\Pr[B]$.
   A and C are independent, since $\Pr[A \cap C] = 2/52 = 1/13 \times 2/4 = \Pr[A]\Pr[C]$.
   B and C are *not* independent, since $\Pr[B \cap C] = 0 \neq 1/4 \times 2/4 = \Pr[B]\Pr[C]$.

   ii. What if a joker is added to the deck? Justify your answer carefully.

   Answer: Let A', B', C' denote the corresponding events when a joker is added to the deck. I assume that the joker has neither suit, rank, nor color, so that the joker is neither a queen, a spade, nor a red card. Then:
   A' and B' are *not* independent, since $\Pr[A' \cap B'] = 1/53 \neq 4/53 \times 13/53 = \Pr[A']\Pr[B']$.
   A' and C' are *not* independent, since $\Pr[A' \cap C'] = 2/53 \neq 4/53 \times 26/53 = \Pr[A']\Pr[C']$.
   B' and C' are *not* independent, since $\Pr[B' \cap C'] = 0 \neq 13/53 \times 26/53 = \Pr[B']\Pr[C']$.

   (b) **Independence (due to H.W. Lenstra)**

   Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two *independent* events. Let $\overline{A} = \Omega - A$ and $\overline{B} = \Omega - B$ (sometimes written $\neg A$ and $\neg B$) denote the complementary events.

   For the purposes of this question, you may use the following definition of independence: Two events A, B are *independent* if $\Pr[A \cap B] = \Pr[A]\Pr[B]$.

   i. Prove or disprove: $\overline{A}$ and $\overline{B}$ are necessarily independent.

Answer: True. $\overline{A}$ and $\overline{B}$ must be independent:

$$\begin{aligned}
\Pr[\overline{A} \cap \overline{B}] &= \Pr[\overline{A \cup B}] && \text{(by De Morgan's law)} \\
&= 1 - \Pr[A \cup B] && \text{(since } \Pr[\overline{E}] = 1 - \Pr[E] \text{ for all } E\text{)} \\
&= 1 - (\Pr[A] + \Pr[B] - \Pr[A \cap B]) && \text{(union of overlapping events)} \\
&= 1 - \Pr[A] - \Pr[B] + \Pr[A]\Pr[B]) \\
& && \text{(using our assumption that } A \text{ and } B \text{ are independent)} \\
&= (1 - \Pr[A])(1 - \Pr[B]) \\
&= \Pr[\overline{A}]\Pr[\overline{B}] && \text{(since } \Pr[\overline{E}] = 1 - \Pr[E] \text{ for all } E\text{)}
\end{aligned}$$

ii. Prove or disprove: A and $\overline{B}$ are necessarily independent.

Answer: True. A and $\overline{B}$ must be independent:

$$\begin{aligned}
\Pr[A \cap \overline{B}] &= \Pr[A - (A \cap B)] \\
&= \Pr[A] - \Pr[A \cap B] \\
&= \Pr[A] - \Pr[A]\Pr[B] \\
&= \Pr[A](1 - \Pr[B]) \\
&= \Pr[A]\Pr[\overline{B}]
\end{aligned}$$

iii. Prove or disprove: A and $\overline{A}$ are necessarily independent.

Answer: False in general. If $0 < \Pr[A] < 1$, then $\Pr[A \cap \overline{A}] = \Pr[\emptyset] = 0$ but $\Pr[A]\Pr[\overline{A}] > 0$, so $\Pr[A \cap \overline{A}] \neq \Pr[A]\Pr[\overline{A}]$; therefore A and $\overline{A}$ are not independent in this case.

iv. Prove or disprove: It is possible that $A = B$.

Answer: True. To give one example, if $\Pr[A] = \Pr[B] = 0$, then $\Pr[A \cap B] = 0 = 0 \times 0 = \Pr[A]\Pr[B]$, so A and B are independent in this case. (Another example: If $A = B$ and $\Pr[A] = 1$, then A and B are independent.)

