1 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1.1 : Solution of LDE with Constant Coefficient

1) The general form of n^{th} order LDE with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^n} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x)$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots a_n$ are constants and $\phi(x)$ is a function of x only. Let $D = \frac{d}{dx}$

$$\therefore (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n) y = \phi(x) \quad \ldots \rightarrow (1)$$

i.e. $f(D)\, y = \phi(x)$ where $f(D)$ is a polynomial in D.

$\therefore$ Auxiliary equation is $f(D) = 0$

The solution of D.E. (1) involves two parts

i) Complementary function (C.F. or y_c)

ii) Particular integral (P.I. or y_p)

$\therefore$ The complete solution of D.E. (1) is $y = y_c + y_p$ = C.F. + P.I.

2) Methods to find C.F.

To find C.F. find the roots of auxiliary equation $f(D) = 0$

i) If $m_1, m_2, m_3 \ldots$ are **real and distinct** roots, then

$$\text{C.F.} = y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \ldots$$

ii) The roots are real and repeated

If $m_1, m_1, m_1, m_2, m_2, m_3$ are real roots then

$$y_c = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + (C_4 + C_5 x) e^{m_2 x} + C_6 e^{m_3 x}$$

iii) The roots are complex and distinct : [Complex roots always occur in complex pairs)

i.e. if $\alpha + i\beta$ is one root then $\alpha - i\beta$ will be another root, then

$$y_c = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

iv) The roots are complex and repeated

i.e. if $\alpha \pm i\beta$ and $\alpha \pm i\beta$ are roots then

$$y_c = e^{\alpha x} [(C_1 + C_2) x \cos \beta x + (C_3 + C_4 x) \sin \beta x$$

3) Particular integrals :

If $f(D)\, y = \phi(x)$ is the LDE with constant coefficients then its particular integral is

$$\text{P.I.} = y_p = \frac{1}{f(D)} \phi(x)$$

4) Shortcut methods to find P.I. of special functions :

Sr. No.	Type of $\phi(x)$	$y_p = \frac{1}{f(D)} \phi(x)$
1.	i) $\phi(x) = e^{ax}$	$y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax};\ f(a) \neq 0$ $y_p = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ $y_p = \frac{1}{\phi(D)(D-a)^r} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{r!} e^{ax};\ \phi(a) \neq 0$
	ii) $\phi(x) = k = k e^{0x}$ where k = constant	$y_p = \frac{1}{f(D)} k e^{0x} = k \frac{1}{f(0)};\ f(0) \neq 0$
	iii) $\phi = a^x = e^{x \log a}$	$y_p = \frac{1}{f(D)} a^x = \frac{1}{f(\log a)} a^x$
2	$\phi = \sin(ax + b)$ or $\cos(ax + b)$	$y_p = \frac{1}{f(D^2)} \frac{\sin}{\cos}(ax + b) = \frac{1}{f(-a^2)} \frac{\sin}{\cos}(ax + b)$, if $f(-a^2) \neq 0$ $y_p = \frac{1}{(D^2 + a^2)^r} \frac{\sin}{\cos}(ax + b) = \left(-\frac{x}{2a}\right)^r \frac{1}{r!} \frac{\sin}{\cos}\left(ax + b + \frac{r\pi}{2}\right)$

3	$\phi = \cosh(ax+b)$ or $\sinh(ax+b)$	$y_p = \frac{1}{f(D^2)}$ $\frac{\cosh}{\sinh}(ax+b) = \frac{1}{f(a^2)}\frac{\cosh}{\sinh}(ax+b)$, if $f(a^2) \neq 0$ If $f(a^2) = 0$ then $\frac{1}{f(D^2)}\frac{\sinh}{\cosh}(ax+b) = \frac{x}{f'(a^2)}\frac{\sinh}{\cosh}(ax+b)$; $f'(a^2) \neq 0$
4	$\phi = x^P$ where P is positive integer.	To find y_p reduce f(D) in any one of the following form i) $(1+D)^n = 1 + nD + \frac{n(n-1)}{2!}D^2 + \frac{n(n-1)(n-2)}{3!}D^3 + \ldots$ ii) $\frac{1}{1+D} = 1 - D + D^2 - D^3 + \ldots$ iii) $\frac{1}{1-D} = 1 + D + D^2 + D^3 + \ldots$
5	$\phi = e^{ax}V$ where V is any function of x	$y_p = \frac{1}{f(D)}e^{ax}V = e^{ax}\left[\frac{1}{f(D+a)}V\right]$
6	$\phi = xV$ where V is any function of x	$y_p = \frac{1}{f(D)}xV = \left\{x - \frac{f'(D)}{f(D)}\right\}\frac{1}{f(D)}V$
7	$\phi = x^n \sin ax$ or $x^n \cos ax$	$y_p = \frac{1}{f(D)}x^n \sin ax = \text{Im part}\left\{\frac{1}{f(D)}x^n e^{iax}\right\}$ $y_p = \frac{1}{f(D)}x^n \cos ax = \text{Real part}\left\{\frac{1}{f(D)}x^n e^{iax}\right\}$

5) General Method :

If it is not possible to apply any of above types then apply general method i.e. if $\phi(x)$ = tan ax, cot ax, cosec ax sec ax, sin e^x, cos e^x, $\frac{1}{1+e^x}$, $\frac{1}{1-e^{-x}}$, $\frac{e^x}{1+e^x}$, then use this method.

To find P.I. use

$$\frac{1}{D-a}\phi(x) = e^{ax}\int \phi(x)\, e^{-ax}\, dx$$

$$\frac{1}{D+a}\phi(x) = e^{-ax}\int \phi(x)\, e^{ax}\, dx$$

Q.1 $(D^3 - 5D^2 + 8D - 4)$
$y = 4e^{2x} + e^x + 2^x + 3$

Ans. : Step 1 : A.E. is $D^3 - 5D^2 + 8D - 4 = 0$

$(D-1)(D^2 - 4D + 4)\ (D-1)(D-2)^2 = 0$

$D = 1, 2, 2$

$\therefore \quad y_c = \text{C.F.} = C_1 e^x + (C_2 + C_3 x)e^{2x}$

Step 2 :

$$y_p = \text{P.I.} = \frac{1}{(D-1)(D-2)^2}(4e^{2x} + e^x + 2^x + 3)$$

$$y_p = \frac{1}{(D-1)(D-2)^2}4e^{2x} + \frac{1}{(D-1)(D-2)^2}e^x + \frac{1}{(D-1)(D-2)^2}2^x + \frac{1}{(D-1)(D-2)^2}3$$

Consider

$$PI_1 = \frac{1}{(D-1)(D-2)^2}4e^{2x}$$

Replace D by a in non zero factor.
i.e. D = 2

$$PI_1 = \frac{1}{(2-1)(D-2)^2}4e^{2x}$$

$$= \frac{4}{1}\cdot\frac{1}{(D-2)^2}e^{2x}$$

$$= \frac{4}{1}\cdot\frac{x^2}{2!}e^{2x}$$

$$PI_1 = 2x^2e^{2x}$$

Now, consider

$$PI_2 = \frac{1}{(D-1)(D-2)^2} e^x$$

Replace D by a in non zero factor i.e. D = 1.

$$PI_2 = \frac{1}{(D-1)(1-2)^2} e^x = \frac{1}{1 \cdot (D-1)} e^x$$

$$PI_2 = \frac{x}{1} e^x$$

Consider

$$PI_3 = \frac{1}{(D-1)(D-2)^2} 2^x$$

For a^x replace D by log a,
(Here D = log 2)

$$PI_3 = \frac{1}{(\log 2-1)(\log 2-2)} 2^x$$

Consider

$$PI_4 = \frac{1}{(D-1)(D-2)^2} 3$$

For X = constant replace D by 0.

$$PI_4 = \frac{1}{(0-1)(0-2)^2} 3$$

$$PI_4 = -\frac{3}{4}$$

Thus P.I. $= PI_1 + PI_2 + PI_3 + PI_4$

$$= 2x^2 e^x + xe^x + \frac{2^x}{(\log 2-1)(\log 2-2)^2} - \frac{3}{4}$$

Step 3 : The complete solution is

$$y = y_c + y_p = C_1 e^x + (C_2 + C_3 x) e^{2x} + 2x^2 e^x + xe^x + \frac{2x}{(\log 2-1)^2 (\log 2-2)^2} - \frac{3}{4}$$

Q.2 $(D^2 + 4) y = \cos 3x \cdot \cos x$ ☞ **[SPPU : May-14]**

Ans. : Step 1 : $(D^2 + 4) y = \cos 3x \cdot \cos x$

A.E. is $D^2 + 4 = 0$ $D^2 = -4 \Rightarrow D = \pm 2i$

$\therefore$ C.F. $= C_1 \cos 2x + C_2 \sin 2x,$

Step 2) Now P.I. $= \frac{1}{D^2+4} (\cos 3x \cos x)$

$$\text{P.I.} = \frac{1}{2} \frac{1}{D^2+4} 2 [\cos 3x \cos x]$$

$$= \frac{1}{2} \frac{1}{D^2+4} (\cos (4x) + \cos (2x))$$

$$= \frac{1}{2} \frac{1}{D^2+4} \cos 4x + \frac{1}{2} \frac{1}{D^2+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{-16+4} \cos 4x + \frac{1}{2} \frac{1}{-4+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{(-12)} \cos 4x + \frac{1}{2} \text{ fail case}$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} \frac{x}{2D} \cos 2x$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} x \frac{\sin 2x}{2}$$

$$= -\frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Step 3 : The complete solution is

$$y = \text{C.F.} + \text{P.I.} = C_1 \cos 2x + C_2 \sin 2x - \frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Q.3 $(D-1)^3 y = e^x + 5^x - 1$ ☞ **[SPPU : May-08, Dec.-11, 12]**

Ans. :

Step 1 : A.E. is $(D-1)^3 = 0$

$\Rightarrow$ D = 1, 1, 1

The complementary function is

$$y_c = (C_1 + C_2 x + C_3 x^2) e^x$$

Step 2 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{(D-1)^3}[e^x + 5^x - 1]$$

$$= \frac{1}{(D-1)^3}e^x + \frac{1}{(D-1)^3}5^x + \frac{1}{(D-1)^3}(-1)$$

$$= \frac{x^3}{3!}e^x + \frac{1}{(\log 5-1)^3}5^x + \frac{1}{(-1)^3}(-1)$$

$$y_p = \frac{x^3}{6}e^x + \frac{1}{(\log 5-1)^3}5^x + 1$$

Step 3 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2x + C_3x^2)e^x + \frac{x^3}{6}e^x + \frac{1}{(\log 5-1)^3}5^x + 1$$

Q.4 Solve $(D^2 - 2D + 5)\,y = 10 \sin x$

☞ **[SPPU : Dec.-05, May-13]**

Ans. : Step 1 :

A.E. is $D^2 - 2D + 5 = 0$

$$D = \frac{-(-2) \pm \sqrt{4-4(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{-b \pm \sqrt{b^2-4a}}{2a}$$

$$D = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The C.F. is $y_c = e^x(C_1 \cos 2x + C_2 \sin 2x)$

Step 2 : The P.I. is $y_p = \dfrac{1}{D^2 - 2D + 5}(10 \sin x)$

$$= (10)\frac{1}{-1-2D+5}\sin x = (10)\frac{1}{4-2D}\sin x$$

$$= 10\frac{4+2D}{16-4D^2}\sin x$$

$$= 10\frac{(4+2D)}{16-4(-1)}\sin x = \frac{1}{2}(4 \sin x + 2 \cos x)$$

$$y_p = 2 \sin x + \cos x$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.5 Solve $(D^3 + 8)y = x^4 + 2x + 1$ ☞ **[SPPU : Dec.-08, 12]**

Ans. :

Step 1 : A.E. is $D^3 + 8 = 0$

By synthetic division method, we get,

$(D+2)(D^2 - 2D + 4) = 0$

$\therefore \quad D = -2,\ D = 1 \pm i\sqrt{3}$

The C.F. is

$$y_c = C_1e^{-2x} + e^x[C_2 \cos\sqrt{3}\,x + C_3 \sin\sqrt{3}\,x]$$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^3+8}[x^4 + 2x + 1]$$

$$= \frac{1}{8}\frac{1}{\left(1+\frac{D^3}{8}\right)}[x^4 + 2x + 1]$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2 \cdots\right][x^4 + 2x + 1]$$

$$= \frac{1}{8}\left[x^4 + 2x + 1 - \underline{\frac{1}{8}(24\,x)} + 0\right]$$

$$= \frac{1}{8}[x^4 - x + 1]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.6 Solve $(D^2 + 6D + 9)\ y = x^{-3}\ e^{-3x}$

☞ **[SPPU : Dec.-14, Marks 4]**

Ans. :

Step 1 : $D^2 + 6D + 9 = 0$ $\therefore$ A.E. is $(D+3)^2 = 0$

$$\Rightarrow D = -3, -3$$

$$\therefore \quad \text{C.F.} = (C_1 x + C_2) e^{-3x}$$

Step 2 : Now

$$\text{P.I.} = \frac{1}{(D+3)^2} x^{-3} e^{-3x} = e^{-3x} \frac{1}{[D-3+3]^2} x^{-3}$$

$$= e^{-3x} \frac{1}{D^2} x^{-3} = e^{-3x} \frac{1}{D}\left(\frac{1}{-2x^2}\right)$$

$$\text{P.I.} = e^{-3x} \frac{1}{2x} = \frac{1}{2x e^{3x}}$$

Step 3 : The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (C_1 x + C_2) e^{-3x} + \frac{1}{2xe^{3x}}$$

Q.7 $(D^2 - 4)y = x \sinh x$ ☞ **[SPPU : May-07, 08, 11, 12]**

Ans. :

Step 1 : A.E. is $D^2 - 4 = 0$

$$\Rightarrow \quad D = \pm 2$$

The C.F. is $y_c = C_1 e^{2x} + C_2 e^{-2x}$

Step 2 : The P.I. is $y_p = \dfrac{1}{D^2-4} x \sinh x$

$$= \left[x - \frac{2D}{D^2-4}\right] \frac{1}{D^2-4} \sinh x$$

$$= \left[x - \frac{2D}{D^2-4}\right]\left(\frac{1}{1-4} \sinh x\right)$$

$$= \left[x - \frac{2D}{D^2-4}\right]\left(-\frac{1}{3} \sinh x\right)$$

$$= -\frac{1}{3} x \sinh x + \frac{2D}{3}\left(-\frac{1}{3} \sinh x\right)$$

$$y_p = -\frac{1}{3} x \sinh x - \frac{2}{9} \cosh x$$

Step 3 : The complete solution is $y = y_c + y_p$

Q.8 $(D^2 - 2D + 1)y = x e^x \sin x$ ☞ **[SPPU : May-05, 13]**

Ans. : Step 1 : The A.E. is $D^2 - 2D + 1 = 0$

$(D-1)^2 = 0 \Rightarrow D = 1, 1$

The C.F. is $y_c = (C_1 + C_2 x) e^x$

Step 2 : The P.I. is $y_p = \dfrac{1}{(D-1)^2} x e^x \sin x$

$$= e^x \frac{1}{[D+1-1]^2} x \sin x$$

$$y_p = e^x \frac{1}{D^2} x \sin x = e^x \left[x - \frac{2D}{D^2}\right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - \frac{2D}{D^2}\right](-\sin x) = -e^x \left[x - \frac{2D}{D^2}\right] \sin x$$

$$y_p = -e^x [x \sin x + 2 \cos x]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.9 $(D^2 + 3D + 2)y = e^x + \cos e^{e^x}$

☞ **[SPPU : Dec.-08,11, May-09]**

Ans. :

Step 1 : A.E. is $D^2 + 3D + 2 = 0$

$(D+2)(D+1) = 0$

$$D = -2, -1$$

The C.F. is $y_c = C_1 e^{-2x} + C_2 e^{-x}$

Step 2 : The P.I. is

$$y_p = \frac{1}{(D+1)(D+2)} [e^{e^x} + \cos e^x]$$

$$= \frac{1}{D+2} e^{-x} \int e^{x} (e^{e^x} + \cos e^x)\, dx$$

$$= \frac{1}{D+2} e^{-x} \left(e^{e^x} + \sin e^x \right)$$

$$= e^{-2x} \int e^{2x} e^{-x} \left(e^{e^x} + \sin e^x \right) dx$$

$$= e^{-2x} \int e^{x} \left(e^{e^x} + \sin e^x \right) dx$$

$$y_p = e^{-2x} \left[e^{e^x} - \cos e^x \right]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.10 $(D^2 + 9)y = \text{cosec } 3x$ ☞ **[SPPU : Dec.-06, May-13]**

Ans. :

Step 1 : A.E. is $D^2 + 9 = 0 \Rightarrow D = \pm 3i$

The C.F. is $y_c = C_1 \cos 3x + C_2 \sin 3x$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^2+9} \text{cosec } 3x = \frac{1}{(D+3i)(D+3i)} \text{cosec } 3x$$

$$y_p = \frac{1}{6i} \left[\frac{1}{D-3i} - \frac{1}{D+3i} \right] \text{cosec } 3x$$

$$= \frac{1}{6i} \left[\frac{1}{D-3i} \text{cosec } 3x - \frac{1}{D+3i} \text{cosec } 3x \right]$$

$$= \frac{1}{6i} [I_1 + I_2]$$

Consider $\quad I_1 = \frac{1}{D-3i} \text{cosec } 3x$

$$= e^{3ix} \int e^{-3ix} \text{cosec } 3x\, dx$$

$$= e^{3ix} \int [\cos 3x - i \sin 3x] \text{cosec } 3x\, dx$$

$$= e^{3ix} \int (\cot 3x - i)\, dx$$

$$= e^{3ix} \left(\frac{\log \sin 3x}{3} - ix \right)$$

Similarly $\quad I_2 = e^{-3ix} \left(\frac{\log \sin 3x}{3} + ix \right)$

$$\therefore y_p = \frac{1}{6i} \left[e^{3ix} \left(\frac{\log \sin 3x}{3} - ix \right) + e^{-3ix} \left(\frac{\log \sin 3x}{3} + ix \right) \right]$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Q.11 $(D^2 - 1)y = e^{-x} \sin e^{-x} + \cos e^{-x}$

☞ **[SPPU : May-11, Dec.- 11]**

Ans. : Step 1 : A.E. $(D-1)(D+1) = 0$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^{x}$$

Step 2 : $\quad y_p = \text{P.I.} = \frac{1}{f(D)} X$

$$\text{P.I.} = \frac{1}{(D-1)(D+1)} (e^{-x} \sin e^{-x} + \cos e^{-x})$$

$$\text{P.I.} = \frac{1}{2} \left[\frac{1}{D-1} - \frac{1}{D+1} \right] (e^{-x} \sin e^{-x} + \cos e^{-x})$$

$$\text{P.I.} = \frac{1}{2} \left\{ \begin{array}{l} \frac{1}{D-1} (e^{-x} \sin e^{-x} + \cos e^{-x}) \\ - \frac{1}{D+1} (e^{-x} \sin e^{-x} + \cos e^{-x}) \end{array} \right\}$$

Consider

$$PI_1 = \frac{1}{2} \left\{ \frac{1}{D-1} e^{-x} \sin e^{-x} + \cos e^{-x} \right\}$$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x} \right) dx$$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x} \right) e^{-x} dx$$

