1 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

Solution:

One counting strategy is based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively. If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the first digit, then it must be preceded by a 1. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.

2 Flippin' Coins

Suppose we have a biased coin, with outcomes H and T, with probability of heads $\Pr[H] = 3/4$ and probability of tails $\Pr[T] = 1/4$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

(f) $\omega \in \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3 \cdot \frac{9}{64} = \frac{27}{64}$.

3 Aces

Consider an ordinary deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:

(a) Inclusion-Exclusion Principle: $\frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$.

(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52} + \frac{13}{52} - 2 \cdot \frac{1}{52} = \frac{15}{52}$.

(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.

(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$.

(e) Complement to getting no aces: $1-\frac{\binom{48}{5}}{\binom{52}{5}}$.

(f) Complement to getting no aces and no hearts: $1-\frac{\binom{36}{5}}{\binom{52}{5}}$.

4 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once.)

Solution:

(a) $18!5!/22! = 1/1463$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

(b) $(2/3)^8 = 256/6561$. This is just by independence.

(c) $(15/20)^5 = 243/1024$. For a single number, we can choose $\{6,7,\ldots,20\}$, so 15 valid outcomes out of 20 total outcomes. Then apply independence as in part (b).