CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Satish Rao — DIS 00b

1. **Polya suggests: learning is better with friends!**

   Solver.

   (a) Prepare: comfortable position, pencil, paper, etc.

   (b) Read hints, suggestions, discuss with partner.

   (c) Read the problem aloud.

   (d) Solve on own. You speak, you solve, parter listens.

   (e) Speak! No need to choose words.

   (f) Go back over problem; “I’m stuck. I better start over.” “No that won’t work.” “Let’s see... hmmm.”

   (g) Try to solve even trivial problems!

   Listener.

   (a) Listener not a critic. “Please elaborate.” “What are you thinking now?” “Can you check that?”

   (b) Role: (a) Demand that PS keep talking but don’t interrupt. (b) Make sure that PS follows the strategy and doesn’t skip any of the steps. (c) Help PS improve his/her accuracy. (d) Help reflect the mental process PS is following. (e) Make sure you understand each step.

   (c) Do not turn away from PS and start to work on problem!!!!!

   (d) Do not let PS continue if:

      i. You don’t understand. “I don’t understand.” or “I don’t follow that.”

      ii. When there is a mistake. “Maybe check that.” or “Does that sound right?”

   (e) No hints! Point out errors, but no correction.

Solution:

2. **Pigeonhole Principle**

   Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

   Solution:

   Suppose this is not the case. Then all the boxes would contain at most 1 apple. Then the maximum number of apples we could have would be n, but this is a contradiction since we have $n+1$ apples.

3. **Contraposition**

   Prove that if $a+b < c+d$, then $a < c$ or $b < d$.

   Solution: Assume $a \geq c$ and $b \geq d$ (note that this is the negation of $a < c \vee b < d$). Then, $a+b \geq c+b \geq c+d$, which is the negation of $a+b < c+d$.

4. **Proof by?**

   (a) Prove that if $x, y \in \mathbb{Z}$, if 10 does not divide xy, then 10 does not divide x and 10 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ \ 10 \nmid xy \implies (10 \nmid x \ \wedge \ 10 \nmid y)$. What proof technique did you use?

   (b) Prove or disprove the contrapositive.

   (c) Prove or disprove the converse.

   Solution:

   (a) We will use proof by contraposition. For any arbitrary given x and y, the statement $10 \nmid xy \implies (10 \nmid x \wedge 10 \nmid y)$ is equivalent using contraposition to $\neg(10 \nmid x \wedge 10 \nmid y) \implies \neg(10 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(10 \mid x \vee 10 \mid y) \implies 10 \mid xy$.

   Now for this part, we give a proof by cases. Assuming that $10 \mid x \vee 10 \mid y$, one of the two cases must be true.

      i. $10 \mid x$: in this case $x = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10ky$ which is a multiple of 10. So $10 \mid xy$.

      ii. $10 \mid y$: in this case $y = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10kx$ which is a multiple of 10. So $10 \mid xy$.

   Therefore assuming $10 \mid x \vee 10 \mid y$ we proved $10 \mid xy$.

   We used proof by cases and proof by contraposition.

   (b) We proved the statement. The contrapositive of a statement has logically equivalent to the statement. So we are done.

   (c) Its not true! The converse is that if 10 does not divide x and does not divide y then 10 does not divide xy. We can choose $x = 2$ and $y = 5$ and see a counterexample to the statement.

5. **Perfect Square**

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

6. **Fermat's Contradiction**

Prove that $2^{1/n}$ is not rational for any integer $n > 3$. (*Hint*: Use Fermat's Last Theorem.)

Solution:

If not, then there exists an integer $n > 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies

$$q^n + q^n = p^n,$$

which is a contradiction to the Fermat's Last Theorem.