SUBJECT CODE : 207006

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SAVITRIBAI PHULE PUNE UNIVERSITY - 2019 SYLLABUS

S.E. (Electrical) Semester - III

ENGINEERING MATHEMATICS - III

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FEATURES

- ☑ Written by Popular Authors of Text Books of Technical Publications
- ☑ Covers Entire Syllabus ☑ Question - Answer Format
- ☑ Exact Answers and Solutions
- ☑ Chapterwise Solved SPPU Questions May-1995 to Dec.-2019
- ☑ Solved Model Question Papers (As Per 2019 Pattern)

SOLVED SPPU QUESTION PAPERS

- May - 2016
- Dec. - 2016
- May - 2017
- Dec. - 2017
- May - 2018
- Dec. - 2018
- May - 2019
- Dec. - 2019

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SYLLABUS

ENGINEERING MATHEMATICS-III - (207006)

Credits	Examination Scheme [Marks]
Th : 03	In Sem : 30 Marks
	End Sem : 70 Marks

Unit I : Linear Differential Equations **(LDE)** and Applications

LDE of n^{th} order with constant coefficients, Complementary Function, Particular Integral,General method, Short methods, Method of variation of parameters, Cauchy's and Legendre's DE, Simultaneous and Symmetric simultaneous DE. Modeling of Electrical circuits. **(Chapters - 1, 2)**

Unit II : Laplace Transform **(LT)**

Definition of LT, Inverse LT, Properties & theorems, LT of standard functions, LT of some special functions viz. Periodic, Unit Step, Unit Impulse. Applications of LT for solving Linear differential equations. **(Chapter - 3)**

Unit III : Fourier and Z - transforms

Fourier Transform **(FT)** : Complex exponential form of Fourier series, Fourier integral theorem, Fourier Sine & Cosine integrals, Fourier transform, Fourier Sine & Cosine transforms and their inverses.

Z - Transform (ZT) : Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations. **(Chapters - 4, 5)**

Unit IV : Statistics and Probability

Measures of central tendency, Measures of dispersion, Coefficient of variation, Moments, Skewness and Kurtosis, Correlation and Regression, Reliability of Regression estimates.

Probability, Probability density function, Probability distributions : Binomial, Poisson, Normal, Test of hypothesis : Chi-square test. **(Chapters - 6, 7)**

Unit V : Vector Calculus

Vector differentiation, Gradient, Divergence and Curl, Directional derivative, Solenoidal and Irrotational fields, Vector identities. Line, Surface and Volume integrals, Green's Lemma, Gauss's Divergence theorem and Stoke's theorem. **(Chapter - 8)**

Unit VI : Complex Variables

Functions of a Complex variable, Analytic functions, Cauchy-Riemann equations, Conformal mapping, Bilinear transformation, Cauchy's integral theorem, Cauchy's integral formula and Residue theorem. **(Chapters - 9, 10)**

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UNIT I

1 Linear Differential Equations with Constant Coefficients

1.1 : Solution of LDE with Constant Coefficient

1) The general form of n^{th} order LDE with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^n} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x)$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots a_n$ are constants and $\phi(x)$ is a function of x only. Let $D = \frac{d}{dx}$

$\therefore (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n) y = \phi(x) \ldots \rightarrow (1)$

i.e. $f(D)\, y = \phi(x)$ where $f(D)$ is a polynomial in D.

$\therefore$ Auxiliary equation is $f(D) = 0$

The solution of D.E. (1) involves two parts

i) Complementary function (C.F. or y_c)

ii) Particular integral (P.I. or y_P)

$\therefore$ The complete solution of D.E. (1) is $y = y_c + y_p$ = C.F. + P.I.

2) Methods to find C.F.

To find C.F. find the roots of auxiliary equation $f(D) = 0$

i) If $m_1, m_2, m_3 \ldots$ are **real and distinct** roots, then

$$\text{C.F.} = y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \ldots$$

ii) The roots are real and repeated

If $m_1, m_1, m_1, m_2, m_2, m_3$ are real roots then

$$y_c = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + (C_4 + C_5 x) e^{m_2 x} + C_6 e^{m_3 x}$$

iii) The roots are complex and distinct : [Complex roots always occur in complex pairs]

i.e. if $\alpha + i\beta$ is one root then $\alpha - i\beta$ will be another root, then

$$y_c = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

iv) The roots are complex and repeated

i.e. if $\alpha \pm i\beta$ and $\alpha \pm i\beta$ are roots then

$$y_c = e^{\alpha x} [(C_1 + C_2) x \cos \beta x + (C_3 + C_4 x) \sin \beta x$$

3) Particular integrals :

If $f(D)\, y = \phi(x)$ is the LDE with constant coefficients then its particular integral is

$$\text{P.I.} = y_p = \frac{1}{f(D)} \phi(x)$$

4) Shortcut methods to find P.I. of special functions :

Sr. No.	Type of $\phi(x)$	$y_p = \frac{1}{f(D)} \phi(x)$
1.	i) $\phi(x) = e^{ax}$	$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax};\ f(a) \neq 0$ $y_p = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ $y_p = \frac{1}{\phi(D)(D-a)^r} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{r!} e^{ax};\ \phi(a) \neq 0$
	ii) $\phi(x) = k = k\, e^{0x}$ where k = constant	$y_p = \frac{1}{f(D)} k e^{0x} = k \frac{1}{f(0)};\ f(0) \neq 0$
	iii) $\phi = a^x = e^{x \log a}$	$y_p = \frac{1}{f(D)} a^x = \frac{1}{f(\log a)} a^x$

2.	$\phi = \sin(ax+b)$ or $\cos(ax+b)$	$y_p = \frac{1}{f(D^2)} \frac{\sin}{\cos}(ax+b) = \frac{1}{f(-a^2)} \frac{\sin}{\cos}(ax+b)$, if $f(-a^2) \neq 0$ $y_p = \frac{1}{(D^2+a^2)^r} \frac{\sin}{\cos}(ax+b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!}$ $\frac{\sin}{\cos}\left(ax+b+\frac{r\pi}{2}\right)$
3.	$\phi = \cosh(ax+b)$ or $\sinh(ax+b)$	$y_p = \frac{1}{f(D^2)}$ $\frac{\cosh}{\sinh}(ax+b) = \frac{1}{f(a^2)} \frac{\cosh}{\sinh}(ax+b)$, if $f(a^2) \neq 0$ If $f(a^2) = 0$ then $\frac{1}{f(D^2)} \frac{\sinh}{\cosh}(ax+b) = \frac{x}{f'(a^2)} \frac{\sinh}{\cosh}(ax+b)$; $f'(a^2) \neq 0$
4.	$\phi = x^p$ where P is positive integer.	To find y_p reduce f(D) in any one of the following form i) $(1+D)^n = 1 + nD + \frac{n(n-1)}{2!}D^2 + \frac{n(n-1)(n-2)}{3!}D^3 + \ldots$ ii) $\frac{1}{1+D} = 1 - D + D^2 - D^3 + \ldots$ iii) $\frac{1}{1-D} = 1 + D + D^2 + D^3 + \ldots$
5.	$\phi = e^{ax} V$ where V is any function of x	$y_p = \frac{1}{f(D)} e^{ax} V = e^{ax}\left[\frac{1}{f(D+a)} V\right]$
6.	$\phi = xV$ where V is any function of x	$y_p = \frac{1}{f(D)} xV = \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)} V$
7.	$\phi = x^n \sin ax$ or $x^n \cos ax$	$y_p = \frac{1}{f(D)} x^n \sin ax = \text{Im part} \left\{\frac{1}{f(D)} x^n e^{iax}\right\}$ $y_p = \frac{1}{f(D)} x^n \cos ax = \text{Real part} \left\{\frac{1}{f(D)} x^n e^{iax}\right\}$

5) General Method :

If it is not possible to apply any of above types then apply general method i.e. if $\phi(x)$ = tan ax, cot ax, cosec ax sec ax, sin e^x, cos e^x, $\frac{1}{1+e^x}$, $\frac{1}{1-e^{-x}}$, $\frac{e^x}{1+e^x}$, then use this method.

To find P.I. use

$$\frac{1}{D-a}\phi(x) = e^{ax}\int \phi(x)\, e^{-ax}\, dx$$

$$\frac{1}{D+a}\phi(x) = e^{-ax}\int \phi(x)\, e^{ax}\, dx$$

Q.1 Solve $D^3y + 3D^2y - 4y = 6e^{-2x}$ **[SPPU : Dec.-17, Marks 4]**

Ans. :

Step 1 : A.E. is $D^3 + 3D^2 - 4 = 0$ ...(1)

D = 1 is one root of equation (1)

1	1	3	0	– 4
		1	4	4
	1	4	4	$\boxed{0}$

$$(D-1)(D^2+4D+4) = 0$$
$$(D-1)(D+2)(D+2) = 0$$
$$D = 1, 2, 2,$$

$\therefore$ $y_c = c_1e^x + (c_2 + c_3x)e^{2x}$ ...(2)

Step 2 : $y_p = \frac{1}{(D-1)(D+2)(D+2)} 6e^{-2x}$

$$= \frac{1}{(-2-1)} \frac{x^2}{2!} 6e^{-2x} = -x^2e^{-2x}$$

Step 3 : The complete solution is

$$y = y_c + y_p = c_1e^x + (c_2+c_3x)e^{2x} - x^2e^{-2x}$$

Q.2 Solve $(D^3 - 5D^2 + 8D - 4)$
$y = 4e^{2x} + e^x + 2^x + 3$

Ans. : Step 1 : A.E. is $D^3 - 5D^2 + 8D - 4 = 0$

$(D-1)(D^2-4D+4)\ (D-1)(D-2)^2 = 0$

$D = 1, 2, 2$

$$\therefore \quad y_c = \text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{2x}$$

Step 2 :

$$y_p = \text{P.I.} = \frac{1}{(D-1)(D-2)^2}(4e^{2x} + e^x + 2^x + 3)$$

$$y_p = \frac{1}{(D-1)(D-2)^2}4e^{2x} + \frac{1}{(D-1)(D-2)^2}e^x + \frac{1}{(D-1)(D-2)^2}2^x + \frac{1}{(D-1)(D-2)^2}3$$

Consider

$$PI_1 = \frac{1}{(D-1)(D-2)^2}4e^{2x}$$

Replace D by a in non zero factor.
i.e. D = 2

$$PI_1 = \frac{1}{(2-1)(D-2)^2}4e^{2x}$$

$$= \frac{4}{1}\cdot\frac{1}{(D-2)^2}e^{2x}$$

$$= \frac{4}{1}\cdot\frac{x^2}{2!}e^{2x}$$

$$PI_1 = 2x^2 e^{2x}$$

Now, consider

$$PI_2 = \frac{1}{(D-1)(D-2)^2}e^x$$

Replace D by a in non zero factor
i.e. D = 1.

$$PI_2 = \frac{1}{(D-1)(1-2)^2}e^x = \frac{1}{1\cdot(D-1)}e^x$$

$$PI_2 = \frac{x}{1}e^x$$

Consider

$$PI_3 = \frac{1}{(D-1)(D-2)^2}2^x$$

For a^x replace D by log a,
(Here D = log 2)

$$PI_3 = \frac{1}{(\log 2-1)(\log 2-2)}2^x$$

Consider

$$PI_4 = \frac{1}{(D-1)(D-2)^2}3$$

For X = constant replace D by 0.

$$PI_4 = \frac{1}{(0-1)(0-2)^2}3$$

$$PI_4 = -\frac{3}{4}$$

Thus
$$\text{P.I.} = PI_1 + PI_2 + PI_3 + PI_4$$
$$= 2x^2 e^x + xe^x + \frac{2^x}{(\log 2-1)(\log 2-2)^2} - \frac{3}{4}$$

Step 3 : The complete solution is

$$y = y_c + y_p$$
$$= C_1 e^x + (C_2 + C_3 x)\, e^{2x} + 2x^2 e^x + xe^x + \frac{2x}{(\log 2-1)^2(\log 2-2)^2} - \frac{3}{4}$$

Q.3 Solve $(D^2 + 4)\, y = \cos 3x \cdot \cos x$ [SPPU : May-14, Marks 4]

Ans. : Step 1 : $(D^2 + 4)\, y = \cos 3x \cdot \cos x$

A.E. is $D^2 + 4 = 0 \quad D^2 = -4 \Rightarrow D = \pm 2i$

$$\therefore \quad \text{C.F.} = C_1 \cos 2x + C_2 \sin 2x,$$

Step 2 : Now $\text{P.I.} = \dfrac{1}{D^2+4} (\cos 3x \cos x)$

$$\text{P.I.} = \frac{1}{2} \frac{1}{D^2+4} 2 [\cos 3x \cos x]$$

$$= \frac{1}{2} \frac{1}{D^2+4} (\cos (4x) + \cos (2x))$$

$$= \frac{1}{2} \frac{1}{D^2+4} \cos 4x + \frac{1}{2} \frac{1}{D^2+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{-16+4} \cos 4x + \frac{1}{2} \frac{1}{-4+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{(-12)} \cos 4x + \frac{1}{2} \text{ fail case}$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} \frac{x}{2D} \cos 2x$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} x \frac{\sin 2x}{2}$$

$$= -\frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Step 3 : The complete solution is

$$y = \text{C.F.} + \text{P.I.} = C_1 \cos 2x + C_2 \sin 2x - \frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Q.4 Solve $(D-1)^3 y = e^x + 5^x - 1$ [SPPU : May-08, Dec.-11, 12, Marks 4]

Ans. :

Step 1 : A.E. is $(D-1)^3 = 0$

$\Rightarrow \quad D = 1, 1, 1$

The complementary function is

$$y_c = (C_1 + C_2 x + C_3 x^2)\, e^x$$

Step 2 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{(D-1)^3} [e^x + 5^x - 1]$$

$$= \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3} 5^x + \frac{1}{(D-1)^3} (-1)$$

$$= \frac{x^3}{3!} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + \frac{1}{(-1)^3} (-1)$$

$$y_p = \frac{x^3}{6} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + 1$$

Step 3 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2 x + C_3 x^2)\, e^x + \frac{x^3}{6} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + 1$$

Q.5 Solve $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$ [SPPU : May-18, Marks 4]

Ans. :

Step 1 : A.E. is $D^3 + 4D = 0$

$$D(D^2+4) = 0$$

$$D(D+2i)(D-2i) = 0$$

$$D = 0, \; -2i, \; 2i$$

$$\therefore \quad y_c = C_1 + C_2 \cos 2x + C_3 \sin 2x \qquad \ldots(1)$$

Step 2 :

The P.I. is $y_p = \dfrac{1}{D(D^2+4)} \sin 2x$

$$y_p = \frac{1}{D^2+4} \frac{1}{D} \sin 2x = \frac{1}{D^2+4} \int \sin 2x dx$$

$$= \frac{1}{D^2+4}\left[\frac{-\cos 2x}{2}\right] = -\frac{1}{2}\frac{x}{2(2)}\sin 2x$$

$$y_p = -\frac{x}{8}\sin 2x$$

Step 3 : The complete solution is

$$y = y_c + y_p = C_1 + C_2\cos 2x + C_3\sin 2x - \frac{x}{8}\sin 2x$$

Q.6 Solve $(D^2 - 2D + 5)\, y = 10 \sin x$ ☞ [SPPU : Dec.-05, May-13, Marks 4]

Ans. : Step 1 :

A.E. is $D^2 - 2D + 5 = 0$

$$D = \frac{-(-2) \pm \sqrt{4-4(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$$D = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The C.F. is $y_c = e^x(C_1 \cos 2x + C_2 \sin 2x)$

Step 2 : The P.I. is $y_p = \frac{1}{D^2 - 2D + 5}(10 \sin x)$

$$= (10)\frac{1}{-1-2D+5}\sin x = (10)\frac{1}{4-2D}\sin x$$

$$= 10\frac{4+2D}{16-4D^2}\sin x$$

$$= 10\frac{(4+2D)}{16-4(-1)}\sin x = \frac{1}{2}(4 \sin x + 2\cos x)$$

$$y_p = 2\sin x + \cos x$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.7 Solve $(D^2 + 2D + 1)\, y = e^{-x} + \cos x$. ☞ [SPPU : May-19, Makrs 4]

Ans. :

Step 1 : A.E is $D^2 + 2D + 1 = 0$

$$(D+1)(D+1) = 0$$

$$D = -1, -1$$

The C.F is $y_c = (c_1 + c_2 x)e^{-x}$

Step 2 : The P.I is

$$y_p = \frac{1}{D^2+2D+1}[e^{-x} + \cos x]$$

$$= \frac{1}{D^2+2D+1}e^{-x} + \frac{1}{D^2+2D+1}\cos x$$

$D \to -1 \qquad D^2 \to -1$

$$= \frac{1}{1-2+1}e^{-x} + \frac{1}{-1+2D+1}\cos x$$

First formula fails

$$= x\frac{1}{2D+2}e^{-x} + \frac{1}{2D}\cos x$$

$D \to -1$

$$= x\frac{1}{-2+2}e^{-x} + 2\int \cos x\,dx$$

First formula fails

$$y_p = x\frac{1}{2}e^{-x} + 2\sin x$$

Step 3 : Complete solution is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{xe^{-x}}{2} + 2\sin x$$

Q.8 Solve $(D^3 + 8)y = x^4 + 2x + 1$ ☞ [SPPU : Dec.-08, 12, Marks 4]

Ans. : Step 1 : A.E. is $D^3 + 8 = 0$

By synthetic division method, we get,

$$(D+2)(D^2 - 2D + 4) = 0$$

$$\therefore \quad D = -2,\ D = 1 \pm i\sqrt{3}$$

The C.F. is

$$y_c = C_1 e^{-2x} + e^x[C_2 \cos\sqrt{3}\,x + C_3 \sin\sqrt{3}\,x]$$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^3+8}[x^4+2x+1]$$

$$= \frac{1}{8}\frac{1}{\left(1+\frac{D^3}{8}\right)}[x^4+2x+1]$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}+\left(\frac{D^3}{8}\right)^2 \cdots\right][x^4+2x+1]$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{1}{8}(24\,x)+0\right]$$

$$= \frac{1}{8}[x^4-x+1]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.9 Solve $(D^2 + 6D + 9)\, y = x^{-3}\, e^{-3x}$ ☞ **[SPPU : Dec.-14, Marks 4]**

Ans. :

Step 1 : $D^2 + 6D + 9 = 0$ ∴ A.E. is $(D + 3)^2 = 0$

$$\Rightarrow D = -3, -3$$

∴ $$\text{C.F.} = (C_1 x + C_2)\, e^{-3x}$$

Step 2 : Now $$\text{P.I.} = \frac{1}{(D+3)^2} x^{-3} e^{-3x} = e^{-3x}\frac{1}{[D-3+3]^2} x^{-3}$$

$$= e^{-3x}\frac{1}{D^2} x^{-3} = e^{-3x}\frac{1}{D}\left(\frac{1}{-2x^2}\right)$$

$$\text{P.I.} = e^{-3x}\frac{1}{2x} = \frac{1}{2x\, e^{3x}}$$

Step 3 : The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (C_1\, x + C_2)\, e^{-3x} + \frac{1}{2\, x\, e^{3x}}$$

Q.10 Solve $(D^2 + D + 1)y = x \sin x$ ☞ **[SPPU : May-17, Dec.-19, Marks 4]**

Ans. :

Step 1 : A.E. is $D^2 + D + 1 = 0$

$$D = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}\,i}{2}$$

$$D = -\frac{1}{2}+i\frac{\sqrt{3}}{2},\quad -\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

∴ $$y_c = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right]$$

Step 2 :

$$y_p = \frac{1}{D^2+D+1} x \sin x$$

$$= \left[x - \frac{2D+1}{D^2+D+1}\right]\frac{1}{D^2+D+1}\sin x$$

$$= \left[x - \frac{2D+1}{D^2+D+1}\right]\frac{1}{-1+D+1}\sin x = \left[x - \frac{2D+1}{D^2+D+1}\right](-\cos x)$$

$$= -\left[x - \frac{2D+1}{D^2+D+1}\right]\cos x = x\cos x + (2D+1)\left(\frac{1}{-1+D+1}\right)\cos x$$

$$y_p = -x\cos x + (2D+1)(\sin x) = -x\cos x + 2\cos x + \sin x$$

Step 3 : The complete solution is $y = y_c + y_p$

$$\therefore\ y = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_1\cos\frac{\sqrt{3}}{2}x\right] - x\cos x + 2\cos x + \sin x$$

Q.11 $(D^2-2D+1)y = x e^x \sin x$ [SPPU : May-05, 13, Marks 8]

Ans. : Step 1 : The A.E. is $D^2-2D+1=0$

$(D-1)^2 = 0 \Rightarrow D = 1, 1$

The C.F. is $y_c = (C_1+C_2x)e^x$

Step 2 : The P.I. is $y_p = \frac{1}{(D-1)^2} x e^x \sin x$

$$= e^x \frac{1}{[D+1-1]^2} x \sin x$$

$$y_p = e^x \frac{1}{D^2} x \sin x = e^x\left[x - \frac{2D}{D^2}\right]\frac{1}{D^2} \sin x$$

$$= e^x\left[x-\frac{2D}{D^2}\right](-\sin x) = -e^x\left[x-\frac{2D}{D^2}\right]\sin x$$

$$y_p = -e^x[x\sin x + 2\cos x]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.12 $(D^2+3D+2)y = e^x + \cos e^{e^x}$ [SPPU : Dec.-08,11, May-09, Marks 4]

Ans. :

Step 1 : A.E. is $D^2+3D+2=0$

$(D+2)(D+1) = 0$

$$D = -2, -1$$

The C.F. is $y_c = C_1e^{-2x} + C_2e^{-x}$

Step 2 : The P.I. is

$$y_p = \frac{1}{(D+1)(D+2)}[e^{e^x} + \cos e^x]$$

$$= \frac{1}{D+2} e^{-x}\int e^x (e^{e^x} + \cos e^x)\,dx$$

$$= \frac{1}{D+2} e^{-x}\left(e^{e^x} + \sin e^x\right)$$

$$= e^{-2x}\int e^{2x} e^{-x}\left(e^{e^x} + \sin e^x\right)dx$$

$$= e^{-2x}\int e^x\left(e^{e^x} + \sin e^x\right)dx$$

$$y_p = e^{-2x}\left[e^{e^x} - \cos e^x\right]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.13 Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ [SPPU : May-15, Marks 4]

Ans. :

Step 1 : A.E. is $D^2+3D+2=0$

$$(D+1)(D+2) = 0$$

$$D = -1, -2$$

$$\therefore \quad y_c = C_1e^{-x} + C_2e^{-2x}$$

Step 2 :

$$y_p = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}$$

$$= e^{-x}\int e^x e^{e^x} dx - e^{-2x}\int e^{2x} e^{e^x} dx$$

Put $e^x = t \Rightarrow e^{e^x} = e^t$ and $e^x dx = dt$

$$= e^{-x}e^t - e^{-2x}[te^t - e^t]$$

$$y_p = e^{-x}e^{e^x} - e^{-2x}[e^x e^{e^x} - e^{e^x}]$$

$$y_p = e^{e^x}[e^{-x} - e^{-x} + e^{-2x}] = e^{-2x}e^{e^x}$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^x}$$

Q.14 Solve $(D^2 - 1)y = e^{-x}\sin e^{-x} + \cos e^{-x}$

[SPPU : May-11, Dec.- 11, Marks 4]

Ans. : Step 1 : A.E. $(D - 1)(D + 1) = 0$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x$$

Step 2 :

$$y_p = \text{P.I.} = \frac{1}{f(D)} X$$

$$\text{P.I.} = \frac{1}{(D-1)(D+1)}(e^{-x}\sin e^{-x} + \cos e^{-x})$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right](e^{-x}\sin e^{-x} + \cos e^{-x})$$

$$\text{P.I.} = \frac{1}{2}\left\{\begin{array}{l}\frac{1}{D-1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right) \\ -\frac{1}{D+1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)\end{array}\right\}$$

Consider

$$PI_1 = \frac{1}{2}\left\{\frac{1}{D-1}e^{-x}\sin e^{-x} + \cos e^{-x}\right\}$$

$$= \frac{1}{2}e^x\int\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx$$

$$= \frac{1}{2}e^x\int\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)e^{-x}dx$$

Put $e^{-x} = t \therefore -e^{-x}dx = dt$ i.e. $e^{-x}dx = -dt$

$$= \frac{1}{2}e^x\int(t\sin t + \cos t)(-dt)$$

$$= \frac{-1}{2}e^x\left[\int t\sin t\ dt + \int \cos t\ dt\right]$$

$$PI_1 = \frac{-e^x}{2}\{[(t)(-\cos t) - (1)(-\sin t)] + \sin t\}$$

$$= \frac{-e^x}{2}\{-t\cos t + 2\sin t\}$$

Put $t = e^{-x}$

$$= \frac{-e^x}{2}\left\{-e^{-x}\cos e^{-x} + 2\sin e^{-x}\right\}$$

$$PI_1 = \frac{1}{2}\cos e^{-x} - e^x\sin e^{-x}$$

Consider

$$PI_2 = \frac{1}{2}\frac{1}{D+1}(e^{-x}\sin e^{-x} + \cos e^{-x})$$

$$= \frac{1}{2}e^{-x}\int -e^x(\cos e^{-x} + e^{-x}\sin e^{-x})dx$$

$$= e^{-x}\sin e^{-x} = \frac{1}{2}e^{-x}\cdot e^x\ \cos e^{-x}$$

$$PI_2 = \frac{1}{2}\cos e^{-x}$$

Thus,

$$PI = PI_1 - PI_2 = -e^x\sin e^{-x}$$

$\therefore$ **Step 3 :** The complete solution is

$\therefore$

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-x} + C_2 e^x - e^x\sin e^{-x}$$

Q.15 Solve $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

[SPPU : Dec.-11, 12, Marks 4]

Ans. : Step - 1 : $D^2 - D - 2 = 0$

$(D - 2)(D + 1) = 0$

$$D = 2, -1$$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{-x}$$

$\therefore$ **Step 2 :**

$$y_p = \text{P.I.} = \frac{1}{f(D)}\, X$$

$$= \frac{1}{(D-2)(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{-3}\left[\frac{1}{D+1} - \frac{1}{D-2}\right]\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

Consider

$$PI_1 = \frac{1}{D+1}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

$$= e^{-x}\int e^{x}\left[\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right] dx$$

$$= e^{-x}\cdot e^{x}\cdot\left(2\log x - \frac{1}{x}\right)$$

$$PI_1 = 2\log x - \frac{1}{x}$$

Consider

$$PI_2 = \frac{1}{D-2}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= e^{2x}\int e^{-2x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

Put $-2x = t \quad \therefore\ x = \frac{-t}{2} \ \therefore dx = \frac{-dt}{2}$

$$= e^{2x}\int e^{t}\left[2\log\left(\frac{t}{-2}\right) - \frac{2}{t} + \frac{4}{t^2}\right]\frac{dt}{-2}$$

$$= \frac{e^{2x}}{-2}\int e^{t}\left[\left(2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right) + \left(\frac{2}{t} + \frac{4}{t^2}\right)\right] dt$$

$$= \frac{e^{2x}}{-2}\cdot e^{t}\left[2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right]$$

Put $t = -2x$

$$= \frac{e^{2x}}{-2}\cdot e^{-2x}\left[2\log\left(\frac{-2x}{2}\right) - \left(\frac{4}{-2x}\right)\right]$$

$$= \frac{1}{-2}\left[2\log x + \frac{2}{x}\right] = -\log x - \frac{1}{x}$$

Thus

$$\text{P.I.} = -\frac{1}{3}\,[PI_1 - PI_2] = -\frac{1}{3}[3\log x]$$

$$= -\log x$$

Step 3 : $\therefore$ The complete solution is

$$\therefore \quad y = \text{C.F.} + \text{P.I.} = C_1 e^{2x} + C_2 e^{-x} - \log x$$

Q.16 Solve $(D^2 - 4D + 3)y = x^3 e^{2x}$ **[SPPU : Dec.-16, Marks 4]**

Ans. : Step 1 :

$$\text{A.E. is } D^2 - 4D + 3 = 0$$

$$(D-1)(D-3) = 0$$

$$D = 1, 3$$

$$\therefore \quad y_c = C_1 e^{x} + C_2 e^{3x} \qquad \ldots(1)$$

Step 2 :

$$y_p = \frac{1}{D^2 - 4D + 3}\, x^3 e^{2x}$$

$$= e^{2x}\frac{1}{(D+2)^2 - 4(D+2) + 3}\, x^3$$

$$= e^{2x}\frac{1}{D^2 - 1}\, x^3$$

$$= -e^{2x}\frac{1}{1 - D^2}\, x^3$$

$$= -e^{2x}[1 + D^2 + D^4 + \ldots..]\, x^3$$

$$y_p = -e^{2x}(x^3 + 6x)$$

Step 3 : The complete solution is $y = y_c + y_p$

$$\therefore \quad y = C_1 e^x + C_2 e^{3x} - e^{2x}(x^3 + 6x)$$

1.2 : Legendre's Differential Equations

1) Legendre's D.E. : The general form of 3rd order D.E. is

$$a_0 (ax+b)^3 \frac{d^3y}{dx^3} + a_1 (ax+b)^2 \frac{d^2y}{dx^2} + a_2 (ax+b)\frac{dy}{dx} + a_3 y = \phi(x)$$

Put $\quad ax + b = e^z \Rightarrow z = \log(ax+b)$

$$\therefore \quad (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\,Dy \quad \text{where} \quad D = \frac{d}{dz}$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

$\therefore$ Given D.E. reduces to LDE with constant coefficients in y and z. Now apply previous methods.

2) Cauchy's D.E. or Homogeneous D.E.

The general form of 3rd order D.E. is

$$a_0 x^3 \frac{d^3y}{dx^3} + a_1 x^2 \frac{d^2y}{dx^2} + a_2 x \frac{dy}{dx} + a_3 y = \phi(x)$$

Put $x = e^z \Rightarrow z = \log x$

and $x\frac{dy}{dx} = Dy, \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$

and $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ where $\frac{d}{dz} \equiv D$

Q.17 Solve $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$

[SPPU : May-16,17, Marks 4]

Ans. :

Step 1 :

$$\text{Put } 2x+3 = e^z$$

$$z = \log(2x+3)$$

$$\therefore \quad (2x+3)\frac{dy}{dx} = 2\,Dy \quad \text{and} \quad D = \frac{d}{dz}$$

$$(2x+3)^2 \frac{d^2y}{dx^2} = 4\,D(D-1)y$$

$\therefore$ DE becomes $[4\,D(D-1) - (2\,D)2 - 12]y = 6\frac{1}{2}(e^z - 3)$

$$(4D^2 - 8D - 12)y = 3(e^z - 3)$$

$$(D^2 - 2D - 3)y = \frac{3}{4}(e^z - 3) \quad \ldots(1)$$

which is the LDE with constant coefficients.

Step 2 :

$$\text{A.E. is } D^2 - 2D - 3 = 0$$

$$(D-3)(D+1) = 0$$

$$\Rightarrow \quad D = -1, 3$$

$$\therefore \quad y_c = C_1 e^{-z} + C_2 e^{3z} \quad \ldots(2)$$

Step 3 :

$$y_p = \frac{1}{D^2 - 2D - 3}\left(\frac{3}{4}e^z - \frac{9}{4}\right)$$

$$= \frac{3}{4}\frac{e^z}{1-2-3} - \frac{9}{4}\frac{1}{0-3}$$

$$\therefore \quad y_p = -\frac{3}{16}e^z + \frac{3}{4}$$

Step 4 : The complete solution is,

$$y = y_c + y_p$$

$$= C_1 e^{-z} + C_2 e^{3z} - \frac{3}{16}e^z + \frac{3}{4}$$

$$y = C_1(2x+3)^{-1} + C_2(2x+3)^3 - \frac{3}{16}(2x+3) + \frac{3}{4}$$

Q.18 Solve $x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 3y = \log x$

☞ [SPPU : Dec.-17, Marks 4]

Ans. : Step 1 : Put $x = e^z, \quad z = \log(x)$

$$x\frac{dy}{dx} = Dy, \quad D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ DE becomes

$$D(D-1)y + 5\,Dy + 3y = z$$

$$(D^2+4D+3)y = z \qquad \ldots(1)$$

Step 2 : A.E. is

$$D^2+4D+3 = 0 \Rightarrow (D+1)(D+3) = 0$$

$$D = -1,\ -3$$

$$y_c = C_1e^{-z} + C_2e^{-3z}$$

Step 3 : $$y_p = \frac{1}{D^2+4D+3}(z) = \left(\frac{\left(\frac{1}{2}\right)}{D+1} - \frac{\left(\frac{1}{2}\right)}{D+3}\right)z$$

$$= \frac{1}{2}\left(\frac{1}{1+D} - \frac{1}{3\left(1+\frac{D}{3}\right)}\right)z$$

$$= \frac{1}{2}\left\{(1-D+D^2-\ldots) - \frac{1}{3}\left(1-\frac{D}{3}+\frac{D^2}{9}-\ldots\right)\right\}(z)$$

$$= \frac{1}{2}\left\{z-1-\frac{1}{3}\left(z-\frac{1}{3}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{2z}{3}-\frac{8}{9}\right\} = \frac{z}{3}-\frac{4}{9}$$

Step 4 : The complete solutions is

$$y = y_c + y_p = C_1e^{-z} + C_2e^{-3z} + \frac{z}{3} - \frac{4}{9}$$

$$= C_1\left(\frac{1}{x}\right) + C_2x^{-3} + \frac{\log x}{3} - \frac{4}{9}$$

Q.19 Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

☞ [SPPU : Dec.-16, Marks 4]

Ans. : Step 1 : Put $x = e^z, \quad z = \log(1+x)$

$$\therefore \quad (x+1)\frac{dy}{dx} = Dy \quad \text{where } D = \frac{d}{dz}$$

$$\text{and} \quad (x+1)^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ D.E. becomes,

$$[D(D-1)+D+1]y = 2\sin z$$

$$(D^2+1)y = 2\sin z \qquad \ldots(1)$$

Step 2 : A.E. is $D^2+1 = 0 \Rightarrow D^2 = -1$

$$D = \pm i$$

$$\therefore \quad y_c = C_1\cos z + C_2\sin z \qquad \ldots(2)$$

Step 3 : $$y_p = \frac{1}{D^2+1}2\sin z = -2\frac{z}{2}\cos z = -z\cos z$$

Step 4 : The complete solution is $y = y_c + y_p$

$$\therefore \quad y = C_1\cos z + C_2\sin z - z\cos z$$

$$y = C_1\cos\log(1+x) + C_2\sin\log(1+x) - (1+x)\cos(1+x)$$

Q.20 Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$

☞ [SPPU : Dec.-15, Marks 4]

Ans. : Step 1 : Put $x = e^z \Rightarrow z = \log x$

$$\therefore \quad x\frac{dy}{dx} = Dy \quad \text{where } D \equiv \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ DE becomes $[D(D-1)+D+1]y = \sin(\log e^z)$

$$(D^2+1) = \sin z$$

By using above example $y_c = C_1\cos z + C_2\sin z$

$$y_p = -\frac{z}{2}\cos z$$

$$\therefore \quad y = C_1\cos z + C_2\sin z - \frac{z}{2}\cos z$$

$$\therefore \quad y = C_1\cos(\log x) + C_2\sin(\log x) - \frac{\log x}{2}\cos(\log x)$$

Q.21 Solve $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = x^2\sin(\log x)$. [SPPU : Dec.-18, Marks 4]

Ans. :

Step 1 : Put $x = e^z \Rightarrow z = \log x$

$$\therefore \quad x\frac{dy}{dx} = Dy \quad \text{where } D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ Given DE becomes

$$D(D-1)y - 3Dy + 3y = e^{2z}\sin z$$

$$(D^2 - 4D + 3)y = e^{2z}\sin z \quad \ldots(1)$$

Step 2 : AE is $D^2 - 4D + 3 = 0$

$$(D-1)(D-3) = 0$$

$$D = 1, 3$$

$$\therefore \quad y_c = C_1e^z + C_2e^{3z}$$

Step 3 :

$$y_p = \frac{1}{(D-1)(D-3)}e^{2z}\sin z$$

$$= e^{2z}\frac{1}{(D+2-1)(D+2-3)}\sin z$$

$$= e^{2z}\frac{1}{(D+1)(D-1)}\sin z = e^{2z}\frac{1}{D^2-1}\sin z$$

$$y_p = e^{2z}\frac{1}{-1-1}\sin z = -\frac{1}{2}e^{2z}\sin z$$

Step 4 : Hence, the complete solution is

$$y = y_c + y_p$$

$$y = C_1e^z + C_2e^{3z} - \frac{1}{2}e^{2z}\sin z$$

$$= C_1x + C_2x^3 - \frac{1}{2}x^2\sin(\log x)$$

Q.22 Solve $(4x+1)^2\dfrac{d^2y}{dx^2} + 2(4x+1)\dfrac{dy}{dx} + y = 2x + 1$

[SPPU : May-18, Marks 4]

Ans. :

Step 1 : Put $4x + 1 = e^z \Rightarrow z = \log(4x+1)$

$$\therefore \quad (4x+1)\frac{dy}{dx} = 4Dy \quad \text{where } D = \frac{d}{dz}$$

$$(4x+1)^2\left(\frac{d^2y}{dx^2}\right) = 16\,D(D-1)y$$

Step 2 : Equation becomes

$$[16D(D-1) + 2(4D) + 1]y = 2\left(\frac{e^z - 1}{4}\right) + 1$$

$$(16D^2 - 8D + 1)y = \frac{1}{2}(e^z - 1) + 1 \quad \ldots(1)$$

$$= \frac{1}{2}e^z + \frac{1}{2}$$

Step 3 : A.E. is

$$16D^2 - 8D + 1 = 0$$

$$(4D-1)(4D-1) = 0$$

$$D = \frac{1}{4}, \frac{1}{4}$$

$$y_c = (C_1 + C_2z)e^{\frac{1}{4}z} \quad \ldots(2)$$

Step 4 :

$$y_p = \frac{1}{(4D-1)(4D+1)}\left[\frac{1}{2}(e^z) + \frac{1}{2}\right]$$

$$= \frac{1}{2}\frac{1}{(4-1)(4+1)}e^z + \frac{1}{2}\frac{1}{(-1)(1)}$$

$$= \frac{1}{30}e^z - \frac{1}{2}$$

Step 5 : The complete solution is $y = y_c + y_p$

$$y = (C_1 + C_2 z)e^{\frac{1}{4}z} + \frac{1}{30}e^z - \frac{1}{2}$$

$$= [C_1 + C_2 \log(4x+1)(4x+1)^{\frac{1}{4}} + \frac{1}{30}(4x+1) - \frac{1}{2}$$

1.3 : Lagranges Method of Variation of Parameters

1) Method of Variation of Parameters to find Particular Integral

Step 1 : If the differential equation is of order two then the complementary function will be of the form C.F. $= C_1 y_1 + C_2 y_2$

Step 2 : Assume

$$\text{P.I.} = uy_1 + vy_2$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \quad \Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}$$

where y_1' and y_2' are derivatives of y_1 and y_2 also X is the R.H.S. of equation. [i.e. f (D) y = X]

Step 4 : Then

$$u = \int \frac{\Delta u}{\Delta} dx, \quad v = \int \frac{\Delta v}{\Delta} dx$$

Step 5 : Substitute u and v in P.I. $= uy_1 + vy_2$

Q.23 Solve $(D^2 + 1)\, y = \operatorname{cosec} x$

☞ [SPPU : Dec.-15, May-16, Marks 4]

Ans. : Step 1 : C.F. $= C_1 \cos x + C_2 \sin x$

Step 2 : Let P.I. $= u\,y_1 + v\,y_2$

Step 3 : Find derivatives of y_1, y_2

$$y_1 = \cos x \quad y_2 = \sin x$$

$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$$

$$X = \operatorname{cosec} x$$

$$\Delta u = -X' y_2 = -\operatorname{cosec} x \sin x = -1$$

$$\Delta v = X y_1 = \operatorname{cosec} x \cos x - \cot x$$

Step 5 : Find u, v

$$u = \int \frac{\Delta u}{\Delta} dx \quad v = \int \frac{\Delta v}{\Delta} dx$$

$$= \int -1\, dx, \quad = \int \cot x\, dx$$

$$= -x, \quad = \log \sin x$$

Step 6 : P.I. $= u\,y_1 + v\,y_2$

$$= -x \cos x + \sin x \log \sin x$$

Step 7 : $y = \text{C.F.} + \text{P.I.}$

$y = C_1 \cos x + C_2 \sin x - x\cos x + \sin x \log \sin x$

is the complete solution.

Q.24 Solve $(D^2 + 9)y = \dfrac{1}{1 + \sin 3x}$ by variation of parameters

☞ [SPPU : May-18, Marks 4]

Ans. : Step 1 : A.E. is $D^2 + 9 = 0 \Rightarrow D^2 = \pm 3i$

$$y_c = C_1 \cos 3x + C_2 \sin 3x$$

Compare y_c with $y_c = C_1 y_1 + C_2 y_2$

$\therefore$ $y_1 = \cos 3x, \quad y_2 = \sin 3x$

Step 2 : Let $y_p = u\,y_1 + v\,y_2$

$$y_1' = -3 \sin 3x, \quad y_2' = 3 \cos 3x$$

Step 3 :

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -Xy_2 = \frac{-1}{1+\sin 3x}(\sin 3x)$$

$$= \frac{-\sin 3x}{1+\sin 3x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = y, X = \frac{\cos 3x}{1+\sin 3x}$$

Step 4 :

$$u = \int \frac{\Delta u}{\Delta} dx$$

$$= \int -\frac{\sin 3x}{3(1+\sin 3x)} dx$$

$$= -\int \frac{\sin 3x(1-\sin 3x)}{3(1-\sin^2 3x)} dx$$

$$= -\int \frac{\sin 3x - \sin^2 3x}{\cos^2 3x} dx$$

$$= -\frac{1}{3}\int (\sec 3x \tan 3x - \tan^2 3x) dx$$

$$= \int \left(-\frac{\sec 3x \tan 3x}{3} + \frac{\sec^2 3x - 1}{3} \right) dx$$

$$u = -\sec 3x + \tan 3x - \frac{1}{3}x$$

$$\text{and } v = \int \frac{\Delta v}{\Delta} dx$$

$$= \int \frac{\cos 3x}{3(1+\sin 3x)} dx$$

$$= \frac{1}{9}\log(1+\sin 3x)$$

Step 5 : $\text{P.I.} = \cos 3x(-\sec 3x + \tan 3x - \frac{1}{3}x) + \frac{\sin 3x}{9}\log(1+\sin 3x)$

Step 6 : The complete solution is $y = y_c + y_p$

Q.25 Solve $(D^2 + 4)y = \sec 2x$ [by variation of parameters]

☞ **[SPPU : May-15, Marks 4]**

Ans. :

Step 1 : A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$\therefore \quad y_c = C_1 \cos 2x + C_2 \sin 2x$$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

$$\therefore \quad y_1 = \cos 2x, \; y_2 = \sin 2x$$

$$y_1' = -2\sin 2x, \; y_2' = 2\cos 2x$$

Step 3 : Assume $\text{P.I.} = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv where

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = 2\cos 2x \cos 2x + 2\sin 2x \sin 2x = 2$$

$$X = \sec 2x$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -Xy_2 = -\sec 2x \sin 2x$$

$$= -\tan 2x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = Xy_1 = \sec 2x \cos 2x = 1$$

Step 3 : $\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

$$= 3\cos^2 3x + 3\sin^2 3x = 3$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -Xy_2 = \frac{-1}{1+\sin 3x}(\sin 3x)$$

$$= \frac{-\sin 3x}{1+\sin 3x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = y_1 X = \frac{\cos 3x}{1+\sin 3x}$$

Step 4 : $u = \int \frac{\Delta u}{\Delta} dx$

$$= \int -\frac{\sin 3x}{3(1+\sin 3x)} dx$$

$$= -\int \frac{\sin 3x(1-\sin 3x)}{3(1-\sin^2 3x)} dx$$

$$= -\int \frac{\sin 3x - \sin^2 3x}{\cos^2 3x} dx$$

$$= -\frac{1}{3}\int (\sec 3x \tan 3x - \tan^2 3x) dx$$

$$= \int \left(-\frac{\sec 3x \tan 3x}{3} + \frac{\sec^2 3x - 1}{3} \right) dx$$

$$u = -\sec 3x + \tan 3x - \frac{1}{3}x$$

and $v = \int \frac{\Delta v}{\Delta} dx$

$$= \int \frac{\cos 3x}{3(1+\sin 3x)} dx$$

$$= \frac{1}{9}\log(1+\sin 3x)$$

Step 5 : P.I. $= \cos 3x(-\sec 3x + \tan 3x - \frac{1}{3}x) + \frac{\sin 3x}{9}\log(1+\sin 3x)$

Step 6 : The complete solution is $y = y_c + y_p$

Q.25 Solve $(D^2 + 4)y = \sec 2x$ [by variation of parameters]

[SPPU : May-15, Marks 4]

Ans. :

Step 1 : A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ $y_c = C_1 \cos 2x + C_2 \sin 2x$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

$\therefore$ $y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$

Step 3 : Assume P.I. $= uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv where

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = 2\cos 2x \cos 2x + 2\sin 2x \sin 2x = 2$$

$$X = \sec 2x$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -Xy_2 = -\sec 2x \sin 2x$$

$$= -\tan 2x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = Xy_1 = \sec 2x \cos 2x = 1$$

Step 5 : $u = \int \frac{\Delta u}{\Delta} dx = \int -\frac{\tan 2x}{2} du = \frac{1}{2}\frac{\log \cos 2x}{2}$

$= \frac{1}{4}\log\cos 2x$

and $v = \int \frac{\Delta v}{\Delta} dx = \int \frac{1}{2} dx = \frac{1}{2}x$

Step 6 : $\therefore$ The P.I. is

$$y_p = \frac{1}{4}\cos 2x \log \cos 2x + (\sin 2x)\frac{1}{2}y$$

Step 7 : The complete solution is

$y = y_c + y_p$

$y = C_1\cos 2x + C_2 \sin 2x + \frac{1}{4}\cos 2x \log \cos 2x + \frac{1}{2}x \sin 2x)$

Q.26 Solve $(D^2 + 3D + 2)y = \sin e^x$ using method of variation of parameters. ☞ [SPPU : May-17, Marks 4]

Ans. : Step 1 : A.E. is $D^2 + 3D + 2 = 0$

$(D+1)(D+2) = 0$

$D = -1, -2$

$\therefore \quad y_c = C_1e^{-2x} + C_2e^{-x}$

Step 2 : Let $y_1 = e^{-2x}, \quad y_2 = e^{-x}, y_1' = -2e^{-2x}, \quad y_2' = -e^{-x}$

Let $y_p = uy_1 + vy_2$

Step 3 : $\Delta = y_1y_2' - y_1'y_2 = -e^{-2x}e^{-x} + 2e^{-x}e^{-2x} = e^{-3x}$

$\Delta u = -Xy_2 = -(\sin e^x)e^{-x}, \quad X = \sin e^x$

$\Delta v = Xy_1 = (\sin e^x)e^{-2x}$

Step 4 : $u = \int \frac{\Delta u}{\Delta} dx = \int -\frac{e^{-x}\sin e^x}{e^{-3x}} dx = -\int e^{2x}\sin e^x dx$

Put $e^x = t$, $e^x dx = dt$

$u = -\int e^x(\sin e^x)e^x dx = \int t \sin t \, dt$

$= -\{(t)(-\cos t) - (1)(-\sin t)\} = t\cos t - \sin t$

$u = e^x\cos e^x - \sin e^x$

$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{e^{-2x}\sin e^x dx}{e^{-3x}} dx = \int e^x \sin e^x dx$

$v = -\cos e^x$

Step 5 :

$\therefore \quad y_p = e^{-2x}(e^x\cos e^x - \sin e^x) - e^{-x}(\cos e^x)$

$y_p = -e^{-2x}\sin e^x$

Step 6 : The complete solution is $y = y_c + y_p$

Q.27 Solve $(D^2 - 1)y = \frac{2}{1+e^x}$ using method of variation of parameters. ☞ [SPPU : Dec.-16, Marks 4]

Ans. :

Step 1 : A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$y_c = C_1e^x + C_2e^{-x}$

Comparing $y_c = C_1y_1 + C_2y_2 \Rightarrow y_1 = e^x, \quad y_2 = e^{-x}$

$y_1' = e^x, \quad y_2' = -e^{-x}$

Step 2 :

Let $y_p = uy_1 + vy_2 = X = \frac{2}{1+e^x}$

Step 3 : $\Delta = y_1y_2' - y_1'y_2 = e^x(-e^{-x}) - e^xe^{-x} = -2$

$\Delta u = -Xy_2 = -\frac{2e^{-x}}{1+e^x}$

$\Delta v = Xy_1 = e^x \frac{2}{1+e^x}$

Step 4 : $$u = \int \frac{\Delta u}{\Delta} = \int \frac{-2e^{-x}}{(-2)(1+e^{x})}dx = \int \frac{e^{-x}}{(1+e^{x})}dx$$

Put $e^{-x} = t \Rightarrow e^{-x}(-dx) = dt$

$$\therefore \quad u = \int \frac{-dt}{1+\frac{1}{t}} = -\int \frac{t}{1+t}dt = -\int \frac{1+t-1}{1+t}dt$$

$$= \int \left[1-\frac{1}{1+t}\right]dt = t-\log(1+t)$$

$$u = e^{-x}-\log(1+e^{-x})$$

$$v = \int \frac{\Delta v}{\Delta}dx = \int \frac{2c^{x}}{-2(1+e^{x})}dx = -\int \frac{c^{x}}{1+e^{x}}dx$$

$$v = -\log(1+c^{x})$$

Step 5 :

$$\therefore \quad y_p = e^{x}[e^{-x}\log(1+e^{-x})]+e^{-x}(-1)\log(1+e^{x})$$

$$y_p = 1-e^{x}\log(1+e^{-x})-e^{x}\log(1+e^{x})$$

Step 6 : The complete solution is $y = y_c + y_p$.

Q.28 Solve by variation of parameters $(D^2 + 2D + 1)y = e^{-x}\log x$.

☞ [SPPU : May-12, 13, 15, Marks 4]

Ans. : Step 1 : A.E. is

$$(D^2+2D+1) = 0$$
$$(D+1)(D+1) = 0$$
$$D = -1, -1$$
$$\therefore \quad y_c = (C_1+C_2x)e^{-x} = C_1e^{-x}+C_2xe^{-x}$$

Step 2 : Comparing y_c with $y_c = C_1y_1+C_2y_2$

$$\therefore \quad y_1 = e^{-x}, \quad y_2 = xe^{-x}$$
$$y_1' = -e^{-x}, \quad y_2' = e^{-x}-xe^{-x}$$

Step 3 : Assume that P.I. $= y_p = uy_1+vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}-xe^{-x} \end{vmatrix}$$

$$= e^{-2x}-xe^{-2x}+xe^{-2x} = e^{-2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\log x & e^{-x}-xe^{-x} \end{vmatrix}$$

$$= -xe^{-2x}\log x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\log x \end{vmatrix}$$

$$= e^{-2x}\log x$$

Step 5 : Find u and v : $u = \int \frac{\Delta u}{\Delta}dx$

$$u = \int \frac{xe^{-2x}\log x}{e^{-2x}}dx = -\int x\log x\,dx$$

$$= -\left[\log x\left(\frac{x^2}{2}\right)-\int \frac{1}{x}\left(\frac{x^2}{2}\right)dx\right]$$

$$= -\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]$$

$$u = \frac{x^2}{4}-\frac{x^2}{2}\log x$$

And $$v = \int \frac{\Delta v}{\Delta}dx = \int \frac{e^{-2x}\log x}{e^{-2x}}dx = \int \log x\,dx$$

$$v = \log x - x$$

Step 6 : $y_p = e^{-x}\left(\frac{x^2}{4}-\frac{x^2}{2}\log x\right)+xe^{-x}(\log x-x)$

Step 7 : The complete solution is

$$y = y_c+y_p$$

Q.29 Solve $(D^2+1)y = 2 \cot x$ [SPPU : Dec.-17, Marks 4]

Ans. :

Step 1 : $$\text{C.F.} = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x,$$

$$y_1' = -\sin x, \quad y_2' = \cos x,$$

Step 2 : $$\text{P.I } y_p = uy_1 + vy_2$$

Step 3 : $$\Delta = y_1y_2' - y_1'y_2 = 1, \quad X = 2 \cot x$$

$$\Delta u = -Xy_2 = -2 \cot x \sin x = 2 \cos x$$

$$\Delta v = Xy_1 = 2 \cot x \cos x$$

Step 4 : $$u = \int \frac{\Delta u}{\Delta} dx = \int 2 \cos x\, dx = 2 \sin x$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{2 \cot x \cos x}{1} dx = 2\int \frac{\cos^2 x}{\sin x} dx$$

$$= 2\int \frac{(1-\sin^2 x)}{\sin x} dx$$

$$= \int (\operatorname{cosec} x - \sin x) dx = \log(\operatorname{cosec} x \cos x \cot x)$$

Step 5 : $\therefore$ $$y_p = uy_1 + vy_2$$

$= 2 \cos x \sin x + \sin x[\log \cos \sec x - \cot x) + \cos x]$

Step 6 : The complete solution is $y = y_C + y_p$

Q.30 Solve $(D+1)^2 y = e^{-x}$ by variation of parameter method. [SPPU : Dec.-18, Marks 4]

Ans. :

Step 1 : A.E. is $(D+1)^2 = 0$

$$D = -1, -1$$

$$\therefore \quad y_c = (C_1 + C_2 x)e^{-x}$$

Step 2 : Comparing y_c with $y_c = C_1y_1 + C_2y_2$

$$\therefore \quad y_1 = e^{-x}, \quad y_2 = xe^{-x}$$

$$y_1' = e^{-x}, \quad y_2' = e^{-x} - xe^{-x}$$

Step 3 : Assume that $y_p = uy_1 + vy_2$

Step 4 : Find $\Delta, \Delta u, \Delta v$

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$= e^{-x}(e^{-x} - xe^{-x}) - (-e^{-x})(xe^{-x})$$

$$= e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -Xy_2 = -e^{-x}xe^{-x} = -xe^{-2x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = Xy_1 = e^{-x}e^{-x} = e^{-2x}$$

Step 5 :

$$u = \int \frac{\Delta u}{\Delta} dx = \int \frac{-xe^{-2x}}{e^{-2x}} dx = -\frac{x^2}{2}$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{e^{-2x}}{e^{-2x}} dx = x$$

$$\therefore \quad y_p = -\frac{x^2}{2}e^{-x} + x^2e^{-x} = \frac{x^2}{2}e^{-x}$$

Step 6 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2 x)\, e^{-x} + \frac{x^2}{2}e^{x}$$

Q.31 Solve by method variation of parameters $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$. [SPPU : May-19, Marks 4]

Ans. :

Step 1 : Auxillary equation is $D^2 - 6D + 9 = 0$

$$(D-3)(D-3) = 0$$

$$D = 3, 3$$

C.F is $y_c = (c_1 + c_2 x)e^{3x}$

Step 2 : Consider particular integral as

$$y_p = (u + vx)e^{3x}$$

Here $y_1 = e^{3x},\ y_2 = xe^{3x}$

$$y_1' = 3e^{3x},\ y_2 = e^{3x} + 3xe^{3x}$$

$$X = \frac{e^{3x}}{x^2}$$

Step 3 :

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$\Delta = e^{3x} \cdot e^{3x} \begin{vmatrix} 1 & x \\ 3 & 1+3x \end{vmatrix}$$

$$= e^{6x}[1 + 3x - 3x]$$

$$\Delta = e^{6x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{3x} \\ \frac{3e^{3x}}{x^2} & e^{3x}(1+3x) \end{vmatrix}$$

$$= e^{3x} \cdot e^{3x} \begin{vmatrix} 0 & x \\ \frac{1}{x^2} & 1+3x \end{vmatrix}$$

$$= \frac{e^{6x}}{x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = \begin{vmatrix} e^{3x} & 0 \\ x\,e^{3x} & \frac{e^{3x}}{x^2} \end{vmatrix}$$

$$= e^{3x} \cdot e^{3x} \begin{vmatrix} 1 & 0 \\ x & \frac{1}{x^2} \end{vmatrix}$$

$$= \frac{e^{6x}}{x^2}$$

Step 4 :

$$u = \int \frac{\Delta u}{\Delta} du = \int \frac{e^{6x}}{x} \times \frac{1}{e^{6x}} dx = \log x$$

$$v = \int \frac{\Delta v}{\Delta} du = \int \frac{e^{6x}}{x^2} \cdot \frac{1}{e^{6x}} dx = \frac{-1}{x}$$

Step 5 : P.I. is $y_p = (u + vx)e^{3x}$

$$y_p = (\log x - 1)e^{3x}$$

Step 6 : Complete solution is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{3x} + (\log x - 1)e^{3x}$$

Q.32 Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

Ans. :

Step 1 : Put $2x + 1 = e^z$, $z = \log(2x + 1)$

$$(2x+1)\frac{dy}{dx} = 2Dy \text{ where } D = \frac{d}{dz}$$

$$(2x+1)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$$

D.E becomes

$$[4D(D-1) - 6 \cdot 2D + 16]y = 8e^{2z}$$

$$(4D^2 - 16D + 16)y = 8e^{2z}$$

$$(D^2 - 4D + 4)\,y = 2e^{2z}$$

Step 2 : A.E. is $D^2 - 4D + 4 = 0$

$$D = 2,\ 2$$

$$y_c = (c_1 + c_2 z)e^{2z}$$

Step 3 : P.I. is

$$y_p = \frac{1}{D^2 - 4D + 4}2e^{2z} \quad D \to 2$$

$$= \frac{1}{4 - 8 + 4}2e^{2z}$$

First formula fails

$$= z\frac{1}{2D - 4}\ 2e^{2z} \quad D \to 2$$

$$= z\,\frac{1}{4 - 4}2e^{2z}$$

First formula fails

$$= \frac{z^2}{2}\cdot 2e^{2z}$$

$$y_p = z^2 e^{2z}$$

G.S is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 z)e^{2z} + z^2 e^{2z}$$

But $\quad z = \log(2x+1)$, $e^z = 2x + 1$

$$y = [c_1 + c_2 \log(2x+1)]\,(2x+1)^2 + (2x+1)^2[\log(2x+1)]^2$$

END...

UNIT I

2 Simultaneous Linear Differential Equations and Applications

2.1 : Simultaneous Linear D.E.

- Method of solving simultaneous LDE is similar to solving algebraic equations

Q.1 The currents x and y in coupled circuits are given by
$$L\frac{dx}{dt} + Rx + R(x - y) = E$$
$$L\frac{dy}{dt} + Ry - R(x - y) = 0$$
where L, R, E are constants. Find x and y in terms of t given x = 0, y = 0, when t = 0. **[SPPU : Dec.-05, 10, 14]**

Ans. : **Step 1 :** Use $D = \frac{d}{dt}$

$$L\,Dx + Rx + Rx - Ry = E$$

$$L\,Dy + Ry - Rx + Ry = 0$$

Collect the terms of x and y.

$$(LD + 2R)\,x - Ry = E \qquad \text{... (1)}$$

$$-Rx + (LD + 2R)\,y = 0 \qquad \text{... (2)}$$

Step 2 : Solving for x using Cramer's rule.

$$\begin{vmatrix} LD + 2R & -R \\ -R & LD + 2R \end{vmatrix} x = \begin{vmatrix} E & -R \\ 0 & LD + 2R \end{vmatrix}$$

$$(L^2D^2 + 4RLD + 4R^2 - R^2)\,x = (LD + 2R)\,E$$

$$(LD + R)(LD + 3R)\,x = 2RE \text{ As } DE = \frac{d}{dt}E = 0$$

A.E. is $(LD + R)(LD + 3R) = 0$

$$D = \frac{-R}{L}, \quad D = \frac{-3R}{L}$$

$$x_c = \text{C.F} = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L}$$

Find P.I for x.

$$x_p = \text{P.I.} = \frac{1}{(LD+R)(LD+3R)} 2RE$$

$$x_p = \text{P.I.} = \frac{2RE}{3R^2} = \frac{2E}{3R}$$

Write $x = x_c + x_p$

$$x = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R} \quad \text{... (3)}$$

Step 3 : Use the equation where the coefficient of y is simple i.e. equation (1).

$$\therefore \quad Ry = (LD + 2R)\, x - E$$

$$Ry = L\frac{dx}{dt} + 2Rx - E$$

Substitute x and $\frac{dx}{dt}$ to find y.

$$Ry = L\left[\frac{-R}{L} C_1 e^{-Rt/L} - \frac{3RC_2}{L} e^{-3Rt/L}\right]$$

$$+ 2R\left[C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R}\right] - E$$

$$\therefore \quad Ry = C_1 R e^{-Rt/L} - RC_2 e^{-3Rt/L} + \frac{1}{3}E$$

$$\therefore \quad y = C_1 e^{-Rt/L} - C_2 e^{-3Rt/L} + \frac{E}{3R} \quad \text{... (4)}$$

Step 4 : Given at t = 0, x = 0 and y = 0

$\therefore$ To find C_1 and C_2 put t = 0, x = 0 in equation (3) and t = 0, y = 0 equation (4).

$$0 = C_1 + C_2 + \frac{2E}{3R}$$

$$0 = C_1 - C_2 + \frac{E}{3R}$$

Find C_1 and C_2.

Adding we get, $\quad 0 = 2C_1 + \frac{E}{R} \Rightarrow C_1 = \frac{-E}{2R}$

Substituting we get $C_2 = -\frac{E}{6R}$

Substitute C_1 and C_2 in equations (3) and (4).

$$\therefore \quad x = \frac{-E}{2R} e^{-Rt/L} - \frac{E}{6R} e^{-3Rt/L} + \frac{2E}{3R}$$

$$y = \frac{-E}{2R} e^{-Rt/L} + \frac{E}{6R} e^{-3Rt/L} + \frac{E}{3R}$$

Q.2 Solve $\frac{dx}{dt} + 2x - 3y = t$

$\frac{dy}{dt} - 3x + 2y = e^{2t}$ ☞ [SPPU : Dec.-09, 11, May-12]

Ans. : Step 1 :

Use $D = \frac{d}{dt}$ hence equations becomes,

$$Dx + 2x - 3y = t$$

$$Dy - 3x + 2y = e^{2t}$$

Collect the terms of x and y.

$$(D+2)\, x - 3y = t \quad \text{... (1)}$$

$$(D+2)\, y - 3x = e^{2t} \quad \text{... (2)}$$

Step 2 : Solving for x

$$\begin{vmatrix} D+2 & -3 \\ -3 & D+2 \end{vmatrix} x = \begin{vmatrix} t & -3 \\ e^{2t} & D+2 \end{vmatrix}$$

$$[(D+2)^2 - 3^2]\, x = (D+2)\, t + 3\, e^{2t} = 1 + 2t + 3e^{2t}$$

$$(D^2 + 4D - 5)\, x = 1 + 2t + 3e^{2t} \quad \text{... (3)}$$

A.E. is $D^2 + 4D - 5 = 0$ gives $D = -5, 1$ hence

$$x_c = C_1 e^{-5t} + C_2 e^t \quad \dots (4)$$

Now $$x_p = \text{P.I.} = \frac{1}{D^2+4D-5}(1+2t) + \frac{3e^{2t}}{D^2+4D-5}$$

$$= -\frac{1}{5}\left[1-\frac{4D+D^2}{5}\right]^{-1}(1+2t) + \frac{3e^{2t}}{4+8-5}$$

$$= -\frac{1}{5}\left(1+\frac{4D}{5}\right)(1+2t) + \frac{3}{7}e^{2t}$$

$$= -\frac{1}{5}\left(\frac{13}{5}+2t\right) + \frac{3e^{2t}}{7} \quad \dots (5)$$

Hence G.S. is given by,

$$x = \text{C.F.} + \text{P.I.} = C_1 e^{-5t} + C_2 e^t - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7} \quad \dots (6)$$

Now $$\frac{dx}{dt} = -5C_1 e^{-5t} + C_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t}$$

Step 3 : Use the equation where coefficient of y is simple.

i.e. $$(D+2)x - 3y = t$$

$$3y = (D+2)x - t$$

Putting values of x and $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{3}\left[\frac{dx}{dt}+2x-t\right]$$

$$= \frac{1}{3}\left[-5C_1 e^{-5t} + C_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t} + 2C_1 e^{-5t} + 2C_2 e^t - \frac{26}{25} - \frac{4t}{5} + \frac{6e^{2t}}{7} - t\right]$$

Simplifying we get,

$$y = -C_1 e^{-5t} + C_2 e^{2t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7} \quad \dots (7)$$

Q.3 $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$, $\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$

[SPPU : May-12]

Ans. : Step 1 :

Use $D = \frac{d}{dt}$ hence system can be written as,

$$D^2x + Dy + 3x = e^{-t}$$

$$D^2y - 4Dx + 3y = \sin 2t$$

Collect the terms of x and y.

$$(D^2+3)x + Dy = e^{-t}$$

$$-4Dx + (D^2+3)y = \sin 2t$$

Step 2 : Solving for x by Cramer's rule.

$$\begin{vmatrix} D^2+3 & D \\ -4D & D^2+3 \end{vmatrix} x = \begin{vmatrix} e^{-t} & D \\ \sin 2t & D^2+3 \end{vmatrix}$$

$$\left[(D^2+3)^2 + 4D^2\right]x = 4e^{-t} - 2\cos 2t$$

$$(D^2+1)(D^2+9)x = 4e^{-t} - 2\cos 2t$$

$(D^2+1)(D^2+9) = 0$ gives $\therefore D = \pm i, \pm 3i$

$\therefore$ C.F. is $x_c = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t$

$$x_p = \text{P.I.} = \frac{1}{(D^2+1)(D^2+9)} 4 \cdot e^{-t} - \frac{1}{(D^2+1)(D^2+9)}(2\cos 2t)$$

$$= \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

$\therefore$ General solution for x = C.F. + P.I.

$$\therefore \quad x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

Step 3 : And similarly solving for y we get,

$$(D^2+1)(D^2+9)\,y = -\sin 2t - 4e^{-t}$$

Auxiliary equation for y is same,

$$(D^2+1)(D^2+9) = 0 \quad \therefore D = \pm i, \ \pm 3i$$

$\therefore$ C.F. is $\quad y_C = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t$

P.I. for $y_p = \dfrac{1}{(D^2+1)(D^2+9)}(-\sin 2t - 4e^{-t})$

$$= (-1)\frac{1}{(D^2+1)(D^2+9)}\sin 2t - 4\frac{1}{(D^2+1)(D^2+9)}e^{-t}$$

$$= +\frac{1}{15}\sin 2t - \frac{1}{5}e^{-t}$$

$\therefore$ General solution for y = C.F. + P.I.

$$\therefore \quad y = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t + \frac{1}{15}\sin 2t - \frac{1}{5}e^{-t}$$

Substituting these values of x and y in any one of given equations we get,

$C_5 - 2C_2$, $C_6 = -2C_1$, $C_7 = -2C_4$, $C_8 = 2C_3$

By comparing the coefficients of functions of t.

Step 4 :

$\therefore$ Required solution for the given system is,

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

and

$$y = 2C_2 \cos t - 2C_1 \sin t - 2C_4 \cos 3t + 2C_3 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

Q.4 Solve $\dfrac{du}{dx} + v = \sin x$, $\dfrac{dv}{dx} + u = \cos x$ at $x = 0$, $u = 1$ and $v = 0$.

[SPPU : Dec.-15, Marks 4]

Ans. : Step 1 :

We have $\quad Du + v = \sin x \quad \ldots(1)$

$$u + Dv = \cos x$$

Step 2 : Solving be Cramer's rule, we get

$$\begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} u = \begin{vmatrix} \sin x & 1 \\ \cos x & D \end{vmatrix}$$

$$(D^2-1)u = D\sin x - \cos x = 0$$

$$(D^2-1)u = 0 \Rightarrow D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$$\therefore \quad u = C_1 e^x + C_2 e^{-x} \quad \ldots(2)$$

Step 3 :

We have $\quad Du + v = \sin x$

$$v = \sin x - Du = \sin x - C_1 e^x + C_2 e^{-x} \quad \ldots(3)$$

At $x = 0$, $u = 1$ and $v = 0$

$$\therefore \quad u = C_1 + C_2 = 1$$

$$\therefore \quad u = u = \frac{-C_1 + C_2 = 0}{\Rightarrow \ 2C_2 = 1}$$

$$C_2 = \frac{1}{2}$$

$$\Rightarrow \quad C_1 = \frac{1}{2}$$

$$u = \frac{1}{2}e^x + \frac{1}{2}e^{-x} \text{ and}$$

$$v = \sin x - \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

2.2 : Symmetrical form of Simultaneous Differential Equations

☞ [SPPU : Dec.-2000, 03, 04, 05, 06, 07, 08, 09, 10, May-05, 06, 07, 08, 09, 10, 13]

General form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are functions of x, y, z are said to be symmetrical simultaneous differential equations. The solution of such a system of equations is given by a pair of relations $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ which are independent of each other. We can solve such a system by following methods.

Q.5 $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$ ☞ [SPPU : Dec.-04, May-10]

Ans. : By combinations,

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2 + y^2 = C_1 \qquad \dots (1)$$

Again by combination,

$$\frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$$

$$\frac{dy}{-1} = \frac{dz}{e^{C_1}}$$

$$e^{C_1} \cdot dy = -dz$$

Integrating we get,

$$y e^{C_1} = -z + C_2$$

$$y e^{x^2+y^2} + z = C_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Q.6 $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - zx^4)} = \frac{dz}{z(x^4 - y^4)}$

☞ [SPPU : Dec.-07, May-06, 13]

Ans. :

Use multipliers $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$

$$\therefore \quad \text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{2}{z}dz}{2y^4 - z^4 + z^4 - 2x^4 + 2x^4 - 2y^4}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{2}{z}dz = 0$$

Integrating we get,

$$\log x + \log y + 2\log z = \log C_1$$

$$\log xyz^2 = \log C_1$$

$$xyz^2 = C_1 \qquad \dots (1)$$

Again use multipliers x^3, y^3, z^3

$$\therefore \text{Each ratio} = \frac{x^3 dx + y^3 dy + z^3 dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)}$$

$$= \frac{x^3 dx + y^3 dy + z^3 dz}{0}$$

$$\Rightarrow \quad x^3dx+y^3dy+z^3dz = 0$$

Integrating we get,

$$\frac{x^4}{4}+\frac{y^4}{4}+\frac{z^4}{4} = C$$

$$x^4+y^4+z^4 = C_2 \qquad \dots (2)$$

Equations (1) and (2) constitute the solution of the system.

Q.7 Solve $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ ☞ **[SPPU : May-05, 09]**

Ans. : Consider $\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$

Integrating, we get, $\log y = \log z + \log C_1$

$$y = C_1 z \qquad \dots (1)$$

Let x, y, z be the set of multipliers for given equations then

$$\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}$$

$$= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

Consider

$$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} = \frac{dy}{2xy}$$

$$\frac{2(xdx+ydy+zdz)}{x^2+y^2+z^2} = \frac{dy}{y}$$

Integrating we get,

$$\log(x^2+y^2+z^2) = \log y + \log C_2$$

$$x^2+y^2+z^2-C_2y = 0 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

2.3 : L-C-R Circuit

Consider the electrical circuit given in Fig. 2.1.

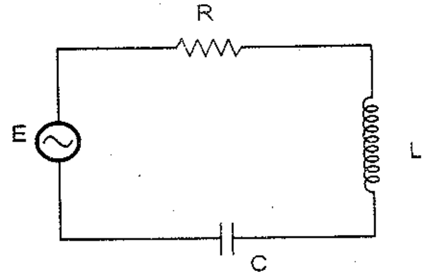

Fig. 2.1

Let

I = Instantaneous current

Q = Instantaneous charge

L = Inductance

C = Capacitor of capacity

R = Resistance

$\therefore$ Voltage drop across R = RI

Voltage drop across L = $L\frac{dI}{dt}$

Voltage drop across C = $\frac{Q}{C}$

We know that $I = \frac{dQ}{dt} \therefore Q = \int I\,dt$

Kirchhoff's law : The algebraic sum of all the voltage drops in an electric circuit is zero. We consider the following cases.

