Numerical Methods

Numerical solution of Algebraic and Transcendental equations :

1) Bisection
2) Secant
3) Regula- Falsi
4) Newton- Raphson and successive approximation methods, Convergence and stability.

Numerical solution of System of linear equations:

1) Gauss elimination method
2) LU decomposition
3) Cholesky
4) Jacobi and Gauss- Seidel Methods.

Intermediate value theorem -

If $f(x)$ is continuous function on some interval $[a, b]$ and $f(a) \cdot f(b) < 0$ [i.e., $f(a)$ $f(b)$ are of opposite signs] then the equation $f(x) = 0$ has at least one real root say ξ in the interval (a, b)

Bisection method -

This is a simple method based upon successive applications of the intermediate value theorem. If function $f(x)$ is continuous in $[a, b]$ and $f(a) f(b) < 0$ then to find the root of $f(x) = 0$ lying in the interval (a, b) we find middle point $c = \frac{a+b}{2}$

If $f(c) = 0$, then $\xi = c$ is the root of the equation.

If $f(c) \neq 0$ then we find further whether, $f(a) f(c) < 0$ OR $f(b) f(c) < 0$.

If $f(a) f(c) < 0$ then the root lies in in the interval $[a, c]$ and if $f(b) \cdot f(c) < 0$, then the root lies in the interval $[b, c]$

Then the interval $[a, b]$ is reduced to half interval $[a_1, b_1]$.

Employing the same line of investigation, the interval $[a_1, b_1]$ is halved to $[a_2, b_2]$ and the above process is repeated.

Finally, at some stage either the exact root or an infinite sequence of nested intervals $[a_1, b_1]$, $[a_2, b_2]$, $\cdots [a_n, b_n]$ is obtained.

Above process ensures that

$$f(a_n) f(b_n) < 0 \qquad n = 1, 2, \cdots$$

and

$$b_n - a_n = \frac{1}{2^n}(b - a)$$

Since $a_1, a_2 \cdots a_n$ form a monotonic non-decreasing bounded sequence, and $b_1, b_2, \cdots b_n$ form monotonic non-increasing bounded sequence their exists a common limit

$$\xi = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.$$

Ex. Using the bisection method, find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places.

Sol^n: $f(x) = x^3 - 4x - 9$

$f(0) = -9$, $f(1) = -12$, $f(2) = -9$, $f(3) = 6$

$\therefore$ Root lies between 2 and 3.

The first approximation to the root by bisection method,

$$\xi_1 = \frac{2+3}{2} = 2.5$$

$f(\xi_1) = -3.375$, Root lies between 2.5 and 3.

$$\xi_2 = \frac{2.5+3}{2} = 2.75$$

$f(\xi_2) = 0.7969$, Root lies between 2.5 and 2.75

$$\xi_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(\xi_3) = -1.4124$, Root lies between 2.625 & 2.75

$$\xi_4 = \frac{2.625+2.75}{2} = 2.6875$$

Continue in this way.

$\xi_{11} = 2.70654$ $\xi_{12} = 2.706295$

$\therefore$ Correct root to the three decimal places is $\underline{2.706}$

Ex- 2) Find a root of the equation $\cos x = x e^{x}$ using the bisection method.

Sol^n: Ans- $\underline{0.5156}$

Secant Method -

The equation of the chord (secant) joining the two points $[a, f(a)]$ and $[b, f(b)]$ is

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{(b - a)}$$

$$\therefore\ y = f(a) = \frac{f(b) - f(a)}{(b - a)}(x - a)$$

Taking point of intersection of chord with x axis (i.e., $y = 0$) as an approximation to the root is given by

$$x = a - \frac{(b - a)}{f(b) - f(a)} f(a)$$

If x_0, x_1 are two initial approximations to the root of $f(x) = 0$ then next approximation x_2 is given by,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} x_1$$

If $f_1 = f(x_1)$. $f_0 = f(x_0)$.

Root lies between 2.7183 and 3

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f_2 - f_1} f_2$$

$$= 2.7183 - \frac{(2.7183 - 3)}{(-0.5053 - 5)}(-0.5053)$$

$$= 2.7442$$

$f(2.7442) = -0.0554$

Root lies between 2.7442 and 3

$$\therefore x_4 = x_3 - \frac{(x_3 - x_2)}{(f_3 - f_2)} \times f_3.$$

$$= 2.7474$$

$f(2.7442) = 0.00094$

$\therefore$ Root lies between 2.7442 and 2.7474

$$x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \cdot f_4$$

$$= 2.7474$$

$\therefore$ Hence the required root is 2.7474

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f_1 - f_0} f_1$$

$$\therefore \quad x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f_i - f_{i-1}} f_i$$

Ex. Use secant method to find root of the equation

$f(x) = x^3 - 5x - 7 = 0$ correct to three decimal places.

Solⁿ: $f(x) = x^3 - 5x - 7$

$f(2) = -9$, $f(2.5) = -3.875$, $f(3) = 5$

$\therefore$ Root lies between 2.5 and 3.

$x_0 = 2.5$, $x_1 = 3$.

We proceed to obtain successive approximations by secand method.

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} f_1$$

$$= 3 - \frac{(3 - 2.5)}{5 - (-3.875)} \times 5$$

$$= \underline{2.7183}$$

$f(2.7183) = -0.5053.$