

Due Thursday March 10th at 10PM

1. (5 points each, 15 total) **Correcting XYZ**

- (a) For any java program P , define $S(P)$ to be the set of all Java programs P' that output the same result as P on all inputs. Formally, $S(P) = \{P' | \forall x: P(x) = P'(x), P'(x) \text{ halts if and only if } P(x) \text{ halts}\}$. XYZ claims that they have built an optimal java-program-shortener. Formally, they claim to have a procedure **optimalShortener** such that:

```
for every java program P:
  let P' = optimalShortener(P)
  then:
    P' is in S(P)
    forall P'' in S(P), length(P'') >= length(P')
```

Prove that XYZ is wrong.

Answer:

We can solve the halting problem with optimalShortener.

```
(using one definition of Halting Problem)
HaltingSolver1(M):
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```
  define M1:
    simulate M(); // suppress any output
    exit 0;

  define M2:
    exit 0;

  if (optimalShortener(M1) == optimalShortener(M2))
    return "halts"
  else
    return "inf loops"
```

```
(using other definition of Halting Problem)
HaltingSolver2(M, x):
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```
  define M1:
```

```

        simulate M(x); // suppress any output
        exit 0;

define M2:
    exit 0;

if (optimalShortener(M1) == optimalShortener(M2))
    return "halts"
else
    return "inf loops"

```

- (b) Having failed with source code shortening, XYZ now tries their luck with runtime optimization. XYZ claims that they have built a new optimizer **optimizer** such that:

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for every java program P:
    let P' = optimizer(P)
    then:
        forall x:
            if P(x) halts
            then P'(x) halts within  $2^{|x|}$  steps
            else P'(x) infinite loops

```

Namely, XYZ claims that their optimizer outputs an equivalent program such that: if $P(x)$ halts, then $P'(x)$ will halt within $2^{|x|}$ steps; if $P(x)$ does not halt, then $P'(x)$ does not halt either. Prove that **optimizer** can not exist.

Answer: We can solve the halting problem with optimizer. To see if $P(x)$ halts, run $P'(x)$ for $2^{|x|} + 1$ steps. If it halts, report that $P(x)$ halts. Otherwise, report that $P(x)$ does not halt.

- (c) Let XYZ-phone be a smart phone with 16GB of storage, 2GB of RAM, and additional state of 1MB (i.e. CPU registers, etc ..).

Prove that for any program P which does not interact with the rest of the world (no user input, no network connection, no wifi, no bluetooth, no sensors), it is possible to determine whether all executions of P halts or whether some execution of P infinite loops. (Different executions of P may behave differently due to the random number generator, the initial state of the phone when P is loaded, and due to non-determinism caused by multi-threading.

Answer: The total state of the machine is less than 17 GB, which is $17 * 2^{32} * 8 < 2^{36}$ bits.

Create a directed graph with 2^{36} nodes, where there is a directed edge from u to v if, in one cpu cycle, the machine can move from state u to state v .

Let S be the set of all nodes reachable by loading program P . We check if there is some node v in S such that v is in some cycle of the graph described above.

2. (10 points) Printing all x where $M(x)$ halts

Prove that it is possible to write a program P which:

- * takes as input M , a java program
- * runs forever, and prints out strings to the console
- * for every x , if $M(x)$ halts, then $P(M)$ eventually prints out x
- * for every x , if $M(x)$ does NOT halt, then $P(M)$ never prints out x

Answer:

```

Starting out, let  $S = \{\}$ 
for  $i = 1$  to infinity:

