Circuits and Transforms

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***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

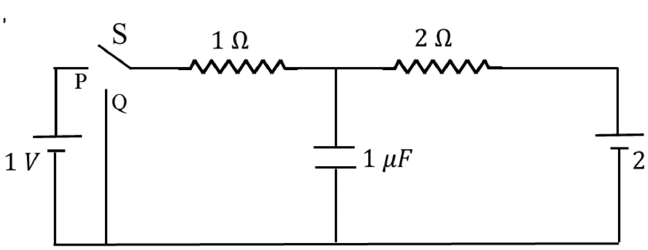

Fig. 2.1

2. Find q_1.
Solution: The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\text{V} \tag{2.2}$$

Hence,

$$q_1 = CV = \frac{4}{3}\,\mu\text{C} \tag{2.3}$$

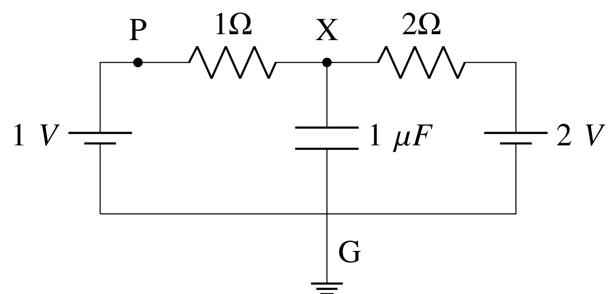

Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We have,

$$u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-st}dt \tag{2.4}$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} e^{-st}dt \tag{2.5}$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \tag{2.6}$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.7}$$

and find the ROC.
Solution: Note that by substituting $s := s + a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.8}$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \tag{2.9}$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

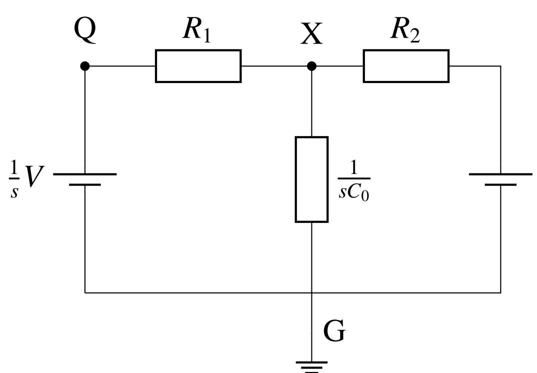

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.10}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.11}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: We see that

$$V_1(s) = \frac{1}{s} V_2(s) = \frac{2}{s} \tag{2.12}$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \tag{2.13}$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \tag{2.14}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.15}$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.16}$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform in (2.16),

$$V(s) \xleftrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2} u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.17}$$

$$= \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.18}$$

The python code `codes/2_6.py` plots the graph below.

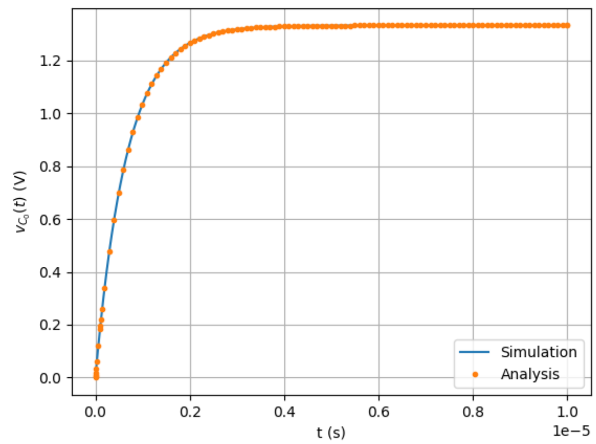

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

7. Verify your result using ngspice.
Solution: The ngspice script `codes/2_7.cir` simulates the given circuit and the generated output is depicted in Fig. (2.4)

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The equivalent circuit at steady state when the switch is at Q is shown below.