Put $e^{-x} = t \quad \therefore -e^{-x} dx = dt$ i.e. $e^{-x} dx = -dt$

$$= \frac{1}{2}e^x \int (t\sin t + \cos t)(-dt)$$

$$= \frac{-1}{2}e^x\left[\int t\sin t\ dt + \int \cos t\ dt\right]$$

$$PI_1 = \frac{-e^x}{2}\{[(t)(-\cos t) - (1)(-\sin t)] + \sin t\}$$

$$= \frac{-e^x}{2}\{-t\cos t + 2\sin t\}$$

Put $t = e^{-x}$

$$= \frac{-e^x}{2}\left\{-e^{-x}\cos e^{-x} + 2\sin e^{-x}\right\}$$

$$PI_1 = \frac{1}{2}\cos e^{-x} - e^x \sin e^{-x}$$

Consider

$$PI_2 = \frac{1}{2}\frac{1}{D+1}(e^{-x}\sin e^{-x} + \cos e^{-x})$$

$$= \frac{1}{2}e^{-x}\int -e^x(\cos e^{-x} + e^{-x}\sin e^{-x})\,dx$$

$$= e^{-x}\sin e^{-x} = \frac{1}{2}e^{-x}\cdot e^x \cos e^{-x}$$

$$PI_2 = \frac{1}{2}\cos e^{-x}$$

Thus,

$$PI = PI_1 - PI_2$$

$$= -e^x \sin e^{-x}$$

$\therefore$ **Step 3)** The complete solution is

$\therefore$ $y = C.F. + P.I.$

$$y = C_1 e^{-x} + C_2 e^x - e^x \sin e^{-x}$$

Q.12 Solve $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

[SPPU : Dec.-11, 12]

Ans. : Step - 1 : $D^2 - D - 2 = 0$

$$(D-2)(D+1) = 0$$

$$D = 2, -1$$

$$C.F. = C_1 e^{2x} + C_2 e^{-x}$$

$\therefore$ **Step 2 :** $y_p = P.I. = \frac{1}{f(D)} X = \frac{1}{(D-2)(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$

$$= \frac{1}{-3}\left[\frac{1}{D+1} - \frac{1}{D-2}\right]\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

Consider

$$PI_1 = \frac{1}{D+1}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= e^{-x}\int e^x\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= e^{-x}\int e^x\left[\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right]dx$$

$$= e^{-x}\cdot e^x\cdot\left(2\log x - \frac{1}{x}\right)$$

$$PI_1 = 2\log x - \frac{1}{x}$$

Consider

$$PI_2 = \frac{1}{D-2}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= e^{2x}\int e^{-2x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

Put $-2x = t$ $\therefore$ $x = \frac{-t}{2}$ $\therefore dx = \frac{-dt}{2}$

$$= e^{2x}\int e^t\left[2\log\left(\frac{t}{-2}\right) - \frac{2}{t} + \frac{4}{t^2}\right]\frac{dt}{-2}$$

$$= \frac{e^{2x}}{-2}\int e^t\left[\left(2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right) + \left(\frac{2}{t} + \frac{4}{t^2}\right)\right]dt$$

$$= \frac{e^{2x}}{-2}\cdot e^t\left[2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right]$$

Put $t = -2x$

$$= \frac{e^{2x}}{-2} \cdot e^{-2x}\left[2\log\left(\frac{-2x}{2}\right) - \left(\frac{4}{-2x}\right)\right]$$

$$= \frac{1}{-2}\left[2\log x + \frac{2}{x}\right] = -\log x - \frac{1}{x}$$

Thus

$$\text{P.I.} = -\frac{1}{3}[PI_1 - PI_2] = -\frac{1}{3}[3\log x]$$

$$= -\log x$$

Step (3) ∴ The complete solution is

∴ $y = \text{C.F.} + \text{P.I.} = C_1 e^{2x} + C_2 e^{-x} - \log x$

1.2 : Legendre's Differential Equations

1) Lagendre's D.E. : The general form of 3^{rd} order D.E. is

$$a_0 (ax+b)^3 \frac{d^3y}{dx^3} + a_1 (ax+b)^2 \frac{d^2y}{dx^2} + a_2 (ax+b)\frac{dy}{dx} + a_3 y = \phi(x)$$

Put $ax + b = e^z \Rightarrow z = \log(ax+b)$

$$\therefore (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\,Dy \quad \text{where } D = \frac{d}{dz}$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)\,y$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)\,y$$

∴ Given D.E. reduces to LDE with constant coefficients in y and z. Now apply previous methods.

2) Cauchy's D.E. or Homogeneous D.E.

The general form of 3^{rd} order D.E. is

$$a_0 x^3 \frac{d^3y}{dx^3} + a_1 x^2 \frac{d^2y}{dx^2} + a_2 x\frac{dy}{dx} + a_3 y = \phi(x)$$

Put $x = e^z \Rightarrow z = \log x$

and $x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)\,y$

and $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y$ where $\frac{d}{dz} \equiv D$

Q.13 $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

[SPPU : May-07, 10, 11, 12, Dec.-10,12]

Ans. : Step 1 : Legendre's differential equation with a = 3, b = 2.

Put $3x + 2 = e^z$ i.e. $z = \log(3x+2)$

$$(3x+2)\frac{dy}{dx} = 3\,D\,y$$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 9D(D-1)\,y$$

Thus the equation becomes

$$9D(D-1)\,y + 3 \cdot 3\,Dy - 36y = 3\left(\frac{e^z - 2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

$$9[D^2 - D + D - 4]\,y = 3\frac{(e^{2z} - 4e^z + 4)}{9} + \frac{4(e^z - 2)}{3} + \frac{3}{3}$$

$$= \frac{e^{2z} - 4e^z + 4 + 4e^z - 8 + 3}{3}$$

$$9(D^2 - 4)\,y = \frac{e^{2z} - 1}{3}$$

$$(D^2 - 4)\,y = \frac{1}{27}(e^{2z} - 1) \quad \text{...(Q.13.1)}$$

Which is linear differential equation with constant coefficient.

Step 2 : A.E. $D^2 - 4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1 e^{2z} + C_2 e^{-2z}$$

Step 3 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f(D)} Z$$

$$\text{P.I.} = \frac{1}{(D-2)(D+2)} \frac{1}{27}(e^{2z} - 1)$$

Separate PI_1 and PI_2.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)(D+2)} e^{2z} - \frac{1}{(D-2)(D+2)} 1\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{(D-2)} \cdot \frac{1}{(2+2)} e^{2z} - \frac{1}{(0-2)(0+2)}\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{4} \cdot \frac{1}{D-2} e^{2z} + \frac{1}{4}\right\} = \frac{1}{108}[z e^{2z} + 1]$$

Step 4 : The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108}\{z\, e^{2z} + 1\}$$

Put $e^z = 3x + 2$ $y = C_1 (3x+2)^2 + C_2 (3x+2)^{-2}$

$$+ \frac{1}{108}\{(3x+2)^2 \log(3x+2) + 1\}$$

Q.14 **Solve** $(2x + 1)^2 \frac{d^2y}{dx^2} - 6\,(2x - 1)\frac{dy}{dx} + 16\,y = 8\,(2\,x + 1)^2$

☞ **[SPPU : May-15]**

Ans. : Step 1 : Put $2x + 1 = e^z$ $\therefore$ $z = \log(2x + 1)$

$$(2x + 1)\frac{dy}{dx} = 2Dy \quad \text{where} \quad \frac{d}{dz}$$

$$(2x + 1)^2 \frac{d^2y}{dx^2} = 2^2 D(D - 1)\, y = 4\,(D^2 - D)\, y$$

$\therefore$ Given D.E. becomes

$$4\,(D^2 - D)\, y - 6\,(2Dy) + 16y = 8\, e^{2z}$$

$$(4D^2 - 16\, D + 16)y = 8\, e^{2z}$$

$$(D^2 - 4D + 4)\, y = 2e^{2z} \qquad \text{...(Q.14.1)}$$

Step 2 : A.E. is $D^2 = 4D + 4 = 0$ $\Rightarrow (D - 2)^2 = 0$

$$D = 2, 2$$

$$\therefore \quad y_c = (C_1 + C_2 z)\, e^{2z}$$

Step 3 : Now $y_p = \frac{1}{D^2 - 4D + 4}(2\, e^{2\,z})$

$$y_p = 2\frac{1}{(D-2)^2} e^{2\,z} = 2\frac{z^2}{2!} e^{2\,z} = z^2\, e^{2\,z}$$

Step 4 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2 z)\, e^{2\,z} + z^2\, e^{2\,z}$$

$$\therefore \quad y = [C_1 + C_2 \log(2x + 1)]\,(2x + 1)^2 + [\log(2x + 1)]^2\,(2x + 1)^2$$

Q.15 **Solved** $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin(\log x)$

☞ **[SPPU : May-08, 10, 11, Dec.-08, 10, 11, 12, 14]**

Ans. :

Step 1 : Here the coefficient of $\frac{d^2y}{dx^2}$ is x^3.

We need x^2 $\therefore$ Divide by x

$$\therefore \quad x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{x}\sin(\log x)$$

Which is homogeneous in y and x.

$\therefore$ Put $x = e^z$ or $z = \log x$

$$\therefore \quad x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D - 1)\, y$$

$\therefore$ The equation becomes

$$D(D-1)\, y + 3\, Dy + y = \frac{1}{e^z}\sin(z)$$

$$(D^2 + 2D + 1)\, y = e^{-z}\sin z \qquad \text{...(Q.15.1)}$$

which is LDE with constant coefficients

Step 2 : Consider A.E.

$$D^2+2D+1 = 0$$
$$(D+1)^2 = 0$$
$$D = -1, -1 \text{ real repeated}$$
$$y_c = \text{C.F.} = (C_1+C_2z)\ e^{-z}$$

Step 3 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D+1)^2}\ e^{-z}\sin z$$
$$y_p = \text{P.I.} = e^{-z}\frac{1}{(D-1+1)^2}\sin z$$
$$= e^{-z}\frac{1}{D^2}\sin z$$
$$\text{P.I.} = e^{-z}\frac{1}{-1}\sin z = e^{-z}\sin z$$

Step 4 : The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$
$$y = (C_1+C_2z)\,e^{-z} - e^{-z}\sin z$$
$$y = \frac{1}{e^z}[C_1+C_2z-\sin z]$$

Put $e^z = x$, $z = \log x$.

$$y = \frac{1}{x}[C_1+C_2\log x-\sin(\log x)]$$

Q.16 $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y = 20\left(x+\dfrac{1}{x}\right)$

☞ **[SPPU : May-09, 12, 14, 15]**

Ans. :

Step 1 : Given D.E. is Cauchy's D.E., so put

$$x = e^z \Rightarrow z = \log x \text{ and } D \equiv \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y \text{ and } x^3\frac{d^3y}{dx^3}y = D(D-1)\,(D-2)y$$

$\therefore$ Given D.E. becomes,

$$[D(D-1)\,(D-2)+2D(D-1)+2]\,y = 20[e^z+e^{-z}] \quad \text{... (Q.16.1)}$$

which is L.D.E with constant coefficients.

Step 2 : The A. E. of equation (Q.16.1) is

$$D^3-D^2+2 = 0 \Rightarrow D = -1,\ 1\pm i$$

The C.F. is

$$y_c = C_1e^{-z}+e^z[C_2\cos z+C_3\sin z]$$

Step 3 : The P.I. is

$$y_p = \frac{1}{D^3-D^2+2}20\,[e^z+e^{-z}]$$
$$= 20\left[\frac{1}{1-1+2}e^z+\frac{1}{3D^2-2D}e^{-z}\right]$$
$$= 10e^z+z\frac{20}{3+2}e^{-z} = 10e^z+4ze^{-z}$$
$$y_p = 10x+\frac{4}{x}\log x$$

Step 4 : The complete solution is

$$y = y_c+y_p$$

1.3 : Lagranges Method of Variation of Parameters

1) Method of Variation of Parameters to find Particular Integral

Step 1 : If the differential equation is of order two then the complementary function will be of the form C.F. $= C_1y_1+C_2y_2$

Step 2 : Assume

$$\text{P.I.} = uy_1+vy_2$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \quad \Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}$$

where y_1' and y_2' are derivatives of y_1 and y_2 also X is the R.H.S. of equation. [i.e. f (D) y = X]

Step 4 : Then

$$u = \int \frac{\Delta u}{\Delta} dx, \quad v = \int \frac{\Delta v}{\Delta} dx$$

Step 5 : Substitute u and v in P.I. $= uy_1 + vy_2$

Q.17 Solve $(D^2+1)y = \operatorname{cosec} x$

☞ **[SPPU : May-02, 12, 13, 14, Dec.-05]**

Ans. : Step 1 : C.F. $= C_1 \cos x + C_2 \sin x$

Step 2 : Let P.I. $= u y_1 + v y_2$

Step 3 : Find derivatives of y_1, y_2

$$y_1 = \cos x \quad y_2 = \sin x$$

$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 4 : Find $\Delta, \Delta u, \Delta v$

$$\Delta = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$$

$$\Delta u = -x y_2 = -\operatorname{cosec} x \sin x = -1$$

$$\Delta v = x y_1 = \operatorname{cosec} x \cos x = \cot x$$

Step 5 : Find u, v

$$u = \int \frac{\Delta u}{\Delta} dx \quad v = \int \frac{\Delta v}{\Delta} dx$$

$$= \int -1\, dx, \quad = \int \cot x\, dx$$

$$= -x, \quad = \log \sin x$$

Step 6 : P.I. $= u y_1 + v y_2$

$$= -x \cos x + \sin x \log \sin x$$

Step 7 : $y = $ C.F. + P.I.

$$y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$$

is the complete solution.

Q.18 Solve $(D^2+4)y = \tan 2x$ **[by variation of parameters]**

☞ **[SPPU : Dec.-05, 07, 08, 11, May-07, 08, 09]**

Ans. :

Step 1 : A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore \quad y_c = C_1 \cos 2x + C_2 \sin 2x$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

$$y_1 = \cos 2x, \; y_2 = \sin 2x$$

$\therefore$

$$y_1' = -2\sin 2x, \; y_2' = 2\cos 2x$$

Step 3 : Assume P.I. $= uy_1 + vy_2$

Step 4 : Find $\Delta, \Delta u, \Delta v$ where

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = 2\cos 2x \cos 2x + 2 \sin 2x \sin 2x = 2$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -\tan 2x \sin 2x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = \tan 2x \cos 2x = \sin 2x$$

Step 5 :

$$u = \int \frac{\Delta u}{\Delta} dx$$

$$= -\int \frac{\sin 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx$$

$$u = -\frac{1}{2}\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{1}{2}\frac{\sin 2x}{2}$$

and

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\sin 2x}{2} dx$$

$$= \frac{-1}{2}\frac{\cos 2x}{2} = -\frac{1}{4}\cos 2x$$

Step 6 : $\therefore$ The P.I. is

$$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

$$+\frac{1}{4}\sin 2x \cos 2x - \frac{1}{4}\sin 2x \cos 2x$$

$$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1\cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Q.19 Solve $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$ (by variation of parameters)

☞ **[SPPU : May-09, 14]**

Ans. :

Step 1 : A.E. is $D^2 - 6D + 9 = 0 \Rightarrow D = 3, 3$

$$y_c = (C_1x + C_2)e^{3x} = C_1x\,e^{3x} + C_2\,e^{3x}$$

Step 2 : Comparing y_c with $y_c = C_1y_1 + C_2y_2$ we get

$$y_1 = xe^{3x},\; y_2 = e^{3x}$$

$$y_1' = 3x\,e^{3x} + e^{3x},\; y_2' = 3e^{3x}$$

Step 3 : Assume that $y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = -e^{6x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -\frac{e^{3x}}{x^2}e^{3x} = \frac{-e^{6x}}{x^2}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = \frac{e^{3x}}{x^2}xe^{3x} = \frac{e^{6x}}{x}$$

Step 5 : Find u, v

$$u = \int \frac{\Delta u}{\Delta}dx = \int -\frac{e^{6x}}{x^2}\frac{1}{-e^{6x}}dx = \int \frac{1}{x^2} = -\frac{1}{x}$$

$$v = \int \frac{\Delta v}{\Delta}dx = \int \frac{e^{6x}}{x}\frac{1}{-e^{6x}}dx = -\int \frac{1}{x}dx$$

$$v = -\log x$$

Step 6 : The P.I. is

$$y_p = -\frac{1}{x}x\,e^{3x} - (\log x)(e^{3x})$$

$$= -e^{3x}(1 + \log x)$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$= C_1x\,e^{3x} + C_2e^{3x} - e^{3x}(1 + \log x)$$

Q.20 Solve $(D^2 - 2D + 2)y = e^x \tan x$ by method of variation of parameters.

☞ **[SPPU : Dec.-11, 15, May-11]**

Ans. :

Step 1 : A.E. is

$$D^2 - 2D + 2 = 0$$

$$D = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

Step 2 : The complementary function is

$$y_c = e^x(C_1\cos x + C_2\sin x)$$

Step 3 : Comparing with $y_p = C_1y_1 + C_2y_2$

where $y_1 = e^x\cos x$, $y_2 = e^x\sin x$

$$y_1' = e^x\cos x - e^x\sin x,$$

$$y_2' = e^x\sin x + e^x\cos x$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$= e^x\cos x\,e^x(\sin x + \cos x)$$

$$- e^x\sin x\,e^x(\cos x - \sin x)$$

$$= e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -e^x\tan x\,e^x\sin x$$

$$= -e^{2x}\frac{\sin^2 x}{\cos x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = e^x \tan x\, e^x \cos x = e^{2x}\sin x$$

Step 5 : $u = \int \frac{\Delta u}{\Delta} dx$

$$= \int \frac{-\sin^2 x}{\cos x} dx = \int -\frac{(1-\cos^2 x)}{\cos x} dx$$

$$u = \int(-\sec x + \cos x)dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$\Delta = \int \frac{\Delta v}{\Delta} dx = \int \sin x \, dx = -\cos x$$

Step 6 : $\therefore$ The complete solution is

$$y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x - \log(\sec x + \tan x)$$

Q.21 **Solve by variation of parameters**

$(D^2 + 2D + 1)y = e^{-x} \log x$. ☞ **[SPPU : May-12, 13, 15]**

Ans. :

Step 1 : A.E. is

$$(D^2 + 2D + 1) = 0$$

$$(D+1)(D+1) = 0$$

$$D = -1, -1$$

$$\therefore \quad y_c = (C_1 + C_2 x)e^{-x} = C_1 e^{-x} + C_2 xe^{-x}$$

Step 2 : Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$

$$\therefore \quad y_1 = e^{-x}, \; y_2 = xe^{-x}$$

$$y_1' = -e^{-x}, \; y_2' = e^{-x} - xe^{-x}$$

Step 3 : Assume that P.I. $= y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$

$$= e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\log x & e^{-x} - xe^{-x} \end{vmatrix}$$

$$= -xe^{-2x}\log x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\log x \end{vmatrix}$$

$$= e^{-2x}\log x$$

Step 5 : Find u and v : $u = \int \frac{\Delta u}{\Delta} dx$

$$u = \int \frac{-xe^{-2x}\log x}{e^{-2x}} dx = -\int x \log x \, dx$$

$$= -\left[\log x\left(\frac{x^2}{2}\right) - \int \frac{1}{x}\left(\frac{x^2}{2}\right)dx\right]$$

$$= -\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]$$

$$u = \frac{x^2}{4} - \frac{x^2}{2}\log x$$

And $$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{e^{-2x}\log x}{e^{-2x}} dx = \int \log x \, dx$$

$$v = \log x - x$$

Step 6 : $y_p = e^{-x}\left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right) + xe^{-x}(\log x - x)$

Step 7 : The complete solution is

$$y = y_c + y_p$$

Q.22 **Solve :** $x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2$ ☞ **[SPPU : Dec.-12, 15]**

Ans. :

Step 1 : Given D.E. is Cauchy's D.E. so we use substitution

$$x = e^z \Rightarrow z = \log x \text{ and } D = \frac{d}{dz}$$

We get $x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given D.E. becomes

$$D(D-1)y - 2Dy - 4y = e^{2z} + 2z$$

$$(D^2 - 3D - 4)y = e^{2z} + 2z \qquad \dots (Q.22.1)$$

Which is the LDE with constant coefficients.

