CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand HW 16

Due Thursday April 28 at 10PM

Before you start your homework, write down your team. Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

1. **Limiting Distribution**

   This problem invites you to test your understanding of the limiting distribution of a Markov chain.

   (a) (5 points) Construct a Markov chain that is not irreducible but that has a unique distribution and is such that its distribution converges to that unique invariant distribution, for any initial distribution.

   (b) (5 points) Show a Markov chain whose distribution converges to a limit that depends on the initial distribution.

   Answer:

   (a) We saw that if a Markov chain is irreducible and aperiodic, its has a unique invariant distribution and its distribution converges to that invariant distribution. The point of this problem is to stress the fact that a Markov chain that is not irreducible may also have these properties.

   A simple example is a Markov chain with two states 0 and 1 and such that $P(0,1) = a > 0$ and $P(1,1) = 1$. This Markov chain is not irreducible since it cannot go from 1 to 0. It has a unique invariant distribution which is $\pi = [0,1]$. Also, it is clear that $\pi_n \to \pi$ as $n \to \infty$, for all choices of π_0.

   (b) The simplest example is a Markov chain that does not move. i.e., whose transition probability matrix is the identity matrix.

2. **Aperiodicity**

   In this problem, we explore the concept of aperiodicity.

   (a) (5 points) Can you find a finite irreducible aperiodic Markov chain whose distribution does not converge?

   (b) (5 points) Construct a finite Markov chain that is a sequence of i.i.d. random variables. Is it necessarily irreducible and aperiodic?

   Answer:

   (a) A key result we saw in the lecture is that there is no such Markov chain.

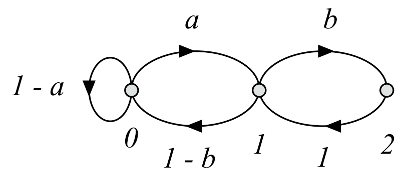

Figure 1: Markov chain for problem 4.

(b) Let $\{X(n), n \geq 0\}$ be a sequence of i.i.d. random variables that take values in some finite set $\{1, \ldots, K\}$ with $P(X(n) = k) = p_k > 0$ for $k = 1, \ldots, K$. This sequence is such that

$$P[X(n+1) = k | X(n), \ldots, X(0)] = p_k, \text{ for } k = 1, \ldots, K \text{ and } n \geq 0.$$

Thus, this sequence is a Markov chain. Also, it is necessarily irreducible, since it can go from every k to every other k' (in fact, in one step). Finally, it is also irreducible since $P(k,k) = p_k > 0$. Its invariant distribution is $\pi = [p_1, \ldots, p_K]$.

3. **Function of a Markov Chain** (5 points)

Show that a function $Y(n) = g(X(n))$ of a Markov chain $X(n)$ may not be a Markov chain.

Answer: Here is a simple example. Consider a Markov chain $X(n)$ on $\{0,1,2\}$ with $P(0,1) = P(1,2) = P(2,0) = 1$. Define $g(0) = g(1) = 0$ and $g(2) = 1$. The claim is that if $\pi_0 = [1/3, 1/3, 1/3]$, then $Y(n) = g(X(n))$ is not a Markov chain. To see this, note that

$$P[Y(2) = 0 | Y(1) = 0, Y(0)] = 0 \neq P[Y(2) = 0 | Y(1) = 0] = 0.5.$$

The intuition is that $g(X(n))$ contains less information than $X(n)$ if $g(\cdots)$ is many-to-one. Thus, although $X(n)$ contains all the information needed to predict its future, the same is not true of $g(X(n))$.

4. **Analyze a Markov Chain** (20 points)

Consider the Markov chain $X(n)$ with the state diagram shown in Figure 1 where $a, b \in (0,1)$.

a) Show that this Markov chain is aperiodic;

b) Calculate $P[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0]$;

c) Calculate the invariant distribution;

d) Let $T_i = \min\{n \geq 0 \mid X(n) = i\}$. Calculate $E[T_2 \mid X(0) = 1]$.