Case 1 : The differential equation of electrical circuit consists of inductance L, capacitance C with emf E is $L\frac{dI}{dt}+\frac{Q}{C} = E$

$$\therefore \quad L\frac{d^2Q}{dt^2}+\frac{Q}{C} = E$$

Case 2 : The differential equation of electrical circuit consists of L - C without emf is,

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C} = 0 \quad \therefore \frac{d^2Q}{dt^2}+\frac{Q}{LC} = 0$$

$$\Rightarrow \quad x^3 dx + y^3 dy + z^3 dz = 0$$

Integrating we get,

$$\frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} = C$$

$$x^4 + y^4 + z^4 = C_2 \quad \dots (2)$$

Equations (1) and (2) constitute the solution of the system.

Q.7 Solve $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ ☞ **[SPPU : May-05, 09]**

Ans. : Consider $\dfrac{dy}{2xy} = \dfrac{dz}{2xz} \Rightarrow \dfrac{dy}{y} = \dfrac{dz}{z}$

Integrating, we get, $\log y = \log z + \log C_1$

$$y = C_1 z \quad \dots (1)$$

Let x, y, z be the set of multipliers for given equations then

$$\frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

Consider

$$\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dy}{2xy}$$

$$\frac{2(xdx + ydy + zdz)}{x^2 + y^2 + z^2} = \frac{dy}{y}$$

Integrating we get,

$$\log(x^2 + y^2 + z^2) = \log y + \log C_2$$

$$x^2 + y^2 + z^2 - C_2 y = 0 \quad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

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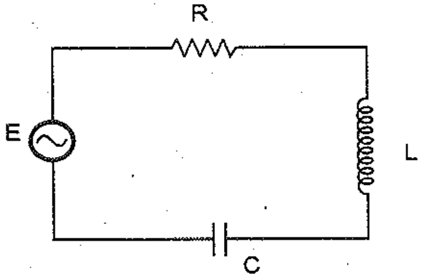
Consider the electrical circuit given in Fig. 2.1.



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Voltage drop across L = $L\dfrac{dI}{dt}$

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Kirchhoff's law : The algebraic sum of all the voltage drops in an electric circuit is zero. We consider the following cases.

Case 1 : The differential equation of electrical circuit consists of inductance L, capacitance C with emf E is $L\dfrac{dI}{dt} + \dfrac{Q}{C} = E$.

$$\therefore \quad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = E$$

Case 2 : The differential equation of electrical circuit consists of L - C without emf is,

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \quad \therefore \quad \frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

Case 3 : The differential equation of electrical circuit consists of inductance L, resistance R and capacitance C with emf E is

$$L\frac{dI}{dt}+RI+\frac{Q}{C}=E$$

or $$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C} = E$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC} = \frac{E}{L}$$

Case 4 : The differential equation of electrical circuit consists of L, R and C without E is

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC}=0$$

Q.8 In an L-C-R circuit, the charge q on the condenser is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \sin \omega t$.

The circuit is tuned to resonate so that $\omega^2 = \frac{1}{LC}$. If initially the current and charge be zero, show that for small values of $\frac{R}{L}$ the current in the circuit at time t is given by $\frac{Et}{2L}$ sin ωt.

☞ **[SPPU : May-16, Marks 4]**

Ans. : Step 1 : The given differential equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E\sin\omega t$$

i.e. $L D^2q + R D q + \frac{q}{C} = E \sin \omega t,$ where $D = \frac{d}{dt}$

i.e. $$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)q = \frac{E}{L}\sin\omega t$$

i.e. $$(D^2 + \omega^2)q = \frac{E}{L}\sin\omega t \quad \dots (1)$$

Step 2 : Note we can neglect $\left(\frac{RD}{L}\right)$ as $\frac{R}{L}$ is small and $\because \frac{1}{LC} = \omega^2$

Step 3 : To find C.F.

A.E. is

$$D^2 + \omega^2 = 0 \text{ i.e. } D^2 = -\omega^2 \quad D = \pm \omega i$$

$$\therefore \quad \text{C.F.} = C_1 \cos\omega t + C_2 \sin\omega t$$

Step 4 : To find P.I.

$$\text{P.I.} = \frac{E}{L}\cdot\frac{1}{(D^2+\omega^2)}\sin\omega t = \frac{-Et}{2L\omega}\cos\omega t$$

Step 5 : $\therefore$ Complete solution is

$$q = C_1\cos\omega t + C_2\sin\omega t - \frac{E}{2L\omega}t\cos\omega t \quad \dots (2)$$

Step 6 :

$$\therefore \quad i = \frac{dq}{dt} = -C_1\omega\sin\omega t + C_2\omega\cos\omega t$$

$$-\frac{E}{2L\omega}(-t\omega\sin\omega t + \cos\omega t) \quad \dots (3)$$

Step 7 : Now initially, at t = 0, q = 0 and i = 0.

$\therefore$ From equations (2) and (3) we get at t = 0

$$0 = C_1 + 0 + 0 \quad \therefore C_1 = 0 \text{ and}$$

$$0 = 0 + C_2\omega - \frac{E}{2L\omega}(0+1)$$

$$\therefore \quad C_2\omega = \frac{E}{2L\omega}$$

$\therefore$ Substituting in equation (3) we get

$$i = 0 + \frac{E}{2L\omega}\cos\omega t - \frac{E}{2L\omega}(-t\omega\sin\omega t + \cos\omega t)$$

$$= \left(\frac{E}{2L\omega}-\frac{E}{2L\omega}\right)\cos\omega t + \frac{Et}{2L}\sin\omega t = \frac{Et}{2L}\sin\omega t$$

Q.9 An uncharged condenser of capacity C is charged by applying an e.m.f of value E $\sin\frac{t}{\sqrt{LC}}$ through the leads of inductance L and negligible resistance. The charge Q on the plate of the condenser satisfies the differential equation $\frac{d^2Q}{dt^2}+\frac{Q}{LC}=\frac{E}{L}\sin\frac{t}{\sqrt{LC}}$. Prove that the charge at any time t is given by $Q=\frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]$

☞ [SPPU : May-15, Marks 4]

Ans. : Step 1 : Let $p^2=\frac{1}{LC}$ ∴ The given differential equation becomes

$$\frac{d^2Q}{dt^2}+p^2Q=\frac{E}{L}\sin pt$$

i.e. $$(D^2+p^2)Q=\frac{E}{L}\sin pt \quad \dots (1)$$

Step 2 : To find C.F. A.E. is $D^2+p^2=0$

i.e. $D^2=-p^2 \quad \therefore D=\pm p$

$$\therefore \quad C.F. = C_1\cos pt+C_2\sin pt$$

Step 3 : To find P.I.

$$P.I. = \frac{E}{L}\frac{1}{(D^2+p^2)}\sin pt=\frac{E}{L}\cdot\frac{(-t)^1}{(2p)^1\,1!}\sin\left(pt+1\frac{\pi}{2}\right)$$

$$= -\frac{Et}{2\,pL}\cos pt$$

Step 4 : The complete solution is

$$\therefore \quad Q = C_1\cos pt+C_2\sin pt-\frac{Et}{2\,pL}\cos pt \quad \dots (2)$$

Step 5 : At t = 0, Q = 0 and i = 0

∴ From equation (2) $0=C_1+0+0 \quad \therefore C_1=0$

$$\therefore \quad Q = C_2\sin pt-\frac{Et}{2pL}\cos pt \quad \dots (3)$$

$$\therefore \quad i=\frac{dQ}{dt}=C_2\,p\cos pt-\frac{E}{2\,pL}(-tp\sin pt+\cos pt)$$

Step 6 : Now, at t = 0, i = 0

$$\therefore \quad 0 = C_2p\,(1)-\frac{E}{2pL}(0+1)$$

$$\therefore \quad C_2p=\frac{E}{2pL} \quad \therefore C_2=\frac{E}{2p^2L}$$

Step 7 : Substituting in Q we get

$$Q=\frac{E}{2p^2L}\sin pt-\frac{Et}{2pL}\cos pt$$

$$=\frac{E}{2\frac{1}{LC}L}\sin pt-\frac{E}{2\frac{1}{\sqrt{LC}}L}t\cos pt \quad \dots \because p=\frac{1}{\sqrt{LC}}$$

$$=\frac{EC}{2}\sin pt-\frac{E}{2}\sqrt{\frac{C}{L}}\,t\cos pt=\frac{EC}{2}\left(\sin pt-\frac{t}{\sqrt{LC}}\cos pt\right)$$

$$=\frac{EC}{2}\left(\sin\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right) \quad \dots \because p=\frac{1}{\sqrt{LC}}$$

Q.10 An uncharged condenser of capacity C is charged by applying emf of value E $\sin\frac{t}{\sqrt{LC}}$ through the leads of inductance L and negligible resistance. Find the charge at any time t.

☞ [SPPU : May-05, 13]

Ans. : The differential equation of L-C circuit is

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=E\sin\frac{t}{\sqrt{LC}}$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{Q}{LC}=\frac{E}{L}\sin\frac{t}{\sqrt{LC}},\ \text{put } \omega^2=\frac{1}{LC}$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\omega^2Q=\frac{E}{L}\sin\omega t$$

$$(D^2+\omega^2)Q=\frac{E}{L}\sin\omega t \quad \dots (1)$$

∴ A.E. is $D^2+\omega^2=0 \Rightarrow D=\pm i\omega$

$\therefore \quad Q_C = C_1 \cos \omega t + C_2 \sin \omega t$

The particular integral is

$$Q_P = \frac{E}{L}\frac{1}{D^2+\omega^2}\sin \omega t = \frac{E}{L}\frac{t}{2D}\sin \omega t = \frac{-E}{L}\frac{t\cos \omega t}{2\omega}$$

$\therefore$ It's general solution is

$$Q = Q_C + Q_P$$

$$= C_1 \cos \omega t + C_2 \sin \omega t - \frac{Et}{2L\omega}\cos \omega t$$

$$Q = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2L}\sqrt{LC}\cos \frac{t}{\sqrt{LC}}$$

$$Q = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos \frac{t}{\sqrt{LC}}$$

Q.11 A electric circuit consists of an inductance L of 0.1 H a resistance R of 20 Ω and a condenser of capacitance C of 100 microfarads. If the differential equation of electric circuit is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ find charge q and current i at any time t given that t = 0, q = 0.05 and i = 0.

☞ [SPPU : May-17, Marks 4]

Ans. : The given D.E. is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$

$$\Rightarrow \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \quad \dots (1)$$

Here $L = 0.1$, $R = 20$, $C = 100\times 10^{-6} = 10^{-4}$

$\therefore$ Equation (1) becomes,

$$\frac{d^2q}{dt^2} + 200\frac{dq}{dt} + 10000\, q = 0 \quad \dots (2)$$

$\therefore$ A.E. is $D^2 + 200\,D + 10000 = 0$

$(D+100)\,(D+100) = 0$

$\Rightarrow \quad D = 100,\ 100$

$\therefore \quad q_C = (C_1 + C_2 t)\, e^{100\,t}$

and $\quad q_P = 0$

$$\therefore \quad q = (C_1 + C_2 t)\, e^{100\,t} \quad \dots (3)$$

Now at $t = 0$, $q = 0.05$, $\frac{dq}{dt} = 0$

$\therefore$ From equation (3) we get,

$$0.05 = C_1 \text{ and } \frac{dq}{dt}$$

$$= (C_1 + C_2 t)\, e^{100\,t}\cdot 100 + e^{100\,t}(C_2)$$

$\therefore \quad D = C_1 \cdot 100 + C_2$

$\Rightarrow \quad C_2 = -100\, C_1 = -100\,(0.05)\ C_2 = -5$

$$q = (0.05 + (-5)t)e^{100t} = (0.05 - 5t)e^{100t}$$

$$\therefore \quad i = \frac{dq}{dt} = 100\,(0.05 - 5\,t)\, e^{100\,t} - 5\, e^{100\,t}$$

$$i = -500t\, e^{100\,t}$$

Q.12 A circuit consists of an inductance L and condenser of capacity C in series. An alternating e.m.f. E sin nt is applied to it at time t = 0, the initial current and charge on the condenser being zero. Find the current flowing in the circuit at any time t for
i) $\omega \neq n$
ii) $\omega = n$ where $\omega^2 = \frac{1}{LC}$.

☞ [SPPU : Dec.-16, Marks 4]

Ans. : By Kirchhoff's law, we have

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = E \sin nt$$

$$\therefore \quad \frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E}{L}\sin nt$$

$$\therefore \quad \frac{d^2Q}{dt^2} + Q\omega^2 = \frac{E}{L}\sin nt \quad \dots (1)$$

$\therefore$ The auxillary equation is $D^2+\omega^2=0 \Rightarrow D=\pm i\omega$

$$\therefore \quad Q_C = C_1\cos\omega t + C_2\sin\omega t$$

The particular integral of equation (1) is

$$Q_P = \frac{1}{D^2+\omega^2}\frac{E}{L}\sin nt = \frac{E}{L}\frac{1}{D^2+\omega^2}\sin nt$$

Case i : If $\omega \neq n$ then

$$Q_P = \frac{E}{L}\frac{1}{\omega^2-n^2}\sin nt$$

$$\therefore \quad Q = Q_C+Q_P = C_1\cos\omega t + C_2\sin\omega t + \frac{E}{L}\frac{1}{\omega^2-n^2}\sin nt$$

Case ii : If $\omega = n$ then

$$Q_P = \frac{E}{L}\frac{t}{2D}\sin nt$$

$$Q_P = -\frac{Et}{2L}\frac{\cos nt}{n}$$

$$\therefore \quad Q = Q_C+Q_P$$

$$= C_1\cos\omega t + C_2\sin\omega t - \frac{Et}{2nL}\cos nt$$

Q.13 A capacitor of 10^{-3} farads is in series with an e.m.f. of 20 V and an inductor of 0.4 H. At t = 0, the charge Q and current I are zero. Find Q at any time t.

Ans. : The differential equation of L-C circuit is

$$L\frac{dI}{dt}+\frac{Q}{C} = E \quad \text{i.e.} \quad L\frac{d^2Q}{dt^2}+\frac{Q}{C}=E$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{1}{LC}Q = \frac{E}{L} \quad \text{i.e.} \quad \frac{d^2Q}{dt^2}+\omega^2Q=\frac{E}{L}$$

$$\left(\because \omega^2=\frac{1}{LC}\right) \quad \ldots (1)$$

$\therefore$ A.E. is $D^2+\omega^2=0 \Rightarrow D=\pm i\omega$

$$\therefore \quad Q_C = C_1\cos\omega t + C_2\sin\omega t$$

And $$Q_P = \frac{1}{D^2+\omega^2}\left(\frac{E}{L}\right)e^{0t} = \frac{1}{\omega^2}\frac{E}{L}=\frac{E}{\omega^2 L}$$

$$\therefore \quad Q = Q_C+Q_P$$

$$= C_1\cos\omega t + C_2\sin\omega t + \frac{E}{\omega^2 L}$$

At $$t=0,\ Q=0 \Rightarrow C_1+\frac{E}{\omega^2 L}=0 \Rightarrow C_1=-\frac{E}{\omega^2 L}$$

We have, $$I = \frac{dQ}{dt} = -\omega C_1\sin\omega t + \omega C_2\cos\omega t$$

At $$t = 0,\ I=0$$

$$\therefore \quad 0 = 0+\omega C_2 \Rightarrow C_2=0$$

$$\therefore \quad Q = -\frac{E}{\omega^2 L}\cos\omega t + \frac{E}{\omega^2 L}$$

$$\therefore \quad Q = \frac{E}{\omega^2 L}[1-\cos\omega t]$$

$$= EC\,[1-\cos\omega t]$$

$$= EC\left(1-\cos\frac{t}{\sqrt{LC}}\right)$$

$$\therefore \quad Q = 20\times 10^{-3}\left(1-\cos\frac{t}{\sqrt{0.4\times 10^{-3}}}\right)$$

Q.14 An inductor of 0.5 henry is connected in series with resistor of 60 ohms. A capacitor of 0.02 farad and generator having alternative voltage given by 24 sin 10 t (t > 0) with a switch k. Forming a differential equation find the current and charge at any time t if charge is zero when switch is closed at t = 0.

[SPPU : May-18, Marks 4]

Ans. : The given D.E. is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = 24\sin 10t$$

$$\therefore \quad \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{CL} = \frac{24\sin 10t}{L} \quad \ldots(1)$$

Here, $L = 0.5, R = 6, C = 0.02$

$\therefore$ Equation (1) becomes,

$$\frac{d^2q}{dt^2}+12\frac{dq}{dt}+100q = 4\sin 10t \quad \ldots(2)$$

A.E. is $D^2+12D+100 = 0$ $\therefore D = -6 \pm 8i$

$$\text{C.F.} = e^{-6t}[A\cos 8t + B\sin 8t] \quad \ldots(3)$$

$$\text{P.I.} = \frac{1}{D^2+12D+100}[48\sin 10t]$$

$$= \frac{1}{-100+12D+100}[48\sin 10t] = \frac{4}{D}\sin 10t$$

$$= -\frac{4}{10}\cos 10t = -\frac{2}{5}\cos 10t$$

$\therefore$ Hence the general solution of (2) is

$$q = e^{-6t}[A\cos 8t + B\sin 8t]-\frac{2}{5}\cos 10t \quad \ldots(4)$$

If we apply initial conditions, we get $Q = 0, I = 0$ at $t = 0$.

$$\Rightarrow \quad A = \frac{2}{5}, B = \frac{3}{10}$$

Hence, $$q = e^{-6t}\left[\frac{2}{5}\cos Bt + \frac{3}{10}\sin 8t\right]-\frac{2}{5}\cos 10t.$$

Q.15 An emf E sin pt is applied at t = 0 to a circuit containing a capacitance C and inductance L. Current I satisfies the equation $L\frac{dI}{dt}+\frac{1}{C}\int I\,dt = E\sin pt$ if $p^2 = \frac{1}{LC}$ and initially the current I and charge Q are zero then show that the current at time t is $\frac{Et}{2L}\sin pt$ where $I = \frac{dQ}{dt}$. [SPPU : Dec.-18, Marks 4]

Ans. : Refer Q.8.

Q.16 A resistance of 50 ohms, an inductor of 2 heneries and a 0.005 farad capacitor are connected in series with e.m.f. of 40 volts and an open switch. Find the instantaneous charge and current after the switch is closed at t = 0, assuming that at that time, charge on capacitor is 4, coloumb. [SPPU : May-19, Marks 4]

Ans. : D.E. of electric circuit is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = 40$$

$$2\frac{d^2q}{dt^2}+50\frac{dq}{dt}+\frac{q}{0.005} = 40$$

$$\frac{d^2q}{dt^2}+25\frac{dq}{dt}+100q = 20$$

$$(D^2+25D+100)q = 20$$

A.E. is $D^2+25D+100 = 0$

$$(D+5)(D+20) = 0$$

$$D = -20, -5$$

C.F. is $$q_C = C_1e^{-20t}+C_2e^{-5t}$$

P.I. is $$q_P = \frac{1}{D^2+25D+100}20$$

$$D \to 0$$

$$= \frac{1}{100}(20) = \frac{1}{5}$$

The complete solution is

$$q = q_C + q_P$$

$$q = C_1e^{-20t}+C_2e^{-5t}+\frac{1}{5}$$

This is charge on capacitor at any time t.

$$i = \frac{dq}{dt} = -20C_1e^{-20t}-5C_2e^{-5t}$$

This is currrent in the circuit at any time t.

END...

UNIT II

3 Laplace Transform (LT)

3.1 : Laplace Transform

I Definition : The Laplace transformation is an operation denoted by 'L' which associates to each function f(t) (t > 0), a unique function $\phi(s)$ called the Laplace Transform of f(t) defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \phi(s)$$

where L is called as Laplace transform operator and s is a parameter, real or complex.

Note : As Laplace transform is a definite integral, therefore all the properties of definite integrals are true for Laplace transforms.

II Condition for Existance of Laplace Transform

A function f(t) is said to have its Laplace transform if

i) f(t) is piece wise continuous.

ii) f(t) is a function of exponential order.

a) Piece wise continuous function : A function f(t) is said to be piece wise continuous in the given interval, if f(t) is continuous in finite number of subintervals which are obtained by subdividing the given interval.

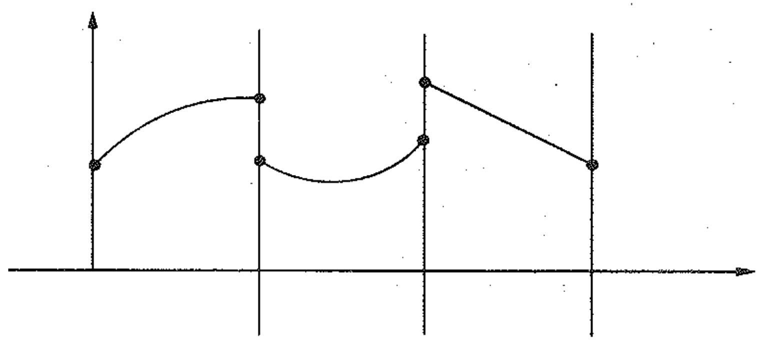

Fig. 3.1

b) Function of exponential order : If there is a constant β with the property that $e^{-\beta t}|f(t)|$ remains bounded as $t \to \infty$ then f(t) is said to be of exponential order.

i.e. f(t) is of exponential order if $e^{-\beta t}|f(t)| < M$ for $t > N$

where M, N, β are all constants.

Note : The above two conditions are sufficient for existance of Laplace transforms but not necessary. Laplace transform exists even if these conditions are not satisfied.

III) Laplace Transform of Some Standard Functions

1) $$f(t) = e^{at}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = 0 - \frac{1}{-(s-a)} \qquad \text{if } s > a$$

$$L[e^{at}] = \frac{1}{s-a} \qquad \text{if } s > a$$

Similarly $L[e^{-at}] = \frac{1}{s+a}$

Note i) $L[1] = L[e^{0t}] = \frac{1}{s+0} = \frac{1}{s}$

ii) $L[a^t] = L[e^{t \log a}] = \frac{1}{s - \log a}$

2) $f(t) = \sinh at$

$$L[f(t)] = L \sinh at$$

$$= L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$L[\sinh at] = \frac{a}{s^2 - a^2}$$

3) $f(t) = \cosh at$

$$L[f(t)] = L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right)$$

$$= \frac{1}{2} L\left(e^{at} + e^{-at}\right) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$L[\cosh at] = \frac{s}{s^2 - a^2}$$

4) $f(t) = \sin at$

$$L[f(t)] = L(\sin at)$$

$$= \int_0^\infty e^{-st} \cdot \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2}(0 - a) = \frac{a}{s^2 + a^2}$$

{formula used $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}$

$(a \sin bx - b \cos bx)$ $e^{-\infty} = 0$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$}

5) $f(t) = \cos at$

$$L[f(t)] = L(\cos at)$$

$$= \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left\{\frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at)\right\}_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2}(-s + 0)$$

$$= \frac{s}{s^2 + a^2}$$

{formula used

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\}$$

6) $f(t) = t^n$

{Recall formulae $\overline{|n} = \int_0^\infty e^{-u} u^{n-1} du$

$$\overline{|n+1} = n\overline{|n}$$

$$\overline{|n+1} = n! \text{ for positive integers}\}$$

$\therefore$ $L[f(t)] = \int_0^\infty e^{-st} t^n \, dt$

Put st = u

s dt = du

t	0	∞
u	0	∞

$$L(t^n) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} \; u^n \; du$$

$$= \frac{\overline{|n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}} \quad \text{(if n positive integer)}$$

Substituting n = 0, 1, 2 in above result we get

7) $L(1) = \frac{1}{s}$

8) $L(t) = \frac{1}{s^2}$

9) $L(t^2) = \frac{2}{s^3}$ and so on.

IV) Properties of Laplace Transforms

1.	First shifting theorem	$L\left[e^{-at} f(t)\right] = \phi(s+a)$
2.	Second shifting theorem	If $g(t) = f(t-a)$, $t > a$, $= 0$, $t < a$ then $L\left[g(t)\right] = e^{-as} \phi(s)$
3.	Change of scale theorem	$L\left[f(at)\right] = \frac{1}{a}\, \phi\left(\frac{s}{a}\right)$

4.	Transforms of Derivatives	$L\left[f'(t)\right] = s\,\phi(s) - f(0)$, $L\left[f''(t)\right] = s^2\,\phi(s) - s\, f(0) - f'(0)$, $L\left[f'''(t)\right] = s^3\,\phi(s) - s^2 f(0) - s\, f'(0) - f''(0)$ Thus, derivates → Multiplication by powers of s
5.	Transform of Integrals	$L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s}\, \phi(s)$ $L\left[\int_0^t \int_0^t f(t)\, dt \cdot dt\right] = \frac{1}{s^2}\, \phi(s)$ Thus, integration → division
6.	Multiplication by t^n	$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\, \phi(s)$ Thus, multiplication → Derivatives
7.	Division by t	$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$ Thus, division → integration
8.	Convolution Theorem	$L\left[f(t) * g(t)\right] = L\left[\int_0^t f(u)\, g(t-u)\, du\right]$ $= F(s) \cdot G(s)$
9.	Initial Value Theorem	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,\phi(s)$
10.	Final Value Theorem	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\phi(s)$

Table 3.1 : Theorems of Laplace Transforms

Q.1 Find Laplace transform of f(t) using definition

$$f(t) = \begin{cases} t/T & 0 \le t \le T \\ 1 & t > T \end{cases}$$

[SPPU : Dec.-03]

Ans. : Step 1 : Given function f(t).

$$f(t) = \begin{cases} t/T & 0 \le t \le T \\ 1 & t > T \end{cases}$$

Step 2 : Use the formula of Laplace Transform.

$$L\,[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Step 3 : Substitute the value of f(t).

$$\therefore \qquad L\,[f(t)] = \int_0^T e^{-st}\left(\frac{t}{T}\right) dt + \int_T^\infty e^{-st}\ 1\ dt$$

Step 4 : Integrate w.r.t t using integration by parts.

$$= \frac{1}{T}\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^T + \left[\frac{e^{-st}}{-s}\right]_T^\infty$$

Step 5 : Substitute the limits of t.

$$= -\frac{e^{-sT}}{s} - \frac{\left(e^{-sT} - 1\right)}{Ts^2} + \frac{e^{-sT}}{s}$$

Step 6 : Simplify.

$$= \frac{1 - e^{-sT}}{Ts^2}$$

Q.2 Find $L\left(e^t 4^t\right)$

Ans. : Step 1 :

$$L\,[4^t] = L\left[e^{t \log 4}\right] = L\left[e^{(\log 4)\,t}\right]$$

$$\therefore \qquad L\left[e^{(\log 4)\,t}\right] = \frac{1}{s - \log 4} \qquad \text{where } s > \log 4$$

Step 2 : Now, using first shifting property.

If, $L\,[f(t)] = \phi\,(s)$ then, $L\ e^{at}\ [f(t)] = \phi\,(s-a)$

Step 3 : $\therefore \quad L\ e^t\ f(t) = \phi\,(s-1)$

$$\therefore \qquad L\left(e^t\, 4^t\right) = \frac{1}{(s-1) - \log 4}$$

Q.3 $L\,[e^{-t}\cos(2t + 3)]$

Ans. : Step 1 : Consider

$$L[\cos(2t + 3)] = L\,(\cos 2t \cos 3 - \sin 2t \sin 3)$$

$$= \cos 3\ L\,(\cos 2t) - \sin 3\ L\,[\sin 2t]$$

$$= \cos 3 \frac{s}{s^2 + 4} - \sin 3 \frac{2}{s^2 + 4}$$

Q.4 $L\ [e^{-t}\cos^3 t]$

Ans. : Step 1 : Consider

$$L\,[\cos^3 t] = L\left[\frac{1}{4}\cos 3t + \frac{3}{4}\cos t\right]$$

$$= \frac{1}{4} L\,[\cos 3t] + \frac{3}{4} L\,[\cos t]$$

$$= \frac{1}{4}\left[\frac{s}{s^2 + 9}\right] + \frac{3}{4}\left[\frac{s}{s^2 + 1}\right]$$

Step 2 : Now, using first shifting property.

If, $L\,[f(t)] = \phi\,(s)$ then, $L\ [e^{at}\ f(t)] = \phi\,(s-a)$

Step 3 :

$$\therefore \qquad L\,[e^{-t}\ f(t)] = \phi\,(s+1)$$

$L\ e^{-t}\ \cos^3 t$ is given by,

$$L\,[e^{-t}\ \cos^3 t] = \frac{1}{4}\frac{(s+1)}{(s+1)^2 + 9} + \frac{3}{4}\frac{(s+1)}{(s+1)^2 + 1}$$

$$= \frac{1}{4}\left[\frac{(s+1)}{(s+1)^2 + 9} + \frac{3\,(s+1)}{(s+1)^2 + 1}\right]$$

Q.5 Find $L\ [g(t)]$

$$g\,(t) = \begin{cases} 0 & t < 3 \\ e^{2\,(t-3)} \sin\ 3\,(t-3) & t \ge 3 \end{cases}$$

Ans. : Step 1 :

$$f(t) = \begin{cases} 0 & t < 3 \\ e^{2(t-3)} \sin 3(t-3) & t \geq 3 \end{cases}$$

Step 2 : We know that, If L [f(t)] = $\phi(s)$ and

$$g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

Then, $L[g(t)] = e^{-as} \phi(s)$

Step 3 : Here $f(t-a) = e^{2(t-3)} \sin 3(t-3)$ where a = 3.

$$f(t) = e^{2t} \sin 3t$$

$$\phi(s) = L\, e^{2t} \sin 3t$$

$$= \frac{3}{(s-2)^2+9} \quad \text{(by 1}^{\text{st}} \text{ shifting property)}$$

Step 4 : $L[g(t)] = e^{-3s} \cdot \phi(s) = e^{-3s} \left[\frac{3}{(s-2)^2+9}\right]$

Q.6 Evaluate $L\left(t \sin^3 t\right)$ [SPPU : May-99, 03, Dec.-2000, 04]

Ans. : Step 1 : Use $\sin 3\theta = 3 \sin\theta - 4 \sin^3 \theta$

$$\therefore \quad L[\sin^3 t] = \left[\frac{3}{4} \sin t - \frac{1}{4} \sin 3t\right] = \frac{3}{4} \frac{1}{s^2+1} - \frac{1}{4} \frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+9)}\right]$$

Step 2 : We know that,

If, L [f(t)] = $\phi(s)$ Then, L t [f(t)] = $-\phi'(s)$

Step 3 :

$$\therefore \quad L[t \sin^3 t] = (-1) \frac{3}{4}\left[\frac{d}{ds}\left(\frac{1}{(s^2+1)} - \frac{1}{(s^2+9)}\right)\right]$$

Step 4 : Find the derivative.

$$= (-1)\frac{3}{4}\left[\frac{-1\,(2s)}{(s^2+1)^2} - \frac{(-1)\cdot 2s}{(s^2+9)^2}\right] = \frac{3}{4}\left[\frac{2s}{(s^2+1)^2} - \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

Q.7 Find the Laplace transform of $\frac{\sin at}{t}$ and hence show that $\int_s^\infty \frac{\sin t}{t} = \frac{\pi}{2}$

Ans. : Step 1 : We have

$$L[\sin at] = \frac{a}{s^2+a^2}$$

Step 2 : We know that if, L [f(t)] = $\phi(s)$

$$\text{Then,} \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\, ds$$

Step 3 : Thus we get

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

Step 4 : Integrate w.r.t. s.

$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Step 5 : When a = 1, we have

$$L\left[\frac{\sin t}{t}\right] = \cot^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \cot^{-1} s \qquad \text{[(By definition)]}$$

by substituting s = 0 we get

$$\int_0^\infty \frac{\sin t}{t}\, dt = \cot^{-1}(0) = \frac{\pi}{2}$$

Q.8 Prove that $L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \log\left(\frac{s+b}{s+a}\right)$ ☞ [SPPU : Dec.-16]

Ans. : Step 1 :

$$L\left(e^{-at} - e^{-bt}\right) = L\, e^{-at} - L\, e^{-bt}$$
$$= \frac{1}{s+a} - \frac{1}{s+b}$$

Step 2 : We know that if, $L\, f(t) = \phi(s)$ Then, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\, ds$

Step 3 : Substitute the value of $\phi(s)$.

$$L\left[\frac{\left(e^{-at} - e^{-bt}\right)}{t}\right] = \int_s^\infty \frac{1}{s+a} - \frac{1}{(s+b)} \cdot ds$$

Step 4 : Integrate w.r.t. s.

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]_s^\infty$$

Step 5 : Substitute the limits of s and simplify.

$$= \log 1 - \log\frac{s+a}{s+b} = 0 - \log\frac{s+a}{s+b}$$
$$= \log\frac{(s+b)}{(s+a)}$$

Q.9 $L\left[\frac{\cos at - \cos bt}{t}\right]$ ☞ [SPPU : Dec.-99, May-01, 05, Dec.-03]

Ans. : Step 1 :

$$L\,[(\cos at - \cos bt)] = L \cos at - L \cos bt$$
$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Step 2 : If, $L\,[f(t)] = \phi(s)$ Then, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\, ds$

Step 3 : Substitute the value of $\phi(s)$.

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

Step 4 : Integrate w.r.t. s.

$$= \int_s^\infty \frac{s}{s^2+a^2} \cdot ds - \int_s^\infty \frac{s}{s^2+b^2} \cdot ds$$
$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2} \cdot ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+b^2} \cdot ds$$
$$= \frac{1}{2}\left[\log\left(s^2+a^2\right)\right]_s^\infty - \frac{1}{2}\left[\log\left(s^2+b^2\right)\right]_s^\infty$$

Step 5 : Substitute limits of s

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$$
$$= \frac{1}{2}\left\{\log 1 - \log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty\right\} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$
$$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

Q.10 Find L of $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$ ☞ [SPPU : May-2000, 10, Dec.-2000]

Ans. : Step 1 :

$$f(t) = \frac{\sin t}{t}$$
$$\therefore \quad f(0) = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

Step 2 : $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty = \left[\frac{\pi}{2}-\tan^{-1}s\right]$$

$$\phi(s) = \cot^{-1} s$$

Step 3 : $L[f'(t)] = -f(0)+s\,\phi(s)$

Step 4 : Using above property simplify.

$$L[f'(t)] = -f(0)+s\,\phi(s) = s\cot^{-1}s-1$$

Q.11 Solve $L\left(\int_0^t \frac{\sin t}{t}\,dt\right)$ **[SPPU : May-17, Marks 4]**

Ans. :

Step 1 : Consider $[f(t)] = \left(\int_0^t \frac{\sin t}{t}\,dt\right)$

Step 2 : If, $L[f(t)] = \phi(s)$ Then $L\int_0^t f(t)dt = \frac{1}{s}\phi(s)$

Step 3 :

$$\therefore \quad L\int_0^t \frac{\sin t}{t} = \frac{1}{s}\,L\,\frac{\sin t}{t} \qquad \ldots (1)$$

Step 4 : Let $f(t) = \dfrac{\sin t}{t}$

$$\therefore \quad L[\sin t] = \frac{1}{s^2+1}$$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^\infty$$

$$L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2}-\tan^{-1} s$$

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$$

Step 5 : Substituting in (1)

$$L\int_0^t \frac{\sin t}{t} = \frac{1}{s}\cot^{-1} s$$

Q.12 Find $L\left(e^{-4t}\int_0^t t\sin 3t\;dt\right)$

Ans. :

Step 1 : Let $f(t) = \int_0^t t\sin 3t\cdot dt$

Step 2 : If, $L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\phi(s)$

Step 3 : $\therefore$ $L[f(t)] = \frac{1}{s}\,L\,[t\sin 3t]$

$$= \frac{-1}{s}\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{-3}{s}\,\frac{-1}{\left(s^2+9\right)^2}\,2s$$

$$= \frac{6}{\left(s^2+9\right)^2}$$

Step 4 : Use 1^{st} shifting property

$$L\,e^{-4t}\int_0^t t\sin 3t = \frac{6}{\left[(s+4)^2+9\right]^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$= \frac{6}{\left(s^2+8s+25\right)^2}$$

Q.13 Evaluate $L\left[t\int_0^t e^{-4t}\sin 3t\;dt\right]$. **[SPPU : May- 18, Marks 4]**

Ans. : Step 1 :

Let $f(t) = \int_0^t e^{-4t}\sin 3t\,dt$

Step 2 :

$$\text{If} \quad L[f(t)] = \phi(s) \text{ then } L\int_0^t f(t)\,dt = \frac{1}{s}\phi(s)$$

Step 3 :

$$L[f(t)] = \frac{1}{s}L[e^{-4t}\sin 3t]$$

$$= \frac{1}{s}\frac{3}{(s+4)^2+3^2} = \frac{3}{s(s^2+8s+25)}$$

Step 4 :

$$\text{Now} \quad L[t\,g(t)] = (-1)\frac{d}{ds}L[g(t)]$$

Step 5 :

$$\therefore \quad L\left[t\int_0^t e^{-4t}\sin 3t\,dt\right] = (-1)\frac{d}{ds}\left[\frac{3}{(s^3+8s^2+25s)}\right]$$

$$= (-1)\left\{\frac{3(3s^2+165+25)}{(s^3+8s^2+25s)^2}\right\}$$

$$= \frac{-3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}.$$

Q.14 Find the Laplace transform of $te^{3t}\cos 2t$.

☞ **[SPPU : May- 16, Marks 4]**

Ans. : Step 1 :

We have $\quad L[\cos 2t] = \dfrac{s}{s^2+4} = \phi(s)$

Step 2 :

$$L[e^{3t}f(t)] = \phi(s-a)$$

Step 3 : $\quad L[e^{3t}\cos 2t] = \dfrac{s-3}{(s-3)^2+4} = \dfrac{s-3}{s^2-6s+13}$

Step 4 : $\quad L[t\,g(t)] = (-1)\dfrac{d}{ds}L[g(t)]$

$$\therefore \quad L[te^{3t}\cos 2t] = (-1)\frac{d}{ds}\frac{s-3}{(s^2-6s+13)}$$

$$= (-1)\left\{\frac{(s^2-6s+13)-(s-3)(2s-6)}{(s^2-6s+13)^2}\right\}$$

$$= (-1)\left\{\frac{s^2-6s+13-2s^2+12s-18)}{(s^2-6s+13)^2}\right\}$$

$$= \frac{s^2-6s+5}{(s^2-6s+13)^2}.$$

Q.15 Find Laplace transform of $L[t\,e^{-4t}\sin 3t]$.

☞ **[SPPU : May-19, Marks 4]**

Ans. :

Step 1 : We have $L\{\sin 3t\} = \dfrac{3}{s^2+9} = \phi(s)$

Step 2 : $\quad L\{e^{-4t}\sin 3t\} = \phi(s+4)$

$$= \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

Stpe 3 : $\quad L\{t\,g(t)\} = -\dfrac{d}{ds}L\{g(t)\}$

$$L\{te^{-4t}\sin 3t\} = -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right]$$

$$= -3\left[\frac{-2s-8}{(s^2+8s+25)^2}\right]$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

Q.16 Evaluate the integral using Laplace Transform $\int_0^\infty te^{-t}\sin t\,dt$.

☞ **[SPPU : May- 17, 19, Marks 4]**

Ans. : $\quad L[\sin t] = \dfrac{1}{s^2+1}$

$$L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$L[te^{-t}\sin t] = (-1)\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$= \frac{2s+2}{(s^2+2s+2)^2}$$

$$L\int_0^\infty te^{-t}\sin t\,dt = \frac{1}{s}\,\frac{25+2}{(s^2+2s+2)^2}.$$

Q.17 Solve Evaluate : $\left[\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt\right]$ ☞ **[SPPU : Dec.-18, Marks 2]**

Ans. : Let $f(t) = \cos 6t - \cos 4t$

$$L[f(t)] = \frac{s}{s^2+36} - \frac{s}{s^2+16}$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+36} - \frac{s}{s^2+16}\right] ds$$

$$= \frac{1}{2}[\log(s^2+36) - \log(s^2+16)]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1+\frac{36}{s^2}}{1+\frac{16}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+36}{s^2+16}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

By the definition of Laplace transform

$$\int_0^\infty e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

Put $s = 0$

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\frac{16}{36} = \log\frac{4}{6} = \log\frac{2}{3}$$

Q.18 Find the Laplace transform of $\frac{1-\cos t}{t}$ ☞ **[SPPU : May-15, Marks 4]**

Ans. :

We have $$L[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1} = \phi(s)$$

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \phi(s)ds = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+1}\right]_s^\infty = \left[\log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \log\left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty = \log[1] - \log\left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log\frac{s}{\sqrt{s^2+1}} = \log\frac{\sqrt{s^2+1}}{s}$$

Q.19 Find L.T. of $f(t) = \begin{cases} t^2 & 0 < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$ ☞ **[SPPU : May-06]**

Ans. : Step 1 : Let $f(t) = \begin{cases} t^2, & 0 < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$

Step 2 : Use definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

Step 3 : We get

$$L[f(t)] = \int_0^{2\pi} t^2 e^{-st} dt + \int_{2\pi}^\infty (\sin t)\, e^{-st}\,dt$$

$$= \left[t^2\left(\frac{e^{-st}}{-s}\right) - (2t)\left(\frac{e^{-st}}{s^2}\right) + (2)\left(\frac{e^{-st}}{s^3}\right) + 0\right]_0^{2\pi}$$

$$+ \left\{\frac{e^{-st}}{s^2+1}[-s \sin t - \cos t\right\}_{2\pi}^{\infty}$$

$$= -\frac{4\pi^2}{s} e^{-2\pi s} - \frac{4\pi e^{-s2\pi}}{s^2} + \frac{2e^{-2\pi s}}{s^3} - \frac{2}{s^3} + \frac{e^{-2\pi s}}{s^2+1}$$

$$L[f(t)] = -\frac{2}{s^3} + e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{4\pi}{s^2} - \frac{4\pi^2}{s} + \frac{2}{s^3}\right]$$

3.2 : Inverse Laplace Transform

I) i.e. If L [f(t)] = ϕ (s) then L^{-1} [ϕ (s)] = f(t).

II) Properties and theorems on inverse Laplace transforms

a) Linearity property :

$$L^{-1}[C_1\phi_1(s) + C_2\phi_2(s)] = C_1\, L^{-1}[\phi_1(s)] + C_2\, L^{-1}[\phi_2(s)]$$

Note : The property of Laplace transform is linear, therefore inverse Laplace is also linear i.e. inverse Laplace transforms are distributive over addition. Also this result can be extended for more functions also.

b) First shifting theorem :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}[\phi(s-a)] = e^{at}\, f(t)$

Proof : We know first shifting theorem of Laplace transforms

i.e. $L[e^{at}\, f(t)] = \phi(s-a)$

Thus $L^{-1}[\phi(s-a)] = e^{at}\, f(t)$

$= e^{at}\, L^{-1}[\phi(s)]$

Note : The replacement of s by s – a in ϕ (s) corresponds to multiplication of original function f(t) by e^{at}.

c) Second shifting theorem :

If $L^{-1}[\phi(s)] = f(t)$ then

$$L^{-1}[e^{-as}\phi(s)] = \begin{cases} f(t-a) & t \geq a \\ 0 & t < a \end{cases}$$

d) Change of scale property :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}[\phi(Ks)] = \frac{1}{K} f\left(\frac{t}{K}\right)$

i.e. $L^{-1}\left[\phi\left(\frac{s}{a}\right)\right] = a\, f(at)$

e) Inverse transform of derivatives :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}[\phi'(s)] = -t\, f(t)$

i.e. $L^{-1}[\phi'(s)] = -t\, L^{-1}\phi(s)$ ... (3.1)

i.e. $\mathbf{L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}\phi'(s)}$

From (3.1) $L^{-1}[\phi''(s)] = (-t)^2\, L^{-1}\phi(s)$

and so on

$$L^{-1}[\phi^n(s)] = (-t)^n\, L^{-1}\phi(s)$$

f) Inverse Laplace transforms of integrals :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}\left[\int_s^{\infty}\phi(s)\, ds\right] = \frac{f(t)}{t}$ and so on.

g) Effect of multiplication by s :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}[s\phi(s)] = f'(t)$ if $f(0) = 0$

Proof : We know that $L[f'(t)] = -f(0) + s\phi(s)$

If $f(0) = 0$ then $L[f'(t)] = s\phi(s)$

i.e. $L^{-1}[s\phi(s)] = f'(t)$

h) Effect of division by s :

If $L^{-1}[\phi(s)] = f(t)$ then $L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t f(t)$

Proof : We know $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\phi(s)$ hence the proof.

i) Convolution theorem : If the function $\phi(s)$ can be expressed as a product of two functions $\phi_1(s)$ and $\phi_2(s)$ whose inverses $f_1(t)$ and $f_2(t)$ respectively are known then the inverse of the product $\phi(s) = \phi_1(s) \cdot \phi_2(s)$ can be calculated by using convolution theorem.

III) Table of Inverse Laplace Transforms

Sr. No.	$\phi(s)$	$L^{-1}[\phi(s)] = f(t)$
1.	$\frac{1}{s-a}$	e^{at}
2.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\lfloor n+1} = \frac{t^n}{n!}$ (n = integer)
3.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
4.	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
5.	$\frac{s}{s^2+a^2}$	$\cos at$
6.	$\frac{s}{s^2-a^2}$	$\cosh at$
7.	$\frac{1}{s}$	1
8.	1	$\delta(t)$
9.	$\phi(s-a)$	$e^{at} f(t) = e^{at} L^{-1}\phi(s)$
10.	$\phi'(s)$	$-t f(t)$
11.	$\int_s^\infty \phi(s)\,ds$	$\frac{f(t)}{t}$
12.	$\frac{1}{s}\phi(s)$	$\int_0^t f(t)\,dt$
13.	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$
14.	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at + at\cos at)$
15.	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at - at\cos at)$
16.	$e^{-as}\phi(s)$	$f(t-a)\,U(t-a)$
17.	$f(a)\,e^{-as}$	$f(t)\delta(t-a)$

Find Laplace inverse of following

Q.20 $L^{-1}\left[\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right]$

Ans. : Step 1 : We have to find

$$L^{-1}\left[3\left(\frac{s}{s^2+4}\right) - 8\left(\frac{1}{s^2+4}\right) - 4\left(\frac{s}{s^2-16}\right) + 24\left(\frac{1}{s^2-16}\right)\right]$$

Step 2 : Simplify

$$= 3L^{-1}\left(\frac{s}{s^2+4}\right) - 8L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$-4L^{-1}\left(\frac{s}{s^2-16}\right) + 24L^{-1}\left(\frac{1}{s^2-16}\right)$$

Step 3 : Use the formula $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$, $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

Step 4 : Simplify

$$f(t) = 3\cos 2t - \frac{8}{2}\sin 2t - 4\cosh 4t + \frac{24}{4}\sinh 4t$$

$$= 3\cos 2t - 4\sin 2t - 4\cosh 4t + 6\sinh 4t$$

Q.21 Find $L^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right]$ **[SPPU : May- 18, Marks 4]**

Ans. :

We have $\quad L^{-1}\int_s^\infty \phi(s)ds = \frac{f(t)}{t}$

$$\therefore \quad L^{-1}\int_s^\infty \frac{2s+1}{(s^2+s+1)^2} = \frac{f(t)}{t}$$

$$L^{-1}\left[\frac{-1}{s^2+s+1}\right]_s^\infty = \frac{f(t)}{t}$$

$$L^{-1}\left[0+\frac{1}{s^2+s+1}\right] = \frac{f(t)}{t}$$

$$L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right] = \frac{f(t)}{t}$$

$$e^{-\frac{t}{2}}L^{-1}\left[\frac{1}{s^2+\frac{3}{4}}\right] = \frac{f(t)}{t}$$

$$e^{-\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{f(t)}{t}$$

$$\therefore \quad f(t) = \frac{2}{\sqrt{3}}\,t\,e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Q.22 Find $L^{-1}\left[\frac{s+7}{(s^2+2s+2)}\right]$ **[SPPU : Dec.- 16]**

Ans. :

Let
$$f(t) = L^{-1}\left[\frac{s+7}{(s+1)^2+1}\right]$$

$$= L^{-1}\left[\frac{(s+1)+6}{(s+1)^2+1}\right]$$

$$= e^{-t}L^{-1}\left[\frac{s+6}{s^2+1}\right]$$

$$= e^{-t}L^{-1}\left[\frac{s}{s^2+1}+\frac{6}{s^2+1}\right]$$

$$f(t) = e^{-t}[\cos t + 6\sin t]$$

Q.23 Find $L^{-1}\left[\frac{s+1}{(s+1)^4}\right]$ **[SPPU : May- 17]**

Ans. :

Let
$$f(t) = L^{-1}\left[\frac{s+1}{(s+1)^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{s}{s^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{-t}\frac{t^2}{21} = \frac{1}{2}t^2e^{-t}.$$

Q.24 Find $L^{-1}\left[\frac{s^3}{s^4-a^4}\right]$

Ans :

Step 1 : Let $f(t) = L^{-1}\frac{s^3}{s^4-a^4}$

$$= L^{-1}\frac{s\cdot s^2}{(s^2-a^2)(s^2+a^2)}$$

Step 2 : Adjusting the terms we get

$$f(t) = L^{-1}\frac{s}{2}\left[\frac{1}{s^2-a^2}+\frac{1}{s^2+a^2}\right]$$

$$= L^{-1}\frac{1}{2}\left[\frac{s}{s^2-a^2}+\frac{s}{s^2+a^2}\right]$$

Step 4 : Use the formula $L^{-1}\frac{s}{s^2-a^2} = \cosh at$ $L^{-1}\frac{s}{s^2+a^2} = \cos at$

$$\therefore \quad f(t) = \frac{1}{2}(\cosh at + \cos at)$$

Q.25 Solve $L^{-1}\left[\frac{2s+1}{(s+4)(5-6)}\right]$.

☞ [SPPU : May- 16]

Ans. :

Let
$$f(t) = L^{-1}\left[\frac{2s+1}{(s+4)(s-6)}\right]$$

$$= L^{-1}\left[\frac{\frac{7}{10}}{s+4}+\frac{\frac{13}{10}}{s-6}\right]$$

$$= \frac{7}{10}e^{-4t}L^{-1}\left[\frac{1}{s}\right]+\frac{13}{10}e^{6t}L^{-1}\left[\frac{1}{s}\right]$$

$$= \frac{7}{10}e^{-4t}(1)+\frac{13}{10}e^{6t}(1)$$

$$f(t) = \frac{1}{10}[7e^{-4t}+13e^{6t}].$$

Q.26 Find inverse Laplace transform of $\left[\frac{1}{s^2(s+1)}\right]$.

☞ [SPPU : May-19, Marks 4]

Ans. :

Step 1 : Let $f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

Step 2 : Let $g(t) = L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t}{1!} = t$

Step 3 : By convolution theorem

$$L^{-1}\{F(s)G(s)\} = f(t)\cdot g(t)$$

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+1}\right\} = e^{-t}t$$

Q.27 Find inverse Laplace transform of $\frac{3s+1}{(s-1)(s^2+1)}$

☞ [SPPU : Dec.-14, 15]

Ans. : Step 1 : We can write as

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1}+\frac{Bs+C}{s^2+1} = \phi(s)$$

Step 2 : Put s = 1, we get $A = \frac{3+1}{1+1} = 2$

$$\therefore \quad \phi(s) = \frac{2}{s-1}+\frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 3s+1 = 2(s^2+1)+(s-1)(Bs+C) \quad \ldots (1)$$

Put $s = 0 \Rightarrow 1 = 2 - C \Rightarrow C = 1$

Put $s = -1$ in (1), We get

$$-3+1 = 2(1+1)+(-2)(-B+C)$$

$$-2 = 4+2B-2$$

$$\Rightarrow 2B = -4 \Rightarrow B = -2$$

$$\therefore \quad \phi(s) = \frac{2}{s-1}+\frac{-2s+1}{s^2+1}$$

Step 3 : $$\phi(s) = \frac{2}{s-1}+(-2)\frac{s}{s^2+1}+\frac{1}{s^2+1}$$

Step 4 : $\therefore$ $$L^{-1}(\phi(s)) = L^{-1}\left(\frac{2}{s-1}\right)-2L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}[\phi(s)] = 2e^t-2\cos t+\sin t$$

Q.28 Find $L^{-1}\left[\frac{s}{(s-1)(s-2)(s-3)}\right]$

☞[SPPU : May-16]

Ans. : Step 1 : We have

$$\frac{s}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

To find A, put $s = 1 \Rightarrow A = \frac{1}{(-1)(-2)} = \frac{1}{2}$

To find B, put $s = 2 \Rightarrow B = \frac{2}{(1)(-1)} = -2$

To find C, put $s = 3 \Rightarrow C = \frac{3}{2\cdot 1} = \frac{3}{2}$

Step 2 : $\therefore$ $$\phi(s) = \frac{\frac{1}{2}}{s-1}+\frac{(-2)}{s-2}+\frac{\frac{3}{2}}{s-3}$$

Step 3 : $\therefore$ $$L^{-1}[\phi(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]-2\,L^{-1}\left[\frac{1}{s-2}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$f(t) = \frac{1}{2}e^{t}-2\,e^{2t}+\frac{3}{2}e^{3t}$$

Q.29 Find $L^{-1}\frac{1}{s^3(s^2+1)}$

☞ **[SPPU : May-15]**

Ans. :

Step 1 : Let $$f(t) = L^{-1}\frac{1}{s^3(s^2+1)}$$

Step 2 : Let $$\phi(s) = \frac{1}{s^2\,(s^2+1)} = \left[\frac{1}{s^2}-\frac{1}{s^2+1}\right]$$

Step 3 : Use the formula given by

$$L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$$

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n\,!}\quad (n = \text{integer})$$

$\therefore$ $$L^{-1}\phi(s) = t-\sin t$$

Step 4 : Now using $\frac{1}{s}\phi(s) = \int_0^t f(t)\,dt$ we get

Step 5 :

$$L^{-1}\frac{1}{s}\phi(s) = \int_0^t (t-\sin t)\,dt$$

$\therefore$ $$f(t) = \left[\frac{t^2}{2}\right]_0^t+[\cos t]_0^t = \frac{t^2}{2}+\cos t-1$$

Q.30 Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$ **by convolution theorem.**

☞ **[SPPU : Dec.- 17]**

Ans. :

Let $$f(t) = L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}+\frac{1}{s^2+a^2}\right]$$

$$f(t) = L^{-1}[\phi_1(s)\cdot\phi_2(s)] \qquad \ldots(1)$$

Let $$f_1(t) = \frac{t}{1},\ f_1(t-u) = t-u$$

$$f_2(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$f_2(u) = \frac{1}{a}\sin au$$

By convolution theorem,

$$L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(t-u)\,f_2(u)\,du$$

$$f(t) = \int_0^t (t-u)\frac{\sin au}{a}\,du$$

$$= \frac{1}{a}\left[\int_0^t t\sin au\,du-\int_0^t u\,\sin au\,du\right]$$

$$= \frac{1}{a}\left\{t\left(\frac{-\cos au}{a}\right)-\left[u\left(\frac{-\cos au}{a}\right)-(1)\left(\frac{-\sin au}{a^2}\right)\right]\right\}_0^t$$

$$= \frac{1}{a}\left[\left(\frac{-t\cos at}{a} + \frac{t\cos at}{a} - \frac{\sin at}{a^2}\right) - \left(\frac{-t}{a} - 0\right)\right]$$

$$f(t) = \frac{t}{a^2} - \frac{\sin at}{a^3}$$

Q.31 $L^{-1}\left[\log \frac{s^2 + a^2}{s^2 + b^2}\right]$

☞ [SPPU : May-15]

Ans. : Step 1 : Let $f(t) = L^{-1} \log \frac{s^2 + a^2}{s^2 + b^2}$

Step 2 :

$$\phi(s) = \log\left(s^2 + a^2\right) - \log\left(s^2 + b^2\right)$$

$$\phi'(s) = \frac{2s}{\left(s^2 + a^2\right)} - \frac{2s}{\left(s^2 + b^2\right)}$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$L^{-1}\phi'(s) = 2(\cos at - \cos bt)$$

Step 4 : Using the formula $L^{-1}\phi(s) = \frac{-1}{t} L^{-1}\phi'(s)$

$$\therefore \quad L^{-1}\phi(s) = \frac{2(\cos bt - \cos at)}{t}$$

Q.32 $L^{-1}\left[\frac{1}{(s+1)\left(s^2+1\right)}\right]$

☞ [SPPU : May-10, Dec.-15]

Ans. : Step 1 : Let $f(t) = L^{-1}\left[\frac{1}{(s+1)\left(s^2+1\right)}\right]$

$$= L^{-1}\frac{1}{s+1}\cdot\frac{1}{s^2+1} = L^{-1}\phi_1(s)\cdot\phi_2(s)$$

$$\therefore \quad f_1(t) = e^{-t}$$

$$f_2(t) = \sin t$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\phi_1(s)\phi_2(s) = \int_0^t f_1(u) f_2(t-u)\, du$$

Step 3 :

$$L^{-1}\phi_1(s)\phi_2(s) = \int_0^t e^{-(t-u)}\sin u \cdot du$$

$$= e^{-t}\int_0^t e^u \sin u \cdot du$$

Step 4 : Integrating w.r.t. u.

$$= e^{-t}\left\{\frac{e^u}{1+1}(\sin u - \cos u)\right\}_0^t$$

Step 5 : Substituting the limits of t.

$$= e^{-t}\left\{\frac{e^t}{2}(\sin t - \cos t) - \frac{1}{2}(-1)\right\}$$

$$= e^{-t}\left\{\frac{e^t}{2}(\sin t - \cos t) + \frac{1}{2}\right\}$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{e^{-t}}{2}$$

$$= \frac{1}{2}\left[\sin t - \cos t + e^{-t}\right]$$

Q.33 $L^{-1}\left[\frac{1}{s^4(s+5)}\right]$ **by convolution theorem.**

☞ [SPPU : Dec.-18, Marks 2]

Ans. :

We have $L\left[\frac{1}{s+5}\right] = e^{-5t}$

and $$L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t f(t)\,dt$$

We have $\phi(s) = \frac{1}{s+5}$ and $f(t) = e^{-5t}$

$$L^{-1}\left[\frac{1}{s(s+5)}\right] = \int_0^t e^{-5t}dt = \left[\frac{e^{-5t}}{-5}\right]_0^t = -\frac{1}{5}[e^{-5t}-1]$$

$$L^{-1}\left[\frac{1}{s^2(s+5)}\right] = \int_0^t -\frac{1}{5}[e^{-5t}-1]dt = -\frac{1}{5}\left[\frac{e^{-5t}}{-5}-t\right]_0^t$$

$$= -\frac{1}{5}\left[\frac{e^{-5t}}{-5}-t+\frac{1}{5}\right] = \frac{1}{25}[e^{-5t}+5t-1]$$

$$L^{-1}\left[\frac{1}{s^3(s+5)}\right] = \int_0^t \frac{1}{25}[e^{-5t}+5t-1]dt$$

$$= \frac{1}{25}\left[\frac{e^{-5t}}{-5}+5\frac{t^2}{2}-t\right]_0^t$$

$$= \frac{1}{25}\left[\frac{e^{-5t}}{-5}+\frac{5}{2}t^2-t+\frac{1}{5}\right]$$

$$L^{-1}\left[\frac{1}{s^4(s+5)}\right] = \int_0^t \frac{1}{25}\left[\frac{e^{-5t}}{-5}+\frac{5}{2}t^2-t+\frac{1}{5}\right]dt$$

$$= \frac{1}{25}\left[\frac{e^{-5t}}{(-5)(-5)}+\frac{5}{2}\frac{t^3}{3}-\frac{t^2}{2}+\frac{1}{5}t\right]_0^t$$

$$= \frac{1}{25}\left[\frac{e^{-5t}}{25}+\frac{5}{6}t^3-\frac{t^2}{2}+\frac{1}{5}t-\frac{1}{25}\right]$$

3.3 : Laplace Transform of Special Functions

1) Heaviside's Unit Step Function : The name of the function itself indicates that the function involves, a step of unit height.

Definition :

$$U(t) \text{ or } H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

Thus the function U (t) is zero when $t < 0$ and one when $t \geq 0$.

i.e.

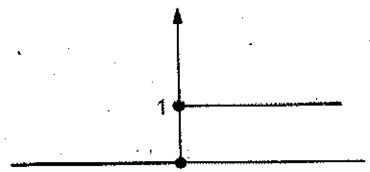

Fig. 3.2

Also the displaced unit step function is denoted by H (t – a) or U (t – a) and is defined by

$$U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

i.e. the function is one after $t \geq a$.

Note : Use of the function

a) Suppose we need certain function f(t) after the point t = a, then it can be obtained by f (t). U (t – a) i.e. by multiplying f (t) by U(t – a). This concept will be more clear by using the figures below.

Let f(t) = sin t

∴ Graph of f(t) is as follows

The graph of $U(t-\pi)$ is as follows.

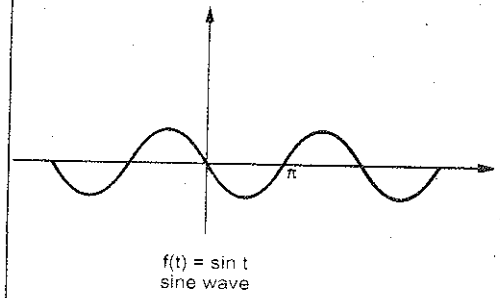

Fig. 3.3 sinewave

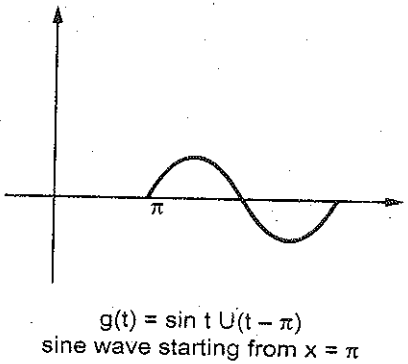

Fig. 3.4

$$U(t-\pi) = \begin{cases} 0 & \text{for } t < \pi \\ 1 & \text{for } t \geq \pi \end{cases}$$

When we consider $g(t) = \sin t \; U(t-\pi)$. As $U(t-\pi)$ is zero for $t < \pi$, thus the value of $g(t)$ will be zero for $t < \pi$ and as $U(t-\pi)$ is one for $t \geq \pi$, thus the value of $g(t)$ will be $\sin t \times (1)$ for $t \geq \pi$

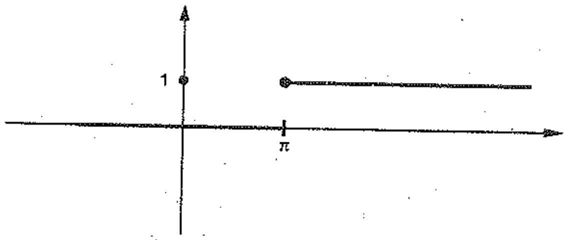

Fig. 3.5

$\therefore$ we can represent $g(t)$

as $$g(t) = \begin{cases} 0 & \text{for } t < \pi \\ \sin t & \text{for } t \geq \pi \end{cases}$$

Thus $g(t) = \sin t \; U(t-\pi)$ exists only for $t \geq \pi$. Generalising this the multiplication of $f(t)$ with $U(t-a)$ i.e. delayed or displaced unit step function gives $f(t)$ after point a i.e. the portion of $f(t)$ for $t < a$ becomes zero.

b) The displaced function f(t − a) U(t − a)

Let $$g(t) = f(t-a)\, U(t-a)$$

i.e. $$g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

which is a displaced function of f (t) by a. Observe the following figures.

Thus if f (t) is the original function then $f(t-a)\,U(t-a)$ is its displaced function.

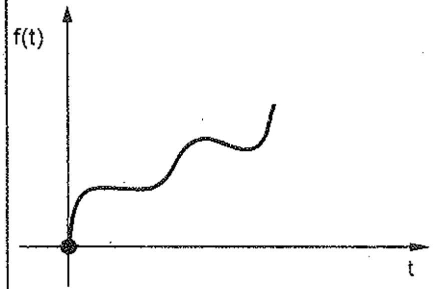

Fig. 3.6 Graph of f(t)

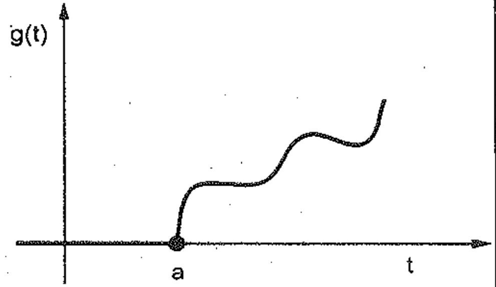

Fig. 3.7 Graph of displaced function g(t) = f(t–a) U(t–a)

c) The function f(t) within the interval $a \leq t \leq b$. Observe Fig. 3.8 on next page.

What will happen if we subtract $U(t-b)$ from $U(t-a)$. We get the portion of $U(t-a)$ after deleting the portion of $U(t-b)$.

Thus $U(t-a) - U(t-b)$ gives the non zero function "1" within the interval $a \leq t \leq b$.

Thus we have succeeded in finding "1" within the interval [a, b].

Now to find any function within this interval just multiply by that function to $[U(t-a) - U(t-b)]$.

Thus if $f(t) = \cos t$ for $\pi/2 \leq t \leq \pi$ we can express f(t) in unit step function as

$$f(t) = \cos t\left[U\left(t-\frac{\pi}{2}\right) - U(t-\pi)\right]$$

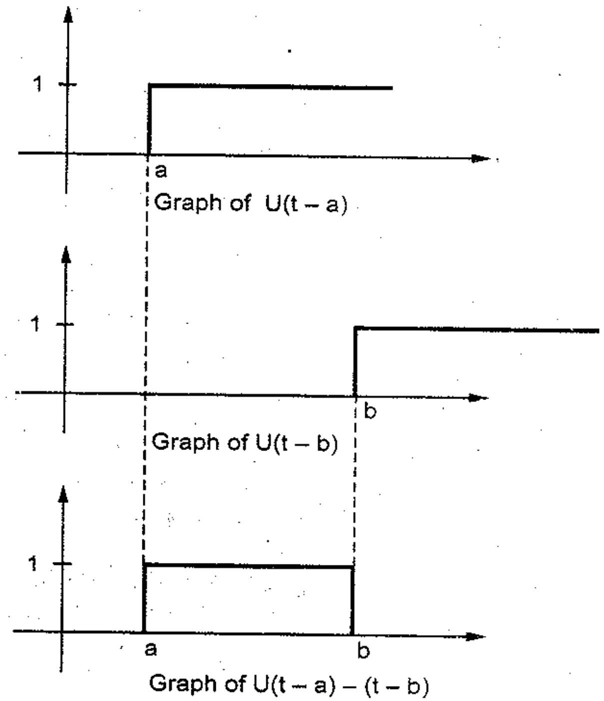

Fig. 3.8

3.4 Laplace Transforms using Heaviside Unit Step Function

a) $L\,[H(t-a)] = \dfrac{e^{-as}}{s}$

Proof : Step 1 : By definition of Laplace transforms

$$L\,[H(t-a)] = \int_0^\infty e^{-st}\, H(t-a)\, dt$$

Step 2 : As there are two values of H(t – a) split the integral into two integrals.

$$= \int_0^a e^{-st}\,(0)\,dt + \int_a^\infty e^{-st}\,(1)\,dt$$

Step 3 : Integrate and substitute the limits

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 - \frac{e^{-as}}{-s} \qquad \left(\text{as } e^{-\infty} = 0\right)$$

$$= \frac{e^{-as}}{s}$$

b) Substituting a = 0 we get

$$L\,(H(t)) = \frac{1}{s}$$

c) $L\,[f(t-a)\,U(t-a)] = e^{-as}\,\phi(s)$ where $L\,f(t) = \phi(s)$

Proof : Step 1 : We know that

$$f(t-a)\,U(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

Step 2 : By definition of Laplace transforms

$$L\,f(t-a)\,U(t-a) = \int_0^\infty e^{-st}\, f(t-a)\cdot U(t-a)\, dt$$

Step 3 : Split the integral in two parts.

$$= \int_0^a e^{-st}\,0\,dt + \int_a^\infty e^{-st}\, f(t-a)\, dt$$

Step 4 : Put (t – a) = u, dt = du

t	a	∞
u	0	∞

$$= 0 + \int_0^\infty e^{-s(a+u)}\, f(u)\, du$$

$$= e^{-as}\int_0^\infty e^{-su}\, f(u)\, du$$

$$= e^{-as}\, L\, f(u) \qquad \text{By definition of LT.}$$

$$= e^{-as}\,\phi(s)$$

i.e. $\boxed{L\,f(t-a)\,U(t-a) \;=\; e^{-as}\,L\,f(t)}$

which is second shifting property from Laplace transforms.

d) $L\ f(t)\ U\ (t-a) = e^{-as}\ L\,f(t+a)$

Proof : Step 1 : By definition of Laplace transforms

$$L\,f(t)\,U(t-a) = \int_0^{\infty} e^{-st}\ f(t)\,U(t-a)\,dt$$

Step 2 : Split the integral

$$= \int_0^{a} e^{-st} f(t)\cdot 0\,dt + \int_a^{\infty} e^{-st}\ f(t)\cdot 1\,dt$$

Step 3 : Put $t - a = u$, $t = a + u$, $dt = du$

t	a	∞
u	0	∞

$$= 0 + \int_0^{\infty} e^{-s(a+u)}\cdot f(u+a)\ du$$

$$= e^{-as}\int_0^{\infty} e^{-su}\ f(u+a)\ du$$

$$= e^{-as}\ L\,f(u+a) \qquad \text{By definition}$$

As the variable is not important in definite integrals.

$$= e^{-as}\ L\,f(t+a)$$

Thus $\boxed{L\ f(t)\,U(t-a) \;=\; e^{-as}\ L\,f(t+a)}$

e) By substituting $a = 0$ we get,

$$L\ [f(t)\ U(t)] = e^{-0s}\ L\,f(t+0) = L\ f(t)$$

$\therefore$ $\boxed{L\ f(t)\,U(t) \;=\; L\,f(t)}$

Q.34 Find Laplace of $e^{-t}\sin t\,U(t-4)$. ☞ [SPPU : Dec.- 16, Marks 4]

Ans. : $L[\sin t\,U(t-4)] = e^{-4s}L[\sin(t+4)]$

$$= e^{-4s}L[\sin t\cos 4+\cos t\ \sin 4] = e^{-4s}\left\{\frac{\cos 4}{s^2+1}+\frac{(\sin 4)(s)}{s^2+1}\right\}$$

By first shifting property $Le^{at}[f(t)] = \phi(s-a)$ using this property we get,

$$L[e^{-t}\sin t\,U(t-4)] = e^{-4(s+1)}\left\{\frac{\cos 4}{(s+1)^2+1}+\frac{(s+1)\sin 4}{(s+1)^2+1}\right\}.$$

Q.35 Find $L\left(1+2t+3t^2+4t^3\right)H(t-2)$

Ans. : $L\ f(t)\ u\ (t-a) = e^{-as}\ L\,[f(t+a)]$

$\therefore L\left(1+2t+3t^2+4+3\right)H(t-2)$

$= e^{-2s}\left[L(1)+2\ (t+2)+3\ (t+2)^2+4\ (t+2)^3\right]$

$= e^{-2s}\left\{L\left[1+2t+4+3\left(t^2+2t+4\right)+4\ (t+2)\ (t+2)\right]\right\}$

$= e^{-2s}\left\{L\left[5+2t+3t^2+6t+12+4\ (t+2)\left(t^2+2t+4\right)\right]\right\}$

$= e^{-2s}\left\{L\left[17+8t+3t^2+4\left(t^3+2t^2+4t+2t^2+4t+8\right)\right]\right\}$

$= e^{-2s}\left\{L\left[49+40t+19t^2+4t^3\right]\right\}$

Now we find the transform using formula $L\left(t^n\right) = \dfrac{n!}{s^{n+1}}$

$$= e^{-2s}\left[\frac{49}{s}+\frac{40}{s^2}+\frac{19\times 2!}{s^3}+\frac{4\times 3!}{s^4}\right] = e^{-2s}\left[\frac{49}{s}+\frac{40}{s^2}+\frac{38}{s^3}+\frac{24}{s^4}\right]$$

Q.36 Find the Laplace transform of $(1+2t-3t^2+4t^3)\,U(t-2)$ ☞ [SPPU : May- 18, Marks 4]

Ans. : We know that,

$$L[f(t)\,U(t-a)] = e^{-as}L[f(t+a)]$$

$$\therefore L[(1+2t-3t^2+4t^3)\,U(t-2)] = e^{-2s}L[1+2(t+2)-3(t+2)^2+4(t+2)^3]$$

$$= e^{-2s}L[1+2t+2-3t^2-12t-12+4t^3+24t^2+48t+32]$$

$$= e^{-2s}L[23+38t+21t^2+4t^3] = e^{-2s}\left[\frac{23}{s}+38\frac{1}{s^2}+21\times\frac{2!}{s^3}+4\frac{3!}{s^4}\right]$$

$$= e^{-2s}\left[\frac{23}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$$

Q.37 Find the Laplace transform of $t^4\,U(t-2)$.