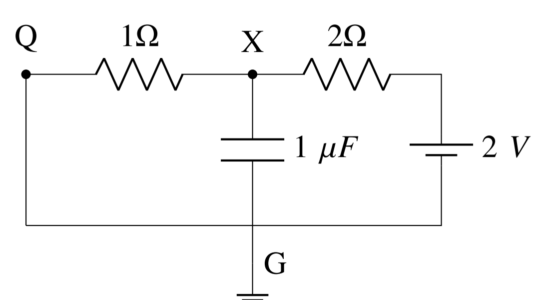

Fig. 3.1

Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1}+\frac{V-2}{2}=0 \implies V=\frac{2}{3}\,\mathrm{V} \quad (3.1)$$

and hence, $q_2 = \frac{2}{3}\,\mu\mathrm{C}$.

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.

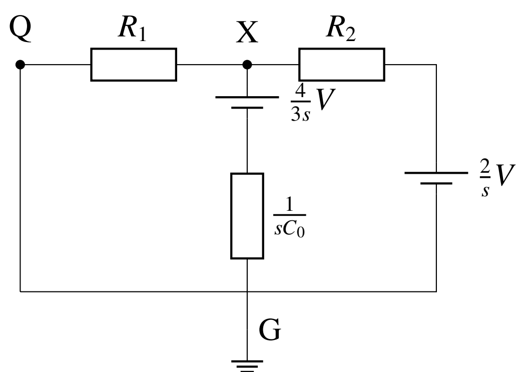

Fig. 3.2

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1}+\frac{V-\frac{2}{s}}{R_2}+sC_0\left(V-\frac{4}{3s}\right)=0 \quad (3.2)$$

$$\implies V_{C_0}(s)=\frac{\frac{2}{sR_2}+\frac{4C_0}{3}}{\frac{1}{R_1}+\frac{2}{R_2}+sC_0} \quad (3.3)$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.3),

$$V_{C_0}(s)=\frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t)=\frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.5)$$

Substituting values gives

$$v_{C_0}(t)=\frac{2}{3}\left(1+e^{-\left(1.5\times10^{6}\right)t}\right)u(t) \quad (3.6)$$

The Python code `codes/3_4.py` plots the graph below.

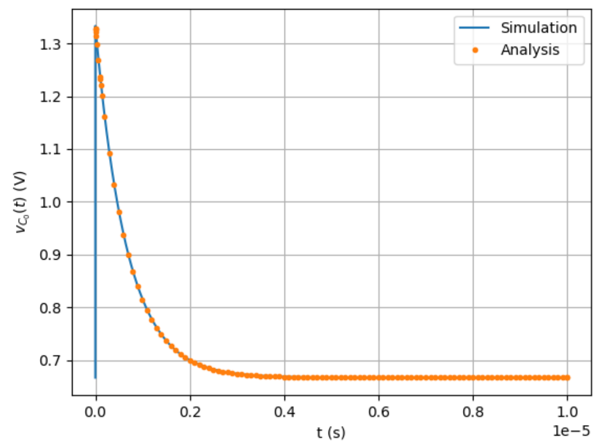

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

5. Verify your result using ngspice.
Solution: The ngspice script `codes/3_5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.3).
6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-)=\frac{q_1}{C_0}=\frac{4}{3}\,\mathrm{V} \quad (3.7)$$

Using (3.6),

$$v_{C_0}(0+)=\lim_{t\to0+}v_{C_0}(t)=\frac{4}{3}\,\mathrm{V} \quad (3.8)$$

$$v_{C_0}(\infty)=\lim_{t\to\infty}v_{C_0}(t)=\frac{2}{3}\,\mathrm{V} \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.
Solution: The equivalent circuit in the t-domain is shown below.

