Numerical differentiation -

Newton's forward difference interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y = y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0 + \cdots$$

Diff. w. r. to x

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \text{where } u = \frac{x-x_0}{h}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \cdots\right]$$

Diff. w. r. to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \cdots\right]$$

Newton's backward difference interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

Diff. w. r. to x

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \text{where } u = \frac{x - x_n}{h}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2u+1)}{2!}\nabla^2 y_n + \frac{(3u^2+6u+2)}{3!}\nabla^3 y_n + \cdots\right]$$

Diff. w. r. to x

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{(6u+6)}{6}\nabla^3 y_n + \cdots\right]$$

Numerical Integration -

Let $f(x)$ be continuous on the interval $[a, b]$ and its antiderivative $F(x)$ is known. Then the definite integral of $f(x)$ from a to b may be evaluated using Newton-Leibnitz formula,

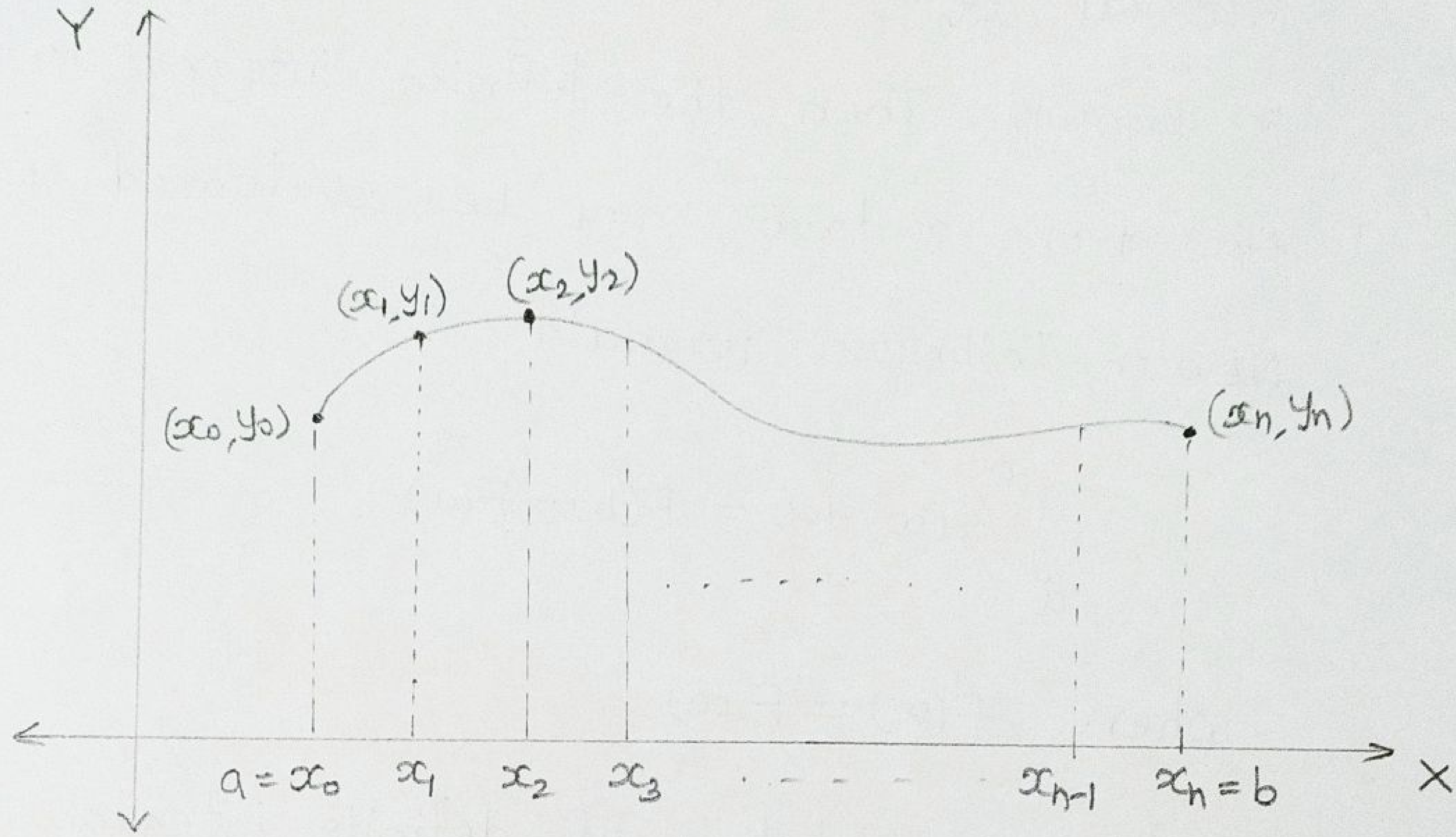
$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (1)$$

where $f'(x) = F(x)$

However, computation of definite integral by (1) becomes difficult or practically impossible when the antiderivative $F(x)$ cannot be found by elementary means or when the integrand $f(x)$ is specified in tabular form.

