CS 70 Discrete Mathematics and Probability Theory Spring 2017 Rao

HW 8

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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2 Story Problems

Prove the following identities by combinatorial argument:

- (a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$
- (b) $n^2 = 2\binom{n}{2} + n$
- (c) $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$

Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(d) $\sum_{k=j}^{n} {n \choose k} {k \choose j} = 2^{n-j} {n \choose j}$ *Hint:* Consider a generalization of the previous part.

Solution:

(a) The left hand side is the number of ways to choose two elements out of 2n. Counting in another way, we first divide the 2n elements (arbitrarily) into two sets of n elements. Then we consider three cases: either we choose both elements out of the first n-element set, both out of the second n-element set, or one element out of each set. The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2 , respectively. Since these three cases are mutually exclusive and cover all the possibilities, summing them must give the same number as the left hand side. This completes the proof.

(b) LHS: There are *n* movies. Choose the best-rated movie and the best-selling movie. There are *n* choices for each title.

RHS: Choose 2 distinct movies, permute them in 2 ways. But what if the best-rated movie is the best selling movie? Then we're just choosing one movie, and there are n ways of doing that.

(c) RHS: From *n* people, pick one team-leader and some (possibly empty) subset of other people on his team.

LHS: First pick *k* people on the team, then pick the leader among them.

(d) RHS: Form a team as follows: Pick j leaders from n people. Then pick some (possibly empty) subset of the remaining people.

LHS: First pick $k \ge j$ people on the team, then pick the j leaders among them.

3 Probability Potpourri

Prove a brief justification for each part.

- (a) For two events A and B in any probability space, show that $Pr(A \setminus B) \ge Pr(A) Pr(B)$.
- (b) If $|\Omega| = n$, how many distinct events does the probability space have?
- (c) Find some probability space Ω and three events A, B, and $C \subseteq \Omega$ such that $\Pr(A) > \Pr(B)$ and $\Pr(A \mid C) < \Pr(B \mid C)$.
- (d) If two events C and D are disjoint and Pr(C) > 0 and Pr(D) > 0, can C and D be independent? If so, provide an example. If not, why not?
- (e) Suppose $Pr(D \mid C) = Pr(D \mid \overline{C})$, where \overline{C} is the complement of C. Prove that D is independent of C.

Solution:

(a) Start with the right side:

$$\begin{aligned} \Pr(A) - \Pr(B) &= \left[\Pr(A \cap B) + \Pr(A \setminus B) \right] - \left[\Pr(A \cap B) + \Pr(B \setminus A) \right] \\ &= \Pr(A \setminus B) - \Pr(B \setminus A) \\ &\leq \Pr(A \setminus B) \end{aligned}$$

- (b) An event is a subset of Ω , and for each outcome, there are 2 options: the outcome is in the event, or it isn't. Since there are n outcomes, there are 2^n events.
- (c) Let $\Omega = \{1,2,3\}$, with each outcome being equally likely. Let $A = \{1,2\}$, $B = \{3\}$, and $C = \{3\}$. Then Pr(A) = 2/3 > Pr(B) = 1/3. But $Pr(A \mid C) = 0 < Pr(B \mid C) = 1$.

- (d) No, C and D cannot be independent. If they are independent, then $Pr(C \cap D) = Pr(C) Pr(D) > 0$, but we know that $Pr(C \cap D) = 0$ because the events are disjoint.
- (e) Using total probability rule:

$$\Pr(D) = \Pr(D \cap C) + \Pr(D \cap \overline{C}) = \Pr(D \mid C) \cdot \Pr(C) + \Pr(D \mid \overline{C}) \cdot \Pr(\overline{C})$$

But we know that $Pr(D \mid C) = Pr(D \mid \overline{C})$, so this simplifies to

$$\Pr(D) = \Pr(D \mid C) \cdot [\Pr(C) + \Pr(\overline{C})] = \Pr(D \mid C) \cdot 1 = \Pr(D \mid C)$$

which defines independence.

4 Parking Lots

Some of the CS 70 staff members founded a start-up company, and you just got hired. The company has twelve employees (including yourself), each of whom drive a car to work, and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

- (a) On any given day, what is the probability that you park next to Professor Rao, who is working there for the summer?
- (b) What is the probability that there are exactly three cars between yours and Professor Rao's?
- (c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

Solution:

- (a) There are 12! possible ways in which the cars get parked (all possible permutations). To count the number of permutations in which your car and Professor Rao's get parked next to each other, do this thought experiment: assume that your car wasn't even there and there were 11 spaces. Now there are 11! ways for the cars to get parked; after this you see where Professor Rao has parked and you create a space (this is a thought experiment, remember), either to the left or to the right of Professor Rao's car, and park your car there. You have two choices, which means that the total number of such arrangements is $11! \times 2$. Therefore the probability is $11! \times 2/12! = 2/12 = 1/6$.
- (b) Again we have to count the number of arrangements and divide by 12!. Either your car is parked to the left of Professor Rao's or to his right. In the first case your car can be in any of these spots (assuming they're numbered from left to right) 1,2,...,8, and Professor Rao's car will always be in the spot whose number is higher by 4. This gives your car and Professor

Rao's car 8 ways to be parked with your car being on the left. By symmetry, there are also 8 ways in which your car is to the right of Professor Rao's car with three cars in between. So in total there are 16 ways for your car and Professor Rao's car to be parked with 3 spaces in between. After the spaces for these two cars are determined, the remaining ones can park in any of the 10! arrangements. So the total number of arrangements is $16 \times 10!$. The probability is therefore $16 \times 10!/12! = 16/(12 \cdot 11) = 4/33$.