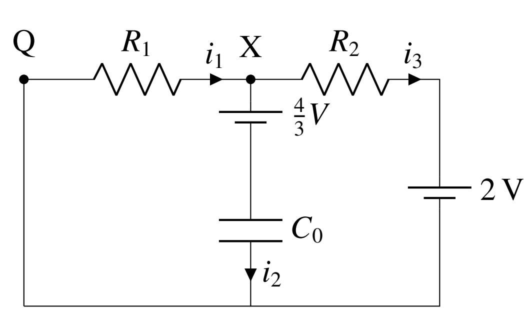

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.10}$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt = 0 \tag{3.11}$$

$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt - i_3 R_2 - 2 = 0 \tag{3.12}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \tag{3.13}$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \tag{3.14}$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \tag{3.15}$$

where $i(t) \overset{\mathcal{L}}{\longleftrightarrow} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s-domain. Equations (3.13) - (3.15) precisely describe Fig. 3.2.

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
Solution: The equivalent circuit in the t-domain is shown below.

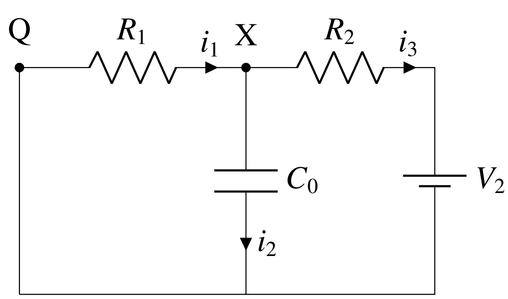

Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$

$$i_1 R_1 + \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.2}$$

$$i_3 R_2 + 2 - \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.3}$$

Differentiating the above equations,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.4}$$

$$R_1 \frac{di_1}{dt} + \frac{i_2}{C_0} = 0 \tag{4.5}$$

$$R_2 \frac{di_3}{dt} - \frac{i_2}{C_0} = 0 \tag{4.6}$$

Using (4.4) and (4.6) in (4.5),

$$R_1\left(\frac{di_2}{dt} + \frac{di_3}{dt}\right) + \frac{i_2}{C_0} = 0 \tag{4.7}$$

$$R_1 \frac{di_2}{dt} + \left(1 + \frac{R_1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.8}$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.9}$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \tag{4.10}$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the RC time constant of the circuit. Note that $i_2(0) = \frac{V_2}{R_2}$ A and $i_2 = C_0 \frac{dV}{dt}$, where V is the voltage of the capacitor. Hence, integrating (4.10),

$$C_0 \frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \tag{4.11}$$

$$\implies \frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \tag{4.12}$$

2. Find $H(s)$ considering the output voltage at the capacitor.
Solution: Transforming Fig. 4.1 to the s-domain,

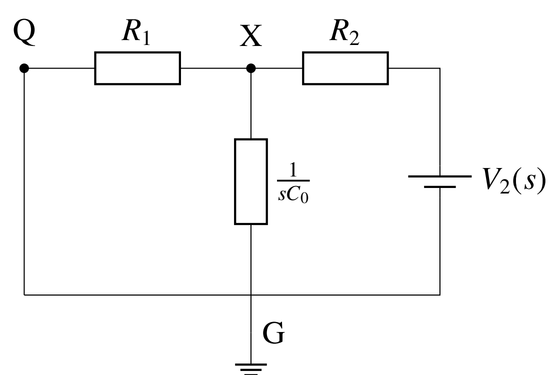

Fig. 4.2

Applying nodal analysis at X, and noting that

$H(s) = \frac{V(s)}{V_2(s)}$,

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \tag{4.13}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.14}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.15}$$

3. Plot $H(s)$. What kind of filter is it?
Solution: The Python code `codes/4_3.py` plots $H(s)$. Clearly, $H(s)$ is a low-pass filter.

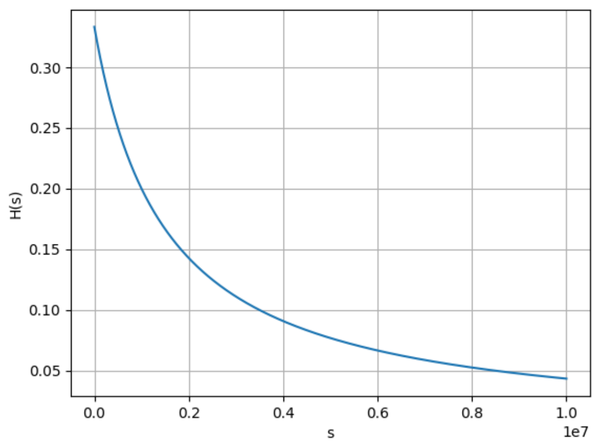

Fig. 4.3: Plot of $H(s)$.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.16}$$