[SPPU : Dec.-17, Marks 4]

Ans. :

We have $L[f(t)U(t-a)] = e^{-as}L[f(t+a)]$

$$\therefore \quad L[(t^4\,U(t-2)] = e^{-2s}L[(t+2)^4]$$

$$= e^{-2s}L\,[t^4+4t^3(2)+6t^2(2)^2+4t(2)^3+2^4]$$

$$= e^{-2s}L[t^4+8t^3+24t^2+32t+16]$$

$$= e^{-2s}\left[\frac{4!}{s^5}+8\cdot\frac{3!}{s^4}+24\cdot\frac{2!}{s^3}+32\frac{1}{s^2}+\frac{16}{s}\right]$$

$$= e^{-2s}\left[\frac{24}{s^5}+\frac{48}{s^4}+\frac{48}{s^3}+\frac{32}{s^2}+\frac{16}{s}\right]$$

Q.38 Find Lf(t) from the following Fig. Q.38.1.

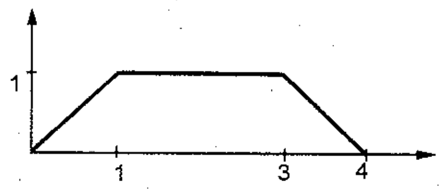

Fig. Q.38.1

Note : In such a problem firstly find equations of all the line segments. i.e. Here find equations of lines joining (0,0) to (1,1), (1, 1) to (3, 1), (3, 1) to (4, 0) these equations are $y = x$, $y = 1$, $y = 4 - x$ respectively then replace y by f(t) and x by t to write f(t) in terms of t.

Ans. : Here

$$f(t) = t \qquad 0 < t < 1$$
$$= 1 \qquad 1 < t < 3$$
$$= 4 - t \qquad 3 < t < 4$$

Here f(t) in terms of unit step function is given by,

$$\therefore\; f(t) = t[u(t)-u(t-1)]+1[u(t-1)-u(t-3)]$$
$$+(4-t)[u(t-3)-u(t-4)]$$

$$\therefore\; f(t) = tu(t)-(t-1)\,u(t-1)-(t-3)\,u(t-3)+(t-4)\,u(t-4)$$

$$L\,f(t) = e^{0s}\frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2} + e^{-4s}\frac{1}{s^2}$$

$$= \frac{1}{s^2}\left(1 - e^{-s} - e^{-3s} + e^{-4s}\right)$$

3.5 : Dirac Delta or Unit Impulse Function

Consider the function

$$f(t) = \begin{cases} 0 & t < a \\ 1/\varepsilon & a \le t \le a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$$

where $\varepsilon > 0$

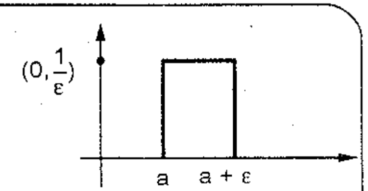

Fig. 3.9

The area under the graph is always unity.

$$\delta(t-a) = \lim_{\varepsilon \to 0} f(t)$$

Properties and Theorems on Dirac Delta Function

a) Shifting property

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a) \text{ Provided } f(t) \text{ is continuous}$$

Proof : Step 1 : Consider $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt$

Step 2 : Split the integral

$$= \int_{-\infty}^{a} f(t)\,\delta(t-a)\,dt + \int_{a}^{a+t} f(t)\,\delta(t-a)\,dt + \int_{a+t}^{\infty} f(t)\,\delta(t-a)\,dt$$

Step 3 : Using the definition of $\delta(t-a)$ we get

$$= 0 + \lim_{t \to 0} \int_a^{a+t} f(t) \cdot \frac{1}{t} dt + 0$$

Step 4 : Let $\int f(t)\, dt = G(t)$ say

$$= \lim_{t \to 0} \frac{[G(t)]_a^{a+t}}{\varepsilon}$$

Step 5 : Substitute the limits

$$= \lim_{t \to 0} \frac{G(a+t) - G(a)}{a}$$

Step 6 : By the definition of derivatives by 1^{st} principal

$$= G'(a)$$

$$= f(a) \qquad \text{As } \int f(t)\, dt = G(t)$$

b) Similarly we can show $\qquad \therefore f(t) = G'(t)\}$

$$\boxed{\int_0^{\infty} f(t)\, \delta(t-a)\, dt = f(a)} \qquad \text{... (i)}$$

Also $\boxed{\int_{-\infty}^{\infty} f(t)\, \delta'(t-a)\, dt = -f'(a)}$ ... (ii)

c) Laplace transform of $\delta(t-a)$ by definition of Laplace

$$L\, \delta(t-a) = \int_0^{\infty} e^{-st}\, \delta(t-a)\, dt$$

$$= e^{-as} \qquad \text{using equation (i)}$$

$$\boxed{L\, \delta(t-a) = e^{-as}}$$

Substituting a = 0 we get

$$\boxed{L(\delta(t)) = 1}$$

d) Laplace transform of $f(t)\,\delta(t-a)$ by definition of Laplace

$$L\, f(t)\,\delta(t-a) = \int_0^{\infty} e^{-st} f(t)\,\delta(t-a)\, dt$$

$$= e^{-as} f(a) \qquad \text{using equation (i)}$$

Thus $\boxed{L\, f(t)\,\delta(t-a) = e^{-as} f(a)}$

Substituting a = 0 we get

$$\boxed{L\, f(t)\,\delta(t) = f(0)}$$

e) Relation between Laplace's of Dirac Delta and unit step function

$$L\,\delta(t-a) = L\,[U'(t-a)]$$

Proof : We know that $L\, f'(t) = -f(0) + s\,\phi(s)$

$$\therefore \quad L\, U'(t-a) = -[U(t-a)]_{t=0} + s\, L\, U(t-a)$$

$$= 0 + s \cdot \frac{e^{-as}}{s} = e^{-as}$$

$$= L\,\delta(t-a)$$

Thus $\boxed{L\, U'(t-a) = L\,\delta(t-a)}$

Similarly $\boxed{L\, U''(t-a) = L\,\delta'(t-a)}$

Q.39 L [sin 2t δ (t – p/4) – t² δ (t – 4)]

Ans. : $L \sin 2t\, \delta(t - \pi/4)$

$$= \int_0^{\infty} e^{-st} \sin 2t\, \delta(t - \pi/4) \cdot dt = e^{-\frac{\pi}{4}s} \sin\frac{\pi}{2} = e^{-\frac{\pi}{4}s}$$

Also $\quad L\, t^2\, \delta(t-4) = \int_0^{\infty} e^{-st}\, t^2\, (t-4) \cdot dt = e^{-4s}\, 16$

$$\therefore \quad L\left[\sin 2t\,\delta\left(t - \frac{\pi}{4}\right) - t^2\,\delta(t-4)\right] = e^{-\frac{\pi s}{4}} - 16\, e^{-4s}$$

Q.40 Find Laplace transform of $t\,u(t-4) - t^3\,\delta(t-2)$

Ans. : $L\left[t\,u(t-4) - t^3\,\delta(t-2)\right]$

Consider $L\,t\,u(t-4) = e^{-4s}\,L(t+4) = e^{-4s}\left[\frac{1}{s^2}+\frac{4}{s}\right]$

Also $\quad L\,t^3\,\delta(t-2) = \int_0^\infty e^{-st}\,t^3\,\delta(t-2) = e^{-2s}\,8$

$$\therefore L\left[t\,U(t-4) - t^3\delta(t-2)\right] = e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right) - 8e^{-2s}$$

Q.41 Solve $L[t\,U(t-4) - t^3\,\delta(t-2)]$ ☞ **[SPPU : May- 15]**

Ans. : We have $L[t\,U(t-4)] - L[t^3\,\delta(t-2)]$

$$= e^{-4s}L(t+4) - e^{-2s}(2)^3 = e^{-4s}\left[\frac{1}{s^2}+\frac{4}{s}\right] - e^{-2s}\,8$$

3.6 Applications of Laplace Transforms

Let $\quad L[y(t)] = \overline{Y}(s)$

$$\therefore \quad L\left[\frac{dy}{dt}\right] = -y(0) + s\,\overline{Y}(s)$$

$$L\left[\frac{d^2y}{dt^2}\right] = -y'(0) - s\,y(0) + s^2\,\overline{Y}(s)$$

$$L\left[\frac{d^3y}{dt^3}\right] = -y''(0) - s\,y'(0) - s^2\,y(0) + s^3\,\overline{Y}(s)$$

$$L\left[\int_0^t y\,dt\right] = \frac{1}{s}\,\overline{Y}(s)$$

Illustrations

Q.42 Find Laplace transform of $\cosh t\;\delta(t-4)$. ☞ **[SPPU : Dec.-18, Marks 4]**

Ans. : We know that

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Now, $L[\cosh t + \delta(t-4)] = \int_0^\infty e^{-st}\cosh t + \delta(t-4)\,dt$

$$= [e^{-st}\cosh t]_{t\to 4} = e^{-4s}\cosh t$$

Q.43 Solve $\frac{dy}{dt} + 2y(t) + \int_0^t y(t)\,dt = \sin t$ given $y(0) = 1$

Ans. : **Step 1 :** Let $L\,y(t) = Y(s)$

$$\therefore \quad L\,\frac{dy}{dt} = -y(0) + s\,Y(s) = -1 + s\,Y(s)$$

$$L\int_0^t y\,dt = \frac{1}{s}\,Y(s)\;, \quad L\,\sin t = \frac{1}{s^2+1}$$

Step 2 : Take Laplace transform of given equation.

$$\therefore \quad L\left[\frac{dy}{dt}\right] + L[2\,y(t)] + L\int_0^t y\,dt = \frac{1}{s^2+1}$$

Step 3 : Substituting the values.

$$-1 + s\,Y(s) + 2\,Y(s) + \frac{1}{s}\,Y(s) = \frac{1}{s^2+1}$$

Step 4 : Collecting the terms of Y(s) on one side

$$Y(s)\left[s+2+\frac{1}{s}\right] = 1 + \frac{1}{s^2+1}$$

$$Y(s)\left(\frac{s^2+2s+1}{s}\right) = 1 + \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+2s+1} + \frac{s}{\left(s^2+2s+1\right)\left(s^2+1\right)}$$

Step 5 : To find inverse use partial fractions or adjustment

$$Y(s) = \frac{s+1-1}{(s+1)^2} + \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2s+1}\right]$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1/2}{s^2+1} - \frac{1/2}{(s+1)^2}$$

$$Y(s) = \frac{1}{s+1} + \frac{1/2}{s^2+1} - \frac{3/2}{(s+1)^2}$$

Step 6 : Take inverse Laplace

$$y(t) = e^{-t} + \frac{1}{2}\sin t - \frac{3}{2}\, t\, e^{-t}$$

Q.44 Solve $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^{-2t}$, $y = 0$, $y' = 0$ at $t = 0$.

☞ **[SPPU : May- 16, Marks 4]**

Ans. : Step 1 : Let $L(y(t)) = Y(s)$

$\therefore$
$$L\,y'(t) = -y(0) + s\,Y(s) = s\,Y(s)$$
$$L\,y''(t) = -y'(0) - s\,y(0) + s^2\,Y(s)$$
$$= s^2\,Y(s)$$

Step 2 : Take Laplace transform of given equation

$$L(y'') - L(2y') + L(y) = L\left(e^{-2t}\right)$$

Step 3 : Substituting the values we get

$$\left[s^2\,Y(s)\right] - 2\left[s\,Y(s)\right] + Y(s) = \frac{1}{s+2}$$

Step 4 : Collecting the terms of Y(s) on one side

$$(s^2 - 2s + 1)\,Y(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s+2)(s-1)^2}$$

Step 5 : Use partial fractions

$$Y(s) = \frac{\frac{1}{9}}{s+2} + \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

Step 6 : Take inverse Laplace transform, we get

$$y(t) = e^{-2t} - 6\,e^{t} + 7\,e^{2t}$$

Q.45 Solve $y'' + 2y' + y = t\,e^{-t}$, $y(0) = 1$, $y'(0) = -2$

☞ **[SPPU : May-17, Marks 4]**

Ans. : Step 1 : Let $L\,y(t) = Y(s)$

$$L\,y'(t) = -y(0) + s\,Y(s) = -1 + s\,Y(s)$$
$$L\,y''(t) = -y'(0) - s\,y(0) + s^2\,Y(s) = +2 - s + s^2\,Y(s)$$

Step 2 : Taking Laplace of given equation

$$L[y''] + 2\,L[y'] + L[y] = L\left[t\,e^{-t}\right]$$

Step 3 : Substituting the values

$$\left[2 - s + s^2\,Y(s)\right] + 2\left[-1 + s\,Y(s)\right] + Y(s) = \frac{1}{(s+1)^2}$$

Step 4 : Collecting the terms of Y(s) on one side

$$\left(s^2 + 2s + 1\right)Y(s) = s + \frac{1}{(s+1)^2}$$

$$\therefore \quad Y(s) = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^4}$$

Step 5 : Adjustment

$$= \frac{s+1-1}{(s+1)^2} + \frac{1}{(s+1)^4} = \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}$$

Step 6 : Take inverse

$$y(t) = e^{-t} - t\,e^{-t} + e^{-t}\cdot\frac{t^3}{3!}$$

Q.46 Solve using Laplace transforms
$\left(D^2+n^2\right) x = a \sin(n t + a)$, $x(0) = x'(0) = 0$ ☞ [SPPU : May-2000]

Ans. : Step 1 : Take Laplace of given equation

$x'' + n^2 x = a \sin(n t + \alpha) = a[\sin nt \cos\alpha + \cos nt \sin\alpha]$

$\therefore L[x''] + n^2 L[x] = a \cos\alpha\, L[\sin nt] + a \sin\alpha\, L[\cos nt]$

i.e. $\left[s^2 X(s) - s\,x(0) - x'(0)\right] + n^2 X(s) =$

$$a\cos\alpha\left[\frac{n}{s^2+n^2}\right] + a\sin\alpha\left[\frac{s}{s^2+n^2}\right]$$

Step 2 : Collect the terms of X(s)

$$\therefore \left(s^2+n^2\right) X(s) = a\cos\alpha\left(\frac{n}{s^2+n^2}\right) + a\sin\alpha\left(\frac{s}{s^2+n^2}\right)$$

$\ldots \because x(0) = x'(0) = 0$

$$\therefore X(s) = a\cos\alpha\left[\frac{n}{\left(s^2+n^2\right)^2}\right] + a\sin\alpha\left[\frac{s}{\left(s^2+n^2\right)^2}\right]$$

Step 3 : Taking inverse Laplace transform we get,

$$\therefore x(t) = L^{-1}[X(s)] = a(\cos\alpha)\, n\, L^{-1}\left[\frac{1}{\left(s^2+n^2\right)^2}\right]$$

$$+ a(\sin\alpha)\, L^{-1}\left[\frac{s}{\left(s^2+n^2\right)^2}\right] \quad \ldots (1)$$

We know

$$L^{-1}\frac{1}{\left(s^2+a^2\right)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$L^{-1}\frac{s}{\left(s^2+a^2\right)^2} = \frac{t}{2a}\sin at$$

Thus $x(t) = a(\cos\alpha)\, n\frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha\,\frac{t\sin nt}{2n}$

$$= \frac{a\cos\alpha}{2n^2}(\sin nt - nt\cos nt) + \frac{a\sin\alpha}{2n}(t\sin nt)$$

Q.47 Solve by Laplace method $\frac{d^2y}{dt^2} + y = t$ with $y(0) = 1$ and $y'(0) = -2$. ☞ [SPPU : Dec.- 16,17]

Ans. : Step 1 : Taking Laplace transform of given equation, we get

$$L\left[\frac{d^2y}{dt^2} + y\right] = L(t)$$

$$[-y'(0) - s\,y(0) + s^2 Y(s)] + Y(s) = \frac{1}{s^2}$$

$$-2 + s^2 Y(s) + Y(s) = \frac{1}{s^2}$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + 2$$

$$Y(s) = \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}$$

$$= \left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] + \frac{2}{s^2+1}$$

Step 2 : Taking inverse Laplace transform, we get

$$y(t) = t - \sin t + 2\sin t$$

$$= t + \sin t$$

Q.48 Solve $\frac{d^2y}{dt^2} + 9y = 18\,t$, $y(0) = 0$, $y(\pi/2) = 0$ ☞ [SPPU : May-18, Marks 4]

Ans. : Step 1 : Taking Laplace transform of given equation,

$$\left[-y'(0) - s\,y(0) + s^2 Y(s)\right] + 9[Y(s)] = 18\frac{1}{s^2}$$

Step 2 : Substituting $y(0) = 0$, $y'(0) = C_1$ (say)

$$\left[-C_1 - 0 + s^2\, Y(s)\right] + 9\,Y(s) = \frac{18}{s^2}$$

$$(s^2+9)\; Y(s) = C_1 + \frac{18}{s^2}$$

$$Y(s) = \frac{C_1}{s^2+9} + \frac{18}{s^2\,(s^2+9)} = \frac{C_1}{s^2+9} + \frac{18}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right]$$

$$= \frac{C_1}{s^2+9} + \frac{2}{s^2} - \frac{2}{s^2+9} = \frac{(C_1-2)}{s^2+9} + \frac{2}{s^2}$$

Step 3 : Taking inverse,

$$y(t) = \left(\frac{C_1-2}{3}\right)\sin 3t + 2t \qquad \dots (1)$$

Put $t = \frac{\pi}{2}$,

$$y\left(\frac{\pi}{2}\right) = \left(\frac{C_1-2}{3}\right)\sin\frac{3\pi}{2} + 2\left(\frac{\pi}{2}\right)$$

$$0 = \left(\frac{C_1-2}{3}\right)(-1) + \pi$$

$$\Rightarrow \qquad \frac{C_1-2}{3} = \pi$$

Substituting in equation (1),

$$y(t) = \pi \sin 3t + 2t$$

Q.49 $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = e^{-t}\sin t,\ x(0) = 0\ x'(0) = 1$

[SPPU : May-15, 19, Marks 4]

Ans. : Step 1 : Taking Laplace transform of given equation

$$L(x'') + 2L(x') + 5L(x) = L(e^{-t}\sin t)$$

$$-x'(0) - s x(0) + s^2 X(s) + 2\left[-x(0) + sX(s)\right] + 5X(s) = \frac{1}{(s+1)^2+1}$$

Step 2 : Substituting $x(0) = 0$, $x'(0) = 1$

$$-1 - 0 + s^2X(s) + 0 + 2sX(s) + 5X(s) = \frac{1}{s^2+2s+2}$$

$$\left[s^2+2s+5\right] X(s) = 1 + \frac{1}{s^2+2s+2}$$

$$X(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)\,(s^2+2s+2)}$$

$$X(s) = \frac{s^2+2s+3}{(s^2+2s+5)\,(s^2+2s+2)}$$

Step 3 : For partial fraction put $s^2 + 2s = u$

$$X(s) = \frac{u+3}{(u+5)\,(u+2)} = \frac{A}{u+5} + \frac{B}{u+2}$$

$$= \frac{2/3}{u+5} + \frac{1/3}{u+2}$$

Again substitute $u = s^2 + 2s$

$$X(s) = \frac{2/3}{s^2+2s+5} + \frac{1/3}{s^2+2s+2} = \frac{1}{3}\left[\frac{2}{(s+1)^2+4} + \frac{1}{(s+1)^2+1}\right]$$

Step 4 : Taking inverse Laplace transform

$$x(t) = \frac{e^{-t}}{3}\, L^{-1}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)$$

$$x(t) = \frac{e^{-t}}{3}\,[\sin 2t + \sin t]$$

END...

UNIT III

4 Fourier Transform(FT)

4.1 : Fourier Transform

1) The following table gives the Fourier transform pairs for ready reference.

Sr. No.	Name of the transform	Expression for the transform F(λ) =	Inverse transform f(x) =
1.	Fourier transform	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du$	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine transform / Fourier transform for even f(x)	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \cos \lambda u\, du$	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(\lambda) \cos \lambda x\, d\lambda$
3.	Fourier sine transform / Fourier transform for odd f(x)	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \sin \lambda u\, du$	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(\lambda) \sin \lambda u\, d\lambda$

We may use the following table also (Note the constant multiples of the integrals)

Sr. No.	Name of the transform	Expression for the transform	Inverse transform
1.	Fourier transform	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine transform / Fourier transform for even f(x)	$F(\lambda) = \int_0^{\infty} f(u) \cos \lambda u du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cos \lambda x d\lambda$
3.	Fourier sine transform / Fourier transform for odd f(x)	$F(\lambda) = \int_0^{\infty} f(u) \sin \lambda u du$	$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda$

Note

1) For solving examples we use above formulae

2) $\int_{-\infty}^{\infty} f(x)\, dx = 0$ if f(x) is odd.

$= 2 \int_{0}^{\infty} f(x)\, dx$ if f(x) is even.

Q.1 Find Fourier cosine transform of
$f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$ ☞ [SPPU : May-12, 14]

Ans. : Consider F.C.T. formula

$$F_c(\lambda) = \int_0^{\infty} f(u) \cos \lambda u\, du$$

$$F_c(\lambda) = \frac{1}{2} \left\{ \int_0^a 2 \cos u \cdot \cos \lambda u\, du + \int_a^{\infty} 0 \right\}$$

Use

$$2 \cos \lambda u \cos u = \cos(\lambda - 1)\, u + \cos(\lambda + 1)\, u$$

$$\therefore \quad F_c(\lambda) = \frac{1}{2} \int_0^a [\cos(\lambda - 1)\, u + \cos(\lambda + 1)\, u] du$$

$$\text{Integrate} = \frac{1}{2}\left[\frac{\sin(\lambda-1)\,u}{\lambda-1}+\frac{\sin(\lambda+1)\,u}{\lambda+1}\right]_0^a$$

Put the limits of u

$$F_c(\lambda) = \frac{1}{2}\left[\frac{\sin(\lambda-1)\,a}{\lambda-1}+\frac{\sin(\lambda+1)\,a}{\lambda+1}\right]$$

Q.2 Find Fourier sine and cosine transform of

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

☞ **[SPPU : May-16, 18, Marks 4]**

Ans. : **Step 1 :** Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$$

Step 2 : Split the integral and substitute the value of f(u)

$$F_s(\lambda) = \int_0^1 u\sin\lambda u\,du+\int_1^2 (2-u)\sin\lambda u\,du$$

Step 3 : Integrating by parts

$$F_s(\lambda) = \left[u\frac{-\cos\lambda u}{\lambda}\right]_0^1-\int_0^1\frac{-\cos\lambda u}{\lambda}\,du$$

$$+\left[(2-u)\frac{-\cos\lambda u}{\lambda}\right]_1^2-\int_1^2 -1\frac{-\cos\lambda u}{\lambda}\,du$$

$$= \left[\frac{-\cos\lambda}{\lambda}+\frac{\sin\lambda}{\lambda^2}+\frac{\cos\lambda}{\lambda}-\frac{\sin 2\lambda}{\lambda^2}+\frac{\sin\lambda}{\lambda^2}\right]$$

$$= \frac{2\sin\lambda}{\lambda^2}-\frac{\sin 2\lambda}{\lambda^2}$$

Step 4 : As $\sin 2\lambda = 2\sin\lambda\cos\lambda$ we get

$$\therefore \quad F_s(\lambda) = \frac{2\sin\lambda(1-\cos\lambda)}{\lambda^2}$$

Now for finding cosine transform

Step 5 : Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

Step 6 : Split the integral and substitute the value of f(u)

$$F_c(\lambda) = \left[\int_0^1 u\cos\lambda u\,du+\int_1^2 (2-u)\cos\lambda u\,du\right]$$

Step 7 : Integrating by parts,

$$F_c(\lambda) = \left[u\frac{\sin\lambda u}{\lambda}\right]_0^1-\int_0^1\frac{\sin\lambda u}{\lambda}\,du$$

$$+\left[(2-u)\frac{\sin\lambda u}{\lambda}\right]_1^2-\int_1^2 -1\frac{\sin\lambda u}{\lambda}\,du$$

$$= \frac{\sin\lambda}{\lambda}+\frac{\cos\lambda}{\lambda^2}-\frac{1}{\lambda^2}-\frac{\sin\lambda}{\lambda}-\frac{\cos 2\lambda}{\lambda^2}+\frac{\cos\lambda}{\lambda^2}$$

Step 8 : Thus $\quad F_c(\lambda) = \dfrac{2\cos\lambda-\cos 2\lambda-1}{\lambda^2}$

Q.3 Find the Fourier sine transform of $f(x) = e^{-x}\cos x;\ x > 0.$

☞ **[SPPU : May-15]**

Ans. : We have $F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$

$$= \int_0^\infty e^{-u}\cos u\sin\lambda u\,du = \frac{1}{2}\int_0^\infty e^{-u}\,2\cos u\sin\lambda u\,du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}\,(\sin(\lambda+1)u+\sin(\lambda-1)u]\,du$$

$$= \frac{1}{2}\left\{\frac{e^{-u}}{1+(\lambda+1)^2}[-\sin(\lambda+1)u-(\lambda+1)\cos(\lambda+1)u]\right.$$

$$\left.+\frac{e^{-u}}{1+(\lambda-1)^2}[-\sin(\lambda-1)u-(\lambda-1)\cos(\lambda-1)u\right\}_0^\infty$$

$$= \frac{1}{2}\left\{[0]-\frac{1}{1+(\lambda+1)^2}[0-(\lambda+1)]-\frac{1}{1+(\lambda-1)^2}[0-(\lambda-1)]\right\}$$

$$F_s(\lambda) = \frac{1}{2}\left[\frac{\lambda+1}{1+(\lambda+1)^2}+\frac{\lambda-1}{1+(\lambda-1)^2}\right]$$

Q.4 Find Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

☞ **[SPPU : Dec.-06, 09, 12, 18, Marks 4, May-08, 12]**

Ans. : As f(x) is an even function of x we find F.C.T. of f(x)

$$\text{F.C.T. } F(\lambda) = \int_0^\infty F(u)\cos\lambda u\,du$$

$$= \int_0^1 \left(1-u^2\right)\cos\lambda u\,du + \int_1^\infty 0$$

$$= \int_0^1 \left(1-u^2\right)\cos\lambda u\,du$$

$$= \left[\left(1-u^2\right)\left(\frac{\sin\lambda u}{\lambda}\right)-(-2u)\left(\frac{-\cos\lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda u}{\lambda^3}\right)\right]_0^1$$

$$= 0-\frac{2\cos\lambda}{\lambda^2}+\frac{2\sin\lambda}{\lambda^3}-(0-0-0)$$

$$F(\lambda) = \frac{-2(\lambda\cos\lambda-\sin\lambda)}{\lambda^3}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x\,d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{-2(\lambda\cos\lambda-\sin\lambda)}{\lambda^3}\cos\lambda x\,d\lambda$$

Put $x = \frac{\pi}{2}$ $\quad f\left(\frac{1}{2}\right) = \frac{-4}{\pi}\int_0^\infty \frac{\lambda\cos\lambda-\sin\lambda}{\lambda^3}\left(\cos\frac{\lambda}{2}\right)d\lambda$

As $\quad f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

$\therefore \quad f\left(\frac{1}{2}\right) = 1-\left(\frac{1}{2}\right)^2 = \frac{3}{4}$ substituting we get

$$\left(\frac{3}{4}\right)\left(\frac{-\pi}{4}\right) = \int_0^\infty \left(\frac{\lambda\cos\lambda-\sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\,d\lambda$$

$\therefore$ As the variable is not important indefinite integrals here replace λ by x

$$\therefore \quad \frac{-3\pi}{16} = \int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx$$

Q.5 Find Fourier sine transform of $e^{-|x|}$**. Hence evaluate** $\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx$

☞ **[SPPU : Dec.-08, May-12]**

Ans. : Consider F.S.T. formula

$$F(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du = \left\{\int_0^\infty e^{-u}\sin\lambda u\,du\right\}$$

$$\therefore \quad F_s(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\sin\lambda u+\lambda\cos\lambda u)\right]_0^\infty$$

$$F_s(\lambda) = \frac{\lambda}{\lambda^2+1}$$

Consider inverse Fourier transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$\therefore \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

Put $x = m$ and then $\lambda = x$ we get

$$\therefore \quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$$

$$\therefore \quad \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-m}$$

Q.6 Using Fourier integral representation show that

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

☞ [SPPU : Dec.-06,12]

Ans. : To prove the result consider the function

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

This function f(x) is defined in $-\infty < x < \infty$ and both $\sin \lambda x$ and $\cos \lambda x$ are present in the integrand so we use the general Fourier transform.

$\therefore$ We should find the F.T. of f(x).

We have $$F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} \, dx$$

$$F(\lambda) = \int_{-\infty}^{0} f(x)\, e^{-i\lambda x} dx + \int_0^\infty f(x)\, e^{-i\lambda x} \, dx$$

$$F(\lambda) = 0 + \int_0^\infty \pi e^{-x} e^{-i\lambda x} \, dx$$

$$\therefore \quad F(\lambda) = \pi \int_0^\infty e^{-x(1+i\lambda)} \, dx = \pi\left[\frac{1}{1+i\lambda}\right] = \pi \frac{(1-i\lambda)}{1+\lambda^2}$$

Consider the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x} \, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi \frac{(1-i\lambda)}{1+\lambda^2} [\cos \lambda x + i \sin \lambda x] \, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda + \frac{1}{2} i \int_{-\infty}^{\infty} \frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2} \, d\lambda$$

$$= \frac{1}{2} 2 \int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda + 0 = \int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda$$

$$\text{Thus } \int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda = f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases} \quad \ldots (1)$$

Putting $x = 0$ we get

$$f(0) = \int_0^\infty \frac{1}{1+\lambda^2} \, d\lambda = \frac{\pi}{2}$$

$\therefore$ We have

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} \, d\lambda = \begin{cases} 0 & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Q.7 Using Fourier integral representation show that

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x \; d\lambda = \begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

[SPPU : May-06, 13]

Ans. : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases} \frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u \, du$$

$$F_s(\lambda) = \int_0^\pi f(u)\sin\lambda u\,du + \int_\pi^\infty f(u)\sin\lambda u\,du$$

$$F_s(\lambda) = \int_0^\pi \frac{\pi}{2}\sin\lambda u\,du + \int_\pi^\infty 0\sin\lambda u\,du$$

$$= \frac{\pi}{2}\int_0^\pi \sin\lambda u\,du + 0 = \frac{\pi}{2}\left[\frac{-\cos\lambda u}{\lambda}\right]_0^\pi$$

$$= \frac{\pi}{2}\left[\frac{-\cos\lambda\pi-(-1)}{\lambda}\right] = \frac{\pi}{2}\,\frac{(1-\cos\lambda\pi)}{\lambda}$$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \frac{\pi}{2}\int_0^\infty \frac{2}{\pi}\,\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\,d\lambda$$

$$f(x) = \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\,d\lambda$$

Q.8 Solve the integral equation

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda & 0\le\lambda\le 1 \\ 0 & \lambda>1 \end{cases}$$

Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

[SPPU : Dec.-06, 07, 09, May-12, 14, 19, Marks 4]

Ans. : As $\cos\lambda x$ is present in the integrand, we use F.C.T.

$$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & 0\le\lambda\le 1 \\ 0 & \lambda>1 \end{cases}$$

To find f(x) consider inverse Fourier cosine transforms

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\cos\lambda x\,d\lambda + \int_1^\infty 0\,d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin\lambda x}{x}\right)-(-1)\left(\frac{-\cos\lambda x}{x^2}\right)\right]_0^1$$

$$f(x) = \frac{2}{\pi}\left[\left(0-\frac{\cos x}{x^2}\right)-\left(0-\frac{1}{x^2}\right)\right] = \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$$

Use $(1-\cos x) = 2\sin^2(x/2) = \frac{2}{\pi}\,\frac{2\sin^2\frac{x}{2}}{x^2}$

$$f(x) = \frac{1}{\pi}\,\frac{\sin^2\frac{x}{2}}{\left(\frac{x^2}{4}\right)} = \frac{1}{\pi}\,\frac{\sin^2(x/2)}{(x/2)^2}$$

Substituting in the given equation, we get

Put $\lambda = 0$

$$\frac{1}{\pi}\int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2}\, 1\, dx = 1$$

Put $x/2 = u,\ x = 2u,\ dx = 2\, du$

$$\therefore \quad \frac{1}{\pi}\int_0^\infty \frac{\sin^2 u}{u^2}\, 2\, du = 1$$

$$\therefore \quad \int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

Q.9 Solve the integral equation

$$\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0$$

[SPPU : Dec.-16, Marks 4]

Ans. : Given

$$\therefore \int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0$$

As $\cos\lambda x$ is present in the integrand we use F.C.T.

$$\therefore \quad F_c(\lambda) = e^{-\lambda},\ \lambda > 0$$

To find f(x) consider inverse Fourier cosine transform.

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x\, d\lambda$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda u + x\sin\lambda x)\right]_0^\infty$$

$$f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$$

Q.10 Solve $\int_0^\infty \frac{\lambda\sin\lambda x}{(4+\lambda^2)(9+\lambda^2)}\, d\lambda = \frac{\pi}{10}(e^{-2x} - e^{-3x})$

[SPPU : Dec. - 17, Marks 4]

Ans. : As $\sin\lambda x$ is present in the integral, we find Fourier sine transform of $f(x) = \frac{\pi}{10}(e^{-2x} - e^{-3x})$.

$$\therefore \quad F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du = \frac{\pi}{10}\int_0^\infty [e^{-2u} - e^{-3u}]\sin\lambda u\, du$$

$$= \frac{\pi}{10}\left\{\int_0^\infty e^{-2u}\sin\lambda u\, du - \int_0^\infty e^{-3u}\sin\lambda u\, du\right\}$$

$$= \frac{\pi}{10}\left\{\frac{e^{-2u}}{\lambda^2+4}(-2\sin\lambda u - \lambda\cos\lambda u)\right\}_0^\infty$$

$$-\frac{\pi}{10}\left\{\frac{e^{-3u}}{\lambda^2+9}(-3\sin\lambda u - \lambda\cos\lambda u)\right\}_0^\infty$$

$$= \frac{\pi}{10}\left\{0 - \frac{1}{\lambda^2+4}(-\lambda)\right\} - \frac{\pi}{10}\left\{0 - \frac{1}{\lambda^2+9}(-\lambda)\right\}$$

$$F_s(\lambda) = \frac{\pi}{10}\left\{\frac{\lambda}{\lambda^2+4} - \frac{1}{\lambda^2+9}\right\}$$

$$= \frac{\pi}{10}\left\{\frac{5\lambda}{(\lambda^2+4)(\lambda^2+9)}\right\} = \frac{\pi}{2}\left\{\frac{\lambda}{(\lambda^2+4)(\lambda^2+9)}\right\}$$

Now Inverse sine transform of f(x) is,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\left\{\frac{\lambda}{(\lambda^2+4)(\lambda^2+9)}\right\}\sin\lambda x\, d\lambda$$

$$f(x) = \int_0^\infty \frac{\lambda}{(\lambda^2+4)(\lambda^2+9)}\sin\lambda x\, d\lambda$$

Q.11 a) Find Fourier transform of $e^{-x^2/2}$.
b) Show that Fourier transform of $e^{-x^2/2}$ is $e^{-\lambda^2/2}$

☞ [SPPU : May-08, 12]

Ans. : a) The Fourier transform of

$f(x) = e^{-x^2/2}$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-i\lambda x}\, dx$$

$$\therefore \quad F(\lambda) = \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}+i\lambda x\right)} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2i\lambda x)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x+i\lambda)^2 - i^2\lambda^2]}\, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2} e^{-\frac{\lambda^2}{2}}\, dx$$

$$= e^{-\frac{\lambda^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2}\, dx$$

The above integrand is even function

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \cdot 2\int_{0}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2}\, dx$$

Substitute $\frac{1}{2}(x+i\lambda)^2 = u$

$\therefore \quad \frac{1}{2}2(x+i\lambda)\, dx = du$

$$\therefore \quad dx = \frac{du}{x+i\lambda} = \frac{du}{\sqrt{2u}}$$

$\therefore$ We get
$$F(\lambda) = e^{-\lambda^2/2} \frac{2}{\sqrt{2}} \int_{0}^{\infty} e^{-u} u^{-\frac{1}{2}}\, du$$

As $\overline{|n} = \int_{0}^{\infty} e^{-u} u^{n-1}\, du$

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \sqrt{2} \overline{\left| \frac{1}{2} \right.} = \sqrt{2} \cdot \sqrt{\pi} \cdot e^{-\lambda^2/2}$$

$$F(\lambda) = \sqrt{2\pi}\; e^{-\lambda^2/2}$$

b) In this example use the formula

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \cdot e^{-\lambda^2/2} \qquad \text{(by part (a))}$$

$$F(\lambda) = e^{-\lambda^2/2}$$

Q.12 Using inverse sine transform, find f(x) if $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$

☞ [SPPU : May-15, 17]

Ans. : Step 1 : Inverse sine transform of $F_s(\lambda)$ is given by (Note the formula for inverse)

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda$$

Step 2 : Put the value of F (λ)

$$\therefore \quad f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{e^{-a\lambda}}{\lambda} \sin \lambda x\, d\lambda$$

Step 3 : [Use the rule of D.U.I.S.]

$$\therefore \quad f'(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\partial}{\partial x}\left(\frac{e^{-a\lambda}}{\lambda} \sin \lambda x\right) d\lambda$$

Step 4 : Differentiate w.r.t. $x = \frac{2}{\pi} \int_{0}^{\infty} e^{-a\lambda} \cos x\lambda\, d\lambda$

Step 5 : Integrate w.r.t. λ $= \frac{2}{\pi}\left[\frac{e^{-a\lambda}}{a^2+x^2}(-a\cos\lambda x + x\sin\lambda x)\right]_0^{\infty}$

Step 6 : Put the limits for λ

$$f'(x) = \frac{2}{\pi}\frac{a}{a^2+x^2} \qquad \text{As } e^{-\infty} = 0,\ e^0 = 1$$

Step 7 : Integrating, w.r.t. x, we get

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + A$$

Step 8 : Put x = 0,

$$f(0) = 0 + A$$

Step 9 : To find f(0), put x = 0 in step 2.

$$f(x) = \left[\frac{2}{\pi}\int_0^{\infty}\frac{e^{-a\lambda}}{\lambda}\sin\lambda x\, d\lambda\right]_{x=0} = A \qquad \therefore A = 0$$

Step 10 : Put A = 0 in step 7

$$\therefore \quad f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

END...

UNIT III

5 Z-Transform (ZT)

5.1 : Z-Transform

I)

1) Definition : The Z-transform of sequence {f(k)} is defined as

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

2) For causal sequence {f(k)}, where $0 \le k < \infty$. The Z-transform is defined by

$$z[f(k)] = \sum_{k=0}^{\infty} f(k)\, z^{-k} = F(z)$$

where $z = x + iy$ is a complex number,

z is the operator of Z-transform and F(z) is the Z-transform of sequence {f(k)}.

Note

The Z-transform of a sequence {f(k)} exists if the series $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ is convergent. i.e. The series tends to a finite value for some values of z. These values of z for which the series is convergent lie within a region known as Region of convergence (ROC) in the z-plane.

II) Z-transforms of Some Standard Sequences

1) Discrete unit step function :

$$u(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \ge 0 \end{cases}$$

$$z[u(k)] = \sum_{k=-\infty}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} u(k)\, z^{-k} + \sum_{k=0}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0\, z^{-k} + \sum_{k=0}^{\infty} 1\, z^{-k}$$

$$= 0 + \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

which is convergent if $\left|\frac{1}{z}\right| < 1$ i.e. $|z| > 1$

Hence $z[u(k)] = \frac{z}{z-1}$ if $|z| > 1$

2) Unit impulse function :

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\therefore \quad z[\delta(k)] = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} (0+\ldots+0+1+0+\ldots+0) z^{-k}$$

$$z[\delta(k)] = 1 \cdot z^0 = 1$$

3) $\{f(k)\} = \{a^k\},\ k \geq 0$

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} (a\, z^{-1})^k$$

$$= 1 + a\, z^{-1} + (a\, z^{-1})^2 + \ldots$$

$$= \frac{1}{1 - az^{-1}} \quad \text{if } \left|az^{-1}\right| < 1$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \text{ if } \left|\frac{a}{z}\right| < 1 \quad \text{i.e. } |z| > |a|$$

Hence the region of convergence is the exterior of circle $x^2 + y^2 = a^2$

4) $\{f(k)\} = \{a^k\}\ :\ k < 0$

By definition

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^k\, z^{-k} + 0$$

Put $k = -r \ \therefore -k = r$

and $-\infty < k \leq -1 \Rightarrow \infty > -k \geq 1$ i.e. $1 \leq r < \infty$

$$\therefore \quad z[f(k)] = \sum_{r=1}^{\infty} a^{-r} z^r = \sum_{r=1}^{\infty} \left(\frac{z}{a}\right)^r$$

$$= \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \ldots$$

$$= \frac{z/a}{1-z/a} \quad \text{provided} \left|\frac{z}{a}\right|<1$$

$$= \frac{z}{a-z}, \quad |z|<|a|$$

Thus $z[f(k)] = z\,[a^k] = \dfrac{z}{a-z}$, $k<0$ and for $|z|<|a|$ and hence region of convergence is the interior of the circle $x^2+y^2 = a^2$.

5) $\{f(k)\} = \{a^{|k|}\}, \quad \forall\ k \in \mathbb{Z}$

By definition,

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{|k|} z^{-k} + \sum_{k=0}^{\infty} a^{|k|} z^{-k} \qquad \dots (5.1)$$

we have $|k| = k$ if $k \geq 0$

$= -k$ if $k<0$

Thus equation (5.1) becomes,

$$z[f(k)] = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \sum_{k=-\infty}^{-1} (az)^{-k} + \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= [az + (az)^2 + (az)^3 + \dots] + \left[1+\frac{a}{z}-\left(\frac{a}{z}\right)^2 + \dots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}} \quad \text{provided } |az|<1 \text{ and } \left|\frac{a}{z}\right|<1$$

$$= \frac{az}{1-az} + \frac{z}{z-a} \ : \ |z| < \frac{1}{|a|} \quad \text{and } |z| > |a|$$

Thus $\quad z\,[a^{|k|}] = \dfrac{az}{1-az} + \dfrac{z}{z-a}, \ \forall\ k$ and $\quad |a| < |z| < \dfrac{1}{|a|}$

Hence ROC is the annulus between the circles

$$x^2+y^2 = a^2 \quad \text{and} \quad x^2+y^2 = \frac{1}{a^2}$$

6) $\{f(k)\} = \{e^{ak}\}$ for $k \geq 0$

$$z[e^{ak}] = \sum_{k=-\infty}^{\infty} e^{ak} z^{-k} = \sum_{k=0}^{\infty} e^{+ak} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^a}{z}\right)^k = \frac{1}{1-e^a/z} \quad \text{for } \left|\frac{e^a}{z}\right| < 1$$

$$= \frac{z}{z-e^a} \quad \text{for } |z| > e^a$$

Thus $\quad z[e^{ak}] = \dfrac{z}{z-e^a} \quad$ for $|z| > e^a$

7) $\{f(k)\} = \sin ak, \quad k \geq 0$

$$z\,[\sin ak] = z\left[\frac{e^{iak}-e^{-iak}}{2i}\right]$$

$$= \frac{1}{2i}\{z\,[e^{iak}] - z\,[e^{-iak}]\}$$

$$= \frac{1}{2i}\left\{\frac{z}{z-e^{ia}} - \frac{z}{z-e^{-ia}}\right\} \text{for } |z| > \left|e^{\pm ia}\right| = 1$$

$$= \frac{1}{2i}\left\{\frac{z^2 - ze^{-ia} - z^2 + ze^{ia}}{z^2-(e^{ia}+e^{-ia})\,z + e^{ia}e^{-ia}}\right\}$$

$$= \frac{z\left(\dfrac{e^{ia} - e^{-ia}}{2i}\right)}{z^2 - 2z\cos a + 1}$$

$$= \frac{z \sin a}{z^2 - 2z\cos a + 1} \quad \text{for} \quad |z| > 1$$

8) $\{f(k)\} = \cos ak, \quad k \geq 0$

$$z\,[\cos ak] = z\left[\frac{e^{iak} + e^{-iak}}{2}\right] = \frac{1}{2}\{z\,[e^{iak}] + z\,[e^{-iak}]\}$$

$$= \frac{1}{2}\left\{\frac{z}{z - e^{ia}} + \frac{z}{z - e^{-ia}}\right\} \quad \text{for } |z| > \left|e^{\pm ia}\right|$$

$$= \frac{z}{2}\left\{\frac{z - e^{-ia} + z - e^{ia}}{z^2 - (e^{ia} + e^{-ia})\,z + e^{ia}\,e^{-ia}}\right\} \quad \text{for } |z| > 1$$

$$= \frac{z\left[z - \left(\dfrac{e^{ia} + e^{-ia}}{2}\right)\right]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z\,[z - \cos a]}{z^2 - 2z\cos a + 1} \quad \text{for} \quad |z| > 1$$

9) $\{f(k)\} = \sinh ak, \quad k \geq 0$

$$z\,[\sinh ak] = z\left[\frac{1}{2}(e^{ak} - e^{-ak})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{a}} - \frac{z}{z - e^{-a}}\right]$$

for $|z| > |e^a|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z - e^{a} - z + e^{a}}{z^2 - (e^{a} + e^{-a})\,z + e^{a}\,e^{-a}}\right]$$

for $|z| >$ maximum of $|e^a|$ and $|e^{-a}|$

$$= \frac{z\left(\dfrac{e^{a} - e^{-a}}{2}\right)}{z^2 - 2z\cosh a + 1}$$

$$= \frac{z \sinh a}{z^2 - 2z\cosh a + 1}$$

for $|z| > \max(|e^a|, |e^{-a}|)$

10) $\{f(k)\} = \cosh ak, \quad k \geq 0$

$$z\,[\cosh ak] = z\left[\frac{e^{ak} + e^{-ak}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{a}} + \frac{z}{z - e^{-a}}\right]$$

for $|z| > |e^a|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z - e^{-a} + z - e^{a}}{z^2 - (e^{a} + e^{-a})\,z + 1}\right]$$

$$= \frac{z\left[z - \left(\dfrac{e^{a} + e^{-a}}{2}\right)\right]}{z^2 - 2z\cosh a + 1} = \frac{z\,[z - \cosh a]}{z^2 - 2z\cosh a + 1}$$

for $|z| >$ max of $|e^a|$ and $|e^{-a}|$

11) a) $\{f(k)\} = {}^nC_k \quad \text{for } 0 \leq k \leq n$

$$z[{}^nC_k] = \sum_{k=-\infty}^{\infty} {}^nC_k\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n} {}^nC_k\, z^{-k} + \sum_{k=n+1}^{\infty} 0$$

$$= \sum_{k=0}^{n} {}^nC_k\, z^{-k}$$

$$= \sum_{k=0}^{n} {}^nC_k \; (z^{-1})^k$$

$$= (1+z^{-1})^n \qquad \text{for } |z^{-1}| < 1$$

$$= \left(1+\frac{1}{z}\right)^n \qquad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \left(\frac{z+1}{z}\right)^n \qquad \text{for } |z| > 1$$

11) b)

$$\{f(k)\} = {}^kC_n \qquad \text{for } k \geq n \geq 0$$

$$z[{}^kC_n] = \sum_{k=-\infty}^{\infty} {}^kC_n \; z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n-1} 0 + \sum_{k=n}^{\infty} {}^kC_n \; z^{-k}$$

$$= \sum_{r=0}^{\infty} {}^{(n+r)}C_n \; z^{-(n+r)} \quad \text{put } k-n=r$$

$$= \sum_{r=0}^{\infty} {}^{n+r}C_r \; z^{-n} \; z^{-r} \quad (\because {}^nC_r = {}^nC_{n-r})$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r \; z^{-r}$$

$$= z^{-n} (1-z^{-1})^{-(n+1)}$$

$$= z^{-n} \left(\frac{z}{z-1}\right)^{n+1} \qquad \text{for } k \geq n \geq 0$$

and $|z| < 1$

$$z[{}^kC_n] = z^{-n} \left(\frac{z}{z-1}\right)^{n+1} \qquad \text{for } |z| > 1$$

$k \geq n \geq 0$

11) c)

$$\{f(k)\} = {}^{k+n}C_n$$

$$Z[{}^{k+n}C_n] = \sum_{k=0}^{\infty} {}^{k+n}C_n \; z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(k+n)(k+n-1)\dots(k+1)}{n!} z^{-k}$$

$$= 1 + (n+1)\left(\frac{1}{z}\right) + \frac{(n+1)(n+2)}{2!}\left(\frac{1}{z}\right)^2 + \dots$$

$$= \left[1-\frac{1}{z}\right]^{-(n+1)} \qquad \text{(by above example)}$$

for $|z| > 1$

$$= \left[\frac{z}{z-1}\right]^{n+1} \qquad \text{for } |z| > 1$$

Examples

Q.1 Find the Z-transform of the following sequence.

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \geq 0 \end{cases}$$

☞ **[SPPU : May-2000, Dec.-05, 07]**

Ans. : We have,

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \geq 0 \end{cases}$$

$\therefore$ By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k}$$

... using f(k)

$$= \sum_{r=\infty}^{1} 2^{-r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^{k}$$

...Putting k = – r in first summation

$$= \sum_{r=1}^{\infty} \left(\frac{z}{2}\right)^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^{k}$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{3z}}, \left|\frac{z}{2}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

Sum of infinite G.P.s $S_\infty = \frac{a}{1-r}, |r| < 1$

$$= \frac{2z}{2-z} + \frac{3z}{3z-1}, |z| < 2,$$

$$|z| > \frac{1}{3} \text{ i.e. } \frac{1}{3} < |z| < 2$$

Q.2 Find Z-transform of
$f(k) = 3(2^k) + 4(-1)^k;\ k \geq 0$ [SPPU : May-13]

Ans. : We have $f(k) = 3(2^k)+4(-1)^k$

$$Z\{f(k)\} = Z\{3(2^k)+4(-1)^k\};\ k \geq 0$$

$$= \frac{3z}{z-2}+\frac{4z}{z+1};\ |z| > 2,\ |z| > 1$$

$$= z\left\{\frac{3}{z-2}+\frac{4}{z+1}\right\};\ |z| > 2$$

$$= z\left\{\frac{3(z+1)+4(z-2)}{(z+1)(z-2)}\right\};\ |z| > 2$$

$$Z\{f(k)\} = \frac{2(7z-5)}{(z+1)(z-2)};\ |z| > 2$$

∴ The region of convergence is exterior of the circle $x^2+y^2=2^2$.

Q.3 Find the Z-transform and region of convergence
$f(k) = \left(\frac{1}{2}\right)^{|k|}$ for all k. [SPPU : May-04, 05, 06, 12, 14, Dec.-10]

Ans. : We have,

$$f(k) = \left(\frac{1}{2}\right)^{|k|}, \quad \forall\ k$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{r=\infty}^{1} \left(\frac{1}{2}\right)^{|-r|} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

... putting k = – r in first

$$= \sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^{r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k}$$

... ∵ k and r are positive

$$= \sum_{r=1}^{\infty} \left(\frac{z}{2}\right)^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2z}\right)^{k}$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{2z}},\ \left|\frac{z}{2}\right| < 1,\ \left|\frac{1}{2z}\right| < 1$$

... sum of infinite G.P.

$$= \frac{z}{2-z} + \frac{2z}{2z-1}, |z| < 2, |z| > \frac{1}{2} \text{ i.e. } \frac{1}{2} < |z| < 2$$

i.e. the ROC is the annulus between the circles $x^2+y^2=\left(\frac{1}{2}\right)^2$ and $x^2+y^2=2^2$ in the z-plane.

Q.4 Find Z-transform of $f(k) = \left(\frac{1}{4}\right)^{|k|} \forall k$

SPPU : Dec.-18, Marks 4]

Ans. : I) We have $f(k) = \left(\frac{1}{4}\right)^{|k|}; \ \forall k$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$$

(Put k = – r in first)

$$= \sum_{r=\infty}^{1} \left(\frac{1}{4}\right)^{r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} z^{-k}$$

$$= \sum_{r=1}^{\infty} \left(\frac{z}{4}\right)^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{4z}\right)^{k}$$

$$= \frac{\frac{z}{4}}{1-\frac{z}{4}} + \frac{1}{1-\frac{1}{4z}}; \ \left|\frac{z}{4}\right| < 1, \ \left|\frac{1}{4z}\right| < 1$$

$$= \frac{z}{4-z} + \frac{4z}{4z-1}; \ |z| < 4 \text{ and } |z| > \frac{1}{4}$$

i.e. The ROC is the annulus between the circles $x^2 + y^2 = \left(\frac{1}{4}\right)^2$ and $x^2 + y^2 = 4^2$ in the z-plane.

5.2 : Properties of Z-transforms

1) Definition

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k)\, z^{-k},$$

for causal sequence $(k \geq 0)$

2) Linearity

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

3) Change of Scale

$$Z\left\{a^{k} f(k)\right\} = F\left(\frac{z}{a}\right)$$

4) Corollary of Change of Scale

$$Z\left\{e^{-ak} f(k)\right\} = F\left(e^{a} z\right)$$

5) Shifting Property

a) For both sided sequence :

$$Z\{f(k+n)\} = z^{n} F(z) \qquad \text{... Left shifting}$$

$$Z\{f(k-n)\} = z^{-n} F(z) \qquad \text{... Right shifting}$$

b) For causal sequence $(k \geq 0)$:

$$Z\{f(k+n)\} = z^{n} F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$$\therefore \quad Z\{f(k+1)\} = z\,F(z) - z\,f(0)$$

$$Z\{f(k+2)\} = z^{2} F(z) - z^{2} f(0) - z\,f(1) \text{ etc.}$$

$$Z\{f(k-n)\} = z^{-n} F(z)$$

$$\therefore \quad Z\{f(k-1)\} = z^{-1} F(z)$$

$$Z\{f(k-2)\} = z^{-2} F(z) \text{ etc.}$$

6) Multiplication by k

$$Z\{k\, f(k)\} = -z \frac{d}{dz} F(z)$$

In general,

$$Z\left\{k^n f(k)\right\} = \left(-z \frac{d}{dz}\right)^n F(z)$$

7) Division by k

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^{\infty} \frac{F(z)}{z} dz$$

8) Initial Value Theorem (for one sided sequence e.g. $k \geq 0$)

$$f(0) = \lim_{z \to \infty} F(z)$$

9) Final Value Theorem (for one sided sequence e.g. $k \geq 0$)

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

10) Convolution Theorem

$$Z[\{f(k)\} * \{g(k)\}] = F(z) \cdot G(z)$$

where, convolution of $\{f(k)\}$ and $\{g(k)\}$ is

$$\{f(k)\} * \{g(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

and for casual sequence ($k \geq 0$)

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\, g(k-m)$$

Examples

Q.5 Find $Z\{f(k)\}$ if

i) $f(k) = 2^k \cos(3k+2)$, $k \geq 0$

ii) $2^{-t} \sin at$, $t \geq 0$ ☞ [SPPU : Dec.-01, 12, May-12, 16, Marks 4]

Ans. :

i) We have, $f(k) = 2^k \cos(3k+2)$, $k \geq 0$

[Form : $Z\{a^k f(k)\}$. Hence, use change of scale property]

$$\cos(3k+2) = \cos 3k \cos 2 - \sin 3k \sin 2$$

$$\therefore \quad z\{\cos(3k+2)\} = \cos 2\, z[\cos 3k] - \sin 2\, z[\sin 3k]$$

$$= \cos 2\left[\frac{2(z-\cos 3)}{z^2 - 2z\cos 3 + 1}\right] - \sin 2\left[\frac{z \sin 3}{z^2 - 2z\cos 3 + 1}\right]$$

$$= G(z) \text{ (say)} \qquad \ldots (1)$$

$\therefore$ By change of scale property

i.e. $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$ we have,

$$Z\{2^k \cos(3k+2)\} = G\left(\frac{z}{2}\right)$$

$$= \cos 2\left[\frac{\frac{z}{2}\left(\frac{z}{2} - \cos 3\right)}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1}\right] - \sin 2\left[\frac{\frac{z}{2}\sin 3}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1}\right]$$

... using G(z) from (1)

$$= \cos 2\left[\frac{\frac{z}{4}(z - 2\cos 3)}{\frac{z^2 - 4z\cos 3 + 4}{4}}\right] - \sin 2\left[\frac{\frac{z}{4}(2\sin 3)}{\frac{z^2 - 4z\cos 3 + 4}{4}}\right]$$

$$= \frac{\cos 2z(z-2\cos 3) - \sin 2(2z)\sin 3}{z^2 - 4z\cos 3 + 4}$$

ii) We have, $f(t) = 2^{-t} \sin at$, $t \geq 0$

$$= \left(\frac{1}{2}\right)^t \sin at,\ t \geq 0$$

$\therefore$ Its Z-transform is

$$Z\{f(t)\} = Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\},\ t \geq 0$$

$$= \frac{\left(\frac{1}{2}\right) z \sin a}{z^2 - 2\left(\frac{1}{2}\right) z\cos a + \left(\frac{1}{2}\right)^2},$$

$|z| > \left|\frac{1}{2}\right|$ $\because$ Standard Result :

$$Z\{c^k \sin \alpha k\} = \frac{cz \sin \alpha}{z^2 - 2\,cz \cos \alpha + c^2},\ |z| > |c|$$

$$= \frac{1}{2} \frac{z \sin a}{\left(z^2 - z \cos a + \frac{1}{4}\right)} = \frac{1}{2} \frac{z \sin a}{\frac{1}{4}\left(4z^2 - 4z \cos a + 1\right)}$$

$$= \frac{2z \sin a}{4z^2 - 4z \cos a + 1},\ |z| > \frac{1}{2}$$

Alternately we can write $Z\{\sin at\}$ first, then use the change of scale property to find $Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\}$.

Q.6 Find $Z\{f(k)\}$ if

$f(k) = e^{-ak} \cos bk,\ k \geq 0$

Ans. : [Form : $Z\{e^{-ak} f(k)\}$. Hence use, the property (case of change of scale, $a \to e^{-a}$).]

We have,

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1},$$

$$|z| > 1 \qquad \text{... (1) Standard result}$$

Now, $\because Z\{e^{-ak} f(k)\} = F(e^a z)$

... Property where, $Z\{f(k)\} = F(z)$

$$\therefore \quad Z\{e^{-ak} \cos bk\} = \frac{e^a z\left(e^a z - \cos b\right)}{\left(e^a z\right)^2 - 2\left(e^a z\right) \cos b + 1},$$

$$\left|e^a z\right| > 1$$

... Replacing z by $(e^a z)$ in (1)

$$= \frac{e^a \cdot e^a z\left(z - e^{-a} \cos b\right)}{e^{2a} \cdot z^2 - 2\,e^a\, z \cos b + 1}$$

$$= \frac{e^{2a} \cdot z\left(z - e^{-a} \cos b\right)}{e^{2a}\left(z^2 - 2\,e^{-a}\, z \cos b + e^{-2a}\right)}$$

$$= \frac{z\left(z - e^{-a} \cos b\right)}{z^2 - 2\,e^{-a}\, z \cos b + e^{-2a}},\ |z| > \frac{1}{e^a}$$

Q.7 Find the Z-transform of the following. Also, find the ROC :
$f(k) = k\,5^k$ **[SPPU : Dec.-02, 04, 12, May-06, 12]**

Ans. : We have,

$$f(k) = k\,5^k, \text{ Assume } k \geq 0$$

Now, $$Z\{5^k\} = \frac{z}{z-5},\ |z| > 5$$

$$\therefore \quad Z\{k\,5^k\} = -z \frac{d}{dz}\left(\frac{z}{z-5}\right) \qquad \text{... Multiplication by k}$$

$$= -z\left[\frac{(z-5) - z}{(z-5)^2}\right] = \frac{5z}{(z-5)^2},\ |z| > 5$$

Q.8 Find $Z\{f(k)\}$ if $f(k) = 4^k e^{-5k} k;\ k \geq 0$ **[SPPU : May-13]**

Ans. : Let $g(k) = k$

$$z\{g(k)\} = z\{k\} = \left(-z \frac{d}{dz}\right) z\{1\}$$

$$= \left(-z \frac{d}{dz}\right)\left(\frac{z}{z-1}\right);\ |z| > 1$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= (-z)\left[\frac{(z-1) - z(1)}{(z-1)^2}\right];\ |z| > 1$$

$$= (-z) \frac{-1}{(z-1)^2}$$

$$z\{g(k)\} = \frac{z}{(z-1)^2};\ |z| > 1$$

$$\therefore \qquad z\{4^k k\} = \frac{\frac{z}{4}}{\left(\frac{z}{4}-1\right)^2}; \ |z| > 1$$

Now $$z\{4^k k\, e^{-5k}\} = \frac{\frac{ze^5}{4}}{\left(\frac{ze^5}{4}-1\right)^2}; \ |z| > 1$$

$$= \frac{4e^5 z}{(ze^5-4)^2}; \ |z| > 1$$

Q.9 Find $Z\{f(k)\}$ if

i) $f(k) = (k+1)\, 2^k$ ☞ [SPPU : Dec.-10, 11, May-11]

ii) $f(k) = k^2 e^{-ak}, \ k \geq 0$ ☞ [SPPU : Dec.-09, May-12]

Ans. : [Here we have the forms $Z\{k f(k)\}$. Hence use the property of multiplication by k.]

i) We have, $$f(k) = (k+1)\, 2^k$$

$$= k\, 2^k + 2^k, \ k \geq 0 \qquad \text{(Assumption)}$$

$$\therefore \qquad Z\{f(k)\} = Z\{k\, 2^k + 2^k\}$$

$$= Z\{k\, 2^k\} + Z\{2^k\} \qquad \text{... Linearity property}$$

$$= \left(-z\frac{d}{dz}\right)[Z\{2^k\}] + \frac{z}{z-2}$$

$$\text{...} \because Z\{k f(k)\} = -z\frac{d}{dz}F(z)$$

and $$Z\{a^k\} = \frac{z}{z-a}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-2}\right) + \frac{z}{z-2}$$

$$= -z\left[\frac{(z-2)\,1 - z(1)}{(z-2)^2}\right] + \frac{z}{z-2}$$

$$= \frac{2z}{(z-2)^2} + \frac{z}{z-2}$$

$$= \frac{2z + z(z-2)}{(z-2)^2} = \frac{z^2}{(z-2)^2}$$

ii) We have, $f(k) = k^2 e^{-ak}, \ k \geq 0$

$$\text{Now, } Z\left\{e^{-ak}\right\} = Z\{(e^{-a})^k\}$$

$$= \frac{z}{z-e^{-a}} \qquad \text{...} \because Z\left\{a^k\right\} = \frac{z}{z-a}$$

$$\therefore \quad Z\left\{k\, e^{-ak}\right\} = -z\frac{d}{dz}Z\left\{e^{-ak}\right\} \quad \text{...} \because Z\{k f(k)\} = -z\frac{d}{dz}F(z)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-e^{-a}}\right)$$

$$= -z\left[\frac{\left(z-e^{-a}\right)1 - z(1)}{\left(z-e^{-a}\right)^2}\right]$$

$$= \frac{z\, e^{-a}}{\left(z-e^{-a}\right)^2} \qquad \text{... (1)}$$

Now, using $Z\{k f(k)\} = -z\dfrac{d}{dz}F(z)$ again, we get

$$Z\left\{k^2 e^{-ak}\right\} = Z\left\{k\left(ke^{-ak}\right)\right\} = -z\frac{d}{dz}Z\left\{ke^{-ak}\right\}$$

$$= -z\frac{d}{dz}\left[\frac{z\, e^{-a}}{\left(z-e^{-a}\right)^2}\right] \qquad \text{... using (1)}$$

$$= -z\, e^{-a}\frac{d}{dz}\frac{z}{\left(z-e^{-a}\right)^2}$$

$$= -z e^{-a}\left[\frac{(z-e^{-a})^2\, 1 - z\; 2(z-e^{-a})\, 1}{(z-e^{-a})^4}\right]$$

$$= -z e^{-a}\left[\frac{(z-e^{-a}) - 2z}{(z-e^{-a})^3}\right]$$

$$= -z e^{-a}\left[\frac{-z-e^{-a}}{(z-e^{-a})^3}\right]$$

$$\therefore \quad Z\left\{k^2 e^{-ak}\right\} = \frac{z e^{-a}(z+e^{-a})}{(z-e^{-a})^3}$$

OR, we can directly write

$$Z\left\{k^2 e^{-ak}\right\} = \left(-z\frac{d}{dz}\right)^2 Z\left\{e^{-ak}\right\} \quad \text{... multiplication by } k^2$$

Q.10 Find the Z-transform of the following
$f(k) = \frac{\sin\alpha\, k}{k}$, $k > 0$
☞ [SPPU : May-2000, 05,12, 15, Dec.-2000, 02, 11, Marks 5]

Ans. : [Form is $Z\left\{\frac{f(k)}{k}\right\}$. Hence, use the property of division by k.]

We have, $\quad f(k) = \frac{\sin\alpha k}{k},\ k > 0 \quad$ i.e. $\ k \geq 1$

Now, $\quad Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1},\quad k \geq 0$

... (1) ... Standard result

which is same for the case $k > 0$ i.e. $k \geq 1$ also.

Now, by property of division by k i.e.

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^\infty \frac{F(z)}{z}\,dz \qquad \text{where } F(z) = Z\{f(k)\}$$

$$\therefore \quad Z\left\{\frac{\sin\alpha k}{k}\right\} = \int_z^\infty \frac{Z\{\sin\alpha k\}}{z}\,dz$$

$$= \int_z^\infty \frac{1}{z}\,\frac{z\sin\alpha}{(z^2 - 2z\cos\alpha + 1)}\,dz \quad \text{... using (1)}$$

$$= \sin\alpha\int_z^\infty \frac{1}{z^2 - 2z\cos\alpha + (\cos^2\alpha + \sin^2\alpha)}\,dz \quad \text{... \textbf{Note this step}}$$

$$= \sin\alpha\int_z^\infty \frac{1}{(z-\cos\alpha)^2 + \sin^2\alpha}\,dz$$

$$= \sin\alpha\left[\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)\right]_z^\infty \quad \text{... } \because \int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

$$\therefore \quad Z\left\{\frac{\sin\alpha k}{k}\right\} = \cot^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

Q.11 Find the Z-transform of the following. Also, find the ROC :
$f(k) = \frac{2^k}{k}$, $k \geq 1$
☞[SPPU : Dec.-16, May-17, Marks 4]

Ans. : We have

$$f(k) = \frac{2^k}{k},\ k \geq 1$$

Now, $\quad Z\left\{2^k\right\} = \sum_{k=1}^{\infty} 2^k z^{-k} \quad$ ... by definition

and note that $k \neq 0$

$$= \sum_1^\infty \left(\frac{2}{z}\right)^k = \frac{\frac{2}{z}}{1-\frac{2}{z}},\ \left|\frac{2}{z}\right| < 1 \quad \text{... sum of GP}$$

$$= \frac{2}{z-2}, |z| > 2$$

$$\therefore \quad Z\left\{\frac{2^k}{k}\right\} = \int_z^\infty \frac{1}{z} Z\left\{2^k\right\} dz \quad \text{... Division by k}$$

$$= \int_z^\infty \frac{1}{z} \cdot \frac{2}{(z-2)} dz = \int_z^\infty \left(\frac{1}{z-2} - \frac{1}{z}\right) dz$$

$$= [\log(z-2) - \log z]_z^\infty = \left[\log\frac{z-2}{z}\right]_z^\infty$$

$$= \left[\log\left(1-\frac{2}{z}\right)\right]_z^\infty$$

$$= \log(1-0) - \log\left(1-\frac{2}{z}\right)$$

$$= -\log\frac{z-2}{z} = \log\frac{z}{z-2}, |z| > 2$$

Q.12 Find $Z\{f(k)\}$ if $f(k) = (k+1)(k+2)a^k, k \geq 0$

[SPPU : May-18, Marks 4]

Ans. : We have,

$f(k) = (k+1)(k+2)a^k = \left(k^2 + 3k + 2\right) a^k, k \geq 0$

$$\therefore \quad Z\{f(k)\} = Z\left\{\left(k^2 + 3k + 2\right) a^k\right\}$$

$$= Z\left\{k^2 a^k\right\} + 3Z\left\{ka^k\right\} + 2Z\left\{a^k\right\} \quad \text{... (1)}$$

... using Linearity property

Now, $$Z\left\{a^k\right\} = \frac{z}{z-a} \quad \text{... (2)}$$

... Standard result

$$\therefore \quad Z\left\{ka^k\right\} = -z\frac{d}{dz}\left(\frac{z}{z-a}\right) \quad \text{... } \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

i.e. Property of multiplication by k

$$= -z\left[\frac{(z-a)1 - z(1)}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2} \quad \text{... (3)}$$

Using the property of multiplication by k again, we get,

$$Z\left\{k^2 a^k\right\} = Z\left\{k\left(k a^k\right)\right\}$$

$$= -z\frac{d}{dz} Z\left\{ka^k\right\}$$

$$= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right] \quad \text{... using (3)}$$

$$= -az\left[\frac{(z-a)^2 1 - z\cdot 2(z-a)}{(z-a)^4}\right]$$

$$= -az\left[\frac{(z-a) - 2z}{(z-a)^3}\right]$$

$$\therefore \quad Z\left\{k^2 a^k\right\} = \frac{az(a+z)}{(z-a)^3} \quad \text{... (4)}$$

$\therefore$ Substituting from (2), (3) and (4) in (1) we get

$$Z\{f(k)\} = \frac{az(a+z)}{(z-a)^3} + 3\frac{az}{(z-a)^2} + 2\frac{z}{z-a}$$

$$= \frac{(a^2 z + az^2) + 3az(z-a) + 2z(z-a)^2}{(z-a)^3}$$

$$= \frac{a^2 z + az^2 + 3az^2 - 3a^2 z + 2z(z^2 - 2az + a^2)}{(z-a)^3}$$

$$= \frac{2z^3}{(z-a)^3}$$

To check : By direct Standard Result,

$$Z\left\{\frac{(k+1)(k+2)a^k}{2!}\right\} = \frac{z^3}{(z-a)^3}$$

$$\therefore Z\{(k+1)(k+2)a^k\} = 2!\frac{z^3}{(z-a)^3} = \frac{2z^3}{(z-a)^3}$$

Q.13 Find Z-transform of $f(k) = \frac{2^k}{k}, k \geq 1$. ☞ [SPPU : May-19, Marks 2]

Ans. :

$$Z\{2^k\} = \sum_{k=1}^{\infty} 2^k z^{-k}$$

$$= \frac{2z^{-1}}{1-2z^{-1}}, \ |z| > 2$$

$$= \frac{2}{z-2}, \ |z| > 2$$

$$\therefore \quad Z\left\{\frac{2^k}{k}\right\} = \int_z^{\infty} z^{-1}\left(\frac{2}{z-2}\right)dz$$

$$= 2\int_z^{\infty} \frac{1}{z(z-2)}dz$$

$$= 2\int_z^{\infty}\left(\frac{-1/2}{z} + \frac{1/2}{z-2}\right)dz$$

$$= [-\log z + \log(z-2)]_z^{\infty}$$

$$= -\log\left(\frac{z-2}{z}\right)$$

$$= -\log(1-2z^{-1}), \ |z| > 2$$

5.3 : Inverse Z-transform

1) If $Z[f(k)] = F(z)$ then

$Z^{-1}F(z) = [f(k)]$ determines the sequence $\{f(k)\}$ which generates the given Z-transform is known as inverse Z-transform.

☞ [SPPU : May-99, 2000, 01, 02, 03, 04, 05, 06, 08, May-09, 11, 12, 13, Dec.-99, 2000, 01, 02, 03, 04, 05, 06, Dec.-07, 08, 09, 10, 11, 12]

For example $Z\{a^k\}\ k \geq 0 = \frac{z}{z-a}$ for $|z| > |a|$

$$\therefore \quad z^{-1}\left\{\frac{z}{z-a}\right\}_{|z|>|a|} = a^k \text{ for } k \geq 0.$$

Using standard results of Z-transform we can form the following table of inverse Z-transforms.

