CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand HW 9

Due Thursday March 31 at 10PM

1. **Balls and Bins**You have n empty bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

   (a) What is the probability that the first ball thrown will cause the first collision?

   Answer: 0

   (b) What is the probability that the second ball thrown will cause the first collision? **Answer:** $\frac{1}{n}$

   (c) What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision? **Answer:** $\frac{2}{n}$

   (d) What is the probability that the third ball thrown will cause the first collision? **Answer:** Basically: $P(\text{Ball 3 collides} \mid \text{Ball 1, 2 do not collide}) \cdot P(\text{Ball 1, 2 do not collide})$, which is $\frac{2}{n} \cdot \frac{n-1}{n}$.

   (e) What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision? **Answer:** $\frac{m-1}{n}$

   (f) What is the probability that the m^{th} ball thrown will cause the first collision? **Answer:** Similar to (d), $\frac{m-1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-m+2}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$.

2. **Flipping coins**

   (a) You have a fair coin, and you flip it 4 times. What is the probability that the number of heads is always ahead of the number of tails in the 4 flips? For example, the sequence HHHT has fits the description, but HTTH does not

   Answer: Let X_i denote the outcome of the i^{th} toss. X_1 must be H, since otherwise you have 1 head and 0 tails. T X_2 must also be H, since otherwise you have 1 head and 1 tail, so #heads is not strictly ahead. If X_3 is H, then X_4 can be either H or T. If X_3 is T, then X_4 must be H (otherwise #heads is not strictly ahead)

   So total probability is

   $$P(X_1 = H)P(X_2 = H)(P(X_3 = H)(P(X_4 = H) + P(X_4 = T)) + P(X_3 = T)(P(X_4) = H))$$

   which evaluates to

   $$\frac{1}{2} \cdot \frac{1}{2}(\frac{1}{2}(\frac{1}{2} + \frac{1}{2}) + \frac{1}{2} \cdot \frac{1}{2}) = \frac{3}{16}$$

   (b) What is the probability of getting 4 heads out of 4 flips, given that there are at least 2 heads?

   Answer:

   $$P(4 \text{ heads}, \text{at least 2 heads}) = P(4 \text{ heads}) = (\frac{1}{2})^4 = \frac{1}{16}$$

$$
\begin{aligned}
&P(\text{at least 2 heads})\\
=&1-P(\text{exactly 1 head})-P(\text{exactly 0 heads})\\
=&1-4\cdot(\frac{1}{2})^4-(\frac{1}{2})^4\\
=&1-\frac{1}{4}-\frac{1}{16}\\
=&\frac{11}{16}
\end{aligned}
$$

Using Bayes rule

$$P(\text{4 heads}|atleast2heads)=\frac{P(\text{4 heads, at least 2 heads}}{P(\text{at least 2 heads})}=\frac{1/16}{11/16}=\frac{1}{11}$$

(c) Now assume that you are given two identical looking coins, but one is fair and the other is loaded, with $P(H)=0.6$. You pick one uniformly at random, and toss it 3 times, getting 3 heads. What is the probability that you picked the loaded coin?

Answer:

$$P(\textbf{loaded},\textbf{3 heads})=P(\textbf{3 heads}|\textbf{loaded})P(\textbf{loaded})=0.6^3\cdot 0.5=0.108$$

$$P(\textbf{fair},\textbf{3 heads})=P(\textbf{3 heads}|\textbf{fair})P(\textbf{fair})=0.5^3\cdot 0.5=0.0625$$

Using bayes rule

$$
\begin{aligned}
&P(\textbf{loaded}|\textbf{3 heads})\\
=&\frac{P(\textbf{loaded},\textbf{3 heads})}{P(\textbf{3 heads})}\\
=&\frac{P(\textbf{loaded},\textbf{3 heads})}{P(\textbf{loaded},\textbf{3 heads})+P(\textbf{fair},\textbf{3 heads})}\\
=&\frac{0.6^3\cdot 0.5}{0.6^3\cdot 0.5+0.5^4}\\
=&0.633\quad\text{(roughly)}
\end{aligned}
$$

3. **Card Game**

A game is played with six double-sided cards. One card has "1" on one side and "2" on the other. Two cards have "2" on one side and "3" on the other. And the last three cards have "3" on one side and "4" on the other. A random card is then drawn and held in a random orientation between two players, each of whom sees only one side of the card. The winner is the one seeing the smaller number. If the card that was drawn was a "2/3" card, compute the probabilities each player thinks he/she has for winning.

