Normal Distribution :

Normal distribution is the probability distribution of a continuous random variable X, known as normal random variable.

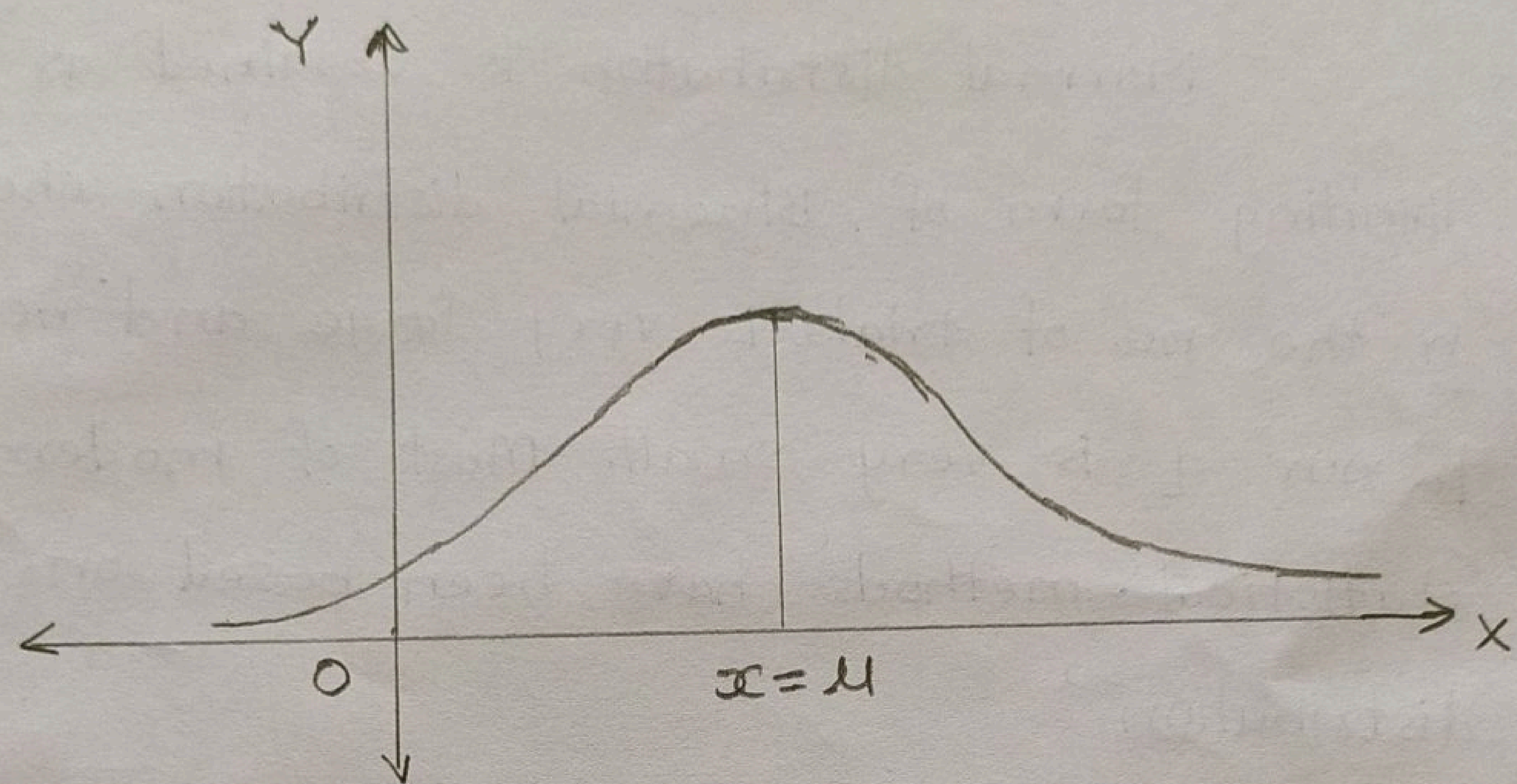
Normal distribution is obtained as limiting form of Binomial distribution when n the no. of trials is very large and neither p nor q is very small. Most of modern statistical methods have been based on this distribution.