Step 2 : A.E. is $D^2 - 3D - 4 = 0$

$$(D-4)(D+1) = 0$$

Step 3 : $D = 4, \ D = -1$

The C.F. is $y_c = C_1 e^{-z} + C_2 e^{4z}$

Step 4 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{f(D)} X = \frac{1}{(D+1)(D-4)}[e^{2z}]$$

$$= \frac{1}{(D+1)(D-4)} e^{2z}$$

$$= \frac{e^{2z}}{(2+1)(2-4)}$$

$$= \frac{e^{2z}}{(-6)}$$

$$y_p = -\frac{1}{6} e^{2z}$$

Step 5 : The complete solution is

$$y = y_c + y_p = C_1 e^{-z} + C_2 e^{4z} - \frac{1}{6} e^{2z}$$

$$= C_1\left(\frac{1}{x}\right) + C_2 x^4 - \frac{1}{6} x^2$$

□□□

2 Simultaneous Linear Differential Equations and Applications

2.1 : Simultaneous Linear D.E.

- Method of solving simultaneous LDE is similar to solving algebraic equations

Q.1 The currents x and y in coupled circuits are given by

$$L\frac{dx}{dt} + Rx + R(x-y) = E$$

$$L\frac{dy}{dt} + Ry - R(x-y) = 0$$

where L, R, E are constants. Find x and y in terms of t given x = 0, y = 0, when t = 0. ☞ **[SPPU : Dec.-05, 10, 14]**

Ans. : Step 1 : Use $D = \frac{d}{dt}$

$$L\,Dx + Rx + Rx - Ry = E$$

$$L\,Dy + Ry - Rx + Ry = 0$$

Collect the terms of x and y.

$$(LD + 2R)\,x - Ry = E \qquad \dots (Q.1.1)$$

$$-Rx + (LD + 2R)\,y = 0 \qquad \dots (Q.1.2)$$

Step 2 : Solving for x using Cramer's rule.

$$\begin{vmatrix} LD+2R & -R \\ -R & LD+2R \end{vmatrix} x = \begin{vmatrix} E & -R \\ 0 & LD+2R \end{vmatrix}$$

$$(L^2D^2 + 4RLD + 4R^2 - R^2)\,x = (LD+2R)\,E$$

$$(LD+R)(LD+3R)\,x = 2RE \quad \text{As } DE = \frac{d}{dt}E = 0$$

A.E. is

$$(LD+R)(LD+3R) = 0$$

$$D = \frac{-R}{L}, \quad D = \frac{-3R}{L}$$

$$x_c = C.F = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L}$$

Find P.I for x.

$$x_p = \text{P.I.} = \frac{1}{(LD+R)(LD+3R)} 2RE$$

$$x_p = \text{P.I.} = \frac{2RE}{3R^2} = \frac{2E}{3R}$$

Write $x = x_c + x_p$

$$x = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R} \quad \text{... (Q.1.3)}$$

Step 3 : Use the equation where the coefficient of y is simple i.e. equation (Q.1.1).

$$\therefore \quad Ry = (LD + 2R)\,x - E$$

$$Ry = L\frac{dx}{dt} + 2Rx - E$$

Substitute x and $\frac{dx}{dt}$ to find y.

$$Ry = L\left[\frac{-R}{L} C_1 e^{-Rt/L} - \frac{3RC_2}{L} e^{-3Rt/L}\right] + 2R\left[C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R}\right] - E$$

$$\therefore \quad Ry = C_1 R e^{-Rt/L} - RC_2 e^{-3Rt/L} + \frac{1}{3}E$$

$$\therefore \quad y = C_1 e^{-Rt/L} - C_2 e^{-3Rt/L} + \frac{E}{3R} \quad \text{... (Q.1.4)}$$

Step 4 : Given at t = 0, x = 0 and y = 0

$\therefore$ To find C_1 and C_2 put t = 0, x = 0 in equation (Q.1.3) and t = 0, y = 0 equation (Q.1.4).

$$0 = C_1 + C_2 + \frac{2E}{3R}$$

$$0 = C_1 - C_2 + \frac{E}{3R}$$

Find C_1 and C_2. Adding we get,

$$0 = 2C_1 + \frac{E}{R} \Rightarrow C_1 = \frac{-E}{2R}$$

Substituting we get $C_2 = -\frac{E}{6R}$

Substitute C_1 and C_2 in equations (Q.1.3) and (Q.1.4).

$$\therefore \quad x = \frac{-E}{2R} e^{-Rt/L} - \frac{E}{6R} e^{-3Rt/L} + \frac{2E}{3R}$$

$$y = \frac{-E}{2R} e^{-Rt/L} + \frac{E}{6R} e^{-3Rt/L} + \frac{E}{3R}$$

Q.2 Solve $\frac{dx}{dt} + 2x - 3y = t$

$\frac{dy}{dt} - 3x + 2y = e^{2t}$ **[SPPU : Dec.-09, 11, May-12]**

Ans. : Step 1 :

Use $D = \frac{d}{dt}$ hence equations becomes,

$$Dx + 2x - 3y = t$$

$$Dy - 3x + 2y = e^{2t}$$

Collect the terms of x and y.

$$(D+2)\,x - 3y = t \quad \text{... (Q.2.1)}$$

$$(D+2)\,y - 3x = e^{2t} \quad \text{... (Q.2.2)}$$

Step 2 : Solving for x

$$\begin{vmatrix} D+2 & -3 \\ -3 & D+2 \end{vmatrix} x = \begin{vmatrix} t & -3 \\ e^{2t} & D+2 \end{vmatrix}$$

$$[(D+2)^2 - 3^2]\,x = (D+2)\,t + 3\,e^{2t} = 1 + 2\,t + 3\,e^{2t}$$

$$(D^2 + 4D - 5)\,x = 1 + 2\,t + 3\,e^{2t} \quad \text{... (Q.2.3)}$$

A.E. is $D^2 + 4D - 5 = 0$ gives $D = -5, 1$ hence

$$x_c = C_1 e^{-5t} + C_2 e^t \quad \text{... (Q.2.4)}$$

Now

$$x_p = \text{P.I.} = \frac{1}{D^2 + 4D - 5}(1+2t) + \frac{3\,e^{2t}}{D^2 + 4D - 5}$$

$$= -\frac{1}{5}\left[1 - \frac{4D + D^2}{5}\right](1 + 2\,t) + \frac{3\,e^{2t}}{4 + 8 - 5}$$

$$= -\frac{1}{5}\left(1+\frac{4D}{5}\right)(1+2t)+\frac{3}{7}e^{2t}$$

$$= -\frac{1}{5}\left(\frac{13}{5}+2t\right)+\frac{3e^{2t}}{7} \qquad \ldots (Q.2.5)$$

Hence G.S. is given by,

$$x = C.F.+P.I. = C_1 e^{-5t}+C_2 e^{t}-\frac{13}{25}-\frac{2t}{5}+\frac{3e^{2t}}{7} \qquad \ldots (Q.2.6)$$

Now $\quad \frac{dx}{dt} = -5C_1 e^{-5t}+C_2 e^{t}-\frac{2}{5}+\frac{6}{7}e^{2t}$

Step 3 : Use the equation where coefficient of y is simple.

i.e. $\quad (D+2)x-3y = t$

$$3y = (D+2)x-t$$

Putting values of x and $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{3}\left[\frac{dx}{dt}+2x-t\right]$$

$$= \frac{1}{3}\left[-5C_1 e^{-5t}+C_2 e^{t}-\frac{2}{5}+\frac{6}{7}e^{2t} + 2C_1 e^{-5t}+2C_2 e^{t}-\frac{26}{25}-\frac{4t}{5}+\frac{6e^{2t}}{7}-t\right]$$

Simplifying we get,

$$y = -C_1 e^{-5t}+C_2 e^{2t}-\frac{12}{25}-\frac{3t}{5}+\frac{4e^{2t}}{7} \qquad \ldots (Q.2.7)$$

Q.3 $\frac{d^2x}{dt^2}+\frac{dy}{dt}+3x = e^{-t}, \frac{d^2y}{dt^2}-4\frac{dx}{dt}+3y = \sin 2t$

☞ **[SPPU : May-12]**

Ans. : Step 1 :

Use $D = \frac{d}{dt}$ hence system can be written as,

$$D^2x+Dy+3x = e^{-t}$$

$$D^2y-4Dx+3y = \sin 2t$$

Collect the terms of x and y.

$$(D^2+3)x+Dy = e^{-t}$$

$$-4Dx+(D^2+3)y = \sin 2t$$

Step 2 : Solving for x by Cramer's rule.

$$\begin{vmatrix} D^2+3 & D \\ -4D & D^2+3 \end{vmatrix} x = \begin{vmatrix} e^{-t} & D \\ \sin 2t & D^2+3 \end{vmatrix}$$

$$\left[(D^2+3)^2+4D^2\right]x = 4e^{-t}-2\cos 2t$$

$$(D^2+1)(D^2+9)x = 4e^{-t}-2\cos 2t$$

$(D^2+1)(D^2+9) = 0 \quad$ gives $\therefore D = \pm i, \pm 3i$

$\therefore$ C.F. is $x_c = C_1\cos t+C_2\sin t+C_3\cos 3t+C_4\sin 3t$

$$x_p = P.I. = \frac{1}{(D^2+1)(D^2+9)}4\cdot e^{-t} - \frac{1}{(D^2+1)(D^2+9)}(2\cos 2t)$$

$$= \frac{1}{5}e^{-t}+\frac{2}{15}\cos 2t$$

$\therefore$ General solution for x = C.F. + P.I.

$$\therefore x = C_1\cos t+C_2\sin t+C_3\cos 3t + C_4\sin 3t+\frac{1}{5}e^{-t}+\frac{2}{15}\cos 2t$$

Step 3 : And similarly solving for y we get,

$$(D^2+1)(D^2+9)y = -\sin 2t-4e^{-t}$$

Auxiliary equation for y is same,

$(D^2+1)(D^2+9) = 0 \quad \therefore D = \pm i, \pm 3i$

$\therefore$ C.F. is $\quad y_C = C_5\cos t+C_6\sin t+C_7\cos 3t+C_8\sin 3t$

P.I. for $y_p = \frac{1}{(D^2+1)(D^2+9)}(-\sin 2t-4e^{-t})$

$$= (-1)\frac{1}{(D^2+1)(D^2+9)}\sin 2t - 4\frac{1}{(D^2+1)(D^2+9)}e^{-t}$$

$$= +\frac{1}{15}\sin 2t - \frac{1}{5}e^{-t}$$

∴ General solution for y = C.F. + P.I.

$$\therefore \quad y = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t + \frac{1}{15}\sin 2t - \frac{1}{5}e^{-t}$$

Substituting these values of x and y in any one of given equations we get,

$C_5 - 2C_2,\ C_6 = -2C_1,\ C_7 = -2C_4,\ C_8 = 2C_3$

By comparing the coefficients of functions of t.

Step 4 :

∴ Required solution for the given system is,

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

and

$$y = 2C_2 \cos t - 2C_1 \sin t - 2C_4 \cos 3t + 2C_3 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

Q.4 Solve $\frac{dx}{dt} - \omega y = a \cos pt,\ \frac{dy}{dt} + \omega x = a \sin pt\ (\omega \neq p)$

☞ **[SPPU : May-07, Dec.-12]**

Ans. : Step 1 : Let $D = \frac{d}{dt}$

∴ Given system becomes,

$$Dx - \omega y = a \cos pt \quad \dots (Q.4.1)$$

$$\omega x + Dy = a \sin pt \quad \dots (Q.4.2)$$

Step 2 : Solving for x by Cramer's rule, we get,

$$\begin{vmatrix} D & -\omega \\ \omega & D \end{vmatrix} x = \begin{vmatrix} a \cos pt & -\omega \\ a \sin pt & D \end{vmatrix}$$

$$(D^2 + \omega^2)\, x = -\, ap \sin pt + a\, \omega \sin pt$$

$$= a(\omega - p)\sin pt \quad \dots (Q.4.3)$$

A.E. of equation (Q.4.3) is $D^2 + \omega^2 = 0$

$$D = \pm i\,\omega$$

The C.F. of equation (Q.4.3) is $x_c = C_1 \cos \omega t + C_2 \sin \omega t$

The P.I. of equation (Q.4.3) is

$$x_p = \frac{1}{D^2+\omega^2} a(\omega - p)\sin pt = a(\omega - p)\frac{1}{-p^2+\omega^2}\sin pt$$

$$= \frac{a}{\omega + p}\sin pt$$

The complete solution of equation (Q.4.3) is

$$x = C_1 \cos \omega t + C_2 \sin \omega t + \frac{a}{\omega + p}\sin pt \quad \dots (Q.4.4)$$

Step 3 : Substituting x in equation (Q.4.1) we get,

$$\omega y = Dx - a \cos pt$$

$$= -C_1\, \omega \sin \omega t + C_2\, \omega \cos \omega t + \frac{ap}{\omega + p}\cos pt - a \cos pt$$

$$\therefore \quad y = -C_1 \sin \omega t + C_2 \cos \omega t + \frac{ap}{\omega(\omega + p)}\cos pt - \frac{a}{\omega}\cos pt \quad \dots (Q.4.5)$$

Step 4 : Equations (Q.4.4) and (Q.4.5) together gives the complete solution of given system.

2.2 : Symmetrical form of Simultaneous Differential Equations

☞ **Dec.-2000, 03, 04, 05, 06, 07, 08, 09, 10, May-05, 06, 07, 08, 09, 10, 13**

General form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are functions of x, y, z are said to be symmetrical simultaneous differential equations. The solution of such a system of equations is given by a pair of relations u (x, y, z) = C_1 and v (x, y, z) = C_2 which are independent of each other. We can solve such a system by following methods.

Q.5 $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$ ☞ **[SPPU : Dec.-04, May-10]**

Ans. : By combinations,

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2 + y^2 = C_1 \quad \ldots (Q.5.1)$$

Again by combination,

$$\frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$$

$$\frac{dy}{-1} = \frac{dz}{e^{C_1}}$$

$$e^{C_1} \cdot dy = -dz$$

Integrating we get,

$$y e^{C_1} = -z + C_2$$

$$y e^{x^2+y^2} + z = C_2 \quad \ldots (Q.5.2)$$

Equations (Q.5.1) and (Q.5.2) together constitute the solution of the system.

Q.6 $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - zx^4)} = \frac{dz}{z(x^4 - y^4)}$ ☞ **[SPPU : Dec.-07, May-06, 13]**

Ans. :

Use multipliers $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$

$$\therefore \quad \text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{2}{z}dz}{2y^4 - z^4 + z^4 - 2x^4 + 2x^4 - 2y^4}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{2}{z}dz = 0$$

Integrating we get,

$$\log x + \log y + 2\log z = \log C_1$$

$$\log xyz^2 = \log C_1$$

$$xyz^2 = C_1 \quad \ldots (Q.6.1)$$

Again use multipliers x^3, y^3, z^3

$$\therefore \text{Each ratio} = \frac{x^3dx + y^3dy + z^3dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)}$$

$$= \frac{x^3dx + y^3dy + z^3dz}{0}$$

$$\Rightarrow \quad x^3dx + y^3dy + z^3dz = 0$$

Integrating we get,

$$\frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} = C$$

$$x^4 + y^4 + z^4 = C_2 \quad \ldots (Q.6.2)$$

Equations (Q.6.1) and (Q.6.2) constitute the solution of the system.

Q.7 Solve $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ ☞ **[SPPU : May-05, 09]**

Ans. : Consider $\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$

Integrating, we get, $\log y = \log z + \log C_1$

$$y = C_1 z \quad \ldots (Q.7.1)$$

Let x, y, z be the set of multipliers for given equations then

$$\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}$$

$$= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

Consider

$$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} = \frac{dy}{2xy}$$

$$\frac{2(xdx+ydy+zdz)}{x^2+y^2+z^2} = \frac{dy}{y}$$

Integrating we get,

$$\log(x^2+y^2+z^2) = \log y + \log C_2$$

$$x^2+y^2+z^2-C_2y = 0 \qquad \ldots (Q.7.2)$$

Equations (Q.7.1) and (Q.7.2) together constitute the solution of the system.

2.3 : L-C-R Circuit

Consider the electrical circuit given in Fig. 2.3.1.

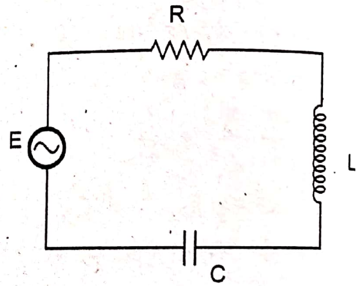

Fig. 2.3.1

Let

I = Instantaneous current

Q = Instantaneous charge

L = Inductance

C = Capacitor of capacity

R = Resistance

$\therefore$ Voltage drop across R $= RI$

Voltage drop across L $= L\frac{dI}{dt}$

Voltage drop across C $= \frac{Q}{C}$

We know that $I = \frac{dQ}{dt} \quad \therefore Q = \int I\,dt$

Kirchhoff's law : The algebraic sum of all the voltage drops in an electric circuit is zero. We consider the following cases.

Case 1 : The differential equation of electrical circuit consists of inductance L, capacitance C with emf E is $L\frac{dI}{dt}+\frac{Q}{C} = E$

$$\therefore \quad L\frac{d^2Q}{dt^2}+\frac{Q}{C} = E$$

Case 2 : The differential equation of electrical circuit consists of L - C without emf is

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C} = 0 \quad \therefore \frac{d^2Q}{dt^2}+\frac{Q}{LC} = 0$$

Case 3 : The differential equation of electrical circuit consists of inductance L, resistance R and capacitance C with emf E is $L\frac{dI}{dt}+RI+\frac{Q}{C}=E$

$$\text{or} \quad L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C} = E$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC} = \frac{E}{L}$$

Case 4 : The differential equation of electrical circuit consists of L, R and C without E is

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC} = 0$$

Q.8 In an L-C-R circuit, the charge q on the condenser is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \sin \omega t$.

The circuit is tuned to resonate so that $\omega^2 = \frac{1}{LC}$. If initially the current and charge be zero, show that for small values of $\frac{R}{L}$ the current in the circuit at time t is given by $\frac{Et}{2L}\sin \omega t$.

