CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao

DIS 07b

1 Let's Talk Probability

(a) When is $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ true? What is the general rule that always holds?

(b) When is $\Pr(A \cap B) = \Pr(A)\Pr(B)$ true? What is the general rule that always holds?

(c) If A and B are disjoint, are they independent?

(d) On the space of a fair roll of a six-sided die, find three events, each of which is independent of the intersection of the other two, such that they are not mutually independent.

(e) If we roll 2 dice, what is the probability that the first roll is a 3? What is the probability that the first roll is a 3 if we know that the sum of the dice is 6?

Solution:

(a) In general, we know $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\Pr(A \cap B) = 0$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ holds.

(b) In general, we know $\Pr(A \cap B) = \Pr(A)\Pr(B \mid A)$. If A and B are independent events, such that $\Pr(B \mid A) = \Pr(B)$, then $\Pr(A \cap B) = \Pr(A)\Pr(B)$ holds.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\Pr(A \cap B) = 0$. But these events are not independent: $\Pr(B \mid A) = 0$, but $\Pr(B) = 1/6$.

Since disjoint events have $\Pr(A \cap B) = 0$, we can see that the only time when A and B are independent is when either $\Pr(A) = 0$ or $\Pr(B) = 0$.

(d) Let A be the event that we roll a 1. Let B be the event that we roll a 2. Let C be the event that we roll a 3. Then $\Pr(A) = \Pr(B) = \Pr(C) = 1/6$, and $\Pr(A)\Pr(B)\Pr(C) = 1/216$. We know $\Pr(A \cap B \cap C) = 0 \neq \Pr(A)\Pr(B)\Pr(C)$, so the events are not mutually independent. However, each of the pairwise intersections is the empty set, such that $\Pr(A \cap B) = \Pr(B \cap C) = \Pr(A \cap C) = 0$, and every event is independent of the empty set. For example, $\Pr(A \cap (B \cap C)) = 0 = \Pr(A) \cdot \Pr(B \cap C)$, and likewise for the other pairs. Thus each event is independent of the intersection of the other two, but the three events are not mutually independent.

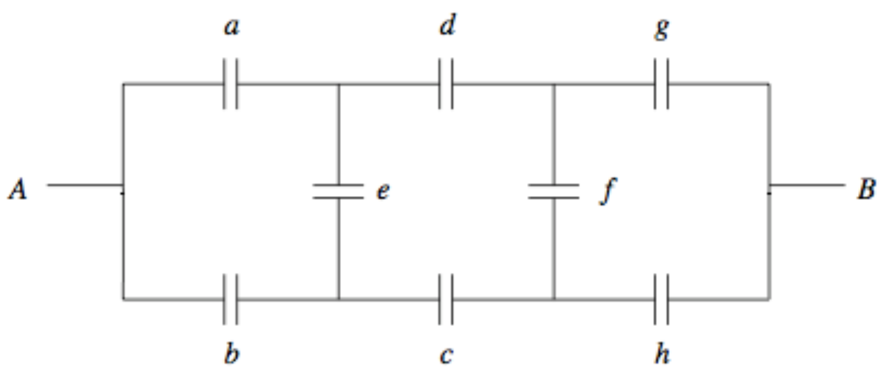
(e) With no prior information, the probability that the first roll is a 3 is $1/6$. Now let A be the event that the sum of the dice is 6, and B be the event that the first roll is a 3. The probability we wish to compute is:

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{1/36}{5/36} = \frac{1}{5}$$

Having additional information about the dice changes the probability that the first roll is a 3.

2 Communication Network

In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



(a) Given that exactly five links have failed, determine the probability that A can still communicate with B.

(b) Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

(c) Given that *a, d* and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

Solution:

(a) There are only two paths of 3 links from A to B. There are $\binom{8}{5}$ ways of the links messing up.

So, the probability is $\dfrac{2}{56} = \dfrac{1}{28}$.

This is because every single case of exactly 5 links being down have the same probability. So it's a uniform distribution over all possibilities.

(b) Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.

So, the probability is $\frac{30}{56} = \frac{15}{28}$.

(c) We would just want the 4 on the only remaining path from A to B not to be down.

The probability of this happening is $(1-p)^4$.

3 Marbles

Box A contains 1 black and 3 white marbles, and box B contains 2 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

(a) What is the probability that the marble is black?

(b) Given that the marble is white, what is the probability that it came from box A?

Solution:

(a)

$$\Pr(\text{black}) = \Pr(\text{black} \mid A)\Pr(A) + \Pr(\text{black} \mid B)\Pr(B) = \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{6} \cdot \frac{1}{2} = \frac{7}{24}.$$

(b)

$$\Pr(A \mid \text{white}) = \frac{\Pr(A \cap \text{white})}{\Pr(\text{white})} = \frac{\Pr(\text{white} \mid A)\Pr(A)}{\Pr(\text{white})} = \frac{3/4 \cdot 1/2}{17/24} = \frac{9}{17}.$$

4 Lie Detector

A lie detector is known to be $4/5$ reliable when the person is guilty and $9/10$ reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only $1/100$ have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

Solution:

Let A denote the event that the test indicates that the person is guilty, and B the event that the person is innocent. Note that

$$\Pr[B] = \frac{99}{100}, \quad \Pr[\overline{B}] = \frac{1}{100}, \quad \Pr[A \mid B] = \frac{1}{10}, \quad \Pr[A \mid \overline{B}] = \frac{4}{5}.$$

Using the Bayesian Inference Rule, we can compute the desired probability as follows:

$$\Pr[B \mid A] = \frac{\Pr[B]\Pr[A \mid B]}{\Pr[B]\Pr[A \mid B] + \Pr[\overline{B}]\Pr[A \mid \overline{B}]} = \frac{(99/100)(1/10)}{(99/100)(1/10) + (1/100)(4/5)} = \frac{99}{107}$$