CS 70 Discrete Mathematics and Probability Theory
Spring 2017 Rao **DIS 13a**

1 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0,1]$.

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbf{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbf{E}(X) = \int_0^\infty \Pr(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbf{E}(Z)$. [*Hint*: Find the CDF.]

Solution:

(a) To calculate $\Pr(Y > y)$, this means that each X_i is greater than y, so $\Pr(Y > y) = (1-y)^n$. We then use the tail sum formula:

$$\begin{aligned}
\mathbf{E}(Y) &= \int_0^1 \Pr(Y > y)\,\mathrm{d}y \\
&= \int_0^1 (1-y)^n\,\mathrm{d}y \\
&= -\frac{1}{n+1}(1-y)^{n+1}\Big|_0^1 \\
&= \frac{1}{n+1}.
\end{aligned}$$

Alternative Solution 1:
As explained above, $Pr[Y \leq y] = 1-(1-y)^n$. This gives us the CDF, and if we take its derivative we'll get the probability density function $f(y) = n(1-y)^{n-1}$.

Then
$$\mathbf{E}(Y) = \int_0^1 y \cdot n(1-y)^{n-1}\,\mathrm{d}y.$$

Perform a u substitution, where $u = 1-y$ and $\mathrm{d}u = -\mathrm{d}y$. We see:

$$\begin{aligned}
\mathbf{E}(Y) &= n \cdot \int_0^1 -(1-u) \cdot u^{n-1}\,\mathrm{d}u \\
&= n \cdot \int_0^1 (u^n - u^{n-1})\,\mathrm{d}u \\
&= n\left[\frac{u^{n+1}}{n+1} - \frac{u^n}{n}\right]_{u=0}^{1} \\
&= n\left[\frac{(1-y)^{n+1}}{n+1} - \frac{(1-y)^n}{n}\right]_{y=0}^{1} \\
&= n\left[0 - \left(\frac{1}{n+1} - \frac{1}{n}\right)\right] \\
&= n\left[\frac{1}{n} - \frac{1}{n+1}\right] \\
&= \frac{1}{n+1}.
\end{aligned}$$

Alternative Solution 2:
Consider adding another independent uniform variable X_{n+1}. $\Pr(X_{n+1} < Y)$ is just the probability that X_{n+1} is the minimum, which is $1/(n+1)$ by symmetry since all the X_i's are identical. It so happens that because X_{n+1} is a uniform variable on [0,1], this probability is equal to $\mathbf{E}(Y)$. Let f_Y denote the pdf of Y.

$$\begin{aligned}
\Pr(X_{n+1} < Y) &= \int_0^1 \Pr(X_{n+1} < y \mid Y = y) f_Y(y)\,\mathrm{d}y \\
&= \int_0^1 \Pr(X_{n+1} < y) f_Y(y)\,\mathrm{d}y && \text{(by independence)} \\
&= \int_0^1 y f_Y(y)\,\mathrm{d}y && \text{(CDF of the uniform distribution)} \\
&= \mathbf{E}(Y).
\end{aligned}$$

Alternative Solution 3:
Since each X_i is i.i.d., their values split the interval $[0,1]$ into $n+1$ sections, and we expect these sections to be of equal length because the X_i are uniformly distributed. Therefore, $\mathbf{E}(Y) = 1/(n+1)$, the position of the smallest indicator.

(b) We could use the tail sum formula, but it turns out that the CDF is in a form that makes it easy to take an integral. If $Z \le z$, each X_i must be less than z, which happens with probability z, so $\Pr[Z \le z] = z^n$. This gives us the CDF, and if we take its derivative we'll get the probability density function $f(z) = nz^{n-1}$. Then

$$
\begin{aligned}
\mathbf{E}(Z) &= \int_0^1 z \cdot n z^{n-1} \, \mathrm{d}z \\
&= \int_0^1 n z^n \, \mathrm{d}z \\
&= \left[n \cdot \frac{z^{n+1}}{n+1} \right]_{z=0}^{1} \\
&= \frac{n}{n+1}.
\end{aligned}
$$

Alternative Solution:
As in the previous part, add another independent uniform random variable X_{n+1}. The probability $\Pr(X_{n+1} > Z)$ is just the probability that X_{n+1} is the maximum, which is $1/(n+1)$ by symmetry.