☞ [SPPU : May-10,11, 14, 15, Dec.-06]

Ans. : Step 1 : The given differential equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E\sin\omega t$$

i.e. $$LD^2q + RDq + \frac{q}{C} = E\sin\omega t, \qquad \text{where } D = \frac{d}{dt}$$

i.e. $$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)q = \frac{E}{L}\sin\omega t$$

i.e. $$(D^2+\omega^2)q = \frac{E}{L}\sin\omega t \qquad \text{... (Q.8.1)}$$

Step 2 : Note we can neglect $\left(\frac{RD}{L}\right)$ as $\frac{R}{L}$ is small and $\because \frac{1}{LC} = \omega^2$

Step 3 : To find C.F.

A.E. is

$$D^2+\omega^2 = 0 \text{ i.e. } D^2 = -\omega^2 \quad D = \pm\,\omega i$$

$\therefore$ $$\text{C.F.} = C_1\cos\omega t + C_2\sin\omega t$$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L}\cdot\frac{1}{(D^2+\omega^2)}\sin\omega t = \frac{-Et}{2L\omega}\cos\omega t$$

Step 4 : $\therefore$ Complete solution is

$$q = C_1\cos\omega t + C_2\sin\omega t - \frac{E}{2L\omega}t\cos\omega t \qquad \text{... (Q.8.2)}$$

Step 5 :

$\therefore$ $$i = \frac{dq}{dt} = -C_1\omega\sin\omega t + C_2\omega\cos\omega t$$

$$-\frac{E}{2L\omega}(-t\omega\sin\omega t+\cos\omega t) \qquad \text{... (Q.8.3)}$$

Step 6 : Now initially, at $t = 0$, $q = 0$ and $i = 0$.

$\therefore$ From equations (Q.8.2) and (Q.8.3) we get at $t = 0$

$$0 = C_1 + 0 + 0 \quad \therefore C_1 = 0 \text{ and}$$

$$0 = 0 + C_2\omega - \frac{E}{2L\omega}(0+1)$$

$\therefore$ $$C_2\omega = \frac{E}{2L\omega}$$

$\therefore$ Substituting in equation (Q.8.3) we get

$$i = 0 + \frac{E}{2L\omega}\cos\omega t - \frac{E}{2L\omega}(-t\omega\sin\omega t + \cos\omega t)$$

$$= \left(\frac{E}{2L\omega}-\frac{E}{2L\omega}\right)\cos\omega t + \frac{Et}{2L}\sin\omega t = \frac{Et}{2L}\sin\omega t$$

Q.9 **An uncharged condenser of capacity C is charged by applying an e.m.f of value E sin $\frac{t}{\sqrt{LC}}$ through the leads of inductance L and negligible resistance. The charge Q on the plate of the condenser satisfies the differential equation $\frac{d^2Q}{dt^2}+\frac{Q}{LC} = \frac{E}{L}\sin\frac{t}{\sqrt{LC}}$. Prove that the charge at any time t is given by $Q = \frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]$**

☞ **[SPPU : Dec.12, 15, May-12]**

Ans. : Step 1 : Let $p^2 = \frac{1}{LC}$ $\therefore$ The given differential equation becomes

$$\frac{d^2Q}{dt^2} + p^2Q = \frac{E}{L}\sin pt$$

i.e. $$(D^2+p^2)Q = \frac{E}{L}\sin pt \qquad \text{... (Q.9.1)}$$

Step 2 : To find C.F. A.E. is $D^2+p^2 = 0$

i.e. $$D^2 = -p^2 \qquad \therefore D = \pm p$$

$\therefore$ $$\text{C.F.} = C_1\cos pt + C_2\sin pt$$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L}\frac{1}{(D^2+p^2)}\sin pt = \frac{E}{L}\cdot\frac{(-t)^1}{(2p)^1\,1!}\sin\left(pt+1\frac{\pi}{2}\right)$$

$$= -\frac{Et}{2pL}\cos pt$$

Step 4 : The complete solution is

$\therefore$ $$Q = C_1\cos pt + C_2\sin pt - \frac{Et}{2pL}\cos pt \qquad \text{... (Q.9.2)}$$

Step 5 : At $t = 0$, $Q = 0$ and $i = 0$

$\therefore$ From equation (Q.9.2) $0 = C_1 + 0 + 0 \therefore C_1 = 0$

$$\therefore \quad Q = C_2 \sin pt - \frac{Et}{2pL} \cos pt \quad \text{... (Q.9...)}$$

$$\therefore \quad i = \frac{dQ}{dt} = C_2 p \cos pt - \frac{E}{2pL} (-tp \sin pt + \cos pt)$$

Step 6 : Now, at $t = 0$, $i = 0$

$$\therefore \quad 0 = C_2 p\,(1) - \frac{E}{2pL}(0+1)$$

$$\therefore \quad C_2 p = \frac{E}{2pL} \quad \therefore C_2 = \frac{E}{2p^2 L}$$

Step 7 : Substituting in Q we get

$$Q = \frac{E}{2p^2 L} \sin pt - \frac{Et}{2pL} \cos pt$$

$$= \frac{E}{2\frac{1}{LC}L} \sin pt - \frac{E}{2\frac{1}{\sqrt{LC}}L} t \cos pt \quad \text{... } \because p = \frac{1}{\sqrt{LC}}$$

$$= \frac{EC}{2} \sin pt - \frac{E}{2}\sqrt{\frac{C}{L}}\, t \cos pt = \frac{EC}{2}\left(\sin pt - \frac{t}{\sqrt{LC}} \cos pt\right)$$

$$= \frac{EC}{2}\left(\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}\right) \quad \text{... } \because p = \frac{1}{\sqrt{LC}}$$

Q.10 An uncharged condenser of capacity C is charged by applying emf of value $E \sin \frac{t}{\sqrt{LC}}$ through the leads of inductance L and negligible resistance. Find the charge at any time t.

[SPPU : May-05, 13]

Ans. : The differential equation of L-C circuit is

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = E \sin \frac{t}{\sqrt{LC}}$$

$$\therefore \quad \frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}, \text{ put } \omega^2 = \frac{1}{LC}$$

$$\therefore \quad \frac{d^2Q}{dt^2} + \omega^2 Q = \frac{E}{L} \sin \omega t$$

$$(D^2 + \omega^2) Q = \frac{E}{L} \sin \omega t \quad \text{... (Q.10.1)}$$

$\therefore$ A.E. is $D^2 + \omega^2 = 0 \Rightarrow D = \pm i\omega$

$$\therefore \quad Q_C = C_1 \cos \omega t + C_2 \sin \omega t$$

The particular integral is

$$Q_P = \frac{E}{L} \frac{1}{D^2 + \omega^2} \sin \omega t = \frac{E}{L} \frac{t}{2D} \sin \omega t = \frac{-E\, t \cos \omega t}{L \quad 2\omega}$$

$\therefore$ It's general solution is

$$Q = Q_C + Q_P$$

$$= C_1 \cos \omega t + C_2 \sin \omega t - \frac{Et}{2L\omega} \cos \omega t$$

$$Q = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2L}\sqrt{LC} \cos \frac{t}{\sqrt{LC}}$$

$$Q = C_1 \cos \frac{t}{\sqrt{LC}} + C_2 \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}}$$

Q.11 A electric circuit consists of an inductance L of 0.1 H a resistance R of 20 Ω and a condenser of capacitance C of 100 microfarads. If the differential equation of electric circuit is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ find charge q and current i at any time t given that $t = 0$, $q = 0.05$ and $i = 0$.

[SPPU : Dec.-06, 08, 14, May-09, 10, 11]

Ans. : The given D.E. is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$

$$\Rightarrow \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \quad \text{... (Q.11.1)}$$

Here $L = 0.1$, $R = 20$, $C = 100 \times 10^{-6} = 10^{-4}$

$\therefore$ Equation (Q.11.1) becomes,

$$\frac{d^2q}{dt^2}+200\frac{dq}{dt}+10000\,q = 0 \quad \dots (Q.11.2)$$

$\therefore$ A.E. is $D^2+200\,D+10000 = 0$

$$(D+100)\,(D+100) = 0$$

$\Rightarrow$ $D = 100,\ 100$

$\therefore$ $q_C = (C_1+C_2 t)\,e^{100\,t}$

and $q_P = 0$

$\therefore$ $q = (C_1+C_2 t)\,e^{100\,t} \quad \dots (Q.11.3)$

Now at $t = 0$, $q = 0.05$, $\frac{dq}{dt} = 0$

$\therefore$ From equation (Q.11.3) we get

$$0.05 = C_1 \text{ and } \frac{dq}{dt} = (C_1+C_2 t)\,e^{100\,t}\cdot 100+e^{100\,t}(C_2)$$

$\therefore$ $D = C_1\cdot 100+C_2$

$\Rightarrow$ $C_2 = -100\,C_1 = -100\,(0.05)\ C_2 = -5$

$$q = (0.05+(-5)t)e^{100t} = (0.05-5t)e^{100t}$$

$\therefore$ $i = \frac{dq}{dt} = 100\,(0.05-5\,t)\,e^{100\,t}-5\,e^{100\,t}$

$$i = -500t\,e^{100\,t}$$

Q.12 A resistance of 50 ohms an inductor of 2H and 0.005 farad capacitor are connected in a series with emf of 40 V and an open switch. Find the instantaneous charge and current after the switch is closed at t = 0, assuming that at that time charge on capacitor is 4 coulomb. **[SPPU : May-15]**

Ans. : The D.E. of L-C-R circuit is

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C} = E$$

$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{1}{LC}Q = \frac{E}{L} \quad \dots (Q.12.1)$$

where $L = 2$ H, $R = 50$, $C = 0.005$ F, $E = 40$

$\therefore$ Equation (Q.12.1) becomes $\frac{d^2Q}{dt^2}+25\frac{dQ}{dt}+100\,Q = 20$

$\therefore$ A.E. is $D^2+25\,D+100 = 0 \quad \Rightarrow \quad (D+20)\,(D+5) = 0$

$$D = -20,\ -5$$

$\therefore$ C.F. $= Q_c = A\,e^{-20\,t}+B\,e^{-5\,t}$

and P.I. $= Q_P = \frac{1}{D^2+25D+100}(20) = \frac{1}{0+0+100}(20) = \frac{1}{5}$

$\therefore$ The complete solution is

$$Q = Q_c+Q_p = A\,e^{-20\,t}+Be^{-5t}+\frac{1}{5} \quad \dots (Q.12.2)$$

$\therefore$ $I = \frac{dQ}{dt} = -20\,A\,e^{-20\,t}-5\,B\,e^{-5t} \quad \dots (Q.12.3)$

At $t = 0$, $I = 0$ and $Q = 4$

$\therefore$ Equation (Q.12.2) and (Q.12.3) $\Rightarrow A+B+\frac{1}{5} = 4 \Rightarrow A+B = \frac{19}{5}$

and $-20\,A-5\,B = 0 \Rightarrow B = 4\,A$

$\therefore$ $A+4A = \frac{19}{5} \Rightarrow 5\,A = \frac{19}{5} \Rightarrow A = \frac{19}{25}$

and $B = \frac{76}{25}$

$\therefore$ $Q = \frac{1}{25}[19e^{-20t}+76e^{-5t}+5]$

and $I = \frac{1}{5}[-76\,e^{-20\,t}+(-76)\,e^{-5t}]$

3 Fourier Transforms

3.1 : Fourier Transform

1) The following table gives the Fourier transform pairs for rea[dy] reference.

Sr. No.	Name of the transform	Expression for the transform $F(\lambda)$ =	Inverse transform $f(x)$ =
1.	Fourier transform	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du$	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \cos\lambda u\, du$	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(\lambda) \cos\lambda x\, d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) \sin\lambda u\, du$	$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(\lambda) \sin\lambda u\, d\lambda$

We may use the following table also (Note the constant multiples [of] the integrals)

Sr. No.	Name of the transform	Expression for the transform	Inverse transform
1.	Fourier transform	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du$	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$F(\lambda) = \int_{0}^{\infty} f(u) \cos\lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c(\lambda) \cos\lambda x\, d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$F(\lambda) = \int_{0}^{\infty} f(u) \sin\lambda u\, du$	$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(\lambda) \sin\lambda x\, d\lambda$

Note

1) For solving examples we use above formulae

2) $\int_{-\infty}^{\infty} f(x)\, dx = 0$ if $f(x)$ is odd.

$= 2 \int_{0}^{\infty} f(x)\, dx$ if f(x) is even.

Q.1 **Find Fourier cosine transform of**

$$f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

[SPPU : May-12, 14]

Ans. : Consider F.C.T. formula

$$F_c(\lambda) = \int_{0}^{\infty} f(u) \cos\lambda u\, du$$

$$F_c(\lambda) = \frac{1}{2} \left\{ \int_{0}^{a} 2 \cos u \cdot \cos\lambda u\, du + \int_{a}^{\infty} 0 \right\}$$

Use

$$2 \cos\lambda u \cos u = \cos(\lambda - 1)\, u + \cos(\lambda + 1)\, u$$

$$\therefore \quad F_c(\lambda) = \frac{1}{2}\int_0^a [\cos(\lambda-1)u + \cos(\lambda+1)u]du$$

Integrate $= \frac{1}{2}\left[\frac{\sin(\lambda-1)u}{\lambda-1} + \frac{\sin(\lambda+1)u}{\lambda+1}\right]_0^a$

Put the limits of u

$$F_c(\lambda) = \frac{1}{2}\left[\frac{\sin(\lambda-1)a}{\lambda-1} + \frac{\sin(\lambda+1)a}{\lambda+1}\right]$$

Q.2 **Find Fourier sine and cosine transform of $f(x) = e^{-mx}$**

☞ [SPPU : Dec.-05, 11, 14]

Ans : Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

Put the value of f(u)

$$= \left\{\int_0^\infty \left(e^{-mx}\right)\cdot \sin\lambda u\, du\right\}$$

$$\therefore \quad F_s(\lambda) = \left[\frac{e^{-mx}}{\lambda^2+m^2}(-m\sin\lambda u - \lambda\cos\lambda u)\right]_0^\infty$$

$$= \frac{\lambda}{m^2+\lambda^2}$$

For finding cosine transform

Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

Put the value of f(u)

$$F_c(\lambda) = \left\{\int_0^\infty (e^{-mx})\cos\lambda u\, du\right\}$$

$$\therefore \quad F_c(\lambda) = \left[\frac{e^{-mx}}{\lambda^2+m^2}(-m\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

$$= \frac{m}{m^2+\lambda^2}$$

Q.3 **Find the Fourier Sine transform of $f(x) = e^{-x}\cos x;\ x > 0$.**

☞ [SPPU : May-15]

Ans. : We have $F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$

$$= \int_0^\infty e^{-u}\cos u\sin\lambda u\, du = \frac{1}{2}\int_0^\infty e^{-u}\, 2\cos u\sin\lambda u\, du$$

$$= \frac{1}{2}\int_0^\infty e^{-u}(\sin(\lambda+1)u + \sin(\lambda-1)u]\, du$$

$$= \frac{1}{2}\left\{\frac{e^{-u}}{1+(\lambda+1)^2}[-\sin(\lambda+1)u - (\lambda+1)\cos(\lambda+1)u]\right.$$

$$\left. + \frac{e^{-u}}{1+(\lambda-1)^2}[-\sin(\lambda-1)u - (\lambda-1)\cos(\lambda-1)u\right\}_0^\infty$$

$$= \frac{1}{2}\left\{[0] - \frac{1}{1+(\lambda+1)^2}[0-(\lambda+1)] - \frac{1}{1+(\lambda-1)^2}[0-(\lambda-1)]\right\}$$

$$F_s(\lambda) = \frac{1}{2}\left[\frac{\lambda+1}{1+(\lambda+1)^2} + \frac{\lambda-1}{1+(\lambda-1)^2}\right]$$

Q.4 **Find Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$**

Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\, dx$

☞ [SPPU : Dec.-06, 09, 12, May-08, 12]

Ans. : As f(x) is an even function of x we find F.C.T. of f(x)

$$\text{F.C.T. } F(\lambda) = \int_0^\infty F(u)\cos\lambda u\, du$$

$$= \int_0^1 (1-u^2)\cos\lambda u\, du + \int_1^\infty 0$$

$$= \int_0^1 (1-u^2)\cos\lambda u\, du$$

$$= \left[(1-u^2)\left(\frac{\sin\lambda u}{\lambda}\right) - (-2u)\left(\frac{-\cos\lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda u}{\lambda^3}\right) \right]_0^1$$

$$= 0 - \frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3} - (0-0-0)$$

$$F(\lambda) = \frac{-2(\lambda\cos\lambda - \sin\lambda)}{\lambda^3}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{-2(\lambda\cos\lambda - \sin\lambda)}{\lambda^3}\cos\lambda x\, d\lambda$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{1}{2}\right) = \frac{-4}{\pi}\int_0^\infty \frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\left(\cos\frac{\lambda}{2}\right) d\lambda$$

As $\quad f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

$\therefore \quad f\left(\frac{1}{2}\right) = 1-\left(\frac{1}{2}\right)^2 = \frac{3}{4}$ substituting we get

$$\left(\frac{3}{4}\right)\left(\frac{-\pi}{4}\right) = \int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\, d\lambda$$

$\therefore$ As the variable is not important indefinite integrals here replace λ by x

$$\therefore \quad \frac{-3\pi}{16} = \int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\, dx$$

Q.5 Find Fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx$

☞ **[SPPU : Dec.-08, May-12]**

Ans. : Consider F.S.T. formula

$$F(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$= \left\{ \int_0^\infty e^{-u}\sin\lambda u\, du \right\}$$

$$\therefore \quad F_s(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\sin\lambda u + \lambda\cos\lambda u) \right]_0^\infty$$

$$F_s(\lambda) = \frac{\lambda}{\lambda^2+1}$$

Consider inverse Fourier transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1}\sin\lambda x\, d\lambda$$

$$\therefore \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1}\sin\lambda x\, d\lambda$$

Put x = m and then λ = x we get

$$\therefore \quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx$$

$$\therefore \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$$

Q.6 Using Fourier integral representation show that

$$\int_0^\infty \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda = \begin{cases} 0 & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

[SPPU : Dec.-06,12]

Ans. : To prove the result consider the function

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

This function f(x) is defined in $-\infty<x<\infty$ and both $\sin\lambda x$ and $\cos\lambda x$ are present in the integrand so we use the general Fourier transform.

$\therefore$ We should find the F.T. of f(x).

