CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Walrand and Rao HW 11

Due Thursday April 14 at 10PM

Before you start your homework, write down your team. Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

1. **Variance**

   This problem will give you practice using the "standard method" to compute the variance of a sum of random variables that are not pairwise independent (so you cannot use "linearity" of variance). If you don't even know what is "linearity" of variance, read the lecture note and slides first.

   (a) (5 points) A building has n floors numbered $1,2,\ldots,n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the *variance* of the number of floors the elevator *does not* stop at? (In fact, the variance of the number of floors the elevator *does* stop at must be the same (do you see why?) but the former is a little easier to compute.)

   (b) (5 points) A group of three friends has n books they would all like to read. Each friend (independently of the other two) picks a random permutation of the books and reads them in that order, one book per week (for n consecutive weeks). Let X be the number of weeks in which all three friends are reading the same book. Compute $\text{Var}(X)$.

   Answer:

   (a) Let X be the number of floors the elevator does not stop at. As in the previous homework, we can represent X as the sum of the indicator variables $X_1,\ldots,X_n$, where $X_i = 1$ if no one gets off on floor i. Thus, we have

   $$\mathbb{E}(X_i) = \Pr[X_i = 1] = \left(\frac{n-1}{n}\right)^m,$$

   and from linearity of expectation,

   $$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n\left(\frac{n-1}{n}\right)^m.$$

   To find the variance, we cannot simply sum the variance of our indicator variables. However, we can still compute $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ directly using linearity of expectation, but now how can we find $\mathbb{E}(X^2)$? Recall that

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}((X_1 + \ldots + X_n)^2) \\
&= \mathbb{E}(\sum_{i,j} X_i X_j) \\
&= \sum_{i,j} \mathbb{E}(X_i X_j) \\
&= \sum_{i}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j).
\end{aligned}$$

The first term is simple to calculate: $\mathbb{E}(X_i^2) = 1^2 \Pr[X_i = 1] = \left(\frac{n-1}{n}\right)^m$, meaning that

$$\sum_{i=1}^{n} \mathbb{E}(X_i^2) = n \left(\frac{n-1}{n}\right)^m.$$

$X_i X_j = 1$ when both X_i and X_j are 1, which means no one gets off the elevator on floor i and floor j. This happens with probability

$$\Pr[X_i = X_j = 1] = \Pr[X_i = 1 \cap X_j = 1] = \left(\frac{n-2}{n}\right)^m.$$

Thus, we can now compute

$$\sum_{i \neq j} \mathbb{E}(X_i X_j) = n(n-1) \left(\frac{n-2}{n}\right)^m.$$

Finally, we plug in to see that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = n \left(\frac{n-1}{n}\right)^m + n(n-1) \left(\frac{n-2}{n}\right)^m - \left(n \left(\frac{n-1}{n}\right)^m\right)^2.$$

(b) Let $X_1, \ldots, X_n$ be indicator variables such that $X_i = 1$ if all three friends are reading the same book on week i. Thus, we have

$$\mathbb{E}(X_i) = \Pr[X_i = 1] = \left(\frac{1}{n}\right)^2,$$

and from linearity of expectation,

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n \left(\frac{1}{n}\right)^2 = \frac{1}{n}.$$

As before, we know that

$$\mathbb{E}(X^2) = \sum_{i}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j).$$

Furthermore, because X_i is an indicator variable, $\mathbb{E}(X_i^2) = 1^2 \Pr[X_i = 1] = \left(\frac{1}{n}\right)^2$, and

$$\sum_{i}^{n} \mathbb{E}(X_i^2) = n \left(\frac{1}{n}\right)^2 = \frac{1}{n}.$$

Again, because X_i and X_j are indicator variables, we are interested in

$$\Pr[X_i = X_j = 1] = \Pr[X_i = 1 \cap X_j = 1] = \frac{1}{(n(n-1))^2},$$

the probability that all three friends pick the same book on week i and week j. Thus,

$$\sum_{i \neq j} \mathbb{E}(X_i X_j) = n(n-1)\left(\frac{1}{(n(n-1))^2}\right) = \frac{1}{n(n-1)}.$$

Finally, we compute

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{n} + \frac{1}{n(n-1)} - \left(\frac{1}{n}\right)^2.$$

