CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand HW 9

Due Thursday March 31 at 10PM

1. **Balls and Bins**You have n empty bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

   (a) What is the probability that the first ball thrown will cause the first collision?

   (b) What is the probability that the second ball thrown will cause the first collision?

   (c) What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?

   (d) What is the probability that the third ball thrown will cause the first collision?

   (e) What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?

   (f) What is the probability that the m^{th} ball thrown will cause the first collision?

2. **Flipping coins**

   (a) You have a fair coin, and you flip it 4 times. What is the probability that the number of heads is always ahead of the number of tails in the 4 flips? For example, the sequence HHHT has fits the description, but HTTH does not

   (b) What is the probability of getting 4 heads out of 4 flips, given that there are at least 2 heads?

   (c) Now assume that you are given two identical looking coins, but one is fair and the other is loaded, with $P(H) = 0.6$. You pick one uniformly at random, and toss it 3 times, getting 3 heads. What is the probability that you picked the loaded coin?

3. **Card Game**

   A game is played with six double-sided cards. One card has "1" on one side and "2" on the other. Two cards have "2" on one side and "3" on the other. And the last three cards have "3" on one side and "4" on the other. A random card is then drawn and held in a random orientation between two players, each of whom sees only one side of the card. The winner is the one seeing the smaller number. If the card that was drawn was a "2/3" card, compute the probabilities each player thinks he/she has for winning.

4. **Boys and Girls**

   There are three children in a family. A friend is told that at least two of them are boys. What is the probability that all three are boys? The friend is then told that the two are the oldest two children. Now what is the probability that all three are boys? Use Bayes' Law to explain this. Assume throughout that each child is independently either a boy or a girl with equal probability.

5. **Birthdays**

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that it takes more than 20 people for this to occur?

(b) What is the probability that it takes exactly 20 people for this to occur?

(c) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

6. **(Alvin's woes)** After a long night of debugging, Alvin has just perfected the new homework party/office hour queue system. CS 70 students sign themselves up for the queue, and TAs go through the queue, resolving requests one by one. Unfortunately, our newest TA (let's call him TA Bob) does not understand how to use the new queue: instead of resolving the requests in order, he always uses the Random Student button, which (as the name suggests) chooses a random student in the queue for him. To make matters worse, after helping the student, Bob forgets to click the Resolve button, so the student still remains in the queue! For this problem, assume that there are n total students in the queue.

(a) Suppose that Bob has already helped k students. What is the probability that the Random Student button will take him to a student who has not already been helped?

(b) Give a description of the probability space Ω. Fully answering this question entails giving a representation for an outcome ω, as well as any conditions on what ω is allowed to be. As an example, the probability space for two coin flips can be described as $\{HH, HT, TH, TT\}$. Each outcome ω is a length-two string of characters from the set $\{H, T\}$, where H represents heads and T represents tails; the first character of ω is the result of the first coin flip, and the second character is the result of the second coin flip. (Hint: Each outcome ω should include information about which students have been helped, along with the total number of Random Student button presses.)

(c) Let X_i^r be the event that TA Bob has not helped student i after pressing the Random Student button a total of r times. What is $\Pr[X_i^r]$? Assume that the results of the Random Student button are independent of each other. Now approximate the answer using the inequality $1 - x \leq e^{-x}$.

(d) Let T_r represent the event that TA Bob presses the Random Student button r times, but still has not been able to help all n students. (In other words, it takes TA Bob longer than r Random Student button presses before he manages to help every student). What is T_r in terms of the events X_i^r? (Hint: Events are subsets of the probability space Ω, so you should be thinking of set operations...)

(e) Using your answer for the previous part, what is an upper bound for $\Pr[T_r]$? (You may leave your answer in terms of $\Pr[X_i^r]$. Use the inequality $1 - x \leq e^{-x}$ from before.)

(f) Now let $r = \alpha n \ln n$. What is $\Pr[X_i^r]$?

(g) Calculate an upper bound for $\Pr[T_r]$ using the same value of r as before. (This is more formally known as a bound on the tail probability of the distribution of button presses required to help every student. This distribution will be explored in more detail later, in the context of random variables.)

(h) What value of r do you need to bound the tail probability by $1/n^2$? In other words, how many button presses are needed so that the probability that TA Bob has not helped every student is at most $1/n^2$?