CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Rao — DIS 6b

1 Clothes and Stuff

(a) Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, etc.). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

(b) It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

(c) If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

(d) We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters.)

(e) Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of d, your answer from the previous part.)

(f) Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

Solution:

(a) 3^5

(b) $14 \cdot 3^4$

(c) $n_1 \cdot n_2 \cdot n_3 \cdots n_k$

(d) $14!/10!$

(e) $\binom{14}{4}$ or written as a function of the previous part, $d/4!$

(f) Within each color category, the hats are indistinguishable, so we will use the stars-and-bars method. Specifically, consider the six choices you have to make to be six "stars", and you must place your stars in one of three color categories. Since there are three categories, we need two "bars" (dividers) to separate the categories. In total, we have $6+2=8$ stars and bars, and so there are $\binom{8}{6}$ ways to choose the positions of the stars (which corresponds to $\binom{8}{6}$ ways to choose our floppy hats). Equivalently, there are $\binom{8}{2}$ ways to choose the positions of the bars.

But let's be serious, you should just keep the black ones – so much more versatile.

2 Charming Star

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same. In this question, only the number of votes for each candidate matters; it does not matter which specific people voted for each candidate.

(a) How many possible voting combinations are there for the 5 candidates?

(b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

Solution:

(a) Let x_i be the number of votes of the i-th candidates. We would like to find all possible combinations of $(x_1, x_2, x_3, x_4, x_5)$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100.$$

It is equivalent to selecting $k = 100$ objects from $n = 5$ categories. The number of possible combinations is:

$$\binom{100+5-1}{100} = \binom{104}{100} = 4598126.$$

(b) Now we have a constraint that one of the x_i should be at least 51. Say, let x_1 be at least 51. It is equivalent to giving the first candidate 51 votes at the beginning and then distributing the remaining 49 votes to them again. The number of possible combinations is $\binom{49+5-1}{49}$. Since one of the 5 candidates could have at least 51 votes, the total number of possible voting combinations such that exactly one candidate gets more than 50 votes is:

$$\binom{5}{1}\binom{49+5-1}{49} = 1464125.$$

3 Combinatorial Proof VII

Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Solution:

LHS: Choose a group of k people out of a total of n people, then select a leader from the chosen group.

RHS: Select a leader out of the n people, then choose an additional $k-1$ people from the $n-1$ unselected people.

4 Combinatorial Proof VI

Prove $$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn.$$

Solution:

LHS: Choose 2 elements from $m+n$ elements.

RHS: Imagine the $m+n$ elements are split into two groups, one of size m and one of size n. To choose two elements, we can either choose both from the group of size m, both from the group of size n, or one from each group. All of these cases are disjoint, so the total number of ways to do this is $\binom{m}{2} + \binom{n}{2} + mn$.