Normal distribution of a continuous random variable X with parameters μ and σ^2 is denoted $X \longrightarrow N(\mu, \sigma^2)$ and given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The graph of the normal distribution curve $y = f(x)$ is bell shaped curve with symmetric about the ordinate $x = \mu$

The mean, median. and mode coincide and therefore normal curve is unimodal (has only one maximum point) Also normal curve is asymptotic to both +ive and -ive x axis.



The total area under the normal curve is unity

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

The probability that the continuous random variable X lies between $X = x_1$ and $X = x_2$ is given by the area under the curve $y = f(x)$, bounded by x axis.

$$P(x_1 \le X \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Introduce $z = \frac{x-\mu}{\sigma}$ $\quad dx = \sigma dz$

$z_1 = \frac{x_1 - \mu}{\sigma}$ $\quad z_2 = \frac{x_2 - \mu}{\sigma}$

$$\therefore \; P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2} dz$$

If $\mu = 0$ and $\sigma = 1$ then standard variable

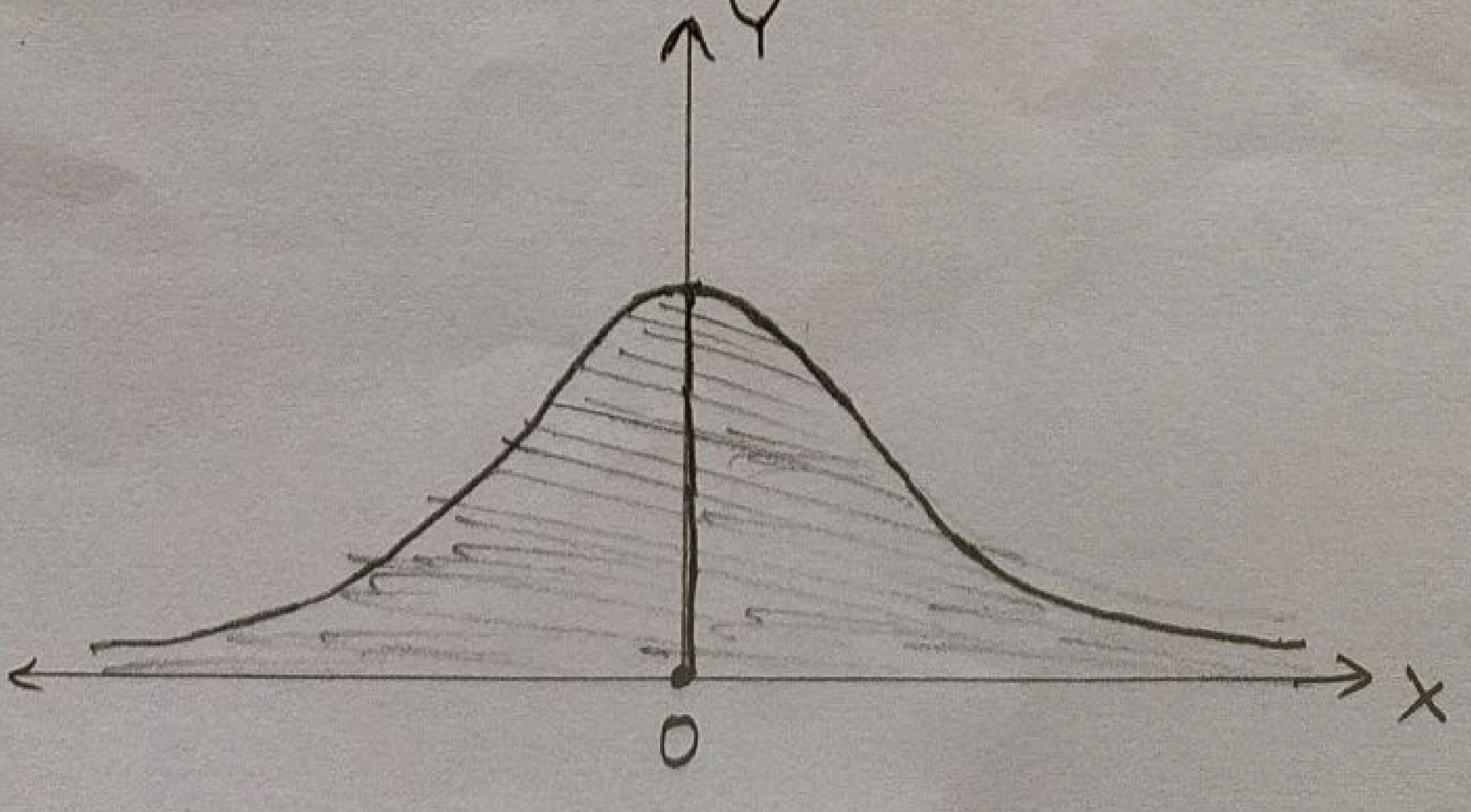
$z = \frac{x-\mu}{\sigma}$ is called standard normal variable

i.e. $N(0, 1)$

The probability density function (pdf) of standard normal variable is given by

$$F(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

This distribution is called standard normal distribution and its normal curve as standard normal curve.



