1 Stable Marriage

Consider the set of men $M = \{1,2,3\}$ and the set of women $W = \{A,B,C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work)

Solution:

The algorithm takes 3 days to produce a matching. The resulting pairing is

$$\{(A,1),(B,2),(C,3)\}.$$

Woman	Day 1	Day 2	Day 3
A	1,3	1	1
B	2	2,3	2
C			3

2 Propose-and-Reject Proofs

Prove the following statements about the traditional propose-and-reject algorithm.

(a) In any execution of the algorithm, if a woman receives a proposal on day i, then she receives some proposal on every day thereafter until termination.

(b) In any execution of the algorithm, if a woman receives no proposal on day i, then she receives no proposal on any previous day j, $1 \leq j < i$.

(c) In any execution of the algorithm, there is at least one woman who only receives a single proposal. (Hint: use the parts above!)

Solution:

1. The idea is to use the Improvement Lemma (remind students of the Improvement Lemma). If a woman receives a proposal on day i, then, by the Improvement Lemma, she will always have someone as good or better proposing to her every day after day i.

2. One way is to use a proof by contradiction. Assume that a woman receives no proposal on day i but did receive a proposal on some previous day j, $1 \leq j < i$. By the Improvement Lemma, since the woman received a proposal on day j, then she will always have someone as good or better proposing to her every day after day j. But then the woman must receive a proposal on day $i > j$. Contradiction.

3. Let's say the algorithm takes k days - then we know that every woman receives a proposal on day k. There is at least one woman w who does not receive a proposal on day $k-1$. (You will prove this in your homework!) Then from part (b), since w did not receive a proposal on day $k-1$, she didn't receive a proposal on any day before k. Since w was not proposed to on days $1,\dots,k-1$ and is proposed to on day k (since we know this is when the algorithm terminates) then w receives only one proposal.

3 Be a Judge

For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

(a) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.

(b) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.

(c) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.

(d) For every $n > 1$, there is a stable marriage instance with n men and n women which has an unstable pairing in which every unmatched man-woman pair is a rogue couple.

Solution:

(a) **False**: If this were to occur it would mean that at the end of the algorithm, every man would have proposed to every woman on his list and has been rejected $n-1$ times. This would also require every woman to reject $n-1$ suitors. We know this is impossible though, as we learned above that at least one woman receives a single proposal. There must be at least one woman who is not proposed to until the very last day.

(b) **True**: We give a simple proof by contradiction. Assume that M and W can put each other at the top of their respective preference lists, but M and W are not paired with each other in some stable pairing. Then we have a stable pairing which includes the pairings (M, W'), (M', W), for some man M' and woman W'. However, M prefers W over his partner in this pairing, since W is at the top of his preference list. Similarly W prefers M over her partner. Thus (M,W) form a rogue couple, so the pairing is not stable. We have arrived at a contradiction.

Therefore if man M and woman W put each other at the top of their respective preference lists, then M must be paired with W in a stable pairing.

(c) **False**: The key here is to realize that this is possible if man M and woman W are at the bottom of everybody else's preference list as well. Consider the following example with the men m and M and the women w and W. Suppose that their preference lists are as follows:

$$\begin{array}{rcl} m & : & w, W \\ M & : & w, W \\ w & : & m, M \\ W & : & m, M \end{array}$$

It is clear that M and W are at the bottom of each other's preference lists; however, it is also true that (m,w) and (M,W) is a stable pairing (indeed, it is the only stable pairing). So, we have a contradiction to the statement: here is a stable marriage instance with at least two men and two women, and man M and woman W put each other at the bottom of their respective preference lists, but yet M and W are paired together in a stable pairing.

(d) **True**: Suppose $n > 1$ and we have men $M_1, \ldots, M_n$ and women $W_1, \ldots, W_n$. Further, assume that for $1 \leq i \leq n$, preference lists are as follows for every man M_i and woman W_i:

$$\begin{array}{ll} & \textit{highest} \Longrightarrow \textit{lowest} \\ M_i : & W_i \;\; W_{i+1} \;\; W_{i+2} \;\; \ldots \;\; W_{i-1} \end{array}$$

$$\begin{array}{ll} & \textit{highest} \Longrightarrow \textit{lowest} \\ W_i : & M_i \;\; M_{i-1} \;\; M_{i-2} \;\; \ldots \;\; M_{i+1} \end{array}$$

Note that the indices are taken modulo n, so if i refers to $n+1$ in the preference lists above, it is really referring to 1. The idea in this construction is that there is a fixed ordering of men into a cycle, and a fixed ordering of women into another cycle. Every man's preference list complies to the ordering of women into the cycle, with the only difference between different men's preferences being where in the ordering the preference list begins. The analogous situation holds for women's preference lists.

Now consider the unstable pairing in which each man M_i, $1 \leq i \leq n$ is paired as (M_i, W_{i-1}). (M_1 is paired to W_n.) We claim every unmatched man-woman pair is a rogue couple.

In this pairing, every man M_i is paired with woman W_{i-1} at the bottom of his preference list, and every woman W_i is paired with man M_{i+1} at the bottom of her preference list. Thus every man prefers any woman he has not been matched to over his partner, and likewise for women. So any unmatched pair (M,W) is a rogue couple.