2) Table of Inverse Z-transform

Sr. No.	F (z)	f (k) = Z^{-1} F(z) ($\|z\| > \|a\|$, $k \geq 0$)	f (k) = Z^{-1} F(z) ($\|z\| < \|a\|$, $k < 0$)
1.	$\frac{z}{z-a}$	a^k	$-a^k$
2.	$\frac{z}{(z-a)^2}$	ka^{k-1}	$-ka^{k-1}$
3.	$\frac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$
4.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(k+1)(k+2)a^k$	$-\frac{1}{2!}(k+1)(k+2)a^k$
5.	In general, $\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}\begin{bmatrix}(k+1)(k+2)\\ \ldots(k+(n-1))\end{bmatrix}a^k$	$-\frac{1}{(n-1)!}\begin{bmatrix}(k+1)(k+2)\\ \ldots(k+(n-1))\end{bmatrix}a^k$

6.	$\frac{1}{z-a}$	$a^{k-1}\,U(k-1)$	$-a^{k-1}\,U(-k)$
7.	$\frac{1}{(z-a)^2}$	$(k-1)a^{k-2}\,U(k-2)$	$-(k-1)\,a^{k-2}$ $U(-k+1)$
8.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(k-2)(k-1)$ $a^{k-3}U(k-3)$	$-\frac{1}{2}(k-2)(k-1)$ $a^{k-3}U(-k+2)$
9.	$\frac{z}{z-1}$	U (k)	
10.	$\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert > 1$	$\sin\alpha k$	
11.	$\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert>1$	$\cos\alpha k$	

Q.14 Show that $Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right] = \{x_k\}$ **for** $|z| > \frac{1}{2}$ **where**

$x_k = 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right],\ k \geq 1$

[SPPU : May-18, Marks 4]

Ans. : We have

$$X(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\ |z| > \frac{1}{2}$$

$$= \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$\text{... Partial Fractions} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$$

where $A = \frac{1}{\frac{1}{2}-\frac{1}{3}} = 6,\ B = \frac{1}{\frac{1}{3}-\frac{1}{2}} = -6$

$$= \frac{6}{z\left(1-\frac{1}{2z}\right)} - \frac{6}{z\left(1-\frac{1}{3z}\right)} \quad \ldots \because |z| > \frac{1}{2} \therefore |z| > \frac{1}{3}$$

$$\therefore \left|\frac{1}{2z}\right| < 1 \text{ and } \left|\frac{1}{3z}\right| < 1$$

$$= \frac{6}{z}\left(1-\frac{1}{2z}\right)^{-1} - \frac{6}{z}\left(1-\frac{1}{3z}\right)^{-1}$$

$$= \frac{6}{z}\left[1+\left(\frac{1}{2z}\right)+\left(\frac{1}{2z}\right)^2+\ldots\right] - \frac{6}{z}\left[1+\left(\frac{1}{3z}\right)+\left(\frac{1}{3z}\right)^2+\ldots\right]$$

$$= 6\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{2^2z^3}+\ldots\right] - 6\left[\frac{1}{z}+\frac{1}{3z^2}+\frac{1}{3^2z^3}+\ldots\right]$$

$$= 6\sum_{k=1}^{\infty}\frac{1}{2^{k-1}z^k} - 6\sum_{k=1}^{\infty}\frac{1}{3^{k-1}z^k} \quad \ldots \text{ General terms}$$

$$= 6\sum_{k=1}^{\infty}\left[\left(\frac{1}{2}\right)^{k-1}z^{-k} - \left(\frac{1}{3}\right)^{k-1}z^{-k}\right]$$

$$= \sum_{k=1}^{\infty} 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right] z^{-k} \quad \ldots (1)$$

$\therefore \quad Z^{-1}X(z) = \{x_k\}$ where

$\{x_k\}$ = coeff. of z^{-k} in (1)

$$= 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right],\quad k \geq 1 \text{ for } |z| > \frac{1}{2}$$

Q.15 Find inverse Z-transform of $f(z) = \dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ **for** $\frac{1}{5} < |z| < \frac{1}{4}$

☞ **[Dec.-18, Marks 4]**

Ans. :

i) We have $F(z) = \dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$

$$\therefore \quad \frac{F(z)}{z} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{5}} = \frac{\frac{1}{20}}{z-\frac{1}{4}} + \frac{-\frac{1}{20}}{z-\frac{1}{5}}$$

$$F(z) = \frac{1}{20}\left[\frac{z}{z-\frac{1}{4}} - \frac{z}{z-\frac{1}{5}}\right]$$

$$\therefore \quad f(k) = z^{-1}[F(z)] = \frac{1}{20}\left\{-\left(\frac{1}{4}\right)^k - \left(\frac{1}{5}\right)^k\right\}$$

$$k \geq 0 \text{ and } \frac{1}{5} < |z| < \frac{1}{4}$$

$$= \frac{1}{20}\left\{-\left(\frac{1}{4}\right)^k - \left(\frac{1}{5}\right)^k\right\}$$

$$(k < 0) \quad (k \geq 0)$$

$$\therefore \quad f(k) = -\frac{1}{20}\left(\frac{1}{4}\right)^k; \quad k < 0$$

$$= -\frac{1}{20}\left(\frac{1}{5}\right)^k; \quad k \geq 0$$

Q.16 Find the Inverse Z-transform of $\dfrac{1}{(z-a)^2}$ **if** $|z| < |a|$

☞ **[SPPU : Dec.-16, Marks 4]**

Ans. : We have

$$F(z) = \frac{1}{(z-a)^2}, \; |z| < |a|, \; \therefore \left|\frac{z}{a}\right| < 1$$

$$\therefore \quad F(z) = \frac{1}{\left[a\left(\frac{z}{a} - 1\right)\right]^2} \quad \text{... Taking out 'a' common}$$

$$= \frac{1}{\left[-a\left(1 - \frac{z}{a}\right)\right]^2}$$

$$= \frac{1}{a^2}\left(1 - \frac{z}{a}\right)^{-2} \quad \text{... } 1 - r = 1 - \frac{z}{a}$$

$$= \frac{1}{a^2}\left[1 + 2\left(\frac{z}{a}\right) + 3\left(\frac{z}{a}\right)^2 + 4\left(\frac{z}{a}\right)^3 + \ldots\right]$$

$$\text{... } \because (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \frac{1}{a^2}\left[1 + \frac{2}{a}z + \frac{3}{a^2}z^2 + \frac{4}{a^3}z^3 + \ldots\right]$$

$$= \frac{1}{a^2} + \frac{2}{a^3}z + \frac{3}{a^4}z^2 + \frac{4}{a^5}z^3 + \ldots$$

$$\therefore \quad F(z) = \sum_{k=0}^{\infty} \frac{(k+1)}{a^{k+2}} z^k \quad \text{... general term}$$

$$\therefore \quad \text{Coefficient of } z^k = \frac{k+1}{a^{k+2}}, \; k \geq 0$$

$$\therefore \quad \text{Coefficient of } z^{-k} = \frac{-k+1}{a^{-k+2}}, \; -k \geq 0 \text{ i.e. } k \leq 0$$

... Replacing k by (– k)

$$\therefore \quad f(k) = \text{Coefficient of } z^{-k} = \frac{1-k}{a^{2-k}}, \; k \leq 0$$

$\therefore \quad Z^{-1}\left[\frac{1}{(z-a)^2}\right] = \{f(k)\} = \left\{\frac{1-k}{a^{2-k}}\right\}, \quad k \le 0$ if $|z| < |a|$

Q.17 Find inverse Z-transform of the following $\frac{3z^2+2z}{z^2-3z+2}$, $1 < |z| < 2$

☞ [SPPU : May-16, Marks 4]

Ans. : We have,

$$F(z) = \frac{3z^2+2z}{z^2-3z+2}, \ 1<|z|<2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{z^2-3z+2} = \frac{3z+2}{(z-2)(z-1)}$$

$$= \frac{8}{z-2} + \frac{(-5)}{z-1}$$

Partial Fractions : $A = \frac{3(2)+2}{2-1} = 8$

$$B = \frac{3(1)+2}{1-2} = -5$$

$$\therefore \quad F(z) = 8\left(\frac{z}{z-2}\right) - 5\left(\frac{z}{z-1}\right)$$

$$\therefore \quad \{f(k)\} = 8\ Z^{-1}\left(\frac{z}{z-2}\right) - 5\ Z^{-1}\left(\frac{z}{z-1}\right)$$

... $|z| < 2, |z| > 1$

$$= 8\{-2^k\} - 5\{1^k\}$$

(k < 0) (k ≥ 0)

$$= -8\{2^k\} - 5\{1\}$$

(k < 0) (k ≥ 0)

i.e. $\quad f(k) = -8 \cdot 2^k, \ k < 0$

$$= -5, \ k > 0$$

Q.18 Find $Z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right]$, $|z| > 1$

☞ [SPPU : Dec.-17, May-17, Marks 4]

Ans. : We have,

$$F(z) = \frac{z(z+1)}{(z-1)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{(z-1)+2}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2}$$

$$\therefore \quad F(z) = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$\therefore$ Inverting, $\quad \{f(k)\} = Z^{-1}\ F(z)$

$$= Z^{-1}\left(\frac{z}{z-1}\right) + 2\ Z^{-1}\left[\frac{z}{(z-1)^2}\right],$$

$|z| > 1$

$$= \{1^k\} + 2\{k(1)^{k-1}\}, \ k \ge 0$$

$$\ldots \because Z^{-1}\ \frac{z}{(z-a)^2} = ka^{k-1}, \ k \ge 0 \text{ for } |z| > |a|$$

$$\therefore \quad \{f(k)\} = \{1+2k\}, \ k \ge 0$$

Q.19 Find $Z^{-1}\left[\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]$, $|z| > 1$

☞ [SPPU : May-15, Marks 4]

$$\therefore \quad Z^{-1}\left[\frac{1}{(z-a)^2}\right] = \{f(k)\} = \left\{\frac{1-k}{a^{2-k}}\right\}, \qquad k \le 0 \text{ if } |z| < |a|$$

Q.17 Find inverse Z-transform of the following $\frac{3z^2+2z}{z^2-3z+2}$, $1 < |z| < 2$

☞ **[SPPU : May-16, Marks 4]**

Ans. : We have,

$$F(z) = \frac{3z^2+2z}{z^2-3z+2}, 1 < |z| < 2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{z^2-3z+2} = \frac{3z+2}{(z-2)(z-1)}$$

$$= \frac{8}{z-2} + \frac{(-5)}{z-1}$$

Partial Fractions : $A = \frac{3(2)+2}{2-1} = 8$

$$B = \frac{3(1)+2}{1-2} = -5$$

$$\therefore \quad F(z) = 8\left(\frac{z}{z-2}\right) - 5\left(\frac{z}{z-1}\right)$$

$$\therefore \quad \{f(k)\} = 8\ Z^{-1}\left(\frac{z}{z-2}\right) - 5\ Z^{-1}\left(\frac{z}{z-1}\right)$$

$$\dots |z| < 2, |z| > 1$$

$$= 8\{-2^k\} - 5\{1^k\}$$

$$(k < 0) \quad (k \ge 0)$$

$$= -8\{2^k\} - 5\{1\}$$

$$(k < 0)\ (k \ge 0)$$

i.e. $$f(k) = -8 \cdot 2^k, k < 0$$

$$= -5, k \ge 0$$

Q.18 Find $Z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right]$, $|z| > 1$

☞ **[SPPU : Dec.-17, May-17, Marks 4]**

Ans. : We have,

$$F(z) = \frac{z(z+1)}{(z-1)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{(z-1)+2}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2}$$

$$\therefore \quad F(z) = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$\therefore$ Inverting, $$\{f(k)\} = Z^{-1}F(z)$$

$$= Z^{-1}\left(\frac{z}{z-1}\right) + 2\ Z^{-1}\left[\frac{z}{(z-1)^2}\right].$$

$$|z| > 1$$

$$= \{1^k\} + 2\{k(1)^{k-1}\}, k \ge 0$$

$$\dots \because Z^{-1}\frac{z}{(z-a)^2} = ka^{k-1}, k \ge 0 \text{ for } |z| > |a|$$

$$\therefore \quad \{f(k)\} = \{1+2k\}, k \ge 0$$

Q.19 Find $Z^{-1}\left[\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]$, $|z| > 1$

☞ **[SPPU : May-15, Marks 4]**

Ans. : We have

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}, \ |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} \quad \text{... (1)}$$

Now,

$$\text{let } \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{(z-1)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2} \quad \text{... (2)}$$

... Partial fractions

∴ Multiplying throughout by the denominator $(z-1)\left(z-\frac{1}{2}\right)^2$ we get

$$z^2 = A\left(z-\frac{1}{2}\right)^2 + B(z-1)\left(z-\frac{1}{2}\right) + C(z-1) \quad \text{... (3)}$$

∴ Putting $z = 1$, $1 = A\left(\frac{1}{2}\right)^2 + 0 + 0$

$$\therefore \quad 1 = \frac{1}{4}A \quad \therefore A = 4$$

Putting $z = \frac{1}{2}$ in (3),

$$\left(\frac{1}{2}\right)^2 = 0 + 0 + C\left(\frac{1}{2} - 1\right)$$

$$\therefore \quad \frac{1}{4} = -\frac{1}{2}C$$

$$\therefore \quad C = -\frac{1}{2}$$

and putting $z = 0$ in (3),

$$0 = A\left(-\frac{1}{2}\right)^2 + B(-1)\left(-\frac{1}{2}\right) + C(-1)$$

$$= \frac{A}{4} + \frac{B}{2} - C$$

$$\therefore \quad \frac{B}{2} = C - \frac{A}{4} = -\frac{1}{2} - \frac{4}{4} = -\frac{3}{2} \quad \text{... } \because C = -\frac{1}{2}, A = 4$$

$$\therefore \quad B = -3$$

∴ Substituting for A, B and C in (2) we get,

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$= \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{1}{2\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad F(z) = 4\left(\frac{z}{z-1}\right) - 3\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2},$$

$$|z| > 1 \Rightarrow |z| > \frac{1}{2}$$

∴ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$

i.e.

$$\{f(k)\} = 4Z^{-1}\left(\frac{z}{z-1}\right) - 3Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$-\frac{1}{2}Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right], |z| > 1, |z| > \frac{1}{2}$$

$$= 4\left\{1^k\right\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \frac{1}{2}\left\{k\left(\frac{1}{2}\right)^{k-1}\right\}$$

$$(k \geq 0) \qquad (k \geq 0) \qquad (k \geq 0)$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = \left\{a^k\right\},$$

$$k \geq 0 \text{ for } |z| > |a|$$

$$\text{and } Z^{-1}\left[\frac{z}{(z-a)^2}\right] = ka^{k-1},$$

$$k \geq 0 \text{ for } |z| > |a|$$

$$= 4\{1\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \left\{k\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$= \left\{4 - 3\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$\therefore \quad \{f(k)\} = \left\{4 - (k+3)\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

Q.20 Find inverse Z-transform of $F(z) = \dfrac{z^2}{(z^2+1)}, |z| > 1$.

☞ [SPPU : May-19, Marks 4]

Ans. : Let

$$F(z) = \frac{z^2}{z^2+1} = \frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$\therefore \quad Z^{-1}(F(z)) = Z^{-1}\left[\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right]$$

$$f(k) = \cos\frac{k\pi}{2},\ k \geq 0$$

5.4 Working Rule for Finding the Poles and Residues

a) To find the poles of the function : $f(z) = \dfrac{\phi(z)}{\psi(z)}$:

1. Consider the equation (from the denominator)

   $\psi(z) = 0$. Its solution, say, $z = a$, $z = b$ ... etc. gives the poles of $f(z)$.

2. If $(z = a)$ is not a repeated root of $\psi(z) = 0$, then $(z = a)$ is a simple pole of $f(z)$.

3. If $(z = b)$ is two times repeated root of $\psi(z) = 0$, then $(z = b)$ is a double pole i.e. a pole of order $(n = 2)$ of $f(z)$. Similarly, if $(z = b)$ is repeated thrice, it is a pole of order $(n = 3)$.

b) To find residue of f(z) at simple pole (z = a) :

1. It is given by,

   $$\boxed{\text{Res } f(a) = \lim_{z \to a} [(z-a)f(z)],}$$ which is a non-zero finite value.

2. Sometimes we use the following result :

   If, $f(z) = \dfrac{\phi(z)}{\psi(z)}$ and $\psi(z) = (z-a)F(z)$ (say) where $F(a) \neq 0$, then

   $$\boxed{\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)} = \left[\frac{\phi(z)}{\frac{d}{dz}\psi(z)}\right]_{z=a}}$$

c) To find the residue of f(z) at the pole (z = b) of order n :

1. It is given by

   $$\boxed{\text{Res } f(b) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-b)^n f(z)\right]\right\}_{z=b}}$$

Thus if (z = b) is a double pole i.e. n = 2 then,

$$\text{Res } f(b) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[(z-b)^2\ f(z)\right]\right\}_{z=b}$$

$$= \left\{\frac{d}{dz}\left[(z-b)^2\ f(z)\right]\right\}_{z=b}$$

and here, $\lim_{z \to b} [(z-b)^2 f(z)]$ is non-zero and finite

and if (z = b) is a pole of order n = 3 then

$$\text{Res } f(b) = \frac{1}{2!}\left\{\frac{d^2}{dz^2}[(z-b)^3\ f(z)]\right\}_{z=b}$$

and here $\lim_{z \to b} [(z-b)^3 f(z)]$ is non-zero and finite.

d) Inversion Integral Method by using Residues

☞ [SPPU : May-2000, 08, 09, 11, 12, 13, Dec.-03, 04, 05, 07, 09, 10, 11]

The Inverse Z-transform of F(z) can be easily obtained by using the Inversion Integral of F (z), given by

$$f(k) = \frac{1}{2\pi i}\int_C F(z) \cdot z^{k-1}\ dz$$

where, C is the closed curve (contour) (drawn according to the given ROC) such that all the poles of F(z) (i.e. the values of z where F(z) becomes infinite) lie inside it.

More, conveniently we have,

$f(k) = \sum$ Residues of $[F(z) \cdot z^{k-1}]$ at the poles of F(z).

e) Working Rule for using Inversion Integral Method

Step 1 : Find the poles of F(z).

Step 2 : Find the expression for $F(z) \cdot z^{k-1}$

Step 3 : Find the residues of $\left[F(z)\, z^{k-1}\right]$ at all the poles of F(z) using proper formulae.

Step 4 : Take algebraic sum of the residues to get f(k).

Q.21 Use inversion integral method to find $Z^{-1}\left[\frac{10\,z}{(z-1)\,(z-2)}\right]$

☞ [SPPU : May-11, 12, Dec.-11, 14, 15, Marks 4]

Ans. : To find poles of F(z) :

We have, $$F(z) = \frac{10\,z}{(z-1)\,(z-2)} \qquad \text{... (1)}$$

$\therefore F(z) \to \infty$ at z = 1, z = 2 $\therefore$ (z = 1) and (z = 2) are simple poles of F(z)

To find $F(z) \cdot z^{k-1}$

Now, using F(z) we have,

$$F(z) \cdot z^{k-1} = \frac{10\,z}{(z-1)\,(z-2)}\, z^{k-1} = 10\frac{z^k}{(z-1)\,(z-2)}$$

Find Residues of $F(z) \cdot z^{k-1}$ at poles

$\because$ (z = 1) and (z = 2) are simple poles

$\therefore$ Residue of $\left[F(z) \cdot z^{k-1}\right]$ at (z = 1)

$$= \left[(z-1)\ F(z)\, z^{k-1}\right]_{z=1}$$

$$= \left[(z-1)\,\frac{10\,z^k}{(z-1)\,(z-2)}\right]_{z=1}$$

$$= 10\left[\frac{z^k}{(z-2)}\right]_{z=1} = 10\left(\frac{1}{1-2}\right)$$

$\therefore$ Residue at (z = 1) = – 10 ... (2)

Similarly,

$$\text{Residue at } (z = 2) = \left[(z-2)\,\frac{10\,z^k}{(z-1)\,(z-2)}\right]_{z=2}$$

$$= 10\left[\frac{z^k}{z-1}\right] = 10 \cdot \frac{2^k}{(2-1)}$$

$$= 10 \cdot 2^k \qquad \text{... (3)}$$

To take algebraic sum of all residues.

$\therefore$ By Inversion integral method.

$$Z^{-1}\,F(z) = \{f\,(k)\} \text{ where}$$

$$f(k) = \sum \text{Residues of } F(z)\,z^{k-1}$$

$$= \text{Res } (z = 1) + \text{Res } (z = 2)$$

$$= -10 + 10 \cdot 2^k,\; k \geq 0$$

... using (2) and (3)

$$= 10\left(2^k - 1\right),\; k \geq 0$$

Q.22 Use Inversion Integral Method to find $Z^{-1}\left[\dfrac{z^3}{(z-3)\,(z-2)^2}\right]$

[SPPU : Dec.-07, 11, May-13]

Ans. :

Step 1 : To find poles of F(z) :

We have, $$F(z) = \frac{z^3}{(z-3)\,(z-2)^2}$$

$\therefore$ $F(z) \rightarrow \infty$ at $z = 3$ and $z = 2$ which is repeated $(r = 2)$ times.

Hence, $z = 3$ is a simple pole of F(z) and $z = 2$ is a double pole of F(z).

Step 2 : To find expression for $F(z) \cdot z^{k-1}$:

Now, we have,

$$F(z) \cdot z^{k-1} = \frac{z^3}{(z-3)\,(z-2)^2} \cdot z^{k-1} = \frac{z^{k+2}}{(z-3)\,(z-2)^2} \qquad \text{... (1)}$$

Step 3 : To find Residues of $F(z) \cdot z^{k-1}$ at the poles of F(z) :

Now, for the simple pole at $(z = 3)$ we have,

Residue of $\left[F(z) \cdot z^{k-1}\right]$ at $(z = 3)$

$$= \left[(z-3)\;F(z) \cdot z^{k-1}\right]_{z=3} \quad \text{... } \because \text{Res } (z = a)$$

$$= \left[(z-a)\;F(z)\;z^{k-1}\right]_{z=a} \text{ for a simple pole}$$

$$= \left[(z-3)\,\frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]_{z=3} \qquad \text{... using (1)}$$

$$= \left[\frac{z^{k+2}}{(z-2)^2}\right]_{z=3} = \frac{3^{k+2}}{(3-2)^2} = 3^{k+2}$$

$$\therefore \qquad \text{Res } (z = a) = 3^{k+2},\; k \geq 0 \qquad \text{... (2)}$$

Also, for the double (i.e. $r = 2$) pole at $(z = 2)$

We have,

Residue of $\left[F(z) \cdot z^{k-1}\right]$ at $(z = 2)$

$$= \frac{1}{(2-1)\,!}\left[\frac{d^{2-1}}{dz^{2-1}}\left[(z-2)^2\;F(z) \cdot z^{k-1}\right]\right]_{z=2} \quad \text{... } \because \text{Res } (z = a)$$

$$= \frac{1}{(r-1)\,!}\left[\frac{d^{r-1}}{dz^{r-1}}\left[(z-a)^r\,F(z) \cdot z^{k-1}\right]\right]_{z=a}$$

for r times repeated pole

$$= \frac{1}{1}\left[\frac{d}{dz}\left[(z-2)^2\,\frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]\right]_{z=2} \qquad \text{... using (1)}$$

$$= \left[\frac{d}{dz}\left(\frac{z^{k+2}}{z-3}\right)\right]_{z=2}$$

$$= \left[\frac{(z-3)\,(k+2)\,z^{k+1} - z^{k+2}\,(1)}{(z-3)^2}\right]_{z=2}$$

$$= \frac{(2-3)\,(k+2)\cdot 2^{k+1} - 2^{k+2}}{(2-3)^2} = -(k+2)\,2^{k+1} - 2^{k+2}$$

$$= -(k+2)\,2^{k+1} - 2\cdot 2^{k+1} = -[(k+2)+2]\,2^{k+1}$$

$\therefore$ Residue $(z = 2) = -(k+4)\,2^{k+1},\ k \geq 0$ ... (3)

Step 4 : Take algebraic sum of all the Residues.

Thus, using (2) and (3) we have, by inversion integral method

$$Z^{-1}[F(z)] = \{f(k)\}$$

where, $f(k) = \sum$ Residues of $\left[F(z)\,z^{k-1}\right]$

at poles of F(z)

$$= 3^{k+2} - (k+4)\,2^{k+1},\ k \geq 0$$

Q.23 Find Inverse Z-transform using inversion integral method, $f(z) = \frac{z^2}{z^2+1}$

☞ [SPPU : Dec.03, 04, 05, 09, 10, 12, 15, May-08, 10, Marks 4]

Ans. : We have,

$$F(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z-i)\,(z+i)}$$ **... Note this step**

$\therefore$ F(z) has simple poles at (z = i) and (z = – i)

$$\text{Now, } F(z)\,z^{k-1} = \frac{z^2\cdot z^{k-1}}{(z-i)\,(z+i)}$$

$$= \frac{z^{k+1}}{(z-i)\,(z+i)} \quad \text{... (1)}$$

$\therefore$ Residue of $[F(z)\,z^{k-1}]$ at (z = i)

$$= [(z-i)\,F(z)\,z^{k-1}]_{z=i}$$

$$= \left[\frac{z^{k+1}}{z+i}\right]_{z=i} \quad \text{... using (1)}$$

$$= \frac{(i)^{k+1}}{i+i} = \frac{1}{2i}(i)^{k+1} = \frac{(i)^k}{2}$$

and Residue at (z = – i)

$$= [[z-(-i)]\,F(z)\,z^{k-1}]_{z=-i}$$

$$= \left[\frac{z^{k+1}}{z-i}\right]_{z=-i}$$

$$= \frac{(-i)^{k+1}}{-i-i} = \frac{(-i)^{k+1}}{2(-i)} = \frac{(-i)^k}{2}$$

$$\therefore \quad Z^{-1}\left(\frac{z^2}{z^2+1}\right) = \{f(k)\}$$

where, $f(k) = \sum$ Residues

$$= \frac{(i)^k}{2} + \frac{(-i)^k}{2}$$

$$= \frac{1}{2}\,[(i)^k + (-i)^k]$$

Now, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $-i = e^{-i\pi/2}$

$$\therefore \quad (i)^k + (-i)^k = e^{i\frac{k\pi}{2}} + e^{-i\frac{k\pi}{2}} = 2\cos\frac{k\pi}{2}$$

$\dots \because e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$\therefore \quad f(k) = \frac{1}{2}\left(2\cos\frac{k\pi}{2}\right) = \cos\frac{k\pi}{2},\ k \geq 0$$

$$= \left[\frac{d}{dz}\left(\frac{z^{k+2}}{z-3}\right)\right]_{z=2}$$

$$= \left[\frac{(z-3)\,(k+2)\,z^{k+1} - z^{k+2}\,(1)}{(z-3)^2}\right]_{z=2}$$

$$= \frac{(2-3)\,(k+2)\cdot 2^{k+1} - 2^{k+2}}{(2-3)^2} = -(k+2)\,2^{k+1} - 2^{k+2}$$

$$= -(k+2)\,2^{k+1} - 2\cdot 2^{k+1} = -[(k+2)+2]\,2^{k+1}$$

$\therefore$ Residue $(z = 2) = -(k+4)\,2^{k+1},\ k \geq 0$ ... (3)

Step 4 : Take algebraic sum of all the Residues.

Thus, using (2) and (3) we have, by inversion integral method

$$Z^{-1}[F(z)] = \{f(k)\}$$

where, $f(k) = \sum$ Residues of $\left[F(z)\,z^{k-1}\right]$ at poles of F(z)

$$= 3^{k+2} - (k+4)\,2^{k+1},\ k \geq 0$$

Q.23 Find Inverse Z-transform using inversion integral method, $\mathbf{f(z)} = \dfrac{z^2}{z^2+1}$

☞ **[SPPU : Dec.03, 04, 05, 09, 10, 12, 15, May-08, 10, Marks 4]**

Ans. : We have,

$$F(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z-i)\,(z+i)}$$ **... Note this step**

$\therefore$ F(z) has simple poles at (z = i) and (z = – i)

$$\text{Now, } F(z)\,z^{k-1} = \frac{z^2 \cdot z^{k-1}}{(z-i)\,(z+i)}$$

$$= \frac{z^{k+1}}{(z-i)\,(z+i)} \quad \text{... (1)}$$

$\therefore$ Residue of $[F(z)\,z^{k-1}]$ at (z = i)

$$= [(z-i)\,F(z)\,z^{k-1}]_{z=i}$$

$$= \left[\frac{z^{k+1}}{z+i}\right]_{z=i} \quad \text{... using (1)}$$

$$= \frac{(i)^{k+1}}{i+i} = \frac{1}{2i}(i)^{k+1} = \frac{(i)^k}{2}$$

and Residue at (z = – i)

$$= [[z-(-i)]\,F(z)\,z^{k-1}]_{z=-i}$$

$$= \left[\frac{z^{k+1}}{z-i}\right]_{z=-i}$$

$$= \frac{(-i)^{k+1}}{-i-i} = \frac{(-i)^{k+1}}{2(-i)} = \frac{(-i)^k}{2}$$

$$\therefore \quad Z^{-1}\left(\frac{z^2}{z^2+1}\right) = \{f(k)\}$$

where, $f(k) = \sum$ Residues

$$= \frac{(i)^k}{2} + \frac{(-i)^k}{2}$$

$$= \frac{1}{2}[(i)^k + (-i)^k]$$

Now, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $-i = e^{-i\pi/2}$

$$\therefore \quad (i)^k + (-i)^k = e^{i\frac{k\pi}{2}} + e^{-i\frac{k\pi}{2}} = 2\cos\frac{k\pi}{2}$$

$$\text{... } \because e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \quad f(k) = \frac{1}{2}\left(2\cos\frac{k\pi}{2}\right) = \cos\frac{k\pi}{2},\ k \geq 0$$

Q.24 Obtain inverse z-transform of $F(z) = \frac{1}{(z-3)(z-4)} |z| > 3$ by inversion integral method.

[SPPU : Dec.-18, Marks 4]

Ans. : We have

$$F(z) = \frac{1}{(z-3)(z-4)}$$

$\therefore F(z) \to \infty$ at $z = 3$ and $z = 4$

$\therefore z = 3$ and $z = 4$ are simple poles of F(z)

Now

$$F(z) z^{k-1} = \frac{z^{k-1}}{(z-3)(z-4)}$$

$$\therefore \text{ Residue of } [F(z) z^{k-1}]_{z=3} = [(z-3)F(z) z^{k-1}]_{z=3}$$

$$= -(3)^{k-1}$$

$$\therefore \text{ Residue of } [F(z) z^{k-1}]_{z=4} = [(z-4)F(z) z^{k-1}]_{z=4}$$

$$= (4)^{k-1}$$

Now $Z^{-1}[F(z)] = \{f(k)\}$

$$\therefore \quad f(k) = -(3)^{k-1} + 4^{k-1} \quad \text{for} \quad k \geq 0$$

5.5 : Solutions of Simple Difference Equations (with constant coefficients) using Z-transforms

[SPPU : Dec.-2000, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12, May-01, 02, 03, 05, 06, 07, 08, 09, 10, 12, 13]

1)

$$Z\{f(k)\} = F(z)$$

$$Z\{f(k+1)\} = z\,F(z) - z\,f(0)$$

$$Z\{f(k+2)\} = z^2 F(z) - z^2 f(0) - z\,f(1) \text{ etc.}$$

$$Z\{f(k-1)\} = z^{-1} F(z)$$

$$Z\{f(k-2)\} = z^{-2} F(z) \text{ etc.}$$

Q.25 Obtain $f(k)$ given that $12\,f(k+2) - 7\,f(k+1) + f(k) = 0,\ k \geq 0,\ f(0) = 0,\ f(1) = 3$

[SPPU : May-16, Dec.-17, Marks 4]

Ans. : We have,

$$12\,f(k+2) - 7f(k+1) + f(k) = 0$$

$$\therefore \quad 12\,Z\{f(k+2)\} - 7Z\{f(k+1)\} + Z\{f(k)\} = 0$$

$$\text{i.e. } 12\left[z^2 F(z) - z^2 f(0) - z\,f(1)\right]$$

$$-7[z\,F(z) - z\,f(0)] + F(z) = 0$$

$$\text{i.e. } 12\left[z^2 F(z) - 0 - 3z\right] - 7[z\,F(z) - 0] + F(z) = 0$$

$\ldots \because f(0) = 0,\ f(1) = 3$

$$\text{i.e. } \left(12z^2 - 7z + 1\right) F(z) = 36\,z$$

$$\therefore \quad F(z) = \frac{36\,z}{12z^2 - 7z + 1}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{36}{(4z-1)(3z-1)}$$

$$= 36 \frac{1}{(4z-1)(3z-1)}$$

$$= 36\left[\frac{-4}{(4z-1)} + \frac{3}{(3z-1)}\right]$$

$$\ldots \text{ Partial fraction } A = \frac{1}{\frac{3}{4}-1} = -4$$

$$B = \frac{1}{\frac{4}{3}-1} = 3$$

$$\therefore \quad F(z) = 36\left[\frac{3\,z}{3\,z-1} - \frac{4\,z}{4\,z-1}\right]$$

$$= 36\left(\frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{4}}\right)$$

$$\therefore \quad \{f(k)\} = Z^{-1} F(z)$$

$$= 36\, Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right) - 36\, Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$= 36\left\{\left(\frac{1}{3}\right)^k\right\} - 36\left\{\left(\frac{1}{4}\right)^k\right\},\ k \geq 0$$

... assuming $|z| > \frac{1}{3}$ since here $k \geq 0$

$$\therefore \quad f(k) = 36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right],\ k \geq 0$$

Q.26 Solve, the following difference equation to find $\{f(k)\}$:

$f(k+2) + 3f(k+1) + 2f(k) = 0,\ f(0) = 0,\ f(1) = 1$

[SPPU : May-15, Marks 4]

Ans. : We have the difference equation

$$f(k+2) + 3f(k+1) + 2f(k) = 0$$

Step 1 : Take the Z-transform of both the sides

$$\therefore \quad Z\{f(k+2)\} + 3Z\{f(k+1)\} + 2\{f(k)\} = 0$$

i.e. $\left[z^2 F(z) - z^2 f(0) - z f(1)\right]$

$$+ 3[z F(z) - z f(0)] + 2F(z) = 0$$

where $F(z) = Z\{f(k)\}$ ...using Standard results

i.e.

$$\left[z^2 F(z) - 0 - z(1)\right] + 3[z F(z) - 0] + 2F(z) = 0$$

... $\because f(0) = 0,\ f(1) = 1$ (given)

Step 2 : Rearrange this algebraic equation to find F(z) :

Thus, $\left(z^2 + 3z + 2\right) F(z) - z = 0$

$$\therefore \quad F(z) = \frac{z}{z^2 + 3z + 2}$$

Step 3 : Find $Z^{-1}F(z)$:

Now, $$F(z) = \frac{z}{z^2 + 3z + 2}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{(z+1)(z+2)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{z+1} - \frac{1}{z+2} \quad \text{... Partial fractions}$$

$$\therefore \quad F(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$\therefore$ Inverting we get,

$$\{f(k)\} = Z^{-1} F(z)$$

$$= Z^{-1}\left[\frac{z}{z-(-1)}\right] - Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

$$= \left\{(-1)^k\right\} - \left\{(-2)^k\right\},\ k \geq 0,$$

... assuming $|z| > 2$ i.e. $|z| > |-2|$ and

$\therefore$ $|z| > |-1|$ and $\because Z^{-1}\left(\frac{z}{z-a}\right) = a^k,\ k \geq 0$ for $|z| > |a|$

$$= \left\{(-1)^k - (-2)^k\right\},\ k \geq 0$$

$$\therefore \quad f(k) = (-1)^k - (-2)^k,\ k \geq 0$$

Q.27 Use Z-transforms to solve

$y_k - \frac{5}{6} y_{k-1} + \frac{1}{6} y_{k-2} = \left(\frac{1}{2}\right)^k,\ k \geq 0$ **[SPPU : May-18, Marks 4]**

Ans. : We have the difference equation,

$$y_k - \frac{5}{6} y_{k-1} + \frac{1}{6} y_{k-2} = \left(\frac{1}{2}\right)^k,\ k \geq 0$$

$$\therefore Z\{y_k\} - \frac{5}{6} Z\{y_{k-1}\} + \frac{1}{6} Z\{y_{k-2}\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

i.e. $Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = \frac{z}{z-\frac{1}{2}}, |z| > \frac{1}{2}$

... $\because$ $Z\{f(k-1)\} = z^{-1} F(z)$ and

$Z\{f(k-2)\} = z^{-2} F(z)$

i.e. $$\left(1 - \frac{5}{6z} + \frac{1}{6z^2}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad \frac{1}{z^2}\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right)}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2} \quad \text{... (1)}$$

Now, consider

$$\frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2} = \frac{A}{\left(z-\frac{1}{3}\right)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2} \quad \text{... Partial fractions}$$

$$\therefore \quad z^2 = A\left(z-\frac{1}{2}\right)^2 + B\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right) + C\left(z-\frac{1}{3}\right) \quad \text{... (2)}$$

Put $z = \frac{1}{2}$

$$\therefore \quad \frac{1}{4} = C\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{C}{6}$$

$$\therefore \quad C = \frac{6}{4} = \frac{3}{2}$$

Put $z = \frac{1}{3}$ in (2),

$$\therefore \quad \frac{1}{9} = A\left(\frac{1}{3} - \frac{1}{2}\right)^2 = A\left(-\frac{1}{6}\right)^2$$

$$\therefore \quad A = \frac{36}{9} = 4$$

Put $z = 0$ in (2),

$$\therefore \quad 0 = A\left(\frac{1}{4}\right) + B\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) + C\left(-\frac{1}{3}\right)$$

$$= 4\left(\frac{1}{4}\right) + \frac{B}{6} + \frac{3}{2}\left(\frac{-1}{3}\right)$$

$$= 1 + \frac{B}{6} - \frac{1}{2} = \frac{1}{2} + \frac{B}{6} \quad \text{... } \because A = 4, C = \frac{3}{2}$$

$$\therefore \quad \frac{B}{6} = -\frac{1}{2}$$

$$\therefore \quad B = -3$$

$\therefore$ Substituting for A, B and C and using (1) we get

$$\frac{Y(z)}{z} = \frac{4}{\left(z-\frac{1}{3}\right)} - \frac{3}{\left(z-\frac{1}{2}\right)} + \frac{3/2}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad Y(z) = 4\left(\frac{z}{z-\frac{1}{3}}\right) - 3\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{3}{2} \frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad \{y_k\} = Z^{-1}\,Y(z) = 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$-3\,Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{3}{2}\,Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right]$$

$$|z| > \frac{1}{2} \Rightarrow |z| > \frac{1}{3}$$

$$= 4\left\{\left(\frac{1}{3}\right)^k\right\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} + \frac{3}{2}\left\{k\left(\frac{1}{2}\right)^{k-1}\right\},\ k \geq 0$$

... Standard results

$$= \left\{4\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k + 3\,k\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$\therefore \quad y_k = 4\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k + 3k\left(\frac{1}{2}\right)^k,\ k \geq 0$$

Q.28 Obtain the output of the system, where the input is U_k and the system is given by $y_k - 4\,y_{k-2} = U_k$

where $U_k = \left(\frac{1}{2}\right)^k,\ k \geq 0$

$= 0,\ k < 0$

Ans. : Here the output of the system is $\{y_k\}$ which is assumed to be causal.

Hence, $y(-1) = y(-2) = \ldots\ldots = 0$

Now, the difference equation of the system is

$$y_k - 4\,y_{k-2} = U_k$$

$\therefore$ Taking Z-transform of both the sides,

$$Z\{y_k\} - 4\,Z\{y_{k-2}\} = Z\{U_k\}$$

i.e. $Y(z) - 4\left[z^{-2}\,Y(z)\right] = Z\left\{\left(\frac{1}{2}\right)^k\right\},\ \because k \geq 0$

where $Y(z) = Z\{y_k\}$ ... using Standard result for $Z\{f(k-2)\}$

and $\because U_k = \left(\frac{1}{2}\right)^k,\ k \geq 0$

i.e. $$\left(1-\frac{4}{z^2}\right)Y(z) = \frac{z}{z-\frac{1}{2}},\ |z| > \frac{1}{2}$$

i.e. $$\left(\frac{z^2-4}{z^2}\right)Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)\left(z^2-4\right)}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)\,(z+2)}$$

$$= \frac{\left(-\frac{1}{15}\right)}{z-\frac{1}{2}} + \frac{\left(\frac{2}{3}\right)}{z-2} + \frac{\left(\frac{2}{5}\right)}{z+2}$$

... Partial fractions : $A = \dfrac{\frac{1}{4}}{\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)} = -\dfrac{1}{15}$

$$B = \frac{4}{\left(\frac{3}{2}\right)(4)} = \frac{2}{3},$$

$$C = \frac{4}{\left(-\frac{5}{2}\right)(-4)} = \frac{2}{5}$$

$$\therefore \quad Y(z) = -\frac{1}{15}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{z}{z-2}\right) + \frac{2}{5}\left(\frac{z}{z+2}\right)$$

$\therefore$ Inverting

$$Z^{-1}Y(z) = \{y_k\} = -\frac{1}{15}\, Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$+\frac{2}{3}Z^{-1}\left(\frac{z}{z-2}\right)+\frac{2}{5}Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

and assume $|z| > 2 \;\therefore |z| > |-2|, |z| > \frac{1}{2}$

$$\therefore \quad \{y_k\} = -\frac{1}{15}\left\{\left(\frac{1}{2}\right)^k\right\} + \frac{2}{3}\{(2)^k\}$$

$$+\frac{2}{5}\{(-2)^k\}, \; k \geq 0$$

$\therefore$ Output of the system is $\{y_k\}$ where

$$y_k = \frac{2}{3}\left(2^k\right) + \frac{2}{5}(-2)^k - \frac{1}{15}\left(\frac{1}{2}\right)^k,$$

$$k \geq 0$$

END...

UNIT IV

6 Statistics

6.1 : Useful Definitions

There are five methods of finding a measure of central tendency.

i) Arithmetic mean ii) Median iii) Mode

iv) Geometric mean v) Harmonic mean.

These are known as measures of central tendency.

A) Arithmetic Mean

1) For ungrouped data :

If $x_1, x_2, \cdots x_n$ are n observations

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{n} = \frac{\text{Sum of the observations}}{\text{Total number of observations}}$$

2) For grouped data : If x_i are class marks and f_i are their respective frequencies for $1 \leq i \leq n$ then

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Note : We can write $\sum f_i = N =$ Total frequency

3) Method of step deviation (to simplify the calculations) :

If $\quad ui = \frac{x_i - A}{h}$, where

A is the working mean or assumed mean of given data and h is the class width or gcd of all $x_i - A$

$$\bar{x} = A + h\,\bar{u}$$

$$= A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right)$$

B) Median

1) For ungrouped data :

It divides total set of data into two equal points.

Median is value of **middle** most terms of series when arranged in **ascending** or **descending** order of magnitude.

If n is odd then median = $\left(\frac{n+1}{2}\right)^{th}$ observation

If n is even then there are two values in the middle so we take mean of these two values

$$\text{median} = \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2} \text{ observation}$$

2) For group frequency distribution :

$$\boxed{\text{Median} = l + \frac{h\left(\frac{N}{2} - C\right)}{f}}$$

l = Lower limit of median class.

f = Frequency of median class.

h = Width of median class.

C = Coefficient of the class preceding the median class.

$N = \sum f_i$

Median class is the class where $\left(\frac{N}{2}\right)^{th}$ observation lies.

C) Mode

1) For ungrouped data : Mode is the most repeated observation in the given set of observation. So mode is not unique. If given data has only one mode, then it is known as unimodal otherwise multimodal.

Ex. 1) Mode of 1, 2, 3, 4, 2, 3, 2 is 2.

2) Mode of 1, 2, 3 is 1, 2, 3.

2) For grouped frequency distribution.

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{2f_1 - f_0 - f_2}$$

l = Lower limit of modal class.

h = Width of the modal class.

f_1 = Frequency of the modal class.

f_0 = Frequency of the class preceding the modal class.

f_2 = Frequency of the class succeeding the modal class.

Modal class is the class with highest frequency.

D) Geometric Mean

i) For ungrouped data : Geometric mean or G.M. of n observations $x_1, x_2 \ldots x_n$ $(x_i \neq 0)$ is the n^{th} root of their product.

i.e. $$\text{G.M.} = (x_1 \cdot x_2 \ldots x_n)^{1/n}$$

Take log on both sides

$$\log(\text{GM}) = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n)$$

$$\therefore \quad \text{GM} = \text{Antilog}\left[\frac{\sum \log x_i}{n}\right]$$

ii) For grouped data : For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1, f_2 \ldots f_n$

$$\text{G.M.} = \left[(x_1)^{f_1} \cdot (x_2)^{f_2} \cdot (x_3)^{f_3} \ldots (x_n)^{f_n}\right]^{1/N}$$

Take log on both sides

$$\log \text{G.M.} = \frac{1}{N}[f_1 \log x_1 + f_2 \log x_2 \ldots + f_n \log x_n]$$

$$\text{G.M.} = \text{Antilog}\left[\frac{\sum f_i \log x_i}{N = \sum f_i}\right]$$

E) Harmonic Mean

Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of the reciprocals of the given values.

i) For ungrouped data :

$$\text{H.M.} = \frac{n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)}$$

i.e. $$\text{H.M.} = \frac{1}{\left(\frac{\sum\left(\frac{1}{x_i}\right)}{n}\right)}$$

ii) For grouped data :

For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1\ f_2 \ldots f_n$

$$\text{H.M.} = \frac{N}{\left(\frac{f_1}{x_1}+\frac{f_2}{x_2}+\ldots+\frac{f_n}{x_n}\right)} = \frac{N}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

where $N = \sum f_i$

6.2 : Dispersion

[SPPU : May-06, Dec.-10]

Meaning of dispersion is scatteredness. To find whether measures of central tendencies are true representative of the data we calculate dispersion.

To measure the scatteredness of data from mean we use

a) Mean deviation

b) Standard deviation

a) Mean Deviation :

i) For ungrouped data :

For variates $x_1, x_2 \ldots x_n$

Deviation from average $A = d_i = x_i - A$

Deviation from mean $= d_i = x_i - \bar{x}$

$$\text{Mean deviation} = \frac{\sum |d_i|}{n}$$

ii) For grouped data :

For variate $x_1, x_2 \ldots x_n$ with corresponding frequencies $f_1, f_2 \ldots f_n$

$$\text{Mean deviation} = \frac{\sum f_i d_i}{\sum f_i} = \frac{\sum f_i |x_i - A|}{\sum f_i}$$

b) Standard Deviation (σ) :

i) For ungrouped data : $x_1, x_2, x_3 \ldots x_n$

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - (\bar{x})^2}$$

ii) For grouped data :

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2} \qquad \text{where } \sum f_i = N$$

i.e. $$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

If we use method of step deviation for simplification of our calculations

i.e. if $u_i = \frac{x_i - A}{h}$ then

i) For ungrouped data

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - \left(\frac{\sum u_i}{n}\right)^2}, \quad \sigma_x = h\sigma_u$$

ii) For grouped data

$$\sigma_u = \sqrt{\frac{\sum f_i u_i^2}{\sum f_i} - \left(\frac{\sum f_i u_i}{\sum f_i}\right)^2}$$

$$\sigma_x = h\sigma_u$$

$$\bar{x} = A + h\bar{u}$$

Note : 1) The square of standard deviation is called variance given by σ^2.

2) The coefficient of variation is given by

$$\text{C.V.} = \frac{\sigma}{\text{A.M.}} \times 100$$

For comparing the variability of two series, we calculate the coefficient of variations for each series. The series having lesser C.V. is said to be more consistent.

Q.1 Goals scored by two teams A and B in a football season were as follows. Determine which team is more consistent.

[SPPU : May-06, Dec.-10]

Number of goals scored	Number of Matches	
	Team A	Team B
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

Ans. : Frequency distribution table for team A.

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	27	– 2	– 54	108
1	9	– 1	– 9	9
2	8	0	0	0
3	5	1	5	5
4	4	2	6	12
	53		– 50	138

Thus for team A

$$\bar{x} = A + \left(\frac{\sum f_i d_i}{\sum f_i}\right) = 2 + \frac{-50}{53} = 1.06$$

$$\sigma_A = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2} = \sqrt{\frac{138}{53} - \left(\frac{-50}{53}\right)^2}$$

$$= 1.31$$

$$\text{C.V.} = \frac{\sigma_A}{\bar{x}} \times 100 = \frac{1.31}{1.06} \times 100 = 123.6$$

Frequency distribution table for team B

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	17	– 2	– 34	68
1	9	– 1	– 9	9
2	6	0	0	0
3	5	1	5	5
4	3	2	6	12
	40		– 32	94

Thus for team B

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i} = 2 + \left(\frac{-32}{40}\right) = 1.2$$

$$\sigma_B = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{94}{40} - \left(\frac{-32}{40}\right)^2} = 1.3$$

$$\text{C.V.} = \frac{\sigma_B}{\bar{x}} \times 100 = \frac{1.3}{1.2} \times 100 = 108.3$$

Since $(\text{C.V.})_B < (\text{C.V.})_A$

$\therefore$ Team B is more consistent.

6.3 : Moments

i) Central moments : Moments about the mean are known as central moments. The arithmetic mean of various powers of the deviation $(x_i - \bar{x})$ is called central moment of the distribution and is denoted by μ_i.

Thus

$$\mu_1 = 0$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$$

$$\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{N}$$

$$\mu_4 = \frac{\sum f_i (x_i - \bar{x})^4}{N}$$

are the first four moments of the distribution about mean.

ii) Raw moments : Moment about any point of observations different from mean is known as raw moments. The r^{th} moment about any number A is denoted by μ_r' and is given by

$$\mu_r' = \frac{\sum f_i (x_i - A)^r}{N}$$

Substituting r = 0, 1, 2, ... we get

$$\mu_0' = 1$$

$$\mu_1' = \frac{\sum f_i (x_i - A)}{N} = \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right) A$$

$$= \bar{x} - A$$

$$\mu_2' = \frac{\sum f_i (x_i - A)^2}{N}$$

$$= s^2 = \text{Mean square deviation}$$

$$\mu_3' = \frac{\sum f_i (x_i - A)^3}{N}$$

and

$$\mu_4' = \frac{\sum f_i (x_i - A)^4}{N}$$

Note : Proper choice of A can reduce that calculations of calculating μ_r' than that of μ_r.

iii) Relations between μ_r' and μ_r :

Thus

$$\mu_0 = 1$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4$$

6.4 : Skewness

To get the idea about the shape of the curve we study skewness. Skewness signifies departure from symmetry.

a) Positive skewness : If the mean lies to the right of mode then the frequency curve stretches to the right then the distribution is right skewed or positively skewed. (Refer Fig. 6.1 on next page)

b) Negative skewness : If the mean lies to the left side of mode then the frequency curve stretches to the left then the distribution is left skewed or negatively skewed. (Refer Fig. 6.2 on next page)

The different measures of skewness are

i) $$\text{Skewness} = \frac{3\,(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

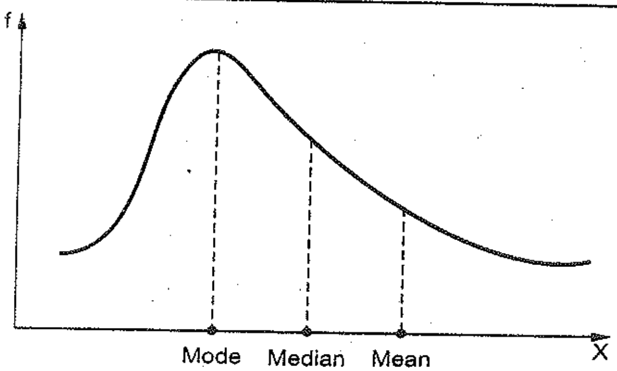

Fig. 6.1

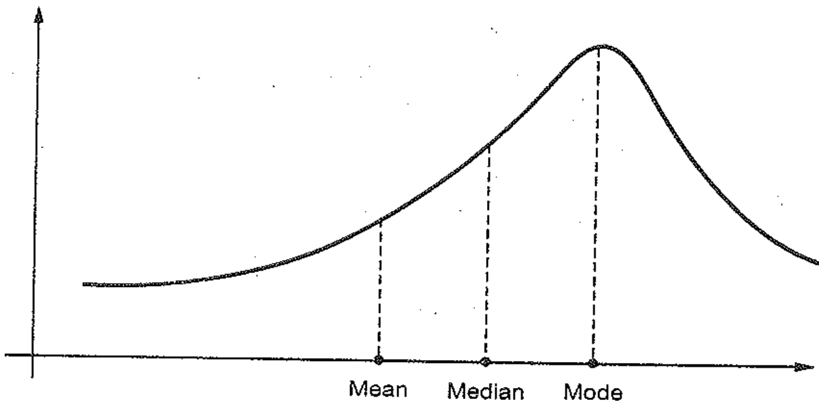

Fig. 6.2

ii) Coefficient of skewness : $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

and $\gamma_1 = +\sqrt{\beta_1}$

6.5 : Kurtosis

☞ [SPPU : Dec.-05, 06, 07, 09, 10, May-06, 08, 10, 13]

If we know the measures of central tendency, dispersion and skewness, we still cannot have a complete idea about the distribution. Observe the Fig. 6.3 there are three curves $C_1 C_2 C_3$ which are symmetrical about

mean and have the same range. Therefore we should know about the flatness or peakedness of the curve.

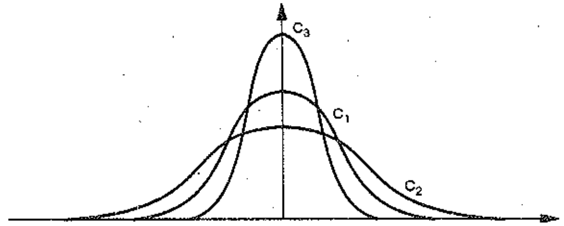

Fig. 6.3

Kurtosis (convexity of curve) is a measure which gives an idea about the flatness or peakedness of the curve. It is measured by the coefficient β_2.

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad \text{or} \quad \gamma_2 = \beta_2 - 3$$

a) Mesokurtic curve : (Normal curve) : The curve C_1 which is neither flat nor peaked is called the normal curve or mesokurtic curve, for which $\beta_2 = 3$ or $\gamma_2 = 0$.

b) Platykurtic curve : The curve (C_2) which is flatter than C_1 is platykurtic curve, for which $\beta_2 < 3$ or $\gamma_2 < 0$.

c) Leptokurtic curve : The curve (C_3) which is more peaked than C_1 is Leptokurtic curve, for which $\beta_2 > 3$ or $\gamma_2 > 0$.

Q.2 The first four moments of a distribution about the value of 4 of the variable are – 1.5, 17, – 30 and 108. Find the moments about mean and β_1 and β_2. ☞ [SPPU : Dec.-06,15, May-11,16]

Ans. : $A = 4, \mu_1' = -1.5, \mu_2' = 17,$

$\mu_3' = -30, \mu_4' = 108$

$$\therefore \mu_2 = \mu_2' - (\mu_1')^2 = 17 - (1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1)^3$$

$$= -30 - 3\,(17)\,(-1.5) + 2\,(-1.5)^3 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4$$

$$= 108 - 4\,(-30)\,(-1.5) + 6\,(17)\,(-1.5)^2 - 3(-1.5)^4$$

$$= 108 - 180 + 229.5 - 15.1875 = 142.3125$$

$$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4926$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.3125}{(14.75)^2} = 0.6543$$

Q.3 Calculate the first four moments about the mean of the given distribution. Also find β_1 and β_2.

x	2	2.5	3	3.5	4	4.5	5
f	5	38	65	92	70	40	10

☞ [SPPU : Dec.-14]

Ans. : Let $u = \frac{x-3.5}{0.5}$ and A = 3.5, h = 0.5

x	f	u	fu	fu^2	fu^3	fu^4
2	5	−3	−15	45	−135	405
2.5	38	−2	−76	152	−304	608
3	65	−1	−65	65	−65	65
3.5	92	0	0	0	0	0
4	70	1	70	70	70	70
4.5	40	2	80	166	320	640
5	10	3	30	90	270	810
Total	**320**	**-**	**24**	**582**	**156**	**2598**

i) Moments about mean A

Assumed mean = A = 3.5

$$\mu_1' = h\frac{\sum fu}{\sum f} = 0.5\left(\frac{24}{320}\right) = 0.0375$$

$$\mu_2' = h^2\frac{\sum fu^2}{\sum f} = (0.5)^2\left(\frac{582}{320}\right) = 0.4546$$

$$\mu_3' = h^3\frac{\sum fu^3}{\sum f} = (0.5)^3\left(\frac{156}{320}\right) = 0.0609$$

$$\mu_4' = h^4\frac{\sum fu^4}{\sum f} = (0.5)^4\left(\frac{2598}{320}\right) = 0.5074$$

$\therefore$ Four moments about mean are $\mu_1 = 0$

$$\mu_2 = \mu_2' - \mu_1'^2 = (0.4546) - (0.0375)^2 = 0.453$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= (0.0609) - 3(0.4546)(0.0375) + 2(0.0375)^3 = 0.0600$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_2' - 3\mu_1'^4$$

$$= (0.5074) - 4(0.0609)(0.0375)$$

$$+6(0.0375)^2(0.4546) - 3(0.0375)^4 = 0.502$$

ii) By definition

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.0600)^2}{(0.453)^3} = 0.0387$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{0.502}{(0.453)^2} = 2.4463$$

Q.4 Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.

☞ [SPPU : May-06, 12, Dec.-13]

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Ans. : We first calculate moments about x = 4 (Assumed mean)

x_i	f_i	$d_i = x_i - 4$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
0	1	− 4	− 4	16	− 64	256
1	8	− 3	− 24	72	− 216	648
2	28	− 2	− 56	112	− 224	448
3	56	− 1	− 56	56	− 56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448

7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f_i = 256$		$\sum f_i d_i = 0$	$\sum f_i d_i^2 = 512$	$\sum f_i d_i^3 = 0$	$\sum f_i d_i^4 = 2816$

We know that

$$\mu_r' = \frac{1}{N}\sum f_i (x_i - 4)^r = \frac{1}{N}\sum f_i d_i^r$$

$$\mu_1' = \frac{1}{N}\sum f_i d_i = 0$$

$$\mu_2' = \frac{1}{N}\sum f_i d_i^2 = \frac{512}{256} = 2$$

$$\mu_3' = \frac{1}{N}\sum f_i d_i^3 = 0$$

$$\mu_4' = \frac{1}{N}\sum f_i d_i^4 = \frac{2816}{256} = 11$$

using the relations between μ_r and μ_r'

$\therefore$ Moments about mean are

$$\mu_1 = 0 \text{ always}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2 - 0$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0 - 3\times 2\times 0 + 2\times 0 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 11 - 4(0)(0) + 6(2)(0) - 3\times 0 = 11$$

$\therefore$ Skewness $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{0}{2^3} = 0$

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{11}{4} = 2.75$

Q.5 Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.

x	2	2.5	3	3.5	4	4.5	5
f	4	36	60	90	70	40	10

☞[SPPU : May-15]

Ans. : Let, A = 3.5, $u_i = \dfrac{x_i - 3.5}{0.5}$ and h = 0.5

x_i	f_i	$u_i = \dfrac{x_i - 3.5}{0.5}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
2.0	4	–3	–12	36	–108	342
2.5	36	–2	–72	144	–288	576
3.0	60	–1	–60	60	–60	60
3.5	90	0	0	0	0	0
4.0	70	1	70	70	70	70
4.5	40	2	80	160	320	640
5.0	10	3	30	90	270	810
Total	**310**		$\sum fu = 36$	$\sum fu^2 = 560$	$\sum fu^3 = 204$	$\sum fu^4 = 2480$

We know

$$\mu_1' = \frac{\sum f_i u_i}{\sum f_i} = \frac{36}{310} = 0.166$$

$$\mu_2' = \frac{\sum f_i u_i^2}{\sum f_i} = \frac{560}{310} = 1.806$$

$$\mu_3' = \frac{\sum f_i u_i^3}{\sum f_i} = \frac{204}{310} = 0.658$$

$$\mu_4' = \frac{\sum f_i u_i^4}{\sum f_i} = \frac{2480}{310} = 8.0$$

Now using relations

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1^2 = 1.806 - 0.013456 = 1.7925$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 0.658 - 0.6258 + 0.003122 = 0.03262$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_2'^2 - 3\mu_1'^4$$

$$= 8.0 - 0.3053 + 0.1458 - 0.000543 = 7.8399$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = \frac{0.001064}{5.7594} = 0.000185$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7.8399}{3.213} = 2.44$$

Q.6 The first four moments of a distribution about the values 5 are 2, 20 , 40 and 50. From the given information obtain the first four central moment, mean, standard deviation and coefficient of skewness and kurtosis. ☞ [SPPU : May-08, 15, Dec.-07]

Ans. : $A = 5, \mu_1' = 2, \mu_2' = 20, \mu_3' = 40$ and $\mu_4' = 50$.

We know that

$$\mu_1' = \bar{x} - A$$

$$\therefore \quad \bar{x} = A + \mu_1' = 5 + 2 = 7$$

To use the relations between μ_r and μ_r'.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = 40 - 3(2)(20) + 2(2)^3$$

$$= 40 - 120 + 16 = -64$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$= 50 - 4(2)(20) + 6(2)^2(2) - 3(2)^4$$

$$= 50 - 160 + 480 - 48 = 322$$

We know

$\therefore$ Variance $= \mu_2 = 16$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sqrt{16} = 4$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-64)^2}{(16)^3} = 1$$

Since $\beta_1 = 1$, the distribution is positively skewed. Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{322}{(16)^2} = 1.26$$

Since the value of β_2 is less than 3, hence the distribution is platykurtic.

Q.7 Calculate the first four moments about the mean of the following distribution. ☞ [SPPU : May-10, Dec.-11]

Marks	No. of students
0 - 10	06
10 - 20	26
20 - 30	47
30 - 40	15
40 - 50	06

Ans. :

Class	Mid pt. x	Freq. f	$\mu = \frac{x-25}{10}$	fu	fu^2	fu^3	fu^4
0 - 10	5	6	– 2	– 12	24	– 48	96
10 - 20	15	26	– 1	– 26	26	– 26	26
20 - 30	25	47	0	0	0	0	0
30 - 40	35	15	1	15	15	15	15
40 - 50	45	6	2	12	24	48	96
Total	-	100	-	– 11	89	– 11	233

The moments about the arbitrary mean A = 25 are

$$\mu_r' = h^r \frac{\sum fu^r}{\sum f} \text{ for } r = 0, 1, 2, 3 \ldots$$

$$\mu_1' = -1.1, \mu_2' = 89, \mu_3' = -110, \mu_4' = 23300$$

$\therefore$ The central moments are

$$\mu_1 = 0, \qquad \mu_2 = 87.79$$

$$\mu_3 = 181.038, \qquad \mu_4 = 23457.7477$$

6.6 : Correlation and Regression

☞ [SPPU : Dec.-06,08,09,10, May-10,13]

I) Karl Pearson's Coefficient of Correlation : To measure the intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called correlation coefficient.

a) Correlation coefficient between two variables x and y is denoted by $r(x, y)$ and is defined as

$$\boxed{r(x, y) = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}}$$

where $\operatorname{cov}(x, y)$ = co-variance of (x, y)

$$= \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y})$$

where, $$\bar{x} = \frac{\sum x_i}{n}$$

$$\bar{y} = \frac{\sum y_i}{n}$$

$$\operatorname{cov}(x, y) = \frac{1}{n} \sum x_i y_i - (\bar{x})(\bar{y})$$

i.e. $$\operatorname{cov}(x, y) = \frac{1}{n} \sum x_i y_i - \left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right)$$

$$\sigma_x^2 = \frac{1}{n} \sum x_i^2 - (\bar{x})^2$$

i.e. $$\sigma_x^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

Similarly $$\boxed{\sigma_y^2 = \frac{\sum y_i^2}{n} - \left(\frac{\sum y_i}{n}\right)^2}$$

b) Method of step deviation

If $u_i = \dfrac{x_i - A}{n}$ and $v_i = \dfrac{y_i - B}{k}$

$$\operatorname{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u} \cdot \bar{v}$$

then $$\sigma_u^2 = \frac{\sum u_i^2}{n} = \bar{u}, \quad \sigma_v^2 = \frac{\sum v_i^2}{n} - \bar{v}$$

$$\bar{u} = \frac{\sum u_i}{n}, \quad \bar{v} = \frac{\sum v_i}{n}$$

and $$r(u, v) = \frac{\operatorname{cov}(u, v)}{\sigma_u \sigma_v}$$

Note that $r(x, y) = r(u, v)$

Note

1) If $r = 0$ then there is lack of relationship between x and y.

2) If $r = \pm 1$ then the relationship between x and y is very strong.

c) Correlation coefficient for bivariate frequency distribution

When a data is presented in bivariate frequency distribution then also

$$r(x, y) = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$$

where $$\boxed{\operatorname{cov}(x, y) = \frac{\sum f_i x_i y_i}{\sum f_i} - \bar{x}\,\bar{y}}$$

$$= \frac{\sum f_i x_i y_i}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)\left(\frac{\sum f_i y_i}{\sum f_i}\right)$$

i.e. $$\operatorname{cov}(x, y) = \frac{\sum f_i x_i y_i}{N} - \left(\frac{\sum f_i x_i}{N}\right)\left(\frac{\sum f_i y_i}{N}\right)$$

Also $$\boxed{\sigma_x^2 = \frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

and $$\boxed{\sigma_y^2 = \frac{\sum f_i y_i^2}{\sum f_i} - (\bar{y})^2}$$

d) Method of step deviation : If $u_i = \frac{x_i - A}{n}$, $v_i = \frac{y_i - B}{k}$

$$\boxed{\begin{aligned} \text{cov}(u, v) &= \frac{\sum f_i u_i v_i}{\sum f_i} - \bar{u}\cdot\bar{v} \\ \sigma_u^2 &= \frac{\sum f_i u_i^2}{\sum f_i} - (\bar{u})^2 \\ \sigma_v^2 &= \frac{\sum f_i v_i^2}{\sum f_i} - (\bar{v})^2 \end{aligned}}$$

where $\bar{u} = \frac{\sum f_i u_i}{\sum f_i}$ $\quad \bar{v} = \frac{\sum f_i v_i}{\sum f_i}$

and $r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$

Substituting all the above values we can write

$$r(x, y) = \frac{\frac{\sum f_i u_i v_i}{N} - \left(\frac{\sum f_i u_i}{N}\right)\left(\frac{\sum f_i v_i}{n}\right)}{\sqrt{\frac{\sum f_i u_i^2}{N} - \left(\frac{\sum f_i u_i}{N}\right)^2}\sqrt{\frac{\sum f_i v_i^2}{N} - \left(\frac{\sum f_i v_i}{N}\right)^2}}$$

i.e. $$\boxed{r(x, y) = \frac{N \cdot \sum f_i u_i v_i - (\sum f_i u_i)(\sum f_i v_i)}{\sqrt{N\sum f_i u_i^2 - (\sum f_i u_i)^2}\sqrt{N\sum f_i v_i^2 - (\sum f_i v_i)^2}}}$$

Q.8 Following are the values of import of raw material and export of finished product in suitable units.

Export (x)	10	11	14	14	20	22	16	12	15	13
Import (y)	12	14	15	16	21	26	21	15	16	14

[SPPU : Dec.-10, May-15]

Find the coefficient of correlation between the import and export values.

Ans. : Let x = Quantity exported and y = Quantity imported.

Consider the following table

x	y	x^2	y^2	xy
10	12	100	144	120
11	14	121	196	154
14	15	196	225	210
14	16	196	256	224
20	21	400	441	420
22	26	484	676	572
16	21	256	441	336
12	15	144	225	180
15	16	225	256	240
13	14	169	196	182
Total = 147	**170**	**2291**	**3056**	**2638**

Here $n = 10, \bar{x} = \frac{\sum x}{N} = \frac{147}{10} = 14.7$

$$\bar{y} = \frac{\sum y}{N} = \frac{170}{10} = 17$$

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{\left(\sum x^2 - n\bar{x}^2\right)\left(\sum y^2 - n\bar{y}^2\right)}}$$

$$= \frac{2638 - 10\times 14.7\times 17}{\sqrt{(2291 - 10\times 14.7^2)(3056 - 10\times 17^2)}}$$

$$r = \frac{139}{\sqrt{130.1\times 166}} = 0.9458$$

$\therefore$ $r = 0.9458$

Q.9 From a group of 10 students marks obtained by each in papers of Mathematics and Applied Mechanics are given as.

x - marks in maths	23	28	42	17	26	35	29	37	16	46
y - marks in ap mech	25	22	38	21	27	39	24	32	18	44

Calculate Karl Pearson's coefficient of correlation.

☞ [SPPU : May-13, Dec.-12,13]

Ans. :

x_i	y_i	$u_i = x_i - 35$	$v_i = y_i - 39$	u_i^2	v_i^2	$u_i v_i$
16	18	– 19	– 21	361	441	399
17	21	– 18	– 18	324	324	324
23	25	– 12	– 14	144	196	168
26	27	– 9	– 12	81	144	108
28	22	– 7	– 17	49	289	119
29	24	– 6	– 15	36	225	90
35	39	0	0	0	0	0
37	32	2	– 7	4	49	– 14
42	38	7	– 1	49	1	– 7
46	44	11	5	121	25	55
		$\sum u = -51$	$\sum v = -100$	$\sum y^2 = 1169$	1694	1242

$$\bar{u} = \frac{\sum u_i}{n}$$

$$\therefore \quad \bar{u} = \frac{-51}{10} = -5.1 \qquad \bar{u}^2 = 26.01$$

$$\therefore \quad \bar{v} = -10 \qquad \bar{v}^2 = 100$$

Now $\operatorname{cov}(u, v) = \frac{1}{n}\sum u_i \; v_i - \bar{u}\,\bar{v}$

$$= \frac{1}{10}(1242) - 51 = 73.2$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - (\bar{u})^2$$

$$= \frac{1169}{10} - 26.01 = 90.89$$

$$\sigma_u = \sqrt{90.89} = 9.5336$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2$$

$$= \frac{1694}{10} - 100 = 69.4$$

$$\sigma_v = \sqrt{69.4} = 8.33$$

$$r(x, y) = r(u, v) = \frac{\operatorname{cov}(u, v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{73.2}{9.534 \times 8.33} = 0.9217$$

Q.10 Find the coefficient of correlation for the following data.

x	10	14	18	22	26	30
f	18	12	24	06	30	36

☞ [SPPU : Dec.-14, May-13]

Ans. : Let, A = 22, $u_i = \frac{x_i - A}{h} = \frac{x_i - 22}{4}$

B = 24 $\quad \therefore v_i = \frac{y_i - B}{k} = \frac{y_i - 24}{6}$

x_i	y_i	$u_i = \frac{x_i - 22}{4}$	$v_i = \frac{y_i - 24}{6}$	u_i^2	v_i^2	$u_i v_i$
10	18	–3	–1	9	1	3
14	12	–2	–2	4	4	4
18	24	–1	0	1	0	0
22	6	0	–3	0	9	0
26	30	1	1	1	1	1

30	36	2	2	4	4	4
Total		**–3**	**–3**	**19**	**19**	**12**

We know $\bar{u} = \frac{\sum u}{n} = \frac{-3}{6} - -\frac{1}{2}$

$$\bar{v} = \frac{\sum v}{n} = \frac{-3}{6} = -\frac{1}{2}$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v} = \frac{12}{6} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 1.75$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2 = \frac{19}{6} - \left(-\frac{1}{2}\right)^2 = 2.9166$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2 = \frac{19}{6} - \left(-\frac{1}{2}\right)^2 = 2.9166$$

We have

$$r = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = \frac{1.75}{\sqrt{(2.9166)(2.9166)}} = \frac{1.75}{2.9166}$$

$$r = \mathbf{0.60}$$

6.7 : Regression

[SPPU : Dec.-03,05,06,07,08, May-95,06,07,09,10]

If x and y are correlated. If the points in scatter diagram lies on some curve then that curve is called curve of regression.

1) The equation of line of regression of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where $\bar{x}, \bar{y}$ are means of distributions for x and y respectively.

2) The equation of line of regression of x on y is given by

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where, $r\frac{\sigma_y}{\sigma_x}$ = regression coefficient of y on x

$= b_{yx}$

$r\frac{\sigma_x}{\sigma_y}$ = regression coefficient of x on y

$= b_{xy}$.

Thus $b_{yx} \cdot b_{xy} = r^2$

Q.11 Obtain regression lines for the following data.