2. **Coupon Collection (10 points)**

Suppose you take a deck of n cards and repeatedly perform the following step: take the current top card and put it back in the deck at a uniformly random position. (I.e., the probability that the card is placed in any of the n possible positions in the deck — including back on top — is $1/n$.) Consider the card that starts off on the bottom of the deck. What is the expected number of steps until this card rises to the top of the deck? (Hint: Let T be the number of steps until the card rises to the top. We have $T = T_n + T_{n-1} + \cdots + T_2$, where the random variable T_i is the number of steps until the bottom card rises from position i to position $i-1$. Thus, for example, T_n is the number of steps until the bottom card rises off the bottom of the deck, and T_2 is the number of steps until the bottom card rises from second position to top position. What is the distribution of T_i?) (More hints: You may use the fact that $\sum_{i=1}^{n} \frac{1}{i} \approx \ln n$.)

Answer: Since a card at location i moves to location $i-1$ when the current top card is placed in any of the locations $i, i+1, \ldots, n$, it will rise with probability $p = \frac{n-i+1}{n}$. Thus, $T_i \sim \mathrm{Geom}(p)$, and $\mathbb{E}(T_i) = \frac{1}{p} = \frac{n}{n-i+1}$. We now can see how this is exactly the coupon collector's problem, but with one fewer term (namely, without T_1). Finally, we can apply linearity of expectation to compute

$$\mathbb{E}(T) = \sum_{i=2}^{n} \mathbb{E}(T_i) = \sum_{i=2}^{n} \frac{n}{n-i+1} = n \sum_{i=2}^{n} \frac{1}{n-i+1} \approx n \ln(n-1)$$

3. **Markov's Inequality and Chebyshev's Inequality**

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}(X) = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) (3 points) $\mathbb{E}(X^2) = 13$.

(b) (3 points) $\Pr[X = 2] > 0$.

(c) (3 points) $\Pr[X \geq 2] = \Pr[X \leq 2]$.

(d) (3 points) $\Pr[X \leq 1] \leq 8/9$.

(e) (3 points) $\Pr[X \geq 6] \leq 9/16$.

(f) (3 points) $\Pr[X \geq 6] \leq 9/32$.

Answer:

(a) TRUE. Since $9 = \text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - 2^2$, we have $\mathbb{E}(X^2) = 9 + 4 = 13$.

(b) FALSE. Construct a random variable X that satisfies the conditions in the question but does not take on the value 2. A simple example would be a random variable that takes on 2 values, where $\Pr[X = a] = \frac{1}{2}, \Pr[X = b] = \frac{1}{2}$, and $a \neq b$. The expectation must be 2, so we have $\frac{1}{2}a + \frac{1}{2}b = 2$. The variance is 9, so $\mathbb{E}(X^2) = 13$ (from part (a)) and $\frac{1}{2}a^2 + \frac{1}{2}b^2 = 13$. Solving for a and b, we get $\Pr[X = -1] = \frac{1}{2}, \Pr[X = 5] = \frac{1}{2}$ as a counterexample.

(c) FALSE. Construct a random variable X that satisfies the conditions in the question but does not have an equal chance of being less than or greater than 2. A simple example would be a random variable that takes on 2 values, where $\Pr[X = a] = p, \Pr[X = b] = 1 - p$. Here, we use the same approach as part (b) except with a generic p, since we want $p \neq \frac{1}{2}$. The expectation must be 2, so we have $pa + (1 - p)b = 2$. The variance is 9, so $\mathbb{E}(X^2) = 13$ and $pa^2 + (1 - p)b^2 = 13$. Solving for a and b, we find the relation $b = 2 \pm \frac{3}{\sqrt{x}}$, where $x = \frac{1-p}{p}$. Then, we can find an example by plugging in values for x so that $a, b \leq 10$ and $p \neq \frac{1}{2}$. One such counterexample is $\Pr[X = -7] = \frac{1}{10}, \Pr[X = 3] = \frac{9}{10}$.

(d) TRUE. Let $Y = 10 - X$. Since X is never exceeds 10, Y is a non-negative random variable. By Markov's inequality,

$$\Pr[10 - X \geq a] = \Pr[Y \geq a] \leq \frac{\mathbb{E}(Y)}{a} = \frac{\mathbb{E}(10 - X)}{a} = \frac{8}{a}.$$

Setting $a = 9$, we get $\Pr[X \leq 1] = \Pr[10 - X \geq 9] \leq \frac{8}{9}$.

