# Unit No 3

# Numerical Methods

1) **Interpolation** - Finite differences, Newton's and Lagrange's interpolation formulae, Numerical differentiation.

2) **Numerical Integration** :- Trapezoidal and Simpson's rules, Bound of truncation error.

3) **Solution of ordinary differential equations** : Euler's, Modified Euler's, Runge-Kutta 4th order methods and Prediction-Corrector method.

**Course outcome:** Obtain interpolating polynomials, numerically differentiate and integrate functions, numerical solution of DEs using single step and multi-step iterative methods used in modern scientific computing.

## Interpolation:

Suppore we are given the following values of $y = f(x)$ for a set of values of $x$:

| $x$ | $x_0$ | $x_1$ | $x_2$ | ..... | $x_n$ |
|---|---|---|---|---|---|
| $y$ | $y_0$ | $y_1$ | $y_2$ | ..... | $y_n$ |

Then the process of finding the values of $y$ corrosponding to any value of $x = x_i$ between $x_0$ and $x_n$ is called <u>interpolation</u>. Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. while the process of computing the value of the function outside the given range is called <u>extrapolation</u>.

## Lagrange's Interpolation :

Lagrange's interpolating polynomial passing through the set of points $(x_i, y_i)$

$i = 0, 1, 2 \ldots n$

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $x_0$ | $x_1$ | $x_2$ | $\ldots$ | $x_n$ |
| $y$ | $y_0$ | $y_1$ | $y_2$ | $\ldots$ | $y_n$ |

is given by

$$y = L_0(x)\, y_0 + L_1(x)\, y_1 + \cdots + L_n(x)\, y_n$$

where

$$L_0(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \quad \ldots$$

$$L_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Here $x_i$'s are unequally spaced.

Ex. 1) Find Lagrange's interpolating polynomial passing through set of points.

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 2 |
| $y$ | 4 | 3 | 6 |

Use it to find $y$ at $x = 1.5$,

$\frac{dy}{dx}$ at $x = 0.5$,

and find $\int_0^3 y\, dx$

Soln: Lagrange's interpolating polynomial passing through $(x_0, y_0)$, $(x_1, y_1)$ $(x_2, y_2)$ is given by

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$\therefore y = \frac{(x-1)(x-2)}{2}(4) + \frac{(x)(x-2)}{(-1)}(3)$$

$$+ \frac{(x)(x-1)}{2}(6)$$

$$\therefore y = 2(x^2-3x+2) - 3(x^2-2x) + 3(x^2-x)$$

$\therefore\ y = 2x^2 - 3x + 4$ ———— (1)

This is required Lagranges interpolating polynomial passing through given points.

Now when $x = 1.5$

$\therefore y = 4$

Diff (1) w.r. to $x$

$$\frac{dy}{dx} = 4x - 3$$

when $x = 0.5$

$$\frac{dy}{dx} = -1$$

Finally,

$$\int_0^3 y\,dx = \int_0^3 (2x^2 - 3x + 4)\,dx$$

$$= 2\left(\frac{x^3}{3}\right)_0^3 - 3\left(\frac{x^2}{2}\right)_0^3 + 4\left(x\right)_0^3$$

$$= 16.5$$

Ex. 2) The velocity distribution of a fluid near a flat surface is given below. $x$ is distance from the surface (mm) and $v$ is the velocity (mm/sec). Use Lagrange's interpolating polynomial to obtain the velocity at $x = 0.4$

| $x$ (mm) | 0.1 | 0.3 | 0.6 | 0.8 |
|---|---|---|---|---|
| $v$ (mm/sec) | 0.72 | 1.81 | 2.73 | 3.47 |

Soln: By Lagrange's interpolation formula,

$$v = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} v_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} v_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} v_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} v_3$$

Here $x = 0.4$

$$v = L_0(0.4) = \frac{(0.4-0.3)(0.4-0.6)(0.4-0.8)}{(0.1-0.3)(0.1-0.6)(0.1-0.8)}$$

$$L_1(0.4) = \frac{(0.4-0.1)(0.4-0.6)(0.4-0.8)}{(0.3-0.1)(0.3-0.6)(0.3-0.8)}$$

$$L_2(0.4) = \frac{(0.4-0.1)(0.4-0.3)(0.4-0.8)}{(0.6-0.1)(0.6-0.3)(0.6-0.8)}$$

$$L_3(0.4) = \frac{(0.4-0.1)(0.4-0.3)(0.4-0.6)}{(0.8-0.1)(0.8-0.3)(0.8-0.6)}$$

$\therefore$ $v = 2.16028$ mm/sec.

**Exercise:** Given $(1.0)^3 = 1.000$

$(1.2)^3 = 1.728$

$(1.3)^3 = 2.197$ and

$(1.5)^3 = 3.375$

Using Lagrange's interpolation formula, evaluate $(1.07)^3$.