(c) **Bonferroni's inequalities**

i. For events A, B in the same probability space, prove that

$$\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1.$$

ii. Generalize part (a) to prove that, for events $A_1, \ldots, A_n$ in the same probability space (and any n),

$$\Pr[A_1 \cap \ldots \cap A_n] \geq \Pr[A_1] + \cdots + \Pr[A_n] - (n-1).$$

Answer:

i. To show this we use the Inclusion-Exclusion theorem. We have that for all events A and B,

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B].$$

Also, we know that the probability of any event is at most 1. Thus, $\Pr[A \cup B] \leq 1$. Using this with the Inclusion-Exclusion theorem, we get

$$\Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A \cup B] \geq \Pr[A] + \Pr[B] - 1.$$

ii. Now we generalize this result to show that:

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] \geq \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_n] - (n-1). \tag{1}$$

For this, we use induction on n.
The base case $n = 1$ just says that $\Pr[A_1] \geq \Pr[A_1] - 0$, which is trivially true.
For our inductive hypothesis, we assume that equation (1) holds for some arbitrary n and any n events.
The inductive step, therefore, is to show that it holds for $n+1$. In other words we need to show:

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}] \geq \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_n] + \Pr[A_{n+1}] - n. \tag{2}$$

Now, let B denote the event $A_n \cap A_{n+1}$. Then we have, by the inductive hypothesis applied to the n events $A_1, A_2, \ldots, A_{n-1}, B$,

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B] \geq \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_{n-1}] + \Pr[B] - (n-1). \tag{3}$$

However, from our proof for the case $n = 2$ in part (a) we have:

$$\Pr[B] = \Pr[A_n \cap A_{n+1}] \geq \Pr[A_n] + \Pr[A_{n+1}] - 1.$$

Substituting this into equation (3) gives us our desired equation (2), which completes the induction proof.

2. **(1/2/2) Cliques in random graphs**

Consider a graph $G(V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G(V,E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two ianswerated vertices.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?

(c) Prove that the probability that the graph contains a k-clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$.

Answer:

(a) There are two choices for each of the $\binom{n}{2}$ pairs of vertices, so the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a xed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\Pr\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \Pr[A_S] = \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{\frac{k(k-1)}{2}}} \le \frac{n^k}{\left(2^{\frac{(k-1)}{2}}\right)^k} \le \frac{n^k}{\left(2^{\frac{(4\log n+1-1)}{2}}\right)^k} = \frac{n^k}{(2^{2\log n})^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \le \frac{1}{n}.$$

3. **(1/2/2) College applications**

There are n students applying to n colleges. Each college has a ranking over all students (i.e. a permutation) which, for all we know, is completely random and independent of other colleges.

College number i will admit the first k_i students in its ranking. If a student is not admitted to any college, he or she might le a complaint against the board of colleges, and colleges want to avoid that as much as possible.

a) If for all i, $k_i = 1$, i.e. if every college only admits the top student on its list, what is the chance that all students will be admitted to at least one college?

Answer: If we consider the first choices of all colleges, there are n^n different possibilities, all of which are equally likely because colleges are indepently sorting students in a random manner. Out of these we want the possibilities that have all students covered, which is the same as those that have no repeated student (because the number of colleges is the same as the number of students). So we are counting permutations, and we know that there are $n!$ of them. So the probability is $\frac{n!}{n^n}$.

b) What is the chance that a particular student, Alice, does not get admitted to any college? Prove that if the average of all k_i's is $2\ln n$, then this probability is at most $1/n^2$. (Hint: use the inequality $1 - x < e^{-x}$)

Answer: The chance that Alice does not get admitted to college i is $1 - \frac{k_i}{n}$. This is because out of all the $n!$ permutations that college i can have on students $k_i \times (n-1)!$ of them result in Alice being one of the top k_i (we first choose Alice's place and then randomly permute the remaining students). So the probability that Alice ends up in the top k_i is k_i/n and the probability that she does not is $1 - \frac{k_i}{n}$.
The probability that she does not get admitted to any college is just

$$\prod_{i=1}^{n}(1 - \frac{k_i}{n})$$

Now using the inequality $1 - x \le e^{-x}$, we get $1 - \frac{k_i}{n} \le e^{-k_i/n}$. Multiplying over all i we get

$$\prod_{i=1}^{n}(1 - \frac{k_i}{n}) < \prod_{i=1}^{n} e^{-k_i/n} = e^{-\sum_{i=1}^{n} k_i/n}$$

