

Pingala Series

Inala Jaahnavi

CONTENTS

1	JEE 2019	1
2	Pingala Series	1
3	Power of the Z transform	3

Abstract—This manual provides a simple introduction to Transforms

1 JEE 2019

Let α, β be the roots of the polynomial $x^2 - x - 1$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

Solution: The following code verifies the above results:

```
wget https://raw.githubusercontent.com/Jaahnavi17/EE3900/master/pingala/codes/1.py
```

The first three results are true while the last one is false.

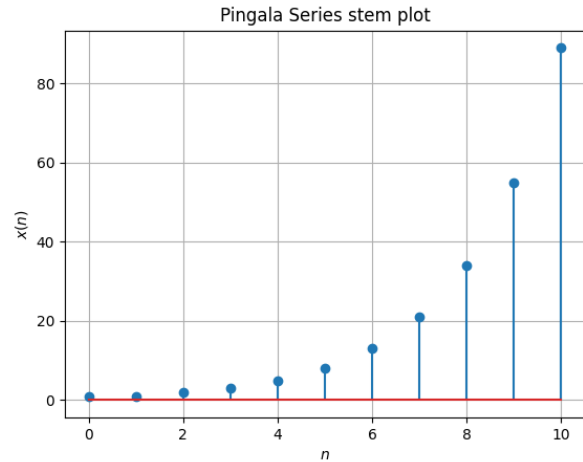


Fig. 2.2: x(n)

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \quad (2.2)$$

Generate a stem plot for $x(n)$.

Solution: The following code plots 2.2

```
wget https://raw.githubusercontent.com/Jaahnavi17/EE3900/master/pingala/codes/2_2.py
```

2.3 Find $X^+(z)$.

Solution:

$$x(n+2) = x(n+1) + x(n) \quad (2.3)$$

$$\Rightarrow \mathcal{Z}^+\{x(n+2)\} = \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n)\} \quad (2.4)$$

$$\Rightarrow z^2 X^+(z) - z^2 - z = zX^+(z) - z + X^+(z) \quad (2.5)$$

$$\Rightarrow X^+(z)(z^2 - z - 1) = z^2 \quad (2.6)$$

$$\begin{aligned} \Rightarrow X^+(z) &= \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.7) \\ &= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad (2.8) \end{aligned}$$

ROC: $|z| > \frac{1+\sqrt{5}}{2}$

2.4 Find $x(n)$.

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad (2.9)$$

$$= \left(\frac{1}{(\alpha - \beta)z^{-1}} \right) \left(\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right) \quad (2.10)$$

$$= \frac{1}{z^{-1}(\alpha - \beta)} \left(\sum_{n=0}^{\infty} (\alpha z^{-1})^n - (\beta z^{-1})^n \right) \quad (2.11)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \quad (2.12)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \quad (2.13)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} z^{-n} \quad (2.14)$$

$$\Rightarrow x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \quad (2.15)$$

$$= a_{n+1} u(n) \quad (2.16)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.17)$$

Solution: The following code plots 2.5

```
wget https://raw.githubusercontent.com/
Jaahnavi17/EE3900/master/pingala/codes
/2_5.py
```

2.6 Find $Y^+(z)$.

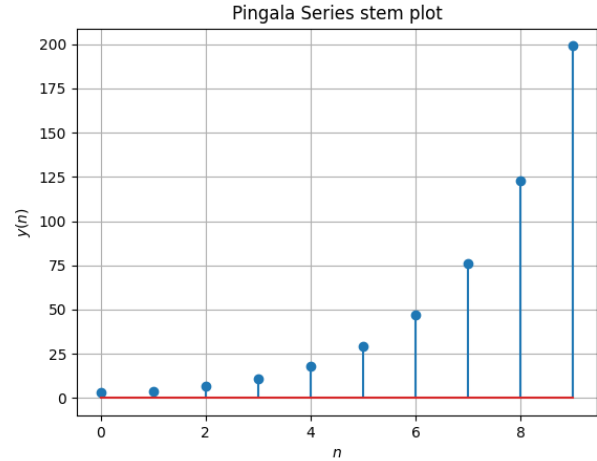


Fig. 2.5: $y(n)$

Solution:

$$\mathcal{Z}^+\{y(n)\} = \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n-1)\} \quad (2.18)$$

$$Y^+(z) = zX^+(z) - z + z^{-1}Z^+(z) \quad (2.19)$$

$$= (z + z^{-1}) \frac{1}{1 - z^{-1} - z^{-2}} - z \quad (2.20)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (2.21)$$

