Solution of ordinary differential Eq^n -

1) Euler's method

2) Modified Euler's method

3) Runge- Kutta method of 4th order

4) Predictor - Corrector method.

Analytical solutions can be obtained only for selected class of ODE & PDE using series solution, Laplace transform, Fourier transform, separation of variable technique etc.

For certain problems, analytical solutions can not be obtained. However numerical (methods) solutions can be obtained to the desired level of accuracy using computers. The advantage is that numerical solutions can be obtained for problems involving irregularly shaped boundaries and is easy to program on computer. The only series disadvantage with these numerical solutions is that they lack the generality of the analytical solutions. When the initial conditions are changed, the numerical solution must be calculated again.

Euler's Method -

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

which is to be solved, subject to initial condition $x = x_0$, $y = y_0$.

Suppose we want to obtain the value of y at $x = x_1 = x_0 + h$

Now given de can be written as

$$dy = f(x, y)\,dx$$

Integration given

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\,dx$$

We assume that for the interval (x_0, x_1)

$f(x, y)$ remains stationary as $f(x_0, y_0)$

$$\therefore \ y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

$$= h f(x_0, y_0)$$

$$\therefore \ y_1 = y_0 + h f(x_0, y_0)$$

Similarly,

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$\vdots$$

$$y_{n+1} = y_n + h f(x_n, y_n).$$

Geometrical Interpretation :

Ex. 1) Use Euler's method to solve the equation

$$\frac{dy}{dx} = 1 + xy$$

subject to the condition at $x = 0$, $y = 0$. and tabulate y for $x = 0.5$ (by taking $h = 0.1$)

Soln:

x	y
0	1
0.1	1.1
0.2	1.211
0.3	1.3352
0.4	1.4753
0.5	1.6343.

Modified Euler's Method -

In modification of Euler's method while integrating

$$dy = f(x, y)\,dx$$

$f(x, y)$ is replaced by $\dfrac{f(x_0, y_0) + f(x_1, y_1)}{2}$ instead of $f(x_0, y_0)$.

$f(x_0, y_0)$ can be computed from known initial condition. Initially y_1 is computed by using Euler's formula.

$y_1 = y_0 + h f(x_0, y_0)$ and then $f(x_1, y_1)$ is computed.

The first modification of y_1 is obtained from the formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

where

$$y_1^{(0)} = y_1$$

Second modification is given by.

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

The nth modification is given by

$$y_1^{(n)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n-1)}) \right]$$

The procedure is terminated when successive modification do not show any change at prefixed decimal place, depending upon the desired accuracy of the result.

Ex. Use modified Euler's method to solve $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ to calculate $y(0.4)$ taking $h = 0.2$.

Ans : $y\big|_{x=0.2} \doteq 0.8481$

$y\big|_{x=0.4} = 0.7760$

Sol^n: We have $\frac{dy}{dx} = x-y^2$ $y(0)=1$

Here $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

$$f(x,y) = x-y^2$$

1) At first, we find y_1 by Euler's formula,

i.e., $y_1 = y_0 + hf(x_0, y_0)$

$= 1 + (0.2)(-1)$

$= 0.8$

By applying modified Euler's method,

First modification:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 1 + 0.1\left[-1 - 0.44\right] = 0.8560$$

Second modification:

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.1\left[-1 - 0.5327\right] = 0.8467$$

Third modification:

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$y_1^{(3)} = 1 + 0.1\left[-1 - 0.5169\right]$$

$$= 0.8483$$

Fourth Modification:

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + 0.1\left[-1 - 0.5196\right]$$

$$= 0.8480$$

Fifth Modification:

$$y_1^{(5)} = 0.8481 \qquad y_1^{(6)} = 0.8481$$

Here $y_1^{(5)} = y_1^{(6)} = 0.8481.$

$$\therefore \; y_1 = y(0.2) = 0.8481.$$

2) Step- 2

Now $x_1 = 0.2$ $\quad y_1 = 0.8481$

$$h = 0.2$$

$$f(x_1, y_1) = 0.2 - (0.8481)^2$$

$$= -0.5193$$

By Euler's method

$$y_2 = y_1 + hf(x_1, y_1) = 0.7962$$

Applying modified Euler's method

First modification -

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

$$= 0.8481 + 0.1\left[-0.5193 - 0.2340\right]$$

$$= 0.7728$$

Second modification:

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 0.8481 + 0.1\left[-0.5193 - 0.1972\right]$$

$$= 0.7765$$

Third modification:

$$y_2^{(3)} = 0.7759$$

Fourth modification:

$y_2^{(4)} = 0.7760$ $\quad\quad$ $y_2^{(5)} = 0.7760$

Here $y_2^{(4)} = y_2^{(5)} = 0.7760$

$$\therefore\ y_2 = y(0.4) = 0.7760$$

Practice Session -

1) Use modified Euler's method to find y satisfying the equation

$$\frac{dy}{dx} = \ln(x+y) \qquad y(1) = 2$$

for $x = 1.2$ and $x = 1.4$ correct to three decimal places by taking $h = 0.2$.