But $\sum_{i=1}^{n} k_i/n$ is simply the average of all k_i. If this average is $2\ln n$, the last expression simply reduces to $e^{-2\ln n}$ which is just $1/n^2$.

c) Prove that when the average k_i is $2\ln n$, then the probability that at least one student does not get admitted to any college is at most $1/n$. (Hint: use the union bound)

Answer: If A_i is the event that student i does not get admitted to any college is at most $1/n^2$ by the previous part. $\cup_{i=1}^{n} A_i$ is the event that at least one of the students does not get admitted to any college.

By using the union bound we get

$$\Pr[\cup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \Pr[A_i] \leq \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n}.$$

4. **Expressions**

- Each subpart is 1 point. 21 points total.
- For each problem, just write down a mathematical expression. There is no need to justify/explain/derive the answer.

(a) **Bayes Rule - Man Speaks Truth**

i. A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $\frac{1}{3}$ of the time and reports it's Heads. What is the probability it is Heads?

ii. A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided dice and reports it comes up 6. What is the probability it is really 6?

Answer:

i. Let E denotes the event the man reports heads, S_1 be the event that the coin comes up heads, and S_2 be the event that the coin comes up tails.
We have: $P(E|S_1) = \frac{3}{4}, P(E|S_2) = \frac{1}{4}, P(S_1) = \frac{1}{3}, P(S_2) = \frac{2}{3}$.
We want to compute $P(S_1|E)$, and let's do so by applying Bayes Rule.
$P(S_1|E) = \frac{P(S_1 E)}{P(E)} = \frac{P(E|S_1)P(S_1)}{P(E|S_1)P(S_1)+P(E|S_2)P(S_2)} = \frac{3/4 \cdot 1/3}{3/4 \cdot 1/3 + 1/4 \cdot 2/3} = \frac{3}{5}$.

ii. Let D be the event that the dice rolls a 6. Let M be the event that the man says 6.
$P(D|M) = \dfrac{P(D \wedge M)}{P(M)} = \dfrac{P(M|D)P(D)}{P(M|D)P(D) + P(M|\neg D)P(\neg D)} = \dfrac{3/4 * 1/6}{3/4 * 1/6 + 1/4 * 1/5 * 5/6} = \dfrac{3/24}{4/24} = \dfrac{3}{4}$

(b) **Unlikely events**

i. Toss a fair coin x times. What is the probability that you never get heads?
Answer: 0.5^x

ii. Roll a fair die x times. What is the probability that you never roll a six?
Answer: $(1 - \frac{1}{6})^x$

iii. Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?
Answer: $(1 - 1/10^6)^x$

iv. How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?
Answer: For coin, want: $0.5^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 0.5} \approx 3.32$

For die, want: $(5/6)^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 5/6} \approx 12.6$

For coin, want: $(1-1/10^6)^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 1-1/10^6} \approx 2*10^6$ Comment on how answer for coin is almost exactly equal to $\log 0.1/(1/10^6)$ using the approximation $(1-x) \approx e^{(-x)}, xbeing 1/10^6$

(c) **Blood Type**

Consider the three alleles, A, B, and O, for human blood types. As each person inherits one of the 3 alleles from each parent, there are 6 possible genotypes: AA, AB, AO, BB, BO, and OO. Blood groups A and B are dominant to O. Therefore, people with AA or AO have type A blood. Similarly, BB and BO result in type B blood. The AB genotype is called type AB blood, and the OO genotype is called type O blood. Each parent contributes one allele randomly. Now, suppose that the frequencies of the A, B, and O alleles are 0.4, 0.25, and 0.35, respectively, in Berkeley. Alice and Bob, two residents of Berkeley are married and have a daughter, Mary. Alice has blood type AB.

i. What is the probability that Bob's genotype is AO?