Trapezoidal Rule:

Let $I = \int_a^b f(x)\,dx$



Divide $[a, b]$ into n equal parts

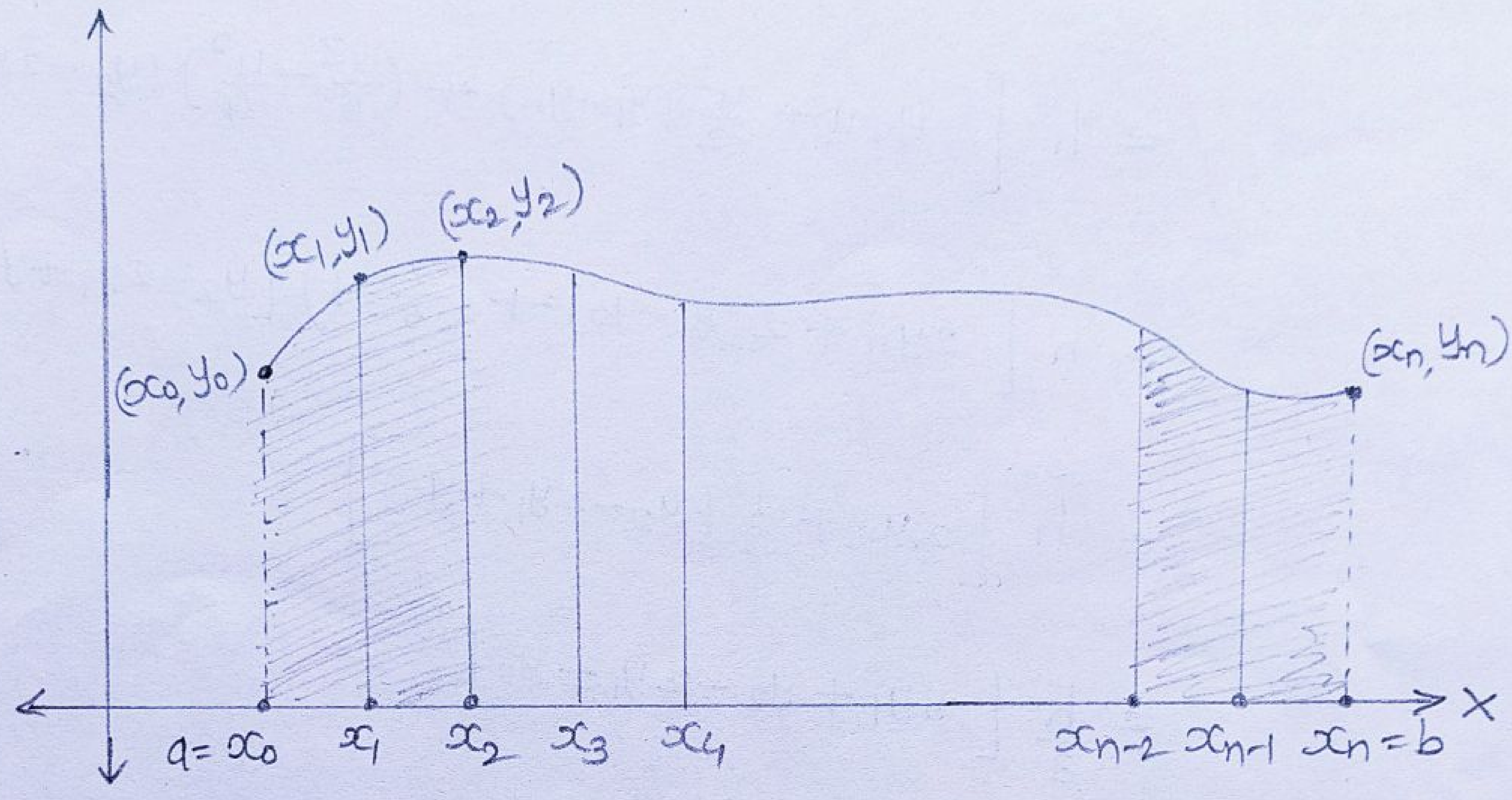
Let us say $h = \dfrac{b-a}{n}$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]}$$

[Here there is no restriction on the number of subintervals]

Simpson's (1/3)rd Rule -

Let $I = \int_a^b f(x)\,dx$



Divide $[a, b]$ into n equal roots

$\therefore\ h = \dfrac{b-a}{n}$ [here n has to be multiple of 2]

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\Big]}$$

Area A_1 of the double strip is

$$A_1 = \int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2} y\,\frac{dx}{du}\,du$$

when $x = x_0$ $\quad u = 0$

$x = x_0$ $\quad u = 2$

$$\therefore \quad A_1 = \int_0^2 \left[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\,\Delta^2 y_0 \right] h\,du$$

$$= h\left[y_0 u + \frac{u^2}{2}(y_1 - y_0) + \left(\frac{u^3}{6} - \frac{u^2}{4}\right)(y_2 - 2y_1 + y_0) \right]_0^2$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{8}{6} - 1\right)(y_2 - 2y_1 + y_0) \right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0) \right]$$

$$= \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0 \right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$

