

Due Thursday March 17th at 10PM

1. **Independence**

2 points per sub problem. 16 points total.

(a) **Independence (due to H.W. Lenstra)**

Suppose we pick a random card from a standard deck of 52 playing cards. Let A represent the event that the card is a queen, B the event that the card is a spade, and C the event that a red card (a heart or a diamond) is drawn.

- i. Which two of A , B , and C are independent? Justify your answer carefully. (In other words: For each pair of events (AB , AC , and BC), state and prove whether they are independent or not.)
- ii. What if a joker is added to the deck? Justify your answer carefully.

(b) **Independence (due to H.W. Lenstra)**

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two *independent* events. Let $\bar{A} = \Omega - A$ and $\bar{B} = \Omega - B$ (sometimes written $\neg A$ and $\neg B$) denote the complementary events.

For the purposes of this question, you may use the following definition of independence: Two events A, B are *independent* if $\Pr[A \cap B] = \Pr[A] \Pr[B]$.

- i. Prove or disprove: \bar{A} and \bar{B} are necessarily independent.
- ii. Prove or disprove: A and \bar{B} are necessarily independent.
- iii. Prove or disprove: A and \bar{A} are necessarily independent.
- iv. Prove or disprove: It is possible that $A = B$.

(c) **Boole's inequalities**

- i. For events A, B in the same probability space, prove that

$$\Pr[A \cap B] \geq \Pr[A] + \Pr[B] - 1.$$

- ii. Generalize part (a) to prove that, for events A_1, \dots, A_n in the same probability space (and any n),

$$\Pr[A_1 \cap \dots \cap A_n] \geq \Pr[A_1] + \dots + \Pr[A_n] - (n - 1).$$

2. **(1/2/2) Cliques in random graphs**

Consider a graph $G(V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v , we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G(V, E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

- (a) What is the size of the sample space?
- (b) A k -clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k -clique?
- (c) Prove that the probability that the graph contains a k -clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$.

3. (1/2/2) College applications

There are n students applying to n colleges. Each college has a ranking over all students (i.e. a permutation) which, for all we know, is completely random and independent of other colleges.

College number i will admit the first k_i students in its ranking. If a student is not admitted to any college, he or she might file a complaint against the board of colleges, and colleges want to avoid that as much as possible.

- a) If for all i , $k_i = 1$, i.e. if every college only admits the top student on its list, what is the chance that all students will be admitted to at least one college?
- b) What is the chance that a particular student, Alice, does not get admitted to any college? Prove that if the average of all k_i 's is $2\ln n$, then this probability is at most $1/n^2$. (Hint: use the inequality $1 - x < e^{-x}$)
- c) Prove that when the average k_i is $2\ln n$, then the probability that at least one student does not get admitted to any college is at most $1/n$. (Hint: use the union bound)

4. Expressions

- Each subpart is 1 point. 21 points total.
- For each problem, just write down a mathematical expression. There is no need to justify/explain/derive the answer.

(a) Bayes Rule - Man Speaks Truth

- i. A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $\frac{1}{3}$ of the time and reports it's Heads. What is the probability it is Heads?
- ii. A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided dice and reports it comes up 6. What is the probability it is really 6?

(b) Unlikely events

- i. Toss a fair coin x times. What is the probability that you never get heads?
- ii. Roll a fair die x times. What is the probability that you never roll a six?
- iii. Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

- iv. How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

(c) **Blood Type**

Consider the three alleles, A, B, and O, for human blood types. As each person inherits one of the 3 alleles from each parent, there are 6 possible genotypes: AA, AB, AO, BB, BO, and OO. Blood groups A and B are dominant to O. Therefore, people with AA or AO have type A blood. Similarly, BB and BO result in type B blood. The AB genotype is called type AB blood, and the OO genotype is called type O blood. Each parent contributes one allele randomly. Now, suppose that the frequencies of the A, B, and O alleles are 0.4, 0.25, and 0.35, respectively, in Berkeley. Alice and Bob, two residents of Berkeley are married and have a daughter, Mary. Alice has blood type AB.

- i. What is the probability that Bob's genotype is AO?
- ii. Assume that Bob's genotype is AO. What is the probability that Mary's blood type is AB?
- iii. Assume Mary's blood type is AB. What is the probability that Bob's genotype is AA?

(d) **Simple probability**

Out of 1000 sophomore EECS students, 400 are taking CS70 (and may concurrently take CS61C), 500 are taking CS61C (and may concurrently take CS70), and 50 are taking both CS70 and CS61C.

- i. Suppose we choose a student uniformly at random. Let C be the event that the student takes CS70 and P the event that the student takes CS61C. Draw a picture of the sample space Ω and the events C and P .
- ii. What is the probability that the student takes CS70?
- iii. What is the probability that the student takes CS61C?
- iv. What is the probability that the student takes CS70 AND CS61C?
- v. What is the probability that the student takes CS70 OR CS61C?

(e) **Roll Dice**

You roll three fair six-sided dice. What is the probability of rolling a triple (all three dice agree)? What is the probability of rolling a double (two of the dice agree with each other)?

(f) **Lie Detector**

A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

(g) **Rain and Wind**

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS70, you are curious to play around with these numbers. Find the probability that

- i. A given day is windy and rainy.
- ii. It rains on a given day.
- iii. Exactly one of any two days is rainy.
- iv. A non-rainy day is also non-windy.

(h) **Chess Squares**

Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

5. Short Answers

- Each subpart is 2 point. 20 points total.
 - For each problem, briefly justify your answer.
- (a) For any probability space, show that $Pr[A \setminus B] \geq Pr[A] - Pr[B]$.
 - (b) Show that $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$.
 - (c) Assume that $|\Omega| = n$. How many distinct events does the probability space have?
 - (d) Assume that $|\Omega| = n$. What is the maximum number of distinct values of $Pr[A]$ can one have for events of the probability space?
 - (e) Can you find a probability space and two events A and B such that $Pr[A|B] = Pr[B]$ and A and B are not independent?
 - (f) Prove that $Pr[A \cap B \cap C] = Pr[A]Pr[B|A]Pr[C|A \cap B]$.
 - (g) Find an example where $Pr[A \cap B \cap C] \neq Pr[A]Pr[B|A]Pr[C|B]$.
 - (h) Find an example where $Pr[A|B] > Pr[A]$, another where $Pr[A|B] < Pr[A]$, and one where $Pr[A|B] = Pr[A]$.
 - (i) Can you find an example where $Pr[A] > Pr[B]$ and $Pr[A|C] < Pr[B|C]$?
 - (j) Can you find an example where $Pr[A] > Pr[B]$ and $Pr[C|A] < Pr[C|B]$?