[SPPU : May-09, Dec.-06, 07, 08,12]

x	6	2	10	4	8
y	9	11	5	8	7

Ans. : To find regression coefficient b_{xy} and b_{yx} prepare the following table.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\sum x_i = 30$	$\sum y_i = 40$	$\sum x_i^2 = 220$	$\sum y_i^2 = 340$	$\sum x_i y_i = 214$

Here $n = 5$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6 \quad \text{and}$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{40}{5} = 8$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{220}{5} - (6)^2$$

$$= 44 - 36 = 8$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2 = \frac{340}{5} - (8)^2$$

$$= 68 - 64 = 4$$

$$\text{cov}(x, y) = \frac{\sum (x_i y_i)}{n} - \bar{x}\,\bar{y}$$

$$= \frac{214}{5} - 6 \times 8$$

$$\text{cov}(x, y) = 42.8 - 48 = -5.2$$

$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{-5.2}{8} = -0.65$$

$$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{-5.2}{6} = -1.3$$

Regression line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 8 = -0.65\,(x - 6)$$

$$y = -0.65\,x + 3.9 + 8$$

$$y = -0.65\,x + 11.9$$

Regression line of X on Y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 6 = -1.3\,(y - 8)$$

$$x - 6 = -1.3\,y + 10.4$$

$$x = -1.3\,y + 10.4 + 6$$

$$x = -1.3\,y + 16.4$$

Q.12 Find lines of regression for the following data.

x	2	3	5	7	9	10	12	15
y	2	5	8	10	12	14	15	16

Find estimate of y when x = 6

☞ [SPPU : Dec.-15, May-16]

Ans. : To find regression lines, we require to calculate regression coefficients σ_{xy} and σ_{yx}. So consider the following table.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
2	2	4	4	4
3	5	9	25	15
5	8	25	64	40
7	10	49	100	70
9	12	81	144	108
10	14	100	196	140
12	15	144	225	180
15	16	225	256	240
Total = 63	**82**	**637**	**1014**	**797**

We have

$$n = 8$$

$$x = \frac{\sum x_i}{n} = \frac{63}{8} = 7.873$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - \bar{x}^2 = \frac{637}{8} - (7.873)^2 = 17.6094$$

$$\bar{y} = \frac{\sum y}{n} = \frac{82}{8} = 10.25,$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - \bar{y}^2 = \frac{1014}{8} - (10.25)^2 = 21.6875$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\,\bar{y} = \frac{797}{8} - (7.873)(10.25)$$

$$= 18.9063$$

$$b_{yx} = \frac{\text{cov}(x,y)}{\sigma_x^2} = \frac{18.9063}{17.6094} = 1.0736$$

$$b_{xy} = \frac{\text{cov}(x,y)}{\sigma_y^2} = \frac{18.9063}{21.6875} = 0.8718$$

i) Regression line of y on x :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 10.25 = 1.0736(x - 7.875)$$

$$y = 1.0736x + 1.7954$$

$\therefore$ At $x = 6,\quad y = 1.0736 \times 6 + 1.7954 = 8.237$

ii) Regression line of x on y :

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 7.875 = (0.8718)(y - 10.25)$$

$$x = 0.8718y - 1.06095$$

Q.13 From record of analysis of correlation data the following results are available variance of x = 9 and lines of regression are given by
8x – 10y + 66 = 0
40 x – 18y = 214
Find out a) Mean values for x and y services. b) Standard deviation of y services. c) Coefficient of correlation between x and y services.

☞ [SPPU : May-06, Dec.-11,15]

Ans. : Let $\bar{x}$, $\bar{y}$ be the means of x and y series then as $(\bar{x}, \bar{y})$ satisfy equations of lines of regression we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \qquad \text{... (1)}$$

$$40\bar{x} - 18\bar{y} - 214 = 0$$

Solving the above equation we get

$$\bar{x} = 13, \bar{y} = 17$$

Expressing equation (1) of lines of regression y in terms of x and x in terms of y, we have

$$8x - 10y + 66 = 0 \text{ as } \begin{cases} y = \dfrac{8}{10}x + \dfrac{66}{10} & \text{...(2)} \\ x = \dfrac{10}{8}y - \dfrac{66}{8} & \text{...(3)} \end{cases}$$

and

$$40x - 18y - 214 = 0 \text{ as } \begin{cases} y = \dfrac{40}{18}x - \dfrac{214}{18} & \text{...(4)} \\ x = \dfrac{18}{40}y + \dfrac{214}{40} & \text{...(5)} \end{cases}$$

Now we take combination of a line y on x with the other line x on y from equation (2), (3), (4) and (5) thus

$$y = \frac{8}{10}x + \frac{66}{10} \quad \text{and} \quad x = \frac{18}{40}y + \frac{214}{40} \qquad \text{... (6)}$$

$$x = \frac{10}{8}y - \frac{66}{8} \quad \text{and} \quad y = \frac{40}{18}x - \frac{214}{18} \qquad \text{... (7)}$$

For pair of lines given by equation (6), we have coefficient of regression as

$$b_{yx} = \frac{8}{10} \quad b_{xy} = \frac{18}{40}$$

$$r^2 = b_{yx}\; b_{xy} = \frac{8 \times 18}{10 \times 40} = 0.3600$$

Hence as r > 0, we consider the equation (6) as the lines of regression for x, y distribution

$$r^2 = 0.36$$

$$\therefore \quad r = \pm 0.6$$

For these lines equation (6) $b_{yx} = \frac{8}{10}$ and $b_{xy} = \frac{18}{40}$ are positive we take r as positive.

$\therefore r = 0.6 =$ correlation coefficient between x and y.

Now $\quad b_{yx} = r\dfrac{\sigma_y}{\sigma_x} \quad$ As $\quad \sigma_x^2 = 9 \quad \therefore \sigma_x = 3$

$$\therefore \quad \frac{8}{10} = 0.6\,\frac{\sigma_y}{3}$$

$$\therefore \quad \sigma_y = \frac{24}{6} = 4$$

END...

UNIT IV

7 Probability and Probability Distributions

7.1 : Probability

1) When an experiment is conducted and each outcome of the experiment has the same chance of appearing as any other then we call the outcomes as equally likely.

$$P(A) = \text{Probability of occurrence of any event A}$$

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

When an event succeeds in "S" ways and fails in "F" ways ($n = F + S$ = Total number of ways) then

$$P(\text{Success}) = \frac{S}{S+F} \quad \text{and} \quad P(\text{Failure}) = \frac{F}{S+F}$$

Generally the probability of success is denoted by p

and the probability of failure is denoted by q.

Obviously $p + q = 1$ i.e. $q = 1 - p$.

2) Theorems on Probability

a) Addition theorem

1) If A and B are mutually exclusive events.

then Prob (A or B)

i.e. $P(A \cup B) = P(A) + P(B)$

If A, B, C are mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

2) When the events are not mutually exclusive then probability that atleast one of the two events A and B will occur is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. $P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$

In case of three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$= -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

b) Multiplication theorem

If A and B are two independent events, then the probability that both will occur is equal to the product of their individual probabilities.

$$P(A \,\&\, B) = P(A) \times P(B)$$

Similarly $P(A, B \,\&\, C) = P(A) \times P(B) \times P(C)$

c) Conditional probability

Multiplication theorem is not applicable when events are dependent.

e.g. when we are computing prob. of a particular event A. When given information about occurrence of B. Such a probability is referred to as conditional probability.

$\therefore$ for two dependent events A and B probability of B, given A has occurred is denoted by

$$P(B|A) = \frac{P(A, B)}{P(A)}$$

Similarly, probability of A given B has occurred is

$$P(A/B) = \frac{P(A, B)}{P(B)}$$

$$\therefore \quad P(A \,\&\, B) = P(A) \times P(B/A)$$
$$P(A \,\&\, B) = P(B) \times P(A/B)$$

for three events A, B and C.

$$P(A, B \,\&\, C) = P(A) \times P(B/A) \times P(C/A, B)$$

Q.1 What is the probability that a leap year will contain 53 Mondays ?

Ans. : A leap year has 366 days.

This contains complete 52 weeks and two more days. These two days may take following combinations.

(i) Monday - Tuesday (ii) Tuesday - Wednesday (iii) Wednesday - Thursday (iv) Thursday - Friday (v) Friday - Saturday (vi) Saturday - Sunday and (vii) Sunday - Monday.

Out of these 7 combinations only 2 contain Monday.

$\therefore$ Required probability $= \frac{2}{7}$

Q.2 Prof. X and Madam Y appear for an interview for two posts. The probability of Prof. X's selection is $\frac{1}{7}$ and that of Madam Y's selection is $\frac{1}{5}$. Find the probability that only one of them is selected. What is probability that at least one of them is selected ? ☞ [SPPU : Dec.-11]

Ans. : $P(X) = \frac{1}{7}$ $\quad P(Y) = \frac{1}{5}$

$P(\overline{X}) = 1-\frac{1}{7}=\frac{6}{7}$ $\quad P(\overline{Y}) = 1-\frac{1}{5}=\frac{4}{5}$

As only one of this is selected

$\Rightarrow$ If X is selected, Y is not selected (Case A)

$\Rightarrow$ If Y is selected, X is not selected (Case B)

$$P(A) = P(X)\times P(\overline{Y}) = \frac{1}{7}\times\frac{4}{5} = \frac{4}{35}$$

$$P(B) = P(\overline{X})\times P(Y) = \frac{6}{7}\times\frac{1}{5} = \frac{6}{35}$$

$\therefore$ Required probability

$$P(A \text{ or } B) = P(A) + P(B)$$

$$\frac{4}{35}+\frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

$$P(\overline{X}) = \frac{6}{7},\ P(\overline{Y}) = \frac{4}{5}$$

$\therefore$ Probability that none is selected $= P(\overline{X})\times P(\overline{Y}) = \frac{6}{7}\times\frac{4}{5}$

$$= \frac{24}{35}$$

$\therefore$ P (at least one is selected) $= 1-\frac{24}{35}$

$$= \frac{11}{35}$$

Q.3 A bag contains 3 red and 5 black balls and a 2^{nd} bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that one is red and other is black.

Ans. : For 1^{st} bag

$$P(r_1) = \frac{3}{8} \quad \text{and } P(b_1) = \frac{5}{8}$$

For 2^{nd} bag

$$P(r_2) = \frac{6}{10} \quad \text{and } P(b_2) = \frac{4}{10}$$

$\therefore$ A ball is drawn from each bag 1 red from 1^{st} and 1 black from 2^{nd} or 1 black from 1^{st} and 1 red from 2^{nd}.

From first case

$$P(r_1 \text{ and } b_2) = P(r_1)\times P(b_2)$$

$$= \frac{3}{8}\times\frac{4}{10} = \frac{12}{80}$$

Second case $P(b_1 \text{ and } r_2) = \frac{5}{8}\times\frac{6}{10} = \frac{30}{80}$

$\therefore$ Required probability $= \frac{12}{80}+\frac{30}{80} = \frac{42}{80}$

Mutually exclusive case.

Q.4 Three coins are tossed once. Find the Probability of getting exactly 2 heads. Exactly two heads are possible in how many ways. HTH, HHT, THH.

Ans. : Probability (H) $= \frac{1}{2}$, Probability (T) $= \frac{1}{2}$

$\therefore$ Probability (HTH) $= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$

Probability (HHT) $= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$

Probability (THH) $= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$

There are mutually exclusive events.

$\therefore$ P(A or B or C) $= \frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{8}$

Q.5 A, B play a game of alternate tossing a coin one who gets head first wins the game. Find the probability that B wins the game if A has a start. [SPPU : Dec.-07]

Ans. : Following are the cases where B wins the game :

1) TH 2) TTTH 3) TTTTTH ....

We know $P(T) = \frac{1}{2}$ $P(H) = \frac{1}{2}$

$$\therefore \quad P((1)) = P(T) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$$

$$P((2)) = P(T)\,P(T)\,P(T)\,P(T)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4}$$

$$P((3)) = P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(H)$$

$$= \frac{1}{2^6}$$

$$\therefore \text{ Required probability} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} \ldots$$

which is a geometric series

$$a + a_r + a_r^2 \ldots\ldots + \frac{a}{1-r} \quad \text{with } a = \frac{1}{2^2} \text{ and } r = \frac{1}{2^2}$$

$$\therefore \text{ Required probability} = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Q.6 A student takes his examination in four subjects $\alpha, \beta, \gamma, \delta$. He estimates his chances of passing α as $\frac{4}{5}$, in β as $\frac{3}{4}$ in γ as $\frac{5}{6}$ in δ as $\frac{2}{3}$. To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies ?

Ans. : Here

$$P(\alpha) = \frac{4}{5}, \quad P(\beta) = \frac{3}{4},$$

$$P(\gamma) = \frac{5}{6}, \text{ and } P(\delta) = \frac{2}{3}$$

$$P(\bar{\alpha}) = 1 - \frac{4}{5} = \frac{1}{5}, \quad P(\bar{\beta}) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(\bar{\gamma}) = 1 - \frac{5}{6} = \frac{1}{6}, \quad P(\bar{\delta}) = 1 - \frac{2}{3} = \frac{1}{3}$$

There are four possibilities of passing at least two subjects,

I) Passing in β, γ and failing in δ

$$= P(\beta) \times P(\gamma) \times P(\bar{\delta}) = \frac{3}{4} \times \frac{5}{6} \times \frac{1}{3} = \frac{5}{24}$$

II) Passing in γ, δ and failing in β

$$= P(\gamma) \times P(\delta) \times P(\bar{\beta}) = \frac{5}{6} \times \frac{2}{3} \times \frac{1}{4} = \frac{5}{36}$$

III) Passing in δ, β and failing in γ

$$= P(\delta) \times P(\beta) \times P(\bar{\gamma})$$

$$= \frac{2}{3} \times \frac{3}{4} \times \frac{1}{6} = \frac{1}{12}$$

IV) Passing in β, γ, δ

$$= P(\beta) \times P(\gamma) \times P(\delta)$$

$$= \frac{3}{4} \times \frac{5}{6} \times \frac{2}{3} = \frac{5}{12}$$

$\therefore$ Probability of passing in at least two other subjects

$$= \frac{5}{24} + \frac{5}{36} + \frac{1}{12} + \frac{5}{12} = \frac{61}{72}$$

$\therefore$ Probability of passing α and at least two other subjects

$$= \frac{4}{5} \times \frac{61}{72} = \frac{61}{90}$$

7.2 : Binomial Distribution

I) Consider the experiment in which we perform a series of n independent trials. Each trial has only two outcomes or two mutually exclusive possibilities, a success or a failure.

Let p = Probability of getting a success

q = Probability of getting a failure

and $p + q = 1$

As r successes and $(n-r)$ failures can occur in ${}^nC_r$ mutually exclusive cases

$\therefore$ P [r successive in n trials] $= {}^nC_r \cdot p^r \cdot q^{n-r}$

Substituting r = 0, 1, 2, 3 ..... n we get the following table.

r	0	1	2	3	......	n
p (r)	${}^nC_0p^0q^n$	${}^nC_1p^1q^{n-1}$	${}^nC_2p^2q^{n-2}$	${}^nC_3p^3q^{n-3}$	......	${}^nC_np^nq^{n-n}$

$$ {}^nC_0 = 1, \quad {}^nC_n = 1 $$

Consider now the Binomial expansion of

$$ (q+p)^n = q^n + {}^nC_1q^{n-1}p + {}^nC_2q^{n-2}p^2 + \ldots + p^n $$

Terms of R.H.S. of this expansion give probability of r = 0, 1, 2, ..., n success. This is the reason for above probability distribution to be called Binomial probability distribution. It is denoted by B (n, p, r).

Thus $\quad B(n, p, r) = {}^nC_r \, p^r \, q^{n-r}$

II) Mean and Variance of the Binomial Distribution

☞ [SPPU : Dec.-04]

1) Mean $\therefore \quad \mu = np \quad$ as $\quad p + q = 1$

$\therefore$ Mean of binomial distribution is np.

2) Variance = npq and S.D. = $\sqrt{npq}$

Q.7 An average box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives.

☞ [SPPU : May-01, 09, Dec.-02, 06]

Ans. : Let $\quad$ p = Probability of box containing defective articles

$= 2/10 = 1/5$

q = Probability on non defective item

$= 4/5$

Probability of box containing three or less defective articles.

$= P(r \le 3) = P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

r = Number of defective items

$$ P(r = 0) = {}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10} = 0.10738 $$

$$ P(r = 1) = {}^{10}C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^{9} = 0.2684 $$

$$ P(r = 2) = {}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{8} = 0.302 $$

$$ P(r = 3) = {}^{10}C_3\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^{7} = 0.2013 $$

$P(r \le 3) = 0.10738 + 0.2684 + 0.302 + 0.2013 = 0.8791$

$100 \times 0.8791 = 87.91$

88 boxes are expected to contain three or less defectives.

Q.8 Mean and variance of binomial distribution are 6 and 2 respectively. Find $P(r \ge 1)$. ☞ [SPPU : Dec.-03, 11, May-13]

Ans. : Mean = np = 6

$$ \text{Variance} = npq = 2 $$
$$ 6 \times q = 2 $$
$$ q = 1/3 $$
$$ P = 1 - \frac{1}{3} = \frac{2}{3} $$

As $np = 6 \Rightarrow n \cdot \frac{2}{3} = 6 \Rightarrow n = 9$

$$ \therefore \; P(r \ge 1) = 1 - \left\{ {}^nC_0 \, p^r \cdot q^{n-r} \right\} $$
$$ = 1 - \left[{}^9C_0\left(\frac{2}{3}\right)^0 \times \left(\frac{1}{3}\right)^9 \right] = 0.9999 $$

Q.9 The incidence of a certain disease is such that on the average 20 % of workers suffer from it. If 10 workers are selected at random, find the probability that

i) Exactly 2 worker suffer from disease.

ii) Not more than 2 workers suffer. ☞ [SPPU : Dec.-14]

Ans. : Let $P = \frac{20}{100} = \frac{1}{5}, \quad q = 1 - P = \frac{4}{5}$

$n = 10$

By Binomial distribution

$$P(r) = {}^{n}C_r \, p^r q^{n-r} = {}^{10}C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{n-r}$$

i) $P(r = 2) = {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{10-2} = 0.302$

ii) $P(r \le 2) = P(r = 0) + P(r = 1) + P(r = 2)$

$$= {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} + {}^{10}C_1 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^9 + 0.302$$

$$= \mathbf{0.678}$$

7.3 : Possion Distribution

1) The distribution with frequencies given by, $\sum_{r=0}^{\infty} \frac{e^{-z} \cdot z^r}{r!}$ corresponding to 0, 1, 2, ... r, success is called Poisson distribution.

2) Variance $\sigma^2 = \lambda = nP$

3) Mean and Variance of Poisson's distribution = nP

Q.10 The accidents per shift in factory are given by,

Acc / Shift	0	1	2	3	4	5
Frequency	142	158	67	27	5	1

Find a Poisson distribution.

Ans. :

x_i	f_i	$f_i x_i$	
0	142	0	
1	158	158	Mean $= \frac{\sum f_i x_i}{\sum f_i}$
2	67	134	$= \frac{398}{400}$
3	27	81	
4	5	20	
5	1	5	$z = 0.995$
	400	398	

x_i	f_i	$f_i x_i$	$P(r) = \frac{z^r e^{-z}}{r!}$	Exp ≠ 400 × P (r)
0	142	0	0.3697	148
1	158	158	0.3678	147
2	67	134	0.183	73
3	27	81	0.607	24
4	5	20	0.0157	6.0
5	1	5	0.003	1

Q.11 If the probability that a concrete cube fails is 0.001. Determine the probability that out of 1000 cubes i) Exactly two ii) More than one cubes will fail. ☞ [SPPU : Dec.-15]

Ans. : Here P = 0.001, n = 1000, np = z = 1

Thus the poisson's distribution is

$$p(r) = \frac{z^r e^{-z}}{r!} = \frac{e^{-1}}{r!}$$

i) Probability that exactly 2 cubes fail

$$= P(r = 2) = \frac{e^{-1}}{r!} = \frac{1}{2e}$$

ii) Probability that more than one cube fail = $P(r \ge 2)$

$$= 1 - P(r < 2) = 1 - P(0) - P(1)$$

$$= 1 - e^{-1} - \frac{e^{-1}}{1} = 1 - \frac{2}{e}$$

$$= \frac{e-2}{e}$$

Q.12 A manufacturer of electronic goods has 4 % of his product defective. He sells the articles in packets of 300 and guarantees 90 % good quality. Determine the probability that a particular packet will violet the quarantee. ☞ [SPPU : May-15]

Ans. : We have P = 0.04, n = 300

$$z = np = 12$$

By Poisson distribution

$$P(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-12} 12^r}{r!}$$

The probability that a particular packet violet the quarantee is that

$$= P(r \geq 1) = 1 - \sum_{r=1}^{r} \frac{e^{-12} 12^r}{r!}$$

Q.13 If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability out of 2000 individuals. I) Exactly 3 II) More than 2 will suffer a bad reaction. ☞ [SPPU : May-13, Dec.-13]

Ans. : Here P = 0.001

$$n = 2000$$

$$\therefore \quad \lambda = nP = 0.001 \times 2000 = 2$$

$$\therefore \quad P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-2}\, 2^r}{r!}$$

$$\therefore \text{I)} \quad P(3) = \frac{e^{-2}\,(2)^3}{3!} = 0.136 \times \frac{8}{6}$$

II) Prob (more than 2) = P(3) + P(4) .... + P(2000)

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-2} \times 2^0}{0!} + \frac{e^{-2} \times 2^1}{1!} + \frac{e^{-2} \times 2^2}{2!}\right]$$

$$= 1 - e^{-2}[1+2+2]$$

$$= 1 - 0.136 \times 5$$

$$= 0.32$$

Q.14 Number of road accidents on a highway during a month follows a poisson distribution with mean 5. Find the probability that in a certain month number of accidents on the highway will be i) Less than 3 ii) Between 3 and 5 ☞ [SPPU : May-14]

Ans. : Let x : number of road accidents on a highway during a month.

We have mean = z = 5

$\therefore$ By Poisson distribution

$$p(r) = \frac{e^{-z} z^r}{r!} \quad ; \quad r = 0, 1, 2, ...$$

$$p(r) = \frac{e^{-5} 5^r}{r!} \quad ; \quad r = 0, 1, 2, ...$$

i) $p(r < 3) = p(r \leq 2)$

$$= p(r = 0) + p(r = 1) + p(r = 2)$$

$$= \frac{e^{-5} 5^0}{0!} + \frac{e^{-5} 5^1}{1!} + \frac{e^{-5} 5^2}{2!}$$

$$= 0.006738 + 0.03690 + 0.084224 = 0.124652$$

ii) $p(3 \leq r \leq 5) = p(3) + p(4) + p(5)$

$$= 0.140374 + 0.175467 + 0.175467$$

$$= 0.491308$$

Q.15 Between 2 p.m. to 3 p.m. the average number of phone calls per minute coming into company are 3. Find the probability that during one particular minute there will be 2 or less calls ☞ [SPPU : May-15, Dec.-14]

Ans. : Given that z = 3

By poisson distribution

$$p(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-3} 3^r}{r!} \quad ; \; r = 0, 1, 2, ...$$

$p(r \leq 2)$ = Probability that during one particular minute there will be 2 or less calls

$$p(r \leq 2) = p(r = 0) + p(r = 1) + p(r = 2)$$

$$= \frac{e^{-3}\,3^0}{0!} + \frac{e^{-3}\,3}{1!} + \frac{e^{-3}\,3^2}{2!}$$

$$= e^{-3}\left(1+3+\frac{9}{2}\right) = e^{-3}\left(\frac{17}{2}\right)$$

$$= e^{-3}\,(8.5) = \left(\frac{8.5}{e^3}\right) = 0.42319$$

7.4 : Normal Distribution

I) The general equation of the normal distribution is given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where the variable 'x' assumes all values from $-\infty$ to ∞ and σ, called the parameters of the distribution respectively are known as mean and standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$ x is called the normal variate and f (x) is probability density function of the normal distribution.

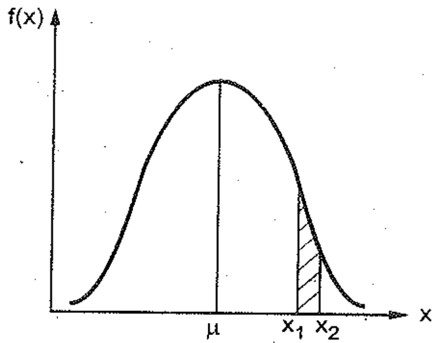

Fig. 7.1

The graph of the normal distribution is called the normal curve (some times known as normal probability curve or normal curve of errors). It is bell - shaped and symmetrical about the mean 'μ' as shown in the figure. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the X-axis respectively and gradually approach the X-axis without ever meeting it. The line $x = \mu$ divides the area under the normal curve above, X-axis into two equals parts. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values failing into the given interval. The total area under the normal curve above the x-axis is '1' i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\text{Thus } P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

II) Standard Form of the Normal Distribution

If 'X' is a normal random variable with mean 'μ' and standard deviation σ, then the random variable $z = \dfrac{X-\mu}{\sigma}$ has the normal distribution with mean '0' and standard deviation 1. The random variable z is called the Standard normal random variable.

Thus probability density function or the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} \quad (-\infty < z < \infty)$$

It is free from any parameter. This is useful to compute areas under the normal probability curve by making use of standard tables.

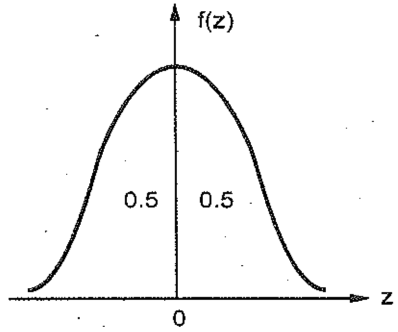

Fig. 7.2

Area under the Normal Curve :

The area under the normal curve is divided into two equal parts by $z = 0$. Left hand side area and right hand side area to $z = 0$ is 0.5.

III) Table of Area

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.6269
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852

0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9653	0.9664	0.9871	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9863	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9998	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9898

Table 7.1 In each row and each colomn 0.5 to be substracted

IV) Important formulae

1) The total area under the curve

= Sum of the probabilities = 1 i.e. $\int_{-\infty}^{\infty} y\,dx$

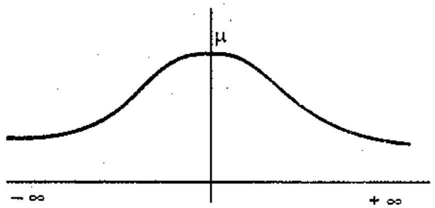

Fig. 7.3

2) $P(x_1 < x < x_2) = \int_{x_1}^{x_2} y\,dx$

= Area under the curve from x_1 to x_2

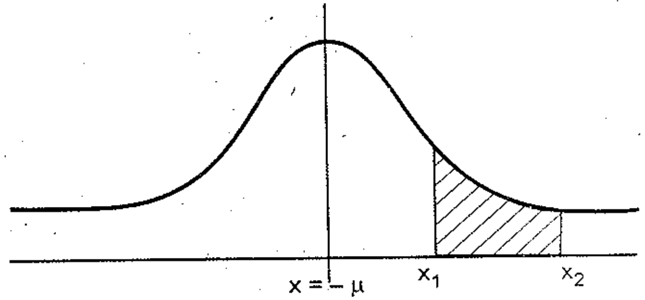

Fig. 7.4

3) $P(\mu < x < x_1) \quad = \quad \int_{\mu}^{x_1} y\,dx$

Put $\quad \dfrac{x-\mu}{\sigma} = z, \quad \dfrac{dx}{\sigma} = dz$

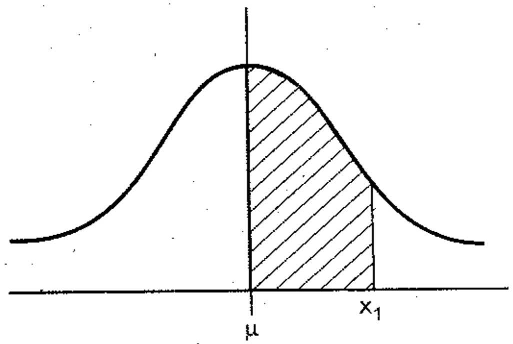

Fig. 7.5

$$P(0 < z < z_1) = \int_0^{z_1} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz$$

$$= \int_0^{z_1} (z)\, dz$$

is known as normal integral gives the area under the standard normal curve between $z = 0$ and $z = z_1$.

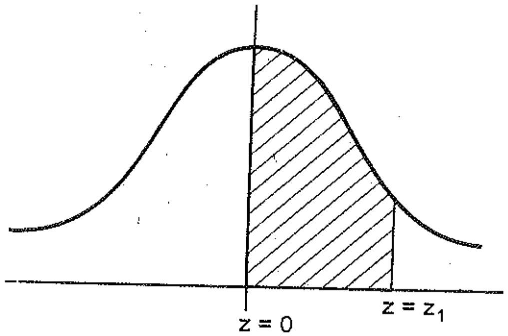

Fig. 7.6

4) $P(z_1 < z < z_2) = A(z_2) - A(z_1)$

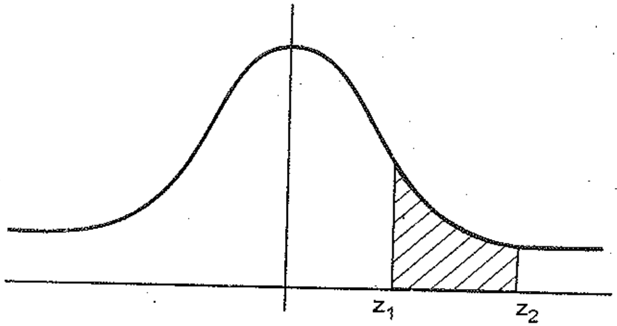

Fig. 7.7

5) $P(z > z_1) = 0.5 - A(z_1)$

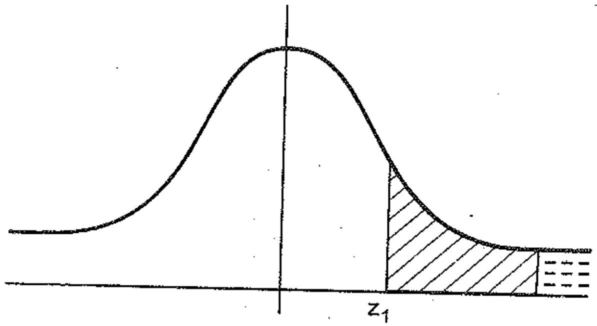

Fig. 7.8

6) $P(z < -z_1) = 0.5 - A(z_1)$

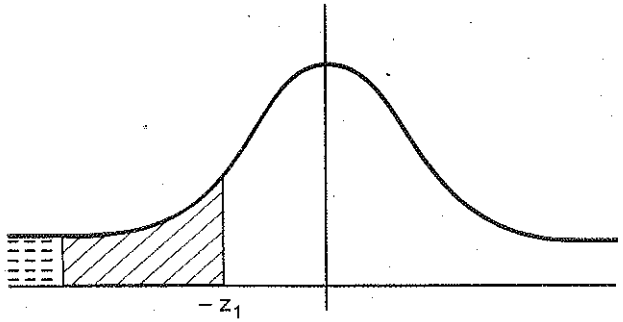

Fig. 7.9

7) $P(-z_1 < z < -z_2) = A(z_1) - A(z_2)$

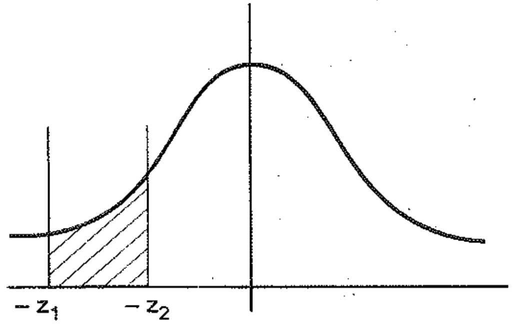

Fig. 7.10

8) $P(-z_1 < z < -z_2) = A(z_1) + A(z_2)$

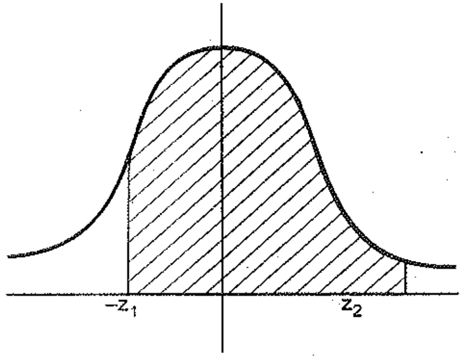

Fig. 7.11

Q.16 Assuming that the diagram of 1000 brass plugs taken consecutively from machine from a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm. How many of the plugs are likely to be approved if the acceptable diagram is 0.752 ± 0.004 cm.

☞ [SPPU : May-09, 16]

Given : A (Z = 2.25) = 0.4878, A (Z = 1.75) = 0.4599

Ans. : $\sigma = 0.0020, \quad \mu = 0.7515$

$$x_1 = 0.752 + 0.004 = 0.756$$

$$x_2 = 0.752 - 0.004 = 0.748$$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{0.756 - 0.7515}{0.0020} = 2.25$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{0.748 - 0.7515}{0.0020} = -1.75$$

A_1 corresponding to z_1 is 0.4878

A_2 corresponding to z_2 is 0.4599

$$p(0.748 < z < 0.756) = 0.4875 + 0.4599 = 0.9477$$

Thus, the number of plugs likely to be approved

$$= 1000 \times 0.9477$$

$$= 947.7 = \mathbf{948 \text{ approximately}}$$

Q.17 In a certain examination test 200 students appeared in subject of statistics. Average marks obtained were 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that marks are distributed normally.

☞ [SPPU : Dec.-05, 06, 12, May-10, 14]

Ans. : $\mu = 0.5, \quad \sigma = 0.05, \quad x_1 = 0.6$

$$z_1 = \frac{0.6 - 0.5}{0.05} = 2$$

A corresponding to z = 2 is 0.4772

$$P(x \geq 6) = 0.5 - 0.4772 = 0.0228$$

Number of students expected to get more than 60 % marks

$$= 0.0228 \times 200$$

$$= 46 \text{ students approximately.}$$

Q.18 In a distribution exactly normal 7 % of the items are under 35 and 89 % are under 63. Find the mean and standard deviation of the distribution.

☞ [SPPU : Dec.-10, May-14]

Ans. : From Fig. Q.18.1 it is clear that 7 % of items are under 35 means area under 35 is 0.07. Similarly area for $x \geq 63$ is 0.11.

$$P(x < 35) = 0.07, \quad P(x > 63) = 0.11$$

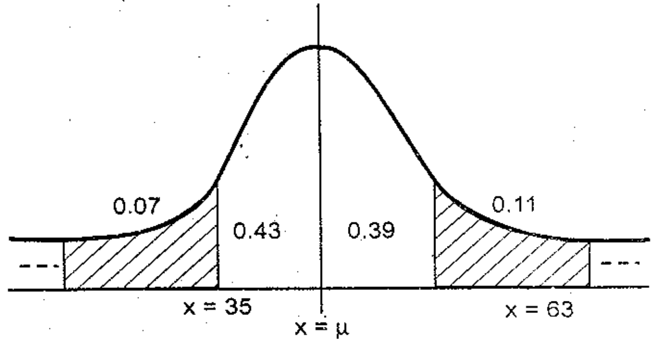

Fig. Q.18.1

x = 35, x = 63 are located as shown in Fig. Q.18.1

When $x = 35, \; z = \dfrac{35 - \mu}{\sigma} = -z_1$ (say)

(Negative sign because x = 35 is to the left of $x = \mu$)

When $x = 63, \; z = \dfrac{63 - \mu}{\sigma} = z_2$ (say)

Area A_2 for $P(0 < z < z_2) = 0.39$

$\therefore$ By Table 7.1, $z_2 = 1.23$

Area A_1 for $P(0 < z < z_1) = 0.43$

Corresponding $z_1 = 1.48$

Thus we get two simultaneous equation

$$\frac{35 - \mu}{\sigma} = -1.48 \quad \ldots (1)$$

$$\frac{63 - \mu}{\sigma} = 1.23 \quad \ldots (2)$$

Solving these two equations we get.

$$\therefore \quad \sigma = 10.33 \quad \mu = 50.3 \text{ (approx)}$$

Q.16 Assuming that the diagram of 1000 brass plugs taken consecutively from machine from a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm. How many of the plugs are likely to be approved if the acceptable diagram is 0.752 ± 0.004 cm.

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$$x_2 = 0.752 - 0.004 = 0.748$$

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A_1 corresponding to z_1 is 0.4878

A_2 corresponding to z_2 is 0.4599

$$p(0.748 < z < 0.756) = 0.4875 + 0.4599 = 0.9477$$

Thus, the number of plugs likely to be approved

$$= 1000 \times 0.9477$$

$$= 947.7 = \textbf{948 approximately}$$

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Ans. : $\mu = 0.5, \quad \sigma = 0.05, \quad x_1 = 0.6$

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A corresponding to z = 2 is 0.4772

$$P(x \geq 6) = 0.5 - 0.4772 = 0.0228$$

Number of students expected to get more than 60 % marks

$$= 0.0228 \times 200$$

$$= 46 \text{ students approximately.}$$

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☞ [SPPU : Dec.-10, May-14]

Ans. : From Fig. Q.18.1 it is clear that 7 % of items are under 35 means area under 35 is 0.07. Similarly area for $x \geq 63$ is 0.11.

$$P(x < 35) = 0.07, \quad P(x > 63) = 0.11$$

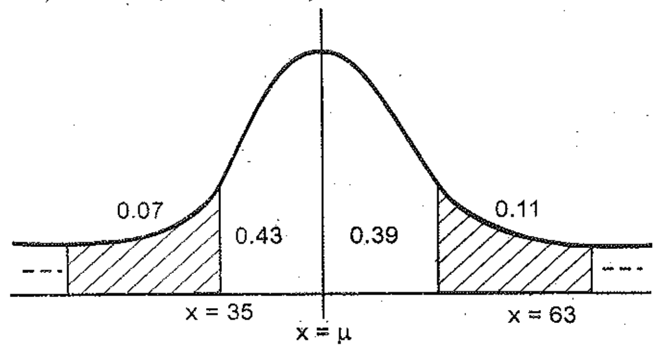

Fig. Q.18.1

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When $x = 35, \; z = \dfrac{35 - \mu}{\sigma} = -z_1$ (say)

(Negative sign because x = 35 is to the left of $x = \mu$)

When $x = 63, \; z = \dfrac{63 - \mu}{\sigma} = z_2$ (say)

Area A_2 for $P(0 < z < z_2) = 0.39$

$\therefore$ By Table 7.1, $z_2 = 1.23$

Area A_1 for $P(0 < z < z_1) = 0.43$

Corresponding $z_1 = 1.48$

Thus we get two simultaneous equation

$$\frac{35 - \mu}{\sigma} = -1.48 \quad \text{... (1)}$$

$$\frac{63 - \mu}{\sigma} = 1.23 \quad \text{... (2)}$$

Solving these two equations we get.

$$\therefore \quad \sigma = 10.33 \quad \mu = 50.3 \text{ (approx)}$$

Q.19 In an intelligence test administered to 1000 students the average score was 42 and standard deviation is 24. Find the number of students with score lying between 30 to 54.

[SPPU : May-15]

Ans. : Here n = 1000, $\bar{x} = u = 42$, $\sigma = 24$

$$z = \frac{x-u}{\sigma} = \frac{x-42}{24}$$

When $x_1 = 30$, $z_1 = \frac{30-42}{24} = \frac{-12}{24} = -\frac{1}{2}$

When $x_2 = 54$, $z_2 = \frac{54-42}{24} = \frac{1}{2}$

Given that for z = 0.5, Area = 0.1915

$$\therefore P(30 < x < 15) = P\left(-\frac{1}{2} < z < \frac{1}{2}\right)$$

$$= A\left(\frac{1}{2}\right) + A\left(-\frac{1}{2}\right)$$

$$= 0.1915 + 0.1915 = \mathbf{0.3830}$$

$\therefore$ The number of students with score lying between 30 to 54 is

= 0.3830 × 1000 = 383 students

Q.20 A random sample of 200 screws is drawn from a population which represents the size of screws. If a sample is distributed normally with a mean 3.15 cm of S.D. 0.025 cm. Find the expected number of screws whose size falls between 3.12 cm and 3.2 cm.

[SPPU : May-05, Dec.-14, 15]

Ans. : Given that $\mu = 3.15$, $\sigma = 0.025$

$$X_1 = 3.12 \text{ and } X_2 = 3.2$$

$$z_1 = \frac{X_1 - \mu}{\sigma} = -1.2, \quad z_2 = \frac{X_2 - \mu}{\sigma} = 2$$

$$\therefore P(3.12 < X < 3.2) = P(-1.2 < z < 2)$$

$$= P(0 < z < 1.2) + P(0 < z < 2)$$

$$= 0.3849 + 0.4772 = 0.8621$$

$\therefore$ Expected number of screws $= 200 \times 0.8621 \cong \mathbf{172}$

7.5 : Chi-square Distribution

1) Level of significance :

Probability of rejecting the null hypothesis H_0 when it is true is called as level of significance. It is denoted by α. Thus it is the probability of committing error of Type I.

If we try to minimize level of significance, the probability of error of Type II increases.

So level of significance cannot be made zero. However we can fix it in advance as 0.01 or 0.05 i.e. (1 % or 5 %). In most of the cases it is 5!.

Degrees of freedom	Distribution of χ^2	
	5 %	1 %
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
6	15.592	16.812
7	15.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
11	19.675	24.725
12	21.026	26.217
13	22.362	27.668
14	23.685	29.141
15	24.996	30.578
16	26.296	32.000
17	27.587	33.409
18	28.869	34.191

19	30.144	36.191
20	31.410	37.566
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638
24	36.415	42.980
25	37.652	44.314
26	38.885	45.642
27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892
40	55.759	63.691
60	79.082	88.379
∞	–	–

Table 7.2

2) Test for goodness of fit of χ^2 distribution :

Consider a frequency distribution, we try to fit some probability distribution.

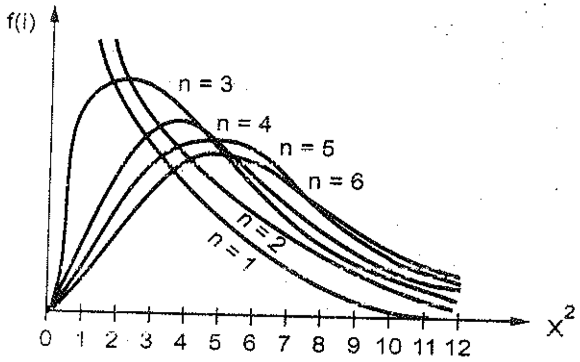


Fig. 7.12

Let H_0 : Fitting of the probability distribution to given data is proper.

As the test is based on χ^2 distribution. $\therefore$ known as χ^2 test of goodness of fit.

Suppose $O_1\ O_2\ \ldots\ O_i\ \ldots\ O_k$ be the observed frequencies and $e_1\ e_2\ \ldots\ e_i\ \ldots\ e_k$ be the expected frequencies or theoretical frequencies. There is no significant difference between observed and theoretical (expected) frequencies.

Let P = Number of parameters estimated for fitting the probability distribution.

$$N = \sum_{i=1}^{k} O_i = \sum_{i=k}^{k} e_i$$

If H_0 is true then the static

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} = \sum_{i=1}^{k} \frac{O_i^2 - 2e_i O_i + e_i^2}{e_i}$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2O_i + e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - \sum_{i=1}^{k} 2O_i + \sum_{i=1}^{k} e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2N + N$$

$$\chi^2 = \sum_{i=1}^{k} \frac{O_i^2}{e_i} - N$$

has Chi-square distribution with (k–p–1) degrees of freedom. If

$$\chi^2_{k-p-1} \geq \chi^2_{k-p-1,\ \alpha}$$

(calculated) (expected or table value)

Thus we reject H_0

Q.21 A set of five similar coins is tossed 210 times and the result is

No. of heads	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*2*	*5*	*20*	*60*	*100*	*31*

Test the hypothesis that the data follow a binomial distribution.

[SPPU : May-05, Dec.-07]

Ans. : Let H_0 : The data follows binomial distribution

Here $N = 210$

$$p_i = \text{Probability of getting head} = \frac{1}{2}$$

$$q_i = \text{Probability of getting tail} = \frac{1}{2}$$

Now $p(r) = \beta(n, p, r) = {}^nC_r \; p^r \; q^{n-r}$

$$= {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^5C_r \left(\frac{1}{2}\right)^5$$

$$\therefore \quad p(0) = {}^5C_0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$p(1) = {}^5C_1 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(2) = {}^5C_2 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(3) = {}^5C_3 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(4) = {}^5C_4 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(5) = {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

As expected frequency $E(x_i) = N \cdot p_i$ thus we prepare the table.

x_i	O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
0	2	1/32	7	– 5
1	5	5/32	35	– 30
2	20	10/32	70	– 50
3	60	10/32	70	– 10
4	100	5/32	35	65
5	37	1/32	7	30
	224			

$k = 6, p = 0$

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_5 = \frac{(-5)^2}{7} + \frac{(-30)^2}{35} + \frac{(-50)^2}{70} + \frac{(-10)^2}{70} + \frac{(+65)^2}{35} + \frac{(30)^2}{7}$$

$$= \frac{25}{7} + \frac{900}{35} + \frac{2500}{70} + \frac{100}{70} + \frac{4225}{35} + \frac{900}{7}$$

$$= 315.7142 \quad \text{(Calculated)}$$

$$\chi^2_{5,\,0.05} = 11.07 \quad \text{(Table value)}$$

$$\therefore \quad \chi^2_5 > \chi^2_{5,\,0.05}$$

$\therefore$ H_0 is rejected.

END...

UNIT V

8 Vector Calculus

8.1 : Algebra of Vectors

Let
$$\bar{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$$
$$\bar{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$$
$$\bar{c} = c_1\,\hat{i} + c_2\,\hat{j} + c_3\,\hat{k}$$

The unit vector in the direction of $\bar{a}$ is given by $\hat{a}$.
$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}, \quad |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

where $\hat{i}\,\hat{j}\,\hat{k}$ are the unit vectors in the direction of x, y, z axis respectively.

a_1, a_2, a_3 are components of $\bar{a}$ in the direction of x, y, z axis respectively.

Dot Product of Two Vectors : $\bar{a}\cdot\bar{b} = ab\cos\theta$

where θ is the angle between $\bar{a}$ and $\bar{b}$.
$$a = |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$b = |\bar{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ $\quad\because \cos 0 = 1$
- $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ $\quad\because \cos 90 = 0$
- $\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$ (Commutative)
- $\bar{a}\cdot\bar{b}$ is a scalar

- From formula of dot product the angle between two vectors $\bar{a}$ and $\bar{b}$ is given as
$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{ab}$$
- If two vectors are perpendicular then $\bar{a}\cdot\bar{b} = 0$.

Cross Product of Two Vectors

$$\bar{a}\times\bar{b} = a\,b\sin\theta\,\hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$.
$$\hat{n} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$$

- $\hat{i}\times\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\times\hat{k} = 0$ $\quad\because \sin 0 = 0$

  $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$ $\quad\because \sin 90 = 1$
- $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- $\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$ (Not commutative).
- Cross product of two vectors is a vector.
- If two vectors are parallel then $\bar{a}\times\bar{b} = 0$.

Scalar Triple Product

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = [\bar{a}\ \bar{b}\ \bar{c}]$$

It is a product of three vectors whose answer is a scalar. It is also known as a box product.
$$\bar{a}\cdot(\bar{b}\times\bar{c}) = [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Scalar triple product gives the volume of parallelopiped with edges a, b, c.
- If $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ then $\bar{a}$, $\bar{b}$, $\bar{c}$ are co-planer i.e. $\bar{a}$, $\bar{b}$, $\bar{c}$ lie in same plane.
- In S.T.P. cyclic changes are allowed $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$.
- In S.T.P. dot and cross are interchangeable i.e. $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$

Vector Triple Product

It is a product of three vectors whose answer is a vector.

$$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\,\bar{b}-(\bar{a}\cdot\bar{b})\,\bar{c}$$

$$\bar{1}\times(\bar{2}\times\bar{3}) = (\bar{1}\cdot\bar{3})\,\bar{2}-(\bar{1}\cdot\bar{2})\,\bar{3}$$

Quadruple Product

- $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d}\\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d}\end{vmatrix}$ (Scalar)
- $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\ \bar{c}-[\bar{a}\ \bar{b}\ \bar{c}]\ \bar{d}$ (vector)

Vector Differentiation

If $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ (Position vector)

then $\dfrac{d\bar{r}}{dt} = \dfrac{dx}{dt}\hat{i}+\dfrac{dy}{dt}\hat{j}+\dfrac{dz}{dt}\hat{k}$ (Velocity vector)

$$\frac{d^2\bar{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i}+\frac{d^2y}{dt^2}\hat{j}+\frac{d^2z}{dt^2}\hat{k} \quad \text{(Acceleration vector)}$$

Standard Results

1) $\dfrac{d}{dt}(\bar{u}\phi) = \bar{u}\dfrac{d\phi}{dt}+\phi\dfrac{d\bar{u}}{dt}$

2) $\dfrac{d}{dt}(\bar{u}\cdot\bar{v}) = \dfrac{d\bar{u}}{dt}\cdot\bar{v}+\bar{u}\cdot\dfrac{d\bar{v}}{dt}$

3) $\dfrac{d}{dt}(\bar{u}\times\bar{v}) = \dfrac{d\bar{u}}{dt}\times\bar{v}+\bar{u}\times\dfrac{d\bar{v}}{dt}$

[Order is important]

4) $\dfrac{d}{dt}[\bar{u}\ \bar{v}\ \bar{w}] = \left[\dfrac{d\bar{u}}{dt}\ \bar{v}\ \bar{w}\right]+\left[\bar{u}\ \dfrac{d\bar{v}}{dt}\ \bar{w}\right]+\left[\bar{u}\ \bar{v}\ \dfrac{d\bar{w}}{dt}\right]$

5) $\dfrac{d}{dt}\{\bar{u}\times(\bar{v}\times\bar{w})\} = \dfrac{d\bar{u}}{dt}\times(\bar{v}\times\bar{w})+\bar{u}\times\left(\dfrac{d\bar{v}}{dt}\times\bar{w}\right)+\bar{u}\times\left(\bar{v}\times\dfrac{d\bar{w}}{dt}\right)$

Q.1 If $\bar{r} = t^2\hat{i}+t\hat{j}-2t^3\hat{k}$ evaluate $\int_1^2 \bar{r}\times\dfrac{d^2\bar{r}}{dt^2}\,dt$

☞ [SPPU : May-12, Dec.-13]

Ans. : **Step 1 :** Consider $\bar{r} = t^2\hat{i}+t\hat{j}-2t^3\hat{k}$

Differentiate w.r.t. t

$$\frac{d\bar{r}}{dt} = 2t\hat{i}+\hat{j}-6t^2\hat{k}$$

Again differentiate w.r.t. t

$$\frac{d^2\bar{r}}{dt^2} = 2\hat{i}+0\hat{j}-12t\hat{k}$$

Step 2 : Now

$$\bar{r}\times\frac{d^2\bar{r}}{dt^2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2 & t & -2t^3\\ 2 & 0 & -12t\end{vmatrix}$$

$$= \hat{i}(-12t^2-0)-\hat{j}(-12t^3+4t^3+\hat{k}(0-2t)) = -12t^2\hat{i}+8t^3\hat{j}-2t\hat{k}$$

Step 3 : Integrating both sides w.r.t. t within the limits 1 to 2.

Thus $\displaystyle\int_1^2\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \int_1^2(-12t^2\hat{i}+8t^3\hat{j}-2t\hat{k})dt$

Step 4 : Integrate w.r.t. t use $\int t^n\,dt = \dfrac{t^{n+1}}{n+1}$

$$= \left[-12\frac{t^3}{3}\hat{i}+8\frac{t^4}{4}\hat{j}-2\cdot\frac{t^2}{2}k\right]_1^2$$

Step 5 : Substitute the upper limit and lower limit

$$= [-4(2^3)\hat{i}+2(2^4)\hat{j}-(2)^2\hat{k}]-[-4\hat{i}+2\hat{j}-\hat{k}]$$

$$= -32\hat{i}+32\hat{j}-4\hat{k}+4\hat{i}-2\hat{j}+\hat{k} = -28\hat{i}+30\hat{j}-3\hat{k}$$

Q.2 Find the angle between the normals to the surface $xy = z^2$ at (1, 2, 1) and at (3, 2, 3)

☞ [SPPU : Dec.-10, May-13]

Ans. : Grad ϕ represents the vector normal to the surface $\phi = C$.

Step 1 : Consider $\phi = xy-z^2$

Step 2 $\nabla\phi = i\dfrac{\partial\phi}{\partial x}+j\dfrac{\partial\phi}{\partial y}+k\dfrac{\partial\phi}{\partial z} = y\hat{i}+x\hat{j}-2z\hat{k}$

Step 3 : As $\nabla\phi$ represents the normal vector,

Let $\overline{N}_1$ and $\overline{N}_2$ be the values of $\nabla\phi$ at (1, 2, 1) and (3, 2, 3) respectively

$\therefore \quad \overline{N}_1 = (\nabla\phi)_{(1,2,1)} = 2\hat{i}+\hat{j}-2\hat{k}$

$\overline{N}_2 = (\nabla\phi)_{(3,2,3)} = 2\hat{i}+3\hat{j}-6\hat{k}$

Step 4 : Let θ be the required angle

$$\therefore \quad \cos\theta = \frac{\overline{N}_1\cdot\overline{N}_2}{|\overline{N}_1|\,|\overline{N}_2|} = \frac{4+3+12}{\sqrt{4+1+4}\sqrt{4+9+36}}$$

$$= \frac{19}{3\times 7} = \frac{19}{21}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$ which is the required angle.

Q.3 Show that tangent at any point of the curve $x = e^t$ cost, $y = e^t$ sint, $z = e^t$ makes a constant angle with z-axis.

☞ [SPPU : Dec.-12]

Ans. : Given that $x = e^t\cos t$, $y = e^t \sin t$, $z = e^t$

$$\therefore \quad \frac{dx}{dt} = e^t\cos t - e^t \sin t, \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t, \quad \frac{dz}{dt} = e^t$$

$$\therefore \quad \text{The tangent} = \overline{T} = \frac{d\overline{r}}{dt} = \frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}$$

$$\therefore \quad \overline{T} = (e^t\cos t - e^t \sin t)\,\hat{i} + (e^t \sin t + e^t \cos t)\,\hat{j} + e^t\,\hat{k}$$

$$\therefore \quad \hat{T} = \frac{(e^t\cos t - e^t\sin t)\,\hat{i}+e^t(\sin t+\cos t)\,\hat{j}+e^t\,\hat{k}}{\sqrt{(e^t\cos t-e^t\sin t)^2+e^{2t}(\cos t+\sin t)^2+e^{2t}}}$$

$$= \frac{e^t(\cos t-\sin t)\,\hat{i}+e^t(\sin t+\cos t)\,\hat{j}+e^t\,\hat{k}}{e^t\sqrt{(\cos^2 t+\sin^2 t)+(\cos^2 t+\sin^2 t)+1}}$$

$$\hat{T} = \frac{(\cos t-\sin t)\,\hat{i}+(\sin t+\cos t)\,\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\therefore \quad \cos\theta = \hat{T}\cdot\overline{k} = \frac{1}{\sqrt{3}}$$

∴ The tangent makes a constant angle with z-axis.

Q.4 For the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$. Find the velocity and acceleration of particle moving on the curve at $t = 0$.

☞ [SPPU : May-19, Marks 4]

Ans. : The displacement of the particle is given by

$$\overline{r} = xi + yj + zk$$

$$\overline{r} = e^t\cos t\, i+e^t\sin t\, j+e^t k$$

Velocity is given by

$$\overline{v} = \frac{d\overline{r}}{dt} = (e^t\cos t - e^t\sin t)i+(e^t\sin t+e^t\cos t)j+e^t k$$

Velocity at t = 0 is

$$\overline{v}\big|_{t=0} = (1-0)i + (0+1)j + k$$

$$= i + j + k$$

Acceleration is given by

$$\overline{a} = \frac{d\overline{v}}{dt}$$

$$= (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)$$

$$+ (e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t)j+e^t k$$

$$\overline{a} = -2e^t\sin t\, i+2e^t\cos t\, j+e^t k$$

Acceleration at t = 0 is

$$\overline{a}\big|_{t=0} = 0 + 2j + k$$

$$\overline{a}\big|_{t=0} = 2j + k$$

Q.5 Show that the necessary and sufficient condition for $\overline{F}$ to have constant magnitude is $\overline{F}\cdot\frac{d\overline{F}}{dt} = 0$.

☞ [SPPU : Dec.-08, 10, 11, 12]

Ans. : Step 1 : To prove the necessary part.

Let $\overline{F}$ be a vector of constant magnitude F.

Step 2 : Consider $\overline{F}\cdot\overline{F}$

$$\overline{F}\cdot\overline{F} = F\,F\cos 0 = F^2$$

$$\overline{F}\cdot\overline{F} = F^2 \quad \text{(Constant)}$$

Step 3 : Differentiate w.r.t. t.

$$\frac{d}{dt}(\overline{u}\cdot\overline{v}) = \frac{d\overline{u}}{dt}\cdot\overline{v}+\overline{u}\cdot\frac{d\overline{v}}{dt}$$

$$\frac{d\overline{F}}{dt}\cdot\overline{F}+\overline{F}\cdot\frac{d\overline{F}}{dt} = 0$$

Step 4 : We know that $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$

Thus

$$2\left(\bar{F} \cdot \frac{d\bar{F}}{dt}\right) = 0$$

$$\therefore \quad \bar{F} \cdot \frac{d\bar{F}}{dt} = 0$$

which proves the necessary part.

Step 5 : To prove the sufficient part i.e. conversely if $\bar{F} \cdot \frac{d\bar{F}}{dt} = 0.$

Step 6 : Again $\bar{F} \cdot \bar{F} = F^2$

Step 7 : Differentiate w.r.t. t, now F is not a constant

$$\frac{d\bar{F}}{dt} \cdot \bar{F} + \bar{F} \cdot \frac{d\bar{F}}{dt} = 2F\frac{dF}{dt}.$$

Step 8 : From step 5 L.H.S. is zero

$$\therefore \quad 0 = 2F\frac{dF}{dt}$$

Thus $\quad \frac{dF}{dt} = 0$

Step 9 : The derivative of F is zero thus F will be constant.

$$\therefore \quad F = \text{Constant}$$

i.e. $\quad |\bar{F}| = \text{Constant}$

i.e. $\bar{F}$ is a vector of constant magnitude which proves the sufficient part.

Q.6 Prove that the necessary and sufficient condition for a vector $\bar{F}$ to have constant direction is $\bar{F} \times \frac{d\bar{F}}{dt} = 0$.

[SPPU : Dec.-07, May-12]

Ans. : To prove the necessary part.

Let $\bar{F}$ be a vector with constant direction $\hat{F}$.

$\therefore$ $\hat{F}$ is a constant vector.

$$\hat{F} = \text{Unit vector in the direction of } \bar{F}$$

$$\therefore \quad \hat{F} = \frac{\bar{F}}{F}$$

$$\therefore \quad \bar{F} = F\hat{F}$$

$$\frac{d\bar{F}}{dt} = \hat{F}\frac{dF}{dt} \quad \text{(As } \hat{F} \text{ is constant keep it as it is)}$$

Consider the cross product of $\bar{F}$ with $\frac{d\bar{F}}{dt}$

$$\bar{F} \times \frac{d\bar{F}}{dt} = F\hat{F} \times \hat{F}\frac{dF}{dt}$$

$$\bar{F} \times \frac{d\bar{F}}{dt} = F\frac{dF}{dt}\left(\hat{F} \times \hat{F}\right)$$

$$\hat{F} \times \hat{F} = 0$$

$$\therefore \quad \bar{F} \times \frac{d\bar{F}}{dt} = 0$$

which proves the necessary part.

To prove the sufficient part.

i.e. conversely if $\bar{F} \times \frac{d\bar{F}}{dt} = 0$

Two vectors are parallel if $\bar{a} \times \bar{b} = 0$

$$\therefore \quad \bar{F} \parallel \frac{d\bar{F}}{dt}$$

Two vectors are parallel then their components are proportional.

If $\quad \bar{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$

$$\frac{d\bar{F}}{dt} = \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k}$$

Thus $\quad \frac{dF_1/dt}{F_1} = \frac{dF_2/dt}{F_2} = \frac{dF_3/dt}{F_3}$

$$\frac{dF_1/dt}{F_1} = \frac{dF_2/dt}{F_2}$$

Integrate w.r.t. t

$$\log F_1 = \log F_2 + \log C_1$$

$$\log F_1 = \log F_2 \, C_1$$

i.e. $$F_1 = F_2 \, C_1$$

or $$F_2 = \frac{F_1}{C_1}$$

i.e. $$F_2 = C_1' \, F_1$$

Thus we have expressed F_2 in terms of F_1.

Similarly we can express F_3 in terms of F_1

$$F_3 = C_2' \, F_1$$

$$\overline{F} = F_1 \, \hat{i} + F_2 \, \hat{j} + F_3 \, \hat{k}$$

$$= F_1 \, \hat{i} + F_1 \, C_1' \, \hat{j} + F_1 \, C_2' \, \hat{k}$$

$$\overline{F} = F_1 \left(\hat{i} + C_1' \, \hat{j} + C_2' \, \hat{k} \right)$$

Thus $\overline{F}$ is a vector with constant direction which proves the sufficient part.

8.2 : Scalar and Vector Point Functions

a) Scalar point function or scalar field : If corresponding to every triplet of values of (x, y, z). If a function u is defined such that $u(x, y, z)$ is a scalar number then the field is called a scalar field and u is called a scalar point function.

b) Vector point function or vector field : If corresponding to every triplet of values of (x, y, z). If a function $\overline{u}$ is defined such that $\overline{u}(x, y, z)$ is a vector then the field is called a vector field and $\overline{u}$ is called a vector point function.

The point functions (scalar or vector) satisfy the general rules of partial differentiation.

i.e. for $u(x, y, z)$ and $\overline{F}(x, y, z)$ we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$d\overline{F} = \frac{\partial \overline{F}}{\partial x} dx + \frac{\partial \overline{F}}{\partial y} dy + \frac{\partial \overline{F}}{\partial z} dz$$

c) Level surface : If $\phi(x, y, z)$ be a scalar point function defined over some region then $\phi(x, y, z) = c$ represents a level surface.

d) The operator ∇ or nabla : The vector differential operator (del)

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

8.3 : Gradient of a Scalar

1) Point function : If $\phi(x, y, z)$ is a scalar point function and the del operator ∇ operates on a scalar point function $\phi(x, y, z)$. We get a vector function $\nabla \phi$ and is called as gradient of a scalar point function ϕ (or Grad ϕ).

Thus

$$\text{Grad } \phi = \nabla \phi$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

Geometrical meaning of gradient :

Thus $(\nabla \phi)_P$ = Vector normal at point P

$$\hat{N} = \frac{\nabla \phi}{|\nabla \phi|} = \text{Unit vector normal.}$$

2) Standard results :

i) $\nabla (u \pm v) = \nabla u \pm \nabla v$

ii) $\nabla (u v) = u \nabla v + v \nabla u$

iii) $\nabla \left(\frac{u}{v} \right) = \frac{v \nabla u - u \nabla v}{v^2}$

iv) $\nabla f(u) = f'(u) \nabla u$

v) $\nabla (m u) = m \nabla u$ where m = Constant

vi) $\nabla f(r) = f'(r) \nabla r$

vii) $\nabla r = \frac{\overline{r}}{r}$

viii) $\nabla f(r) = f'(r) \frac{\overline{r}}{r}$

ix) $\nabla (\overline{a} \cdot \overline{r}) = \overline{a}$

where $\overline{a}$ is a constant vector

Q.7 Prove the following results.

i) $\nabla f(u) = f'(u) \nabla u$

ii) $\nabla f(r) = f'(r) \dfrac{\bar{r}}{r}$

iii) $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$

Ans. : i) Consider $\nabla f(u)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(u)$$

$$= \hat{i}\frac{\partial}{\partial x} f(u) + \hat{j}\frac{\partial}{\partial y} f(u) + \hat{k}\frac{\partial}{\partial z} f(u)$$

$$= \hat{i} f'(u)\frac{\partial u}{\partial x} + \hat{j} f'(u)\frac{\partial u}{\partial y} + \hat{k} f'(u)\frac{\partial u}{\partial z}$$

$$= f'(u)\left[i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right]$$

$$= f'(u) \nabla u$$

which gives the first result.

ii) From above result $\nabla f(r) = f'(r) \nabla r$

Now $\quad \nabla r = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) r = \hat{i}\dfrac{\partial r}{\partial x} + \hat{j}\dfrac{\partial r}{\partial y} + \hat{k}\dfrac{\partial r}{\partial z}$

We know that $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Similarly $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Thus $\quad \nabla r = \dfrac{x}{r}\hat{i} + \dfrac{y}{r}\hat{j} + \dfrac{z}{r}\hat{k} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$

$$\nabla r = \frac{\bar{r}}{r}$$

Thus $\quad \nabla f(r) = f'(r) \nabla r = f'(r) \dfrac{\bar{r}}{r}$

which gives the second result.

Let $\quad \bar{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \quad \bar{a} \cdot \bar{r} = a_1 x + a_2 y + a_3 z$$

$$\therefore \quad \frac{\partial}{\partial x}(\bar{a} \cdot \bar{r}) = a_1$$

$$\frac{\partial}{\partial y}(\bar{a} \cdot \bar{r}) = a_2 \quad \text{and} \quad \frac{\partial}{\partial z}(\bar{a} \cdot \bar{r}) = a_3$$

iii) Now consider $\nabla(\bar{a} \cdot \bar{r})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\bar{a} \cdot \bar{r})$$

$$= \hat{i}\frac{\partial}{\partial x}(\bar{a} \cdot \bar{r}) + \hat{j}\frac{\partial}{\partial y}(\bar{a} \cdot \bar{r}) + \hat{k}\frac{\partial}{\partial z}(\bar{a} \cdot \bar{r})$$

$$= a_1 i + a_2 j + a_3 k = \bar{a}$$

which proves the third result.

Q.8 Find constant a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface. $4x^2 y + z^3 = 4$ at $(1, -1, 2)$.

[SPPU : Dec.-06, 07, 12, May-06, 11]

Ans. :

Step 1 : Consider $\phi_1 = 0$, $\phi_2 = 0$ as two surfaces.

$$\phi_1 = a x^2 - b y z - (a + z) x = 0$$

$$\phi_2 = 4 x^2 y + z^3 - 4 = 0$$

Step 2 : Find $\nabla\phi_1$ and $\nabla\phi_2$

$(\nabla\phi_1) = (2ax - a - 2) i - bz j - (by) k = 0$

$(\nabla\phi_2) = 8 xy i + 4 x^2 j + 3 z^2 k - 0 = 0$

Step 3 : Substituting the point in $\nabla\phi_1$ and $\nabla\phi_2$ we get

$\bar{N}_1 = (\nabla\phi_1)_{(1, -1, 2)} = (2a - a - 2) i - 2 b j + b k = 0$

$\overline{N}_2 = (\nabla\phi_2)_{(1,-1,2)} = -8\,i + 4\,j + 12\,k = 0$

Step 4 : As two surfaces are orthogonal.

$$\overline{N}_1 \cdot \overline{N}_2 = 0$$

$$\cos\theta = 0 \qquad \theta = 90°$$

Consider

$$\overline{N}_1 \cdot \overline{N}_2 = -16\,a + 8\,a + 16 - 8\,b + 12\,b$$

$$0 = -8\,a + 4\,b + 16$$

$$0 = 4(-2\,a + b + 4)$$

$$\therefore \quad 2a = b + 4 \qquad \dots (1)$$

Step 5 : The point $(1, -1, 2)$ lies on

$ax^2 - by\,z = (a+2)x$

The point will satisfiy the equation

$$a\,x^2 - b\,y\,z = (a+2)\,x$$

$$a(1)^2 - b(-1)(2) = (a+3)$$

$$a + 2b = a + 2$$

$$b = 1 \qquad \dots (2)$$

Step 6 : Substitute the value of b in equation (1)

$$2a = b + 4$$

$$2a = 5$$

$$a = \frac{5}{2}$$

8.4 : Directional Derivative

1) The directional derivative of a scalar point function ϕ (x, y, z) in the direction of a vector $\overline{u}$, is a component of $\nabla\phi$ in the direction of $\overline{u}$.

If θ is the angle between $[\nabla\phi]_{\text{at P}}$ and $\overline{u}$ direction of $\overline{u}$ is $[\nabla\phi]_P \cdot \cos\theta = (\nabla\phi)_P \cdot \hat{u}$

Thus the directional derivative of ϕ at a point P is a scalar product of $(\nabla\phi)_P$ and $\hat{u}$ unit vector in the direction of $\overline{u}$.

As a component of a vector is maximum in its own direction. Therefore directional derivative is maximum in the direction of $\nabla\phi$ only, and its maximum magnitude is given by $|\nabla\phi|$.

2) Procedure for solving problems :

Note : For finding directional derivative we need

i) Scalar function ϕ (x, y, z) ii) Co-ordinates of point P

iii) Direction $\overline{u}$

If the above three things are not given then firstly find above three points and then use the following procedure.

Step 1 : From given ϕ (x, y, z) find $\nabla\phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Step 2 : Find $(\nabla\phi)$ at a point P

Step 3 : From $\overline{u}$ find $\hat{u} = \dfrac{\overline{u}}{|\overline{u}|}$

Step 4 : The required D.D. $= (\nabla\phi)_P \cdot \hat{u}$.

Q.9 Find D.D. of $\phi = x\,y^2 + y\,z^3$ at $(2, -1, 1)$ along the line $2(x-2) = (y+1) = z-1$. ☞ **[SPPU : Dec.-08, 12, May-15, 19]**

Ans. : Step 1 : Consider ϕ, $\phi = x\,y^2 + y\,z^3$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = y^2\,\hat{i} + \left(2xy + z^3\right)\hat{j} + \left(3yz^2\right)\hat{k}$$

Step 3 : Find $\nabla\phi$ at $(2, -1, 1)$

$$(\nabla\phi)_P = \hat{i} - 3\,\hat{j} - 3\,\hat{k}$$

Step 4 : Here $\overline{u}$ is not given.

To find $\overline{u}$ consider equation of line joining two points.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

i.e. $$\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$$

where
$$\alpha = x_2 - x_1$$
$$\beta = y_2 - y_1$$
$$\gamma = z_2 - z_1$$

are known as the direction ratios of the line and its direction is given by $\alpha\,\hat{i} + \beta\,\hat{j} + \gamma\,\hat{k}$.

∴ Consider the given equation of line

$$2(x-2) = y+1 = z-1$$

Divide by 2

$$\therefore \quad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{2}$$

Thus $\alpha = 1, \ \beta = 2, \ \gamma = 2$

∴ Direction $\bar{u} = \hat{i} + 2\hat{j} + 2\hat{k}$

Step 5 : Find $\hat{u}$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

Step 6 : D.D. $= (\nabla\phi)_P \cdot \hat{u}$

$$= (i-3j-3k)\cdot\frac{\hat{i}+2\hat{j}+2\hat{k}}{3} = \frac{1-6-6}{3} = -\frac{11}{3}$$

Q.10 Find D.D. of $\phi = e^{2x}\cos yz$ at the origin in the direction tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = at$ at $t = \pi/4$.

☞ [SPPU : May-06, 07, 11, 13, Dec.-10, 12, 16]

Ans. : Step 1 : Consider ϕ

$$\phi = e^{2x}\cos yz$$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = \left(2e^{2x}\cdot\cos yz\right)\hat{i} + \left(e^{2z}(-z)\sin yz\right)\hat{j} + \left(e^{2x}(-y)\sin yz\right)\hat{k}$$

Step 3 : Find $\nabla\phi$ at point P = (0 0 0)

$$(\nabla\phi)_P = 2\hat{i}+0\hat{j}+0\hat{k}$$

Step 4 : Here $\bar{u}$ is not given. To find $\bar{u}$.

The given direction is tangential to the curve at $t = \pi/4$.

i.e. $\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ at $t = \pi/4$

i.e. $(a\cos t)\hat{i} + (-a\sin t)\hat{j} + a\hat{k}$ at $t = \pi/4$

i.e. $\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

∴ $\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$ is the required direction say $\bar{u}$.

Step 5 : Find $\bar{u}$

$$\bar{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{a\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}\right)}{a\sqrt{\frac{1}{2}+\frac{1}{2}+1}}$$

$$= \frac{\left(\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{i}}{\sqrt{2}} + \hat{k}\right)}{\sqrt{2}} = \frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$$

Step 6 : D.D. $= (\nabla\phi)_P \cdot \hat{u}$

$$\text{D.D.} = \left(2\hat{i}-0\hat{j}+0\hat{k}\right)\cdot\left(\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right) - 0 - 0 = 1$$

Q.11 Find directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2, – 1, 2) along a line equally inclined with co-ordinate axes.