Answer:

(a) The Markov chain is irreducible because $a, b \in (0,1)$. Also, $P(0,0) > 0$, so that

$$\text{g.c.d.}\{n > 0 | P^n(0,0) > 0\} = \text{g.c.d.}\{1, 2, 3, \ldots\} = 1,$$

which shows that the Markov chain is aperiodic.

(b) We see that the probability is

$$P(0,1)P(1,0)P(0,0)P(0,1) = a(1-b)(1-a)a.$$

(c) The balance equations are

$$\begin{aligned}\pi(0) &= (1-a)\pi(0) + (1-b)\pi(1)\\ \pi(1) &= a\pi(0) + \pi(2).\end{aligned}$$

After some simple manipulations, we see that they imply the following equations:

$$\begin{aligned}a\pi(0) &= (1-b)\pi(1)\\ b\pi(1) &= \pi(2).\end{aligned}$$

These equations express the equality of the probability of a jump from i to $i+1$ and from $i+1$ to i, for $i=0$ and $i=1$, respectively. These relations are called the 'detailed balance equations.' From these equations we find successively that

$$\pi(1) = \frac{a}{1-b}\pi(0) \text{ and } \pi(2) = b\pi(1) = \frac{ab}{1-b}\pi(0).$$

The normalization equation is

$$\begin{aligned}1 &= \pi(0) + \pi(1) + \pi(2) = \pi(0)[1 + \frac{a}{1-b} + \frac{ab}{1-b}]\\ &= \pi(0)\frac{1-b+a+ab}{1-b},\end{aligned}$$

so that

$$\pi(0) = \frac{1-b}{1-b+a+ab}.$$

Thus,

$$\pi = \frac{1}{1-b+a+ab}[1-b, a, ab].$$

(c) We define

$$\beta(i) = E[T_2|X(0) = i], i = 0,1,2.$$

The FSE are $\beta(2) = 0$ and

$$\begin{aligned}\beta(0) &= 1 + (1-a)\beta(0) + a\beta(1)\\ \beta(1) &= 1 + (1-b)\beta(0).\end{aligned}$$

The first equation is equivalent to

$$\beta(0) = \frac{1}{a} + \beta(1).$$

Substituting this expression in the second equation, we get

$$\beta(1) = 1 + (1-b)[\frac{1}{a} + \beta(1)] = (1-b)\beta(1) + \frac{1+1-b}{a},$$

so that

$$\beta(1) = \frac{1+1-b}{ab}.$$

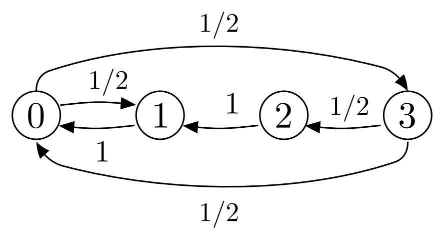

Figure 2: Markov chain for problem 5.

5. **Period of States** (10 points)
Calculate explicitly $d(0), d(1), d(2)$ and $d(3)$, defined as

$$d(i) = \text{g.c.d}\{n > 0 \mid Pr[X_n = i | X_0 = i] > 0\}$$

for the Markov chain of Figure 2. That is, for each state i, identify the set $\{n > 0 \mid Pr[X_n = i | X_0 = i] > 0\}$ and finds its g.c.d.

Answer:

(a) We see that

$$d(0) = \text{g.c.d.}\{2, 4, 6, \ldots\} = 2.$$

(b) Similarly

$$d(1) = \text{g.c.d.}\{2, 4, 6, \ldots\} = 2.$$

(c) One has

$$d(2) = \text{g.c.d.}\{4, 6, 8, \ldots\} = 2.$$

(d) Finally,

$$d(3) = \text{g.c.d.}\{2, 4, \ldots\} = 2.$$

Since the Markov chain is irreducible, all the states have the same period.