Answer: Let B_{1A}, B_{1B} and B_{1O} be the events that Bob's first allele is A, B, and O, respectively. Let B_{2A}, B_{2B} and B_{2O} be the events that Bob's second allele is A, B, and O respectively. Bob's blood type can be AA, AB, AO, BB, or BO. Let B_{AA} be the event that Bob has type AA blood, B_{AB} be the event that Bob has type AB blood, B_{AO} be the event that Bob has type AO blood, B_{BB} be the event that Bob has type BB blood, and B_{BO} be the event that Bob has type BO blood. The sample space is $\Omega = \{B_{AA}, B_{AB}, B_{AO}, B_{BB}, B_{BO}\}$. Note that we have

$$\begin{aligned}
B_{AA} &= B_{1A} \cap B_{2A} \\
B_{AB} &= (B_{1A} \cap B_{2B}) \cup (B_{1B} \cap B_{2A}) \\
B_{AO} &= (B_{1A} \cap B_{2O}) \cup (B_{1O} \cap B_{2A}) \\
B_{BB} &= B_{1B} \cap B_{2B} \\
B_{BO} &= (B_{1B} \cap B_{2O}) \cup (B_{1O} \cap B_{2B}).
\end{aligned}$$

Since the first allele and second allele don't know about each other, the occurrence of the first allele will not affect the second, and vice versa. Therefore, using the rules that $P(A \cap B) = P(A|B)P(B)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

$$\begin{aligned}
P(B_{AA}) &= P(B_{1A}|B_{2A})P(B_{2A}) = P(B_{1A})P(B_{2A}) = (.4)(.4) = .16 \\
P(B_{AB}) &= P(B_{1A}|B_{2B})P(B_{2B}) + P(B_{1B}|B_{2A})P(B_{2A}) - P((B_{1A} \cap B_{2B}) \cap (B_{1B} \cap B_{2A})) \\
&= P(B_{1A})P(B_{2B}) + P(B_{1B})P(B_{2A}) = 2(.4)(.25) = .2 \\
P(B_{AO}) &= P(B_{1A}|B_{2O})P(B_{2O}) + P(B_{1O}|B_{2A})P(B_{2A}) - P((B_{1A} \cap B_{2O}) \cap (B_{1O} \cap B_{2A})) \\
&= P(B_{1A})P(B_{2O}) + P(B_{1O})P(B_{2A}) = 2(.4)(.35) = .28 \\
P(B_{BB}) &= P(B_{1B}|B_{2B})P(B_{2B}) = P(B_{1B})P(B_{2B}) = (.25)(.25) = .0625 \\
P(B_{BO}) &= P(B_{1B}|B_{2O})P(B_{2O}) + P(B_{1O}|B_{2B})P(B_{2B}) - P((B_{1B} \cap B_{2O}) \cap (B_{1O} \cap B_{2B})) \\
&= P(B_{1B})P(B_{2O}) + P(B_{1O})P(B_{2B}) = 2(.25)(.35) = .175.
\end{aligned}$$

Therefore $P(B_{AO}) = .28$.

ii. Assume that Bob's genotype is AO. What is the probability that Mary's blood type is AB?

Answer: Since Alice has type AB and Bob has type AO, the sample space of possible genotypes for Mary is $\{AA, AO, AB, BO\}$. Since there is uniform probability of inheriting either allele from a given parent, there is a $1/4$ chance that Mary will have type AB blood.

iii. Assume Mary's blood type is AB. What is the probability that Bob's genotype is AA?

Answer: Bob's blood type can be AA, AB, AO, BB, or BO. As in part (a), let B_{AA} be the event that Bob has type AA blood, B_{AB} be the event that Bob has type AB blood, B_{AO} be the event that Bob has type AO blood, B_{BB} be the event that Bob has type BB blood, and B_{BO} be the event that Bob has type BO blood. We already computed the probability of Bob having these blood types in part (a):

$$\begin{aligned}
Pr(B_{AA}) &= .16\\
Pr(B_{AB}) &= .2\\
Pr(B_{AO}) &= .28\\
Pr(B_{BB}) &= .0625\\
Pr(B_{BO}) &= .175.
\end{aligned}$$