(e) TRUE. Chebyshev's inequality says $\Pr[|X - \mathbf{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$. If we set $a = 4$, we have

$$\Pr[|X - 2| \geq 4] \leq \frac{9}{16}.$$

Now we simply observe that $\Pr[X \geq 6] \leq \Pr[|X - 2| \geq 4]$, because the event $X \geq 6$ is a subset of the event $|X - 2| \geq 4$.

(f) FALSE. We use the same approach as in part (c), except we find a counterexample that fits the inequality $\Pr[X \geq 6] \leq 9/32$. One example is $\Pr[X = 0] = \frac{9}{13}, \Pr[X = \frac{13}{2}] = \frac{4}{13}$.

4. **Umbrella Store**

Bob has a store that sells umbrellas. The number of umbrellas that Bob sells on a rainy day is a random variable Y with mean 25 and standard deviation $\sqrt{105}$. But if it is a clear day, Bob doesn't sell any umbrellas at all. The weather forecast for tomorrow says it will rain with probability $\frac{1}{5}$. Let Z be the number of umbrellas that Bob sells tomorrow.

(a) (3 points) Let X be an indicator random variable that it will rain tomorrow. Write Z in terms of X and Y.

(b) (4 points) What is the mean and standard deviation of Z?

(c) (3 points) Use Chebyshev's inequality to bound the probability that Bob sells at least 25 umbrellas tomorrow.

Answer:

(a) $Z = XY$.

(b) We have

$$\mathbb{E}(Z) = \Pr[X=1]\cdot\mathbb{E}(Y) + \Pr[X=0]\cdot 0 = \frac{1}{5}\mathbb{E}(Y) = \frac{1}{5}\cdot 25 = 5$$

and

$$\mathbb{E}(Z^2) = \Pr[X=1]\cdot\mathbb{E}(Y^2) + \Pr[X=0]\cdot 0 = \frac{1}{5}\mathbb{E}(Y^2) = \frac{1}{5}\cdot 730 = 146$$

since $\mathbb{E}(Y^2) = \mathrm{Var}(Y) + \mathbb{E}(Y)^2 = 105 + 625 = 730$. So

$$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = 146 - 25 = 121.$$

Therefore, the mean of Z is 5 and the standard deviation is $\sqrt{121} = 11$.

(c) Since $\mathbb{E}(Z) = 5$ and $\mathrm{Var}(Z) = 121$,

$$\Pr[Z \ge 25] \;=\; \Pr[Z - \mathbb{E}(Z) \ge 20] \;\le\; \Pr[|Z - \mathbb{E}(Z)| \ge 20] \;\le\; \frac{\mathrm{Var}(Z)}{400} \;=\; \frac{121}{400}.$$

5. **Casino wins**

A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability $1/6$, a hand of black jack with probability $1/2$, and a hand of stud poker with probability $1/5$. Assume the outcomes of the card games are mutually independent.

(a) (3 points) What is the expected number of hands the gambler wins in a day?

(b) (3 points) What is the variance in the number of hands won per day?

(c) (3 points) What would the Markov bound be on the probability that the gambler will win 108 hands on a given day?

(d) (3 points) What would the Chebyshev bound be on the probability that the gambler will win 108 hands on a given day?

Answer:

(a) Let R be the number of games won. By linearity of expectation:

$$\mathbf{E}[R] = 120\cdot\frac{1}{6} + 60\cdot\frac{1}{2} + 20\cdot\frac{1}{5} = 54.$$

(b) The variance can also be calculated using linearity of variance, since the $120+60+20 = 200$ indicator r.v.'s, one for each hand of some game, are mutually independent. For an individual hand, the variance is $p(1-p)$ where p is the probability of winning (it's just a single "Bernoulli trial"). Therefore, $\mathrm{Var}(R) = 120(1/6)(5/6) + 60(1/2)(1/2) + 20(1/5)(4/5) = 50/3 + 15 + 16/5 = 523/15 = 34\frac{13}{15}$.

(c) The expected number of games won is 54, and the number of games played is non-negative, so by Markov, $\Pr[R \ge 108] \le 54/108 = 1/2$.

(d) The Chebyshev bound yields:

$$\Pr[R - 54 \ge 54] \le \Pr[|R - 54| \ge 54] \le \frac{\mathrm{Var}(R)}{54^2} = \frac{523/15}{54^2} \le 0.012$$

Note that the first inequality in this case is actually an equality, but that's irrelevant here.