☞ [SPPU : May-08, Dec.-11, 14, 15]

Ans. : Step 1 : Given that $\phi = 4xz^3 - 3x^2y^2z$

Step 2 : Find $\nabla\phi$ at point $(2,-1,2)$

$$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$(\nabla\phi)_{(2,-1,2)} = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

Step 3 : D.D. along line equally inclined with co-ordinate axes

$$\therefore \quad \bar{u} = m\hat{i} + m\hat{j} + m\hat{k}$$

Step 4 : $\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

Step 5 : Directional derivative $= (\nabla\phi)_P \cdot \hat{u}$

$$= (8\hat{i}+48\hat{j}+84\hat{k})\cdot\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\text{D.D.} = \frac{140}{\sqrt{3}}$$

Q.12 If the directional derivative of $\phi = axy + byz + czx$ at (1, 1, 1) has maximum magnitude 4 in a direction parallel to x-axis. Find values of a, b, c. ☞ [SPPU : Dec.-08, 12, May-13, 16]

Ans. : Step 1 : Consider $\phi = axy + byz + czx$

$\therefore \nabla\phi = (ay + cz)\,\hat{i} + (ax + bz)\hat{j} + (by + cx)\hat{k}$

Step 2 : $(\nabla\phi)_{(1,1,1)} = (a+c)\,\hat{i} + (a+b)\,\hat{j} + (b+c)\,\hat{k}$

Step 3 : Given that

$(a+c)\,\hat{i} + (a+b)\,\hat{j} + (b+c)\,\hat{k} = 4\,\hat{i}$

$\therefore$ $a + c = 4$, $a + b = 0$, $b + c = 0$

Solving above equation, we get

$a = 2$, $b = -2$, $c = 2$

8.5 : Divergence of a Vector Point Function

Let $\overline{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ be any V.P.F. defined in a certain field.

The expression

$$\nabla \cdot \overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar point function, called the divergence of $\overline{F}$.

Thus

$$\boxed{\text{div } \overline{F} = \nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If $\nabla \cdot \overline{F} = 0$ then $\overline{F}$ is said to be **solenoidal.**

8.6 : Curl of a Vector Point Function

☞ [SPPU : Dec.-05, 06, 09, 10]

1) Let $\overline{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ be any V.P.F. defined in a certain field.

The expression

$$\nabla \times \overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right) - j\left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right) + k\left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)$$

which is a vector point function called the curl of $\overline{F}$.

If $\nabla \times \overline{F} = 0$ then E is said to be irrotational or conservative.

In this case there exists a scalar point function ϕ such that $\overline{F} = \nabla\phi$, where ϕ is called as scalar potential of $\overline{F}$.

The formula for obtaining scalar potential ϕ for an irrotational field $\overline{F}$ is

$$d\phi = F_1\,dx + F_2\,dy + F_3\,dz$$

Integration gives ϕ

$$\phi = \int_{\substack{y,\,z \\ \text{constant}}} F_1\,dx + \int_{\substack{z \text{ constant} \\ \text{free } x}} F_2\,dy + \int_{\substack{\text{free} \\ x,\,y}} F_3\,dz$$

> **Note :** The work done in moving an object from P to Q in an irrotational field is $[\phi]_P^Q$ where ϕ is scalar potential of $\overline{F}$.

2) Standard results :

If $\bar{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ (constant vector) and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (position vector) then

i) $\nabla \cdot \bar{a} = 0$ ii) $\nabla \times \bar{a} = 0$

iii) $\nabla \cdot \bar{r} = 3$ iv) $\nabla \times \bar{r} = 0$

v) $\nabla \cdot r^n \, \bar{r} = (n+3) r^n$ vi) $\nabla \times r^n \, \bar{r} = 0$

Proofs :

i) $$\nabla \cdot \bar{a} = \frac{\partial}{\partial x} a_1 + \frac{\partial}{\partial y} a_2 + \frac{\partial}{\partial z} a_3 = 0 + 0 + 0$$

ii) $$\nabla \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ a_1 & a_2 & a_3 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

iii) $$\nabla \times \bar{r} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 1 + 1 + 1 = 3$$

iv) $$\nabla \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

v) Step 1 : To find $\nabla \cdot r^n \, \bar{r}$ consider the formula

$$\nabla \cdot \phi \bar{F} = \nabla \phi \cdot \bar{F} + \phi (\nabla \cdot \bar{F})$$

Put $\phi = r^n$ and $\bar{F} = \bar{r}$

$$\therefore \quad \nabla \cdot r^n \, \bar{r} = \nabla r^n \cdot \bar{r} + r^n (\nabla \cdot \bar{r})$$

Step 2 : Use $\nabla f(r) = f'(r) \dfrac{\bar{r}}{r}$ and $\nabla \cdot \bar{r} = 3$

$$\nabla \cdot r^n \, \bar{r} = n \, r^{n-1} \frac{\bar{r}}{r} \cdot \bar{r} + r^n (3) = n \, r^{n-2} (\bar{r} \cdot \bar{r}) + r^n (3)$$

Step 3 : Put $\bar{r} \cdot \bar{r} = r^2$

$$\nabla \cdot r^n \, \bar{r} = n \, r^{n-2} (r^2) + 3 r^n = n r^n + 3 r^n$$

$$= (n+3) r^n$$

Proved.

vi) Step 1 : To find $\nabla \times r^n \, \bar{r}$ use the formula

$$\nabla \times \phi \bar{F} = \nabla \phi \times \bar{F} + \phi (\nabla \times \bar{F})$$

Put $\phi = r^n \quad \bar{F} = \bar{r}$

$$\therefore \quad \nabla \times r^n \, \bar{r} = \nabla r^n \times \bar{r} + r^n (\nabla \times \bar{r})$$

Step 2 : Use $\nabla f(r) = f'(r) \dfrac{\bar{r}}{r}$, $\nabla \times \bar{r} = 0$

$$\nabla \times r^n \, \bar{r} = n \, r^{n-1} \frac{\bar{r}}{r} \times \bar{r} + 0 = n \, r^{n-2} (\bar{r} \times \bar{r})$$

Step 3 : $\bar{r} \times \bar{r} = 0$

$$\therefore \quad \nabla \times r^n \, \bar{r} = 0$$

Q.13 Prove that $F = (2xy + z^3)\hat{i} + x^2 \hat{j} + 3xz^2 \hat{k}$ is conservative force field. Find its scalar potential and the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4).

[SPPU : Dec.-09, 10, May-06, 13]

Ans. : Consider $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0-0)\hat{i} - (3z^2 - 3z^2)\hat{j} + (2x - 2x)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

As $\nabla \times \bar{F} = 0$ $\therefore$ $\bar{F}$ is irrotational there exists a scalar point function and such that

$$\bar{F} = \nabla \phi$$

The scalar potential is given by

$$\phi = \int_{\substack{y,z \\ \text{const}}} f_1 \, dx + \int_{\substack{z \text{ const} \\ \text{free } x}} f_2 \, dy + \int_{\substack{\text{free} \\ x,\, y}} f_3 \, dz$$

Substituting the values of f_1, f_2, f_3

$$\phi = \int (2xy + z^3) \, dx + \int 0 \, dy + \int 0 \, dz$$

Integrate $\quad \phi = x^2 y + x z^3$

The work done in moving an object in an irrotational field is

$$= [\phi]_P^Q \qquad (\phi = \text{scalar potential of } \overline{F})$$

$$\therefore \quad \text{Work done} = \left[x^2 y + x z^3\right]_{(1,-2,1)}^{(3,1,4)} = 202 \quad \text{units}$$

Q.14 Show that $\overline{F} = r^2 \,\overline{r}$ is conservative. Obtain the scalar potential associated with it. ☞ **[SPPU : May-15, 16]**

Ans. : Consider $\nabla \times \overline{F} = \nabla \times r^2 \overline{r}$

$$= \nabla r^2 \times \overline{r} + f(r)(\nabla \times \overline{r}) = \frac{2r}{r}\overline{r} \times \overline{r} + f(r)(0)$$

$$\nabla \times \overline{F} = 0$$

$\therefore \overline{F}$ is irrotational.

$\therefore$ We can find ϕ such that $\overline{F} = \nabla \phi$

To find the scalar potential ϕ,

Consider $d\phi = F_1\, dx + F_2\, dy + F_3\, dz$

Now $\quad r = x\hat{i} + y\hat{j} + 2\hat{k},\ d\overline{r} = dx\hat{i} + dy\hat{j} + dz\,\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz$$

$$r dr = x dx + y dy + z dz$$

and $\quad \overline{F} = r^2 x\hat{i} + r^2 y\hat{j} + r^2 z\hat{k}$

$\therefore \quad d\phi = r^2 x\, dx + r^2 y\, dy + r^2 z\, dz$

$$= r^2 (x dx + y dy + z dz) = r^2 (r dr)$$

$$d\phi = r^3\, dr$$

Integrating, we get

$$\boxed{\phi = \frac{r^4}{4} + C}$$

Q.15 Show that $\overline{F} = (x^2 - yz)\,\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\,\hat{k}$ is irrotational. Also find the ϕ such that $\overline{F} = \nabla \phi$ ☞ **[SPPU : May 14]**

Ans. :

$$\text{Consider } \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \hat{i}(-x + x) - \hat{j}(-y + y) + \hat{k}(-z + z) = 0$$

$\therefore \quad \overline{F}$ is irrotational

To find the scalar potential ϕ

$$\phi = \int_{\substack{y,z\\ \text{const}}} (x^2 - yz)dx + \int_{\text{free } x} y^2\, dy + \int_{\substack{\text{free}\\ x,y}} z^2 dz + C$$

$$\phi = \frac{x^3}{3} - xyz + \frac{y^2}{2} + \frac{z^3}{3} + C$$

Q.16 A fluid motion is given by $\overline{v} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$. Is the motion irrotational. If so, find the velocity potential. ☞ **[SPPU : Dec.-11, 13, 14, 15, Marks 6]**

Ans. :

Given that $\overline{v} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$

$$\text{Consider } \nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= i\,(x \cos z + 2y - 2y - x \cos z) - j\,(y \cos z - y \cos z)$$

$$+ k\,(\sin z - \cos z)$$

$$\nabla \times \overline{F} = 0$$

Therefore $\overline{F}$ is irrotational

The velocity potential is given by

$$\phi = \int_{y,\,z \text{ constant}} F_1 dx + \int_{\substack{y,\,z \text{ constant}\\ \text{free } x}} F_2 dy + \int_{\text{free } x,\,y} F_3 dz$$

$$= \int (y\sin z - \sin x)dx + \int 2yz\,dy + \int 0\, dz$$

$$\phi = xy \sin z + \cos x + y^2 z + C$$

which is the required scalar potential.

Q.17 Show that $\overline{F} = (y\, e^{xy} \cos z)\, \hat{i} + (x\, e^{xy} \cos z)\, \hat{j} - (e^{xy} \sin z)\, \hat{k}$ is irrotational and find the scalar potential ϕ show that $\overline{F} = \nabla\phi$

[SPPU : May-10, 12, Dec.-12]

Ans. : Consider

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y e^{xy} \cos z & x e^{xy} \cos z & -e^{xt} y \sin z \end{vmatrix}$$

$$\nabla \times \overline{F} = \overline{0}$$

$\therefore$ $\overline{F}$ is irrotational.

$\therefore$ $\exists$ a scalar point function ϕ show that

$$\overline{F} = \nabla\phi$$

The scalar potential is given by

$$\phi = \int\limits_{\substack{y\, z \\ \text{constant}}} f_1\, dx + \int\limits_{\substack{z \text{ constant} \\ \text{free } x}} f_2\, dy + \int\limits_{\text{free } x,\, y} f_3\, dz + c$$

$$\therefore\ \phi = \int y\, e^{xy} \cos z\, dx + \int 0\, dy + \int 0\, dz + c$$

$$\phi = e^{xy} \cos z + c$$

8.7 : Vector Identities

1) Laplacian of ϕ : [SPPU : May-13]

The expression $\text{div}(\text{grad } f) = \nabla \cdot (\nabla\phi)$

$$= \left(\hat{i}\, \frac{\partial}{\partial x} + \hat{j}\, \frac{\partial}{\partial y} + \hat{k}\, \frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\, \frac{\partial\phi}{\partial x} + \hat{j}\, \frac{\partial\phi}{\partial y} + \hat{k}\, \frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

is known as Laplacian of ϕ

Here the operator $\nabla^2 = \nabla \cdot \nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is known as Laplacian or Laplace's operator.

The equation $\nabla^2 \phi = 0$

i.e. $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

is known as Laplace's equation in three dimensions.

2) Curl grad ϕ : $(\nabla \times \nabla\, \phi)$ [SPPU : May-13]

$$\text{Grad } \phi = \frac{\partial\phi}{\partial x}\, \hat{i} + \frac{\partial\phi}{\partial y}\, \hat{j} + \frac{\partial\phi}{\partial z}\, \hat{k}$$

$$\therefore \quad \nabla \times \nabla\, \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ \partial\phi / \partial x & \partial\phi / \partial y & \partial\phi / \partial z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\, \partial z} - \frac{\partial^2\phi}{\partial z\, \partial y}\right) + j\left(\frac{\partial^2\phi}{\partial z\, \partial x} - \frac{\partial^2\phi}{\partial x\, \partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\, \partial y} - \frac{\partial^2\phi}{\partial y\, \partial x}\right)$$

$$= 0\, \hat{i} + 0\, \hat{j} + 0\, \hat{k}$$

$$\boxed{\nabla \times \nabla\, \phi = 0}$$

$\therefore$ Curl grad $\phi = 0$

$\therefore$ Grad ϕ is an irrotational vector whatever scalar point function ϕ may be.

3) Divergence curl $\overline{F}$: $\nabla \cdot (\nabla \times \overline{F})$

Let $\overline{F} = F_1\, \hat{i} + F_2\, \hat{j} + F_3\, \hat{k}$

$$\therefore \quad \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\therefore$ Divergence curl $\overline{F} = \nabla \cdot (\nabla \times \overline{F})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_1}{\partial z\partial y} = 0$$

$$\therefore \quad \boxed{(\nabla\times\bar{F}) = 0}$$

As $\boxed{\operatorname{div}(\operatorname{curl}\bar{F}) = 0}$

curl $\bar{F}$ is a solenoidal. Vector, whatever vector point function $\bar{F}$ may be.

4) Curl (curl $\bar{F}$) :

Step 1 : Consider $\nabla\times(\nabla\times\bar{F})$ and use suffixes

$$\nabla\times(\nabla\times\bar{F}) = \nabla_1\times(\nabla_2\times\bar{F})$$

Step 2 : Use $\bar{1}\times(\bar{2}\times\bar{3}) = \bar{2}(\bar{1}\cdot\bar{3})-(\bar{1}\cdot\bar{2})\bar{3}$

$$= \nabla_2(\nabla_1\cdot\bar{F})-(\nabla_1\cdot\nabla_2)\bar{F}$$

Step 3 : Dropping the suffixes and use $\nabla\cdot\nabla = \nabla^2$

$$= \left[\nabla(\nabla\cdot\bar{F})-\nabla^2\bar{F}\right]$$

Thus $\boxed{\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F})-\nabla^2\bar{F}}$

Note that : $d(xy) = y\,dx + x\,dy$

$$= d_y(xy)+d_x(xy)$$

where suffixes of d are to be treated as a constant in each expression.

5) Divergence $(\phi\bar{F})$:

It is convenient to employ the symbolic method as follows

$$\nabla\cdot(\phi\bar{F}) = \nabla_\phi\cdot(\phi\bar{F})+\nabla_F\cdot(\phi\bar{F})$$

where the suffix of del (∇) is to be treated as a constant in each expression.

$$= \phi(\nabla_\phi\cdot\bar{F})+\nabla_F\,\phi\cdot\bar{F}$$

Dropping the suffixes

$$= \phi(\nabla\cdot\bar{F})+\nabla\phi\cdot\bar{F}$$

$$\therefore \quad \nabla\cdot(\phi\bar{F}) = \nabla\phi\cdot\bar{F}+\phi(\nabla\cdot\bar{F})$$

6) Curl $(\phi\bar{F})$: It is convenient to employ the symbolic method as follows.

$$\nabla\times(\phi\bar{F}) = \nabla_\phi\times(\phi\bar{F})+\nabla_F\times(\phi\bar{F})$$

where the suffix of del (∇) is to be treated as a constant in each expression.

$$= \phi(\nabla_\phi\times\bar{F})+\nabla_F\,\phi\times\bar{F}$$

Dropping the suffixes

$$= \phi(\nabla\times\bar{F})+\nabla\phi\times\bar{F}$$

$$\therefore \quad \boxed{\nabla\times(\phi\bar{F}) = \nabla\phi\times\bar{F}+\phi(\nabla\times\bar{F})}$$

7) Divergence $(\bar{A}\times\bar{B})$:

$$\nabla\cdot(\bar{A}\times\bar{B}) = \nabla_A\cdot(\bar{A}\times\bar{B})+\nabla_B\cdot(\bar{A}\times\bar{B})$$

Now, $\nabla_A\cdot(\bar{A}\times\bar{B}) = \bar{A}\cdot(\bar{B}\times\nabla_A) = -\bar{A}\cdot(\nabla_A\times\bar{B})$

and $\nabla_B\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla_B\times\bar{A})$

$\therefore$ $\nabla\cdot(\bar{A}\times\bar{B}) = -A\cdot(\nabla_A\times\bar{B})+\bar{B}\cdot(\nabla_B\times\bar{A})$

$= \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$

$\therefore$ $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$

$$\boxed{\text{Divergence }(\bar{A}\times\bar{B}) = \bar{B}\cdot(\text{curl})\,\bar{A}-\bar{A}\cdot\text{ curl }\bar{B}}$$

8) Curl $(\bar{u}\times\bar{v})$:

$\therefore$ $\nabla\times(\bar{u}\times\bar{v}) = \nabla_u\times(\bar{u}\times\bar{v})+\nabla_v\times(\bar{u}\times\bar{v})$

$\therefore$ $\nabla_u\times(\bar{u}\times\bar{v}) = (\nabla_u\cdot\bar{v})\,\bar{u}-(\bar{u}\cdot\nabla_u)\,\bar{v}$

$\nabla_v\times(\bar{u}\times\bar{v}) = (\bar{v}\cdot\nabla_v)$ we get $\bar{u}(\nabla_v\cdot\bar{u})\,\bar{v}$

$\therefore$ Dropping the suffixes and substituting

$$\boxed{\{\nabla\times(\bar{u}\times\bar{v}) = (\nabla\cdot\bar{v})\,\bar{u}-(\bar{u}\cdot\nabla)\,\bar{v}+(\bar{v}\cdot\nabla)\,\bar{u}-(\nabla\cdot\bar{u})\}}$$

9) Grad $(\bar{u}\cdot\bar{v})$:

$\nabla(\bar{u}\cdot\bar{v}) = \nabla_u(\bar{u}\cdot\bar{v})+\nabla_v(\bar{u}\cdot\bar{v})$

$\therefore$ $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\,\bar{b}-(\bar{a}\cdot\bar{b})\,\bar{c}$

$\therefore$ $\bar{b}(\bar{a}\cdot\bar{c}) = \bar{a}\times(\bar{b}\times\bar{c})+(\bar{a}\cdot\bar{b})\,\bar{c}$

$\therefore$ $\nabla_u(\bar{u}\cdot\bar{v}) = \bar{u}\times(\nabla_u\times\bar{v})+(\bar{u}\cdot\nabla_u)\,\bar{v} = \bar{u}\times(\nabla\times\bar{v})+(\bar{u}\cdot\nabla)\,\bar{v}$

Similarly

$$\nabla_v(\bar{u}\cdot\bar{v}) = \bar{v}\times(\nabla\times\bar{u})+(\bar{v}\cdot\nabla)\,\bar{u}$$

$$\boxed{\left\{\nabla(\bar{u}\cdot\bar{v}) = \bar{u}\times(\nabla\times\bar{v})+(\bar{u}\cdot\nabla)\,\bar{v}+\bar{v}\times(\nabla\times\bar{u})+(\bar{v}\cdot\nabla)\,\bar{u}\right\}}$$

Q.18 $\nabla^2 \mathbf{F(r)} = \mathbf{F''(r)}+\frac{2}{r}\mathbf{F'(r)}$

☞ [SPPU : Dec.-05, 06, 07, 08, 09, 11, 12, May-06, 07, 08, 09, 10, 11, 12, 13]

Ans. : Use $\nabla^2 = \nabla\cdot\nabla$

$$\nabla^2 F(r) = \nabla\cdot\nabla f(r)$$

Use $\nabla f(r) = \frac{f'(r)}{r}\,\bar{r} = \nabla\cdot\frac{f'(r)}{r}\,\bar{r}$

Use $\nabla\cdot\phi\bar{F} = \nabla\phi\cdot\bar{F}+\phi(\nabla\cdot\bar{F})$

Put $\phi = \frac{f'(r)}{r}$ $\quad \bar{F} = \bar{r}$

$$= \left(\nabla\frac{f'(r)}{r}\right)\cdot\bar{r}+\frac{F'(r)}{r}(\nabla\cdot\bar{r})$$

$$= \left[\frac{r\nabla f'(r)-f'(r)\nabla r}{r^2}\right]\cdot\bar{r}+\frac{f'(r)}{r}(3)$$

$$= \frac{1}{r^2}\left[r\,\frac{f''(r)}{r}\,\bar{r}-f'(r)\,\frac{\bar{r}}{r}\right]\cdot\bar{r}+3\,\frac{f'(r)}{r}$$

Take $\bar{r}$ common

$$= \frac{1}{r^2}\left[f''(r)-\frac{f'(r)}{r}\right](\bar{r}\cdot\bar{r})+3\,\frac{f'(r)}{r}$$

$$= f''(r)-\frac{1}{r}\,f'(r)+\frac{3}{r}\,f'(r)$$

Thus $\nabla^2 f(r) = f''(r)+\frac{2}{r}\,f'(r)$

Q.19 $\nabla^2\frac{(\bar{a}\cdot\bar{b})}{r} = 0$ ☞ [SPPU : Dec.-05, 12, 13, May-15]

Ans. : As $\bar{a}\cdot\bar{b}$ is a constant therefore we can take $\bar{a}\cdot\bar{b}$ outside

$$= (\bar{a}\cdot\bar{b})\left(\nabla^2\frac{1}{r}\right)$$

Use $\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r) = (\bar{a}\cdot\bar{b})\left[\left(\frac{1}{r}\right)''+\frac{2}{r}\left(\frac{1}{r}\right)'\right]$

Find the derivatives

$$= (\bar{a}\cdot\bar{b})\left[\frac{2}{r^3}+\frac{2}{r}\left(\frac{-1}{r^2}\right)\right]$$

$$= (\bar{a}\cdot\bar{b})\left[\frac{2}{r^3}-\frac{2}{r^3}\right] = (\bar{a}\cdot\bar{b})\cdot 0 = 0$$

Q.20 $\nabla^2(\mathbf{r^2 e^r}) = (\mathbf{r^2+6r+6})\,\mathbf{e^r}$ ☞ [SPPU : Dec.-10, May-12]

Ans. :

Let $f(r) = r^2 e^r$ and find derivatives

$$f(r) = r^2 e^r$$
$$f'(r) = 2re^r+r^2e^r$$
$$f''(r) = 2e^r+2re^r+2re^r+r^2e^r$$
$$= (2+4r+r^2)\,e^r$$

Use $\nabla f(r) = f''(r)+\frac{2}{r}\,f'(r)$

$$= (2+4r+r^2)\,e^r+\frac{2}{r}(2r+r^2)\,e^r$$
$$= (6+6r+r^2)\,e^r$$

Q.21 $\nabla^4\ (\mathbf{r^2\log r}) = 6/\mathbf{r^2}$

☞ [SPPU : Dec.-06, 07, 11, 14, 16, May-11]

Ans. : Consider $\nabla^2 r^2\log r$ and use

$$\nabla^2(r^2\log r) = (r^2\log r)''+\frac{2}{r}(r^2\log r)'$$

Find derivatives of $r^2\log r$ and substitute

$$\nabla^2(r^2\log r) = (3+2\log r)+\frac{2}{r}(r+2r\log r)$$

Simplify

$$\nabla^2 \left(r^2 \log r\right) = 5+6 \log r$$

Consider $\nabla^4 \left(r^2 \log r\right)$

$$\nabla^4 \left(r^2 \log r\right) = \nabla^2 \left(\nabla^2 r^2 \log r\right)$$

$$= \nabla^2 (5+6 \log r)$$

Use

$$\nabla^2 f(r) = f''(r)+\frac{2}{r} f'(r)$$

$$\nabla^2 (5+6 \log r) = (5+6 \log r)'' + \frac{2}{r} (5+6 \log r)'$$

$$= \left(-\frac{6}{r^2}\right)+\frac{2}{r}\left(0+\frac{6}{r}\right) = -\frac{6}{r^2}+\frac{12}{r^2}$$

$$= \frac{6}{r^2}$$

Q.22 Prove that

$$\nabla \times \left(\frac{\bar{a}\times \bar{r}}{r^n}\right) = \frac{(2-n)\,\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$$

☞ [SPPU : Dec.-05, 08, 09, 10, 11, 12,13, May-06, 07, 11,15]

Ans. : Use formula :

$$\bar{1}\times\left(\bar{2}\times\bar{3}\right) = \left(\bar{1}\cdot\bar{3}\right)\bar{2}-\left(\bar{2}\cdot\bar{1}\right)\bar{3}$$

$$\therefore \nabla\times\left(\bar{a}\times\frac{\bar{r}}{r^n}\right) = \left(\nabla\cdot\frac{\bar{r}}{r^n}\right)\bar{a}-(\bar{a}\cdot\nabla)\frac{\bar{r}}{r^n}$$

Use formula $\nabla\cdot r^n\,\bar{r} = (n+3)\,r^n$

$$= (-n+3)\,r^{-n}\,\bar{a}-(\bar{a}\cdot\nabla)\frac{1}{r^n}\,\bar{r}$$

Use $(\bar{a}\cdot\nabla)\,\phi\,\bar{F} = \phi(\bar{a}\cdot\nabla)\,\bar{F}+\bar{F}(\bar{a}\cdot\nabla)\,\phi$

product rule for $\bar{a}\cdot\nabla$

$$= (-n+3)\,r^{-n}\,\bar{a}-\frac{1}{r^n}(\bar{a}\cdot\nabla)\,\bar{r}-\bar{r}(\bar{a}\cdot\nabla)\frac{1}{r^n}$$

Use $(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$

$$= \frac{(3-n)\,\bar{a}}{r^n}-\frac{\bar{a}}{r^n}-\bar{r}\left(\bar{a}\cdot\nabla\frac{1}{r^n}\right)$$

Use $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ i.e. $\nabla\frac{1}{r^n} = \frac{-n}{r^{n+1}}\frac{\bar{r}}{r}$

$$= \frac{(2-n)\,\bar{a}}{r^n}-\bar{r}\left(\bar{a}\cdot\frac{-n}{r^{n+1}}\frac{\bar{r}}{r}\right)$$

$$= \frac{(2-n)\,\bar{a}}{r^n}+\frac{n\,\bar{r}(\bar{a}\cdot\bar{r})}{r^{n+2}}$$

Q.23 Prove that $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

☞ [SPPU : Dec.-06, 07, May-11, 12]

Ans. :

Use $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u-u\nabla v}{v^2}$

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{r^n\,\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\bar{r})\nabla r^n}{r^{2n}}$$

Use $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ and $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

$$\nabla r^n = n\,r^{n-1}\frac{\bar{r}}{r} = n\,r^{n-2}\,\bar{r}$$

$$\therefore \nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{r^n\,\bar{a}-(\bar{a}\cdot\bar{r})\,n\,r^{n-2}\,\bar{r}}{r^{2n}} = \frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$$

Q.24 Prove that $\nabla\times[\bar{a}\times(\bar{b}\times\bar{r})] = \bar{a}\times\bar{b}$ ☞ [SPPU : May-10, 11, 13]

Ans. : Consider L.H.S. $\nabla\times\left[\bar{a}\times\left(\bar{b}\times\bar{r}\right)\right]$

Use formula :

$$\left[\bar{a}\times\left(\bar{b}\times\bar{c}\right) = (\bar{a}\cdot\bar{c})\,b-(\bar{a}\cdot b)\,\bar{c}\right]$$

$$\left[1\times(2\times 3) = (1\cdot 3)\,2-(1\cdot 2)\,3\right]$$

$$\text{L.H.S.} = \nabla\times\left[(a\cdot r)\,\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{r}\right]$$

Open the [ ].

$$= \nabla\times(\bar{a}\cdot\bar{r})\,\bar{b}-\nabla\times(\bar{a}\cdot\bar{b})\,\bar{r}$$

Use formula : $\nabla\times\phi\bar{F}=\phi(\nabla\times\bar{F})+\nabla\phi\times\bar{F}$ for both the terms.

Put $\phi=(\bar{a}\cdot\bar{r})$ $\bar{F}=\bar{b}$ and $\phi=(\bar{a}\cdot\bar{b})$, $\bar{F}=\bar{r}$

$$=\left[(\bar{a}\cdot\bar{r})(\nabla\times\bar{b})+\nabla(\bar{a}\cdot\bar{r})\times b\right]-\left[(\bar{a}\cdot\bar{b})(\nabla\times\bar{r})+\nabla(\bar{a}\cdot\bar{b})\times\bar{r}\right]$$

As $\nabla\times\bar{b}=0$, $\nabla(\bar{a}\cdot\bar{r})=a$, $\nabla\times\bar{r}=0$, $\nabla(\bar{a}\cdot\bar{b})=0$

$$=0+\bar{a}\times\bar{b}-0-0$$

$$=\bar{a}\times\bar{b}$$

Q.25 $\bar{b}\times\nabla\,[\bar{a}\cdot\nabla\log r]$

$$=\frac{\bar{b}\times\bar{a}}{r^2}-\frac{2(\bar{a}\cdot\bar{r})}{r^4}(\bar{b}\times\bar{r})$$

☞ [SPPU : May-05, Dec.-13, 15]

Solution :

$$\text{Use } \nabla f(r) = f'(r)\frac{\bar{r}}{r}$$

$$\therefore \quad \nabla\log r = \frac{1}{r}\frac{\bar{r}}{r}=\frac{\bar{r}}{r^2}$$

Consider

$$\bar{a}\cdot\nabla\log r = \frac{\bar{a}\cdot\bar{r}}{r^2}$$

Operate ∇ on both sides

$$\nabla[\bar{a}\cdot\nabla\log r] = \nabla\left[\frac{(\bar{a}\cdot\bar{r})}{r^2}\right] = \frac{r^2\,\nabla(a\cdot r)-(\bar{a}\cdot\bar{r})\nabla r^2}{r^4}$$

Use $\nabla(\bar{a}\cdot\bar{r})=\bar{a}$, $\nabla r^2 = 2r\frac{\bar{r}}{r}=2\bar{r}$

$$=\frac{r^2\,\bar{a}-(\bar{a}\cdot\bar{r})\,2\bar{r}}{r^4}$$

$$\therefore\nabla[\bar{a}\cdot\nabla\log r] = \frac{\bar{a}}{r^2}-\frac{2(\bar{a}\cdot\bar{r})}{r^4}\bar{r}$$

As cross product exists only for vectors.

Take x with $\bar{b}$ on both sides.

$$\bar{b}\times[\bar{a}\cdot\nabla\log r] = \frac{b\times\bar{a}}{r^2}-\frac{2(\bar{a}\cdot\bar{r})}{r^4}(\bar{b}\times\bar{r})$$

Q.26 For solenoidal. Vector $\bar{E}$ show that $\nabla\times\nabla\times\nabla\times\nabla\times\bar{E}=\nabla^4\bar{E}$.

☞ [SPPU : May-97, Dec.-15]

Ans. Consider $\nabla\times(\nabla\times\bar{E})$

Use $\nabla\times\nabla\times\bar{F} = \nabla(\nabla\cdot\bar{F})-\nabla^2\bar{F}=\nabla(\nabla\cdot\bar{E})-\nabla^2\bar{E}$

Given $\nabla\cdot\bar{E}=0$ As $\bar{E}$ solenoidal.

$$=0-\nabla^2\bar{E}$$

Let $\bar{F}=(-\nabla^2\bar{E})$

Consider L.H.S.

$$\nabla\times\nabla\times\nabla\times\nabla\times\bar{E}$$

Put $\nabla\times\nabla\times\bar{E}=\bar{F}$

$$=\nabla\times\nabla\times\bar{F}$$

Use vector identity

$$=\nabla(\nabla\cdot\bar{F})-\nabla^2\bar{F}$$

As $\nabla\cdot\bar{E}=0$ Thus $\nabla\cdot\bar{F}=0$

Put $\bar{F}=-\nabla^2\bar{E}$

$$=-\nabla^2(-\nabla^2\bar{E})$$

$$=\nabla^4\bar{E}$$

Q.27 Show that $\bar{F}=\frac{\bar{a}\times\bar{r}}{r^n}$ is solenoidal.

☞ [SPPU : May-12, Dec.-12]

Ans. : Consider $\nabla\times\bar{F}$

$$\nabla\times\bar{F} = \nabla\times\left(\bar{a}\times\frac{\bar{r}}{r^n}\right)$$

Formula

$$\nabla(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A})-\bar{A}\cdot(\nabla\times\bar{B})$$

Put $\bar{A}=\bar{a}$ and $\bar{B}=\frac{\bar{r}}{r_n}$ Thus

$$\nabla\cdot\left(\bar{a}\times\frac{\bar{r}}{r^n}\right) = \frac{\bar{r}}{r^n}\cdot(\nabla\times\bar{a})-\bar{a}\cdot\left(\nabla\times\frac{\bar{r}}{r^n}\right)$$

Use $\nabla \times \bar{a} = 0$ and $\nabla \times \frac{\bar{r}}{r^n} = 0$

$$= 0 - 0 = 0$$

$\therefore$ Solenoidal.

Q.28 Prove that

$$\nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right] = \frac{n(n-2)}{r^{n+1}}.$$

☞[SPPU : May-06, 07, 14, Dec.-08]

Ans. : Consider $\nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right]$

$$= \nabla \cdot \left[r \frac{(-n)}{r^{n+1}} \frac{\bar{r}}{r} \right] \qquad \left(\because \nabla f(r) = f'(r) \frac{\bar{r}}{r} \right)$$

$$= \nabla \cdot \left[\frac{(-n)\,\bar{r}}{r^{n+1}} \right] = (-n) \left\{ \nabla \cdot \frac{1}{r^{n+1}} \bar{r} \right\}$$

$$= (-n) \left\{ \nabla \frac{1}{r^{n+1}} \cdot \bar{r} + \frac{1}{r^{n+1}} \nabla \cdot \bar{r} \right\}$$

$$= (-n) \left\{ \frac{-(n+1)}{r^{n+2}} \frac{\bar{r}}{r} \cdot \bar{r} + \frac{3}{r^{n+1}} \right\}$$

$$= (-n) \left\{ \frac{-(n+1)}{r^{n+1}} + \frac{3}{r^{n+1}} \right\} = (-n) \left\{ \frac{-n+2}{r^{n+1}} \right\}$$

$$= \frac{n(n-2)}{r^{n+1}}$$

Q.29 Prove that $\nabla^4 r^4 = 120$.

☞[SPPU : May-14]

Ans. : We have $\nabla^4 r^4 = \nabla^4 (\nabla^2 r^4)$

$$= \nabla^2 \left[12 r^2 + \frac{2}{r} (4 r^3) \right] = \nabla^2 \; [12 \, r^2 + 8 \, r^2]$$

$$= \nabla^2 (20 \, r^2) = 40 + \frac{2}{r} (40 \, r) = 40 + 80$$

$$= 120$$

$$\therefore \quad \nabla^4 r^4 = 120$$

Q.30 Prove that

$$\bar{a} \cdot \nabla \left(\bar{b} \cdot \nabla \frac{1}{r} \right) = \frac{3(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})}{r^5} - \frac{\bar{a} \cdot \bar{b}}{r^3}$$

☞[SPPU : May-11, Dec.-12]

Ans. : We have

$$\nabla \frac{1}{r} = -\frac{1}{r^2} \frac{\bar{r}}{r} = \frac{-1}{r^3} \bar{r}$$

$$\bar{b} \cdot \nabla \frac{1}{r} = -r^{-3} (\bar{b} \cdot \bar{r})$$

$$\nabla \left(\bar{b} \cdot \nabla \frac{1}{r} \right) = \nabla (-r^{-3} (\bar{b} \cdot \bar{r}))$$

$$= -[(\nabla r^{-3})(\bar{b} \cdot \bar{r}) + r^3 \nabla (\bar{b} \cdot \bar{r})]$$

$$= -[(-3r^{-5})\bar{r}(\bar{b} \cdot \bar{r}) + r^3 \bar{b}]$$

Now

$$\bar{a} \cdot \nabla \left(\bar{b} \cdot \nabla \frac{1}{r} \right) = \frac{3(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})}{r^2} - \frac{\bar{a} \cdot \bar{b}}{r^3}$$

Q.31 Prove that $\nabla^2 [\nabla \cdot (r^{-2} \bar{r})] = 2r^{-4}$. ☞[SPPU : May-19, Marks 4]

Ans. :

$$\text{LHS} = \nabla^2 \left[\nabla \cdot \frac{\bar{r}}{r^2} \right]$$

$$= \nabla^2 \left[\frac{1}{r^2} (\nabla \cdot \bar{r}) + \nabla (r^{-2}) \cdot \bar{r} \right]$$

$$= \nabla^2 \left[\frac{1}{r^2} (3) + (-2 r^{-4} \bar{r}) \cdot \bar{r} \right]$$

$$= \nabla^2 \left[\frac{3}{r^2} - \frac{2}{r^4} (r^2) \right]$$

$$= \nabla^2 \left[\frac{1}{r^2} \right]$$

Using $\nabla^2 [f(r)] = f''(r) + \frac{2}{r} f'(r)$

Here $f(r) = \frac{1}{r^2}$, $f'(r) = \frac{-2}{r^3}$, $f''(r) = \frac{6}{r^4}$

$$\text{LHS} = \nabla^2\left(\frac{1}{r^2}\right)$$

$$= \frac{6}{r^4} + \frac{2}{r}\left(\frac{-2}{r^3}\right)$$

$$= \frac{2}{r^4} = \text{RHS}$$

Q.32 Prove that $\nabla \times \left[\frac{1}{r}(r^2\bar{a} + (\bar{a}\cdot\bar{r})\bar{r})\right] = 0.$

☞ [SPPU : May-19, Marks 4]

Ans. : $\text{LHS} = \nabla \times \left[\frac{1}{r}(r^2\bar{a} + (\bar{a}\cdot\bar{r})\bar{r})\right]$

$$= \nabla \times \left[r\bar{a} + \frac{1}{r}(\bar{a}\cdot\bar{r})\bar{r}\right]$$

$$= \nabla(r)\times\bar{a} + r(\nabla\times\bar{a}) + \nabla\left(\frac{1}{r}\right)(\bar{a}\cdot\bar{r})\times\bar{r} + \frac{1}{r}\nabla(\bar{a}\cdot\bar{r})\times\bar{r} + \frac{1}{r}(\bar{a}\cdot\bar{r})(\nabla\times\bar{r})$$

$$= \bar{r}^{-1}\,\bar{r}\times\bar{a} + r(0) + (-r^{-3}\bar{r})(\bar{a}\cdot\bar{r})\times\bar{r} + \frac{1}{r}\bar{a}\times\bar{r} + \frac{1}{r}(\bar{a}\cdot\bar{r})(0)$$

$$= \frac{1}{r}(\bar{r}\times\bar{a}) - \frac{1}{r^3}(\bar{a}\cdot\bar{r})(\bar{r}\times\bar{r}) + \frac{1}{r}(\bar{a}\times\bar{r})$$

$$= \frac{1}{r}(\bar{r}\times\bar{a}) - \frac{1}{r^3}(\bar{a}\cdot\bar{r})(0) - \frac{1}{r}(\bar{r}\times\bar{a})$$

$$= 0$$

$$= \text{RHS}$$

8.8 : Line Integral

I) Work done by $\bar{F}$

If $\bar{F}$ represents the force field, then $\bar{F}\cdot\hat{T}$ represents a force acting along the tangent to the curve C, under this force say the particle is displaced by distance say ds along C.

$\therefore \quad (\bar{F}\cdot\hat{T})\,ds = \text{Force} \times \text{Displacement} = \text{Work done}$

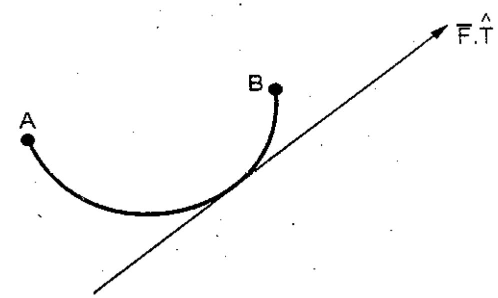


Fig. 8.1

$\therefore$ Total work done in moving a particle along C from A to B is given by

$$\text{Work done} = \int_A^B \bar{F}\cdot\hat{T}\,ds = \int_A^B \bar{F}\cdot d\bar{r}$$

II)

> **Note :** **Circulation** In fluid dynamics, if $\bar{V}$ represents the velocity of the fluid particle and C is the closed curve, then the integral $\oint_C \bar{V}\cdot d\bar{R}$ is called circulation of $\bar{V}$ round the curve C.

> **Note :** In evaluation of line integrals we have to express the line integral $\int_C \bar{F}\cdot d\bar{r}$ in terms of one variable either x or y or z or t or θ and then integrate within corresponding limits.

III) Reduction Formulae

1) $\int_0^{2\pi} \sin^m\theta\cos^n\theta\,d\theta = 0$ if both are not even.

$$\int_0^{2\pi} \sin^m\theta\cos^n\theta\,d\theta = 4\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta$$

if m, n both even.

2) $\int_0^{\pi} \sin^m \theta \cos \theta \, d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos \theta \, d\theta$

if n is even and any m.

$\int_0^{\pi} \sin^m \theta \cos \theta \, d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos \theta \, d\theta = 0$

if n is odd.

3) $\int_0^{\pi/2} \sin^m \theta \, d\theta = \int_0^{\pi/2} \cos^m \theta \, d\theta$

$\int_0^{\pi/2} \sin^m \theta \, d\theta = \left[\frac{(m-1) \ldots 2 \text{ or } 1}{(m) \ldots (2)}\right] \times \frac{\pi}{2}$ if m even

4) $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{[(m-1) \ldots][(n-1) \ldots]}{[(m+n) \ldots]} \times \left[\frac{\pi}{2}\right]$ if both even

5) $\int_0^{\pi} \sin \theta \cos \theta \, d\theta = \int_0^{\pi/2} \sin \theta \cos^m \theta \, d\theta = \frac{1}{m+1}$

Note :

1) For single integral over the circle $x^2 + y^2 = a^2$

Put $x = a \cos \theta$, $y = a \sin \theta$

$\therefore$ $dx = -a \sin \theta \, d\theta$, $dy = a \cos \theta \, d\theta$

and the limits of θ are 0 to 2π.

2) For single integral over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Put $x = a \cos \theta$, $y = b \sin \theta$

$\therefore$ $dx = -a \sin \theta \, d\theta$, $dy = b \cos \theta \, d\theta$

and the limits of θ are 0 to 2π.

Q.33 If $\bar{F} = x^2 \hat{i} + (x-y)\hat{j} + (y+z)\hat{k}$ displaces a particle from A(1, 0, 1) to B (2, 1, 2) along the straight line AB, find the work done.

☞ [SPPU : Dec.-15]

Ans. : We have $\bar{F} \cdot d\bar{r} = x^2 dx + (x-y)\, dy + (y+z)\, dz$

The equation of line joining (1, 0, 1) to (2, 1, 2) is

$$\frac{x-1}{2-1} = \frac{y-0}{1-0} = \frac{z-1}{2-1} \Rightarrow \frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{1} = t$$

$\therefore x = 1 + t,\ y = t,\ z = 1 + t$

$dx = dt,\ dy = dt,\ dz = dt$

Limits are from t = 0 to t = 11

$$\therefore \text{Work done} = \int_0^1 \bar{F} \cdot d\bar{r}$$

$$= \int_0^1 [(1+t)^2 dt + (1+t-t)dt + (t+1+t)dt]$$

$$= \int_0^1 [t^2 + 2t + 1 + 1 + 2t + 1]\, dt$$

$$= \int_0^1 [t^2 + 4t + 3]\, dt$$

$$= \left[\frac{t^3}{3} + 4\frac{t^2}{2} + 3t\right]_0^1$$

$$= \left(\frac{1}{3} + \frac{4}{2} + 3\right)(-0)$$

$$= \frac{2+12+18}{6} = \frac{32}{6} = \frac{16}{3}$$

$$\therefore \quad \text{Work done} = \frac{16}{3}$$

Q.34 Evaluate $\int_C \bar{F} \cdot d\bar{r}$ for $\bar{F} = 3x^2 \bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the following curve $x = \alpha t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$, $t = 1$.

☞ [SPPU : Dec.-18, Marks 6]

Ans. : We have $\bar{F} = 3x^2 \bar{i} + (2xz - y)\bar{j} + z\bar{k}$

$\therefore$ $\bar{F} \cdot d\bar{r} = 3x^2 dx + (2xz - y)\, dy + z\, dz$

Now, $x = \alpha t^2,\ y = t,\ z = 4t^2 - t$

$$dx = 2\alpha t\,dt\ ;\ dy = dt\ ;\ dz = (8t - 1)dt$$

$$\therefore \bar{F}\cdot d\bar{r} = 3\alpha^2 t^2\, 2\alpha t\,dt + (2\alpha t^2)(4t^2 - t)dt - t\,dt + (4t^2 - t)(8t - 1)dt$$

$$= 6\alpha^3 t^3 dt + (8\alpha t^4 - 2\alpha t^3)dt - t\,dt + (32t^3 - 12t^2 + t)dt$$

$$= [6\alpha^3 t^3 + 8\alpha t^4 - 2\alpha t^3 + 32t^3 - 12t^2]dt$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} [6\alpha^3 t^3 + 8\alpha t^4 - 2\alpha t^3 + 32t^3 - 12t^2]dt$$

$$= \left[6\alpha^3 \frac{t^4}{4} + 8\alpha\frac{t^5}{5} - 2\alpha\frac{t^4}{4} + 32\frac{t^4}{4} - 12\frac{t^3}{3}\right]_0^1$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \frac{6\alpha^3}{4} + \frac{8\alpha}{5} - \frac{\alpha}{2} + 8 - 4 = \frac{3}{2}\alpha^3 + \frac{11}{5}\alpha + 4$$

Q.35 Evaluate the line integral of vector point function $\bar{F} = (x^2 - y^2)\,i + 2xy\,j$ along the curve $y^2 = x$ from the point (0, 0) to (1, 1) in XY - plane. ☞ [SPPU : May-19, Marks 6]

Ans. : Line integral is

$$I = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_C \{(x^2 - y^2)i + 2xyj\}\cdot\{dxi + dyj\}$$

$$= \int_C (x^2 - y^2)dx + 2xy\,dy$$

Curve C is parabola $y^2 = x$ from point A(0, 0) to B(1, 1)

Let $y = t \Rightarrow x = y^2 = t^2$

Parametric equation of parabola is

$$x = t^2,\ y = t \Rightarrow dx = 2t\,dt,\ dy = dt$$

limits are

$$t = 0 \Rightarrow x = 0,\ y = 0 \quad \text{i.e. } A(0, 0)$$

$$t = 1 \Rightarrow x = 1,\ y = 1 \quad \text{i.e. } B(1, 1)$$

$$I = \int_C (x^2 - y^2)dx + 2xy\,dy$$

$$= \int_{t=0}^{1} (t^4 - t^2)(2t\,dt) + 2t^2\cdot t\cdot dt$$

$$= \int_{t=0}^{1} (2t^5 - 2t^3 + 2t^3)\,dt$$

$$= \int_{t=0}^{1} 2t^5 dt$$

$$= \left.\frac{t^6}{3}\right|_{t=0}$$

$$I = \frac{1}{3}$$

Q.36 If $\bar{F} = (2x + y^2)\,\hat{i} + (3y - 4x)\,\hat{j}$ then evaluate $\int_C \bar{F}\cdot d\bar{r}$ around the parabolic arc $y^2 = x$ joining (0, 0) to (1, 1). ☞ [SPPU : Dec.-11, May-15]

Ans. : We have $\bar{F}\cdot d\bar{r} = (2x + y^2)\,dx + (3y - 4x)\,dy$

Put $x = y^2 \Rightarrow dx = 2y\,dy$

$$\therefore \bar{F}\cdot d\bar{r} = (2y^2 + y^2)(2y\,dy) + (3y - 4y^2)\,dy$$

$$= 6y^3\,dy + (3y - 4y^2)\,dy$$

$$\bar{F}\cdot d\bar{r} = (6y^3 - 4y^2 + 3y)\,dy$$

As the curve is parabola $y^2 = x$ from (0, 0) to (1, 1). Thus limits of y are 0 to 1.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_0^1 (6y^3 - 4y^2 + 3y)\,dy = \left[6\frac{y^4}{4} - 4\frac{y^3}{3} + 3\frac{y^2}{2}\right]_0^1$$

$$= \frac{6}{4} - \frac{4}{3} + \frac{3}{2} - 0$$

$$= \frac{18 - 16 + 18}{12} = \frac{20}{12} = \frac{5}{3}$$

Q.37 Find the work done in moving a particle from (0, 1, – 1) to $\left(\frac{\pi}{2}, -1, 2\right)$ in a force field $\overline{F} = (y^2 \cos x + z^3)\,\hat{i} + (2y \sin x - 4)\,\hat{j} + (3xz^2 + 2)\,\hat{k}$.

☞ [SPPU : Dec.-14]

Ans. : We have

$$\overline{F}\cdot d\overline{r} = (y^2 \cos x + z^3)\,dx + (2y \sin x - 4)\,dy + (3xz^2 + 2)\,dz$$

$$= d\,(y^2 \sin x + xz^3 - 4y) + 2dz$$

Consider

$$\text{Work done} = \int \overline{F}\cdot d\overline{r}$$

$$= \int_{(0,1,-1)}^{\left(\frac{\pi}{2},-1,2\right)} [d(y^2 \sin x + xz^3 - 4y) + 2dz]$$

$$= \left[y^2 \sin x + xz^3 - 4y + 2z\right]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left[(-1)^2 \sin\frac{\pi}{2} + \left(\frac{\pi}{2}\right)(2)^3 - 4(-1) + 2(2)\right]$$

$$- [1^2 \sin 0 + 0\,(-1)^3 - 4\,(1) + 2(-1)]$$

$$= [1 + 4\pi + 4 + 4] - [-6]$$

$$= 15 + 4\pi$$

Q.38 Find work done in moving a particle once around ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ in plane z = 0, where $\overline{F} = (3x - 2y)\,\hat{i} + (2x + 8y)\,\hat{j} + y^2\hat{k}$.

☞ [SPPU : Dec.-08, May-09, 13]

Ans. : Consider

$$\overline{F} = (3x - 2y)\,\hat{i} + (2x + 8y)\,\hat{j} + y^2\hat{k}$$

Consider

$$\int_C \overline{F}\cdot d\overline{r} = \int (3x - 2y)\,dx + (2x + 8y) + \left(y^2\right) dz$$

a = 4 and b = 3 for ellipse.

$x = 4 \cos\theta \;\therefore\; dx = -4 \sin\theta\, d\theta$

$y = 3 \sin\theta \;\therefore\; dy = 3 \cos\theta\, d\theta$

$$\therefore \int \overline{F}\cdot d\overline{r} = \int [3\,(4\cos\theta) - 2\,(3\sin\theta)](-4\sin\theta\, d\theta)$$

$$+[2\,(4\cos\theta) + 8\,(3\sin\theta)](3\cos\theta\, d\theta) + [9\sin^2\theta]\,(0)$$

$$= \int_0^{2\pi} \begin{pmatrix} -48\sin\theta\cos\theta\, d\theta + 24\sin^2\theta\, d\theta + 24\cos^2\theta\, d\theta \\ +72\sin\theta\cos\theta\, d\theta \end{pmatrix}$$

$$= \int_0^{2\pi} 24\left(\sin^2\theta + \cos^2\theta\right) d\theta + 24\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{2\pi} 24\,(1)\,d\theta + 0 \quad \text{\{using conversion formula\}}$$

$$= 48\pi$$

Q.39 Find the work done in moving a particle along x = a cos θ, y = a sin θ, z = b θ from $\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$ under a field of force given by $\overline{F} = -3a \sin^2\theta \cos\theta\,\hat{i} + a\,(2 \sin\theta - 3 \sin^3\theta)\,\hat{j} + b \sin 2\theta\,\hat{k}$

☞ [SPPU : Dec.-13, May-14]

Ans. : We have x = a cos θ, y = a sin θ, z = bθ

$$dy = a \cos\theta\, d\theta,\; dz = b\,d\theta$$

$$\therefore \int \overline{F}\cdot d\overline{r} = \int_{\pi/4}^{\pi/2} \{-3a\sin^2\theta\cos\theta\,(-a\sin\theta\, d\theta)$$

$$+ (a)\,(2\sin\theta - 3\sin^3\theta)\,(a\cos\theta\, d\theta) + b\sin 2\theta\,(b\,d\theta)\}$$

$$= \int_{\pi/4}^{\pi/2} \{3a^2\sin^3\theta\cos\theta\, d\theta + 2a^2\sin\theta\cos\theta$$

$$- 3a^2\sin^3\theta\cos\theta\, d\theta + b^2\sin 2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \{2a^2\sin\theta\cos\theta + b^2\sin 2\theta\}\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} (a^2+b^2)\sin 2\theta\, d\theta = (a^2+b^2)\left[-\frac{\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= (a^2+b^2)\frac{1}{2}\left[-\cos\pi + \cos\frac{\pi}{2}\right]$$

$$= \frac{a^2+b^2}{2}[-(-1)+0] = \frac{a^2+b^2}{2}$$

Q.40 Evaluate $\int_C \overline{F}\cdot d\overline{r}$ for $\overline{F} = 3x^2\hat{i}+(2xz-y)\hat{j}+z\hat{k}$ along the path straight line joining (0, 0, 0) and (2, 1, 3).

☞ [SPPU : May-10, Dec.-12]

Ans. : Given that $\overline{F} = 3x^2\hat{i}+(2xz-y)\hat{j}+z\hat{k}$

Consider

$$\overline{F}\cdot d\overline{r} = 3x^2 dx + (2xz-y)dy + z\,dz$$

$$\text{Work done} = \int_C \overline{F}\cdot d\overline{r}$$

$$= \int_C [3x^2 dx + (2xz-y)dy + z\,dz] \quad \ldots(1)$$

The equation of line joining (0, 0, 0) to (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$\therefore$ $x = 2t,\ y = t,\ z = 3t \therefore t = 0 \text{ to } t = 1$

Equation (1) becomes

$$\text{Work done} = \int_0^1 [3(4t^2)2\,dt + (12t^2-t)dt + 3t3dt]$$

$$= \int_0^1 [36t^2+8t]dt = \left[36\frac{t^3}{3}+8\frac{t^2}{2}\right]_0^1$$

$$= 36\left(\frac{1}{3}\right)+\frac{8}{2}(1) = \mathbf{16}$$

Q.41 Find the work done in moving a particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and $z=0$ under the force field $\overline{F} = (2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$.

☞ [SPPU : Dec.-05, 11, 12, 15, May-10]

Ans. : Step 1 : We have

$$\overline{F} = (2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$$

Step 2 : Let $x = 5\cos\theta$, $y = 4\sin\theta$ and $z = 0$

$$\therefore \overline{F} = (10\cos\theta-4\sin\theta)\hat{i}+(5\cos\theta+4\sin\theta)\hat{j}+(15\cos\theta-8\sin\theta)\hat{k}$$

and $\overline{r} = x\hat{i}+y\hat{j}+z\hat{k} = 5\cos\theta\,\hat{i}+4\sin\theta\,\hat{j}+0\,\hat{k}$

$$d\overline{r} = (-5\sin\theta\,\hat{i}+4\cos\theta\,\hat{i})\,d\theta$$

Step 3 : Work done $= \int_0^{2\pi} \overline{F}\cdot d\overline{r}$

$$= \int_0^{2\pi} [-50\sin\theta\cos\theta+20\sin^2\theta+16\cos\theta\sin\theta+2\cos^2\theta]\,d\theta$$

$$= \int_0^{2\pi} [-34\sin\theta\cos\theta+20]\,d\theta = \int_0^{2\pi}[-17\sin 2\theta+20]\,d\theta$$

$$= \left[17\frac{\cos 2\theta}{2}+20\theta\right]_0^{2\pi} = 40\pi$$

8.9 Green's Theorem

1) Cartesian Form :

If R is a closed region of the XY-plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivation in R, then

$$\oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

where C is traversed in the positive (counter clockwise) direction.

2) Green's Theorem in Plane in Vector Notation

Green's theorem in the XOY plane can be written as

$$\oint_C \overline{F} \cdot d\overline{r} = \iint_R (\nabla \times \overline{F}) \cdot \hat{k}\, dx\, dy$$

Q.42 Use Green's theorem to evaluate the integral $\oint_C (xy\ dx + y^2\,dy)$. over the area bounded by curves $y = x^2$ and line $y = x$ in first quadrant. [SPPU : May-19, Marks 6]

Ans. : To evaluate

$$I = \oint_C xy\ dx + y^2\ dy$$

C is closed curve bounded by parabola $y = x^2$ and line $y = x$

Since Curve C is closed so Green's Lemma is applicable

By Green's Lemma

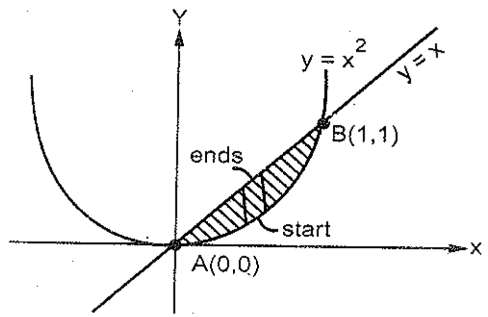


Fig. Q.42.1

$$\oint_C u\,dx + v\,dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy$$

Here $u = xy,\ v = y^2$

$$\frac{\partial u}{\partial y} = x,\ \frac{\partial v}{\partial x} = 0$$

$$I = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy$$

$$= \iint_A x\ dx\ dy$$

A : Area bounded by parabola $y = x^2$ and straight line $y = x$

limits are $x = 0,\ x = 1$

$y = x^2,\ y = x$

$$I = \int_{x=0}^{1} \int_{y=x^2}^{x} x\{dy\}dx$$

$$= \int_{x=0}^{1} x\left\{ y\Big|_{y=x^2}^{x} \right\}dx$$

$$= \int_{x=0}^{1} x\ \{x - x^2\}dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4}\Bigg|_{x=0}^{1}$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - (0)$$

$$I = \frac{1}{12}$$

Q.43 Verify Green's theorem in plane for $\oint_C [(3x^2 - 8y^2)\ dx + (4y - 6xy)\ dy]$ where C is the boundary of the region defined by $y = \sqrt{x},\ y = x^2$. [SPPU : Dec.-11]

Ans. : $y^2 = x,\ y = x^2$, intersects at O(0, 0), A(1, 1) we have

$M = 3x^2 - 8y^2,\ N = 4y - 6xy$

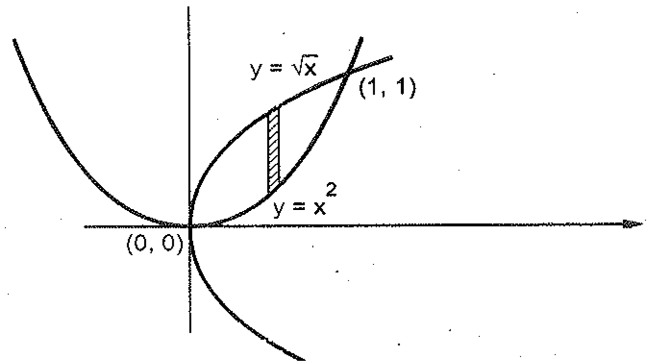


Fig. Q.43.1

$$\frac{\partial M}{\partial y} = -16\,y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10\,y$$

Let R be the region bounded by two parabolas.

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10\,y\;dy\;dx$$

$$= 5\int_0^1 [y^2]_{x^2}^{\sqrt{x}}\,dx = 5\int_0^1 (x - x^4)\,dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2} \qquad \dots (1)$$

Next $\oint_C M\,dx + N\,dy = \oint_{C_1} (M\,dx + N\,dy) + \oint_{C_2} (M\,dx + N\,dy)$

Along C_1; $x^2 = y \Rightarrow dy = 2x\,dx$, x varies from 0 to 1.

$$\therefore \quad M\,dx + N\,dy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= (3x^2 - 8y^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= (3x^2 + 8x^3 - 20x^4)\,dx$$

$$\therefore \int_{C_1} M\,dx + N\,dy = \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= [x^3 + 2x^2 - 4x^4]_0^1$$

$$= -1$$

Along C_2; $y^2 = x \Rightarrow dx = 2y\,dy$

$$\therefore \quad M\,dx + N\,dy = (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= (4y - 22y^3 + 6y^5)\,dy$$

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 (4y - 22y^3 + 6y^5)\,dy$$

$$= \left[2y^2 - \frac{22}{4}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$$\therefore \oint_C M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2} \qquad \dots (2)$$

From equations (1) and (2), Green's theorem is verified.

Q.44 Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)\,dx + (x^2 + y^2)\,dy$ where C is the boundary of the area enclosed by the axis and the upper half of the circle. $x^2 + y^2 = 16$. [SPPU : May-15]

Ans. : Green's theorem states that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad \dots (1)$$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$$\frac{\partial N}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial N}{\partial y} = 2\,(x + y)$$

$\therefore$ Equation (1) becomes

$$I = \oint M dx + N dy = \iint_R 2(x+y)\,dx\,dy$$

Given that $x^2 + y^2 = 16$ $\therefore$ $x = r\cos\theta$

$y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$

$$\therefore \quad I = \int_{\theta=0}^{\pi} \int_{r=0}^{4} 2(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} 2(\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^4 d\theta$$

$$= 2\left(\frac{4^3}{3}\right)\int_0^{\pi}[\cos\theta+\sin\theta]d\theta$$

$$= \frac{2(4)^3}{3}[\sin\theta-\cos\theta]_0^{\pi}$$

$$= 2\times\frac{64}{3}[0-(-1)-0+1]=\frac{256}{3}$$

Q.45 Using Green's theorem evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F}=x\overline{i}+y\overline{j}$ over the first quadrant of the circle $x^2+y^2=a^2$.

☞ [SPPU : Dec.-18, Marks 6]

Ans. : Green's Theorem states that

$$\oint \overline{F}\cdot d\overline{r} = \iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy \quad \ldots(1)$$

Let $M = x,\ N = y$

$$\frac{\partial M}{\partial y} = 0\ ;\ \frac{\partial N}{\partial y}=0$$

$$\therefore \quad \oint \overline{F}\cdot d\overline{r} = \iint 0\ dx\,dy = 0$$

8.10 Stoke's Theorem

I) If S be an open surface bounded by a closed curve C and $\overline{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C \overline{F}\cdot d\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds, \text{ where}$$

$\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.

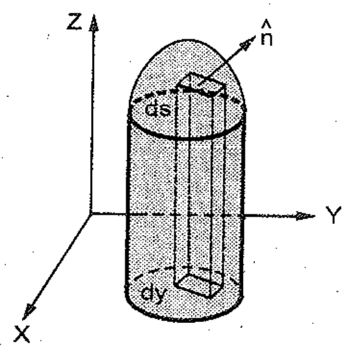


Fig. 8.2

Note

1) The abbreviation of $\hat{N}\,ds$ or $\hat{n}\,ds$ is $d\overline{s}$.

2) If the surface is in XOY plane then $\hat{N}=k$,
ds = dx dy.

3) If the surface is closed then bounding curve does not exist therefore we cannot evaluate Stoke's theorem.

4) If $\overline{F}$ is irrotational then $\nabla\times\overline{F}=0$.
$\therefore \int_C \overline{F}\cdot d\overline{r}=0$.

5) If S_1 and S_2 are two surfaces with the same bounding curve C then
$$\iint_{S_1}(\nabla\times\overline{F})\cdot d\overline{s}_1 = \iint_{S_2}(\nabla\times\overline{F})\cdot d\overline{s}_2$$

6) If the surface is not in XOY plane then $\hat{N}=\frac{\nabla\phi}{|\nabla\phi|}$ then use the projection formula $ds=\frac{dx\,dy}{|\hat{N}\cdot k|}$ (where $\phi=C$ is the surface) for the projection of surface element ds on XOY plane.

Q.46 Verify Stokes' theorem for $\overline{F}=xy^2\overline{i}+y\overline{j}+z^2x\overline{k}$ for the surface of rectangular lamina bounded by x = 0, y = 0, x = 1, y = 2, z = 0.

☞ [SPPU : May-10, 13, Dec.-12]

Ans. : Step 1 : Given that

$$\overline{F} = xy^2\,\hat{i}+y\hat{j}+z^2x\hat{k}$$

Step 2 :

$$\int_C \overline{F}\cdot d\overline{r} = \int_C [xy^2dx + y\,dy]$$

$$= \int_{OA}\overline{F}\cdot d\overline{r}+\int_{AB}\overline{F}\cdot d\overline{r}+\int_{BC}\overline{F}\cdot d\overline{r}+\int_{CO}\overline{F}\cdot d\overline{r}$$

$$= 0 + \int_0^2 y\,dy + \int_1^0 4x\,dx + \int_0^2 y\,dy$$

$$= \int_0^2 y\,dy + \left[\frac{4x^2}{2}\right]_1^0 - \int_0^2 y\,dy = -2$$

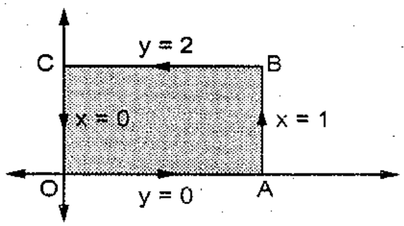


Fig. Q.46.1

Step 3 : Now, to obtain surface integral

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y & 0 \end{vmatrix} = -2xy\,\hat{k}$$

$\therefore\ \hat{n} = \hat{k}$ and $dS = dx\,dy$

Step 4 :

$$\iint_S (\nabla \times \overline{F}) \cdot \hat{n}\,dS = \iint_S (-2xy)\,\hat{k}\cdot\hat{k}\,dx\,dy$$

$$= \int_{x=0}^{1} \int_0^2 (-2xy)\,dy\,dx$$

$$= -2\int_0^1 \left[x\frac{y^2}{2}\right]_0^2 dx$$

$$= -4\int_0^1 x\,dx = -4\left[\frac{x^2}{2}\right]_0^1$$

$$= -2$$

Step 5 : Thus $\int_C \overline{F}\cdot d\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,dS = -2$

Hence Stoke's theorem is verified.

Q.47 Verify Stoke's theorem $\overline{F} = -y^3\hat{i} + x^3\hat{j}$ over ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

☞ [SPPU : Dec.-14, for a = b = 1)

Ans. : Step 1 : Consider

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0-0) - \hat{j}\,(0-0) + \hat{k}\,(3x^2 + 3y^2)$$

Step 2 : As the surface is in X-O-Y plane.

$\therefore\ \hat{N} = \hat{k}$, $ds = dx\,dy$

Step 3 : Consider $\iint (\nabla\times\overline{F})\cdot\hat{N}\,ds$

Step 4 : Substitute the values.

$$= \iint 3\left(x^2 + y^2\right) dx\,dy$$

Step 5 : Consider

$$\iint x^2\,dx\,dy \text{ over } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Step 6 : Put $x = a\,r\cos\theta$, $y = b\,r\sin\theta$

$dx\,dy = a\,b\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 a^2 r^2 \cos^2\theta \cdot a\,b\,r\,dr\,d\theta$$

Step 7 : Integrate

$$= \int_0^{2\pi} a^3 b\left(\frac{r^4}{4}\right)_0^1 \cdot \cos^2\theta\,d\theta$$

Using conversion formula

$$= \frac{a^3 b}{4} \cdot 4 \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

Using reduction formula

$$= a^3 b \left(\frac{1}{2} \cdot \frac{\pi}{2}\right) = a^3 b \frac{\pi}{4}$$

Step 8 : Similarly we can show

$$\therefore \iint y^2 \, dx \, dy = ab^3 \frac{\pi}{4}$$

Step 9 : Thus $\iint_S (\nabla \times \overline{F}) \cdot \hat{N} \, ds$

$$= 3\left[a^3 b \frac{\pi}{4} + ab^3 \frac{\pi}{4}\right] = 3\, ab \frac{\pi}{4} [a^2 + b^2] \quad \ldots (1)$$

Step 10 : To find $\int_C \overline{F} \cdot d\overline{r}$

Consider

$$\int_C \overline{F} \cdot d\overline{r} = \int_C -y^3 dx + x^3 \, dy$$

over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Step 11 : The parametric equations of ellipse are

$x = a \cos \theta$, $y = b \sin \theta$, $z = 0$

$\therefore dx = -a \sin \theta \, d\theta$, $dy = b \cos \theta \, d\theta$, $dz = 0$

Step 12 : Substituting we get

$$\int_C \overline{F} \cdot d\overline{r} = \int_0^{2\pi} -b^3 \sin^3 \theta (-a \sin \theta) \, d\theta$$

$$+ a^3 \cos^3 \theta \, b \cos \theta \, d\theta$$

$$= \int_0^{2\pi} \left(ab^3 \sin^4 \theta + a^3 b \cos^4 \theta\right) d\theta$$

Step 13 : Using conversion formula

$$= 4 \int_0^{\pi/2} \left(ab^3 \sin^4 \theta + a^3 b \cos^4 \theta\right) d\theta$$

Step 14 : Using reduction formula

$$= 4\left[ab^3 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right) + a^3 b \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)\right]$$

$$= 3ab \frac{\pi}{4} \left(a^2 + b^2\right) \quad \ldots (2)$$

From equations (1) and (2) Stoke's theorem is verified.