Answer: For the player that sees the "2" side:

$$
\begin{aligned}
&\Pr[\text{Win}\mid\text{Player sees 2}]=\frac{\Pr[\text{Win}\cap\text{Player sees 2}]}{\Pr[\text{Player sees 2}]}=\frac{\Pr[\text{Card is "2/3"}\cap\text{Player sees 2}]}{\Pr[\text{Player sees 2}]}\\
&=\frac{\Pr[\text{Card is "2/3"}]\cdot\Pr[\text{Player sees 2}\mid\text{Card is "2/3"}]}{\Pr[\text{Card has a "2"}]\cdot\Pr[\text{The side with "2" is chosen}\mid\text{Card has a "2"}]}\\
&=\frac{\frac{2}{6}\times\frac{1}{2}}{\frac{3}{6}\times\frac{1}{2}}=\frac{2}{3}
\end{aligned}
$$

For the player that sees the "3" side:

$$\Pr[\text{Win} \mid \text{Player sees 3}] = \frac{\Pr[\text{Win} \cap \text{Player sees 3}]}{\Pr[\text{Player sees 3}]} = \frac{\Pr[\text{Card is "3/4"} \cap \text{Player sees 3}]}{\Pr[\text{Player sees 3}]}$$
$$= \frac{\Pr[\text{Card is "3/4"}] \cdot \Pr[\text{Player sees 3} \mid \text{Card is "3/4"}]}{\Pr[\text{Card has a "3"}] \cdot \Pr[\text{The side with "3" is chosen} \mid \text{Card has a "3"}]}$$
$$= \frac{\frac{3}{6} \times \frac{1}{2}}{\frac{5}{6} \times \frac{1}{2}} = \frac{3}{5}$$

4. **Boys and Girls**

There are three children in a family. A friend is told that at least two of them are boys. What is the probability that all three are boys? The friend is then told that the two are the oldest two children. Now what is the probability that all three are boys? Use Bayes' Law to explain this. Assume throughout that each child is independently either a boy or a girl with equal probability.

Answer: Let A be the information that you are told and B the event that all three are boys. By Bayes' rule, we have $\Pr[B \mid A] = \frac{\Pr[B] \cdot \Pr[A|B]}{\Pr[A]}$. However, note that whether $A =$ "at least two boys" or $A =$ "oldest two are boys", $\Pr[A \mid B]$ is simply 1. So in either case we have $\Pr[B \mid A] = \frac{\Pr[B]}{\Pr[A]}$.

Hence,

$$\Pr[\text{All 3 boys} \mid \text{At least 2 boys}] = \frac{\Pr[\text{All 3 boys}]}{\Pr[\text{At least 2 boys}]} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

$$\Pr[\text{All 3 boys} \mid \text{Oldest 2 are boys}] = \frac{\Pr[\text{All 3 boys}]}{\Pr[\text{Oldest 2 are boys}]} = \frac{\frac{1}{8}}{\frac{2}{8}} = \frac{1}{2}.$$

In this case we saw Bayes' rule simplify to $\Pr[B \mid A] = \frac{\Pr[B]}{\Pr[A]}$. Since B is a subset of A, the formula directly shows that this conditional probability depends only on the number of possibilities contained in A. When we are told that at least two children are boys, any of the three children could be a girl. In contrast, if we are told that the oldest two children are boys, then only the youngest child has the possibility of being a girl. Therefore the latter case has fewer possibilities and therefore larger conditional probability.

5. **Birthdays**

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that it takes more than 20 people for this to occur?