Solution: Integrating (4.12) between limits n to $n+1$ and applying the trapezoidal formula,

$$v(n+1) - v(n) + \frac{v(n) + v(n+1)}{2\tau} = \frac{V_2\left(u(n) + u(n+1)\right)}{C_0R_2} \tag{4.17}$$

$$v(n)\left(2\tau + 1\right) + v(n-1)\left(2\tau - 1\right) = \frac{V_2\tau\left(u(n) + u(n-1)\right)}{C_0R_2} \tag{4.18}$$

for $n > 0$, where $v(0) = 0$.

5. Find $H(z)$.
Solution: Note that for the input voltage, $v_i(n) = 2u(n)$ and so, $V_i(z) = \frac{2}{1-z^{-1}}$. Applying the Z-transform on both sides of (4.18),

$$V(z)\left[(2\tau + 1) - z^{-1}(2\tau - 1)\right] = \frac{\tau\left(1 + z^{-1}\right)V_i(z)}{C_0R_2} \tag{4.19}$$

Hence,

$$H(z) = \frac{\tau\left(1 + z^{-1}\right)}{C_0R_2\left((2\tau + 1) - (2\tau - 1)\, z^{-1}\right)} \tag{4.20}$$

since $\left|\frac{2\tau-1}{2\tau+1}\right| < 1$, the ROC is $|z| > 1$.

6. How can you obtain $H(z)$ from $H(s)$?
Solution: We use the bilinear transformation. Setting

$$s := \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.21}$$

we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T}\frac{1-z^{-1}}{1+z^{-1}}} \tag{4.22}$$

$$= \frac{T\tau\left(1 + z^{-1}\right)}{C_0R_2\left((2\tau + T) - (2\tau - T)\, z^{-1}\right)} \tag{4.23}$$

Setting $T = 1$ gives (4.20).

7. Find $v(n)$. Verify using ngspice and the differential equation.
Solution: We have,

$$V(z) = H(z)V_i(z) \tag{4.24}$$

$$= \frac{TV_2\tau\left(1 + z^{-1}\right)}{C_0R_2\left(1 - z^{-1}\right)\left((2\tau + T) - (2\tau - T)\, z^{-1}\right)} \tag{4.25}$$

$$= \frac{V_2\tau\,(z+1)}{2C_0R_2}\sum_{k=-\infty}^{\infty}\left(1 - p^k\right)u(k)z^{-k} \tag{4.26}$$

where $p := \frac{2\tau-T}{2\tau+T}$. Thus,

$$v(n) = \frac{V_2\tau}{C_0R_2}\left[u(n)\left(1 - p^n\right) + u(n+1)\left(1 - p^{n+1}\right)\right] \tag{4.27}$$

where $p := \frac{2\tau-1}{2\tau+1}$. We take $T = 10^{-7}$ as the sampling interval. The python code `codes/4_7.py` verifies these equalities.

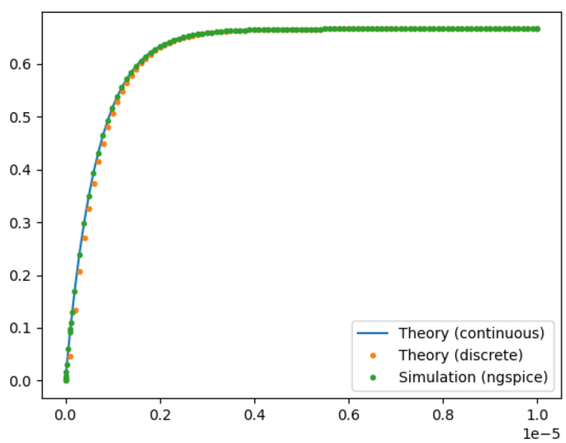

Fig. 4.4: Representation of output across C_0.