We have

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$F(\lambda) = \int_{-\infty}^{0} f(x)\, e^{-i\lambda x} dx + \int_0^\infty f(x)\, e^{-i\lambda x}\, dx$$

$$F(\lambda) = 0+\int_0^\infty \pi e^{-x}\, e^{-i\lambda x}\, dx$$

$$\therefore \quad F(\lambda) = \pi \int_0^\infty e^{-x(1+i\lambda)}\, dx = \pi\left[\frac{1}{1+i\lambda}\right] = \pi\frac{(1-i\lambda)}{1+\lambda^2}$$

Consider the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\frac{(1-i\lambda)}{1+\lambda^2}\,[\cos\lambda x+i \sin\lambda x]\, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda + \frac{1}{2} i \int_{-\infty}^{\infty} \frac{-\lambda \cos\lambda x+\sin\lambda x}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{2}\, 2\int_0^\infty \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda + 0$$

$$= \int_0^\infty \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda$$

Thus

$$\int_0^\infty \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda = f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases} \quad \text{... (Q.6.1)}$$

Putting x = 0 we get

$$f(0) = \int_0^\infty \frac{1}{1+\lambda^2}\, d\lambda = \frac{\pi}{2}$$

$\therefore$ We have

$$\int_0^\infty \frac{\cos\lambda x+\lambda \sin\lambda x}{1+\lambda^2}\, d\lambda = \begin{cases} 0 & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Q.7 Using Fourier integral representation show that

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\, \sin\lambda x\, d\lambda = \begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

[SPPU : May-06, 13]

Ans. : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases} \frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$F_s(\lambda) = \int_0^\pi f(u)\sin\lambda u\, du + \int_\pi^\infty f(u)\sin\lambda u\, du$$

$$F_s(\lambda) = \int_0^\pi \frac{\pi}{2}\sin\lambda u\, du + \int_\pi^\infty 0\sin\lambda u\, du$$

$$= \frac{\pi}{2}\int_0^\pi \sin\lambda u\, du + 0 = \frac{\pi}{2}\left[\frac{-\cos\lambda u}{\lambda}\right]_0^\pi$$

$$= \frac{\pi}{2}\left[\frac{-\cos\lambda\pi-(-1)}{\lambda}\right] = \frac{\pi}{2}\frac{(1-\cos\lambda\pi)}{\lambda}$$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda$$

Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \frac{\pi}{2}\int_0^\infty \frac{2}{\pi}\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda$$

$$f(x) = \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda$$

Q.8 Solve the integral equation.

$$\int_0^\infty f(x)\cos\lambda x\, dx = \begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$

☞ [SPPU : Dec.-06, 07, 09, May-12, 14]

Ans. : As $\cos\lambda x$ is present in the integrand, we use F.C.T.

$$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx = \begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

To find f(x) consider inverse Fourier cosine transforms

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\cos\lambda x\, d\lambda + \int_1^\infty 0\, d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin\lambda x}{x}\right)-(-1)\left(\frac{-\cos\lambda x}{x^2}\right)\right]_0^1$$

$$f(x) = \frac{2}{\pi}\left[\left(0-\frac{\cos x}{x^2}\right)-\left(0-\frac{1}{x^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$$

Use $(1-\cos x) = 2\sin^2(x/2)$

$$= \frac{2}{\pi}\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$f(x) = \frac{1}{\pi}\frac{\sin^2\frac{x}{2}}{\left(\frac{x^2}{4}\right)} = \frac{1}{\pi}\frac{\sin^2(x/2)}{(x/2)^2}$$

Substituting in the given equation, we get

$$\int_0^\infty \frac{1}{\pi}\frac{\sin^2(x/2)}{(x/2)^2}\cos\lambda x\, dx = \begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

Put $\lambda = 0$

$$\frac{1}{\pi}\int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2}1\, dx = 1$$

Put $x/2 = u$, $x = 2u$, $dx = 2\, du$

$$\therefore \quad \frac{1}{\pi}\int_0^\infty \frac{\sin^2 u}{u^2}\, 2\, du = 1$$

$$\therefore \quad \int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

Q.9 Solve the integral equation

$$\int_0^\infty f(x) \cos\lambda x\, dx = e^{-\lambda},\ \lambda>0$$

☞ [SPPU : May-05, 13, Dec.-08, 11, 12]

Ans. : Given

$$\therefore \int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda>0$$

As $\cos\lambda x$ is present in the integrand we use F.C.T.

$$\therefore \quad F_c(\lambda) = e^{-\lambda},\ \lambda>0$$

To find f(x) consider inverse Fourier cosine transform.

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x\, d\lambda$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda u + x\sin\lambda x)\right]_0^\infty$$

$$f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$$

Q.10 Solve the integral equation

$$\int_0^\infty f(x)\sin\lambda x\, dx = e^{-\lambda},\ \lambda>0$$

☞ [SPPU : Dec.-12]

Ans. : Given

$$\int_0^\infty f(x)\sin\lambda x\, dx = e^{-\lambda},\ \lambda>0$$

As $\sin\lambda x$ is present in the integrand we use F.S.T.

$$\therefore \quad F_s(\lambda) = e^{-\lambda}$$

To find f(x) we use inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\sin\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\sin\lambda x - x\cos\lambda x)\right]_0^\infty$$

$$f(x) = \frac{2}{\pi}\,\frac{x}{1+x^2}$$

Q.11 a) Find Fourier transform of $e^{-x^2/2}$.

b) Show that Fourier transform of $e^{-x^2/2}$ is $e^{-\lambda^2/2}$

☞ [SPPU : May-08, 12]

Ans. : The Fourier transform of

$$f(x) = e^{-x^2/2} \text{ is given by}$$

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-x^2/2}\cdot e^{-i\lambda x}\, dx$$

$$\therefore \quad F(\lambda) = \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}+i\lambda x\right)} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2i\lambda x)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x+i\lambda)^2 - i^2\lambda^2]} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2} e^{-\frac{\lambda^2}{2}} dx$$

$$= e^{-\frac{\lambda^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2} dx$$

The above integrand is even function

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \cdot 2\int_0^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2} dx$$

Substitute $\frac{1}{2}(x+i\lambda)^2 = u$

$$\therefore \quad \frac{1}{2} 2(x+i\lambda)\, dx = du$$

$$\therefore \quad dx = \frac{du}{x+i\lambda} = \frac{du}{\sqrt{2u}}$$

$\therefore$ We get

$$F(\lambda) = e^{-\lambda^2/2} \frac{2}{\sqrt{2}} \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du$$

$$\text{As } \lceil n = \int_0^{\infty} e^{-u} u^{n-1} du$$

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \sqrt{2} \left\lceil \frac{1}{2} \right. = \sqrt{2} \cdot \sqrt{\pi} \cdot e^{-\lambda^2/2}$$

$$F(\lambda) = \sqrt{2\pi}\, e^{-\lambda^2/2}$$

b) In this example use the formula

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \cdot e^{-\lambda^2/2} \qquad \text{(by part (a))}$$

$$F(\lambda) = e^{-\lambda^2/2}$$

Q.12 **Find Fourier transform of $e^{-|x|}$ hence show that**

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda = \pi e^{-|x|}.$$

☞ **[SPPU : Dec.-15]**

Ans. : The Fourier transform of given function is

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx = \int_{-\infty}^{\infty} e^{-|x|} (\cos\lambda x - i \sin\lambda x)\, dx$$

As $e^{-|x|}$ is an even function, hence we get,

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-|x|} \cos\lambda x\, dx = 2\int_0^{\infty} e^{-x} \cos\lambda x\, dx$$

$$F_c(\lambda) = 2\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda \sin\lambda x)\right]_0^{\infty}$$

$$F_c(\lambda) = \left[\frac{2}{1+\lambda^2}\right]$$

To find f(x) taking inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x} d\lambda$$

Put the value of $F(\lambda)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \frac{1}{1+\lambda^2} e^{i\lambda x} d\lambda$$

$$\pi f(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda = \pi e^{-|x|}$$

☐☐☐

4 Z-Transform

4.1 : Z-Transform

I)

1) **Definition :** The Z-transform of sequence $\{f(k)\}$ is defined as

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

2) For causal sequence $\{f(k)\}$, where $0 \le k < \infty$. The Z-transform is defined by

$$z[f(k)] = \sum_{k=0}^{\infty} f(k)\, z^{-k} = F(z)$$

where $z = x + iy$ is a complex number,

z is the operator of Z-transform and F(z) is the Z-transform of sequence $\{f(k)\}$.

Note

The Z-transform of a sequence $\{f(k)\}$ exists if the series $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ is convergent. i.e. The series tends to a finite value for some values of z. These values of z for which the series is convergent lie within a region known as Region of convergence (ROC) in the z-plane.

II) Z-transforms of Some Standard Sequences

1) Discrete unit step function :

$$u(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \ge 0 \end{cases}$$

$$z[u(k)] = \sum_{k=-\infty}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} u(k)\, z^{-k} + \sum_{k=0}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 z^{-k} + \sum_{k=0}^{\infty} 1 z^{-k}$$

$$= 0 + \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

which is convergent if $\left|\frac{1}{z}\right| < 1$ i.e. $|z| > 1$

Hence $z[u(k)] = \dfrac{z}{z-1}$ if $|z| > 1$

2) Unit impulse function :

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \ne 0 \end{cases}$$

$$\therefore \quad z[\delta(k)] = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} (0+\ldots+0+1+0+\ldots+0) z^{-k}$$

$$z[\delta(k)] = 1 \cdot z^{0} = 1$$

3) $\{f(k)\} = \{a^k\},\ k \ge 0$

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} (a\,z^{-1})^k$$

$$= 1 + a\,z^{-1} + (a\,z^{-1})^2 + \dots$$

$$= \frac{1}{1-az^{-1}} \quad \text{if } \left|az^{-1}\right| < 1$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \text{ if } \left|\frac{a}{2}\right| < 1 \quad \text{i.e. } |z| > |a|$$

Hence the region of convergence is the exterior of circle $x^2 + y^2 = a^2$

4) $\{f(k)\} = \{a^k\} \; : \; k < 0$

By definition

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^k\, z^{-k} + 0$$

Put $k = -r \;\therefore -k = r$

and $-\infty < k \leq -1 \Rightarrow \infty > -k \geq 1$ i.e. $1 \leq r < \infty$

$$\therefore \quad z[f(k)] = \sum_{r=1}^{\infty} a^{-r} z^{r}$$

$$= \sum_{r=1}^{\infty} \left(\frac{z}{a}\right)^{r}$$

$$= \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \dots$$

$$= \frac{z/a}{1-z/a} \quad \text{provided } \left|\frac{z}{a}\right| < 1$$

$$= \frac{z}{a-z}, \quad |z| < |a|$$

Thus $z[f(k)] = z\,[a^k] = \dfrac{z}{a-z}$, $k < 0$ and for $|z| < |a|$ and hence region of convergence is the interior of the circle $x^2 + y^2 = a^2$.

5) $\{f(k)\} = \{a^{|k|}\}, \; \forall \; k \in \mathbf{Z}$

By definition,

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{|k|}\, z^{-k} + \sum_{k=0}^{\infty} a^{|k|} z^{-k} \quad \dots (4.1)$$

we have $\quad |k| = k \;\text{ if } k \geq 0$

$\qquad\qquad = -k \text{ if } k < 0$

Thus equation (4.1) becomes,

$$z[f(k)] = \sum_{k=-\infty}^{-1} a^{-k}\, z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \sum_{k=-\infty}^{-1} (az)^{-k} + \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= [az + (az)^2 + (az)^3 + \dots] + \left[1 + \frac{a}{z} - \left(\frac{a}{z}\right)^2 + \dots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}} \quad \text{provided } |az| < 1 \text{ and } \left|\frac{a}{z}\right| < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a} \; : \; |z| < \frac{1}{|a|}$$

and $|z| > |a|$

Thus $z[a^{|k|}] = \frac{az}{1-az} + \frac{z}{z-a}, \ \forall\ k$ and

$$|a| < |z| < \frac{1}{|a|}$$

Hence ROC is the annulus between the circles

$x^2 + y^2 = a^2$ and $x^2 + y^2 = \frac{1}{a^2}$

6) $\{f(k)\} = \{e^{ak}\}$ for $k \geq 0$

$$z[e^{ak}] = \sum_{k=-\infty}^{\infty} e^{ak} z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{+ak} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^a}{z}\right)^k$$

$$= \frac{1}{1-e^a/z} \quad \text{for} \left|\frac{e^a}{z}\right| < 1$$

$$= \frac{z}{z-e^a} \quad \text{for } |z| > e^a$$

Thus $z[e^{ak}] = \frac{z}{z-e^a}$ for $|z| > e^a$

7) $\{f(k)\} = \sin ak, \ k \geq 0$

$$z[\sin ak] = z\left[\frac{e^{iak} - e^{-iak}}{2i}\right]$$

$$= \frac{1}{2i}\{z[e^{iak}] - z[e^{-iak}]\}$$

$$= \frac{1}{2i}\left\{\frac{z}{z-e^{ia}} - \frac{z}{z-e^{-ia}}\right\} \quad \text{for } |z| > \left|e^{\pm ia}\right| = 1$$

$$= \frac{1}{2i}\left\{\frac{z^2 - ze^{-ia} - z^2 + ze^{ia}}{z^2 - (e^{ia} + e^{-ia})z + e^{ia}e^{-ia}}\right\}$$

$$= \frac{z\left(\frac{e^{ia} - e^{-ia}}{2i}\right)}{z^2 - 2z\cos a + 1}$$

$$= \frac{z \sin a}{z^2 - 2z\cos a + 1} \quad \text{for } |z| > 1$$

8) $\{f(k)\} = \cos ak, \ k \geq 0$

$$z[\cos ak] = z\left[\frac{e^{iak} + e^{-iak}}{2}\right]$$

$$= \frac{1}{2}\{z[e^{iak}] + z[e^{-iak}]\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-e^{ia}} + \frac{z}{z-e^{-ia}}\right\} \quad \text{for } |z| > \left|e^{\pm ia}\right|$$

$$= \frac{z}{2}\left\{\frac{z - e^{-ia} + z - e^{ia}}{z^2 - (e^{ia} + e^{-ia})z + e^{ia}e^{-ia}}\right\} \quad \text{for } |z| > 1$$

$$= \frac{z\left[z - \left(\frac{e^{ia} + e^{-ia}}{2}\right)\right]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[z - \cos a]}{z^2 - 2z\cos a + 1} \quad \text{for } |z| > 1$$

9) $\{f(k)\} = \sinh ak, \ k \geq 0$

$$z[\sinh ak] = z\left[\frac{1}{2}(e^{ak} - e^{-ak})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^a} - \frac{z}{z-e^{-a}}\right]$$

for $|z| > |e^a|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z - e^a - z + e^a}{z^2 - (e^a + e^{-a})z + e^a e^{-a}}\right]$$

for $|z| >$ maximum of $|e^a|$ and $|e^{-a}|$

$$= \frac{z\left(\frac{e^{a}-e^{-a}}{2}\right)}{z^2-2z\cosh a+1}$$

$$= \frac{z\sinh a}{z^2-2z\cosh a+1} \quad \text{for } |z| > \max(|e^{a}|, |e^{-a}|)$$

10) $\{f(k)\} = \cosh ak, \ k \geq 0$

$$z[\cosh ak] = z\left[\frac{e^{ak}+e^{-ak}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{a}}+\frac{z}{z-e^{-a}}\right]$$

for $|z| > |e^{a}|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z-e^{-a}+z-e^{a}}{z^2-(e^{a}+e^{-a})z+1}\right]$$

$$= \frac{z\left[z-\left(\frac{e^{a}+e^{-a}}{2}\right)\right]}{z^2-2z\cosh a+1}$$

$$= \frac{z[z-\cosh a]}{z^2-2z\cosh a+1}$$

for $|z| > \max$ of $|e^{a}|$ and $|e^{-a}|$

11) a) $\{f(k)\} = {}^{n}C_k$ for $0 \leq k \leq n$

$$z[{}^{n}C_k] = \sum_{k=-\infty}^{\infty} {}^{n}C_k\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n} {}^{n}C_k\, z^{-k} + \sum_{k=n+1}^{\infty} 0$$

$$= \sum_{k=0}^{n} {}^{n}C_k\, z^{-k}$$

$$= \sum_{k=0}^{n} {}^{n}C_k\, (z^{-1})^{k}$$

$$= (1+z^{-1})^{n} \quad \text{for } |z^{-1}| < 1$$

$$= \left(1+\frac{1}{z}\right)^{n} \quad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \left(\frac{z+1}{z}\right)^{n} \quad \text{for } |z| > 1$$

b) $\{f(k)\} = {}^{k}C_n$ for $k \geq n \geq 0$

$$z[{}^{k}C_n] = \sum_{k=-\infty}^{\infty} {}^{k}C_n\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n-1} 0 + \sum_{k=n}^{\infty} {}^{k}C_n\, z^{-k}$$

$$= \sum_{r=0}^{\infty} {}^{(n+r)}C_n\, z^{-(n+r)} \quad \text{put } k-n=r$$

$$= \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-n}\, z^{-r} \quad (\because {}^{n}C_r = {}^{n}C_{n-r})$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-r}$$

$$= z^{-n}(1-z^{-1})^{-(n+1)}$$

$$= z^{-n}\left(\frac{z}{z-1}\right)^{n+1} \quad \text{for } k \geq n \geq 0$$

and $|z| < 1$

$$z[{}^{k}C_n] = z^{-n}\left(\frac{z}{z-1}\right)^{n+1} \quad \text{for } |z| > 1$$

$k \geq n \geq 0$

c) $\{f(k)\} = {}^{k+n}C_n$

$$Z[{}^{k+n}C_n] = \sum_{k=0}^{\infty} {}^{k+n}C_n \, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(k+n)(k+n-1)\dots(k+1)}{n!} z^{-k}$$

$$= 1+(n+1)\left(\frac{1}{z}\right)+\frac{(n+1)(n+2)}{2!}\left(\frac{1}{z}\right)^2+\dots$$

$$= \left[1-\frac{1}{z}\right]^{-(n+1)} \quad \text{(by above example)}$$

for $|z| > 1$

$$= \left[\frac{z}{z-1}\right]^{n+1} \quad \text{for } |z| > 1$$

Examples

Q.1 **Find the Z-transform of the following sequence.**

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \geq 0 \end{cases}$$

☞ **[SPPU : May-2000, Dec.-05, 07]**

Ans. : We have,

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \geq 0 \end{cases}$$

∴ By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k z^{-k}$$

... using f (k)

$$= \sum_{r=\infty}^{1} 2^{-r} z^r + \sum_{k=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^k$$

... Putting $k = -r$ in first summation

$$= \sum_{r=1}^{\infty}\left(\frac{z}{2}\right)^r + \sum_{k=0}^{\infty}\left(\frac{1}{3z}\right)^k$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{3z}}, \left|\frac{z}{2}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

Sum of infinite G.P.s $S_\infty = \frac{a}{1-r}, |r| < 1$

$$= \frac{2z}{2-z} + \frac{3z}{3z-1}, |z| < 2,$$

$$|z| > \frac{1}{3} \text{ i.e. } \frac{1}{3} < |z| < 2$$

Q.2 **Find Z-transform of**
$f(k) = 3(2^k)+4(-1)^k; \; k \geq 0$ ☞ **[SPPU : May-13]**

Ans. : We have $f(k) = 3(2^k)+4(-1)^k$

$$Z\{f(k)\} = Z\{3(2^k)+4(-1)^k\}; \; k \geq 0$$

$$= \frac{3z}{z-2}+\frac{4z}{z+1}; \; |z| > 2, |z| > 1$$

$$= z\left\{\frac{3}{z-2}+\frac{4}{z+1}\right\}; \; |z| > 2$$

$$= z\left\{\frac{3(z+1)+4(z-2)}{(z+1)(z-2)}\right\}; \; |z| > 2$$

$$Z\{f(k)\} = \frac{2(7z-5)}{(z+1)(z-2)}; \; |z| > 2$$

∴ The region of convergence is exterior of the circle $x^2+y^2 = 2^2$.