    Let  $N_i$  = a new machine loaded with program  $M$  and input  $i$ 
     $S = S + \{ N_i \}$ 

    simulate every machine in  $S$  for 1 cycle

    forall  $N_x$  in  $S$  that has halted:
        print out  $x$ 
        remove  $N_x$  from  $S$ 

```

Consider any x , such that $M(x)$ halts after n steps. For some k , at stage k , $M(x)$ is added to S . At state $k + n + 1$, $M(x)$ halts, and x is then printed out.

For any x where $M(x)$ does not halt, x is never printed out.

3. (10 points) Lexicographical output is impossible

Lexicographical ordering of strings means (1) shorter strings are in front of longer strings (2) for two strings of the same length, they are sorted in alphabetical order.

Prove that it's impossible to solve the above problem if we require the output be in lexicographical order.

Answer:

Let M be any Java program. We show how to construct a program which can decide whether $M(x)$ halts for any x . Consider the program $M2$ defined as follows:

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 $M2(2 * k) = \text{simulate } M(k)$ 
 $M2(2 * k + 1) = \text{halt}$ 

```

Now, suppose there is a lexicographical enumerator for $M2$. Call this enumerator E .

To decide whether $M(x)$ halts, we run E until we see $2k + 1$ printed on the output tape. It must eventually be printed since $M2(2k + 1)$ halts.

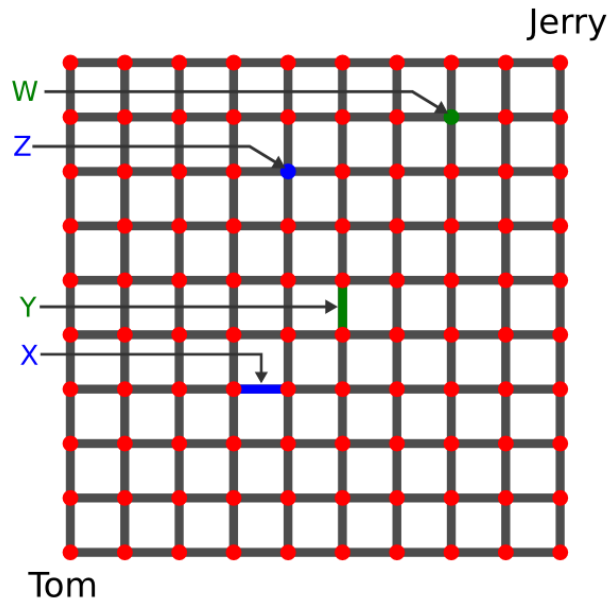
Then, we see whether $2k$ was printed out the tape already.

If so, we know that $M2(2k)$ halted, meaning that $M(k)$ halted.

If not, we know that $M2(2k)$ infinite looped.

4. (2,2,3,3) Maze

Let's assume that Tom is located at the bottom left corner of the maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



- a) How many such shortest paths exist?

Answer: Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHV VVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.

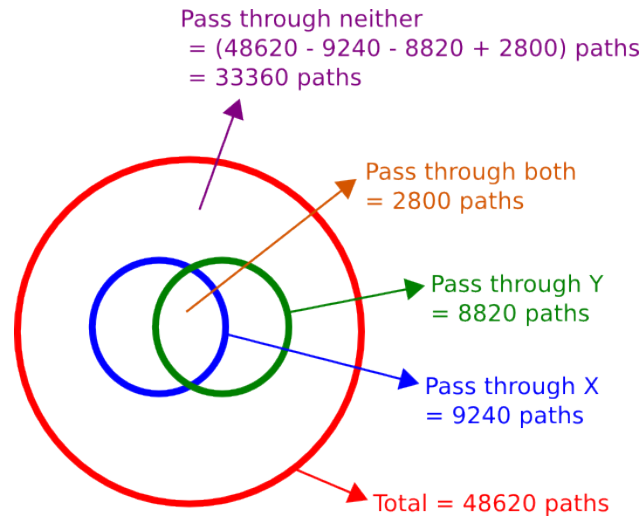
- b) How many shortest paths pass through the edge labelled X? The edge labelled Y? Both the edges X and Y? Neither edge X nor edge Y?

Answer: For a shortest path to pass through the edge X, it has to first get to the left vertex of X. So the first portion of the path has to start at the origin, and end at the left vertex of X. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of X to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge X is $20 \times 462 = 9240$.