Now, let the event that Mary has blood type AB be M_{AB}. The problem asks us to nd $Pr(B_{AA}|M_{AB})$. We can compute this using Bayes' formula, which says that

$$Pr(B_{AA}|M_{AB}) = \frac{Pr(M_{AB}|B_{AA}) \cdot Pr(B_{AA})}{Pr(M_{AB})}.$$

To nd $Pr(M_{AB})$, we can use the Law of Total Probability, which says that

$$\begin{aligned}
Pr(M_{AB}) = {} & Pr(M_{AB}|B_{AA}) \cdot Pr(B_{AA}) + Pr(M_{AB}|B_{AB}) \cdot Pr(B_{AB}) + Pr(M_{AB}|B_{AO}) \cdot Pr(B_{AO})\\
& + Pr(M_{AB}|B_{BB}) \cdot Pr(B_{BB}) + Pr(M_{AB}|B_{BO}) \cdot Pr(B_{BO}).
\end{aligned}$$

To calculate this, we must nd the conditional probabilities that Mary has AB blood given Bob's blood type. Recall that Alice has type AB blood.

- If Bob has AA blood, the possible combinations of their alleles are AA, AA, AB, and AB, so $Pr(M_{AB}|B_{AA}) = 1/2$.
- If Bob has AB blood, the possible combinations of their alleles are AA, AB, AB, and BB, so $Pr(M_{AB}|B_{AB}) = 1/2$.
- If Bob has AO blood, the possible combinations of their alleles are AA, AO, AB, and BO, so $Pr(M_{AB}|B_{AO}) = 1/4$.
- If Bob has BB blood, the possible combinations of their alleles are AB, AB, BB, and BB, so $Pr(M_{AB}|B_{BB}) = 1/2$.
- If Bob has BO blood, the possible combinations of their alleles are AB, AO, BB, and BO, so $Pr(M_{AB}|B_{BO}) = 1/4$.

We now have all the information we need to plug in and answerve. By the Law of Total Probability above, we have

$$\begin{aligned}
Pr(M_{AB}) &= Pr(M_{AB}|B_{AA}) \cdot Pr(B_{AA}) + Pr(M_{AB}|B_{AB}) \cdot Pr(B_{AB}) + Pr(M_{AB}|B_{AO}) \cdot Pr(B_{AO})\\
&\quad + Pr(M_{AB}|B_{BB}) \cdot Pr(B_{BB}) + Pr(M_{AB}|B_{BO}) \cdot Pr(B_{BO})\\
&= (.5)(.16) + (.5)(.2) + (.25)(.28) + (.5)(.0625) + (.25)(.175)\\
&= .325,
\end{aligned}$$

and plugging in to Bayes' formula, we nd that

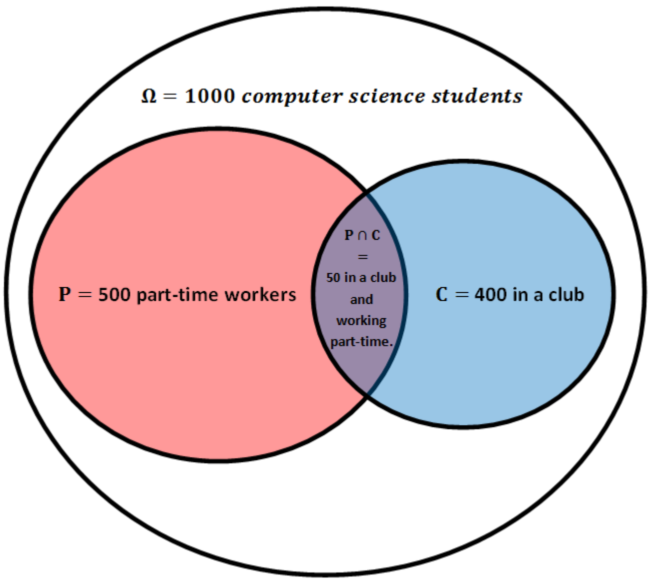
$$Pr(B_{AA}|M_{AB}) = \frac{Pr(M_{AB}|B_{AA}) \cdot Pr(B_{AA})}{Pr(M_{AB})} = \frac{(.5)(.16)}{.325} = .246.$$

(d) **Simple probability**

Out of 1000 sophomore EECS students, 400 are taking CS70 (and may concurrently take CS61C), 500 are taking CS61C (and may concurrently take CS70), and 50 are taking both CS70 and CS61C.

i. Suppose we choose a student uniformly at random. Let C be the event that the student takes CS70 and P the event that the student takes CS61C. Draw a picture of the sample space Ω and the events C and P.