6. **Find the right key**
   A man has a set of n keys, one of which fits the door to his apartment. He tries the keys until he finds the correct one. Give the expected number and variance for the number of trials until success if

   (a) (6 points) he tries the keys at random (possibly repeating a key tried earlier).

   (b) (6 points) he chooses keys randomly from among those he has not yet tried.

   Answer:

   (a) There are $n-1$ wrong keys and 1 right key. Each time the man chooses a key, he tries it and then places it back into the key set. Let T be the waiting time for the man to pick the right key. $T = k$ means that on the k-th trial the man picks the right key. The probability of picking the right key on the first trial is $1/n$, and the probability of picking the wrong key is $(n-1)/n$.

   This is the same as number of trials until the first head comes up if you keep tossing a coin with probability $p = 1/n$ of coming up heads.

   Thus, T is a geometric random variable with parameter $p = 1/n$, so $\mathbf{E}[T] = 1/p = n$ and $\mathrm{Var}(T) = (1-p)/p^2 = \frac{1-1/n}{1/n^2} = n^2 - n$.

   (b) $T = k$ means that the man picks the wrong key on the first trial, and he picks the wrong key on the second trial, etc, and he picks the right key on the k-th trial. Let K_i be the indicator random variable for the i-th trial, i.e. $K_i = 1$ if he picks the right key on the i-th trial, and 0 otherwise. Then

$$\begin{aligned}\Pr[T=k] &= \Pr[(K_1=0)\wedge(K_2=0)\wedge\cdots\&(K_{k-1}=0)\wedge(K_k=1)]\\ &= \left(\prod_{i=1}^{k-1}\Pr[K_i=0|K_1=0\wedge\ldots\wedge K_{i-1}=0]\right)\cdot\Pr[K_k=1|K_1=0\wedge\ldots\wedge K_{k-1}=0]\end{aligned}$$

   Conditioned on the first $i-1$ trials having failed, the correct key must be among the remaining $n-i+1$ keys that haven't been tried yet, so the conditional probability of the i'th trial succeeding is $\frac{1}{n-i+1}$, and the conditional probability of it failing is $\frac{n-i}{n-i+1}$. So:

$$\begin{aligned}\Pr[T=k] &= \left(\prod_{i=1}^{k-1}\frac{n-i}{n-i+1}\right)\cdot\frac{1}{n-k+1}\\ &= \frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdot\frac{n-3}{n-2}\cdots\frac{n-k+1}{n-k+2}\cdot\frac{1}{n-k+1}\\ &= \frac{1}{n}.\end{aligned}$$

   A much simpler way to see that these probabilities are uniform is to consider the process as picking a permutation of keys uniformly at random from the set of all permutations, and having the man try keys in the permutation's order until he finds the right one. The probability that a random permutation of n objects has a specific object in position i is $\frac{(n-1)!}{n!} = \frac{1}{n}$.

   The expectation and variance are now easy to compute from the definitions.

$$\mathbf{E}[T] = \sum_{k=1}^{n} k \Pr[T=k] = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

$$\begin{aligned}
\mathrm{Var}(T) &= \mathbf{E}[T^2] - (\mathbf{E}[T])^2 \\
&= \sum_{k=1}^{n} k^2 \Pr[T=k] - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n}\sum_{k=1}^{n} k^2 - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{n^2-1}{12}.
\end{aligned}$$

7. **Extra Credit (1 point)**
   I think of two distinct real numbers between 0 and 1 but do not reveal them to you. I now choose one of the two numbers at random and give it to you. Can you give a procedure for guessing whether you were shown the smaller or the larger of the two numbers, such that your guess is correct with probability *strictly* greater than 0.5 (although exactly how much better than 0.5 may depend on the actual values of the two numbers?

   Answer: Let the two numbers be x and y and assume without loss of generality that $x > y$. Since the numbers are between 0 and 1, we can think of them as probabilities. Given a number, say, x, we say that it is the larger one with probability x.

   With probability 0.5, we get x, in which case we are correct with probability x.With probability 0.5, we get y, in which case we are correct with probability $1-y$. So in total we are correct with probability $\frac{x}{2} + \frac{1}{2}(1-y) = \frac{1}{2} + \frac{x-y}{2}$ which is a positive advantage since $x \neq y$.