Q.3 **Find the Z-transform and region of convergence**

$$f(k) = \left(\frac{1}{2}\right)^{|k|} \quad \text{for all k.}$$

[SPPU : May-04, 05, 06, 12, 14, Dec.-10]

Ans. : We have,

$$f(k) = \left(\frac{1}{2}\right)^{|k|}, \quad \forall k$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{r=\infty}^{1} \left(\frac{1}{2}\right)^{|-r|} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$$

... putting $k = -r$ in first

$$= \sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^{r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k}$$

... $\because$ k and r are positive

$$= \sum_{r=1}^{\infty} \left(\frac{z}{2}\right)^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{2z}\right)^{k}$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{2z}}, \quad \left|\frac{z}{2}\right| < 1, \quad \left|\frac{1}{2z}\right| < 1$$

... sum of infinite G.P.

$$= \frac{z}{2-z} + \frac{2z}{2z-1}, \quad |z| < 2, \quad |z| > \frac{1}{2} \quad \text{i.e.} \quad \frac{1}{2} < |z| < 2$$

i.e. the ROC is the annulus between the circles $x^2 + y^2 = \left(\frac{1}{2}\right)^2$ and $x^2 + y^2 = 2^2$ in the z-plane.

4.2 : Properties of Z-transforms

1) Definition

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k)\, z^{-k},$$

for causal sequence ($k \geq 0$)

2) Linearity

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

3) Change of Scale

$$Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$$

4) Corollary of Change of Scale

$$Z\{e^{-ak} f(k)\} = F(e^{a} z)$$

5) Shifting Property

a) For both sided sequence :

$Z\{f(k+n)\} = z^n F(z)$ ... Left shifting

$Z\{f(k-n)\} = z^{-n} F(z)$ ... Right shifting

b) For causal sequence ($k \geq 0$) :

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$\therefore Z\{f(k+1)\} = z\,F(z) - z\,f(0)$

$Z\{f(k+2)\} = z^2 F(z) - z^2 f(0) - z\,f(1)$ etc.

$Z\{f(k-n)\} = z^{-n} F(z)$

$\therefore Z\{f(k-1)\} = z^{-1} F(z)$

$Z\{f(k-2)\} = z^{-2} F(z)$ etc.

6) Multiplication by k

$$Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

In general,

$$Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$$

7) Division by k

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^{\infty}\frac{F(z)}{z}\,dz$$

8) Initial Value Theorem (for one sided sequence e.g. $k \geq 0$)

$$f(0) = \lim_{z\to\infty} F(z)$$

9) Final Value Theorem (for one sided sequence e.g. $k \geq 0$)

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z)$$

10) Convolution Theorem

$$Z[\{f(k)\} * \{g(k)\}] = F(z)\cdot G(z)$$

where, convolution of $\{f(k)\}$ and $\{g(k)\}$ is

$$\{f(k)\} * \{g(k)\} = \sum_{m=-\infty}^{\infty} f(m)\,g(k-m)$$

and for casual sequence ($k \geq 0$)

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\,g(k-m)$$

Examples

Q.4 Find $Z\{f(k)\}$ if

i) $f(k) = 3^k \cos(4k+5),\ k \geq 0$

ii) $2^{-t}\sin at,\ t \geq 0$ ☞ **[SPPU : Dec.-01, 12, May-12]**

Ans. :

i) We have, $f(k) = 3^k \cos(4k+5),\ k \geq 0$

[Form : $Z\{a^k f(k)\}$. Hence, use change of scale property]

Now,

$$\cos(4k+5) = \cos 4k \cos 5 - \sin 4k \sin 5$$

$$\therefore Z\{\cos(4k+5)\} = Z\{\cos 5\cdot\cos 4k - \sin 5 \sin 4k\}$$

$$= \cos 5\cdot Z\{\cos 4k\} - \sin 5\, Z\{\sin 4k\}$$

... Linearity Property

$$= \cos 5\left[\frac{z(z-\cos 4)}{z^2-2z\cos 4+1}\right] - \sin 5\left[\frac{z\sin 4}{z^2-2z\cos 4+1}\right]$$

... Standard Result

$$= \text{say } G(z) \qquad \text{... (Q.4.1)}$$

$\therefore$ By change of scale property

i.e. $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$ we have,

$$Z\{3^k \cos(4k+5)\} = G\left(\frac{z}{3}\right)$$

$$= \cos 5\left[\frac{\frac{z}{3}\left(\frac{z}{3}-\cos 4\right)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos 4 + 1}\right] - \sin 5\left[\frac{\frac{z}{3}\sin 4}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos 4 + 1}\right]$$

... using G(z) from (Q.4.1)

$$= \cos 5\left[\frac{\frac{z}{9}(z-3\cos 4)}{\frac{z^2-6z\cos 4+9}{9}}\right] - \sin 5\left[\frac{\frac{z}{9}(3\sin 4)}{\frac{z^2-6z\cos 4+9}{9}}\right]$$

$$= \frac{(\cos 5)\cdot z(z-3\cos 4) - (\sin 5)\cdot 3z\sin 4}{z^2-6z\cos 4+9}$$

ii) We have, $f(t) = 2^{-t} \sin at,\ t \geq 0$

$$= \left(\frac{1}{2}\right)^t \sin at,\ t \geq 0$$

$\therefore$ Its Z-transform is

$$Z\{f(t)\} = Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\},\ t \geq 0$$

$$= \frac{\left(\frac{1}{2}\right) z \sin a}{z^2 - 2\left(\frac{1}{2}\right) z \cos a + \left(\frac{1}{2}\right)^2},$$

$|z| > \left|\frac{1}{2}\right|$ $\because$ Standard Result :

$$Z\{c^k \sin \alpha k\} = \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2},\ |z| > |c|$$

$$= \frac{1}{2} \frac{z \sin a}{\left(z^2 - z \cos a + \frac{1}{4}\right)} = \frac{1}{2} \frac{z \sin a}{\frac{1}{4}\left(4z^2 - 4z \cos a + 1\right)}$$

$$= \frac{2z \sin a}{4z^2 - 4z \cos a + 1},\ |z| > \frac{1}{2}$$

Alternately we can write $Z\{\sin at\}$ first, then use the change of scale property to find $Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\}$.

Q.5 **Find $Z\{f(k)\}$ if**
$f(k) = e^{-ak} \cos bk,\ k \geq 0$

Ans. : [Form : $Z\left\{e^{-ak} f(k)\right\}$. Hence use, the property (case of change of scale, $a \to e^{-a}$).]

We have,

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1},$$

$|z| > 1$ ... (Q.5.1) Standard Result

Now, $\because Z\left\{e^{-ak} f(k)\right\} = F\left(e^a z\right)$

... Property where,
$Z\{f(k)\} = F(z)$

$$\therefore Z\{e^{-ak} \cos bk\} = \frac{e^a z\left(e^a z - \cos b\right)}{\left(e^a z\right)^2 - 2\left(e^a z\right) \cos b + 1},$$

$\left|e^a z\right| > 1$

... Replacing z by $\left(e^a z\right)$ in (Q.5.1)

$$= \frac{e^a \cdot e^a z\left(z - e^{-a} \cos b\right)}{e^{2a} \cdot z^2 - 2e^a z \cos b + 1}$$

$$= \frac{e^{2a} \cdot z\left(z - e^{-a} \cos b\right)}{e^{2a}\left(z^2 - 2e^{-a} z \cos b + e^{-2a}\right)}$$

$$= \frac{z\left(z - e^{-a} \cos b\right)}{z^2 - 2e^{-a} z \cos b + e^{-2a}},$$

$|z| > \frac{1}{e^a}$

Q.6 **Find z-transform of**
$f(x) = e^{-3k} \sin 2k \cos 2k,\ k \geq 0$ ☞ **[SPPU : May-13]**

Ans. : We have

$$f(k) = e^{-3k} \sin 2k \cos 2k$$

$$f(k) = \frac{1}{2} e^{-3k} \sin 4k$$

We know that

$$z\{\sin 4k\} = \frac{z \sin 4}{z^2 - 2z\cos 4 + 1}\ ; |z| > 1$$

$$\therefore \quad z\left\{\frac{1}{2}e^{-3k}\sin 4k\right\} = \frac{1}{2}\frac{(e^3 \sin 4)(z)}{(e^3 z)^2 - 2(e^3 z)\cos 4 + 1}$$

$$Z\{f(k)\} = \frac{(e^3 \sin 4)\, z}{2[e^6 z^2 - 2e^3 z\cos 4 + 1]}\ ; |z| > e^{-3}$$

Q.7 Find the Z-transform of the following. Also, find the ROC :
$f(k) = \alpha^k \cos(\beta k + \gamma),\ k \geq 0$

[SPPU : May-06, 07, 08, Dec.-07, 10]

Ans. : We have,

$$f(k) = \alpha^k \cos(\beta k + \gamma),\ k \geq 0$$

$$= \alpha^k (\cos\beta k \cos\gamma - \sin\beta k \sin\gamma)$$

$$= \cos\gamma\left(\alpha^k \cos\beta k\right) - \sin\gamma\left(\alpha^k \sin\beta k\right)$$

$$\therefore \quad Z\{f(k)\} = \cos\gamma\, Z\left\{\alpha^k \cos\beta k\right\} - \sin\gamma\, Z\left\{\alpha^k \sin\beta k\right\}$$

$$= \cos\gamma\left[\frac{z(z - \alpha\cos\beta)}{z^2 - 2\alpha z\cos\beta + \alpha^2}\right]$$

$$-\sin\gamma\left[\frac{\alpha z \sin\beta}{z^2 - 2\alpha z\cos\beta + \alpha^2}\right], |z| > |\alpha|$$

$$\ldots \because Z\left\{c^k \cos ak\right\} = \frac{z(z - c\cos a)}{z^2 - 2cz\cos a + c^2}$$

$$\text{and } Z\left\{c^k \sin ak\right\} = \frac{cz \sin a}{z^2 - 2cz\cos a + c^2}, |z| > |c|$$

$$= \frac{z^2\cos\gamma - \alpha z\cos\beta\cos\gamma - \alpha z\sin\beta\sin\gamma}{z^2 - 2\alpha z\cos\beta + \alpha^2}$$

$$= \frac{z^2\cos\gamma - \alpha z\cos(\beta - \gamma)}{z^2 - 2\alpha z\cos\beta + \alpha^2}, |z| > |\alpha|$$

Q.8 Find the Z-transform of the following. Also, find the ROC :
$f(k) = k\, 5^k$

[SPPU : Dec.-02, 04, 12, May-06, 12]

Ans. : We have,

$$f(k) = k\, 5^k,\ \text{Assume } k \geq 0$$

Now,

$$Z\left\{5^k\right\} = \frac{z}{z-5}, |z| > 5$$

$$\therefore \quad Z\left\{k\, 5^k\right\} = -z\frac{d}{dz}\left(\frac{z}{z-5}\right) \qquad \ldots \text{Multiplication by k}$$

$$= -z\left[\frac{(z-5) - z}{(z-5)^2}\right]$$

$$= \frac{5z}{(z-5)^2}, |z| > 5$$

Q.9 Find Z{f(k)} if
$f(k) = 4^k e^{-5k} k;\ K \geq 0$

[SPPU : May-13]

Ans. : Let $g(k) = k$

$$z\{g(k)\} = z\{k\} = \left(-z\frac{d}{dz}\right)z\{1\}$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right); |z| > 1$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= (-z)\left[\frac{(z-1) - z(1)}{(z-1)^2}\right]; |z| > 1$$

$$= (-z)\frac{-1}{(z-1)^2}$$

$$z\{g(k)\} = \frac{z}{(z-1)^2}\ ; |z| > 1$$

$$\therefore \quad z\{4^k k\} = \frac{\frac{z}{4}}{\left(\frac{z}{4}-1\right)^2};\ |z| > 1$$

Now

$$z\{4^k k e^{-5k}\} = \frac{\frac{ze^5}{4}}{\left(\frac{ze^5}{4}-1\right)^2};\ |z| > 1$$

$$= \frac{4e^5 z}{(ze^5-4)^2};\ |z| > 1$$

Q.10 **Find $Z\{f(k)\}$ if**

i) $f(k) = (k+1)\ 2^k$ ☞ **[SPPU : Dec.-10, 11, May-11]**

ii) $f(k) = k^2 e^{-ak},\ k \geq 0$ ☞ **[SPPU : Dec.-09, May-12]**

Ans. : [Here we have the forms $Z\{k\,f(k)\}$. Hence use the property of multiplication by k.]

i) We have, $f(k) = (k+1)\ 2^k$

$$= k\ 2^k + 2^k,\ k \geq 0 \qquad \text{(Assumption)}$$

$$\therefore \quad Z\{f(k)\} = Z\{k\ 2^k + 2^k\}$$

$$= Z\{k\ 2^k\} + Z\{2^k\} \qquad \text{... Linearity Property}$$

$$= \left(-z\frac{d}{dz}\right)[Z\{2^k\}] + \frac{z}{z-2}$$

$$\text{... } \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

and $\quad Z\{a^k\} = \dfrac{z}{z-a}$

$$= -z\frac{d}{dz}\left(\frac{z}{z-2}\right) + \frac{z}{z-2}$$

$$= -z\left[\frac{(z-2)\,1 - z(1)}{(z-2)^2}\right] + \frac{z}{z-2}$$

$$= \frac{2z}{(z-2)^2} + \frac{z}{z-2}$$

$$= \frac{2z + z(z-2)}{(z-2)^2}$$

$$= \frac{z^2}{(z-2)^2}$$

ii) We have, $f(k) = k^2 e^{-ak},\ k \geq 0$

Now, $Z\{e^{-ak}\} = Z\{(e^{-a})^k\}$

$$= \frac{z}{z-e^{-a}} \qquad \text{... } \because Z\{a^k\} = \frac{z}{z-a}$$

$$\therefore \quad Z\{k e^{-ak}\} = -z\frac{d}{dz}\,Z\{e^{-ak}\} \qquad \text{... } \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-e^{-a}}\right)$$

$$= -z\left[\frac{\left(z-e^{-a}\right)1 - z(1)}{\left(z-e^{-a}\right)^2}\right]$$

$$= \frac{z\,e^{-a}}{\left(z-e^{-a}\right)^2} \qquad \text{... (Q.10.1)}$$

Now, using $Z\{k\,f(k)\} = -z\dfrac{d}{dz}F(z)$ again, we get

$$Z\{k^2 e^{-ak}\} = Z\{k\left(ke^{-ak}\right)\} = -z\frac{d}{dz}\,Z\{ke^{-ak}\}$$

$$= -z\frac{d}{dz}\left[\frac{z\,e^{-a}}{\left(z-e^{-a}\right)^2}\right] \qquad \text{... using (Q.10.1)}$$

$$= -z e^{-a} \frac{d}{dz} \frac{z}{\left(z - e^{-a}\right)^2}$$

$$= -z e^{-a} \left[\frac{\left(z - e^{-a}\right)^2 1 - z\, 2\left(z - e^{-a}\right) 1}{\left(z - e^{-a}\right)^4} \right]$$

$$= -z e^{-a} \left[\frac{\left(z - e^{-a}\right) - 2z}{\left(z - e^{-a}\right)^3} \right]$$

$$= -z e^{-a} \left[\frac{-z - e^{-a}}{\left(z - e^{-a}\right)^3} \right]$$

$$\therefore \quad Z\left\{k^2 e^{-ak}\right\} = \frac{z e^{-a}\left(z + e^{-a}\right)}{\left(z - e^{-a}\right)^3}$$

OR, we can directly write

$$Z\left\{k^2 e^{-ak}\right\} = \left(-z \frac{d}{dz}\right)^2 Z\left\{e^{-ak}\right\} \quad \text{... multiplication by k}$$

Q.11 Find the Z-transform of the following

$f(k) = \dfrac{\sin \alpha k}{k}, k > 0$

☞ [SPPU : May-2000, 05,12, 15, Dec.-2000, 02, 11]

Ans. : [Form is $Z\left\{\dfrac{f(k)}{k}\right\}$. Hence, use the property of division by k.]

We have, $f(k) = \dfrac{\sin \alpha k}{k}, k > 0$ i.e. $k \geq 1$

Now $Z\{\sin \alpha k\} = \dfrac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \quad k \geq 0$

... (Q.11.1) ... Standard result

which is same for the case $k > 0$ i.e. $k \geq 1$ also.

Now, by property of division by k i.e.

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^\infty \frac{F(z)}{z} dz \qquad \text{where } F(z) = Z\{f(k)\}$$

$$\therefore Z\left\{\frac{\sin \alpha k}{k}\right\} = \int_z^\infty \frac{Z\{\sin \alpha k\}}{z} dz$$

$$= \int_z^\infty \frac{1}{z} \frac{z \sin \alpha}{\left(z^2 - 2z \cos \alpha + 1\right)} dz \qquad \text{... using (Q.11.1)}$$

$$= \sin \alpha \int_z^\infty \frac{1}{z^2 - 2z \cos \alpha + \left(\cos^2 \alpha + \sin^2 \alpha\right)} dz \qquad \text{... Note this step}$$

$$= \sin \alpha \int_z^\infty \frac{1}{(z - \cos \alpha)^2 + \sin^2 \alpha} dz$$

$$= \sin \alpha \left[\frac{1}{\sin \alpha} \tan^{-1}\left(\frac{z - \cos \alpha}{\sin \alpha}\right) \right]_z^\infty \qquad \text{... } \because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \tan^{-1} \infty - \tan^{-1}\left(\frac{z - \cos \alpha}{\sin \alpha}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z - \cos \alpha}{\sin \alpha}\right)$$

$$\therefore Z\left\{\frac{\sin \alpha k}{k}\right\} = \cot^{-1}\left(\frac{z - \cos \alpha}{\sin \alpha}\right)$$

4.3 : Inverse Z-transform

1) If $Z[f(k)] = F(z)$ then

$Z^{-1}F(z) = [f(k)]$ determines the sequence $\{f(k)\}$ which generates the given Z-transform is known as inverse Z-transform.

☞ [SPPU : May-99, 2000, 01, 02, 03, 04, 05, 06, 08, May-09, 11, 12, 13, Dec.-99, 2000, 01, 02, 03, 04, 05, 06, Dec.-07, 08, 09, 10, 11, 12]

For example $Z\{a^k\}\ k \geq 0 = \dfrac{z}{z-a}$ for $|z|>|a|$

$$\therefore \quad z^{-1}\left\{\frac{z}{z-a}\right\}_{|z|>|a|} = a^k \text{ for } k \geq 0.$$

Using standard results of Z-transform we can form the following table of inverse Z-transforms.

