Due Thursday April 7 at 10PM

Before you start your homework, write down your team. Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

1. (10 points) Suppose that Allen and Alvin are flipping coins for fun. Allen flips a fair coin k times and Alvin flips $n-k$ times. In total there are n coin flips. Prove that the probability that Allen and Alvin flip the same number of heads is equal to the probability that there are a total of k heads.

2. Suppose X follows a binomial distribution with parameters n and p.

   (a) (8 points) Prove that as x goes from 0 to n, $\mathbf{P}(X=x)$ first increases monotonically. After it reaches its largest value, $\mathbf{P}(X=x)$ then decreases monotonically.

   (b) (5 points) What is the value of x that $\mathbf{P}(X=x)$ reaches its largest value?

3. Suppose that Allen and Fan are playing a series of games. Assume all games are independent, Allen has probability of p winning each game and Fan has probability of $1-p$ winning. The winner of the series is the first one to win k games.

   (a) (8 points) If $k=4$, what's the probability that a total of 7 games are played?

   (b) (5 points) What's the maximum probability from previous part? What's the value of p when probability is maximized?

   (c) (10 points) If $k=3$, what's the expected number of games that are played?

   (d) (5 points) What's the maximum number from previous part? What's the value of p at that point?

4. (5 points) Two faulty machines, M_1 and M_2, are repeatedly run synchronously in parallel (i.e., both machines execute one run, then both execute a second run, and so on). On each run, M_1 fails with probability p_1 and M_2 fails with probability p_2, all failure events being independent. Let the random variables X_1, X_2 denote the number of runs until the first failure of M_1, M_2 respectively; thus X_1, X_2 have geometric distributions with parameters p_1, p_2 respectively. Let X denote the number of runs until the first failure of *either* machine. Show that X also has a geometric distribution, with parameter $p_1+p_2-p_1p_2$.

5. In this question, we will use another approach to calculate the expectation of the geometric distribution.

   (a) (8 points) If $S=\sum_{i=1}^{\infty} ir^{i-1}=1+2r+3r^2+\ldots$ where $-1<r<1$, prove

   $$S=\frac{1}{(1-r)^2}. \tag{1}$$

   (Hint: what is $S-rS$?)

(b) (8 points) Given a random variable X having the geometric distribution with parameter p where $0 < p \leq 1$, *i.e.*,

$$\mathbf{P}(X = i) = (1-p)^{i-1}p \qquad \text{for } i = 1, 2, 3, \ldots,$$

use Equation (1) to prove $\mathbf{E}(X) = \frac{1}{p}$.

6. Suppose Professor Walrand now offers you a game: you bet any amount of homework points in this homework, then you flip a coin and if it comes up heads you win that amount, and if it comes up tails you lose that amount. Suppose you follow this strategy: you start with a bet of 1 homework point, and if you lose you increase your bet to 2 homework points, and again if you lose you double your bet to 4 homework points, and so on. As soon as you win, you take your winnings and you go out (i.e. you bet 0 homework points for the next rounds). So in round n you bet 2^{n-1} homework points if you have lost all the previous rounds, and you bet 0 homework points if you have won any of the previous rounds.

   (a) (5 points) What is your net winnings (i.e. subtract your losses from your wins) if you win after the n-th coin flip.

   (b) (5 points) What is the expected value of your winnings on round i?

   (c) (5 points) What is you expected net winnings after round n?

7. (10 points) Suppose we put a total of r balls, independently one at a time, in k boxes. Probability that each ball goes into box i is p_i, such that $\sum_{i=1} p_i = 1$. Collision occurs when we put a ball in a nonempty box. There are 9 collisions if a bin is filled with 10 balls. What's the expected number of collisions.

8. Suppose you roll a fair die n times. What is the expectations of each of the following random variables?

   (a) (5 points) A is the random variable that denotes the sum of the numbers in those rolls.

   (b) (5 points) B is the random variable that denotes the maximum number in the those rolls.

   (c) (8 points) C is the random variable that denotes the sum of the largest two numbers in the first three rolls.

   (d) (8 points) D is the random variable that denotes the number of multiples of three in those rolls.

   (e) (8 points) E is the random variable that denotes the number of faces which fail to appear in those rolls.

   (f) (5 points) F is the random variable that denotes the number of distinct faces that appear in those rolls.

9. (10 points) Suppose you flip a biased coin until two of the most recent three flips are heads. The probability that you see a head is p. X is the random variable that denotes the number of total flips. If the first two flips are heads, then $X = 2$. Find the expected value of X.

10. Suppose you flip a biased coin n times ($n > 4$). The probability that you see a head is p. Let's define the concept of a *run* of three heads. It can be the first four flips in the pattern $HHHT$, the last four flips in the pattern $THHH$, or elsewhere in the flips in the pattern $THHHT$. Let $R(3,n)$ denotes the number of runs of three heads in the n trials.

(a) (8 points) Find the expectation of $R(3,n)$.

(b) (8 points) Let's now define $R(m,n)$ as the number of heads of length m in n flips, similarly for $1 \leq m \leq n$. Find the expectation of $R(m,n)$.

(c) (8 points) Let $R(n)$ be the total number of non-overlapping head runs in n trials, counting runs of any length between 1 and n. Find the expectation of $R(n)$ by using the result of part b.

(d) (8 points) Find the expectation of $R(n)$ another way by considering for each $1 \leq j \leq n$ the number of runs that start on the jth trial. Check that the two methods give the same answer.