Answer: $\Pr[\text{it takes more than 20 people}] = \Pr[\text{20 people don't have the same birthday}] = \frac{\frac{365!}{(365-20)!}}{365^{20}} = \boxed{\frac{\frac{365!}{345!}}{365^{20}}} \approx .589$

Another explanation that does not use counting:

Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes more than 20 people}]\\
=&\Pr[b_{20} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19] \times \Pr[b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
=&\Pr[b_{20} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19] \times \Pr[b_{19} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 18] \times\\
&\quad \cdots \times \Pr[b_3 \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 2] \times \Pr[b_2 \neq b_1]\\
=&\frac{365-19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365}\\
\approx&.589
\end{aligned}$$

(b) What is the probability that it takes exactly 20 people for this to occur?
Answer: $\Pr[\text{it takes exactly 20 people}] =$

$\Pr[\text{first 19 have different birthdays } \textbf{and } 20^{\text{th}}\text{person shares a birthday with one of the first 19}]$.
How total ways can the birthdays be chosen for 20 people? 365^{20}. How many ways can the birthdays be chosen so the first 19 have different birthdays and the 20^{th} person shares a birthday with the first 19? Well, the firs person has 365 choices, the second has 364 choices left, and so on until the nineteenth person has $(365-19+1) = 347$ choices left. Then, the 20^{th} person has 19 choices for his birthday. So in total, there are $365 \cdot 364 \cdot \cdots \cdot 348 \cdot 347 \cdot 19 = \frac{365!}{346!} \cdot 19$ ways of getting what we want. So $\Pr[\text{it takes exactly 20 people}] = \frac{365 \cdot 364 \cdot \cdots \cdot 348 \cdot 347 \cdot 19}{365^{20}} = \boxed{\frac{\frac{365!}{346!} \cdot 19}{365^{20}}} \approx .032$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes exactly 20 people}]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19] \times\\
&\Pr[b_i\text{'s are all different } \forall 1 \leq i \leq 19]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \leq i \leq 19] \times\\
&\Pr[b_{19} \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 18] \times \cdots \times\\
&\Pr[b_3 \neq b_i \mid b_i\text{'s are all different } \forall 1 \leq i \leq 2] \times \Pr[b_2 \neq b_1]\\
=&\frac{19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365}\\
\approx&.032
\end{aligned}$$

(c) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?
Answer: $\Pr[\text{it takes exactly 20 people}] =$

$\Pr[\text{first 19 don't have your birthday } \textbf{and } 20^{\text{th}}\text{person has your birthday}]$.
Similar to the last problem, there are 364 choices for the first person's birthday to be different than yours, 364 for the second person, and so on until the nineteenth person has 364 choices. Then, the 20^{th} person has exactly 1 choice to have your birthday. So the total number of ways to get what we want is $364^{19} \cdot 1$. There are 365^{20} possibilities total. So $\Pr[\text{it takes exactly 20 people}] =$

$\boxed{\frac{364^{19}}{365^{20}}} \approx .0026$

Another explanation that does not use counting:

$$
\begin{aligned}
\Pr[\text{it takes exactly 20 people}] = & \Pr[\text{the 1st person does not have the same birthday as yours}] \times \\
& \Pr[\text{the 2nd person does not have the same birthday as yours}] \times \\
& \cdots \times \Pr[\text{the 19th person does not have the same birthday as yours}] \times \\
& \Pr[\text{the 20th person has the same birthday as yours}] \\
= & \frac{364}{365} \times \frac{364}{365} \times \cdots \frac{364}{365} \times \frac{1}{365} \\
= & \frac{364^{19} \times 1}{365^{20}} \\
\approx & 0.0026
\end{aligned}
$$

6. **(Alvin's woes)** After a long night of debugging, Alvin has just perfected the new homework party/office hour queue system. CS 70 students sign themselves up for the queue, and TAs go through the queue, resolving requests one by one. Unfortunately, our newest TA (let's call him TA Bob) does not understand how to use the new queue: instead of resolving the requests in order, he always uses the Random Student button, which (as the name suggests) chooses a random student in the queue for him. To make matters worse, after helping the student, Bob forgets to click the Resolve button, so the student still remains in the queue! For this problem, assume that there are n total students in the queue.