2) Table of Inverse Z-transform

No.	F (z)	f (k) = $Z^{-1}F(z)$ $(\lvert z\rvert > \lvert a\rvert,\ k \geq 0)$	f (k) = $Z^{-1}F(z)$ $(\lvert z\rvert < \lvert a\rvert,\ k < 0)$
1.	$\frac{z}{z-a}$	a^k	$-a^k$
2.	$\frac{z}{(z-a)^2}$	ka^{k-1}	$-ka^{k-1}$
3.	$\frac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$
4.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(k+1)(k+2)a^k$	$-\frac{1}{2!}(k+1)(k+2)a^k$
5.	In general, $\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}\left[(k+1)(k+2)\ldots(k+(n-1))\right]a^k$	$-\frac{1}{(n-1)!}\left[(k+1)(k+2)\ldots(k+(n-1))\right]a^k$
6.	$\frac{1}{z-a}$	$a^{k-1}U(k-1)$	$-a^{k-1}U(-k)$
7.	$\frac{1}{(z-a)^2}$	$(k-1)a^{k-2}U(k-2)$	$-(k-1)a^{k-2}U(-k+1)$
8.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(k-2)(k-1)a^{k-3}U(k-3)$	$-\frac{1}{2}(k-2)(k-1)a^{k-3}U(-k+2)$
9.	$\frac{z}{z-1}$	U (k)	
10.	$\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert > 1$	$\sin\alpha k$	
11.	$\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert>1$	$\cos\alpha k$	

Q.12 **Find inverse Z-transform of**

$$F(z) = \frac{1}{z^2-z-6};\ |z| > 3$$

[SPPU : May-13]

Ans. : By partial fractions

$$F(z) = \frac{\frac{1}{5}}{z-3} - \frac{\frac{1}{5}}{z+2};\ |z| > 3$$

$$= \frac{1}{5}\frac{1}{z\left(1-\frac{3}{z}\right)} - \frac{1}{5}\frac{1}{z\left(1+\frac{2}{z}\right)};\ 1 > \left|\frac{3}{z}\right| \text{ and } 1 > \left|\frac{2}{z}\right|$$

$$= \frac{1}{5z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\ldots\right] - \frac{1}{5z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\right]$$

$$F(z) = \frac{1}{5}\left[\frac{1}{z}+\frac{3}{z^2}+\frac{3^2}{z^3}+\ldots\right] - \frac{1}{5}\left[\frac{1}{z}-\frac{2}{z^2}+\frac{2^2}{z^3}+\ldots\right]$$

$\therefore$ The coefficient of z^{-k} in first series is $= \frac{1}{5}3^{k-1}$

and the coefficient of z^{-k} in second series is $= -\frac{1}{5}(-1)^{k+1}2^{k-1}$

$$f(k) = \frac{1}{5}\left(3^{k-1}-(-1)^{k+1}\ 2^{k-1}\right)$$

Q.13 Find $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$ if

i) $|z| > \frac{1}{4}$ ☞ [SPPU : May-02, Dec.-14]

ii) $\frac{1}{5} < |z| < \frac{1}{4}$ ☞ [SPPU : Dec.-08, 09, May-12]

iii) $|z| < \frac{1}{5}$ ☞ [SPPU : May-11, 14, Dec.-11]

Ans. : We have

$$F(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$$

Let us use Partial Fractions Method which is more convenient here

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{5}}$$

Partial fractions $A = \dfrac{1/4}{1/4-1/5} = 5$ and

$$B = \frac{1/5}{1/5-1/4} = -4 = \frac{5}{z-\frac{1}{4}} - \frac{4}{z-\frac{1}{5}}$$

$$\therefore \quad F(z) = 5\frac{z}{z-\frac{1}{4}} - 4\frac{z}{z-\frac{1}{5}}$$

$\therefore$ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$

$$= 5Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right) \quad \text{... (Q.13.1)}$$

Case I :

$|z| > \frac{1}{4} \Rightarrow |z| > \frac{1}{5}$

$$\therefore \quad \{f(k)\} = 5\,Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| > \frac{1}{4},\ |z| > \frac{1}{5}$$

$$= 5\left\{\left(\frac{1}{4}\right)^k\right\} - 4\left\{\left(\frac{1}{5}\right)^k\right\}$$

$(k \geq 0)$ $(k \geq 0)$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = a^k,\ k \geq 0 \text{ for } |z| > |a|$$

$$= \left\{5\left(\frac{1}{4}\right)^k - 4\left(\frac{1}{5}\right)^k\right\},\ k \geq 0$$

Case II :

$\frac{1}{5} < |z| < \frac{1}{4}$ $\therefore$ From (Q.13.1) we have

$$\{f(k)\} = 5\,Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| < \frac{1}{4},\ |z| > \frac{1}{5}$$

$$= 5\left\{-\left(\frac{1}{4}\right)^k\right\} - 4\left\{\left(\frac{1}{5}\right)^k\right\}$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = -a^k,\ k < 0$$

$$\text{for } |z| < |a|$$

$a^k,\ k \geq 0$ for $|z| > |a|$

$$= \quad (k<0) \quad (k \geq 0)$$

$$= \left\{-5\left(\frac{1}{4}\right)^k\right\} + \left\{-4\left(\frac{1}{5}\right)^k\right\}$$

$$(k<0) \qquad (k \geq 0)$$

Thus, $$Z^{-1}\left[\frac{z^{-1}}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right] = \{f(k)\}$$

where, $$f(k) = -5\left(\frac{1}{4}\right)^k,\ k<0$$

$$= -4\left(\frac{1}{5}\right)^k,\ k \geq 0$$

Case III :

$$|z| < \frac{1}{5} \Rightarrow |z| < \frac{1}{4}$$

$\therefore$ from (Q.13.1) we have,

$$\{f(k)\} = 5Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| < \frac{1}{4},\ |z| < \frac{1}{5}$$

$$= 5\left\{-\left(\frac{1}{4}\right)^k\right\} - 4\left\{-\left(\frac{1}{5}\right)^k\right\}$$

$$(k<0) \qquad (k<0)$$

$$\ldots Z^{-1}\left(\frac{z}{z-a}\right) = -a^k,\ k<0 \text{ for } |z| < |a|$$

$$= \left\{4\left(\frac{1}{5}\right)^k - 5\left(\frac{1}{4}\right)^k\right\},\ k<0$$

Q.14 **Find Inverse Z-transform of the following**

$$\frac{3z^2+2z}{z^2-3z+2},\ 1<|z|<2$$

[SPPU : Dec.-2000, 02, 03, 04, 06, 10, 12, 15, May-01, 06, 08, 12]

Ans. : We have,

$$F(z) = \frac{3z^2+2z}{z^2-3z+2},\ 1<|z|<2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{z^2-3z+2} = \frac{3z+2}{(z-2)(z-1)}$$

$$= \frac{8}{z-2} + \frac{(-5)}{z-1}$$

Partial Fractions : $A = \frac{3(2)+2}{2-1} = 8$

$$B = \frac{3(1)+2}{1-2} = -5$$

$$\therefore \quad F(z) = 8\left(\frac{z}{z-2}\right) - 5\left(\frac{z}{z-1}\right)$$

$$\therefore \quad \{f(k)\} = 8\,Z^{-1}\left(\frac{z}{z-2}\right) - 5\,Z^{-1}\left(\frac{z}{z-1}\right)$$

$$\ldots |z|<2,\ |z|>1$$

$$= 8\{-2^k\} - 5\{1^k\}$$

$$(k<0) \quad (k \geq 0)$$

$$= -8\{2^k\} - 5\{1\}$$

$$(k<0)\ (k \geq 0)$$

i.e. $$f(k) = -8 \cdot 2^k,\ k<0$$

$$= -5,\ k \geq 0$$

Q.15 Find $Z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right], |z| > 1$

☞ [SPPU : May-99, 02, 14, Dec.-11]

Ans. : We have,

$$F(z) = \frac{z(z+1)}{(z-1)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{(z-1)+2}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2}$$

$$\therefore \quad F(z) = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$\therefore$ Inverting,

$$\{f(k)\} = Z^{-1} F(z)$$

$$= Z^{-1}\left(\frac{z}{z-1}\right) + 2 Z^{-1}\left[\frac{z}{(z-1)^2}\right],$$

$|z| > 1$

$$= \{1^k\} + 2\{k(1)^{k-1}\}, k \geq 0$$

$$\ldots \because Z^{-1}\frac{z}{(z-a)^2} = ka^{k-1}, k \geq 0 \text{ for } |z| > |a|$$

$$\therefore \quad \{f(k)\} = \{1+2k\}, k \geq 0$$

Q.16 Find $Z^{-1}\left[\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right], |z| > 1$

☞ [SPPU : May-2000, 01, 02, 04, 05, 09, 12, Dec.-2000, 02, 04, 12]

Ans. : We have

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} \qquad \ldots \text{(Q.16.1)}$$

Now,

$$\text{let } \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{(z-1)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2} \qquad \ldots \text{(Q.16.2)}$$

... Partial fractions

$\therefore$ Multiplying throughout by the denominator $(z-1)\left(z-\frac{1}{2}\right)^2$ we get

$$z^2 = A\left(z-\frac{1}{2}\right)^2 + B(z-1)\left(z-\frac{1}{2}\right) + C(z-1) \qquad \ldots \text{(Q.16.3)}$$

$\therefore$ Putting $z = 1$, $1 = A\left(\frac{1}{2}\right)^2 + 0 + 0$

$$\therefore \quad 1 = \frac{1}{4}A \quad \therefore A = 4$$

Putting $z = \frac{1}{2}$ in (Q.16.3),

$$\left(\frac{1}{2}\right)^2 = 0 + 0 + C\left(\frac{1}{2} - 1\right)$$

$$\therefore \quad \frac{1}{4} = -\frac{1}{2}C$$

$$\therefore \quad C = -\frac{1}{2}$$

and putting $z = 0$ in (Q.15.3),

$$0 = A\left(-\frac{1}{2}\right)^2 + B(-1)\left(-\frac{1}{2}\right) + C(-1)$$

$$= \frac{A}{4} + \frac{B}{2} - C$$

$$\therefore \quad \frac{B}{2} = C - \frac{A}{4} = -\frac{1}{2} - \frac{4}{4} = -\frac{3}{2}$$

$$\ldots \because C = -\frac{1}{2},\ A = 4$$

$$\therefore \quad B = -3$$

$\therefore$ Substituting for A, B and C in (Q.15.2) we get,

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$= \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{1}{2\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad F(z) = 4\left(\frac{z}{z-1}\right) - 3\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2},$$

$$|z| > 1 \Rightarrow |z| > \frac{1}{2}$$

$\therefore$ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$

i.e. $$\{f(k)\} = 4Z^{-1}\left(\frac{z}{z-1}\right) - 3Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$-\frac{1}{2}Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right],\ |z| > 1,\ |z| > \frac{1}{2}$$

$$= 4\left\{1^k\right\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \frac{1}{2}\left\{k\left(\frac{1}{2}\right)^{k-1}\right\}$$

$$(k \geq 0) \qquad (k \geq 0) \qquad (k \geq 0)$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = \left\{a^k\right\},$$

$$k \geq 0 \text{ for } |z| > |a|$$

$$\text{and } Z^{-1}\left[\frac{z}{(z-a)^2}\right] = ka^{k-1},$$

$$k \geq 0 \text{ for } |z| > |a|$$

$$= 4\{1\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \left\{k\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$= \left\{4 - 3\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$\therefore \quad \{f(k)\} = \left\{4 - (k+3)\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

Q.17 **Find Inverse Z-transform of the following functions :**

$$\frac{z^2}{(z^2+1)},\ |z| > 1$$

☞ **[SPPU : Dec.-01, 02, 04, 06, 12, May-02]**

Ans. : We have,

$$F(z) = \frac{z^2}{z^2+1}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)}$$

... Note this step

$$= \frac{1}{2}\left(\frac{1}{z-i} + \frac{1}{z+i}\right) \quad \text{... Partial fractions}$$

$$\therefore \quad F(z) = \frac{1}{2}\frac{z}{(z-i)} + \frac{1}{2}\frac{z}{[z-(-i)]}$$

$$\text{...} |i| = \sqrt{0+1} = 1, |-i| = 1,$$

$$\therefore |z| > 1 \Rightarrow |z| > |i|, |z| > |-i|$$

$\therefore$ Inverting we get,

$$\{f(k)\} = Z^{-1} F(z)$$

$$= \frac{1}{2} Z^{-1}\left(\frac{z}{z-i}\right) + \frac{1}{2} Z^{-1}\left(\frac{z}{z-(-i)}\right),$$

$$|z| > |i|, |z| > |-i|$$

$$= \frac{1}{2}\{i^k\} + \frac{1}{2}\{(-i)^k\}, k \geq 0$$

$$= \frac{1}{2}\left\{i^k + (-i)^k\right\}, k \geq 0$$

Now, $\because$ $\quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $\quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

$$\therefore \quad \{f(k)\} = \frac{1}{2}\left\{e^{\frac{i\pi k}{2}} + e^{-\frac{i\pi k}{2}}\right\}$$

$$= \frac{1}{2}\left\{\left(2\cos\frac{\pi k}{2}\right)\right\} = \left\{\cos\frac{k\pi}{2}\right\}, k \geq 0$$

Alternately, we can write

$$F(z) = \frac{z^2}{z^2+1} = \frac{z(z-0)}{z^2-2z(0)+1} \quad \text{... Note this step}$$

$$= \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\left(\cos\frac{\pi}{2}\right)+1}, \quad \text{...} \because \cos\frac{\pi}{2} = 0$$

... Note this step

$$\therefore \quad \{f(k)\} = Z^{-1}\left[\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1}\right], |z| > 1$$

$$= \cos\left(\frac{\pi}{2}k\right), k \geq 0$$

$$\text{...} \because Z^{-1}\left[\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}\right] = \cos\alpha k, k \geq 0 \text{ for } |z| > 1$$

Type 1 (b) If linear real factorisation of denominator is not possible.

Q.18 Show that $Z^{-1}\left[\frac{z+2}{z^2-2z+1}\right] = \{x_k\}$ **for** $|z| > 1$ **where**

$$x_k = 0, k < 1$$
$$= 3k-1, k \geq 1$$

☞ **[SPPU : Dec.-15, May-03]**

Ans. : We have,

$$X(z) = \frac{z+2}{z^2-2z+1}, |z| > 1$$

$$= \frac{(z-1)+3}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{3}{(z-1)^2}, |z| > 1$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{z^2\left(1-\frac{1}{z}\right)^2}$$

$$\because |z| > 1 \therefore \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{3}{z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\frac{1^2}{z^2}+\frac{1^3}{z^3}+\dots\right]$$

$$+ \frac{3}{z^2}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\dots\right]$$

$$= \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right]+3\left[\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\dots\right]$$

$$\therefore \quad X(z) = \sum_{k=1}^{\infty}\frac{1}{z^k} + 3\sum_{k=1}^{\infty}\frac{k}{z^{k+1}}$$

$$= \sum_{k=1}^{\infty} z^{-k} + \sum_{k=1}^{\infty} 3k\,z^{-(k+1)}$$

$\therefore$ coefficient of z^{-k} in 1st series = 1

$\therefore \quad f_1(k) = 1, k \geq 1$

coefficient of $z^{-(k+1)}$ in 2nd series = $3k, k \geq 1$

Put $(k+1 = n) \therefore k = n-1$

$\therefore$ coefficient of z^{-n} in 2nd series = $3(n-1), n-1 \geq 1$ i.e. $n \geq 2$

$\therefore$ coefficient of z^{-k} in 2nd series = $3(k-1), k \geq 2$,

... Replacing n by k

$\therefore \quad f_2(k) = 3k-3, k \geq 2$

$\therefore$ Combining we get

$$Z^{-1}X(z) = \{x_k\} = \{f_1(k)\} + \{f_2(k)\}$$

$$= \{1\} + \{3k-3\}$$

$$(k \geq 1) \quad (k \geq 2)$$

$\therefore \quad x_k = 0, k < 1$

$= 1, k = 1$

$= 1 + (3k-3) = 3k-2, k \geq 2$

Thus, $\quad x_k = 0, k < 1$

$= 3k-2, k \geq 1 \quad$ ... $\because$ at $k = 1$, $3(k)-2$

$= 3-2 = 1$

4.4 Working Rule for Finding the Poles and Residues

a) To find the poles of the function : $f(z) = \frac{\phi(z)}{\psi(z)}$:

1. Consider the equation (from the denominator) $\psi(z) = 0$. Its solution, say, $z = a$, $z = b$ ... etc. gives the poles of $f(z)$.
2. If $(z = a)$ is not a repeated root of $\psi(z) = 0$, then $(z = a)$ is a simple pole of $f(z)$.
3. If $(z = b)$ is two times repeated root of $\psi(z) = 0$, then $(z = b)$ is a double pole i.e. a pole of order $(n = 2)$ of $f(z)$. Similarly, if $(z = b)$ is repeated thrice, it is a pole of order $(n = 3)$.

b) To find residue of f(z) at simple pole (z = a) :

1. It is given by,

$$\boxed{\text{Res } f(a) = \lim_{z \to a} [(z-a)f(z)],}$$ which is a non-zero finite value.

2. Sometimes we use the following result :

If, $f(z) = \frac{\phi(z)}{\psi(z)}$ and $\psi(z) = (z-a)\,F(z)$ (say) where $F(a) \neq 0$,

then

$$\boxed{\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)} = \left[\frac{\phi(z)}{\frac{d}{dz}\psi(z)}\right]_{z=a}}$$

c) To find the residue of f(z) at the pole (z = b) of order n :

1. It is given by

$$\boxed{\text{Res } f(b) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-b)^n f(z)\right]\right\}_{z=b}}$$

Thus if (z = b) is a double pole i.e. n = 2 then,

$$\text{Res } f(b) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[(z-b)^2 f(z)\right]\right\}_{z=b}$$

$$= \left\{\frac{d}{dz}\left[(z-b)^2 f(z)\right]\right\}_{z=b}$$

and here, $\lim_{z \to b} [(z-b)^2 f(z)]$ is non-zero and finite

and if (z = b) is a pole of order n = 3 then

$$\text{Res } f(b) = \frac{1}{2!}\left\{\frac{d^2}{dz^2}[(z-b)^3 f(z)]\right\}_{z=b}$$

and here $\lim_{z \to b} [(z-b)^3 f(z)]$ is non-zero and finite.

d) Inversion Integral Method by using Residues

☞ [SPPU : May-2000, 08, 09, 11, 12, 13, Dec.-03, 04, 05, 07, 09, 10, 11]

The Inverse Z-transform of F(z) can be easily obtained by using the Inversion Integral of F (z), given by

$$f(k) = \frac{1}{2\pi i}\int_C F(z) \cdot z^{k-1}\, dz$$

where, C is the closed curve (contour) (drawn according to the given ROC) such that all the poles of F(z) (i.e. the values of z where F(z) becomes infinite) lie inside it.

More, conveniently we have,

$f(k) = \sum$ Residues of $[F(z) \cdot z^{k-1}]$ at the poles of F(z).

e) Working Rule for using Inversion Integral Method

Step 1 : Find the poles of F(z).

Step 2 : Find the expression for $F(z) \cdot z^{k-1}$

Step 3 : Find the residues of $\left[F(z)\, z^{k-1}\right]$ at all the poles of F (z) using proper formulae.

Step 4 : Take algebraic sum of the residues to get f(k).

Q.19 Use Inversion Integral Method to find $Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$

☞ [SPPU : May-11, 12, Dec.-11, 14, 15]

Ans. : To find poles of F(z) :

We have, $$F(z) = \frac{10z}{(z-1)(z-2)} \quad \text{... (Q.19.1)}$$

$\therefore F(z) \to \infty$ at z = 1, z = 2 $\therefore$ (z = 1) and (z = 2) are simple poles of F(z)

To find $F(z) \cdot z^{k-1}$

Now, using F(z) we have,

$$F(z) \cdot z^{k-1} = \frac{10z}{(z-1)(z-2)} z^{k-1} = 10\frac{z^k}{(z-1)(z-2)}$$

Find Residues of $F(z) \cdot z^{k-1}$ at poles

$\because$ (z = 1) and (z = 2) are simple poles

$\therefore$ Residue of $\left[F(z) \cdot z^{k-1}\right]$ at (z = 1)

$$= \left[(z-1)\, F(z)\, z^{k-1}\right]_{z=1}$$

$$= \left[(z-1)\frac{10z^k}{(z-1)(z-2)}\right]_{z=1}$$

$$= 10\left[\frac{z^k}{(z-2)}\right]_{z=1} = 10\left(\frac{1}{1-2}\right)$$

$\therefore$ Residue at (z = 1) = – 10 ... (Q.19.2)

Similarly,

$$\text{Residue at } (z=2) = \left[(z-2)\frac{10\,z^k}{(z-1)(z-2)}\right]_{z=2}$$

$$= 10\left[\frac{z^k}{z-1}\right] = 10\cdot\frac{2^k}{(2-1)}$$

$$= 10\cdot 2^k \qquad \text{... (Q.19.3)}$$

To take algebraic sum of all residues.

