CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Satish Rao — DIS 0a

1. **Summations**

   (a) What is $\sum_{i=4}^{7} i$?

   (b) Let $S = \{1,2,20\}$, what is $\sum_{x \in S} x^2$?

   (c) Let $S = \{1,2,4\}$, is $1 + \sum_{x \in S} x$ divisible by 2?

   (d) Let $S = \{0,\ldots,n\}$, is $1 + \sum_{x \in S} 2^x$ divisible by 2? ($n \geq 0$.)

   (e) Let $S = \{0,\ldots,n\}$, is $1 + \Pi_{x \in S} 2^x$ divisible by 2? ($n \geq 0$)

   (f) Let $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$, what is $\sum_{x \in A \cup B} x + \Pi_{x \in A \cap B} x$?

   (g) Let $A = \{1,2,3\}$, $B = \{2,3,4,5\}$, and $C = \{4,5,6\}$ what is $|(A \cap B) \cup (B \cap C)|$?

   Solution:

   (a) $\sum_{i=4}^{7} i = 4+5+6+7 = 22$.

   (b) $\sum_{x \in S} x^2 = 1^2 + 2^2 + 20^2 = 405$.

   (c) Yes. $1 + \sum_{x \in S} = 1+1+2+4 = 8$, which is divisible by 2.

   (d) Yes. This is 2^{n+1}. Briefly, for $n = 0$, the sum is 2 which is 2^1, we assume that $2^n = 1 + \sum_{x \in S \setminus \{n\}} 2^x$ and notice the $1 + \sum_{x \in S} 2^x = 1 + \sum_{x \in S \setminus \{n\}} 2^x + 2^n = 2^n + 2^n = 2^{n+1}$. This is called induction, which we will get more into shortly in the semester. Do not worry too much if you don't get it now.

   (e) Yes, for $n = 0$, as we have $1+1$. No, for $n > 0$, as the product is even (it contains a factor of 2).

   (f) $A \cup B = \{1,2,3,4,5\}$ and $A \cap B = \{2,3\}$. Thus, $\sum_{x \in A \cup B} x = 15$ and $\Pi_{x \in A \cap B} x = 6$, the total is 21.

   (g) $S = A \cap B = \{2,3\}$ and $T = B \cap C = \{4,5\}$. Then $S \cup T = \{2,3,4,5\}$ or B, which has size 4.

2. **Writing in propositional logic**

   For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

   (a) The square of a nonzero integer is positive.

   (b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

   (c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

(d) For any two distinct real numbers, we can find a rational number in between them.

Solution:

(a) We can rephrase the sentence as "if n is a nonzero integer, then $n^2 > 0$", which can be written as

$$\forall n \in Z, (n \neq 0) \rightarrow (n^2 > 0)$$

or equivalently as

$$\forall n \in Z, (n = 0) \vee (n^2 > 0).$$

The latter is easier to negate, and its negation is given by

$$\exists n \in Z, (n \neq 0) \wedge (n^2 \leq 0)$$

(b) The sentence is

$$\forall x, y \in Z, x^2 - y^2 \neq 10.$$

The negation is

$$\exists x, y \in Z, x^2 - y^2 = 10$$

(c) Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x." Or,

$$\exists x \in \mathbb{R}, (p(x) = 0) \wedge (\forall y \in \mathbb{R}, (p(y) = 0) \implies (x = y)).$$

Its negation is given by

$$\forall x \in \mathbb{R}, (p(x) \neq 0) \vee (\exists y \in \mathbb{R}, (p(y) = 0) \wedge (x \neq y)).$$

(d) The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$\forall x, y \in \mathbb{R}, (x \neq y) \implies (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

Equivalently,

$$\forall x, y \in \mathbb{R}, (x = y) \vee (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

The negation is

$$\exists x, y \in \mathbb{R}, (x \neq y) \wedge (\forall z \in \mathbb{Q}, ((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

3. **Implication**

Which of the following implications are true? Give a counterexample for each false assertion.

(a) $\forall x, \forall y, P(x, y) \implies \forall y, \forall x, P(x, y)$.

(b) $\exists x, \exists y, P(x, y) \implies \exists y, \exists x, P(x, y)$.

(c) $\forall x, \exists y, P(x, y) \implies \exists y, \forall x, P(x, y)$.

(d) $\exists x, \forall y, P(x, y) \implies \forall y, \exists x, P(x, y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedant is true and the consequence is false, those the entire implication statement is false.

(d) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note that the two statements are not equivalent as the converse of this is statement 3, which is false.