CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Rao — DIS 8a

1 Balls and Bins

Throw n balls into n bins.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Give an upper bound on the probability that at least k bins are empty.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution:

(a) $\left(\frac{n-1}{n}\right)^n$.

(b) $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. Then

$$\Pr(A) = \Pr\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} \Pr(A_i)$$

where A_i is the event that the ith set of k bins is empty. We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\Pr(A_i) = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\Pr(A) \leq m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using probability rules:

$$\begin{aligned}\Pr[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\Pr[\text{2nd bin empty} \cap \text{1st bin empty}]}{\Pr[\text{1st bin empty}]} \\ &= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\ &= \left(\frac{n-2}{n-1}\right)^n\end{aligned} \tag{1}$$

Alternate solution:
We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $(n-2)/(n-1)$. For n total balls, this probability is $[(n-2)/(n-1)]^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $[(n-2)/(n-1)]^n$. The probability that the second bin is empty (without any prior information) is $[(n-1)/n]^n$. Since these probabilities are not equal, the events are dependent.

2 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

Solution:

(a) $\Pr(\text{chocolate egg}) = \frac{1}{3}$.

(b) Long calculation using Total Probability Rule: let C_i denote the event that the ith egg is chocolate, and P_i denote the event that the ith egg is plastic. We have

$$\begin{aligned}\Pr(C_2) &= \Pr(C_1 \cap C_2) + \Pr(P_1 \cap C_2) \\ &= \Pr(C_1)\Pr(C_2 \mid C_1) + \Pr(P_1)\Pr(C_2 \mid P_1) \\ &= \frac{1}{3}\cdot\frac{19}{59} + \frac{2}{3}\cdot\frac{20}{59} \\ &= \frac{1}{3}.\end{aligned} \tag{2}$$

Short calculation: By symmetry, this is the same probability as part (a), $1/3$. This is because we don't know what type of egg was picked on the first draw, so the distribution for the second egg is the same as that of the first.

(c) By symmetry, since we don't know any information about the 2nd, 3rd, or 4th eggs, $\Pr(\text{5th egg = plastic} \mid \text{1st egg = plastic}) = \Pr(\text{2nd egg = plastic} \mid \text{1st egg = plastic}) = 39/59$.

3 Head Count

Consider flipping a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if $\omega = (H,T)$ and 0 otherwise. What is the distribution of Y?

(d) Define a third random variable $Z = X + Y$. What is the distribution of Z?

Solution:

(a) $\{(T,T),(H,T),(T,H),(H,H)\}$.

(b)
$$X = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .5 \\ 2 & \text{w.p.} \quad .25 \end{cases}$$

(c)
$$Y = \begin{cases} 0 & \text{w.p.} \quad .75 \\ 1 & \text{w.p.} \quad .25 \end{cases}$$

(d) Let's determine the values Z can take and the corresponding probabilities:

- $Z = 0$: $\Pr(Z = 0) = \Pr(X = 0 \cap Y = 0) = \Pr(X = 0) \cdot \Pr(Y = 0 \mid X = 0) = .25 \cdot 1 = .25$
- $Z = 1$:
$$\begin{aligned} \Pr(Z = 1) &= \Pr(X = 0 \cap Y = 1) + \Pr(X = 1 \cap Y = 0) \\ &= \Pr(X = 0) \cdot \Pr(Y = 1 \mid X = 0) + \Pr(X = 1) \cdot \Pr(Y = 0 \mid X = 1) \\ &= .25 \cdot 0 + .5 \cdot .5 = .25 \end{aligned} \tag{3}$$
- $Z = 2$:
$$\begin{aligned} \Pr(Z = 2) &= \Pr(X = 1 \cap Y = 1) + \Pr(X = 2 \cap Y = 0) \\ &= \Pr(X = 1) \cdot \Pr(Y = 1 \mid X = 1) + \Pr(X = 2) \cdot \Pr(Y = 0 \mid X = 2) \\ &= .5 \cdot 5 + .25 \cdot 1 = .5 \end{aligned} \tag{4}$$

- $Z = 3$: $\Pr(Z = 3) = \Pr(X = 2 \cap Y = 1) = \Pr(X = 2) \cdot \Pr(Y = 1 \mid X = 2) = .25 \cdot 0 = 0$

$$Z = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .25 \\ 2 & \text{w.p.} \quad .5 \end{cases}$$

4 Head Count II

Now consider a new coin with $\Pr(\text{Heads}) = 2/5$. We'll flip the coin 20 times.

(a) As before, define X to be the number of heads. What is $\Pr(X = 7)$?

(b) What is $\Pr(X \geq 1)$?

(c) What is $\Pr(12 \leq X \leq 14)$?

Solution:

(a) X is a binomially distributed random variable.

$$\Pr(X = 7) = \binom{20}{7} \left(\frac{2}{5}\right)^7 \left(\frac{3}{5}\right)^{13}.$$

(b)

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(c)

$$\Pr(12 \leq X \leq 14) = \binom{20}{12} \left(\frac{2}{5}\right)^{12} \left(\frac{3}{5}\right)^{8} + \binom{20}{13} \left(\frac{2}{5}\right)^{13} \left(\frac{3}{5}\right)^{7} + \binom{20}{14} \left(\frac{2}{5}\right)^{14} \left(\frac{3}{5}\right)^{6}.$$