Newton's Forward difference interpolation formula -

Newton's forward difference interpolating polynomial passing through a set of points (x_i, y_i) where $i = 0, 1, \cdots n$ is given by

$$\boxed{y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n y_0}$$

where $u = \dfrac{x - x_0}{h}$

$$(u-1) = \frac{x-x_0}{h} - 1 = \frac{x - x_0 - h}{h}$$

$$= \frac{x - x_1}{h}$$

Similar we can show that

$$(u-2) = \frac{x - x_2}{h}$$

Ex. 1) Find Newton's interpolating polynomial satisfying the data.

x	0	1	2	3	4
y	-4	-4	0	14	44.

Sol^n: Prepare forward difference table for given (x_i, y_i)

We get

$y_0 = -4 \quad \Delta y_0 = 0$

$\Delta^2 y_0 = 4 \quad \Delta^3 y_0 = 6 \quad \Delta^4 y_0 = 0$

and $h = 1$

By using Newton's forward difference interpolation formula,

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

Here $u = \dfrac{x - x_0}{h} = \dfrac{x-0}{1} = x$

$$\therefore\ y = -4 + 0 + \frac{x(x-1)}{2!}(4) + \frac{x(x-1)(x-2)}{3!}(6) + 0$$

$$= -4 + 2x(x-1) + x(x-1)(x-2)$$

$y = x^3 - x^2 - 4$ which is the required interpolating polynomial.

Practice session :

1) Find a polynomial passing through the points $(0,1)$, $(1,1)$, $(2,7)$, $(3,25)$, $(4,61)$, $(5,121)$ using Newton's interpolation formula and hence find y and $\frac{dy}{dx}$ at $x = 0.5$

Soln :-

Ans $y = 1 - x + x^3$

$y|_{0.5} = 0.625$

$\frac{dy}{dx}\Big|_{0.5} = -0.25$

2) Estimate the number of students who secured marks between 40 and 45 from the following table.

Marks x	30-40	40-50	50-60	60-70	70-80
No. of students y	31	42	51	35	31

Operator Theory -

1) Forward difference operator : (Δ)

$$\Delta f(x) = f(x+h) - f(x)$$

2) Backward difference operator (∇)

$$\nabla f(x) = f(x) - f(x-h)$$

3) Central difference operator (δ)

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

4) Average operator (μ)

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}$$

Practice examples :

1) Show that

$$\Delta\nabla = \nabla\Delta = \delta^2$$

2) Show that

$$\mu^2 = 1 + \frac{\delta^2}{4}$$

Shift operator :- (E)

$$E f(x) = f(x+h)$$

$$E^{r} f(x) = f(x+rh)$$

$$E^{-r} f(x) = f(x-rh)$$

Important Relations :-

1) $\Delta = E - 1$

2) $\nabla = 1 - E^{-1}$

3) $\delta = E^{1/2}\nabla$

4) $\delta = E^{-1/2}\Delta$

Newton's backward difference interpolation -

Newton's backward difference interpolating polynomial which passes through (x_i, y_i) $i = 0, 1, 2 \ldots n$ is given by

$$y = y_n + u \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n$$

$$\ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!} \nabla^n y_n.$$

Ex. 1) Estimate the population in 1895 and 1925 from the following statistics.

Year x :	1891	1901	1911	1921	1931
Population y :	46	66	81	93	101

Soln : Newton's forward / backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$\therefore$ Forward differences : 20, −5, 2, −3

Backward differences : 8, −4, −1, −3.

Newton's forward difference interpolation :

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

Here $y_0 = 46$, $\Delta y_0 = 20$, $\Delta^2 y_0 = -5$, $\Delta^3 y_0 = 2$

$\Delta^4 y_0 = -3$

$$u = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

$\therefore$ $y = 54.85$

Newton's ~~backdwfard~~ backward difference interpolation :

$$y = y_4 + q\nabla y_4 + \frac{q(q+1)}{2!}\nabla^2 y_4 + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_4$$

$$+ \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_4$$

Here $y_4 = 101$, $\nabla y_4 = 8$ $\nabla^2 y_4 = -4$ $\nabla^3 y_4 = -1$

$\nabla^4 y_4 = -3$

$$q = \frac{x - x_3}{h} = \frac{1925 - 1931}{10} = -0.6$$

$y = 96.84$

Practice session :

1) For the following tabulated data.

x	1	2	3	4	5
y	3.47	6.92	11.25	16.75	22.94

Find y at $x = 4.5$

Ans: 19.81875