Using similar logic, any shortest path that passes through Y has to consist of 2 legs, the first leg going from the origin to the bottom vertex of Y, and the second leg going from the top vertex of Y

to the top right corner of the maze. The first leg will consist of exactly 5 “H”s and 4 “V”s, while the second leg will consist of exactly 4 “H”s and 4 “V”s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

By a similar argument, let’s try to figure out how many paths will pass through both X and Y . Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 “H”s and 3 “V”s (going from the origin to the left edge of X), the second leg consisting of 1 “H” and 1 “V” (going from the right vertex of X to the bottom vertex of Y), and the third leg consisting of 4 “H”s and 4 “V”s (going from the top vertex of Y to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.



Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through X , 8820 pass through Y , and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).

- c) How many shortest paths pass through the vertex labelled Z ? The vertex labelled W ? Both the vertices Z and W ? Neither vertex Z nor vertex W ?

Answer: This part is very similar in spirit to the previous one, except that in this case, each leg of the path we consider begins exactly where the previous leg ended, and *not* to the right or to the top of where the previous leg ended.

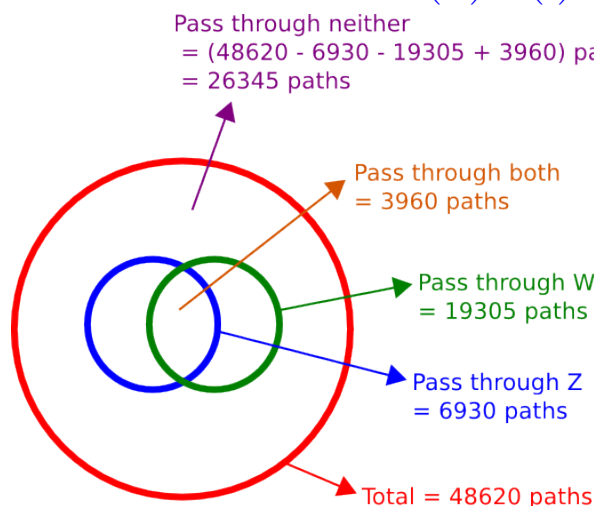
For concreteness, let’s find out how many shortest paths pass through vertex Z . Observe that for a shortest path to pass through Z , it has to first get to Z . So the first portion of the path has to start at the origin, and end at Z . Using the same logic as above, there are exactly $\binom{11}{4} = 330$ ways to complete this “first leg” of the path (consisting of 4 “H” edges and 7 “V” edges). Having chosen one of these 330 ways, the path has to then go from Z to the top right corner of the maze. This second leg will consist of 5 “H” edges and 2 “V” edges, and so there are exactly $\binom{7}{2} = 21$ possibilities. Therefore, the total number of shortest paths that pass through the vertex Z is $330 \times 21 = 6930$.

Using similar logic, any shortest path that passes through W has to consist of 2 legs, the first leg going from the origin to W , and the second leg going from W to the top right corner of the maze. The first leg will consist of exactly 7 “H”s and 8 “V”s, while the second leg will consist of exactly

2 “H”s and 1 “V”. So the total number of such shortest paths will be $\binom{15}{7} \times \binom{3}{1} = 19305$.

By a similar argument, let’s try to figure out how many paths will pass through both Z and W. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 4 “H”s and 7 “V”s (going from the origin to Z), the second leg consisting of 3 “H”s and 1 “V” (going from Z to W), and the third leg consisting of 2 “H”s and 1 “V” (going from W to the top right corner of the maze).

The total number of such shortest paths is therefore $\binom{11}{4} \times \binom{4}{1} \times \binom{3}{1} = 3960$.



Finally, we know that there are 48620 shortest paths in all, of which 6930 pass through Z, 19305 pass through W, and 3960 pass through both. So the number of paths that pass through neither is 26345 (see the figure above for an intuitive explanation).

5. (1 point each, 20 total) Counting practice!

The only way to learn counting is to practice, practice, practice—so here is your chance to do so. No need to justify your answers or show your calculations on this problem. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

- (a) How many 10-bit strings are there that contain exactly 4 ones?

