Binomial Probability Distribution:

Consider the experiment or a trial which has only two outcomes, a success or failure with p as the probability of success and q as the probability of failure

Since there are only two outcomes,

$$p+q=1$$

Let us consider series of n such independent trials each of which either results in success or failure.

To find probability of r successes in n trials,

Consider one run of outcomes.

$$\underbrace{SSS\cdots S}_{r}\ \underbrace{FFF\cdots F}_{n-r}$$

in which there are r consecutive successes and $(n-r)$ failures.

Probability of this event $P(r$ success in n trials$)$ is given by

$$P(SS\cdots SFF\cdots F) = P(S)P(S)\cdots P(S)\,P(F)P(F)\cdots P(F)$$

$$= p^r q^{n-r}$$

r successes and $n-r$ failures can occur.

in ${}^nC_r$ mutually exclusive cases each of which has the probability $p^r q^{n-r}$.

$\therefore$ Probability of r success in n trials is

${}^nC_r\, p^r q^{n-r}$

r	0	1	2	$\cdots$	n
$P(r)$	${}^nC_0\, p^0 q^n$	${}^nC_1\, p^1 q^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	$\cdots$	${}^nC_n\, p^n q^0$

$$(p+q)^n = p^n + {}^nC_1\, p^{n-1} q + {}^nC_2\, p^{n-2} q^2 + \cdots + {}^nC_n\, p^n q^0$$

$$\therefore q^n + {}^nC_1\, q^{n-1} p + {}^nC_2\, q^{n-2} p^2 + \cdots + p^n = 1$$

The above probability distribution (discrete) is called Binomial probability distribution.

$$\therefore \quad B(n, p, r) = {}^nC_r\, p^r q^{n-r}$$

Mean and Variance of Binomial Distribution -

Mean $= n \times p$

Variance $= n \times p \times q$

where n is number of trials

p - is probability of success

q - is probability of failure.

Ex. 1) An unbiased coin is thrown 10 times. Find the probability of getting a) exactly 6 heads

b) at least 6 heads.

Soln: Here $p = \frac{1}{2}$ $q = \frac{1}{2}$ $n = 10$

Probability of getting 6 heads

$$= {}^{10}C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^4$$

Now let us calculate probability of getting at least 6 heads.

$$P(r \geq 6) = {}^{10}C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^4 + {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + \dots + {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_6 + {}^{10}C_7 + \cdots + {}^{10}C_{10} \right]$$

$$= 0.3769.$$

Ex. 2) A random variable X follows Binomial distribution.

$$X \longrightarrow B(n = 6, p)$$

Find p if $9P(r=4) = P(r=2)$

Soln: We have

$$P(r) = {}^{n}C_r \, p^r q^{n-r} \qquad n = 6$$

$$\therefore 9 \cdot \binom{6}{4} \cdot p^4 q^2 = \cdot \binom{6}{2} p^2 q^4$$

$$\therefore 9p^2 = q^2$$

$$9p^2 = (1-p)^2$$

$$\therefore p = \frac{1}{4} \text{ OR } p = -\frac{1}{2}$$

$p = -\frac{1}{2}$ which is not possible.

$$\therefore p = \frac{1}{4} \qquad q = \frac{3}{4}.$$

Ex. 3) The mean and variance of Binomial distribution are 6 and 2 respectively. Find $p(r \geq 1)$.

Sol^n: We have

$$\text{mean} = np = 6$$

$$\text{variance} = npq = 2$$

$$\therefore q = \frac{2}{6} = \frac{1}{3}$$

$$p = \frac{2}{3} \qquad \therefore n = 9$$

$$\therefore p(r \geq 1) = 1 - p(r=0)$$

$$= 1 - q^n$$

$$= 1 - \left(\frac{1}{3}\right)^9$$

$$p(r \geq 1) = 0.9999$$

4) On an average a box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives?

Sol^n: p = Probability of box containing defective articles

$= \frac{2}{10} = \frac{1}{5}$

q = Probability of non-defective items

$= \frac{4}{5}$

Now Probability of box containing three or less defective articles/items

$$p(r \leq 3) = p(r=0) + p(r=1) + p(r=2) + p(r=3)$$

$$= {}^{10}C_0 \left(\tfrac{1}{5}\right)^0 \cdot \left(\tfrac{4}{5}\right)^{10} + \cdots + {}^{10}C_3 \left(\tfrac{1}{5}\right)^3 \left(\tfrac{4}{7}\right)^7$$

$$= 0.1074 + 0.2684 + 0.302 + 0.2013$$

$\therefore$ $p(r \leq 3) = 0.8791$

$\therefore$ No of boxes contains three OR less defective articles

$= 0.8791 \times 100$

$= 87.91$

≈ 88 boxes

5) 20 % of bolts produced by a machine are defective. Determine the probability that out of 4 bolts chosen at random

i) 1 is defective

ii) No one is defective

iii) at most 2 bolts are defective.

Sol^n: Let p be probability of success of getting defective bolts

$$\therefore p = 20\% = \frac{20}{100} = \frac{1}{5} = 0.2$$

$$\therefore q = 0.8 \qquad \underline{n = 4}$$

i) $p(r=1) = 4C_1 (0.2)^1 (0.8)^3$

$= 0.4096$

ii) $p(r=0) = 4C_0 (0.2)^0 (0.8)^4$

$= 0.4096$

iii) $p(r \leq 2) = p(r=0) + p(r=1) + p(r=2)$

$= 0.9728$

Exercise:

1) Out of 2000 families with 4 children each, how many would you expect to have

a) at least a boy

b) 2 boys

c) 1 OR 2 girls

d) No girls ?

2) Probability of man now aged 60 years will live up to 70 years of age is 0.65

Find the probability that out of 10 men 60 years old, 6 OR more will live up to the age of 70 years.