(c) We know that 5 spots from the 11 that are not occupied by your car have been freed. All choices of these 5 spots are equally likely to happen. So we can count the arrangements of the 5 free spots to compute the probability. In total there are $\binom{11}{5}$ possible ways to choose the free spots. To count to number of ways that free both spaces next to your car, we pick those two spaces and then choose 3 more spots from the remaining 9. So there are $\binom{9}{3}$ such arrangements. Therefore the desired probability is $\binom{9}{3}/\binom{11}{5}$ which is equal to

$$\frac{9!/(3!6!)}{11!/(5!6!)} = \frac{5 \times 4}{11 \times 10} = \frac{2}{11}.$$

- 5 Calculate These... or Else
- (a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order, i.e. $\{8,9,10,J,Q\}$ is a straight. Values do not loop around, so $\{Q,K,A,2,3\}$ is not a straight. However, an ace counts as both a low card and a high card, so both $\{A,2,3,4,5\}$ and $\{10,J,Q,K,A\}$ are considered straights. When drawing a 5 card hand, what is the probability of drawing a straight from a standard 52-card deck?
- (b) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?
- (c) Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?
- (d) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column).
- (e) A bag has two quarters and a penny. If someone removes a coin, the Coin-Replenisher will come and drop in 1 of the coin that was just removed with 3/4 probability and with 1/4 probability drop in 1 of the opposite coin. Someone removes one of the coins at random. The Coin-Replenisher drops in a penny. You randomly take a coin from the bag. What is the probability you take a quarter?

Solution:

(a) The probability space is uniform over all possible 5-card hands, so we can use counting to solve this problem. There are $\binom{52}{5}$ possible hands, so that is our denominator. To count the

number of possible straights, note that there are 4 choices of suit for each of the cards for a total of 4^5 suit choices. Also, observe that once we pick a starting card for the straight, the rest of the cards are determined (e.g. if we choose 3 as the first card, then our straight must be $\{3,4,5,6,7\}$). Therefore, we need to multiply by the number of possible starting cards.

Note: Originally the question did not clearly specify whether $\{10, J, Q, K, A\}$ was a straight. Hence, both of the following answers are accepted:

$$\frac{9 \cdot 4^5}{\binom{52}{5}}$$
 or $\frac{10 \cdot 4^5}{\binom{52}{5}}$.

The first answer arises if you assume that the ace is only a low card, so $\{10, J, Q, K, A\}$ is not considered a straight. In this case, the possible starting cards for the straight are

$${A,1,2,3,4,5,6,7,8,9}$$

for a total of 9 choices.

The second answer arises if you include the straight $\{10, J, Q, K, A\}$, which is more aligned with traditional poker rules. This is also the answer to the clarified statement of the problem.

- (b) $13^4 \cdot 48$ counts twice the total number of combinations of 1 card from each suit. So the final probability is $13^4 \cdot 24/\binom{52}{5}$.
- (c) In 64 squares, there are:
 - (1) 4 at-corner squares, each shares ONLY 2 sides with other squares.
 - (2) $6 \cdot 4 = 24$ side squares, each shares ONLY 3 sides with other squares.
 - (3) $6 \cdot 6 = 36$ inner squares, each shares 4 sides with other squares.

Notice that the three cases are mutually exclusive. So we just sum up the probabilities.

$$\frac{4}{64} \cdot \frac{2}{63} + \frac{24}{64} \cdot \frac{3}{63} + \frac{36}{64} \cdot \frac{4}{63} = \frac{1}{18}.$$

- (d) $8!/\binom{64}{8}$. This counts safe arrangements (8 choices in first row, 7 in second row, etc) over total arrangements, and since this is a uniform probability space, this gives the probability no rooks are threatening one another.
- (e) The two possibilities are either the penny was removed first or one of the quarters. Let the former event be *P*. Let the event the Coin-Replenisher dropped in a penny be *C*. Use Bayes rule.

$$Pr(P \mid C) = \frac{Pr(P) Pr(C \mid P)}{Pr(C)} = \frac{Pr(P) Pr(C \mid P)}{Pr(P) Pr(C \mid P) + Pr(\bar{P}) Pr(C \mid \bar{P})}$$
$$= \frac{(1/3) \cdot (3/4)}{(1/3) \cdot (3/4) + (2/3) \cdot (1/4)} = \frac{3}{5}.$$

Thus, there's a 3/5 chance there are two quarters and one penny, and a 2/5 chance there are two pennies and one quarter. The chance you pick a quarter is then 8/15.