If $A(z)$ denotes the area under the normal curve $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ from 0 to z_1

z_1 being any number

$\therefore$ $p(z)$

$$p(0 \leq z \leq z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}} dz$$

$$\therefore p(z) = A(z) = \frac{1}{\sqrt{2\pi}} \int_0^{z} e^{-\frac{z^2}{2}} dz \quad \text{——— (1)}$$

The above definite integral is called normal probability integral (or error function)

Now

$$p(x_1 \leq X \leq x_2) = p(z_1 \leq z \leq z_2)$$

$$= p(z_2) - p(z_1)$$

Properties of Normal Distribution -

1) The normal curve is

a) symmetric about $x = \mu$

b) it is bell shaped

c) the mean, median. and mode coincide and therefore is unimodal (has only one maximum point.

d) ~~No~~ has inflections points at $\bar{x} \pm \sigma$

e) asymptotic to both positive X axis and negative X axis.

2) Area under the normal curve is unity.

3) Probability that continuous rv X lies betn x_1 and x_2 is denoted by

$$p(x_1 \le X \le x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Since the above integral depends on μ and σ we get different normal curves for different values of μ and σ. and it is impracticable task to plot all such normal curves.

$$\therefore \quad z = \frac{x-\mu}{\sigma}$$

Here z is called standardized normal variable.

4) Change of scale from x axis to z axis

$$p(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$p(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} \, e^{-z^2/2} \, dz$$

where $\quad z_1 = \frac{x_1-\mu}{\sigma} \qquad z_2 = \frac{x_2-\mu}{\sigma}$

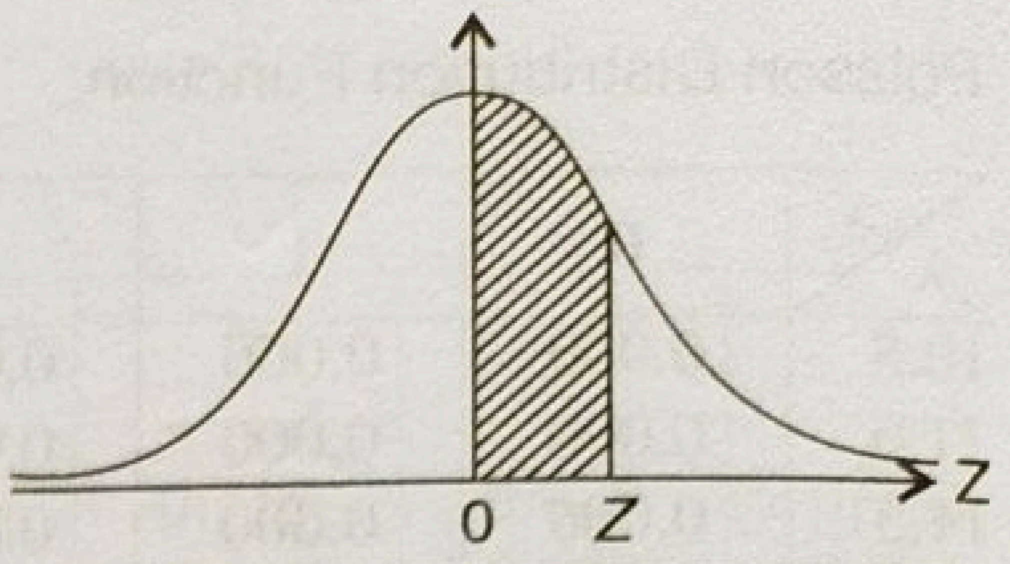
5) Error function or probability integral is defined as

$$p(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} \, dz$$

$$p(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} \, e^{-z^2/2} \, dz$$

$$= p(z_2) - p(z_1)$$

3. Areas under the Standard Normal Curve from 0 to z
(Normal Tables)



z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1256	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1916	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2649
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4654	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4979	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4098	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Ex. Find the area A under the normal curve.

a) to the left of $z = -1.78$

b) to the left of $z = 0.56$

c) to the right of $z = -1.45$

d) corrosponding to $z \geq 2.16$

e) corrosponding to $-0.80 \leq z \leq 1.53$

f) to the left of $z = -2.52$ and to the right of $z = 1.83$

Ans: a) 0.0375

b) 0.7123

c) 0.9265

d) 0.0154

e) 0.7251

f) 0.0395

6) Area under normal curve is distributed as follows.