ROC for the above z transform will again be

$$|z| > \frac{1+\sqrt{5}}{2}$$

2.7 Find $y(n)$.

Solution:

$$Y^+(z) = X^+(z) + \frac{2}{(z - \alpha)(z - \beta)} \quad (2.22)$$

$$= X^+(z) + \frac{2}{\alpha - \beta} \left(\frac{1}{1 - \alpha z^{-1}} + \frac{1}{1 - \beta z^{-1}} \right) \quad (2.23)$$

$$= X^+(z) + \frac{2}{\alpha - \beta} \left(\sum_{n=0}^{\infty} (\alpha^n + \beta^n) z^{-n} \right) \quad (2.24)$$

$$\Rightarrow y(n) = x(n) + 2u(n) \frac{\alpha^n + \beta^n}{\alpha - \beta} \quad (2.25)$$

$$= \frac{\alpha^{n+1} + 2\alpha^n - \beta^{n+1} + 2\beta^n}{\alpha - \beta} u(n) \quad (2.26)$$

$$= \frac{\alpha^n(\alpha + 1) + \alpha^n - \beta^n(\beta + 1) + \beta^n}{\alpha - \beta} u(n) \quad (2.27)$$

$$= \frac{\alpha^{n+2} - \beta^{n+2} - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta} u(n) \quad (2.28)$$

$$= \frac{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1})}{\alpha - \beta} u(n) \quad (2.29)$$

$$\Rightarrow y(n) = \alpha^{n+1} + \beta^{n+1} \quad (2.30)$$

Solution: Using the definition of $x(n)$ from (2.16)

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} a_{k+1} \quad (3.2)$$

$$= \sum_{k=0}^{n-1} x(n) \quad (3.3)$$

$$= \sum_{k=0}^{n-1} x(k) + x(n) \times 0 \quad (3.4)$$

$$= \sum_{k=0}^{n-1} x(k)u(n-1-k) + x(n) \times u(-1) \quad (3.5)$$

$$= \sum_{k=0}^n x(k)u(n-1-k) \quad (3.6)$$

$$= x(n) * u(n-1) \quad (3.7)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.8)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.9)$$

Solution: The above expression can be written for $n \geq 0$ as:

$$a_{n+1} - 1, \quad n \geq 0 \quad (3.10)$$

$$= x(n) - 1, \quad n \geq 0 \quad (3.11)$$

$$= (x(n) - 1)u(n) \quad (3.12)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+ (10) \quad (3.13)$$

3 POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \quad (3.1)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}} \quad (3.14)$$

$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \quad (3.15)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.16)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} x(k) 10^{-k} \quad (3.17)$$

$$= \frac{X^+(10)}{10} \quad (3.18)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.19)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.20)$$

and find $W(z)$.

Solution: Replacing n with $n + 1$:

$$\alpha^{n+1} + \beta^{n+1}, \quad n \geq 0 \quad (3.21)$$

$$= (\alpha^{n+1} + \beta^{n+1})u(n) = w(n) \quad (3.22)$$

The z-transform can be computed as follows:

$$W(z) = \sum_{n=-\infty}^{\infty} w(n)z^{-n} \quad (3.23)$$

$$= \sum_{n=0}^{\infty} \alpha^{n+1}z^{-n} + \beta^{n+1}z^{-n} \quad (3.24)$$

$$= \alpha \sum_{n=0}^{\infty} \alpha^n z^{-n} + \beta \sum_{n=0}^{\infty} \beta^n z^{-n} \quad (3.25)$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \quad (3.26)$$

$$= \frac{\alpha + \beta - 2\alpha\beta z^{-1}}{1 + \alpha\beta z^{-2} - (\alpha + \beta)z^{-1}} \quad (3.27)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.28)$$

ROC: $|z| > \frac{1+\sqrt{5}}{2}$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.29)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}} \quad (3.30)$$

$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \quad (3.31)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.32)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \quad (3.33)$$

$$= \frac{Y^+(10)}{10} \quad (3.34)$$

3.6 Solve the JEE 2019 problem.

Solution:

a) From (3.7)

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.35)$$

Taking the positive z transform on the RHS:

$$\mathcal{Z}\{x(n) * u(n-1)\} = X^+(z)z^{-1} \frac{1}{1 - z^{-1}} \quad (3.36)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \quad (3.37)$$

$$= z \left(\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right) \quad (3.38)$$

$$= z \sum_{n=0}^{\infty} (x(n) - 1)z^{-n} \quad (3.39)$$

$$= \sum_{n=0}^{\infty} (x(n) - 1)z^{-n+1} \quad (3.40)$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1)z^{-n} \quad (3.41)$$

Taking the Inverse Z transform on (3.41):

$$x(n) * u(n-1) = (x(n+1) - 1)u(n) \quad (3.42)$$

$$\Rightarrow \sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (3.43)$$

b) From (3.18)

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{X^+(10)}{10} \quad (3.44)$$

$$= \frac{100}{100 - 10 - 1} \times \frac{1}{10} \quad (3.45)$$

$$= \frac{10}{89} \quad (3.46)$$

c) Using (1.2), (2.16), (2.30) we get:

$$b_n = a_{n+1} + a_{n-1} \quad (3.47)$$

$$= x(n) + x(n-2) \quad (3.48)$$

$$= y(n-1) \quad (3.49)$$

$$= \alpha^n + \beta^n \quad (3.50)$$

d) From (3.34)

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{Y^+(10)}{10} \quad (3.51)$$

$$= \frac{100 + 20}{100 - 10 - 1} \times \frac{1}{10} \quad (3.52)$$

$$= \frac{12}{89} \neq \frac{8}{89} \quad (3.53)$$

Thus, options (a),(b) and (c) are correct