Q.48 Evaluate $\iint_C (e^x \, dx + 2y dy - dz)$ where C is the curve $x^2 + y^2 = 4$, $z = 2$. ☞ [SPPU : Dec.-15]

Ans. : Let $I = \int (e^x dx + 2y dy - dz)$ and $x^2 + y^2 = 2$, $z = 2$

$\therefore$ By stokes theorem

$$\oint_C \overline{F} \cdot d\overline{r} = \iint_S (\nabla \times \overline{F}) \cdot \hat{N} \, ds \quad \ldots (1)$$

where $\overline{F} = e^x \hat{i} + 2y\hat{j} - \hat{k}$

$$\therefore \quad \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

$\therefore$ Equations (1) becomes

$$I = \oint \overline{F} \cdot d\overline{r} = \iint 0 \, d\overline{s} = 0$$

Q.49 Using stoke's theorem, evaluate the integral $\iint_S \nabla \times \overline{F} \cdot d\,\overline{s}$ where : $\overline{F} = yi + zj + xk$ and 'S' is the surface of paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ above X-Y-plane. ☞ [SPPU : May-19, Marks 7]

Ans. :

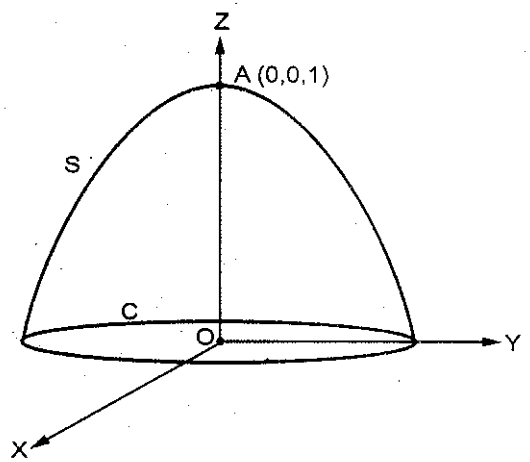


Fig. Q.49.1

By stokes theorem

$$\iint_S \nabla \times \bar{F} \cdot d\bar{s} = \oint_C \bar{F} \cdot d\bar{r}$$

Where $S \rightarrow$ Surface of the paraboloid

$z = 1 - x^2 - y^2$ above xy plane

$C \rightarrow$ Circumference of circle C which is intersection of $z = 1 - x^2 - y^2$ and $z = 0$ i.e. $x^2 + y^2 = 1$

$$I = \oint_C \bar{F} \cdot d\bar{r}$$

$$= \oint_C y\,dx + z\,dy + x\,dz$$

C : Circumference of $x^2 + y^2 = 1$, $z = 0$

Since $z = 0 \Rightarrow dz = 0$

$$I = \oint_C y\,dx$$

C : Circumference of $x^2 + y^2 = 1$

Parametric equation of circle is

$$x = \cos\theta,\ y = \sin\theta$$

$$dx = -\sin\theta\,d\theta,\ dy = \cos\theta\,d\theta$$

limits $\theta \rightarrow 0$ to 2π

$$I = \oint_C y\,dx$$

$$= \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta\,d\theta)$$

$$= -\int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta$$

$$= -4\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= -4\left[\frac{1}{2}\frac{\pi}{2}\right]$$

$$\boxed{I = -\pi}$$

Q.50 Using Stokes's theorem, evaluate $\iint_S (\nabla \times \bar{F}) \cdot d\bar{s}$ where $\bar{F} = 3y\hat{i} - xz^2\hat{j} + yz^2\hat{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$. [SPPU : May-15]

Ans. : By Stoke's theorem

$$I = \iint_S (\nabla \times \bar{F}) \cdot \hat{N}\,d\bar{s} = \oint_C \bar{F} \cdot d\bar{r} \qquad \ldots(1)$$

The equation of paraboloid is

$$x^2 + y^2 = 2z \text{ and } z = 2$$

$$\therefore \quad x^2 + y^2 = 2 \times 2 = 4$$

$\therefore$ Put $x = 2\cos\theta,\ y = 2\sin\theta$

$$dx = -2\sin\theta\,d\theta, \quad dy = 2\cos\theta\,d\theta$$

$\therefore$ Equation (1) becomes

$$I = \oint_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi} (3ydx - xzdy + yz^2dz)$$

$$= \int_0^{2\pi} (3\times 2\sin\theta)\ (-2\sin\theta\, d\theta) - 2\ (2\cos\theta)\ (2\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} [-12\sin^2\theta - 8\cos^2\theta]\ d\theta$$

$$= 4\int_0^{\pi/2} [-12\sin^2\theta - 8\cos^2\theta]\ d\theta$$

$$= 4\left(-12\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) - 8\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\right) = -12\pi - 8\pi = -20\pi$$

Q.51 Evaluate using Stoke's theorem, $\oint_C [(x^2+y^2)i + (x^2-y^2)j]\cdot d\overline{r}$ **where 'C' is the boundary of region bounded by circles,** $x^2+y^2 = 4$ **and** $x^2+y^2 = 16$ **in X-Y plane.**

[SPPU : May-19, Marks 7]

Ans. : By Stokes theorem

$$\oint_C \overline{F}\cdot d\overline{r} = \iint_A (\nabla\times\overline{F})\cdot d\overline{s}$$

where C : Boundary of circle $x^2 | y^2 = 4$ and $x^2+y^2 = 16$

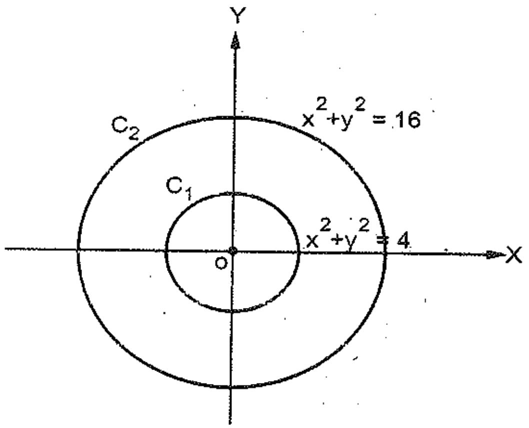


Fig. Q.51.1

A : Area between above two circles

We have $\quad I = \iint_A (\nabla\times\overline{F})\cdot d\overline{s}$

$$= \iint_A (\nabla\times\overline{F})\cdot \hat{n}\, ds$$

$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & x^2-y^2 & 0 \end{vmatrix}$$

$$= (0)\ i + (0)\ j + k\ (2x - 2y)$$

$$\nabla\times\overline{F} = 2(x-y)k$$

$\hat{n}$ = Outword normal to circles and it is along z-axis

$$\hat{n} = k$$

$$\Rightarrow \quad (\nabla\times\overline{F})\cdot\hat{n} = 2(x-y)$$

$$I = \iint_A 2(x-y)\, dx\, dy$$

A : Area between the circles $x^2+y^2 = 4$ and $x^2+y^2 = 16$.

Converting integral into polar form as

$$x = r\cos\theta,\ y = r\sin\theta$$

$$dx\, dy = r\, dr\, d\theta,\ r^2 = x^2+y^2$$

limits $\quad r \to 2$ to 4

$\theta \to 0$ to 2π

$$I = 2\int_{\theta=0}^{2\pi}\int_{r=2}^{4} (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{2\pi} (\cos\theta - \sin\theta)\left\{\frac{r^3}{3}\Big|_{r=2}^{4}\right\} d\theta$$

$$= 2\frac{64}{3}\int_{\theta=0}^{2\pi} (\cos\theta - \sin\theta)\, d\theta$$

$$= \frac{128}{3}(0) \text{ using reduction formula}$$

$$= 0$$

Q.52 Using Stokes theorem, evaluate $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$ where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and $x + y = 2a$. [SPPU : May-15]

Ans. : We have $\overline{F} = (x+y)\,\hat{i} + (2x-z)\,\hat{j} + (y+z)\,\hat{k}$

$$\therefore \quad \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(2-1) = 2\hat{i} + k$$

Here the surface S is the intersection of

$x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and $x + y = 2a$

$\therefore \hat{N}$ is the unit normal to the plane $x + y = 2a$

$$\hat{N} = \frac{\nabla(x+y-2a)}{|\nabla(x+y-2a)|} = \frac{\hat{i}+\hat{j}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$\nabla \times \overline{F} \cdot \hat{N} = \frac{1}{\sqrt{2}}(2+0+0) = \sqrt{2}$$

$$\therefore \iint_S \nabla \times \overline{F} \cdot \hat{N}\, ds = \sqrt{2} \iint ds$$

$$= \sqrt{2}\,[\text{Area of s}]$$

$$= \sqrt{2}\,[\text{Area of given circle}] \quad \text{...(1)}$$

We have $x^2 + y^2 + z^2 - 2ax - 2ay = 0$

$(x-a)^2 + (y-0)^2 + (z-0)^2 = 2a^2$

$\therefore$ Centre $= (a, a, 0)$ and radius $= \sqrt{2}\,a$

$\therefore$ The plane $x + y = 2a$ passes through $(a, a, 0)$

$\therefore$ Radius of circle = Radius of sphere

$\therefore$ Equation (1) becomes $= a\sqrt{2}$

$$\oint_C \overline{F} \cdot d\overline{r} = \iint_R \nabla \times \overline{F} \cdot \hat{n}\, ds = \sqrt{2}\,\pi\,[a\sqrt{2}]^2 = 2\sqrt{2}\,\pi\,a^2$$

Q.53 Use Stoke's theorem for $\int_C (4yi + 2zj + 6yk) \cdot d\overline{r}$ where $C \equiv$ intersection of $x^2 + y^2 + z^2 = 2z$ and $x = z - 1$. [SPPU : Dec.-04, 06, 12, 13, 14, May-14]

Ans. : Step 1 : Consider $\overline{F}$ and find $\nabla \times \overline{F}$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix}$$

$$= \hat{i}(6-2) + 0\hat{j} + (0-4)\,\hat{k}$$

$$= 4\hat{i} + 0\hat{j} - 4\hat{k}$$

Step 2 : Here surface 'S' is the intersection of circle

$x^2 + y^2 + z^2 - 2z = 0$ and $x = z - 1$

$\hat{N}$ is unit normal to the plane $x - z + 1 = 0$

$$\therefore \quad \hat{N} = \frac{\nabla(x-z+1)}{|\nabla(x-z+1)|} = \frac{\hat{i}-\hat{k}}{\sqrt{1^2+(-1)^2}} = \frac{\hat{i}-\hat{k}}{\sqrt{2}}$$

$$\therefore \quad \nabla \times \overline{F} \cdot \hat{N} = \left(4\hat{i} - 4\hat{k}\right) \cdot \frac{\left(\hat{i}-\hat{k}\right)}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = \frac{8}{\sqrt{2}}$$

Step 3 :

$$\therefore \quad \iint_S \nabla \times \overline{F} \cdot \hat{N}\, ds = \frac{8}{\sqrt{2}} \iint_S ds = \frac{8}{\sqrt{2}}\,(\text{Area of S})$$

$$= \frac{8}{\sqrt{2}}\,(\text{Area of given circle})$$

$$= \frac{8}{\sqrt{2}}\left(\pi r^2\right) \quad r \text{ is radius of given circle} \quad \text{... (1)}$$

Step 4 : The given sphere is

$$x^2 + y^2 + z^2 - 2z = 0$$

i.e. $$x^2 + y^2 + (z-1)^2 = 1$$

$\therefore$ Centre = (0, 0, 1), Radius = 1

As the centre satisfies the equation x = z − 1.

$\therefore$ The plane x = z − 1 passes through the centre of the given sphere.

$\therefore$ The circle obtained is of maximum radius.

Step 5 : i.e. Radius of circle = Radius of sphere = 1

Step 6 : Hence from equation (1) the required integral is

$$= \frac{8}{\sqrt{2}}\pi(1)^2 = \frac{8\pi}{\sqrt{2}}$$

Q.54 Evaluate $\iint_S (\nabla \times \overline{F}) \cdot d\overline{s}$ where $\overline{F} = (x^3 - y^3)\hat{i} - xyz\hat{j} + y^2\hat{k}$ and S is the surface $x^2 + 4y^2 + z^2 - 2x = 4$ above the plane $x = 0$

☞ [SPPU : May-11, 12, Dec.-12, 13, 15]

Ans. : By Stoke's theorem

$$\iint_S (\nabla \times \overline{F}) \cdot d\overline{s} = \int_C \overline{F} \cdot d\overline{r}$$

$$= \int_C [(x^3 - y^3)\,dx - xyz\,dy + y^2\,dz]$$

where C is the curve $4y^2 + z^2 = 4$

$$\therefore \quad \frac{y^2}{1^2} + \frac{z^2}{2^2} = 1 \text{ which is an ellipse.}$$

$\therefore$ Put $y = \cos\theta,\ z = 2\sin\theta,\ x = 0$

$$\therefore \quad \int_C \overline{F} \cdot d\overline{r} = \int_{\theta=0}^{2\pi} y^3\,dz = \int_{\theta=0}^{2\pi} \cos^2\theta\,(2\cos\theta)\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi} \cos^3\theta\,d\theta$$

$$= 2 \times 0 = 0$$

8.11 : Gauss Divergence Theorem

If $\overline{F}$ is a continuously differentiable vector function in the region v bounded by the closed surface S, then

$$\iint_S \overline{F} \cdot \hat{n}\,ds = \iiint_V \text{div}\,\overline{F}\,dv$$

where $\hat{n}$ is the unit outward drawn normal to the surface S.

i.e. $$\boxed{\iiint_V \nabla \cdot \overline{F}\,dv = \oiint_S \overline{F} \cdot \hat{n}\,ds}$$

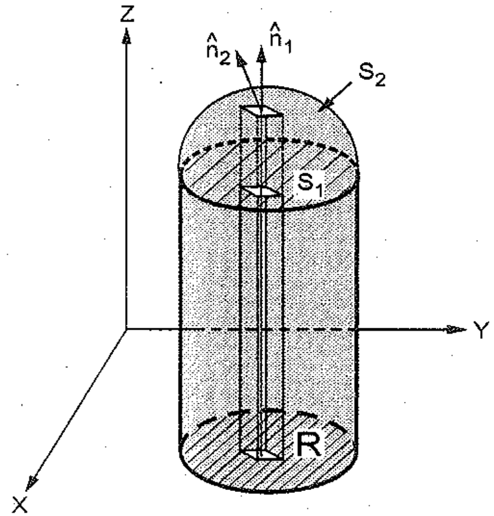


Fig. 8.3

Q.55 Verify Gauss Divergence theorem $\overline{F} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ over the volume of tetrahedron bounded by co-ordinate planes and plane $2x + y + 2z = 6$. ☞ [SPPU : Dec.-14]

Ans. : Step 1 : Consider $\overline{F}$ and find $\nabla \cdot \overline{F}$

$$\nabla \cdot \overline{F} = 1 - 0 + 2y$$

$$= 1 + 2y$$

Step 2 : Consider $\iiint \nabla \cdot \bar{F}\ dv = \iiint 1+2y\ dx\,dy\,dz$

over the volume bounded by $x = 0,\ y = 0,\ z = 0,\ 2x+y+2z = 6$

Now consider $2x+y+2z = 6$

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1$$

$\therefore$ Put $x = \frac{x}{3},\quad y = \frac{y}{6},\ z = \frac{z}{3}$ then

$dx\,dy\,dz = 3 \cdot 6 \cdot 3\ dx\,dy\,dz$ and the volume is bounded by $x = 0$, $y = 0,\ z = 0,\ x+y+z = 1$.

Thus the integral becomes

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+2 \cdot 6\,y)\ 3 \cdot 6 \cdot 3\ dz\,dy\,dx$$

$$= 54 \int_0^1 \int_0^{1-x} (1+12\,y)(1-x-y)\,dy\,dx$$

Put $1-x = h$ for convenience

$$= 54 \int_0^1 \int_0^h (1+12\,y)(h-y)\,dy\,dx$$

$$= 54 \int_0^1 \int_0^h [(h-y)+12\,(h\,y-y^2)]\,dy\,dx$$

$$= 54 \int_0^1 \left\{\frac{(h-y)^2}{-2} + 12\left[h\frac{y^2}{2} - \frac{y^3}{3}\right]\right\}_0^h dx$$

$$= 54 \int_0^1 \left\{0+12\left[\frac{h^3}{2} - \frac{h^3}{3}\right]\right\} - \left\{\frac{h^2}{-2}+0\right\} dx$$

Put $h = 1-x$

$$= 54 \int_0^1 2(1-x)^3 + \frac{(1-x)^2}{2}\ dx$$

$$= 54 \left\{\frac{2(1-x)^4}{-4} + \frac{(1-x)^3}{-3 \cdot 2}\right\}_0^1$$

$$= 54 \left\{[0-0] - \left[-\frac{2}{4} - \frac{1}{6}\right]\right\}$$

$$= 54 \left\{\frac{1}{2} + \frac{1}{6}\right\} = 54 \cdot \frac{4}{6} = 36$$

Step 3 : For double integral draw the Fig. Q.55.1.

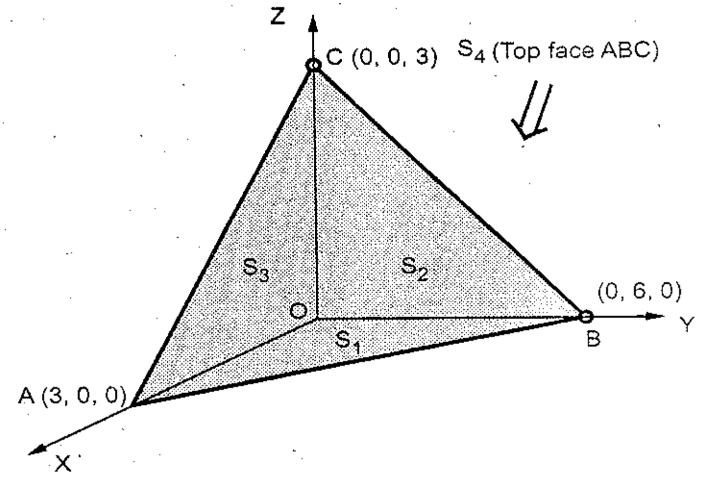


Fig. Q.55.1

Step 4 : For S_1 OAB $\hat{N} = -\hat{k}$, $ds = dx\,dy$, $z = 0$, $2x+y = 6$, $\bar{F} \cdot \hat{N} = -2\,yz = 0$

For S_2 OBC $\hat{N} = -\hat{i}$, $ds = dy\,dz$, $x = 0$, $y + 2z = 6$, $\bar{F} \cdot \hat{N} = -\left(x+y^2\right) = -y^2$

For S_3 OAC $\hat{N} = -\hat{j}$, $ds = dx\,dz$,

$y = 0,\ 2x + 2z = 6,\ x + z = 3,\ \bar{F} \cdot \hat{N} = 2x$

Step 5 : $\iint_{S_1} \bar{F} \cdot \hat{N}\,ds = \iint 0\,dx\,dy = 0$

Step 6 : $\iint_{S_2} \overline{F}\cdot\hat{N}\, ds = \iint -y^2\, dy\, dz$

over the triangle bounded by y = 0, z = 0, y + 2z = 6.

$$= \int_0^3 \int_0^{6-2z} -y^2\, dy\, dz = \int_0^3 -\frac{(6-2z)^3}{3}\, dz$$

$$= \left[-\frac{(6-2z)^4}{3\cdot 4(-2)}\right]_0^3$$

$$= 0 - \frac{6^4}{3\cdot 4\cdot 2} = -\frac{6^3}{3} = -54$$

Step 7 : $\iint_{S_3} \overline{F}\cdot\hat{N}\, ds = \iint 2x\, dx\, dz$

over the triangle bounded by x = 0, z = 0, x + z = 3

$$= \int_0^3 \int_0^{3-z} 2x\, dx\, dz = 2\int_0^3 \frac{(3-z)^2}{2}\, dz$$

$$= \left[\frac{(3-z)^3}{-3}\right]_0^3 = \frac{3^3}{3} = 9$$

Step 8 : Now S_4 : ABC

Let $\phi = 2x+y+2z-6=0$

$$\nabla\phi = 2\hat{i}+\hat{j}+2\hat{k}$$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

$$\overline{F}\cdot\hat{N} = \frac{2\left(x+y^2\right)-2x+4yz}{3} = \frac{2}{3}\left(y^2+2yz\right)$$

The projection of S_4 on XOY plane is triangle OAB.

$$ds = \frac{dx\, dy}{|\hat{N}\cdot k|} = \frac{dx\, dy}{|2/3|}$$

$$\therefore \iint_{S_4} \overline{F}\cdot\hat{N}\, ds = \iint \frac{2}{3}\left(y^2+2yz\right)\frac{dx\, dy}{2/3}$$

$$= \iint \left(y^2+2yz\right) dx\, dy$$

On S_4 $\quad 2z = 6-2x-y = \iint y^2+y(6-2x-y)\, dx\, dy$

$$= \iint y(6-2x)\, dx\, dy$$

Over the triangle OAB bounded by

x = 0, y = 0, 2x + y = 6

$$= \int_0^3 \int_0^{6-2x} y(6-2x)\, dy\, dx$$

$$= \int_0^3 \frac{(6-2x)^2}{2}(6-2x)\, dx$$

$$= \left[\frac{1}{2}\frac{(6-2x)^4}{4(-2)}\right]_0^3 = 0 + \frac{6^4}{2\cdot 4\cdot 2} = 81$$

Step 9 : Adding on the integrals we get

$$\iint_S \overline{F}\cdot\hat{N}\, ds = 0+(-54)+9+81$$

$$= 36$$

G.D.T. is verified.

Q.56 Evaluate $\iint \left(2xy\hat{i}+yz^2\hat{j}+xz\hat{k}\right)\cdot d\overline{s}$ over the total surface of region bounded by x = 0, y = 0, z = 0, y = 3 and x + 2z = 6.

[SPPU : Dec.-10, 15]

Ans. : **Step 1 :** Consider

$$\nabla\cdot\overline{F} = 2y+z^2+x$$

Step 2 : Now by G.D.T.

$$\iint_S \overline{F}\cdot\hat{N}\, ds = \iiint \nabla\cdot\overline{F}\, dv$$

$$= \iiint \left(2y+z^2+x\right) dx\, dy\, dz$$

Step 3 : Over the volume bounded by x = 0, y = 0, z = 0, y = 3, x + 2z = 6

$$= \int_{z=0}^{3}\int_{y=0}^{3}\int_{x=0}^{6-2z}\left(2y+z^2+x\right)dx\,dy\,dz$$

$$= \int_0^3\int_0^3\left[\begin{array}{l}2y(6-2z)+z^2(6-2z)\\ +\dfrac{(6-2z)^2}{2}\end{array}\right]dy\,dz$$

$$= \int_0^3\left[\begin{array}{l}(3^2)(6-2z)+(6z^2-2z^3)(3)\\ +\dfrac{(6-2z)^2}{2}(3)\end{array}\right]dz$$

$$= \left[\begin{array}{l}9\dfrac{(6-2z)^2}{2(-2)}+3\left(6\cdot\dfrac{z^3}{3}-2\dfrac{z^4}{4}\right)\\ +\dfrac{(6-2z)^3}{2\cdot 3(-2)}(3)\end{array}\right]_0^3$$

$$= \left\{\begin{array}{l}\left[0+3\left(2\cdot 3^3-\dfrac{3^4}{2}\right)+0\right]\\ -\left[9\cdot\dfrac{6^2}{-4}+0-\dfrac{6^3}{2\cdot 3\cdot 2}(3)\right]\end{array}\right\}$$

$$= 3^4\left(2-\frac{3}{2}\right)+\frac{6^2\cdot 9}{4}+\frac{6^3}{4}$$

$$= \frac{81}{2}+82+54=\frac{351}{2}$$

Q.57 Evaluate $\iint_S (z^2-x)\,dy\,dz - xy\,dz\,dx + 3z\,dx\,dy$ **where S is the closed surface of region bounded by x = 0, x = 3, z = 0, z = $4-y^2$.**

[SPPU : Dec.-18, Marks 6]

Ans. :

We have, $\bar{F} = (z^2-x)\bar{i} - xy\,\bar{j} + 3z\,\bar{k}$

$$\nabla\cdot\bar{F} = -1 - x + 3 = 2 - x$$

We have, $I = \iiint \nabla\cdot\bar{F}\,dv = \iiint (2-x)\,dx\,dy\,dz$

$$= \iint\int_{z=0}^{4-y^2}(2-x)\,dz\,dx\,dy = \iint_R (2-x)(4-y^2)\,dx\,dy$$

$$I = \int_{x=0}^{3}\int_{y=-2}^{2}(2-x)(4-y^2)\,dx\,dy$$

$$= \int_{x=0}^{3}(2-x)\,dx\int_{y=-2}^{2}(4-y^2)\,dy$$

$$= \left[2x-\frac{x^2}{2}\right]_0^3\left[4y-\frac{y^3}{3}\right]_{-2}^{2}$$

$$= \left(6-\frac{9}{2}\right)\left(8-\frac{8}{3}-(-8)-\frac{8}{3}\right)$$

$$= \left(\frac{3}{2}\right)\left(16-\frac{16}{3}\right) = \frac{3}{2}\times\frac{48-16}{3} = 16$$

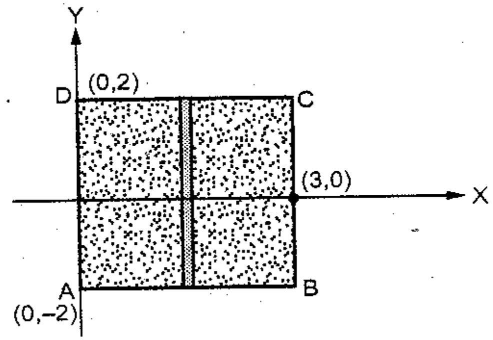


Fig. Q.57.1

Q.58 Find $\iint_S \bar{F} \cdot d\bar{S}$ where S is the sphere $x^2 + y^2 + z^2 = 9$ and

$\bar{F} = (4x + 3yz^2)\,\bar{i} - (x^2 z^2 + y)\,\bar{j} + (y^3 + 2z)\,\bar{k}$.

☞ [SPPU : Dec.-18, Marks 6]

Ans. : By Gauss divergence theorem

$$I = \iint \bar{F} \cdot \hat{n}\, ds = \iiint \nabla \cdot \bar{F}\, dv \qquad \ldots(1)$$

We have, $\nabla \cdot \bar{F} = 4 - 1 + 2 = 5$

$$\therefore \quad I = \iiint 5\, dxdydz = 5 \iiint dxdydz$$

$$= [5 \times \text{ Volume of } x^2 + y^2 + z^2 = 9]$$

$$I = 5 \times \frac{4}{3}\pi (3)^3 = 180\pi$$

Q.59 Evaluate $\iint \bar{r} \cdot d\bar{s}$ over the surface of sphere $x^2 + y^2 + z^2 = 1$.

☞ [SPPU : May-12]

Ans. : By G.D.T.

$$\iint \bar{r} \cdot d\bar{s} = \iiint \nabla \cdot \bar{r}\, dv$$

$$= \iiint 3\, dv$$

$$= 3\ (\text{vol})$$

$$= 3 \cdot \frac{4}{3}\pi a^3 = a = 1$$

$$= 4\pi$$

Q.60 Evaluate $\iint \left(x^3\, \hat{i} + y^3\, \hat{j} + z^3\, \hat{k}\right) \cdot d\bar{s}$ over the surface of sphere $x^2 + y^2 + z^2 = a^2$.

☞ [SPPU : May-06, 10, 13, 19, Dec.-11, 12, 13, 15]

Ans. : Step 1 : By G.D.T.

$$\iint \bar{F} \cdot \hat{N}\, ds = \iiint \nabla \cdot \bar{F}\, dv$$

$$= \iiint 3\left(x^2 + y^2 + z^2\right) dx\, dy\, dz$$

Over $x^2 + y^2 + z^2 = a^2$

Step 2 : Use spherical polar co-ordinate for sphere

Put
$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$
$$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\phi\, d\theta$$

Here the value of $x^2 + y^2 + z^2$ is r^2.

Step 3 : Use the standard limits for full sphere of radius.

$\therefore$ The integral becomes

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{a} 3 r^2\, r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$= 3 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{a} r^4\, dr$$

$$= 3\, [-\cos\theta]_0^{\pi}\, (\phi)_0^{2\pi} \left(\frac{r^5}{5}\right)_0^a$$

$$= 3(2) \cdot 2\pi \cdot \frac{a^5}{5} = \frac{12\pi}{5} a^2$$

Q.61 Use G.D.T. to evaluate $\iint_S \bar{F} \cdot \hat{N}\, ds$

$\bar{F} = y^2 z^2\, \hat{i} + x^2 z^2\, \hat{j} + x^2 y^2\, \hat{k}$ where S is the surface of hemisphere $x^2 + y^2 + z^2 = 9$ above XOY plane.

☞ [SPPU : May-05, Dec.-14]

Ans. : Consider the hemisphere

S_1 = Surface of hemisphere

S_2 = Surface of circle

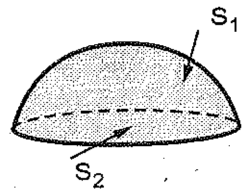


Fig. Q.61.1

$\therefore$ To find the integral over S_1 by G.D.T.

$$\iint_S \bar{F} \cdot \hat{N}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$$

But S is combination of S_1 and S_2

$$\therefore \quad \iint_{S_1} \bar{F} \cdot \hat{N}\, ds + \iint_{S_2} \bar{F} \cdot \hat{N}\, ds = \iiint \nabla \cdot \bar{F}\, dv \qquad \ldots(1)$$

Now $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}\left(y^2 z^2\right) + \frac{\partial}{\partial y}\left(x^2 z^2\right) + \frac{\partial}{\partial z}\left(x^2 y^2\right)$

$= 0 + 0 + 0 = 0$

Thus equation (1) becomes

$$\iint_{S_1} \bar{F} \cdot \hat{N}\, ds + \iint_{S_2} \bar{F} \cdot \hat{N}\, ds = 0$$

$$\therefore \iint_{S_1} \bar{F} \cdot \hat{N}\, ds = -\iint_{S_2} \bar{F} \cdot \hat{N}\, ds \quad \ldots (2)$$

For S_2 $\hat{N} = -k$, $z = 0$, $ds = dx\, dy$ and $x^2 + y^2 = 9$

$\therefore \bar{F} \cdot \hat{N} = -x^2 y^2$

Consider the surface integral over S_2

$$\therefore \iint_{S_2} \bar{F} \cdot \hat{N}\, ds = \iint -x^2 y^2\, dx\, dy$$

over $x^2 + y^2 = 9$

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, dr\, d\theta$ and substitute the standard limits for circle of radius 3.

$$= \int_0^{2\pi} \int_0^{3} -r^2 \cos^2\theta\, r^2 \sin^2\theta\, r\, dr\, d\theta$$

Integrate w.r.t. r and use conversion formula for θ.

$$= -4 \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta \left(\frac{r^6}{6}\right)_0^3$$

Use reduction formula for θ.

$$= -4 \frac{[1][1]}{[4 \cdot 2]} \frac{\pi}{2} \left(\frac{3}{6}\right) = -\frac{\pi}{4} \cdot \frac{3^6}{6} = -\frac{\pi\, 3^6}{24}$$

- Thus from equation (2)

$$\iint_{S_1} \bar{F} \cdot \hat{N}\, ds = \frac{\pi \cdot 3^6}{24}$$

Q.62 Evaluate $\iint_s \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) \cdot d\bar{s}$, where s is the curved surface of the cone $x^2 + y^2 = z^2$, $z = 4$ [SPPU : May-14, Marks 4]

Ans. : To evaluate $\iint \left(yz\hat{i} + zx\hat{j} + xyk\right) \cdot ds$, using divergence theorem.

$$\iint \bar{F} \cdot \hat{n}\, ds = \iiint \nabla \cdot \bar{F} \cdot dv$$

Consider $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$

$\therefore \iiint \nabla \cdot \bar{F} \cdot dv = 0$

Q.63 Prove that $\iiint_V \frac{1}{r^2}\, dv = \iint_S \frac{1}{r^2}\, \bar{r} \cdot d\bar{s}$, where s is closed surface enclosing the volume v.

Hence evaluate $\iint_S \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2} \cdot d\bar{s}$, where s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. [SPPU : May-14, Marks 4]

Ans. :

Let $\bar{F} = \frac{\bar{r}}{r^2}$

$\therefore \nabla \cdot \bar{F} = \nabla \cdot \frac{\bar{r}}{r^2} = (-2 + 3)\, r^{-2} = \frac{1}{r^2}$

Thus by G.D.T.

$$\iint \frac{\bar{r}}{r^2} \cdot \hat{N}\, ds = \iiint \cdot \frac{1}{r^2}\, dv$$

To evaluate

$\iint_S \frac{xi + yj + zk}{r^2} \cdot d\bar{s}$ s is the surface of the sphere $x^2 + y^2 + z^2 = u^2$

From above

$$\Rightarrow \iint \frac{\bar{r}}{r^2}\, ds = \iiint \frac{1}{r^2}\, dv$$

$$\therefore \iint \frac{xi + yj + zk}{r^2}\, ds = \iiint \frac{1}{r^2}\, dv$$

over sphere $x^2 + y^2 + z^2 = a^2$

$x = a \sin\theta \cos\phi$, $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$y = a \sin\theta \sin\phi$, $z = a\cos\theta$

$$= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \frac{1}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi, \; r = 0 \text{ to } a, \; \theta = 0 \text{ to } \pi/2, \; \phi = 0 \text{ to } 2n$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \int_0^a dr = a \int_0^{2\pi} [-\cos\theta]_0^{\pi/2} \, d\phi$$

$$= -a \int_0^{2\pi} (0-1)\, d\phi = + a\,(\phi)_0^{2\pi} = 2\pi a$$

Q.64 Show that $\iint \frac{\bar{r}}{r^2} \cdot \hat{n} \, ds = \iiint \frac{dv}{r^2}$

☞ [SPPU : May-08, 13, Dec.-13]

Ans. : Let $\bar{F} = \frac{\bar{r}}{r^2}$

$$\therefore \quad \nabla \cdot \bar{F} = \nabla \cdot \frac{\bar{r}}{r^2} = (-2+3)\, r^{-2} = \frac{1}{r^2}$$

Thus by G.D.T.

$$\iint \frac{\bar{r}}{r^2} \cdot \hat{N} \, ds = \iiint \frac{1}{r^2} \, dv$$

Q.65 Use divergence theorem to evaluate $\iint_S \left(x\hat{i} - 2y^2 \hat{j} + z^2 \hat{k} \right) \cdot d\bar{s}$ **where S is the surface bounded by** $x^2 + y^2 = 1$ **and** $z = 0$, $z = 1$.

☞ [SPPU : May-15, 19, Marks 6]

Ans. : We have $\bar{F} = x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 1 + 4y + 2z$$

By Gauss divergence theorem

$$I = \iint_S \bar{F} \cdot d\bar{s} = \iiint \nabla \cdot \bar{F} \, dv$$

$$= \iint \int_{z=0}^{1} (1 + 4y + 2z) \, dx \, dy \, dz = \iint_R (z + 4yz + z^2)_0^1 \, dx \, dy$$

$$= \iint_R [(1 + 4y + 1) - 0] \, dx \, dy = \iint_R (2 + 4y) \, dx \, dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ and $r = 0$ to 1 and $\theta = 0$ to 2π

$$\therefore \quad I = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} [2 + 4r\sin\theta] \, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[2\frac{r^2}{2} + \frac{4r^3}{3} \sin\theta \right]_0^1 d\theta = \int_0^{2\pi} \left[1 + \frac{4}{3}\sin\theta \right] d\theta$$

$$= \left[\theta + \frac{4}{3}(-\cos\theta) \right]_0^{2\pi} = 2\pi - \frac{4}{3}\cos 2\pi - 0 + \frac{4}{3}\cos 0$$

$$= 2\pi \frac{4}{3} + \frac{4}{3} = 2\pi$$

Q.66 Show that $\iiint_V \frac{2}{r} \, dv \iint_S \frac{\bar{r} \cdot \hat{n}}{r} \, ds$

☞ [SPPU : May-15]

Ans. : By divergence theorem

$$\iint_S \frac{\bar{r} \cdot \hat{n}}{r} \, ds = \iiint \nabla \cdot \left(\frac{\bar{r}}{r} \right) dv \qquad \ldots(1)$$

Now $\quad \nabla \cdot \frac{\bar{r}}{r} = \nabla \cdot (\bar{r}\, r^{-2})$

$$\therefore \quad \nabla \cdot \frac{\bar{r}}{r} = (\nabla \cdot \bar{r}) r^{-1} + \nabla (r^{-1}) \cdot \bar{r}$$

$$= \frac{3}{r} + \left(-\frac{1}{r^2} \right) \frac{1}{r} \bar{r} \cdot \bar{r} = \frac{3}{r} + \left(-\frac{1}{r^3} \right) \bar{r} \cdot \bar{r}$$

$$= \frac{3}{r} - \frac{1}{r^3} r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$\therefore$ Equation (1) becomes

$$\iint_S \frac{\bar{r} \cdot \hat{n}}{r} \, ds = \iiint_V \frac{2}{r} \, dv$$

END...

UNIT VI

9 Complex Differentiation

9.1 : Analytic Function

1) A function f(z) is said to be analytic in a region R if it is analytic at every point of a region R. Analytic function is also known as a holomorphic or regular or monogenic.

A point at which the function ceases to be analytic is called as a singular point of function.

2) Entire function : A function which is analytic everywhere in the complex plane is known as entire function.

3) Necessary and sufficient conditions for $f(z) = u + i\,v$ to be analytic in a region R are

i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.

ii) $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \text{ and } \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}}$ at all points of R.

The conditions in (ii) are known as **Cauchy - Riemann equations** or in briefly C.R. equations.

iii) Cauchy-Riemann equations in polar form :

Let $\quad f(z) = u(x, y) + i\,v(x, y) = u + i\,v$

and $\quad z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

$\therefore \quad f(z) = f(r\,e^{i\theta}) = u + iv$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

which are C.R. equations in polar form.

iv) If $f(z) = u + iv$ is analytic in a region R, then u and v satisfy C.R. equations in R.

i.e. $u_x = v_y$ and $v_x = -u_y$

v) C.R. equations are necessary but not sufficient conditions for function to be an analytic. These C.R equations are sufficient conditions for analytic function if $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions.

i.e. f analytic $\Rightarrow$ C-R equations

But C.R. equations may or may not $\Rightarrow$ Analytic function.

C.R. equations + Continuous partial derivatives $\Rightarrow$ Analyticity.

4) Harmonic function : A function $\phi(x, y)$ is said to be harmonic function if it is continuous and has continuous first and second order partial derivatives and satisfies Laplace equation $\boxed{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0.}$

> **Note :** If $f(z) = u + iv$ is analytic then both u and v are harmonic functions.

5) Conjugate harmonic function :

[SPPU : Dec.-05, 06, 07, 08, 09, 10, 11, 12]

If $f(z) = u(x, y) + iv(x, y)$ is analytic function, then the real part u of f(z) is known as conjugate harmonic function of imaginary part v of f(z) and vice versa.

6) Construction of analytic function :

A)Theorem : If $f(z) = u + iv$ is an analytic function of z then f(z) is independent of $\bar{z}$.

Proof : Let $f(z) = u(x, y) + i\,v(x, y)$ ... (9.1)

be an analytic function of $z = x + iy$

Now $\bar{z} = x - iy$

$\therefore$ $z + \bar{z} = 2x \Rightarrow x = \frac{z+\bar{z}}{2}$

And $z - \bar{z} = 2iy \Rightarrow y = \frac{z-\bar{z}}{2i}$

Substituting x and y in equation (9.1) we get,

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Here f(z) is a function of two complex variables z and $\bar{z}$. Now u, v are functions of x, y. And x and y are functions of z and $\bar{z}$. We have

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u+iv) = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left[\frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right] + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \text{... (9.2)}$$

As $f(z) = u + iv$ is analytic function, u and v satisfy C-R equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ substituting these relations in equation (9.2), we get $\frac{\partial f}{\partial \bar{z}} = 0 + i\,0 = 0$ which means that f(z) is independent of $\bar{z}$.

Remarks : i) If f(z) is analytic function of z, then it consist of expressions in z only. i.e. if f(z) is analytic function of z then x and y occurs in f(z) in the combination of $x + iy$.

ii) If $f(z) = u(x, y) + iv(x, y)$ is analytic function then substitute $x = z$ and $y = 0$ in f(z), we get f(z) in terms of z.

B) Milne - Thompson Method :

i) To find analytic function $f(z) = u + iv$ whose real part u (x, y) is given :

$\Rightarrow$ If u is known $\therefore u_x$ and u_y are obtained.

Now $f'(z) = u_x + i\,v_x = u_x + i(-u_y)$ by C-R equations

$$f'(z) = u_x(x, y) - i\,u_y(x, y) \quad \text{... (9.3)}$$

As f(z) is analytic, f'(z) is also analytic.

$\therefore$ By substituting $x = z$ and $y = 0$ in equation (9.3), we get,

$$f'(z) = u_x(z, 0) - i\,u_y(z, 0)$$

Integrating both sides with respect to z we get

$$f(z) = \int u_x(z, 0)\,dz - i\int u_y(z, 0)\,dz + C$$

which is analytic function.

Here we get f(z) in terms of z. To find v(x, y) if u(x, y) is given, substitute $z = x + iy$ in f(z) and then find imaginary part of f(z).

ii) To find analytic function $f(z) = u + iv$ whose imaginary part v(x, y) is given :

$\Rightarrow$ $f'(z) = u_x + i\,v_x = v_y + i\,v_x = v_y(x, y) + i\,v_x(x, y)$

Substituting $x = z$ and $y = 0$, and then integrating we get,

$$f(z) = \int v_y(z, 0)\,dz + i\int v_x(z, 0)\,dz + C$$

To find u (x, y) substitute $z = x + iy$ and then find real part of f(z).

> **Note** In polar form
> $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$

C) i) Suppose that $f(z) = u + iv$ is an analytic function and u(x, y) is known.

Claim : To determine v(x, y)

Since u and v satisfy C-R equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{... (9.4)}$$

We have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \ldots (9.5)$$

But u is known $\therefore \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be calculated and hence dv can be obtained using equation (9.5).

We write $\quad M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$\therefore$ Equation (9.5) becomes,

$$dv = M\,dx + N\,dy \quad \ldots (9.6)$$

Now $\quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0$

i.e. $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Equation (9.6) is exact

$$\therefore \quad v = \int_{y\ \text{constant}} M\,dx + \int \begin{bmatrix}\text{terms in N not} \\ \text{containing x}\end{bmatrix} dy + C$$

$$\text{i.e. } v = \int_{y\ \text{constant}} \left(-\frac{\partial u}{\partial y}\right) dx + \int \begin{bmatrix}\text{terms in } \frac{\partial u}{\partial x} \text{ not} \\ \text{containing x}\end{bmatrix} dy + C$$

ii) Suppose that $f(z) = u + iv$ is an analytic function and $v(x, y)$ is known.

Claim : To determine u(x, y)

We have $\quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

which is exact differential equation.

$\therefore$ It's general solution is

$$u = \int_{y\ \text{constant}} \frac{\partial v}{\partial y}dx + \int \begin{bmatrix}\text{terms in} \left(-\frac{\partial v}{\partial x}\right) \text{not} \\ \text{containing x}\end{bmatrix} dy + C$$

Q.1 If f(z) is analytic function then show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

[SPPU : Dec.-10, 13, 15, May-06, 12]

Ans. : Let

$$f(z) = u + iv \therefore |f(z)|^2 = u^2 + v^2 = g$$

$$\therefore \quad \frac{\partial}{\partial x}g = \frac{\partial}{\partial x}(u^2 + v^2) = 2uu_x + 2vv_x$$

$$\frac{\partial^2 g}{\partial x^2} = 2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2$$

Similarly, $\quad \frac{\partial^2 g}{\partial y^2} = 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2$

Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)g$$

$$= 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + v_y^2 + u_y^2] \quad \ldots (1)$$

Since f(z) is analytic, u, v satisfy Laplace equation

$$\therefore \quad u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

and $\quad f'(z) = u_x + i\,v_x = u_y + i\,v_y$

$$\therefore \quad |f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$$

$\therefore$ Equation (1) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2[2\,|f'(z)|^2] = 4|f'(z)|^2$$

Q.2 Find the condition under which $u = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic.

☞ [SPPU : Dec.-16, May-19, Marks 4]

Ans. : We have $u = ax^3 + bx^2y + cxy^2 + dy^3$

$$\therefore \quad \frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by$$

$$\frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2$$

$$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

Now if u is harmonic, it must satisfy Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \quad 6ax + 2by + 2cx + 6dy = 0$

$\therefore \quad (6a + 2c)\,x + (2b + 6d)\,y = 0$

$\Rightarrow \quad 6a + 2c = 0$ and $2b + 6d = 0$

$\Rightarrow \quad 3a + c = 0$ and $b + 3d = 0$

which are required conditions

Q.3 Find 'a' such that the function $f(z) = r^2\cos 2\theta + ir^2\sin(a\theta)$ is analytic.

☞ [SPPU : May- 18, Marks 4]

Ans. : f(z) is analytic

$\therefore$ f(z) satisfies C.R. equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$$\Rightarrow \quad 2r\cos 2\theta = \frac{1}{r} r^2 a\cos a\theta \text{ and}$$

$$2r\sin(a\theta) = -\frac{1}{r}(-r^2\, 2\sin 2\theta)$$

$$\Rightarrow \quad a = 2$$

Q.4 If $u = \frac{1}{2}\log(x^2 + y^2)$ and $f(z) = u + iv$ is analytic then find f(z) in terms of z and hence find v.

☞ [SPPU : May-12]

Ans. : Let $f(z) = u + iv$ be an analytic function and

$$u = \frac{1}{2}\log(x^2 + y^2)$$

$\therefore$ f(z) satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \dots (1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2 + y^2}\cdot(2x) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

Now $\quad f'(z) = u_x + i v_x = u_x - i u_y$

$$f'(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

As f(z) is analytic, f′(z) is also analytic.

$\therefore$ By substituting $x = z$ and $y = 0$, we get

$$f'(z) = \frac{z}{z^2} + i\,0 = \frac{1}{z}$$

$\therefore$ Integrating w.r.t z we get

$$f(z) = \int \frac{1}{z}dz + C = \log z + C$$

where $\quad C = C_1 + i\,C_2$

To find v substitute $z = x + iy$ in f(z), we get,

$$f(z) = \log(x + iy) + C_1 + i\,C_2$$

$$= \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x} + C_1 + i\,C_2$$

$$= \log\sqrt{x^2 + y^2} + i\left(\tan^{-1}\frac{y}{x} + C_2\right)$$

where $$C_1 = 0 = \frac{1}{2}\log(x^2+y^2)+i\left(\tan^{-1}\frac{y}{x}+C_2\right)$$

$$\therefore \quad v(x, y) = \tan^{-1}\frac{y}{x}+C_2$$

Q.5 If v = sinh x cos y find u such that u + iv is analytic function.

[SPPU : May- 18, Marks 4]

Ans. : Given that $f(z) = u + iv$ an analytic function and $v = \sinh x \cos y$

$\therefore$ f(z) satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}\right)$$

$$= -\sinh x \sin y+i\cosh x \cos y \quad \ldots(1)$$

As f(z) is an analytic function, f′(z) is also analytic

substituting x = z and y = 0, we get

$$f'(z) = i\cosh z$$

Integrating w.r.t. z, we get

$$f(z) = i\sinh z+C \quad \ldots(2)$$

To find u put $z = x + iy$ in equation (2)

$$\therefore \quad f(z) = i\sinh(x+iy)+C_1+iC_2$$

$$= i[\sin x \cosh(iy)+\cosh x \sinh iy]+C_1+iC_2$$

$$= i[\sinh x \cos y+i \cosh x \sin y]+C_1+iC_2$$

$$= (-\cosh x \sin y+C_1)+i(\sinh x \cos y+C_2)$$

$$= u + iv$$

$$\therefore \quad u = -\cosh x \sin y+C_1.$$

Q.6 Find the imaginary part of the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$. **[SPPU : May-15]**

Ans. : Let $f(z) = u + iv$ be an analytic function.

and $$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

$\therefore$ f(z) must satisfies C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$u_x = 3x^2 - 3y^2 + 6x \text{ and } u_y = -6xy - 6y$$

$$\therefore \quad f'(z) = u_x + i\,v_x = u_x - i\,u_y$$

$$\therefore \quad f'(z) = (3x^2 - 3y^2 + 6x) - i(-6xy - 6y) \quad \ldots(1)$$

As f(z) is analytic, f′(z) is also analytic.

$\therefore$ By putting x = z, y = 0, we get

$$\therefore \quad f'(z) = 3z^2 + 6z + i(0) = 3z^2 + 6z$$

Integrating we get

$$f(z) = 3\frac{z^3}{3}+6\frac{z^2}{2}+C \quad \text{where } C = C_1 + iC_2$$

$$= z^3 + 3z^2 + C$$

To find v put z = x + iy, we get

$$f(z) = (x + iy)^3 + 3(x + iy)^2 + C_1 + iC_2$$

$$u + iv = [x^3 + 3x^2\, iy + 3\,x\,(iy)^2 + (iy)^3]$$

$$+ 3[x^2 - y^2 + 2i\,xy] + C_1 + iC_2$$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + C_1)$$

$$+ i(3x^2y - y^3 + 6xy + C_2)$$

$$\therefore \quad v = 3x^2y - y^3 + 6xy + C_2$$

which is the required imaginary part of z.

Q.7 Show that u(x, y) = y + e^xcos y is harmonic function. Find it's harmonic conjugate. **[SPPU : Dec.- 17, Marks 4]**

Ans. :

We have $$u = y+e^x\cos y,$$

$$\therefore \quad u_x = e^x \cos y$$
$$u_{xx} = e^x \cos y$$
$$u_y = 1 - e^x \sin y$$
$$u_{yy} = -e^x \cos y$$
$$\therefore \quad u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$$

$\therefore$ u is a Harmonic function.

Let f(z) = u + iv be an analytic function.

$\therefore$ f(z) satisfies C. R. equations,

$$u_x = v_y, \; v_x = -u_y$$
$$\therefore \quad f'(z) = u_x + iv_x = u_x - iu_y$$
$$= e^x \cos y - i(1 - e^x \sin y) \quad \ldots(1)$$

As f(z) is an analytic function, f′(z) is also analytic.

$\therefore$ By substituting x = z and y = 0 in equation (1), we get

$$f'(z) = e^z - i(1) = e^z - i$$

Integrating w.r.t. z, we get,

$$f(z) = e^z - iz + C$$

put

$$z = x + iy$$

$$\therefore \quad f(z) = e^{x+iy} - i(x+iy) + C$$
$$= e^x e^{iy} - ix + y + C_1 + iC_2$$
$$= e^x [\cos y + i \sin y] - ix + y + C_1 + iC_2$$
$$= (e^x \cos y + y + C_1) + i(e^x \sin y - x + C_2)$$
$$= u + iv$$
$$\therefore \quad v = e^x \sin y - x.$$

Q.8 If $w = \phi + i\psi$ represents the complex potential for an electric field and $\phi = -2xy + \dfrac{y}{x^2 + y^2}$ determine the function ψ.

☞[SPPU : May-15, 17, 19, Marks 4]

Ans. : We have

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$
$$= -\frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy \quad \ldots(1)$$

Now

$$\frac{\partial \phi}{\partial y} = -2x + \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2}$$
$$= -2x + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$
$$\frac{\partial \phi}{\partial x} = -2y + \frac{y(-1)}{(x^2 + y^2)^2}(2x)$$
$$= -2y - \frac{2xy}{(x^2 + y^2)^2}$$
$$d\psi = \left[2x - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dx - \left[2y + \frac{2xy}{(x^2 + y^2)^2}\right] dy$$

which is exact D.E.

$$\therefore \quad \psi = \int \left[2x^2 - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dx + \int (-2y)\, dy + C$$
$$= x^2 - y^2 - \int \left[\frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dx + C$$
$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2} + C$$

Q.9 If f(z) = u + iv is analytic and $v = \dfrac{-y}{x^2 + y^2}$. Find f(z) in terms of z.

☞[SPPU : Dec.-16, 17, Marks 4]

Ans. : Given that f(z) = u + iv is an analytic and $v = \dfrac{-y}{x^2 + y^2}$

$\therefore$ f(z) satisfies C.R. equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= -\left[\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}\right] + i\left[\frac{y(2x)}{(x^2+y^2)^2}\right]$$

$$f'(z) = -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] + i\frac{2xy}{(x^2+y^2)^2}$$

As f(z) is analytic, f′(z) is also analytic.

∴ By substituting $x = z$ and $y = 0$ in $f'(z)$

We get $$f'(z) = -\frac{z^2}{(z^2+0)^2} + i0 = -\frac{1}{z^2}$$

Integrating, we get

$$\boxed{f(z) = \frac{1}{z} + C}$$

Q.10 If $u - v = x^3 + 3x^2y - 3xy^2 - y^3$, find an analytic function $f(z) = u + iv$. ☞ [SPPU : Dec.-18, Marks 4]

Ans. : Given that $u - v = x^3 + 3x^2y - 3xy^2 - y^3$ ...(1)

Differentiating w.r.t. x, we get

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \quad \ldots (2)$$

Differentiating equation (1) w.r.t. y, we get

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2 \quad \ldots (2)$$

By Cauchy Riemann equations,

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2 \quad \ldots (3)$$

Add equation (2) with equation (3) ⇒

$$-2\frac{\partial v}{\partial x} = 6x^2 - 6y^2 = 6(x^2-y^2)$$

Subtract equation (3) from equation (2) ⇒

$$2\frac{\partial u}{\partial x} = 12\,xy \Rightarrow \frac{\partial u}{\partial x} = 6xy$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 6xy - 3(x^2-y^2)i$$

$$= 6xy - 3i(x^2-y^2) = -3i(x^2+2ixy-y^2)$$

$$= -3i(x+iy)^2 = -3iz^2$$

∴ $$f(z) = -3i\frac{z^3}{3} + C = -iz^3 + C$$

Q.11 Find analytic function $f(z) = u + iv$ where $u = r^3 \cos 3\theta + r \sin \theta$. ☞ [SPPU : May-15]

Ans. : We have C.R. equations in polar form

$$u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta$$

$$u_r = 3r^2 \cos 3\theta + \sin\theta$$

Here $$u_\theta = -3r^2 \sin 3\theta + r\cos\theta$$

Now $$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = 3r^3 \cos 3\theta + r\sin\theta$$

Integrating w.r.t. θ and treating r as a constant,

$$v = 3r^3\frac{\sin 3\theta}{3} - r\cos\theta + \phi(r)$$

$$= r^3 \sin 3\theta - r\cos\theta + \phi(r)$$

$$\frac{\partial v}{\partial r} = 3r^2 \sin 3\theta - \cos\theta + \phi'(r) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\Rightarrow \quad 3r^2 \sin 3\theta - \cos\theta + \phi'(r) = 3r^2 \sin 3\theta - \cos\theta$$

$$\Rightarrow \quad \phi'(r) = 0 \Rightarrow \phi(r) = \text{Constant} = C$$

$$\therefore \quad v = r^3 \sin 3\theta - r\cos\theta + C$$

and $\quad f(z) = u + iv - r^3 \cos 3\theta + r \sin\theta + i$

$$(r^3 \sin 3\theta - r \cos\theta + C)$$

$$= r^3(\cos 3\theta + i \sin 3\theta) - ir(\cos\theta + i\sin\theta)$$

$$r^3 e^{3i\theta} - i r e^{i\theta} + iC$$

$$= z^3 - iz + C_1 \quad (\because z = re^{i\theta})$$

Q.12 Find an analytic function f(z) whose imaginary part is $r^n \sin n\theta$. ☞ [SPPU : Dec.-18, Marks 4]

Ans. : Given that, $\quad v = r^n \sin n\theta$

$$v_\theta = n r^n \cos n\theta; \; V_r = n r^{n-1} \sin n\theta$$

By C. R. equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r} n r^n \cos n\theta = n r^{n-1} \cos n\theta$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = -r n r^{n-1} \sin n\theta = -n r^n \sin n\theta$$

Intergrating w.r.t. θ and treating r as constant

$$\therefore \quad u = -n r^n\left(-\frac{\cos n\theta}{n}\right) + \phi(r)$$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta + \phi'(r)$$

But $\quad \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$

$$\phi'(r) + n r^{n-1} \cos n\theta = \frac{1}{r} n r^n \cos n\theta$$

$$\psi'(r) = 0 \rightarrow \quad \phi(r) = \text{constant}$$

$$\therefore \quad u = r^n \cos n\theta + C$$

And $\quad f(z) = u + iv = r^n \cos n\theta + C + i r^n \sin n\theta$

$$= r^n (\cos n\theta + i \sin n\theta) + C$$

$$= r^n e^{n\theta} + C \qquad [\because z = r e^{i\theta}]$$

$$f(z) = z^n + C$$

Q.13 Let f(z) = u + iv be an analytic function and u = v^2 then show that f(z) is constant. ☞ [SPPU : Dec.-11, 15]

Ans. : Let f(z) = u + iv be an analytic function

$\therefore$ f(z) satisfies C.R. equations

$$\therefore \quad u_x = v_y \text{ and } v_x = -u_y$$

We have

$$u = v^2 \Rightarrow u - v^2 = 0$$

Differentiating w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} = 0$$

By C.R. equations, we get

$$\frac{\partial v}{\partial y} - 2v\frac{\partial v}{\partial x} = 0 \qquad \ldots (1)$$

and $\quad -\dfrac{\partial v}{\partial x} - 2v\dfrac{\partial v}{\partial y} = 0 \qquad \ldots (2)$

Multiplying equation (1) by 2v, we get

$$2v\frac{\partial v}{\partial y} - 4v^2\frac{\partial v}{\partial x} = 0 \qquad \ldots (3)$$

Equation (3) + Equation (2) $\Rightarrow$

$$\Rightarrow \quad -\frac{\partial v}{\partial x}(1 + 4v^2) = 0$$

$\Rightarrow \dfrac{\partial v}{\partial x} = 0$ or $1+4v^2 = 0$

$\Rightarrow v^2 = -\dfrac{1}{4}$ which is impossible as v is a real function.

$\Rightarrow \dfrac{\partial v}{\partial x} = 0$ i.e. v is independent of x

$$\therefore \qquad v = \phi(y)$$

Now from $\dfrac{\partial v}{\partial x} = 0 \Rightarrow \dfrac{\partial u}{\partial y} = 0 \Rightarrow u = \psi(x)$

Now substituting u and v in equation $u = v^2$ we get

$$\psi(x) = \phi^2(y)$$

which is possible only when

$$\psi(x) = \phi^2(y) = \text{Constant}$$

$\Rightarrow$ u and v are constant

$\Rightarrow$ f(z) = u + iv is a constant function.

Q.14 Prove that analytic function with constant argument (or amplitude) is constant. [SPPU : May-15, 17, Marks 4]

Ans. : Let f(z) = u + iv be analytic function with constant argument and let arg(f(z)) = k = constant.

$$\therefore \qquad \arg(u+iv) = k$$

$$\tan^{-1}\left(\frac{v}{u}\right) = k$$

$$\therefore \qquad v = u \tan k \qquad \dots (1)$$

Case i : Suppose k = 0, then $v = 0 \Rightarrow v_x = v_y = 0$

By C.R. equations we get $u_x = u_y = 0$

$\therefore$ u is a constant function.

$\therefore$ f(z) = u + iv = constant function.

Case ii : Differentiating equation (1) w.r.t. x, we get

$$v_x = u_x \tan k$$

$$\therefore \quad u_x \tan k - v_x = 0 \qquad \dots (2)$$

Differentiating equation (1) w.r.t. y, we get,

$$v_y = u_y \tan k$$

By C.R. equations, we get,

$$u_x = -v_x \tan k$$

$$\therefore \quad -u_x + v_x \tan k = 0 \qquad \dots (3)$$

Multiplying equation (2) by tan k, we get,

$$u_x \tan^2 k - v_x \tan k = 0 \qquad \dots (4)$$

Adding equations (3) and (4), we get,

$$u_x (1+\tan^2 k) = 0$$

$$u_x = 0$$

Similarly, we get $v_x = 0$

$$\therefore \qquad f'(z) = u_x + i v_x = 0$$

$\therefore$ f(z) is a constant function.

Q.15 Prove that analytic function with constant modulus is constant. [SPPU : May-16, Marks 4]

Ans. : Let f(z) = u + iv be an analytic function with constant modulus.

Suppose $\quad |f(z)| = k = \text{Constant}$

$$\therefore \qquad |u+iv| = k \Rightarrow \sqrt{u^2+v^2} = k$$

$$\Rightarrow \qquad u^2+v^2 = k^2 \qquad \dots (1)$$

Case 1 : Suppose $k = 0$ then $u^2 + v^2 = 0 \Rightarrow u = 0$ and $v = 0$

$$\therefore \quad f(z) = u + iv = 0 + i0 = 0$$

Thus f(z) is a constant function.

Case 2 : Suppose $k \neq 0$

Differentiating equation (1) partially w.r.t. x, we get

$$2uu_x + 2vv_x = 0$$

$$uu_x + vv_x = 0 \quad \ldots (2)$$

Differentiating equation (1) w.r.t. y we get

$$2\,uu_y + 2vv_y = 0 \quad \Rightarrow \quad uu_y + vv_y = 0$$

By C.R. equations, we get $-uv_x + vu_x = 0$ ... (3)

Multiplying equation (2) by u and equation (3) by v we get

$$u^2u_x + uvv_x = 0 \quad \ldots (4)$$

$$-uvv_x + v^2u_x = 0 \quad \ldots (5)$$

Adding equations (4) and (5), we get

$$(u^2 + v^2)u_x = 0$$

As $u^2 + v^2 = k^2 \neq 0$, u_x must be zero.

Similarly, we get $v_x = 0$

$$\therefore \quad f'(z) = u_x + iv_x = 0 + i0 = 0$$

$\therefore$ f(z) is a constant function.

9.2 : Bilinear Transformation

The transformation $w = \dfrac{az+b}{cz+d}$

where a, b, c, d are complex constants and $ad-bc \neq 0$ is known as the bilinear transformation or Mobius transformation or linear fractional transformation.

Now $\dfrac{dw}{dz} = \dfrac{(cz+d)\,a-(az+b)\,c}{(az+d)^2} = \dfrac{ad-bc}{(az+d)^2} \neq 0$

$(\because ad-bc \neq 0)$

Hence bilinear transformation is conformal.

Note 1) There is 1-1 correspondence between all points in z-plane and w-plane.

2) If z maps into itself in the w-plane i.e. $w = z$ then

$$z = \frac{az+b}{cz+d} \Rightarrow c\,z^2 + (d-a)\,z - b = 0 \quad \ldots (9.7)$$

The roots of equation (9.7) are the invariant or fixed points of the bilinear transformation. But equation (9.7) has exactly two roots, so bilinear transformation has at most two invariant points. If two roots of equation (9.7) are equal then the corresponding bilinear transformation is said to be parabolic.

3) The bilinear transformation is a combination of translation, rotation and inversion.

4) We have $w = \dfrac{az+b}{cz+d} = \dfrac{\left(\dfrac{a}{d}\right)z + \dfrac{b}{d}}{\left(\dfrac{c}{d}\right)z+1} = \dfrac{Az+B}{Cz+2}$

Thus to define bilinear transformation, we require 3 pairs points only.

5) The cross ratio of four points z_1, z_2, z_3, z_4 is denoted by (z_1, z_2, z_3, z_4) and defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

The cross ratio is invariant under bilinear transformation.

i.e. $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

This property is useful to find bilinear transformation $w = f(z)$ if $z_i \to \infty$ or $w_i \to \infty$, $i = 1, 2, 3..$

Note For simplicity, we use the following formula.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

If $z_1 = \infty$ then $\lim\limits_{|z_1| \to \infty} \frac{z-z_1}{z_2-z_1} = \lim\limits_{|z_1| \to \infty} \frac{\left(\frac{z}{z_1}-1\right)}{\left(\frac{z_2}{z_1}-1\right)} = 1$

Similarly if z_2 or $z_3 \to \infty$ then $\frac{z_2-z_3}{z_2-z_1}$ or $\frac{z_2-z_3}{z-z_3} \to 1$

6) Matrix method : Let $w = T(z) = \frac{az+b}{cz+d}$ the associated matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the inverse of the linear fractional or bilinear transformation is $T^{-1}(w) = \frac{dw-b}{-cw+a}$ and the associated matrix is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \text{adj } A$ where adj A is the adjoint matrix of A.

Q.16 Find bilinear transformation which maps the points z = 1, i, – 1 to the points 0, 1, ∞ respectively. [SPPU : May-12, Dec.-11, 15]

Ans. : Using the cross ratio property

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \ldots (1)$$

Here $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ as $w_3 = \infty$, $\frac{w_2-w_3}{w-w_3} = 1$

$\therefore$ Equation (1) becomes

$$\frac{w-0}{(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\therefore \quad w = \frac{(z-1)(i+1)}{(z+1)(i-1)} \times \frac{(-i-1)}{(-i-1)}$$

$$= \frac{(z-1)(1-2i-1)}{(z+1)(1+1)} = \frac{-iz+i}{z+1}$$

$$\therefore \quad w = \frac{-iz+i}{z+1}$$

Q.17 Find the bilinear transformation which maps points 1, i, – 1 of z-plane onto i, 0, – i of w-plane. [SPPU : May- 16, 18, Marks 4]

Ans. : By cross ratio property

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Here $z_1 = 1$, $z_2 = i$, $z_3 = -1$,

$w_1 = i$, $w_2 = 0$, $w_3 = -i$

$$\therefore \quad \frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \quad \frac{1+iw}{1-iw} = \frac{z(1+i)-(1+i)}{z(i-1)+(i-1)}$$

$$\frac{iw+1}{-iw+1} = \frac{z(1+i)-(1+i)}{z(i-1)+(i-1)}$$

$$S(w) = T(z) \Rightarrow w = S^{-1}(T(z))$$

Let $w = \frac{az+b}{cz+d}$

$$\therefore \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \right\} \begin{bmatrix} 1+i & -1-i \\ i-1 & i-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} 1+i & -1-i \\ i-1 & i-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+i-i+1 & -1-i-i+1 \\ i-1-1-i & -i+1-1-i \end{bmatrix} = \begin{bmatrix} 2 & -2i \\ -2 & -2i \end{bmatrix}$$

$$\therefore \quad w = \frac{2z-2i}{-2z-2i}$$

Q.18 Find the bilinear transformation, which maps the points − 1, 0, 1 onto the points 0, i, 3i respectively.

☞ [SPPU : May-06, 11, 15, Dec.-14, 15]

Ans. : By using cross ratio property, we get

$$\frac{(w-0)\,(i-3\,i)}{(w-3\,i)\,(i-0)} = \frac{(z+1)\,(0-1)}{(z-1)\,(0+1)}$$

$$\therefore \quad \frac{w\,(-2\,i)}{(w-3\,i)\,i} = \frac{-z-1}{z-1}$$

$$\therefore \quad \frac{-2\,w}{w-3\,i} = \frac{-z-1}{z-1}$$

i.e. $S(w) = T(z)$

$$\therefore \quad w = S^{-1}\,(T(z))$$

Let
$$w = \frac{a\,z+b}{c\,z+d}$$

$$\therefore \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj}\begin{bmatrix} -2 & 0 \\ 1 & -3\,i \end{bmatrix} \right\} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3\,i & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3\,i & 3\,i \\ -1 & 3 \end{bmatrix}$$

$$\therefore \quad w = \frac{3\,i\,(z+1)}{-z+3}$$

Q.19 Find the bilinear transformation which maps the points − i, 0, (2 + i) of z-plane onto the points 0, −2i, 4 of the w-plane.

☞ [SPPU : Dec.-18, Marks 4]

Ans. : By using cross ratio property, we get

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

where $w_1 = 0$, $w_2 = -2i$, $w_3 = 4$, $z_1 = -i$, $z_2 = 0$, $z_3 = 2+i$

$$\therefore \quad \frac{(w-0)(-2i-4)}{(w-4)(-2i-0)} = \frac{(z+i)(0-2-i)}{(z-2-i)(0+i)}$$

$$\frac{(-2i-4)\,w}{-2\,iw+8i} = \frac{(-2-i)\,z+(1-2i)}{zi+(1-2i)}$$

i.e.
$$S(w) = T(z)$$

$$\therefore \quad w = S^{-1}[T(z)]$$

Let
$$w = \frac{az+b}{cz+d} \text{ where}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj}\begin{bmatrix} -2i-4 & 0 \\ -2i & 8i \end{bmatrix} \right\} \begin{bmatrix} -2-i & 1-2i \\ i & 1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 8i & 0 \\ 2i & -2i-4 \end{bmatrix} \begin{bmatrix} -2-i & 1-2i \\ i & 1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 8-16i & 16+8i \\ 2-4i+2-4i & 4+2i-8+16i \end{bmatrix}$$

$$= \begin{bmatrix} 8-16i & 16+8i \\ 4-8i & -4+8i \end{bmatrix}$$

$$\therefore \quad w = \frac{az+b}{cz+d} = \frac{(8-16i)z+(16+8i)}{(4-8i)z+(-4+8i)}$$

$$w = \frac{(2-4i)\,z+(4+2i)}{(1-2i)\,z+(-1+2i)}$$

which is the required transformation.

Q.20 Find the bilinear transformation which maps the points 1, 0, i of the z-plane onto the points ∞, −2, $-\frac{1}{2}(1+i)$ of the w-plane.

☞ [SPPU : May-15, 17, 19, Marks 4]

Ans. : Consider $w = \dfrac{az+b}{cz+d}$

When $\quad z = 1,\ w = \infty \ \Rightarrow \infty = \dfrac{a+b}{c+d} \ \Rightarrow\ c+d = 0 \Rightarrow c = -d$

When $\quad z = 0,\ w = -2 \ \Rightarrow -2 = \dfrac{b}{d} \ \Rightarrow\ b = -2d$

When $\quad z = i,\ w = -\dfrac{1}{2}(1+i) \ \Rightarrow -\dfrac{1}{2}(1+i) = \dfrac{ia+b}{ic+d}$

$$\therefore \quad -\frac{1}{2}(1+i) = \frac{ia-2b}{-id+d}$$

$$\therefore \quad -\frac{1}{2}(1+i)\ (1-i)\ d = ai-2d$$

$$-(1+1)\ d = 2\ (ai-2d)$$

$$-d = ai-2d$$

$$d = ai \Rightarrow a = -di$$

$$\therefore \quad w = \frac{-2-iz}{1-z} = \frac{2+iz}{z-1}$$

Q.21 Find the bilinear transformation which maps the points 1, i, 2i on the points –2i, 0, 1, in w-plane. [SPPU : Dec.-17, Marks 4]

Ans. : By using cross property, we have

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

where $z_1 = 1,\ z_2 = i,\ z_3 = 2i,\ w_1 = -2i,\ w_2 = 0$

$w_3 = 1$

$\therefore$ We get

$$\frac{(w+2i)(0-1)}{(w-1)(0+2i)} = \frac{(z-i)(i-2i)}{(z-2i)(i-1)}$$

$$\frac{-w-2i}{2iw-2i} = \frac{-iz+i}{(i-1)z+2+2i}$$

i.e. $\quad S(w) = T(z)$

$$\therefore \quad w = S^{-1}[T(z)]$$

Let $\quad w = \dfrac{az+b}{cz+d}$

Then the associated matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \text{adj}\begin{bmatrix} -1 & -2i \\ 2i & -2i \end{bmatrix} \right\} \begin{bmatrix} -i & i \\ i-1 & 2+2i \end{bmatrix}$$

$$= \begin{bmatrix} -2i & 2i \\ -2i & -1 \end{bmatrix} \begin{bmatrix} -i & i \\ i-1 & 2+2i \end{bmatrix}$$

$$= \begin{bmatrix} -2-2-2i & 2-4+4i \\ -2-i+1 & 2-2-2i \end{bmatrix} = \begin{bmatrix} -4-2i & -2+4i \\ -1-i & -2i \end{bmatrix}$$

$$\therefore \quad w = \frac{(-4-2i)z+(-2+4i)}{(-1-i)z+(-2i)}$$

which is the required transformation.

9.3 : Special Conformal Transformations

[SPPU : May-05, 06, 07, 08, 09, 10, 11, 13, Dec.-05, 06, 07, 08, 09, 10, 11, 12]

1) The transformation $w = z^2$ [SPPU : May-08]

We have $\quad u+iv = z^2 = (x+iy)^2 = (x^2-y^2)+2ixy$

Equating real and imaginary parts, we get

$$u = x^2-y^2 \quad \text{and} \quad v = 2xy$$

If u = Constant say a, then $x^2-y^2 = a$ which represents a rectangular hyperbola. If v = Constant say b, then $2xy = b \therefore xy = \frac{b}{2}$ which is also represents a rectangular hyperbola. Differentiating equations $x^2-y^2 = a$ and $xy = \frac{b}{2}$ w.r.t. x, we get

$$2x-2y\frac{dy}{dx} = 0 \quad \text{and} \quad x\frac{dy}{dx}+y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y} \quad \text{and} \quad \frac{dy}{dx} = -\frac{y}{x}$$

$\therefore$ Slope of tangent of $x^2-y^2 = a$ is $\frac{x}{y}$ and $xy = \frac{b}{2}$ is $-\frac{y}{x}$.

$\therefore$ The product of slopes of tangents $= \left(\frac{x}{y}\right)\left(-\frac{y}{x}\right) = -1$

$\therefore$ These two hyperbolas are also orthogonal to each other.

Thus a pair of lines u = a and v = b which are orthogonal to each other in w-plane are map into a pair of orthogonal rectangular hyperbolas in the z-plane.

Again if $x = c$ then $y = \frac{v}{2}$ and $y^2 = c^2 - u$ eliminating y, we get $v^2 = 4c^2\ (c^2 - u)$ which represents a parabola.

Similarly if y = d then we get parabola

$$v^2 = 4\,d^2(d^2 + u)$$

Hence pair of orthogonal lines x = c and y = d are map into orthogonal parabolas in w-plane.

Now $$\frac{dw}{dz} = 2\,z = 0 \text{ if } z = 0$$

$\therefore$ z = 0 is the critical point of the transformation.

If $z = r\,e^{i\theta}$ and $w = z^2 = r^2\ e^{i2\theta} = Re^{i\phi}$

Thus upper half of the z-plane $0 < \theta < \pi$ transforms into the entire w-plane $0 \le \phi \le 2\pi$ same for lower half z-plane.