Answer: We must select 4 of the 10 bits to set to 1. Since we don’t care about the order of the selection (i.e. selecting bit 3 before bit 0 is no different from selecting bit 0 before bit 3), the answer is $\binom{10}{4}$. Equivalently, we could choose 6 bits out of the 10 to set to 0. There are $\binom{10}{6}$ ways to do this. The two expressions are the same.

- (b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

Answer: Since the order of the cards in the hand is irrelevant, we again choose 13 of the 52 cards. The answer is $\binom{52}{13}$.

- (c) How many different 13-card bridge hands are there that contain no aces?

Answer: There are 48 cards that contain no aces and out of these we choose 13, so the answer is $\binom{48}{13}$. In more detail, there are 48 ways to select the first card since we don’t want an ace, 47 ways to select the second card, etc. If the order of the choices mattered, we would have $48!/35!$

ways to select the cards, but since order doesn't matter, we divide by $13!$ and get $\frac{48!}{13!35!} = \binom{48}{13}$.

- (d) How many different 13-card bridge hands are there that contain all four aces?

Answer: We know that 4 of the cards in our hand will be aces, so we only have to select the remaining 9. Thus, we choose 9 out of 48 and get the answer $\binom{48}{9}$.

- (e) How many different 13-card bridge hands are there that contain exactly 6 spades?

Answer: We first choose the 6 spades from the 13 total spades, then we must choose 7 remaining cards from the 39 non-spades. There are thus $\binom{13}{6}\binom{39}{7}$ ways total.

- (f) How many 99-bit strings are there that contain more ones than zeros?

Answer: 2^{98} . There are 2^{99} 99-bit strings total. Since a 99-bit string cannot have equal number of ones and zeroes, by symmetry, there are $2^{99}/2 = 2^{98}$ possible different strings. To see the symmetry, notice that if we have a string of more ones than zeroes, we can flip all the bits and obtain a string of more zeroes than ones and vice versa. Hence, there is a 1-1 correspondence between the strings with more ones and the strings with more zeroes.

Put another way, if S denotes the set of 99-bit strings with more ones than zeros, and T the set of 99-bit strings with more zeros than ones, we see that $S \cap T = \emptyset$ and that $S \cup T =$ the set of all 99-bit strings. By the sum rule, $|S| + |T| = 2^{99}$. Moreover because S can be put into bijective correspondence with T , $|S| = |T|$. Plugging this into the equation above, we see $2 \times |S| = 2^{99}$, so $|S| = 2^{98}$. To put it yet another way, half of all 99-bit strings have more ones than zeros, so the answer is $\frac{1}{2} \times 2^{99} = 2^{98}$.

- (g) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

Answer: There are 52 ways to select the topmost card, 51 ways to select the 2nd topmost, etc. There are thus $52!$ ways total to order the cards.

- (h) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

Answer: First we pretend that the 104 cards are all distinguishable, then there are $104!$ ways to order them. But we have 52 pairs of identical cards and have counted each configuration twice for each pair. Hence, there are $\frac{104!}{2^{52}}$ possible ways to order the cards.

Here's another way to think about this problem. Out of 104 possible positions, we first choose 2 positions to place the pairs of Ace of Spades; there are $\binom{104}{2}$ ways to do this. Then we have 102 positions left from which we choose 2 positions to place the pairs of Ace of Hearts; there are $\binom{102}{2}$ ways. Using this reasoning, we get the expression $\binom{104}{2}\binom{102}{2}\binom{100}{2}\dots\binom{2}{2} = \frac{104!}{2^{52}}$.

This is very much like counting the number of anagrams of BONBON, except with 104 letters instead of 6 letters. There are $6!/(2! \cdot 2! \cdot 2!) = 6!/2^3$ anagrams of BONBON. Similarly, there are $8!/2^4$ anagrams of FROUFROU, $10!/2^5$ anagrams of INTESTINES, and ... well, you get the idea.

Note that interestingly enough, if we want to rephrase this problem in term of balls and bins, we would let the position be the balls and let the card face be the bins even though we intuitively think we should be placing the cards into a position. So, this problem is really the same as finding the number of ways of throwing 104 labelled balls into 52 bins such that every bin has exactly 2 balls.