Answer: The following is the sample space. (Belong to a club = taking CS70, work part time = taking CS61C)



ii. What is the probability that the student takes CS70?

Answer: $\Pr[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \boxed{\mathbf{.4}}$

iii. What is the probability that the student takes CS61C?

Answer: $\Pr[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \boxed{\mathbf{.5}}$

iv. What is the probability that the student takes CS70 AND CS61C?

Answer: $\Pr[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \boxed{\mathbf{.05}}$

v. What is the probability that the student takes CS70 OR CS61C?

Answer: $\Pr[P \cup C] = \Pr[P] + \Pr[C] - \Pr[P \cap C] = \boxed{\mathbf{.85}}$. This answer also comes from something called the Inclusion-exclusion principle.

(e) **Roll Dice**

You roll three fair six-sided dice. What is the probability of rolling a triple (all three dice agree)? What is the probability of rolling a double (two of the dice agree with each other)?

Answer: The sample space Ω consists of all possible outcomes of rolling 3 dies. Therefore, the size of it is: $|\Omega| = 6^3$. Let A be the event of rolling a triple, B be the event of rolling a double. The size of A is 6 since A consists of three ones', three twos', etc.

$$P[\text{rolling a triple}] = \frac{|A|}{|\Omega|} = \frac{6}{6^3}$$

The size of B is $6 \cdot 5 \cdot \frac{3!}{2!1!}$ because you have 6 ways to choose a number that appears twice in the roll, 5 ways to choose a number that is different the previous number and appears once in the roll. And you have $\frac{3!}{2!1!}$ possible different arrangement for two identical number and one distinct number. So we have

$$P[\text{rolling a double}] = \frac{|B|}{|\Omega|} = \frac{6\cdot 5\cdot 3}{6^3} = \frac{5}{12}.$$

(f) **Lie Detector**

A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

Answer: Let A denote the event that the test indicates that the person is guilty, and B the event that the person is innocent. Note that

$$\Pr[B] = 0.99, \quad \Pr[\overline{B}] = 0.01, \quad \Pr[A \mid B] = 0.05, \quad \Pr[A \mid \overline{B}] = 0.8$$

Using the Bayesian Inference Rule, we can compute the desired probability as follows:

$$\Pr[B \mid A] = \frac{\Pr[B]\Pr[A \mid B]}{\Pr[B]\Pr[A \mid B] + \Pr[\overline{B}]\Pr[A \mid \overline{B}]} = \frac{0.99\cdot 0.05}{0.99\cdot 0.05 + 0.01\cdot 0.8} \approx 0.86$$

(g) **Rain and Wind**

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS70, you are curious to play around with these numbers. Find the probability that

i. A given day is windy and rainy.

Answer: Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $P(R|W) = 0.3$, $P(R|W^C) = 0.8$ and $P(W) = 0.2$. Then probability that a given day is both rainy and windy is $P(R\cap W) = P(R|W)P(W) = 0.3\times 0.2 = 0.06$

ii. It rains on a given day. **Answer:** Probability that it rains on a given day is $P(R) = P(R|W)P(W) + P(R|W^C)P(W^C) = 0.3\times 0.2 + 0.7\times 0.8 = 0.62$

iii. Exactly one of any two days is rainy. **Answer:** Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $P(R_1) = P(R_2) = P(R)$. The required probability is $P(R_1)P(R_2^C) + P(R_1^C)P(R_2) = 2*0.62*0.38 = 0.4712$

iv. A non-rainy day is also non-windy. **Answer:** Probability that a non-rainy day is non-windy is $P(W^C|R^C) = \frac{P(W^C\cap R^C)}{P(R^C)} = \frac{P(R^C|W^C)(W^C)}{P(R^C)} = \frac{0.2\times 0.8}{0.38} = \frac{8}{19}$

(h) **Chess Squares**

Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

Answer: In 64 squares, there are:

(1) 4 at-corner squares, each has ONLY 2 squares each having a side in common with.

(2) $6\cdot 4 = 24$ side squares, each has ONLY 3 squares such that each has a side in common with.