2) The transformation $w = e^z$

We have $$w = Re^{i\phi} \text{ and } z = x + i\,y$$

$$\therefore \quad w = e^z \Rightarrow Re^{i\phi} = e^x\ e^{iy}$$

$$\Rightarrow \quad R = e^x \text{ and } \phi = y$$

If x is a constant say x = a $\therefore$ $R = e^a$ = constant which represents a circle with centre at (0, 0) and radius R. Thus the lines parallel to Y-axis in the z-plane are mapped into concentric circles with centre at (0, 0) in the w-plane.

If y = b = constant then we have $\phi = b$ = constant which represents a straight line passing through origin makes angle b with positive U axis in UV plane. Thus the lines parallel to X-axis in the z-plane are mapped into radial lines in the w-plane : If $0 \le y \le 2\pi$, we have $0 \le \phi \le 2\,\pi$ i.e. any horizontal strip of width 2π in z-plane will cover the entire w-plane.

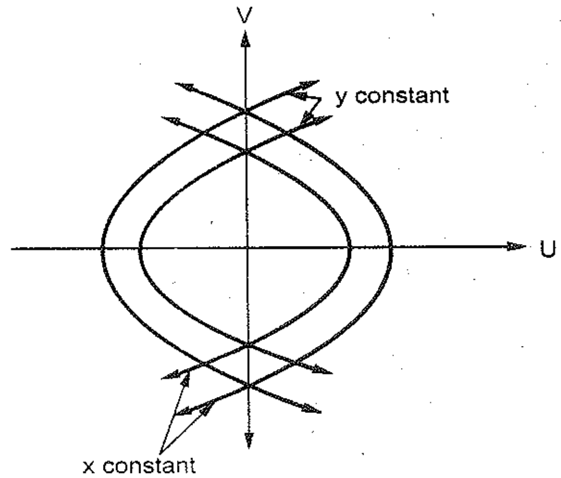
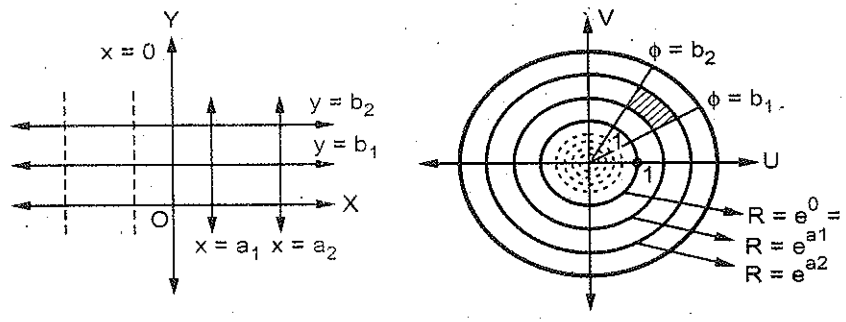


Fig. 9.1

The rectangular region $a_1 \le x \le a_2$ and $b_1 \le y \le b_2$ in the z-plane transforms into the region $e^{a1} \le R \le e^{a2}$ and $b_1 \le \phi \le b_2$ in the w-plane.



(a) (b)

Fig. 9.2

- The left half of XOY plane is mapped into the interior of the unit circle and the right half is mapped into the exterior of the unit circle in w-plane.

- $x = a$ and $y = b$ are mutually orthogonal. Also their map circles and radial lines are also mutually orthogonal. Thus proves the conformal nature of $w = e^z$ everywhere.

3) The transformation $w = z + \frac{1}{z}$ is known as Joukowski's transformation. ☞ [SPPU : Dec.-05]

Let $z = r(\cos\theta + i\sin\theta)$ and $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$

We have
$$w = u + iv = z + \frac{1}{z}$$
$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$
$$\therefore \quad u = \left(r + \frac{1}{r}\right)\cos\theta \text{ and } v = \left(r - \frac{1}{r}\right)\sin\theta$$

As $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \quad \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad \ldots (9.8)$$

and
$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$$
$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad \ldots (9.9)$$

- From equation (9.8) it follows that the circles r = Constant in z-plane map into a family of ellipses in the w-plane. These ellipses are confocal for $\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$ i.e. a constant.

In particular if $r = 1$ then we get $u = 2\cos\theta$ and $v = 0$ i.e. the segment $-2 \le u \le 2$ of real axis in w-plane. Thus exterior of the circle $r = 1$ maps into the entire w-plane.

- The radical line θ = constant = b in z-plane maps into a hyperbola $\frac{u^2}{4\cos^2 b} - \frac{v^2}{4\sin^2 b} = 1$ which is a confocal in the w-plane.

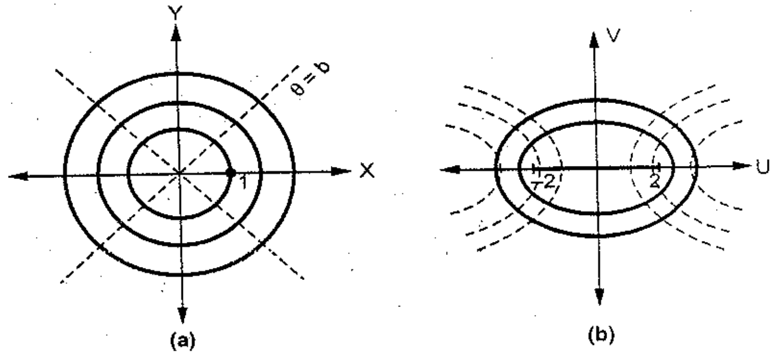


Fig. 9.3

Here $r = a$ and $\theta = b$ are orthogonal to each other in z-plane. Also their maps ellipses and hyperbola's are orthogonal. This proves the conformal nature of $w = z + \frac{1}{z} = \frac{z^2 + 1}{z}$ except at points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ in w-plane.

$\left(\because \frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2} = 0 \text{ at } z = 1 \text{ and } z = -1\right)$

4) The transformation $w = \sinh z$

☞ [SPPU : May-05, Dec.-11]

We have $w = \sinh z = \sinh(x + iy)$

$$\therefore \quad u + iv = \sinh x \cos y + i\cosh x \sin y$$
$$\therefore \quad u = \sinh x \cos y, \quad v = \cosh x \sin y$$
$$\therefore \quad \cos y = \frac{u}{\sinh x}, \quad \sin y = \frac{v}{\cosh x}$$

Now $\cos^2 y + \sin^2 y = 1$

$$\Rightarrow \quad \frac{u^2}{\sinh^2 x}+\frac{v^2}{\cosh^2 x} = 1 \qquad \text{... (9.10)}$$

And $\quad \cosh^2 x-\sinh^2 x = 1$

$$\Rightarrow \quad \frac{v^2}{\sin^2 y}-\frac{u^2}{\cos^2 y} = 1 \qquad \text{... (9.11)}$$

- The straight line parallel to Y-axis i.e. x = a maps to ellipse $\frac{u^2}{\sinh^2 a}+\frac{v^2}{\cosh^2 a}=1$ in the w-plane.

Thus family of straight lines x = a in z-plane map into the family of ellipses in the w-plane.

- The family of straight lines y = b in z-plane map into the family of hyperbola $\frac{v^2}{\sin^2 b}-\frac{u^2}{\cos^2 b}=1$ in the w-plane.
- The rectangular region $a_1 \le x \le a_2$ and $b_1 \le y \le b_2$ in the z-plane maps into the shaded area bounded by corresponding ellipse and hyperbola in the w-plane.
- The lines x = a and y = b are mutually orthogonal in the z-plane.

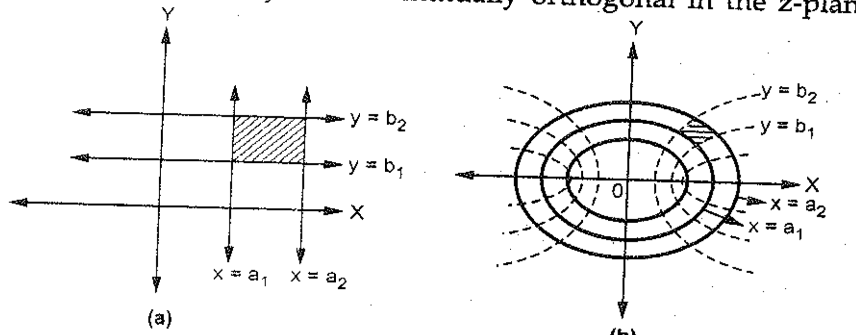


Fig. 9.4

Also the corresponding mapped family of ellipses and hyperbola in w-plane are mutually orthogonal.

This proves the conformality of w = sinh z.

5) The transformation w = cosh z   [SPPU : May-09]

We have

u + i v = cosh (x + i y) = cosh x cos y + i sinh x sin y

$\therefore$ u = cosh x cos y, v = sinh x sin y

Eliminating x, we get $\frac{u^2}{\cos^2 y}-\frac{v^2}{\sin^2 y} = 1$ ... (9.12)

while eliminating y, we get $\frac{u^2}{\cosh^2 x}+\frac{v^2}{\sinh^2 x} = 1$ ... (9.13)

Thus

- The family of straight lines x = a map into ellipses $\frac{u^2}{\cosh^2 a}+\frac{v^2}{\sinh^2 a}=1$ in the w-plane.
- The family of straight lines y = b map into hyperbolae $\frac{u^2}{\cos^2 b}-\frac{v^2}{\sin^2 b}=1$ in the w-plane.
- The rectangular region $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane maps into the region in the w-plane bounded by the corresponding hyperbolae and ellipses. Thus if the lines x = a and y = b are mutually orthogonal. Also the corresponding mapped family of ellipses and hyperbolae are mutually orthogonal. This proves the conformality of w = cosh z .

Q.22 Find the map of straight line y = 3x under the transformation $w=\frac{z-1}{z+1}$.   [SPPU : Dec.-16, May-19, Marks 5]

Ans. : We have

$$w = \frac{z-1}{z+1}$$

$zw+w=z-1$ or $z(w-1)=-1-w \Rightarrow z=\frac{w+1}{1-w}$

Let z = x + i y and w = u + i v

$\therefore$ We get

$$x+iy = \frac{(u+iv)+1}{1-(u+iv)} = \frac{(u+1)+iv}{(1-u)-iv} \times \frac{(1-u)+iv}{(1-u)+iv}$$

$$x+iy = \frac{(1-u^2-v^2)+i[v(1-u)+v(u+1)]}{(1-u)^2+v^2}$$

$$x = \frac{1-u^2-v^2}{(1-u)^2+v^2}, \quad y = \frac{2v}{(1-u)^2+v^2}$$

$\therefore$ The map of $y = 3x$ is

$$\frac{2v}{(1-u)^2+v^2} = \frac{3(1-u^2-v^2)}{(1-u)^2+v^2}$$

$$\Rightarrow \quad 2v = 3(1-u^2-v^2)$$

$$3u^2+3v^2+2v-3 = 0$$

$$\Rightarrow u^2+v^2+\frac{2}{3}v-1 = 0$$

which is the required equation of circle.

Q.23 Show that the bilinear transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x = 0$ onto the straight line $4u+3 = 0$

[SPPU : Dec.-09, 10, 11, May-11]

Ans. : The given transformation is $w = \frac{2z+3}{z-4}$ ... (1)

The inverse transformation of (1) is

$$z = \frac{4w+3}{w-2}$$

Thus $$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$$

The equation of the given circle is

$$x^2+y^2-4x = 0 \text{ can be written as}$$

$$z\bar{z}-2(z+\bar{z}) = 0$$

Substituting the values of z and $\bar{z}$, we get

$$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right)-2\left(\frac{4w+3}{w-2}+\frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$\therefore 16w\bar{w}+12w+12\bar{w}-9$$

$$-2(4w\bar{w}-8w+3\bar{w}-6)$$

$$-2(4w\bar{w}-8\bar{w}+3w-6) = 0$$

$$22(w+\bar{w})+33 = 0$$

$$22(2u)+33 = 0$$

$$(\because w = u+ir, \therefore u = 2(w+\bar{w})$$

$$4u+3 = 0$$

Which is the required equation of the straight line.

Q.24 Show that $w = \frac{z-i}{1-iz}$ maps upper half of z-plane onto interior of unit circle in w-plane. **[SPPU : Dec. 14, 15]**

Ans. : We have $w = \frac{z-i}{1-iz}$. Let $z = x+iy$ and $w = u+iv$

$$\therefore \quad w(1-iz) = z-i$$

$$-izw-z = -i-w \qquad \Rightarrow z = \frac{w+i}{1+iw}$$

$$\therefore \quad x+iy = \frac{u+iv+i}{1+i(u+iv)} = \frac{u+(v+1)i}{(1-v)+iu}$$

$$= \frac{u+(v+1)i}{(1-v)+iu}\times\frac{(1-v)-iu}{(1-v)-iu}$$

$$= \frac{u(1-v)+(v+1)u+i[-u^2+(1+v)(1-v)]}{(1-v)^2+u^2}$$

$$x+iy = \frac{u+u+i(-u^2-v^2+1)}{(1-v)^2+u^2}$$

But upper half of z plane is $y > 0$

$$\therefore \frac{-u^2-v^2+1}{(1-v)^2+u^2} > 0$$

$$\Rightarrow -u^2-v^2+1 > 0$$

$$u^2+v^2 < 1$$

which is the interior of unit circle in w-plane.

Q.25 Show that the transformation ω = sin z transforms the straight lines x = C of z plane into hyperbolas in the ω plane.

[SPPU : May-14]

Ans. : Given transformation

$$\omega = \sin z$$

$$\omega = u + iv, \quad z = x + iy$$

$$u + iv = \sin (x + iy) = \sin x \cosh y + i\cos x \sinh y$$

$$u = \sin x \cosh y, \quad v = \cos x \sinh y$$

$$\frac{u}{\sin x} = \cosh y \quad \frac{v}{\cos x} = \sinh y$$

$\therefore \quad \cosh^2 y - \sinh^2 y = 1$

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

When x = C

$\Rightarrow \dfrac{u^2}{\sin^2 C} - \dfrac{v^2}{\cos^2 C} = 1$ which is hyperbola

$\therefore$ Straight lines x = C of z plane is mapped on hyperbola in the ω-plane.

Q.26 Find the map of the circle $|z - i| = 1$ under the transformation $w = \frac{1}{z}$ into w-plane.

[SPPU : Dec.-14, May-12]

Ans. : Let z = x + iy and w = u + iv.

Given that $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow \quad x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

To find the image of $|z - i| = 1$

$$|x + iy - i| = 1$$

$$\therefore \quad |x + i(y - 1)| = 1$$

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + y^2 = 2y$$

$\therefore$ Putting values of x and y, we get

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = \frac{-2v}{u^2 + v^2}$$

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{-2v}{u^2 + v^2}$$

$$1 = -2v$$

$\Rightarrow \quad v = -\frac{1}{2}$ which is a straight line

Thus the image of given circle is a straight line $v = -\frac{1}{2}$ in w-plane.

Q.27 Show that under the transformation $w = \frac{i - z}{i + z}$, x-axis in z-plane is mapped onto the circle $|w| = 1$. [SPPU : Dec.-06, 12, May-07, 13]

Ans. : Let z = x + i y and w = u + i v

and $\quad w = \dfrac{i - z}{i + z} \quad \therefore w(i + z) = i - z$

$$\therefore \quad z = \frac{i(1 - w)}{1 + w}$$

$$\Rightarrow \quad x + iy = \frac{i(1 - w)}{1 + w} = \frac{i(1 - u - iv)}{1 + u + iv} \times \frac{1 + u - iv}{1 + u - iv}$$

$$x + iy = \frac{i[(1 - u^2) - v^2 + i\{(1 - u)(-v) + (-v)(1 + u)\}]}{(1 + u)^2 + v^2}$$

Equating corresponding real and imaginary parts we get,

$$x = \frac{2v}{(1+u)^2+v^2} \quad \text{and}$$

$$y = \frac{1-u^2-v^2}{(1+u)^2+v^2}$$

Therefore the x-axis i.e. $y = 0$ maps to

$$\frac{1-u^2-v^2}{(1+u)^2+v^2} = 0$$

$$\Rightarrow \qquad u^2+v^2 = 1 \text{ i.e. } |w| = 1$$

Q.28 Find the map of the straight line $2y = x$ under the transformation $w = \frac{2z-1}{2z+1}$. [SPPU : Dec.- 17, Marks 4]

Ans. : The given transformation is $w = \frac{2z-1}{2z+1}$

$$\therefore \quad w(2z+1) = 2z-1$$

$$2zw + w - 2z = -1$$

$$z(2w-2) = -1-w$$

$$z = \frac{-1-w}{2w-2} = \frac{w+1}{2-2w}$$

Let $z = x+iy$ and $w = u+iv$

$\therefore$ We get

$$x+iy = \frac{u+iv+1}{2-2u-2iv} = \frac{(u+1)+iv}{(2-2u)+i(-2v)}$$

$$\therefore \quad x+iy = \frac{(u+1)+iv}{(2-2u)-i(2v)} \times \frac{(2-2u)+i(2v)}{(2-2u)+i(2v)}$$

$$= \frac{2(u+1)-2u(u+1)+2iv(1-u)+i2v(u+1)-2v^2}{(2-2u)^2+(2v)^2}$$

$$x+iy = \frac{(-2u^2+2-2v^2)+i(4v)}{4-8u+4u^2+4v^2}$$

$$\therefore \quad x = \frac{-2u^2+2-2v^2}{4-8u+4u^2+4v^2},$$

$$y = \frac{4v}{4-8u+4u^2+4v^2}$$

But given that $2y = x$

$$\therefore \quad \frac{8v}{(4-8u+4u^2+4v^2)} = \frac{-2u^2+2-2v^2}{4-8u+4u^2+4v^2}$$

$$8v = -2u^2+2-2v^2$$

$$2u^2+2v^2-8v = 2$$

$$u^2+v^2-4v = 2$$

$$(u-0)^2+(v-2)^2 = 6$$

which is the circle with centre at (0, 2) and radius $\sqrt{6}$.

Q.29 Find the image of hyperbola $x^2-y^2 = 1$ under the transformation $w = \frac{1}{z}$. [SPPU : May- 17, Marks 15, Marks 4]

Ans. :

We have

$$z = \frac{1}{w} \Rightarrow x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\therefore \quad y = \frac{-v}{u^2+v^2} \text{ and } x - \frac{u}{u^2+v^2}$$

But $x^2-y^2 = 1$

$$\Rightarrow \quad \frac{u^2}{(u^2+v^2)^2} - \frac{v^2}{(u^2+v^2)^2} = 1$$

$$\frac{u^2-v^2}{(u^2+v^2)^2} = 1$$

$$\Rightarrow \quad u^2-v^2 = (u^2+v^2)^2$$

$$\Rightarrow \quad u^2-v^2 = u^4+v^4+2u^2v^2$$

$$\Rightarrow \quad u^4+y^4+2u^2v^2 + u^2-v^2 = 0$$

which is the required image in w-plane.

Q.30 Show that the transformation $w = z+\frac{1}{z}-2i$ maps the circle $|z| = 2$ onto an ellipse. Find the centre and semi-major and semi-minor axes of ellipse. [SPPU : May-15,16, Marks 5]

Ans. : Let $z = x+iy$ and $w = u+iv$

$\therefore \quad w = z + \frac{1}{z} - 2i \Rightarrow$

$$u + iv = x + iy + \frac{1}{x+iy} - 2i = x + iy + \frac{x - iy}{x^2+y^2} - 2i$$

$$u + iv = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - z - \frac{y}{x^2+y^2}\right)$$

Equating corresponding real and imaginary parts of above equation we get

$$u = x + \frac{x}{x^2+y^2}, \quad v = y - z - \frac{y}{x^2+y^2}$$

For the circle $|z| = 2 \Rightarrow x^2 + y^2 = 4$

$\therefore$ We get

$$u = x + \frac{x}{4} = \frac{5x}{4} \text{ and}$$

$$v = y - 2 - \frac{y}{4} = \frac{3y}{4} - 2$$

$$\Rightarrow \quad x = \frac{4u}{5} \text{ and } y = \frac{4}{3}(v+2)$$

Thus $x^2 + y^2 = 4$ gives

$$\therefore \quad \frac{16u^2}{25} + \frac{16(v+2)^2}{9} = 4$$

$$\therefore \quad \frac{u^2}{\left(\frac{5}{2}\right)^2} + \frac{(v+2)^2}{\left(\frac{3}{2}\right)^2} = 1$$

which represents the ellipse with centre at $(0, -2)$ and semi-major axis $= \frac{5}{2}$ and semi-minor axis $= \frac{3}{2}$.

END...

UNIT VI

10 Complex Integration

10.1 : Line Integral

I) Let $f(z) = u + iv$ and $z = x + iy \therefore dz = dx + idy$ then we have,

$$I = \int_C f(z)\, dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

This shows that the evaluation of the line integral of the complex function can be reduced to the evaluation of two line integrals of real function.

II) Basic properties :

1. $\int_C (k_1 f(z) + k_2 g(z))dz = k_1 \int_C f(z)\, dz + k_2 \int_C g(z)\, dz$

2. $\int_A^B f(z)\, dz = -\int_B^A f(z)\, dz$

3. If A < B < C then

$$\int_A^C f(z)\, dz = \int_A^B f(z)\, dz + \int_B^C f(z)\, dz$$

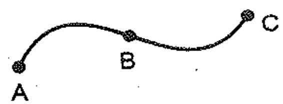


Fig. 10.1

4. Let $|f(z)| \leq M$ then

$$\left|\int_C f(z)\, dz\right| \leq M\,L$$ where L is the length of the curve C.

Remark : Real definite integrals are interpreted as area, but no such interpretation is possible for complex definite integrals.

Q.1 Evaluate $\int_{2+4i}^{5-5i} (z+1)\, dz$

i) Along the path $x = t^2+1,\ y = 3t+1$

ii) Along the straight line joining $2+4i$ **and** $5-5i$.

[SPPU : May-17,19, Marks 5]

Ans. :

Let $\quad I = \int_{2+4i}^{5-5i} (z+1)\, dz$

Let $\quad z = x+iy \quad \therefore dz = dx + idy$

$$\therefore \quad I = \int_{2+4i}^{5-5i} [(x+1)+iy]\,[dx+idy]$$

$$= \int_{2+4i}^{5-5i} [(x+1)dx - y\,dy] + i\int_{2+4i}^{5-5i} [(x+1)\,dy + y\,dx] \quad \dots (1)$$

i) Along the path $x = t^2+1,\quad y = 3t+1$

$\therefore \quad x = 2,\ y = 4 \quad \Rightarrow t = 1$

and $\quad x = 5$ and $y = -5 \Rightarrow t = -2$

$\therefore \quad dx = 2t\,dt,\ dy = 3dt$

$\therefore$ Equation (1) becomes

$$I = \int_{1}^{-2} [(t^2+2)\,2t\,dt - (3t+1)\,3\,dt]$$

$$+ i\int_{1}^{-2} [(t^2+2)\,3\,dt + (3t+1)\,2t\,dt]$$

$$= \int_{1}^{-2} (2t^3 - 5t - 3)\,dt + i\int_{1}^{-2} (9t^2+2t+6)\,dt$$

$$= \left[2\frac{t^4}{4} - 5\frac{t^2}{2} - 3t\right]_1^{-2} + i\left[9\frac{t^3}{3} + 2\frac{t^2}{2} + 6t\right]_1^{-2}$$

$$= 9 - 42i$$

ii) Along the straight line joining $2+4i$ and $5-5i$

We have $\quad \dfrac{x-2}{3} = \dfrac{y-4}{-9} = t$

$\therefore \quad x = 3t+2,\quad y = -9t+4$

$\therefore \quad dx = 3dt,\quad dy = -9\,dt \quad$ Here $t = 0$ to $t = 1$

$$\therefore \quad I = \int_0^1 [(3t+3)\,3\,dt + (9t-4)(-9\,dt)]$$

$$+ i\int_0^1 [(3t+3)(-9\,dt) + (-9t+4)\,3\,dt]$$

$$= \int_0^1 [-72t+45]\,dt + i\int_0^1 (-54t-15)\,dt$$

$$I = \left[-72\frac{t^2}{2} + 45t\right]_0^1 + i\left[-54\frac{t^2}{2} - 15t\right]_0^1$$

$$I = 9 - 42i$$

10.2 : Cauchy's Theorem

1) Cauchy's theorem : If $f(z)$ is an analytic function in a simply connected domain D, then for every simple closed curve C in D, $\oint_C f(z)\,dz = 0$.

Proof : Let R be region bounded by the closed curve C and

let $f(z) = u(x,y) + i\,v(x,y)$

and $z = x+iy \quad \therefore dz = dx + idy$

then,

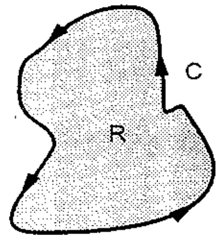


Fig. 10.2

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C [u\,dx - v\,dy] + i\oint_C [u\,dy + v\,dx] \quad \dots (10.1)$$

As $f'(z)$ is continuous, the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are also continuous in R.

$\therefore$ Applying Green's theorem on both the integrals of R.H.S. of equation (10.1), we get

$$\oint_C f(z)\,dz = \iint_R \left[-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right] dxdy + i \iint_R \left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right] dxdy \qquad \text{... (10.2)}$$

Since f(z) is analytic, C.R. equations must be satisfied

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ By using C.R. equations both the integrals of R.H.S. of equation (10.2) become zero.

$$\therefore \qquad \oint_C f(z)\,dz = 0$$

> **Note**
>
> 1) Cauchy's theorem is also known as Cauchy's integral theorem.
>
> 2) In a Cauchy's theorem the condition that f be analytic in D is sufficient rather than necessary.

e.g. $\oint_C \frac{dz}{z^2} = 0$ where C is a unit circle.

But $f(z) = \frac{1}{z^2}$ is not analytic at z = 0.

Corollary 1 :

Let f(z) be an analytic function then $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path joining the points z_1 and z_2.

We can prove this corollary by using Cauchy's theorem.

Corollary 2 :

If f(z) is analytic in the region R between two simple closed contours C_1 and C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

where C_1 and C_2 are described in a same direction.

Corollary 3 :

In a Cauchy's theorem the condition that the domain D be simply connected is quite essential. In other words by Cauchy's theorem, if f(z) is analytic on and within a simple closed curve C then $\oint_C f(z)\,dz=0$

e.g. If C is the annulus $\frac{1}{2}<|Z|<\frac{3}{2}$ where

$f(z) = \frac{1}{z}$ is analytic but this domain is not simply connected, so we cannot apply Cauchy's theorem, Here $\oint_C \frac{1}{z}\,dz = 2\pi i$.

10.3 : Cauchy's Integral Formula

[SPPU : May-05, 06, 08, 09, 10, 11, 13, Dec.-05, 06, 08, 10, 11, 12]

Let f(z) is analytic on and within a closed curve C and if a is any point within C then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Proof : Consider the function $\frac{f(z)}{z-a}$ which is analytic at all points within C except a.

Draw a small circle C_1 around point a with the point z = a as a centre and radius r.

$\therefore$ $\frac{f(z)}{z-a}$ is analytic in the region R bounded by C_1 and C_2 (i.e. Dotted region).

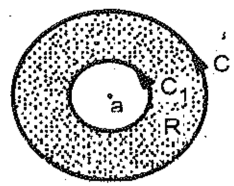


Fig. 10.3

$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \qquad \text{... (10.3)}$$

Consider R.H.S. = $\oint_{C_1} \frac{f(z)}{z-a}\,dz$, For any point on C_1 $z - a = re^{i\theta}$

$$\therefore \qquad dz = rie^{i\theta}\,d\theta$$

$$\therefore \qquad \text{R.H.S.} = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\,rie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta \qquad \text{... (10.4)}$$

In the limiting form, as C_1 shrinks to the point a i.e. as $r \to 0$ equation (10.4) becomes,

$$\oint_{C_1} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = i\,f(a)\,2\pi = 2\pi i\,f(a)$$

$$\therefore \qquad \oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$$

$\therefore$ Equation (10.3) becomes

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$$

$$\therefore \qquad f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

which is the desired Cauchy's Integral formula.

Corollary : We have $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$

Differentiating both sides w.r.t. a under integral sign, we get.

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left(\frac{f(z)}{z-a}\right)dz$$

$$= \frac{1}{2\pi i}\oint_C -\frac{f(z)}{(z-a)^2}(-1)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

Again differentiating w.r.t. a we get

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$$

In general

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

By above results, we observed that if f(z) is analytic on the simple closed curve C then the values of function and all it's derivatives can be determined at any point of C. Also it follows that **an analytic function possesses analytic derivatives of all orders**.

Q.2 Evaluate $\int_C \frac{4z^2+z}{(z-1)^2}\,dz$ where C is the circle $|z - 1| = 2$

[SPPU : Dec.-17]

Ans. : Let $I = \int_C \frac{4z^2+z}{(z-1)^2}\,dz$

$z = +1$ is a pole of order 2 which lies inside $|z - 1| = 2$

$f(z) = 4z^2 + z$ is analytic everywhere inside $|z - 1| = 2$

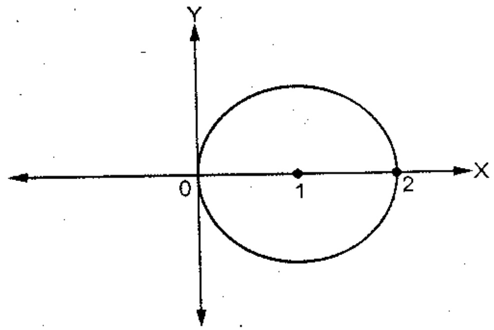


Fig. Q.2.1

$\therefore$ By Cauchy's Integral theorem

$$I = \int \frac{4z^2+z}{(z-1)^2} dz$$

$$I = 2\pi i f'(1)$$

$$I = 2\pi i [82+1]_{z=1} = 9(2\pi i) = 18\pi i$$

Q.3 Evaluate $\int_C \frac{2z^3+z+5}{(z-5)^3} dz$ **where** $\frac{x^2}{16}+\frac{y^2}{4} = 1$

[SPPU : Dec.-17]

Ans. : Let $I = \int_C \frac{2z^3+z+5}{(z-5)^3} dz$ ...(1)

$$f(z) = \frac{2z^3+z+5}{(z-5)^3}$$

is an analytic everywhere inside $\frac{x^2}{16}+\frac{y^2}{4} = 1$

$\therefore$ By Cauchy's integral theorem

$$I = \int \frac{2z^3+z+5}{(z-5)^3} dz = 0$$

Q.4 Evaluate $\oint \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz$ **where C is** $|z| = 2$.

[SPPU : Dec.-08, 14 May-09, 11]

Ans. :

Let $I = \oint \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz$

$z = -\frac{\pi}{3}$ is pole of order 4

$z = -\frac{\pi}{3}$ lies inside of $|z| = 2$

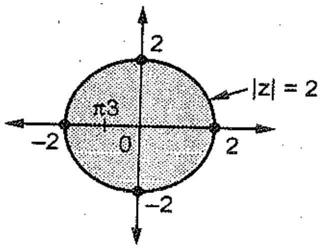


Fig. Q.4.1

$\therefore$ By Cauchy's integral theorem

$$I = \oint \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz = \frac{2\pi i}{3!} f'''\left(-\frac{\pi}{3}\right) \qquad ...(1)$$

where $f(z) = \sin 2z$

$$f'(z) = 2\cos 2z$$

$$f''(z) = -4\sin 2z$$

$$f'''(z) = -8\cos 2z$$

$$\therefore \quad f'''\left(-\frac{\pi}{3}\right) = -8\cos\left(-\frac{\pi}{6}\right) = -8\cos\frac{\pi}{6} = -8\times\frac{1}{2} = -4$$

$\therefore$ Equation (1) becomes.

$$I = \frac{2\pi i}{3\times 2}\times(-4) = -\frac{4\pi i}{3}$$

Q.5 Evaluate :
$\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ **where C is the circle** $|z| = \frac{3}{2}$

[SPPU : Dec.-18, Marks 5]

Ans. : Let $f(z) = \sin^2 z,\ f'(z) = 2\sin z\cos z = \sin 2z$

$$f''(z) = 2\cos 2z$$

$$f''\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

Let $I = \oint \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$

$z = \frac{\pi}{6}$ lies inside of $|z| = \frac{3}{2}$

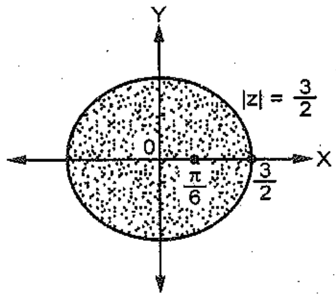


Fig. Q.5.1

$\therefore$ By Cauchy's integral formula,

$$I = \oint \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz = 2\pi i f''\left(\frac{\pi}{6}\right)$$

$$I = 2\pi i(1) = 2\pi i$$

Q.6 Evaluate $\oint_C \frac{z^2 + \cos^2 z}{\left(z - \frac{\pi}{4}\right)^3}$ dz where C is a circle $x^2 + y^2 = 1$

[SPPU : May-14]

Ans. :

Let $$I = \oint_C \frac{z^2 + \cos^2 z}{\left(z-\frac{\pi}{4}\right)^3} dz \quad \ldots(1)$$

$$z = \frac{\pi}{4} \text{ is a pole of order 3.}$$

$\therefore$ $$I = \frac{2\pi i}{2!} f''\left(\frac{\pi}{4}\right) \quad \ldots(2)$$

Let $$f(z) = z^2 + \cos^2 z$$

$$f'(z) = 2z - 2\cos z \sin z$$

$$= 2z - \sin 2z \Rightarrow f''(z) = 2 - 2\cos 2z$$

$$f''\left(\frac{\pi}{4}\right) = 2 - 2\cos 2\frac{\pi}{4} = 2$$

$\therefore$ $$I = \frac{2\pi i}{2}(2) = 2\pi i$$

Q.7 Evaluate $\oint_C \frac{z^2+1}{z^2-1}$ dz where C is the circle $|z - 1| = 1$.

[SPPU : May-15]

Ans. : Let $I = \oint_C \frac{z^2+1}{z^2-1} dz$

$\therefore z^2 - 1 = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$

$\therefore$ z = 1, – 1 are two singularities and out of these singularities and out of these singularities z = 1 lies inside of $|z - 1| = 1$ and z = – 1 lies outside of $|z - 1| = 1$

$\therefore$ $$f(z) = \frac{z^2+1}{z+1} \text{ is analytic everywhere in } |z-1| = 1$$

$\therefore$ By Cauchy's Integral formula

$$I = 2\pi i f(1) = 2\pi i \left[\frac{1+1}{1+1}\right] = 2\pi i$$

Q.8 Evaluate $\oint_C \frac{2z^2+z+5}{\left(z-\frac{3}{2}\right)^2}$ dz where C is $\frac{x^2}{4} + \frac{y^2}{9} = 1$

[SPPU : May-10, Dec.-15]

Ans. : Let $f(z) = 2z^2 + z + 5$

f(z) is analytic everywhere and $z = \frac{3}{2}$ lies within $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

$\therefore$ By Cauchy's integral formula

$$I = \oint \frac{f(z)}{\left(z-\frac{3}{2}\right)^2} dz = 2\pi i f'\left(\frac{3}{2}\right)$$

$$= 2\pi i[f'(z)]_{z=\frac{3}{2}} = 2\pi i[4z+1]_{z=\frac{3}{2}}$$

$$= 2\pi i[6+1] = 14\pi i$$

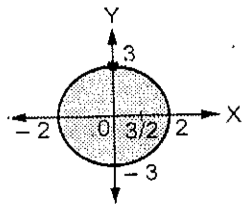


Fig. Q.8.1

Q.9 By using Cauchy's integral formula evaluate $\oint \frac{e^z}{z^2+1}$ dz over $|z - 1| = 1$.

[SPPU : May-13]

Ans. :

Let $$f(z) = \frac{e^z}{z^2+1}, \quad |z-1| = 1$$

$$= \frac{e^z}{(z-i)(z+i)}$$

f(z) is analytic within and on circle $|z - 1| = 1$. The singular points $z = \pm i$ lies outside of $|z - 1| = 1$

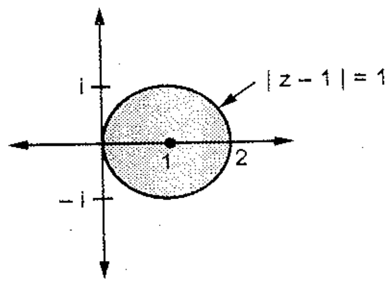


Fig. Q.9.1

$\therefore$ By Cauchy's theorem

$$\oint \frac{e^z}{(z^2+1)} dz = 0$$

Q.10 Evaluate $\oint_C \frac{z+2}{z^2+1} dz$ **where C is the circle** $|z+i| = \frac{1}{2}$

☞ [SPPU : Dec.-18, Marks 5]

Ans. : Let
$$I = \oint \frac{z+2}{(z^2+1)} dz$$

Now
$$z^2+1 = 0 \Rightarrow z^2 = -1 = i^2 \Rightarrow z = \pm i$$

$z = i$ and $z = -i$ are simple poles.

Given that, $|z+i| = \frac{1}{2}$ is a circle with centre at $-i$ and radius $\frac{1}{2}$.

$z = -i$ lies inside of $|z+i| = \frac{1}{2}$ and $z = i$ lies outside

$\therefore$ By Residue theorem,

$$I = \int \frac{z+2}{(z^2+1)} dz = 2\pi i (R_1)$$

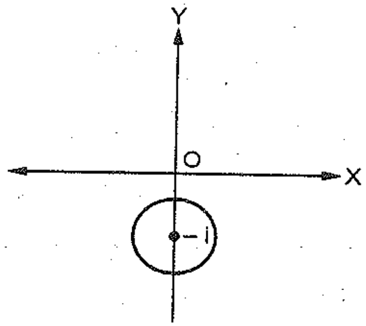


Fig. Q.10.1

where R_1 = Residue of f(z) at $z = -i$

$$= \left[\frac{z+2}{(z^2+1)}(z+i)\right]_{z=-i}$$

$$= \left[\frac{z+2}{z-i}\right]_{z=-i}$$

$$= \frac{-i+2}{-i-i} = \frac{-i+2}{-2i} = \frac{1}{2} - \frac{1}{i} = \frac{1}{2} + i$$

$$\therefore I = 2\pi i\left(\frac{1}{2}+i\right) = \pi i - 2\pi = \pi(i-2)$$

Q.11 Evaluate $\int_C \frac{1+z}{z(z-2)} dz$ **where c is the circle** $|z| = 1$

☞ [SPPU : May-18]

Ans. : Let $f(z) = \frac{1+z}{z-2}$ and $z = 0$ lies inside $|z| = 1$

$\therefore$ By Cauchy's Integral formula, a = 0

$$I = \int \frac{1+z}{z(z-2)} dz = 2\pi i f(0)$$

$$= 2\pi i\left(\frac{1}{-2}\right)$$

$$= -\pi i$$

Q.12 Evaluate $\frac{z+4}{(z+1)^2(z+2)^2} dz$ **where C is the circle** $|z+1| = \frac{1}{2}$

☞ [SPPU : May-17]

Ans. :
$$\text{Let } I = \frac{z+4}{(z+1)^2(z+2)^2} dz \quad \ldots(1)$$

Let $f(z) = \frac{z+4}{(z+2)^2}$ is analytic everywhere

and $z = -i$ lies inside $|z+1| = \frac{1}{2}$

$\therefore$ By Cauchy's Integral theorem

$$I = \oint \frac{f(z)}{(z+1)^2} dz$$

$$= 2\pi i f'(-1) \quad \ldots(2)$$

$$f'(2) = \frac{(z+2)^2(1)-(z+4)2(z+2)}{(z+2)^4}$$

$$\therefore f'(-1) = \frac{1-(3)2(1)}{1} = -5$$

$$\therefore I = 2\pi i(-5) = -10\pi i$$

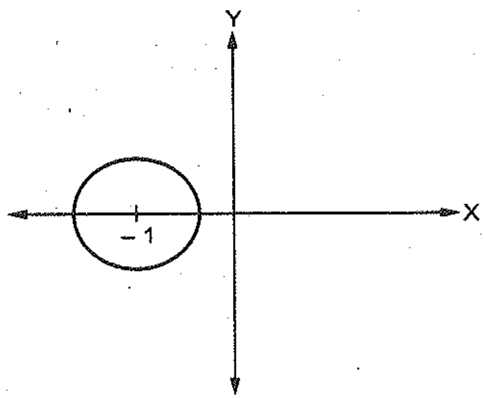


Fig. Q.12.1

Q.13 Evaluate $\int_C \frac{e^{3z}}{(z-\log 2)^4} dz$ **where C is the square with vertics** $\pm 1, \pm i$ **[SPPU : May-16]**

Ans. :

$$\text{Let } I = \int_C \frac{e^{32}}{(z-\log 2)^4} dz \quad \ldots(1)$$

Let $f(z) = e^{3z}$ is an analytic function everywhere inside the square $\pm 1, \pm i$.

$\therefore$ By Cauchy's theorem

$$I = 0$$

10.4 : Zeroes, Singular Points, Poles and Residues

[SPPU : Dec.-06, 07, 09, 10, 11, 12, May-05, 07, 12, 13]

A] Zero : A zero of an analytic function f(z) is that value of z for which $f(z) = 0$.

If f(z) is analytic in the neighbourhood of a point

$z = a$, then by Taylor's series.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ldots$$

where
$$a_n = \frac{f^{(n)}(a)}{n!}$$

If $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$, then f(z) is said to have a zero of order m at $z = a$. If $m = 1$ then zero is said to be simple.

B] Singularities of an analytic function :

A point at which function f(z) ceases to be analytic is said to be singular point of f(z).

C] Pole : If the principal part of the Laurent's series contain n terms

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \ldots + \frac{a_{-n}}{(z-a)^n}$$

then the singular point $z = a$ is called the pole of f(z) of order n.

If there is only one term $\frac{a_{-1}}{z-a}$ then $z = a$ is called the simple pole.

If $n = 2$ then $z = a$ is called double pole.

Note : If $f(z) = \frac{p(z)}{q(z)}$ then the poles of f(z) are the zeroes of q(z).

D] Residues :

The coefficient of $(z-a)^{-1}$ in the expansion of f(z) around an isolated singularity is called the residue of f(z) at that point.

$\therefore$ In a Laurent series expansion a_{-1} is the residue of f(z) at $z = a$.

$\therefore$ Residue of f(z) at $(z = a) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$

$$\therefore \oint_C f(z)\, dz = 2\pi i \operatorname{Res} f(a)$$

E] Cauchy's residue theorem

If f(z) is analytic on and within a closed contour C except at finite number of isolated singular points within 'C' then

$$\oint_C f(z)\, dz = 2\pi i \begin{bmatrix} \text{Sum of residues at these} \\ \text{singular points} \end{bmatrix}$$

$$= 2\pi i\, [r_1 + r_2 + r_3 + \ldots + r_n]$$

where $r_1, r_2, \ldots, r_n$ are the residues at the singular points within C.

F) Working rules to find poles and residues :

1) If f(z) has a simple pole at $z = a$ then

$$\operatorname{Res} f(a) = \lim_{z \to a} (z-a)\, f(z)$$

2) If $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at $z = a$ then

$$\text{Res } f(a) = \frac{p(a)}{q'(a)} = \left[\frac{p(z)}{q'(z)}\right]_{z=a}$$

3) If f(z) has a double poles at z = a then

$$\text{Res } f(a) = \lim_{z \to b} \frac{d}{dz} (z-a)^2 f(z)$$

4) If f(z) has a pole of order n at z = a then

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]\right]_{z=a}$$

Q.14 Evaluate $\oint_C \frac{z}{z^4 + 13z^2 + 36} dz$ where C is the circle $|z| = 2.5$.

[SPPU : May-07, Dec.-15]

Ans. : Let

$$I = \oint_C f(z)\, dz$$

and

$$f(z) = \frac{z}{z^4 + 13z^2 + 36}$$

$$f(z) = \frac{z}{(z^2+9)(z^2+4)}$$

$$= \frac{z}{(z+3i)(z-3i)(z+2i)(z-2i)}$$

(z) has simple poles at z = 3i, – 3i, 2i, – 2i. Out of these 4 poles only two poles z = 2i and – 2i lie inside |z| = 2.5.

∴ By Residue theorem, we have

$$\oint_C f(z)\, dz = 2\pi i [r_1 + r_2]$$

where r_1, r_2 are residues of f(z) at z = 2i and – 2i respectively.

Now

$$r_1 = \left[\frac{z}{(z+2i)(z-3i)(z+3i)}\right]_{z=2i} = \frac{1}{10}$$

$$r_2 = \left[\frac{z}{(z-2i)(z-3i)(z+3i)}\right]_{z=-2i} = \frac{1}{10}$$

$$\therefore \quad I = \oint_C f(z)\, dz = 2\pi i\left[\frac{1}{10} + \frac{1}{10}\right] = \frac{2\pi i}{5}$$

Q.15 Evaluate : $\oint_C \frac{z+4}{z^2 + 2z + 5} dz$ where 'C' is the circle $|z - 2i| = \frac{3}{2}$.

[SPPU : May-19]

Ans. :

Let

$$I = \oint_C f(z) dz$$

where,

$$f(z) = \frac{z+4}{z^2+2z+5}$$

$$= \frac{z+4}{(z+1+2i)(z+1-2i)}$$

f(z) has simple poles at $z_1 = -1 - 2i$ and $z_2 = -1 + 2i$. Out of these poles only $z_2 = -1 + 2i$ lies inside the circle $|z - 2i| = 3/2$.

∴ By residue theorem

$$\oint_C f(z) dz = 2\pi i\, r_1$$

where r_1 is residue of f(z) at $z = z_2$

$$r_1 = [(z - z_2) f(z)]\big|_{z=z_2}$$

$$= \left[(z+1-2i)\frac{z+4}{(z+1+2i)(z+1-2i)}\right]\Bigg|_{z=-1+2i}$$

$$= \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$$\Rightarrow \quad \oint_C f(z) dz = 2\pi i\left[\frac{3+2i}{4i}\right]$$

$$= \frac{\pi}{2}(3+2i)$$

Q.16 Evaluate $\int \frac{15z+9}{z(z+3)} dz$ where C is the circle $|z-1| = 3$

[SPPU : May-18]

Ans. : Let $I = \oint \frac{15z+9}{z(z+3)} dz$

$\therefore$ f(z) has a simple pole at z = 0, –3, Out of these poles only z = 0 lies inside $|z-1| = 3$.

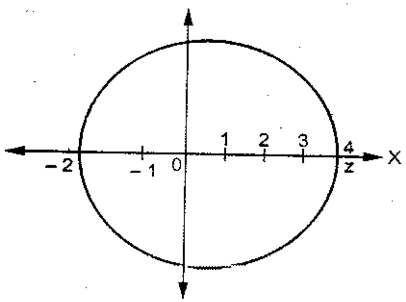


Fig. Q.16.1

$\therefore$ By Residue theorem, we have

$$I = \oint_C f(z)\, dz$$

$$= 2\pi i (R_1) \qquad \ldots(1)$$

Where R_1 is the residue of f(z) at z = 0.

$$\therefore \qquad R_1 = \left[\frac{15z+9}{z+3}\right]_{z=0} = \frac{9}{3} = 3$$

$$\therefore \qquad I = \oint_C f(z)\, dz = 2\pi i\,(3) = \pi i$$

Q.17 Evaluate the following integral using residue theorem $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$ where C is circle $|z| = \frac{3}{2}$

[SPPU : Dec.-15]

Ans. : Let $\quad I = \oint \frac{4-3z}{z(z-1)(z-2)} dz$

Let $\qquad f(z) = \frac{4-3z}{z(z-1)(z-2)}$

$\therefore$ f(z) has a simple pole at z = 0, 1, 2 out of these poles z = 0, 1 lie inside and z = 2 lies outside of $|z| = \frac{3}{2}$

$$\therefore \qquad I = \oint_C f(z)\, dz = 2\pi i\,[r_1 + r_2] \qquad \ldots(1)$$

where $\qquad r_1$ = Residue of f(z) at z = 0

$$= \left[z\frac{(4-3z)}{z(z-1)(z-2)}\right]_{z=0} = \frac{4}{(-1)(-2)} = 2$$

r_2 = Residue of f(z) at z = 1

$$r_2 = \left[(z-1)\frac{(4-3z)}{z(z-1)(z-2)}\right]_{z=1}$$

$$r_2 = \left[\frac{4-3}{1(1-2)}\right] = \frac{1}{-1} = -1$$

$\therefore$ Equation (1) becomes

$$I = 2\pi i\,[2 + (-1)] = 2\pi i$$

Q.18 By using Cauchy's residue theorem evaluate $\oint \frac{e^{2z}}{z(z-1)^2} dz$ over $|z| = 3$.

[SPPU : Dec.-12, 15, May-13]

Ans. : Let $f(z) = \frac{e^{2z}}{z(z-1)^2}$

$\therefore$ f(z) has a simple pole at z = 0 and double Pole at z = 1. All these poles lie within $|z| = 3$.

$\therefore$ By Residue theorem, we have

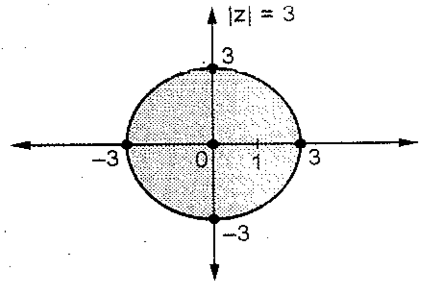


Fig. Q.18.1

$$\oint \frac{e^{2z}}{z(z-1)^2} dz = 2\pi i[r_1 + r_2] \qquad \ldots(1)$$

where r_1 and r_2 are residue of f(z) at z = 0 and z = 1 respectively.

Now $\qquad r_1$ = Residue of f(z) at z = 0

$$= [(z-0)f(z)]_{z=0}$$

$$= \left[z \times \frac{e^{2z}}{z(z-1)^2}\right]_{z=0} = \left[\frac{e^{2z}}{(z-1)^2}\right]_{z=0}$$

$$r_1 = \frac{1}{1} = 1$$

and

$$r_2 = \text{Residue of } f(z) \text{ at } z = 1$$

$$= \left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$$

$$= \left[\frac{d}{dz}(z-1)^2 \frac{e^{2z}}{z(z-1)^2}\right]_{z=1}$$

$$= \left[\frac{d}{dz}\frac{e^{2z}}{z}\right]_{z=1} = \left[\frac{ze^{2z}2 - e^{2z}}{z^2}\right]_{z=1}$$

$$r_2 = \left[\frac{2e^2 - e^2}{1}\right] = e^2$$

$\therefore$ Equation (1) becomes

$$\oint \frac{e^{2z}}{z(z-1)^2} dz = 2\pi i\,[1+e^2]$$

Q.19 Evaluate $\int \frac{5z-2}{z(z-1)} dz$ **where C is** $|z| = 3$ **[SPPU : May-16]**

Ans. :

$$\text{Let } I = \int_C \frac{5z-2}{z(z-1)} dz \qquad \ldots(1)$$

$$\text{Let } f(z) = \frac{5z-2}{z(z-1)}$$

$\therefore$ f(z) has simple poles at z = 0 and 1

All these poles lie inside $|z| = 3$.

$\therefore$ By Cauchy's Residue theorem

$$I = \oint f(z)dz = 2\pi i\,[R_1 + R_2] \qquad \ldots(2)$$

Where R_1 = Residue of f(z) at z = 0

$$= \left[\frac{z(5z-2)}{z(z-1)}\right]_{z=0} = \left[\frac{5z-2}{z-1}\right]_{z=0} = 2$$

$$R_2 = \text{Residue of } f(z) \text{ at } z = 1$$

$$= \left[\frac{(z-1)(5z-2)}{z(z-1)}\right]_{z=1} = \left[\frac{5z-2}{z}\right]_{z=1} = 3$$

$$\therefore \quad I = 2\pi i[R_1 + R_2] = 2\pi i[2+3] = 10\,\pi i$$

Q.20 Evaluate $\int \frac{z+4}{z^2+2z+5} dz$ **where C is a circle** $|z - 2i| = \frac{3}{2}$ **[SPPU : Dec.-16]**

Ans. :

$$\text{Let } I = \int \frac{z+4}{z^2+2z+5} dz = \int \frac{z+4}{(z-a)\,(z-b)} dz$$

$$z^2+2z+5 = 0 \quad \Rightarrow \quad z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\text{Let } f(z) = \frac{z+4}{z^2+2z+5} \text{ has simple poles at } -1 \pm 2i$$

Out of these poles z = –1 + 2i lies inside $|z - 2i| = \frac{3}{2}$

$\therefore$ By Residue's theorem $I = \oint f(z)dz = 2\pi i\,(R)$ ...(1)

where R = Residue of f(2) at z = –1 + 2i

$$= \left[\frac{(z+4)\,(z-(1+2i))}{(z-(-1+2i))\,(z-(-1-2i))}\right]_{z=-1+2i}$$

$$= \left[\frac{z+4}{z+1+zi}\right]_{z=-1+2i} = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$$R = \frac{1}{2} - i\frac{3}{4}$$

$$\therefore \quad I = 2\pi i(R) = 2\pi i\left(\frac{1}{2} - \frac{3i}{4}\right) = \pi i\left(1 - \frac{3i}{2}\right)$$

$$I = \frac{3\pi}{2} - i\pi$$

10.5 : Evaluation of Real Definite Integrals

[SPPU : Dec.-05, 06, 07, 08, 09, May-05, 06, 07, 08, 09, 10, 11, 12]

Consider the integral of the type

$$I = \oint_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \qquad \dots (10.5)$$

where the integrand $f(\cos\theta, \sin\theta)$ is rational function of $\sin\theta$, $\cos\theta$.

Let $\quad z = e^{i\theta} \quad \therefore dz = i e^{i\theta} dz = i z\, d\theta$

$$\therefore \quad d\theta = \frac{dz}{iz}$$

Now $$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Here as θ varies from 0 to 2π, z describes the unit circle $|z| = 1$ in the anticlockwise sense. Substituting in equation (10.5), we get,

$$I = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]$$

$$\frac{dz}{iz} = \oint_{|z|=1} \phi(z)\, dz$$

where $\phi(z)$ is a rational function of z. Now, use Cauchy residue theorem to evaluate I.

> **Note** If $f(2a-x) = f(x)\ \forall\ x$ then
> $$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

Q.21 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ [SPPU : Dec.-16]

Ans. : Let $$I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

Also as θ varies from 0 to 2π, z describes the unit circle $|z| = 1$ in the anticlockwise sense.

$$\therefore \quad I = \int_{|z|=1} \frac{1}{5+\frac{3}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{3z^2+10z+3}$$

$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{(3z+1)(z+3)}$$

Let $$f(z) = \frac{1}{(3z+1)(z+3)}$$

f(z) has a simple poles at $z = -\frac{1}{3}$ and $z = -3$.

Out of these two poles only $z = -\frac{1}{3}$ lies inside $|z| = 1$.

$\therefore$ By Cauchy's residue theorem

$$I = \frac{2}{i}\cdot 2\pi i\left[\text{Residue of } f(z) \text{ at } z = -\frac{1}{3}\right]$$

$$\text{Res } f\left(-\frac{1}{3}\right) = \left[\left(z+\frac{1}{3}\right)\cdot\frac{1}{(3z+1)(z+3)}\right]_{z=-\frac{1}{3}}$$

$$= \frac{1}{3}\cdot\frac{1}{\left(-\frac{1}{3}+3\right)} = \frac{1}{8}$$

$$\therefore \quad I = 4\pi\cdot\frac{1}{8} = \frac{\pi}{2}$$

Q.22 Using Contour integration, evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\sin\theta} d\theta$.

[SPPU : Dec.-05, May-11, 12]

Ans. : Let
$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\sin\theta} d\theta$$

Substituting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$\therefore \quad I = \oint_C \frac{-\frac{1}{4}\left(z-\frac{1}{z}\right)^2}{5+4\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)} \frac{dz}{iz}$$

where C is $|z| = 1$

$$= r\frac{1}{i}\oint_C \frac{\left(-\frac{1}{4}\right)(z^2-1)^2}{5z^3+2i\left(\frac{z^2-1}{z}\right)} dz = -\frac{1}{4i}\oint_C \frac{(z^2-1)^2}{5z^3-2i\,z^4+2i\,z^2} dz$$

$$= -\frac{1}{4i}\oint_C \frac{(z^2-1)^2}{z^2\{-2i\,z^2+5z+2i\}} dz$$

Let
$$f(z) = \frac{(z^2-1)^2}{z^2\{-2i\,z^2+5z+2i\}} = \frac{(z^2-1)^2}{-z^2(2i)\left[z^2-\frac{5}{2i}z-1\right]}$$

$$= \frac{(z^2-1)^2}{(-2i)\,z^2\left(z-\frac{1}{2i}\right)(z+2i)}$$

$\therefore$ f(z) has simple poles at $z = \frac{1}{2i}$ and $z = -2i$ and double pole at $z = 0$.

Out of these poles $z = 0$ and $z = \frac{1}{2i}$ lie within a circle $|z| = 1$.

$$\text{Res } f(0) = \left\{\frac{d}{dz}\left[\frac{(z^2-1)^2}{(-2i)\left(z-\frac{1}{2i}\right)(z+2i)}\right]\right\}_{z=0}$$

$$= \left\{\frac{\left(z-\frac{1}{2i}\right)(z+2i)\,2(z^2-1)\,(2z)-(z^2-1)^2\left(2z-\frac{5}{2i}\right)}{(-2i)\left(z-\frac{1}{2i}\right)^2(z+2i)^2}\right\}_{z=0}$$

$$= \left\{\frac{0-1\left(-\frac{5}{2i}\right)}{(-2i)\left(-\frac{1}{2i}\right)^2(2i)^2}\right\} = \frac{5}{2i}\cdot\frac{1}{(-2i)\left(-\frac{1}{4}\right)(-4)} = \frac{5}{4}$$

$$\text{Res } f\left(\frac{1}{2i}\right) = \left[\left(z-\frac{1}{2i}\right)\frac{(z^2-1)^2}{(-2i)\,z^2\left(z-\frac{1}{2i}\right)(z+2i)}\right]_{z=\frac{1}{2i}}$$

$$= \frac{\left(-\frac{1}{4}-1\right)^2}{(-2i)\left(-\frac{1}{4}\right)\left(\frac{1}{2i}+2i\right)} = \frac{25}{16}\times\frac{1}{(-2i)\left(-\frac{1}{4}\right)\left(-\frac{3}{2i}\right)}$$

$$= \frac{25}{16}\cdot\frac{1}{\left(-\frac{3}{4}\right)} = -\frac{25}{12}$$

$$\therefore \quad I = \oint_C f(z)\,dz = \frac{2\pi i}{-4i}\left[\text{Res } f(0)+\text{Res } f\left(\frac{1}{2i}\right)\right]$$

$$= \frac{2\pi i}{-4i}\left[\frac{5}{4}-\frac{25}{12}\right] = \frac{2\pi i}{-4i}\left(\frac{-10}{12}\right)$$

$$I = \frac{5\pi}{12}$$

END...

SOLVED MODEL QUESTION PAPER (In Sem)
Engineering Mathematics - III
S.E. (Electrical) Semester - III [As per 2019 Pattern]

Time : 1 Hour] **[Maximum Marks : 30**

N. B. :

i) Attempt Q.1 or Q.2, Q.3 or Q.4.
ii) Neat diagrams must be drawn wherever necessary.
ii) Figures to the right side indicate full marks.
iv) Assume suitable data, if necessary.

Q.1 a) *Solve any 2*

i) $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

(Refer Q.15 of Chapter - 1)

ii) $(D^2 + 2D + 1)y = e^{-x}\log x$ **(Refer Q.28 of Chapter - 1)**

iii) $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

(Refer similar Q.17 of Chapter - 1) **[10]**

b) *A 0.1 henry inductor, a 0.004 Farad capacitor and a generator having e.m.f. given by 180 cos(40t), t ≥ 0 are connected in series : find instantaneous charge Q and current I, if I = Q = 0 at t = 0.* **(Refer similar Q.14 of Chapter - 2)** **[5]**

OR

Q.2 a) *Solve any 2*

i) $(D^2 + 9)y = x^2 + 2x + \cos 3x$

(Refer similar Q.25 of Chapter - 1)

ii) $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$

(Refer similar Q.18 of Chapter - 1)

iii) $(D^2 + 1)y = \sec x \tan x$ *(By method of variation of parameters)* **[10]**

Ans. :

Step 1 : A.E. is $D^2 + 1 = 0 \Rightarrow D^2 - 1 \Rightarrow D = \pm i$

$$y_c = C_1 \cos x + C_2 \sin x$$

Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$

$$y_1 = \cos x,\ y_2 = \sin x,\ x = \sec x \tan x$$

Step 2 : Let $y_p = uy_1 + vy_2$

$$y_1' = -\sin x,\ y_2' = \cos x$$

Step 3 :

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}$$

$$= -\sin x \sec x \tan x = -\tan^2 x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix}$$

$$= \cos x \cdot \sec x \tan x = \tan x$$

Step 4 :

$$u = \int \frac{\Delta u}{\Delta} dx = \int \frac{-\tan^2 x}{1} dx = -\int (\sec^2 x - 1)\, dx$$

$$u = x - \tan x$$

Also,
$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\tan x}{1} dx$$

$$v = \log(\sec x)$$

Step 5 : Particular integral is

$$y_p = uy_1 + vy_2$$

$$y_p = (x - \tan x) \cos x + \sin x \log (\sec x)$$

$$y_p = x \cos x - \sin x + \sin x \log (\sec x)$$

Step 6 : General solution is

$$y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x + x\cos x - \sin x + \sin x \log (\sec x)$$

b) *Solve* $\frac{dx}{dt} - \omega y = a\cos(pt)$, $\frac{dy}{dt} + \omega x = a\sin(pt)$. **[5]**

Ans. : Step 1 : Let $D \equiv \frac{d}{dt}$ hence the system can be written as

$$Dx - \omega y = a \cos(pt)$$

$$\omega x + Dy = a \sin(pt)$$

Step 2 : Solving for x by cramer's rule

$$\begin{vmatrix} D & -\omega \\ \omega & D \end{vmatrix} x = \begin{vmatrix} a\cos(pt) & -\omega \\ a\sin(pt) & D \end{vmatrix}$$

$$(D^2 + \omega^2)x = -ap\sin(pt) + a\omega\sin(pt)$$

$$(D^2 + \omega^2)x = a(\omega - p)\sin(pt)$$

General solution is $x = x_c + x_p$

C.F. A.E. is $D^2 + \omega^2 = 0$

$$\Rightarrow D^2 = -\omega^2$$

$$\Rightarrow D = \pm \omega i$$

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

P.I.

$$x_p = \frac{1}{D^2 + \omega^2} a(\omega - p)\sin(pt)$$

$$D^2 = -p^2$$

$$= \frac{1}{-p^2 + \omega^2} a(\omega - p)\sin(pt)$$

$$x_p = \frac{a}{\omega + p}\sin(pt)$$

General solution is

$$x = x_c + x_p$$

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \left(\frac{a}{\omega + p}\right)\sin(pt)$$

Step 3 : Since $Dx - \omega y = a\cos(pt)$

$$\omega y = -a\cos(pt) + Dx$$

$$= -a\cos(pt) + D\left[C_1\cos(\omega t) + C_2\sin(\omega t) + \left(\frac{a}{\omega + p}\right)\sin(pt)\right]$$

$$\omega y = -a\cos(pt) + C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t) + \left(\frac{ap}{\omega + p}\right)\cos(pt)$$

$$\omega y = -C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t) + \left(\frac{a\omega}{\omega + p}\right)\cos(pt)$$

$$y = -C_1\sin(\omega t) + C_2\cos(\omega t) - \left(\frac{a}{\omega + p}\right)\cos(pt)$$

Q.3 a) *Attempt any 2*

i) Find Laplace transform of $t\int_0^t e^{-4t}\sin 3t\,dt$

(Refer Q.13 of Chapter - 3)

ii) Use Laplace transform to evaluate $\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right)dt$

(Refer Q.17 of Chapter - 3)

iii) Find inverse Laplace transform of $\frac{s+7}{s^2+2s+2}$

(Refer Q.22 of Chapter - 3) **[10]**

b) *Solve* $\frac{dy}{dt} + 2y(t) + \int_0^t y(t)dt = \sin t$ *given* $y(0) = 1$

(Refer Q.43 of Chapter - 3) **[5]**

OR

Q.4 a) *Attempt any 2*

i) Find Laplace transform of $t^4 \, U(t-2)$

(Refer Q.37 of Chapter - 3)

ii) Find inverse Laplace transform using convolution theorem of $\frac{1}{s^4(s+5)}$ **(Refer Q.33 of Chapter - 3)**

iii) $L^{-1}\left[\frac{1}{(s+1)\left(s^2+1\right)}\right]$ **(Refer Q.32 of Chapter - 3)** **[10]**

b) *Solve* $\frac{d^2y}{dt^2}+y = t$ *with* $y(0) = 1$ *and* $y'(0) = -2$

(Refer Q.47 of Chapter - 3) **[5]**

SOLVED MODEL QUESTION PAPER (End Sem)
Engineering Mathematics - III
S.E. (Electrical) Semester - III [As per 2019 Pattern]

Time : 2 $\frac{1}{2}$ Hours] **[Maximum Marks : 70**

N. B. :

i) Attempt Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.
ii) Neat diagrams must be drawn wherever necessary.
ii) Figures to the right side indicate full marks.
iv) Assume suitable data, if necessary.

Q.1 a) *Find Fourier Sine transform of* $e^{-|x|}$. *Hence evaluate* $\int_0^\infty \frac{x \sin mx}{1+x^2}dx$ **(Refer Q.5 of Chapter - 4)** **[6]**

b) *Solve the integral equation*

$$\int_0^\infty f(x)\cos \lambda x dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$ **(Refer Q.8 of Chapter - 4)** **[5]**

c) *Find inverse Z-transform of* $\frac{3Z^2+2Z}{Z^2-3Z+2}$, $1 < |z| < 2$

(Refer Q.17 of Chapter - 5) **[6]**

OR

Q.2 a) *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

(Refer Q.6 of Chapter - 4) **[5]**

b) *Find Z-transform of following functions*

i) $f(k) = 3(2^k)+4(-1)^k$, $k \ge 0$ **(Refer Q.2 of Chapter - 5)**

ii) $f(k) = k5^k$, $k \ge 0$ **(Refer Q.7 of Chapter - 5)** **[6]**

c) *Use Z-transforms to solve*

$$y_k - \frac{5}{6}y_{k-1} + \frac{1}{6}y_{k-2} = \left(\frac{1}{2}\right)^k, \; k \ge 0$$

(Refer Q.27 of Chapter - 5) **[6]**

Q.3 a) *Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.*

x	2	2.5	3	3.5	4	4.5	5
f	4	36	60	90	70	40	10

(Refer Q.5 of Chapter - 6) **[6]**

b) *From record of analysis of correlation data the following results are available variance of* $x = 9$ *and lines of regression are given by*

$8x - 10y + 66 = 0$

$40x - 18y = 214$

Find out a) Mean values for x and y services. b) Standard *deviation of y services. c) Coefficient of correlation between x and y services.* **(Refer Q.13 of Chapter - 6)** **[6]**

c) *The incidence of a certain disease is such that on the average 20 % of workers suffer from it. If 10 workers are selected at random, find the probability that*
i) Exactly 2 worker suffer from disease. ii) Not more than 2 workers suffer. **(Refer Q.9 of Chapter - 7)** **[6]**

OR

Q.4 a) *Find the coefficient of correlation for the following data.*

x	10	14	18	22	26	30
f	18	12	24	06	30	36

(Refer Q.10 of Chapter - 6) **[6]**

b) *In an intelligence test administered to 1000 students the average score was 42 and standard deviation is 24. Find the number of students with score lying between 30 to 54.*
(Refer Q.19 of Chapter - 7) **[6]**

c) *Fit a Poisson's distribution to the following data and test the goodness of fit by applying the x^2 test* **[6]**

x	0	1	2	3	4
F	155	157	58	22	8

Given $x^2_{4,0.05} = 7.815$

Ans. : Fitting of Poissions distribution

We have,

x_i	0	1	2	3	4	Sum
f_i	155	157	58	22	8	400
$f_i x_i$	0	157	116	66	32	371

Arithmetic mean

$$Z = \frac{\sum f_i x_i}{\sum f_i} = \frac{371}{400} = 0.9275$$

According to Poisson's distribution, probability of r success is

$$P(r) = \frac{e^{-Z} Z^r}{r!} = \frac{e^{-0.9275} (0.9275)^r}{r!}$$

To test goodness of fit using x^2 test

Define H_0 : Given data follows Poissions distribution at 5 % los.

H_1 : Give data doesnot follow Poissons distribution at 5 % los.

x_i	Observed frequency $f_i = o_i$	Probability according to Poisson's distribution $P(x_i)$	Expected frequency $e_i = P(x_i) \times 400$	$\frac{(o_i - e_i)^2}{e_i}$
0	155	0.39554	158.21	0.0654
1	157	0.36686	146.75	0.7166
2	58	0.17013	68.05	1.4851
3	22	0.05259	21.04	0.0438
4	8	0.01219	4.88	1.9971
sum	400	-	-	4.3080

Thus, the calculated value of x^2 variable is

$$x_4^2 = 4.3080$$

since $\underset{\text{calculated value}}{x_4^2 = 4.3080} < \underset{\text{Table value}}{x^2_{4,0.05} = 7.815}$

H_0 is accepted.

Conclusion :

Given data follows Poissions distribution at 5 % los.

Q.5 a) *Find D.D. of* $\phi = e^{2x}\cos yz$ *at the origin in the direction tangent to the curve* $x = a\sin t$, $y = a\cos t$, $z = at$ *at* $t = \pi/4$. **(Refer Q.10 of Chapter - 8)** **[6]**

b) *Prove that* $F = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ *is conservative force field. Find its scalar potential and the work done in moving an object in this field from* $(1, -2, 1)$ *to* $(3, 1, 4)$. **(Refer Q.13 of Chapter - 8)** **[6]**

c) *Using Stokes's theorem, evaluate* $\iint_S (\nabla \times \overline{F}) \cdot d\overline{s}$ *where* $\overline{F} = 3y\hat{i} - xz^2\hat{j} + yz^2\hat{k}$ *and S is the surface of the paraboloid* $2z = x^2 + y^2$ *bounded by* $z = 2$. **(Refer Q.50 of Chapter - 8)** **[6]**

OR

Q.6 a) *Use Green's theorem to evaluate the integral* $\oint_C (xy\,dx + y^2\,dy)$ *over the area bounded by curves* $y = x^2$ *and line* $y = x$ *in first quadrant.* **(Refer Q.42 of Chapter - 8)** **[6]**

b) *Prove following identities*

i) $\nabla \cdot \left[r\nabla\left(\frac{1}{r^n}\right)\right] = \frac{n(n-2)}{r^{n+1}}$ **(Refer Q.28 of Chapter - 8)**

ii) $\nabla^2(r^2 e^r) = (r^2 + 6r + 6)e^r$ **(Refer Q.20 of Chapter - 8)** **[6]**

c) *Evaluate* $\int_C \overline{F} \cdot d\overline{r}$ *for* $\overline{F} = 3x^2\overline{i} + (2xz - y)\overline{j} + z\overline{k}$ *along the following curve* $x = \alpha t^2$, $y = t$, $z = 4t^2 - t$ *from* $t = 0$, $t = 1$. **(Refer Q.34 of Chapter - 8)** **[6]**

Q.7 a) *If* $u = \frac{1}{2}\log(x^2 + y^2)$ *and f(z) = u + iv is analytic then find f(z) in terms of z and hence find v.*
(Refer Q.4 of Chapter - 9) **[5]**

b) *By using Cauchy's integral formula evaluate* $\oint \frac{e^z}{z^2 + 1} dz$ *over* $|z - 1| = 1$. **(Refer Q.9 of Chapter - 10)** **[6]**

c) *Find bilinear transformation which maps the points* $z = 1, i, -1$ *to the points* $0, 1, \infty$ *respectively.*
(Refer Q.16 of Chapter - 9) **[6]**

OR

Q.8 a) *If* $w = \phi + i\psi$ *represents the complex potential for an electric field and* $\phi = -2xy + \frac{y}{x^2 + y^2}$ *determine the function* ψ.
(Refer Q.8 of Chapter - 9) **[5]**

b) *Evaluate* $\oint_C \frac{z + 4}{z^2 + 2z + 5} dz$ *where 'C' is the circle* $|z - 2i| = \frac{3}{2}$.
(Refer Q.15 of Chapter - 10) **[6]**

c) *Show that the bilinear transformation* $w = \frac{2z + 3}{z - 4}$ *maps the circle* $x^2 + y^2 - 4x = 0$ *onto the straight line* $4u + 3 = 0$
(Refer Q.23 of Chapter - 9) **[6]**

END...