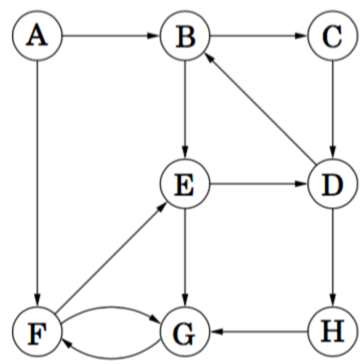
CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao
DIS 2a

1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$
ii. $\{(F,G),(G,F)\}$
iii. $\{(A,B),(B,C),(C,D),(D,B)\}$
iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$
ii. $\{(E,G),(G,F)\}$
iii. $\{(F,G),(G,F)\}$
iv. $\{(A,B),(B,C),(C,D)\}$
v. $\{(E,G),(G,F),(F,G),(G,F)\}$
vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(A,B),(B,C),(C,D)\}$

v. $\{(E,G),(G,F),(F,G),(G,F)\}$

vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$$V = \{A,B,C,D,E,F,G,H\},$$
$$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),(H,G)\}.$$

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$\{(B,C),(C,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,G),(G,F)\}$ (length = 3)

The last one listed above has the shortest path.

(d) A cycle should be a path that starts and ends at the same point, so iii is not a cycle. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)\}$ should be distinct, so iv is not a cycle. The correct answers are i and ii.

(e) All of them. A walk can end on the same vertex on which it begins or on a different vertex. A walk can travel over any edge and any vertex any number of times.

(f) iii. A tour is a walk which starts and ends at the same vertex, and has no repeated edges.

(g) True. In a cycle, every vertex has degree at least 2.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If such a process succeeds, then the part of our walk from the first time we visited w until the second time is a cycle, so it remains only to argue this process succeeds. Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Bipartite Graph

Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a graph is bipartite if and only if it no cycles of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no cycles of odd length.

Let us start traveling the cycle from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the set connects our start node n_0 to the a node n_1 in R. The 2nd edge in the cycle must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our cycle in L, the cycle must end with an even number of edges.

Prove the reverse direction: A undirected graph with no cycles of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are connected, then we have a cycle of odd length: (v, u_1) (odd), (u_1, u_2) (odd), and (u_2, v) (odd). This means no two vertices in R are connected. We can pick any vertex in R to repeat, and we have that L, R are disjoint.

3 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since $v_1 v_2$ is not an edge, by property T it must be that $v_1 v$ and $v_2 v$ where $v \in \{v_5, v_6, v_7\}$ are edges. Similarly for $v_3, v_4, v_3 v$ and $v_4 v$ where $v \in \{v_5, v_6, v_7\}$ are edges. So now any three vertices in $\{v_1, v_2, v_3, v_4\}$ on one side and $\{v_5, v_6, v_7\}$ on the other form an instance of $K_{3,3}$. Contradiction.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with > 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.