Problem Set 3

Advanced Logic

21st September 2022

1. (40%) Suppose that A is a set; R is an injective relation on A; and B is a subset of A such that whenever $x \in B$, it is not the case that Rxx. Let C be the closure of B under R. Prove that whenever $x \in C$, it is not the case that Rxx.

   (a) To use induction, we need to have 2 steps:

   (b) first, show that every element of B does not have property Rxx

   (c) second, !Rxx is preserved.

   (d) Since B is that whenever x that is an element of B, x is also an element of A. Thus, we have if $x \in B$, $x \in A$, as B is a subset of A, and it is not the case that Rxx

   (e) Thus, every member of B has this property

   (f) Since C is the closure of B under R, it is that case the property of B is preserved.

   (g) Thus, the inductive steps worked.

Hint: this will be a proof by induction.

3. Prove that the following hold for all $m, n \in \mathbb{N}$
   (a) (20%)
   double n =n+n
   Base case: double 0 = 0 (Dz)
   induction step:
   Suppose double n = n + n, Then
   double(suc(n)) = suc(n) + suc(n), Since
   suc(n) + suc(n) = suc(suc(n+n)), and
   double(suc(n)) = suc(suc(double n)). (Ds)
   Therefore, double(n) = n + n

   (b) (10%)
   m+n =n+m
   Base case: 0 + 0 = 0 + 0
   Base case can be proved by using (Az) and (As),
   assume that m is suc(n), then n + suc(n) = suc(n) + n
   as n + suc(n) = suc(n + n).

Since n is just 0 it is proved.
Induction Step:
Suppose that m + n = n + m
by(As), n + suc(m) = suc(n+m)
Since n + 0 = n,
suc(n+m) + 0 = suc(n + m) substitute n with 0, then
0 + suc(m) = suc(0+m) suc(m) = suc(m),
Thus, m + n = n + m
(c) (10%)
double $n = 2 \times n$
base case: double(0) = 2 ×0 By (Mz), Since $n \times 0 = 0$, 2 = suc(suc(0)).
Thus,
2 is a case of n.
So, (Mz) can be applied to the base case
Induction Step:
As I showed that n is just any suc(0).
As, suc is a property that holds for all natural numbers
and (Ms), $n \times suc(m) = (n \times m) + n$
and (Ds), $double(suc(n)) = suc(suc(doublen))$ since 2 is just suc(suc(0))
substitues 2 we can get double n = $2 \times n$

(d) (5%)
(e) (5%)
(f) (5%)
(g)(5%)

$$0 \times n = 0$$
$$(\text{ suc } n) \times m = (n \times m) + m$$
$$n \times m = m \times n$$

If double $m =$ double n, then $m = n$

Note: in these proofs you may assume the Axiom of Numbers and the following principles about addition, multiplication, and the 'double' function (which follow from their standard recursive definitions):

(Dz)
(Ds)
(Az)
(As)
(Mz)
(Ms)

$$\text{double } 0 = 0$$
$$\text{double } (\text{suc}\, n) = \text{suc}(\text{suc}(\text{ double } n))$$
$$n + 0 = n$$
$$n + \text{suc}\, m = \text{suc}(n + m)$$
$$n \times 0 = 0$$
$$n \times \text{suc}\, m = (n \times m) + n$$

for any $n, m \in \mathbb{N}$. You may also assume the facts I give proofs of in the supplementary document called 'Proofs by induction: a guide'. Also, some of the later proofs will require on earlier ones. Remember too that ' 2 ' abbreviates 'suc(suc 0)$'$ '.

5. (10% extra credit) Open the Lean Natural Numbers Game at https://www . ma.imperial. ac. uk/ buzzard/xena/natural_number_game/ and play through (at least) the levels 'Tutorial World', 'Addition World', and 'Multiplication World'. To show that you've completed the levels, send us a screenshot of the last level of Multiplication World open on your computer screen, with your name showing somewhere in the screenshot (e.g. in a text editor window).

Note that the facts you're proving in these levels of the game overlap a lot with the ones you're asked to prove in problem 2. So, you might find it helpful to play the game first and tackle problem 2 afterwards. Whichever order you do it in, it should be instructive to look at your proofs in problem 2 with your solutions to the game, and see how they have the same mathematical content.