6 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: \overline{A} and \overline{B} are necessarily independent.

(b) Prove or disprove: A and \overline{B} are necessarily independent.

(c) Prove or disprove: A and \overline{A} are necessarily independent.

(d) Prove or disprove: It is possible that A = B.

Solution:

(a) True. \overline{A} and \overline{B} must be independent:

$$\Pr[\overline{A} \cap \overline{B}] = \Pr[\overline{A \cup B}] \qquad \text{(by De Morgan's law)}$$

$$= 1 - \Pr[A \cup B] \qquad \text{(since } \Pr[\overline{E}] = 1 - \Pr[E] \text{ for all } E)$$

$$= 1 - (\Pr[A] + \Pr[B] - \Pr[A \cap B]) \qquad \text{(union of overlapping events)}$$

$$= 1 - \Pr[A] - \Pr[B] + \Pr[A] \Pr[B] \qquad \text{(using our assumption that } A \text{ and } B \text{ are independent)}$$

$$= (1 - \Pr[A])(1 - \Pr[B])$$

$$= \Pr[\overline{A}] \Pr[\overline{B}] \qquad \text{(since } \Pr[\overline{E}] = 1 - \Pr[E] \text{ for all } E)$$

(b) True. A and \overline{B} must be independent:

$$\begin{aligned} \Pr[A \cap \overline{B}] &= \Pr[A - (A \cap B)] \\ &= \Pr[A] - \Pr[A \cap B] \\ &= \Pr[A] - \Pr[A] \Pr[B] \\ &= \Pr[A] (1 - \Pr[B]) \\ &= \Pr[A] \Pr[\overline{B}] \end{aligned}$$

- (c) False in general. If $0 < \Pr[A] < 1$, then $\Pr[A \cap \overline{A}] = \Pr[\varnothing] = 0$ but $\Pr[A] \Pr[\overline{A}] > 0$, so $\Pr[A \cap \overline{A}] \neq \Pr[A] \Pr[\overline{A}]$; therefore A and \overline{A} are not independent in this case.
- (d) True. To give one example, if Pr[A] = Pr[B] = 0, then $Pr[A \cap B] = 0 = 0 \times 0 = Pr[A] Pr[B]$, so *A* and *B* are independent in this case. (Another example: If A = B and Pr[A] = 1, then *A* and *B* are independent.)

7 Bag of Coins

Your friend Forest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forest picks a fair coin, and let B be the event that Forest picks a biased coin. Forest draws three coins from the bag, but he does not know which are biased and which are fair.

- (a) What is the probability of FFB?
- (b) What is the probability that the third coin he draws is biased?
- (c) What is the probability of picking at least two fair coins?
- (d) Given that Forest flips the second coin and sees heads, what is the probability that this coin is biased?

Solution:

(a) The probability of picking F for the first coin is (n-k)/n. The probability of picking F for the second coin, after picking one fair coin already is (n-k-1)/(n-1). The probability of picking B for the third coin is k/(n-2). Thus, the probability of picking the exact sequence FFB is

$$\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

(b) One approach is to condition on the possible outcomes for the first and second coins

$$\{FF, FB, BF, BB\}$$

such that

$$Pr(T) = Pr(T \cap FF) + Pr(T \cap FB) + Pr(T \cap BF) + Pr(T \cap BB)$$

where *T* is the event that the third coin is biased.

A simpler approach is to use the notion of symmetry. Since we don't know any information about the first and second coins, the probability that the third coin is biased is the same as the probability that the first coin is biased, which is k/n.

(c) Note that the probability of picking any sequence of two fair coins and a biased coin is the same. It is in fact the probability from part (a). We need to multiply by the number of arrangements of biased and fair coins, however. So, the probability of picking any sequence with two fair coins is

$$\binom{3}{1} \frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}$$
.

We additionally need to consider the probability of getting 3 fair coins.

$$\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

We simply sum the two to get our answer:

$$\binom{3}{1} \frac{(n-k)(n-k-1)k}{n(n-1)(n-2)} + \frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

(d) We can apply Bayes Rule. Let *H* denote the event that Forest sees heads.

$$Pr(B \mid H) = \frac{Pr(H \mid B) Pr(B)}{Pr(H)}$$

Note that $Pr(H \mid B) = p$ and that Pr(B) = k/n. We can now compute the denominator. Using the law of total probability, we can expand Pr(H).

$$Pr(H) = Pr(H \mid B) Pr(B) + Pr(H \mid F) Pr(F)$$

$$= p \frac{k}{n} + \frac{1}{2} \frac{n - k}{n}$$

$$= \frac{2pk + n - k}{2n}$$

We now combine both parts to get our answer:

$$\frac{p \cdot (k/n)}{(2pk+n-k)/(2n)} = \frac{2pk}{2pk+n-k}.$$