68.27 % area lies between $\mu-\sigma$ to $\mu+\sigma$

i.e., $-1 \leq z \leq 1$

94.45% area lies between $\mu-2\sigma$ to $\mu+2\sigma$

i.e., $-2 \leq z \leq 2$

99.73 % area lies between $\mu-3\sigma$ to $\mu+3\sigma$

i.e., $-3 \leq z \leq 3$

Ex. 2 If z is normally distributed with mean 0 and variance 1, Find

a) $p(z \geq -1.64)$

b) $p(-1.96 \leq z \leq 1.96)$

c) $p(z \leq 1)$

d) $p(z \geq 1)$

Soln: a) 0.9495

b) 0.9500

c) 0.8413

d) 0.1587

Ex. 3) The mean weight of 500 students is 63 kgs and the standard deviation is 8 kgs. Assuming that the weights are normally distributed. Find how many students weigh 52 kgs. The weights are recorded to the nearest kg.

Soln: We have $\mu = 63$

$\sigma = 8$

$$\therefore f(x) = \frac{1}{8\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\cdot\left(\frac{x-63}{8}\right)^2}$$

Since the weights are recorded to the nearest kg, the students weighing 52 kgs have their actual weights between $x_1 = 51.5$ and $x_2 = 52.5$ kg.

$$z = \frac{x-\mu}{\sigma}$$

$$z_1 = \frac{x_1-\mu}{\sigma} = -1.44$$

$$z_2 = \frac{x_2-\mu}{\sigma} = -1.31$$

$\therefore P(z_1 \le z \le z_2) = P(z_2) - P(z_1) = 0.4251 - 0.4049 = 0.0202$

No. of students weighing 52 kg $= 500(0.020) = \underline{\underline{10}}$

Ex. Let $X \to N(4, 16)$ Find

i) $p(x>5)$

ii) $p(x<2)$

iii) $p(x>0)$

iv) $p(6<x<8)$

v) $p(|x|>6)$

Sol^n: i) 0.40129

ii) 0.30854

iii) 0.8413

iv) 0.1498

v) 0.3147

Ex. In a distribution exactly normal, 7 % of the items are under 35 and 89 % are under 63. Find the mean and SD of the distribution.

Soln: 7 % of items are under 35 means area under 35 is 0.07.

Similarly area for $x > 63$ is 0.11

$p(x < 35) = 0.07$

$p(x > 63) = 0.11$

when $x = 35$

$z = \frac{35-\mu}{\sigma} = -z_1$ (-ive sign because $x=35$ to the left of $x=\mu$)

when $x = 63$

$z = \frac{63-\mu}{\sigma} = z_2$ (+ive sign for $x=63$ lies to the right of $x=\mu$)

Area $A_1 = p(0 < z < z_1) = 0.48$ $\qquad z_1 = 1.48$ approx.

$A_2 = p(0 < z < z_2) = 0.39$ $\qquad z_2 = 1.23$ approx.

$\therefore \frac{35-\mu}{\sigma} = -1.48$

$\frac{63-\mu}{\sigma} = 1.23$

$\therefore \mu = 50.3$ approx.

$\sigma = 10.33$ approx.

Ex. Suppose heights of students follows normal distribution with mean 190 cm and variance 80 cm^2. In a school of 1000 students how many would you expect to be above 200 cm tall ?

Sol^n: Let x - height of students

$$x \longrightarrow N(190, 80)$$

Proportion of students having height above 200 cm

$$= p(x>200)$$

$$= p(z>1.1180)$$

$$= 0.1314.$$

$\therefore$ Number of students out of 1000 having height above 200 cm

$$= 1000 \times 0.1314$$

$$= 131 \text{ students approx.}$$