Similarly you can show that

$$A_2 = \int_{x_2}^{x_4} y\,dx$$

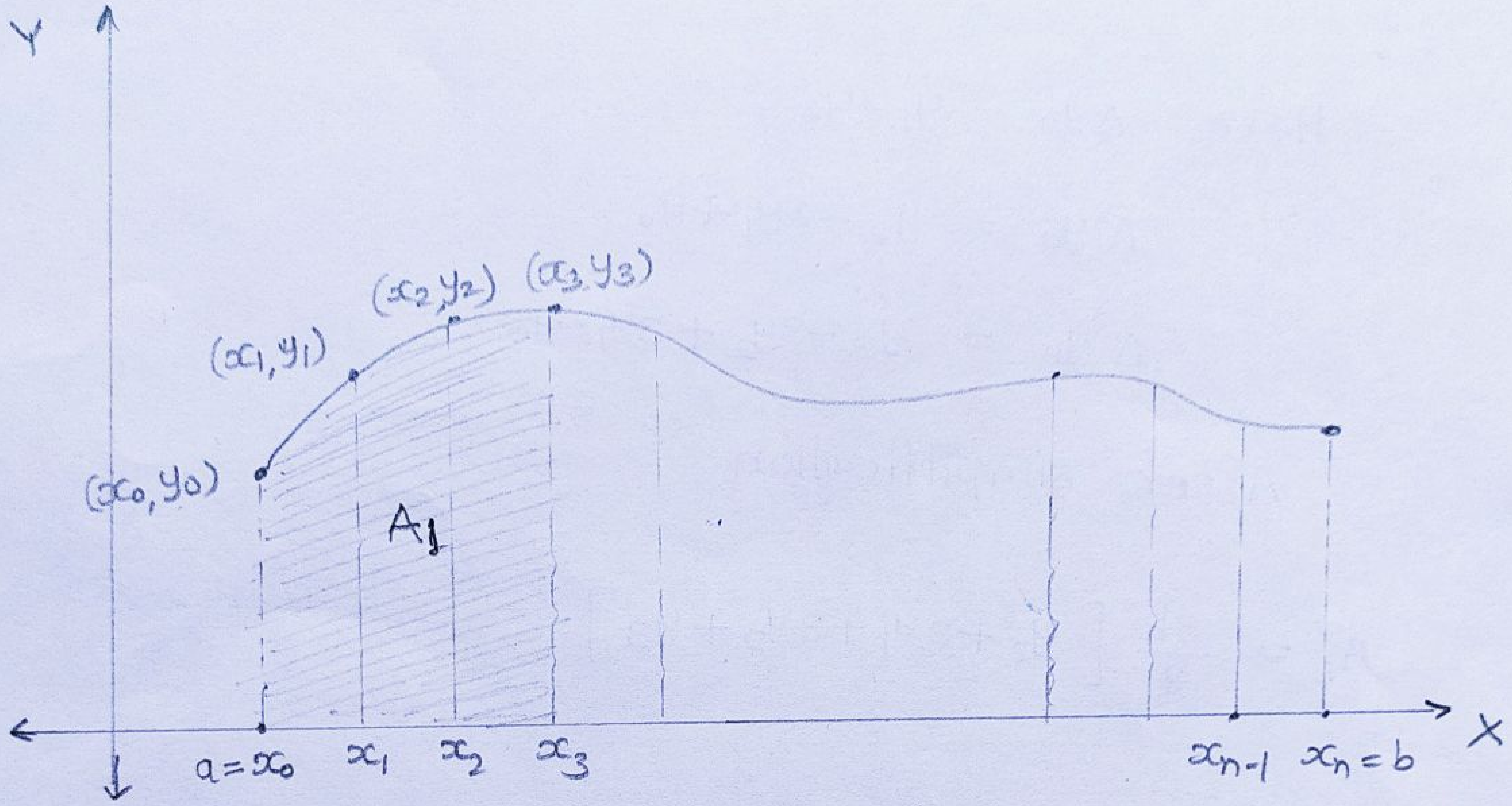
$$= \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right]$$

$\vdots$

$$A_n = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$

Simpson's $(\frac{3}{8})$th Rule -

Let $I = \int_a^b f(x)\,dx$



Divide $[a, b]$ into n equal parts

$\therefore h = \frac{b-a}{n}$ [Here n has to be multiple of 3]

$$\boxed{\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1} + y_{n-2}) + 2(y_3 + y_6 + y_9 + \dots + y_{n-3})\right]}$$

Area A_1 of the casecutive first 3 strips is

$$A_1 = \int_{x_0}^{x_3} y\,dx = \int_{x_0}^{x_3} y \cdot \frac{dx}{du}\,du$$

when $x = x_0 \quad u = 0, \quad x = x_3 \quad u = 3$

$$\therefore A_1 = \int_0^3 \left[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \right] h\,du$$

Here $\Delta y_0 = y_1 - y_0$

$\Delta^2 y_0 = y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

After simplification

$$A_1 = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3 \right]$$

Similarly

$$A_2 = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6 \right]$$

$\vdots$

$$A_n = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n \right]$$