CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao

DIS 2b

1 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learn from lecture note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies either of these two properties then it must be a tree:

(c) Prove that if every pair of vertices in a graph are connected by exactly one simple path, then the graph must be a tree.

(d) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

Solution:

(a) Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a simple cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a simple cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) Assume we have a graph with the property that there is a unique simple path between every pair of vertices. We will show that the graph is a tree, namely, it is connected and acyclic. First, the graph is connected because every pair of vertices is connected by a path. Moreover, the graph is acyclic because there is a unique path between every pair of vertices. More explicitly, if the graph has a cycle, then for any two vertices x, y in the cycle there are at least two simple paths between them (obtained by going from x to y through the right or left half of the cycle), contradicting the uniqueness of the path. Therefore, we conclude the graph is a tree.

(d) Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If $\{x,y\}$ is an edge, then clearly there is a path from x to y. Otherwise, if $\{x,y\}$ is not an edge, then by assumption, adding the edge $\{x,y\}$ will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge $\{x,y\}$ from this cycle. Therefore, we conclude the graph is a tree.

2 Hamiltonian Tour in a Hypercube

An alternative type of tour to an Eulerian Tour in graph is a Rudrata Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Rudrata cycle, for hypercubes of dimension $n \geq 2$.

Solution:

We will strengthen the inductive hypothesis.

Stronger Inductive Claim: There exists a tour in an n-dimensional hypercube that uses the edge: $(0^n, 10^{n-1})$.

Base Case: $n = 2$, the hypercube is just a four cycle, which is a cycle that contains the edge $(00, 10)$ as required.

Inductive Hypothesis: We assume the claim holds for dimension n.

Inductive Step: The recursive definition of an $n+1$ dimensional hypercube is to take two n dimensional hypercubes, relabel each vertex x in one "subcube" as $0x$ and relabel each vertex in the other "subcube" as $1x$ and add edges $(0x, 1x)$ for each $x \in \{0,1\}^n$.

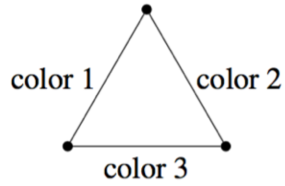
Use the inductive hypothesis to form seperate tours of each subcube which in the 0th subcube contains the edge $(00^{n-1}, 010^{n-2})$ and the 1th subcube contains $(10^{n-1}, 110^{n-1})$. We remove these edges then add the edges between the subcubes; $(00^{n-1}, 10^{n-1})$ and $(010^{n-2}, 110^{n-2})$.

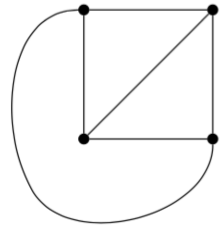
Notice we do not change the degrees of any node in this swap thus the degree of all the nodes is two.

Moreover, the tour is connected as one can reach every node from all zeros in the first cube using the inductive tour, and in the second cube using the edge to the second cube and the rest of the inductive tour.

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

(b) How many colors are required to edge color a 3-dimensional hypercube?

(c) Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

(d) Show that any tree with at least one vertex has a degree 1 vertex. (You may use any definition of a tree that we provided in the notes, homeworks or lectures to prove this fact.)

(e) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Add the fourth vertex, notice that each edge has a different color available from the set of three colors.

(b) 3. Recall that edges connect vertices that differ in a dimension. And each vertex is incident to exactly one edge for each dimension. Thus, the entire set of edges for a specific dimension can be colored with a single color.

(c) By induction on the number of edges. We will use a set of $2d-1$ colors. Remove an edge and $2d-1$ color the remaining graph from our set. This can be done by the induction hypothesis as the remaining graph's degree is no bigger than d and the graph has fewer edges. The edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts.

(d) The number of edges in an n-vertex tree is $n-1$, so the total degree is $2(n-1)$, and the average degree is at most $2-2/n$, thus there must be a vertex of degree at most 1.

(e) By induction on the number of vertices. Base case is a single vertex, which has no edges to color, and thus can be colored with 0 colors. Remove the degree 1 vertex, v. Color the remaining tree with d colors. Note that vertex v's neighboring vertices has degree at most $d-1$ without the edge to v and thus its incident edges use at most $d-1$ colors. Thus, there is a color available for coloring the edge incident to this vertex.