$$
\begin{aligned}
\Pr(X_{n+1} > Z) &= \int_0^1 \Pr(X_{n+1} > z \mid Z = z) f_Z(z) \, \mathrm{d}z \\
&= \int_0^1 \Pr(X_{n+1} > z) f_Z(z) \, \mathrm{d}z \\
&= \int_0^1 (1 - z) f_Z(z) \, \mathrm{d}z \\
&= \int_0^1 f_Z(z) \, \mathrm{d}z - \int_0^1 z f_Z(z) \, \mathrm{d}z \\
\frac{1}{n+1} &= 1 - \mathbf{E}(Z) \\
\mathbf{E}(Z) &= \frac{n}{n+1}
\end{aligned}
$$

Alternative Solution 2:
Since each X_i is i.i.d., their values split the interval $[0,1]$ into $n+1$ sections, and we expect these sections to be of equal length because the X_i are uniformly distributed. The expectation of the smallest X_i is $1/(n+1)$, the expectation of the second smallest is $2/(n+1)$, etc. Therefore, $\mathbf{E}(Z) = n/(n+1)$, the position of the largest indicator.

2 Conditioning on Exponentials

Let X_i be i.i.d. $\text{Expo}(\lambda)$ random variables.

(a) Compute $\mathbf{E}[Y \mid Z]$, where $Y = \max\{X_1, X_2\}$ and $Z = \min\{X_1, X_2\}$.

(b) Compute $\mathbf{E}[X_1 + X_2 \mid Z]$. [*Hint*: Use part (a).]

(c) Use part (b) to compute $\mathbf{E}[Z]$.

(d) Compute $\mathbf{E}[X_1 + X_2 \mid X_1 + X_2 + X_3]$.

Solution:

(a) View the exponential distributions as light bulbs burning out. Conditioned on Z, we are looking for the expected time until the next light bulb burns out, which by the memoryless property is $1/\lambda$. Therefore, $\mathbf{E}[Y \mid Z] = Z + 1/\lambda$.

(b) Observe that $X_1 + X_2 = \min\{X_1, X_2\} + \max\{X_1, X_2\}$.

$$\mathbf{E}[X_1 + X_2 \mid Z] = \mathbf{E}[\min\{X_1, X_2\} \mid Z] + \mathbf{E}[\max\{X_1, X_2\} \mid Z] = Z + Z + \frac{1}{\lambda} = 2Z + \frac{1}{\lambda}.$$

(c) Take expectations of both sides and apply iterated expectation.

$$\mathbf{E}[X_1 + X_2] = \mathbf{E}[\mathbf{E}[X_1 + X_2 \mid Z]] = 2\mathbf{E}[Z] + \frac{1}{\lambda}$$

Note that $\mathbf{E}[X_1 + X_2] = 2/\lambda$ by linearity, so we find that $\mathbf{E}[Z] = 1/(2\lambda)$. (In fact, we can prove a stronger result: the minimum of two independent exponentials with parameter λ is exponential with parameter 2λ.)

(d) Given $X_1 + X_2 + X_3$, the i.i.d. nature of the exponentials tells us that we should expect

$$\mathbf{E}[X_1 + X_2 \mid X_1 + X_2 + X_3] = \frac{2}{3}(X_1 + X_2 + X_3).$$

3 Bayesian Darts

You play a game of darts with your friend. You are better than he is, and the distances of your darts to the center of the target are i.i.d. $U[0,1]$ whereas his are i.i.d. $U[0,2]$. To make the game fair, you agree that you will throw one dart and he will throw two darts. The dart closest to the center wins the game. What is the probability that you will win? *Note*: The distances *from the center of the board* are uniform.

Solution:

Let X be the distance of your closest dart to the center and Y that of the closest of your friend's darts. Then, for $x \in [0,1]$ and $y \in [0,2]$,

$$\Pr[X > x] = (1 - x) \quad \text{and} \quad \Pr[Y > y] = \left(1 - \frac{y}{2}\right)^2.$$

Hence,

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - \Pr[X > x]) = -\frac{\mathrm{d}}{\mathrm{d}x}(1 - x) = 1, \qquad x \in [0,1].$$

Also,

$$\Pr[Y > X \mid X = x] = \left(1 - \frac{x}{2}\right)^2.$$

Thus,

$$\Pr[Y > X] = \int_0^1 \Pr(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \mathbf{E}\left[\left(1 - \frac{X}{2}\right)^2\right] = \mathbf{E}\left[1 - X + \frac{X^2}{4}\right]$$
$$= 1 - \frac{1}{2} + \frac{1}{12} = \frac{7}{12}.$$

since $\mathbf{E}[X] = 1/2$ and $\mathbf{E}[X^2] = \int_0^1 x^2\,\mathrm{d}x = 1/3$.