- (i) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

Answer: Since the order of the letters matter and all of the letters are distinct, there are $7!$ different anagrams.

- (j) How many different anagrams of ALASKA are there?

Answer: If we first pretend that the 3 A's are all distinct (i.e. subscripted), then there are $6!$ anagrams. But since the 3 A's are identical, we counted each anagram an extra $3!$ ways. Hence, there are $6!/3!$ anagrams total. Another way to think about this: we first choose 3 of out of the 6 possible positions to place the A, there are $\binom{6}{3}$ choices. There are then 3 positions left to place the L, 2 positions to place the S, and one position to place the A, so there are $\binom{6}{3}(3!) = 6!/3!$ anagrams total.

- (k) How many different anagrams of ALABAMA are there?

Answer: Similar to previous problem, there are $7!/4!$ anagrams total.

- (l) How many different anagrams of MONTANA are there?

Answer: If we pretend the N's are distinct, then there would be $7!/2!$ anagrams. But since the N's are identical, we counted each configuration twice and must divide by an additional $2!$. So our final answer is $\frac{7!}{2!2!}$.

- (m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins?

Answer: Each ball can go into any one of the 27 bins. So there are 27^9 possible ways. One can also view this problem as asking for the number of functions that map the balls into the bins.

- (n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty?

Answer: The answer is the same as the number of ways to distribute 2 balls among 7 bins. Once we've distributed those 2 balls in any way whatsoever, then we can add one ball to each bin, yielding a configuration with 9 balls in 7 bins such that no bin is empty. There is bijective correspondence between ways to distribute 2 balls among 7 bins and ways to distribute 9 balls among 7 bins such that no bin is empty. There are 7 ways to distribute 2 balls into the bins so that both balls fall into the same bin, and $\binom{7}{2}$ ways to distribute 2 balls into the bins so that both balls fall into different bins, so the total number of ways is $7 + \binom{7}{2}$.

Alternate solution: This is a stars and bars problem. The bars represent the dividers between the bins, and each star represents one ball. We require there to be at least one star between every bar and the leftmost and rightmost character cannot be a bar. Consequently we are actually counting the number of ways to create a binary string of length $6 + 9 = 15$ where there are 6 one-bits and no two one-bits are adjacent. We can use problem 2 to get $\binom{8}{2} = \binom{8}{6} = \binom{7}{5} + \binom{7}{6}$ as our final answer.

Alternate solution: In every configuration, there can either be 5 bins with 1 ball in each and 2 bins with 2 balls in each, or there could be 6 bins with 1 ball and 1 bin with 3 balls. In the first case, there are $\binom{7}{5}$ ways to choose the 5 1-ball bins and once the 1-ball bins are chosen,

we have no more choices for the 2-ball bins so we are done. In the second case, there are $\binom{7}{6}$ ways to choose the 6 1-ball bins and hence the total ways to distribute the 9 balls is just $\binom{7}{5} + \binom{7}{6}$.

- (o) How many different ways are there to throw 9 identical balls into 27 bins?

Answer: This is a stars and bars problem. The bars represent dividers between the bins, and each star represents one ball. We wish to insert 26 bars in between the 9 stars, which corresponds to choosing a 35-bit string that has exactly 9 zero-bits. There are thus $\binom{35}{9}$ ways to distribute the balls. You should contrast this with the distinguishable balls case in part (m).

- (p) There are exactly 132 students currently enrolled in CS70. How many different ways are there to pair up the 132 CS70 students, so that each student is paired with one other student?

Answer: This problem is equivalent to throwing 132 balls into 66 **identical** bins such that every bin has two balls. We first imagine that the bins are not identical; then there would be $\frac{132!}{2^{66}}$ ways to throw the balls as shown before. But since we can't distinguish between the case where ball A and B are in bin 1 and ball C is in bin 4 vs. the case where ball A and B are in bin 4 and ball C is in bin 1, we can permute the content of the bins and still get the same configuration. Thus we know that we over-counted by a factor of $66!$ and derive $\frac{132!}{2^{66}66!}$ as our final answer.