(3) $6\cdot 6 = 36$ inner squares, each has 4 squares such that each has a side in common with.

Notice that the three cases are mutually exclusive. So we just sum up the probabilities.

$\frac{4}{64}\cdot\frac{2}{63} + \frac{24}{64}\cdot\frac{3}{63} + \frac{36}{64}\cdot\frac{4}{63} = \frac{1}{18}$

5. **Short Answers**

- Each subpart is 2 point. 20 points total.
- For each problem, briefly justify your answer.

(a) For any probability space, show that $Pr[A \setminus B] \geq Pr[A] - Pr[B]$.

Answer:

$$\begin{aligned} RHS &= P[A] - P[B] && (4) \\ &= (P[A \cap B] + P[A \setminus B]) - (P[A \cap B] + P[B \setminus A]) && (5) \\ &= P[A \setminus B] - P[B \setminus A] && (6) \\ &\leq P[A \setminus B] && (7) \\ & && (8) \end{aligned}$$

(b) Show that $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$.

Answer:

$$\begin{aligned} RHS &= (P[A \setminus B] + P[A \cap B]) + (P[B \setminus A] + P[A \cap B]) - (P[A \setminus B] + P[B \setminus A] + P[A \cap B]) && (9) \\ &= P[A \cap B] && (10) \end{aligned}$$

(c) Assume that $|\Omega| = n$. How many distinct events does the probability space have?

Answer: There's a bijection between events and binary strings of length n. Thus, there are 2^n possible events.

(d) Assume that $|\Omega| = n$. What is the maximum number of distinct values of $Pr[A]$ can one have for events of the probability space?

Answer:

Given there are at most 2^n events, we can not achieve more than 2^n values.

We can construct 2^n values as follows:

Let the points be $p_0, \ldots, p_{n-1}$ where $P[p_i] = \frac{2^i}{2^n - 1}$.

Then, for all $0 \leq k \leq 2^n - 1$, to achieve $\frac{k}{2^n - 1}$, construct a set S_k where $p_i \in S_k$ iff the binary representation of k has a 1 at bit i.

(e) Can you find a probability space and two events A and B such that $Pr[A|B] = Pr[B]$ and A and B are not independent?

Answer:

Let $\Omega = \{1,2,3,4\}$.

Let $A = \{1\}$.

Let $B = \{1,2\}$.

$P[A|B] = 1/2 = P[B]$.

A and B are not independent: $P[A \cap B] = 1/4 \neq P[A] * P[B]$.

(f) Prove that $Pr[A \cap B \cap C] = Pr[A]Pr[B|A]Pr[C|A \cap B]$.

Answer:

$P[A \cap B \cap C] = P[C|A \cap B]P[A \cap B] = P[C|A \cap B]P[B|A]P[A]$

(g) Find an example where $Pr[A \cap B \cap C] \neq Pr[A]Pr[B|A]Pr[C|B]$.

Answer:

Let $\Omega = \{1,2\}$.
Let $B = \{1,2\}$.
Let $A = \{1\}$.
Let $C = \{2\}$.
LHS is 0. RHS is 1/4.

(h) Find an example where $Pr[A|B] > A$, another where $Pr[A|B] < Pr[A]$, and one where $Pr[A|B] = Pr[A]$.

Answer:

$Pr[A|B] > Pr[A]$: $\Omega = \{1,2\}$. $A = \{1\}$, $B = \{1\}$.
$Pr[A|B] < Pr[A]$: $\Omega = \{1,2\}$. $A = \{1\}$, $B = \{2\}$.
$Pr[A|B] = Pr[A]$: $\Omega = \{1,2\}$. $A = \{1\}$, $B = \{1,2\}$.

(i) Can you find an example where $Pr[A] > Pr[B]$ and $Pr[A|C] < Pr[B|C]$?

Answer:

```
Omega = {1, 2, 3}
A = {1, 2}
B = {3}
C = {3}
```

(j) Can you find an example where $Pr[A] > Pr[B]$ and $Pr[C|A] < Pr[C|B]$?

Answer:

```
Omega = {1, 2, 3}
A = {1, 2}
B = {3}
C = {3}
```