Ans: $y(1.2) = 2.2332$

$y(1.4) = 2.492$

2) Use modified Euler's method to find y satisfying the equation

$$\frac{dy}{dx} = \frac{1}{x+y} \qquad x_0 = 0, \; y_0 = 1$$

for $x = 0.4$ taking $h = 0.2$

Ans:

Runge - Kutta Methods-

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

Second order Runge-Kutta method -

If we replace

$y_1 = y_0 + hf(x_0, y_0)$ in the RHS

of modified Euler's formula, we get

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0+hf(x_0, y_0))\right]$$

$$= y_0 + \frac{1}{2}\left[hf(x_0, y_0) + hf(x_0+h, y_0+hf(x_0, y_0))\right]$$

If we say $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0+h, y_0+k_1)$

and $k = \frac{1}{2}(k_1+k_2)$

$\therefore$ $y_1 = y_0 + k$ which is second

order Runge-Kutta formula.

Fourth order Runge - Kutta method -

A more accurate and commonly used method of great practical importance is the classical Runge- Kutta method of fourth order which we call briefly Runge- Kutta method.

Fourth order Runge - Kutta method formulae are

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

and $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Then the required approximate value is

$$y_1 = y_0 + k$$

Ex. 1) Apply Runge - Kutta fourth order method to find an approximate value of y when $x = 0.2$, given that

$$\frac{dy}{dx} = x + y \quad \text{and} \quad y = 1 \text{ when } x = 0.$$

Sol^n: We have

$$\frac{dy}{dx} = x + y$$

here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$

$h = 0.2$

Now $y_1 = y_0 + k$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_0, y_0)$$

$$= 0.2\,[x_0 + y_0] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\, f(0.1, 1.1) = 0.24$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\, f(0.1\ 1.12) = 0.244$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_4 = 0.2 f(0.2, 1.244)$$

$$= 0.2888$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 0.48 + 0.4880 + 0.2888)$$

$$= 0.2468$$

$$\therefore \quad y_1 = y_0 + k$$

$$= 1 + 0.2468$$

$$= 1.2468$$

Hence the required approximate value of y is 1.2428.

Ex. 2) Using Runge-Kutta method of fourth order.

Solve

$$\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2} \quad \text{with } y(0) = 1 \text{ at}$$

$x = 0.2,\ 0.4.$

Soln: We have

$$\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2} \qquad \therefore f(x,y) = \frac{y^2-x^2}{y^2+x^2}$$

$x_0 = 0,\ y_0 = 1 \qquad h = 0.2$

Step 1 To calculate $y(0.2)$

$$k_1 = hf(x_0, y_0) = 0.2 f(0,1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f(0.1, 1.1) = 0.1967$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f(0.1, 1.09836)$$

$$= 0.1967$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

$$= 0.2 f(0.2, 1.1967)$$

$$= 0.1891$$

Now $k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.2 + 2(0.1967) + 2(0.1967) + 0.1891]$

$= 0.1959$

$\therefore y_1 = y_0 + k$

$= 1 + 0.1959$

$y_1 = 1.196$

$\therefore y(0.2) = 1.196$

Step-2 To calculate $y(0.4)$

$x_1 = x_0 + h = 0.2 \quad y_1 = 1.196 \quad h = 0.2$

$k_1 = hf(x_1, y_1) = 0.2f(0.2, 1.196) = 0.1891$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.2f(0.3, 1.2906) = 0.1795$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.2f(0.3, 1.2858) = 0.1793$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753)$

$= 0.1688$

$\therefore k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.1792$

$\therefore y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$

Practice Session -

1) Apply Runge- Kutta method to find an approximate value of y for $x=0.2$ in steps of 0.1 if

$$\frac{dy}{dx} = x+y^2$$ given that $y=1$ where $x=0$

Ans : $k_1 = 0.1$ $k_2 = 0.1152$, $k_3 = 0.1168$

$k_4 = 0.1347$

$\therefore$ $k = 0.1165$

$\therefore$ $y(0.1) = 1.1165$

Step - 2 $x_1 = 0.1$, $y_1 = 1.1165$ $h=0.1$

$k_1 = 0.1347$

$k_2 = 0.1551$

$k_3 = 0.1576$

$k_4 = 0.1823$

$\therefore$ $k = 0.1571$

$\therefore$ $y(0.2) = 1.2736$

2) Use Fourth order Runge- Kutta method to find y at $x=0.1$ given that

$$\frac{dy}{dx} = 3e^{x}+2y \,, \quad y(0)=0 \quad \text{and}$$

$h=0.1$.

3) Using R-K method of fourth order find $y(0.2)$ for the equation

$$\frac{dy}{dx} = \frac{y-x}{y+x} \qquad y(0)=1$$

(Take $h=0.1$).

4) Use Runge-Kutta method of 4th order to find y when $x=1.2$ in steps of 0.1 given that

$$\frac{dy}{dx} = x^2+y^2 \quad \text{and} \quad y(1)=1.5$$