   (a) Suppose that Bob has already helped k students. What is the probability that the Random Student button will take him to a student who has not already been helped?

   (b) Give a description of the probability space Ω. Fully answering this question entails giving a representation for an outcome ω, as well as any conditions on what ω is allowed to be. As an example, the probability space for two coin flips can be described as $\{HH, HT, TH, TT\}$. Each outcome ω is a length-two string of characters from the set $\{H, T\}$, where H represents heads and T represents tails; the first character of ω is the result of the first coin flip, and the second character is the result of the second coin flip. (Hint: Each outcome ω should include information about which students have been helped, along with the total number of Random Student button presses.)

   (c) Let X_i^r be the event that TA Bob has not helped student i after pressing the Random Student button a total of r times. What is $\Pr[X_i^r]$? Assume that the results of the Random Student button are independent of each other. Now approximate the answer using the inequality $1 - x \leq e^{-x}$.

   (d) Let T_r represent the event that TA Bob presses the Random Student button r times, but still has not been able to help all n students. (In other words, it takes TA Bob longer than r Random Student button presses before he manages to help every student). What is T_r in terms of the events X_i^r? (Hint: Events are subsets of the probability space Ω, so you should be thinking of set operations...)

   (e) Using your answer for the previous part, what is an upper bound for $\Pr[T_r]$? (You may leave your answer in terms of $\Pr[X_i^r]$. Use the inequality $1 - x \leq e^{-x}$ from before.)

   (f) Now let $r = \alpha n \ln n$. What is $\Pr[X_i^r]$?

(g) Calculate an upper bound for $\Pr[T_r]$ using the same value of r as before. (This is more formally known as a bound on the tail probability of the distribution of button presses required to help every student. This distribution will be explored in more detail later, in the context of random variables.)

(h) What value of r do you need to bound the tail probability by $1/n^2$? In other words, how many button presses are needed so that the probability that TA Bob has not helped every student is at most $1/n^2$?

Answer:

(a) There are $n-k$ students who have not been helped, and the probability that one of these students is chosen is $(n-k)/n$ or $1-k/n$.

(b) One way to represent Ω is through a list of tuples, where each tuple is a (Student, Time) pair. Here, Time refers to the number of button presses it took for Bob to help the student. Here is an example of a particular outcome ω (for the case of $n=3$ students): [(Fan, 2), (Allen, 3), (Alex, 6)] The restrictions on ω are that the list must be of length n (all students must be helped) and the number of button presses for each student must be strictly increasing. Note that the total number of button presses can be inferred from the list as the maximum number of button presses for any student. This question was intended to solidify your understanding of probability spaces and their representations for complicated problems.

(c) The probability that student i is chosen by the Random Student button is $1/n$, so the complement of this probability is $1-1/n$. Using the assumption of independence:

$$\Pr[X_i^r] = \left(1-\frac{1}{n}\right)^r \le e^{-r/n}$$

(d) T_r is the event that TA Bob has pressed the button r times, but has not been able to help either student 1, or student 2, or student 3, ... This is the union: $T_r = \bigcup_{i=1}^{n} X_i^r$.

(e) Use the union bound. $\Pr[T_r] \le n \cdot \Pr[X_i^r] \le ne^{-r/n}$.

(f) Plug in for r. $\Pr[X_i^r] \le e^{-r/n} = e^{-\alpha \ln n} = n^{-\alpha}$.

(g) A quick application of the union bound derived in the previous parts: $\Pr[T_r] \le n \cdot \Pr[X_i^r] = n^{1-\alpha}$.

(h) Set $1-\alpha = -2$, which is $\alpha = 3$. This gives $r = 3n\ln n$. (Side-note: This problem is more commonly known as the coupon collector's problem. Once we cover random variables, we will see that the expected number of button presses required to help every student is $\Theta(n \log n)$.)