

## Due Thursday April 14 at 10PM

Before you start your homework, write down your team. Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

### 1. Variance

This problem will give you practice using the “standard method” to compute the variance of a sum of random variables that are not pairwise independent (so you cannot use “linearity” of variance). If you don't even know what is “linearity” of variance, read the lecture note and slides first.

- (a) (5 points) A building has  $n$  floors numbered  $1, 2, \dots, n$ , plus a ground floor  $G$ . At the ground floor,  $m$  people get on the elevator together, and each gets off at a uniformly random one of the  $n$  floors (independently of everybody else). What is the *variance* of the number of floors the elevator *does not* stop at? (In fact, the variance of the number of floors the elevator *does* stop at must be the same (do you see why?) but the former is a little easier to compute.)
- (b) (5 points) A group of three friends has  $n$  books they would all like to read. Each friend (independently of the other two) picks a random permutation of the books and reads them in that order, one book per week (for  $n$  consecutive weeks). Let  $X$  be the number of weeks in which all three friends are reading the same book. Compute  $\text{Var}(X)$ .

### 2. Coupon Collection (10 points)

Suppose you take a deck of  $n$  cards and repeatedly perform the following step: take the current top card and put it back in the deck at a uniformly random position. (I.e., the probability that the card is placed in any of the  $n$  possible positions in the deck — including back on top — is  $1/n$ .) Consider the card that starts off on the bottom of the deck. What is the expected number of steps until this card rises to the top of the deck? (Hint: Let  $T$  be the number of steps until the card rises to the top. We have  $T = T_n + T_{n-1} + \dots + T_2$ , where the random variable  $T_i$  is the number of steps until the bottom card rises from position  $i$  to position  $i - 1$ . Thus, for example,  $T_n$  is the number of steps until the bottom card rises off the bottom of the deck, and  $T_2$  is the number of steps until the bottom card rises from second position to top position. What is the distribution of  $T_i$ ?) (More hints: You may use the fact that  $\sum_{i=1}^n \frac{1}{i} \approx \ln n$ .)

### 3. Markov's Inequality and Chebyshev's Inequality

A random variable  $X$  has variance  $\text{Var}(X) = 9$  and expectation  $\mathbb{E}(X) = 2$ . Furthermore, the value of  $X$  is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

- (a) (3 points)  $\mathbb{E}(X^2) = 13$ .

- (b) (3 points)  $\Pr[X = 2] > 0$ .
- (c) (3 points)  $\Pr[X \geq 2] = \Pr[X \leq 2]$ .
- (d) (3 points)  $\Pr[X \leq 1] \leq 8/9$ .
- (e) (3 points)  $\Pr[X \geq 6] \leq 9/16$ .
- (f) (3 points)  $\Pr[X \geq 6] \leq 9/32$ .

#### 4. Umbrella Store

Bob has a store that sells umbrellas. The number of umbrellas that Bob sells on a rainy day is a random variable  $Y$  with mean 25 and standard deviation  $\sqrt{105}$ . But if it is a clear day, Bob doesn't sell any umbrellas at all. The weather forecast for tomorrow says it will rain with probability  $\frac{1}{5}$ . Let  $Z$  be the number of umbrellas that Bob sells tomorrow.

- (a) (3 points) Let  $X$  be an indicator random variable that it will rain tomorrow. Write  $Z$  in terms of  $X$  and  $Y$ .
- (b) (4 points) What is the mean and standard deviation of  $Z$ ?
- (c) (3 points) Use Chebyshev's inequality to bound the probability that Bob sells at least 25 umbrellas tomorrow.

#### 5. Casino wins

A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability  $1/6$ , a hand of black jack with probability  $1/2$ , and a hand of stud poker with probability  $1/5$ . Assume the outcomes of the card games are mutually independent.

- (a) (3 points) What is the expected number of hands the gambler wins in a day?
- (b) (3 points) What is the variance in the number of hands won per day?
- (c) (3 points) What would the Markov bound be on the probability that the gambler will win 108 hands on a given day?
- (d) (3 points) What would the Chebyshev bound be on the probability that the gambler will win 108 hands on a given day?

#### 6. Find the right key

A man has a set of  $n$  keys, one of which fits the door to his apartment. He tries the keys until he finds the correct one. Give the expected number and variance for the number of trials until success if

- (a) (6 points) he tries the keys at random (possibly repeating a key tried earlier).
- (b) (6 points) he chooses keys randomly from among those he has not yet tried.

#### 7. Extra Credit (1 point)

I think of two distinct real numbers between 0 and 1 but do not reveal them to you. I now choose one of the two numbers at random and give it to you. Can you give a procedure for guessing whether you were shown the smaller or the larger of the two numbers, such that your guess is correct with probability *strictly* greater than 0.5 (although exactly how much better than 0.5 may depend on the actual values of the two numbers)?