∴ By Inversion integral method.

$$Z^{-1}F(z) = \{f(k)\} \text{ where}$$

$$f(k) = \sum \text{Residues of } F(z)\, z^{k-1}$$

$$= \text{Res}\,(z=1) + \text{Res}\,(z=2)$$

$$= -10 + 10\cdot 2^k,\ k \geq 0$$

... using (Q.19.2) and (Q.19.3)

$$= 10\left(2^k - 1\right),\ k \geq 0$$

Q.20 Find inverse Z-transform for

$$F(z) = \frac{z}{(z-2)(z+4)^2}$$ **by inversion integral method.**

☞ [SPPU : May-13]

Ans. : To find poles of F(z)

We have $F(z) = \dfrac{z}{(z-2)(z+4)^2}$

∴ $F(z) \to \infty$ at $z = 2$ and $z = -4$

Hence $z = 2$ is a simple pole and $z = -4$ is a double pole of F(z).

We have $F(z)z^{k-1} = \dfrac{z\,z^{k-1}}{(z-2)(z+4)^2} = \dfrac{z^k}{(z-2)(z+4)^2}$

Now, find residue of $F(z)\,z^{k-1}$ at poles.

So residue of $F(z)\,z^{k-1}$ at $z = 2$ is

$$= [(z-2)F(z)z^{k-1}]_{z=2} = \left[(z-2)\frac{z^k}{(z-2)(z+4)^2}\right]_{z=2}$$

$$= \frac{1}{36}2^k;\ k = 0.$$

Now, we have

Residue of $[F(z)z^{k-1}]$ at $z = -4$ is

$$= \frac{1}{(2-1)!}\left[\frac{d}{dz}(z+4)^2 F(z)z^{k-1}\right]_{z=-4}$$

$$= \left[\frac{d}{dz}\frac{z^k}{z-2}\right]_{z=-4} = \left[\frac{(z-2)\,k - z^{k-1}z^k}{(z-2)^2}\right]_{z=-4}$$

$$= \frac{1}{36}[-6k\,(-4)^k - (-4)^{k-1}]$$

Thus, $f(k) = \dfrac{1}{36}[2^k - 6k(-4)^{k-1} - (-4)^k]$

Q.21 Use Inversion Integral Method to find

$$Z^{-1}\left[\frac{z^3}{(z-3)\,(z-2)^2}\right]$$

☞ [SPPU : Dec.-07, 11, May-13]

Ans. :

Step 1 : To find poles of F(z) :

We have, $F(z) = \dfrac{z^3}{(z-3)\,(z-2)^2}$

∴ $F(z) \to \infty$ at $z = 3$ and $z = 2$ which is repeated $(r = 2)$ times.

Hence, $z = 3$ is a simple pole of F(z) and $z = 2$ is a double pole of F(z).

Step 2 : To find expression for $F(z)\cdot z^{k-1}$:

Now, we have,

$$F(z)\cdot z^{k-1} = \frac{z^3}{(z-3)\,(z-2)^2}\cdot z^{k-1} = \frac{z^{k+2}}{(z-3)\,(z-2)^2} \qquad \text{... (Q.21.1)}$$

Step 3 : To find Residues of $F(z) \cdot z^{k-1}$ at the poles of $F(z)$:

Now, for the simple pole at $(z = 3)$ we have,

Residue of $\left[F(z) \cdot z^{k-1}\right]$ at $(z = 3)$

$$= \left[(z-3)\,F(z) \cdot z^{k-1}\right]_{z=3} \quad \text{... } \because \text{Res } (z = a)$$

$$= \left[(z-a)\,F(z)\,z^{k-1}\right]_{z=a} \text{ for a simple pole}$$

$$= \left[(z-3)\,\frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]_{z=3} \quad \text{... using (Q.21.1)}$$

$$= \left[\frac{z^{k+2}}{(z-2)^2}\right]_{z=3} = \frac{3^{k+2}}{(3-2)^2} = 3^{k+2}$$

$\therefore$ Res $(z = a) = 3^{k+2}, k \geq 0$ ... (Q.21.2)

Also, for the double (i.e. $r = 2$) pole at $(z = 2)$ we have,

Residue of $\left[F(z) \cdot z^{k-1}\right]$ at $(z = 2)$

$$= \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left[(z-2)^2\,F(z) \cdot z^{k-1}\right]\right]_{z=2} \quad \text{... } \because \text{Res } (z = a)$$

$$= \frac{1}{(r-1)!}\left[\frac{d^{r-1}}{dz^{r-1}}\left[(z-a)^r F(z) \cdot z^{k-1}\right]\right]_{z=a}$$

for r times repeated pole

$$= \frac{1}{1}\left[\frac{d}{dz}\left[(z-2)^2\,\frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]\right]_{z=2} \quad \text{... using (Q.21.1)}$$

$$= \left[\frac{d}{dz}\left(\frac{z^{k+2}}{z-3}\right)\right]_{z=2}$$

$$= \left[\frac{(z-3)\,(k+2)\,z^{k+1} - z^{k+2}\,(1)}{(z-3)^2}\right]_{z=2}$$

$$= \frac{(2-3)\,(k+2) \cdot 2^{k+1} - 2^{k+2}}{(2-3)^2} = -(k+2)\,2^{k+1} - 2^{k+2}$$

$$= -(k+2)\,2^{k+1} - 2 \cdot 2^{k+1} = -\left[(k+2)+2\right]2^{k+1}$$

$\therefore$ Residue $(z = 2) = -(k+4)\,2^{k+1}, k \geq 0$ ... (Q.21.3)

Step 4 : Take algebraic sum of all the Residues.

Thus, using (Q.21.2) and (Q.21.3) we have, by inversion integral method

$$Z^{-1}[F(z)] = \{f(k)\}$$

where, $f(k) = \sum$ Residues of $\left[F(z)\,z^{k-1}\right]$ at poles of $F(z)$

$$= 3^{k+2} - (k+4)\,2^{k+1}, k \geq 0$$

Q.22 Find Inverse Z-transform using Inversion Integral Method,

$$f(z) = \frac{z^2}{z^2+1}$$

☞ [SPPU : Dec.03, 04, 05, 09, 10, 12, 15, May-08, 10]

Ans. : We have,

$$F(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z-i)\,(z+i)} \quad \textbf{... Note this step}$$

$\therefore$ F (z) has simple poles at $(z = i)$ and $(z = -i)$

$$\text{Now, } F(z)\,z^{k-1} = \frac{z^2 \cdot z^{k-1}}{(z-i)\,(z+i)}$$

$$= \frac{z^{k+1}}{(z-i)\,(z+i)} \quad \text{... (Q.22.1)}$$

$\therefore$ Residue of $[F(z)\,z^{k-1}]$ at $(z = i)$

$$= [(z-i)\,F(z)\,z^{k-1}]_{z=i}$$

$$= \left[\frac{z^{k+1}}{z+i}\right]_{z=i} \quad \text{... using (Q.22.1)}$$

$$= \frac{(i)^{k+1}}{i+i} = \frac{1}{2i}(i)^{k+1} = \frac{(i)^k}{2}$$

and Residue at $(z = -i)$

$$= [[z-(-i)]\,F(z)\,z^{k-1}]_{z=-i}$$

$$= \left[\frac{z^{k+1}}{z-i}\right]_{z=-i}$$

$$= \frac{(-i)^{k+1}}{-i-i} = \frac{(-i)^{k+1}}{2(-i)} = \frac{(-i)^k}{2}$$

$$\therefore Z^{-1}\left(\frac{z^2}{z^2+1}\right) = \{f(k)\}$$

where, $f(k) = \sum \text{Residues}$

$$= \frac{(i)^k}{2} + \frac{(-i)^k}{2}$$

$$= \frac{1}{2}[(i)^k + (-i)^k]$$

Now, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $-i = e^{-i\pi/2}$

$$\therefore (i)^k + (-i)^k = e^{i\frac{k\pi}{2}} + e^{-i\frac{k\pi}{2}} = 2\cos\frac{k\pi}{2}$$

$$\text{... } \because e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \quad f(k) = \frac{1}{2}\left(2\cos\frac{k\pi}{2}\right) = \cos\frac{k\pi}{2},\ k \geq 0$$

4.5 : Solutions of Simple Difference Equations (with constant coefficients) using Z-transforms

[SPPU : Dec.-2000, 03, 04, 05, 06, 07, 08, 09, 10, 11, 12, May-01, 02, 03, 05, 06, 07, 08, 09, 10, 12, 13]

1)

$$Z\{f(k)\} = F(z)$$
$$Z\{f(k+1)\} = z\,F(z) - z\,f(0)$$
$$Z\{f(k+2)\} = z^2\,F(z) - z^2\,f(0) - z\,f(1) \text{ etc.}$$
$$Z\{f(k-1)\} = z^{-1}\,F(z)$$
$$Z\{f(k-2)\} = z^{-2}\,F(z) \text{ etc.}$$

Q.23 **Solve, the following difference equation to find $\{f(k)\}$:**
$f(k+2) - 3f(k+1) + 2f(k) = 0,\ f(0) = 0,\ f(1) = 1$

[SPPU : May-07, 09, 12, 15, Dec.-03, 05]

Ans. : We have the difference equation

$$f(k+2) - 3f(k+1) + 2f(k) = 0$$

Step 1 : Take the Z-transform of both the sides

$$\therefore \quad Z\{f(k+2)\} - 3Z\{f(k+1)\} + 2\{f(k)\} = 0$$

i.e. $\left[z^2 F(z) - z^2 f(0) - z\,f(1)\right]$

$$- 3\left[z\,F(z) - z\,f(0)\right] + 2F(z) = 0$$

where $F(z) = Z\{f(k)\}$ ...using Standard results

i.e.

$$\left[z^2 F(z) - 0 - z(1)\right] - 3\left[z\,F(z) - 0\right] + 2F(z) = 0$$

$$\text{... } \because f(0) = 0,\ f(1) = 1 \text{ (given)}$$

Step 2 : Rearrange this algebraic equation to find F(z) :

Thus, $\left(z^2 - 3z + 2\right) F(z) - z = 0$

$$\therefore \quad F(z) = \frac{z}{z^2 - 3z + 2}$$

Step 3 : Find $Z^{-1}F(z)$:

Now, $$F(z) = \frac{z}{z^2 - 3z + 2}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} \qquad \text{... Partial fractions}$$

$$\therefore \quad F(z) = \frac{z}{z-1} - \frac{z}{z-2}$$

$\therefore$ Inverting we get,

$$\{f(k)\} = Z^{-1}F(z)$$

$$= Z^{-1}\left[\frac{z}{z-(1)}\right] - Z^{-1}\left[\frac{z}{z-(2)}\right]$$

$$= \{(+1)^k\} - \{(2)^k\},\ k \ge 0,$$

... assuming $|z| > 2$ i.e. $|z| > |-2|$ and

$\therefore |z| > |-1|$ and $\because Z^{-1}\left(\frac{z}{z-a}\right) = a^k,\ k \ge 0$ for $|z| > |a|$

$$= \{(1)^k - (2)^k\},\ k \ge 0$$

$$\therefore \quad f(k) = (1)^k - (2)^k,\ k \ge 0$$

$$\therefore \quad f(k) = 1^k - 2^k,\ k \ge 0$$

Q.24 Obtain $f(k)$, given that

$$f(k+1) + \frac{1}{4}f(k) = \left(\frac{1}{4}\right)^k,\ k \ge 0,\ f(0) = 0$$

☞ [SPPU : May-08, 11, 12, 13, Dec.-15]

Ans. : We have, the difference equation :

$$f(k+1) + \frac{1}{4}f(k) = \left(\frac{1}{4}\right)^k,\ k \ge 0$$

Step 1 : Taking Z-transform of both the sides, we get

$$Z\{f(k+1)\} + \frac{1}{4}Z\{f(k)\} = Z\left\{\frac{1}{4}\right\}^k$$

$$\text{i.e. } [zF(z) - zf(0)] + \frac{1}{4}F(z) = \frac{z}{z-\frac{1}{4}},\ |z| > \frac{1}{4}$$

... using Standard results

$$\text{i.e. } [zF(z) - 0] + \frac{1}{4}F(z) = \frac{z}{z-\frac{1}{4}} \qquad \text{... } \because f(0) = 0 \text{ (given)}$$

Step 2 : To find F(z) :

$$\therefore \quad \left(z+\frac{1}{4}\right)F(z) = \frac{z}{z-\frac{1}{4}}$$

$$\therefore \quad F(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)} \qquad \text{... } |z| > \frac{1}{4}$$

Step 3 : To find $Z^{-1}F(z)$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$$

$$= \frac{\frac{1}{2}}{\left(z-\frac{1}{4}\right)} + \frac{\left(-\frac{1}{2}\right)}{\left(z+\frac{1}{4}\right)} \qquad \text{... Partial fractions}$$

$$\therefore \quad F(z) = \frac{1}{2}\frac{z}{\left(z-\frac{1}{4}\right)} - \frac{1}{2}\frac{z}{\left(z+\frac{1}{4}\right)}$$

$\therefore$ Inverting,

$$Z^{-1}F(z) = \{f(k)\} = \frac{1}{2}Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

$$\text{... } |z| > \frac{1}{4} \text{ i.e. } |z| > \left|-\frac{1}{4}\right|$$

Step 3 : Find $Z^{-1}F(z)$:

Now, $F(z) = \dfrac{z}{z^2 - 3z + 2}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{(z-1)(z-2)}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ ... Partial fractions

$\therefore \quad F(z) = \dfrac{z}{z-1} - \dfrac{z}{z-2}$

$\therefore$ Inverting we get,

$$\{f(k)\} = Z^{-1}F(z)$$

$$= Z^{-1}\left[\frac{z}{z-(1)}\right] - Z^{-1}\left[\frac{z}{z-(2)}\right]$$

$$= \{(+1)^k\} - \{(2)^k\},\ k \geq 0,$$

... assuming $|z| > 2$ i.e. $|z| > |-2|$ and

$\therefore |z| > |-1|$ and $\because Z^{-1}\left(\dfrac{z}{z-a}\right) = a^k,\ k \geq 0$ for $|z| > |a|$

$$= \{(1)^k - (2)^k\},\ k \geq 0$$

$\therefore \quad f(k) = (1)^k - (2)^k,\ k \geq 0$

$\therefore \quad f(k) = 1^k - 2^k,\ k \geq 0$

Q.24 Obtain $f(k)$, given that

$$f(k+1) + \frac{1}{4}f(k) = \left(\frac{1}{4}\right)^k,\ k \geq 0,\ f(0) = 0$$

☞ [SPPU : May-08, 11, 12, 13, Dec.-15]

Ans. : We have, the difference equation :

$$f(k+1) + \frac{1}{4}f(k) = \left(\frac{1}{4}\right)^k,\ k \geq 0$$

Step 1 : Taking Z-transform of both the sides, we get

$$Z\{f(k+1)\} + \frac{1}{4}Z\{f(k)\} = Z\left\{\frac{1}{4}\right\}^k$$

i.e. $\left[zF(z) - zf(0)\right] + \dfrac{1}{4}F(z) = \dfrac{z}{z-\frac{1}{4}},\ |z| > \dfrac{1}{4}$ ... using Standard results

i.e. $\left[zF(z) - 0\right] + \dfrac{1}{4}F(z) = \dfrac{z}{z-\frac{1}{4}}$ ... $\because f(0) = 0$ (given)

Step 2 : To find F(z) :

$\therefore \left(z+\dfrac{1}{4}\right)F(z) = \dfrac{z}{z-\frac{1}{4}}$

$\therefore \quad F(z) = \dfrac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$ ... $|z| > \dfrac{1}{4}$

Step 3 : To find $Z^{-1}F(z)$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$

$$= \frac{\frac{1}{2}}{\left(z-\frac{1}{4}\right)} + \frac{\left(-\frac{1}{2}\right)}{\left(z+\frac{1}{4}\right)}$$ ... Partial fractions

$\therefore \quad F(z) = \dfrac{1}{2}\dfrac{z}{\left(z-\frac{1}{4}\right)} - \dfrac{1}{2}\dfrac{z}{\left(z+\frac{1}{4}\right)}$

$\therefore$ Inverting,

$$Z^{-1}F(z) = \{f(k)\} = \frac{1}{2}Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

... $|z| > \dfrac{1}{4}$ i.e. $|z| > \left|-\dfrac{1}{4}\right|$

$$\therefore \quad \{f(k)\} = \frac{1}{2}\left\{\left(\frac{1}{4}\right)^k\right\} - \frac{1}{2}\left\{\left(-\frac{1}{4}\right)^k\right\}, \; k \geq 0$$

$$\therefore \quad f(k) = \frac{1}{2}\left[\left(\frac{1}{4}\right)^k - \left(-\frac{1}{4}\right)^k\right], \; k \geq 0$$

Q.25 Obtain f(k) given that, $f(k+2) + 5f(k+1) + 6f(k) =$ ... $k \geq 0$, $f(0) = 0$, $f(1) = 2$ by using Z transform.

☞ [SPPU : May-14]

Ans. : Given $f(k+2) + 5f(k+1) + 6f(k) = 0$

$$z\{f(k+2)\} + 5z\{f(k+1)\} + 6z\{f(k)\} = 0$$

$$\left[z^2 f(z) - z^2 f(0) - z f(1)\right] + 5\left[z f(1) - z f(0)\right] + 6 f(z) = 0$$

$$\left[z^2 f(z) - 0 - 2z\right] + 5\left[z f(z) - 0\right] + 6 f(z) = 0$$

$$[z^2 + 5z + 6]\, f(z) - 2z = 0$$

$$f(z) = \frac{2z}{z^2 + 5z + 6}$$

$$\frac{f(z)}{z} = \frac{2}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$A = 2 \quad B = -2$$

$$\therefore \quad \frac{f(z)}{z} = \frac{2}{(z+2)} - \frac{2}{(z+3)}$$

$$f(z) = \frac{2z}{(z+2)} - \frac{2z}{(z+3)}$$

$$z^{-1} f(z) = 2z^{-1}\left\{\frac{z}{z+2}\right\} - 2z^{-1}\left\{\frac{z}{z+3}\right\}$$

$$f(k) = 2\{(-2)^k\} - 2\{(-3)^k\}, \; k \geq 0$$

□□□

5 Interpolation, Numerical Differentiation and Integration

5.1 : Finite Differences

1. Forward Differences

The first forward difference is defined by $\Delta f(x) = f(x+h) - f(x)$ where Δ is called the forward or descending difference operator. (Δ = Delta)

In particular $\Delta f(x_0) = f(x_0 + h) - f(x_0)$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta f(x_0 + h) = f(x_0 + 2h) - f(x_0 + h)$$

$$\Delta y_1 = y_2 - y_1$$

....

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The differences of first forward differences are called the second forward differences.

Thus

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

In particular

$$\Delta^2 f(x_1) = f(x_1 + 2h) - 2f(x_1 + h) + f(x_1)$$

i.e. $\quad \Delta^2 y_1 = y_3 - 2y_2 + y_1$

and $\quad \Delta^2 y_2 = y_4 - 2y_3 + y_2$

The third forward difference is given by

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x)) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$