Alternative solution: Another way to approach this problem is to first send 66 students to the moon. The selection order doesn't matter in this first choice so there are $\binom{132}{66}$ ways to do this. Then we match the remaining 66 students on Earth one by one with the students on the moon. There are $66!$ ways to perform the matching. But in this scheme, we counted sending student Andy to the moon and matching him with student Betty on Earth as different configuration from sending student Betty to the moon and matching her with student Andy on Earth. Since we double counted for each pair of students, we divide by 2^{66} to get $(\binom{132}{66}66!)/2^{66} = \frac{132!}{2^{66}66!}$ as our final expression.

Alternative solution: Here's a naive approach. We have 132 students in the classroom. Pick any two of them ($\binom{132}{2}$ ways to do this), pair them up, and send them home. Now there are 130 students. Pick two of them ($\binom{130}{2}$ ways), pair 'em, and send 'em home. Repeat until there's no one left anymore. In total there are $\binom{132}{2}\binom{130}{2}\binom{128}{2}\cdots\binom{2}{2}$ ways to do this.

But wait! This overcounts shamelessly. Suppose we had four students. Picking Alice and Betty first followed by picking Carol and Dave would be equivalent to first picking Carol and Dave followed by picking Alice and Betty. So, with four students, we'd be overcounting by a factor of two. With $2n$ students, we're overcounting by a factor of $n!$, since there are $n!$ different orders in which we could have chosen the n pairs and they all lead to the same pairing. Consequently in the original problem we've overcounted by a factor of $66!$. So the final answer is $\binom{132}{2}\binom{130}{2}\binom{128}{2}\cdots\binom{2}{2}/66!$ ways to do this.

- (q) How many ways are there to arrange n 1s and k 0s into a sequence?

Answer: $\binom{n+k}{k}$

- (r) How many solutions does

$$x_0 + x_1 + \dots + x_k = n$$

have, if all x s must be non-negative integers?

Answer: $\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

(s) How many solutions does

$$x_0 + x_1 = n$$

have, if all x s must be *strictly positive* integers?

Answer: $n - 1$. It's easy just to enumerate the solutions here. x_0 can take values $1, 2, \dots, n - 1$ and this uniquely fixes the value of x_1 . So, we have $n - 1$ ways to do this. But, this is just an example of the more general question below.

(t) How many solutions does

$$x_0 + x_1 + \dots + x_k = n$$

have, if all x s must be *strictly positive* integers?

Answer: $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k + 1$ variables, and $k + 1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

6. (2 points each, 10 total) Prove the following identities by combinatorial argument:

(a)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Answer: The left hand side is the number of ways to choose k elements out of n . Looking at this another way, we look at the first element and decide whether we are going to choose it or not. If we choose it, then we need to choose $k - 1$ more elements from the remaining $n - 1$. If we don't choose it, then we need to choose all our k elements from the remaining $n - 1$. We are not double counting, since in one of our cases we chose the first element and in the other, we did not.

(b)

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Answer: The left hand side is the number of ways to choose two elements out of $2n$. Counting in another way, we first divide the $2n$ elements (arbitrarily) into two sets of n elements. Then we consider three cases: either we choose both elements out of the first n -element set, both out of the second n -element set, or one element out of each set. The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2 , respectively. Since these three cases are mutually exclusive and cover all the possibilities, summing them must give the same number as the left hand side. This completes the proof.

(c)

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

Answer: RHS: From n people, pick one team-leader and some (possibly empty) subset of other people on his team.

LHS: First pick k people on the team, then pick the leader among them.

(d)

$$\sum_{k=j}^n \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}$$

Answer: RHS: Form a team as follows: Pick j leaders from n people. Then pick some (possibly empty) subset of the remaining people.

LHS: First pick $k \geq j$ people on the team, then pick the j leaders among them.

(e)

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

Answer: We'll start with the right-hand side. This is counting the number of ways to pick k things from $m+n$ objects. The left-hand side is summing up, for all possible values of i , the ways to pick i things from m and $k-i$ things from n . We see that the two are equivalent.