Solutions to Problem Set 10

Advanced Logic

3rd December 2022

1. (50%) Show that the following are definable in the standard string structure.

   (a) (10%) the set of all strings whose length is an even number.

   $$\forall x(\mathrm{EvenLength}(x) \leftrightarrow \exists y \exists z(x = y \oplus z \land x \approx y))$$

   where we define $\approx$ ('are equal in length') by

   $$\forall x \forall y(x \approx y \leftrightarrow (x \leq y \land y \leq x))$$

   (b) (10%) the function that maps every ordered pair of strings $\langle s, t\rangle$ to its first element.

   $$\forall x \forall y \forall z(z = \mathrm{First}(x, y) \leftrightarrow z = x)$$

   (c) (10%) the set of all ordered pairs of strings $\langle s, t\rangle$ such that s is a length-one substring of t.

   $$\forall x \forall y(\mathrm{CharOf}(x, y) \leftrightarrow \exists z \exists z'(y = z \oplus (x \oplus z') \land x \approx \texttt{"?"})$$

   (d) (10%) the function that maps each string to an equally long string comprised entirely of spaces.

   $$\forall x \forall y(y = \mathrm{Spaces}(x) \leftrightarrow (y \approx x \land \forall z(\mathrm{CharOf}(z, y) \to z = \texttt{" "}))$$

   (e) (10%) the relation that holds between two strings s and t when s is a *line* of t---i.e., s would appear as a line if we pasted t into a text editor. That is: s doesn't contain any newlines, and either s is t, or s is an initial substring of t that's followed by a newline, or s is a final substring of t that's preceded by a newline, or s is a substring of t that's both preceded and followed by a newline.

   $$\begin{aligned}\forall x \forall y[\mathrm{LineOf}(x, y) \leftrightarrow\ & [\forall z(\mathrm{CharOf}(z, y) \to z \neq \mathrm{new}) \land \\ & (y = x \\ & \ \lor \exists z(y = x \oplus \mathrm{new} \oplus z) \\ & \ \lor \exists z(y = z \oplus \mathrm{new} \oplus x) \\ & \ \lor \exists z \exists z'(y = z \oplus \mathrm{new} \oplus x \oplus \mathrm{new} \oplus z'))]]\end{aligned}$$

2. (25%) Suppose set X and binary relations R and R' are definable in a certain structure S with domain D_S. Show that the following are also definable:

   (a) (5%) $D_S \setminus X$

(b) (5%) R^{-1}

(c) (5%) $R \cup R'$

(d) (5%) $R \circ R'$

(e) (5%) $\{d \mid Rdd\}$

> Let F be a 1-ary predicate that defines X in a definitional expansion of S, and let G and H be binary predicates that respectively define R and R'. Then the listed relations are defined, respectively, by the following formulae:
>
> | | |
> |---|---|
> | (a) | $\neg Fx$ |
> | (b) | $G(y, x)$ |
> | (c) | $G(x, y) \vee H(x, y)$ |
> | (d) | $\exists z(H(x, z) \wedge G(z, y))$ |
> | (e) | $G(x, x)$ |

3. (25%) Here is a list of expressions in the language of strings, specified using various shorthands that have been introduced. For each one, say

   (i) what string it is (write out in full, with no shorthands like omitting parentheses or infix notation)

   (ii) whether it is a term or a formula

   (iii) what its free variables are (if any)

   (iv) if it's a term, what it denotes in the standard string structure on an assignment function that maps the variable x to the string `cat`.

   (v) if it's a formula, whether it is true in the standard string structure on an assignment function that maps the variable x to the string `cat`.

   (vi) if it's a formula: whether it is valid, inconsistent, or neither.

   (a) ""

   (b) $x = \text{"}x\text{"}$

   (c) $x \oplus \text{""} \oplus x$

   (d) $\text{"}x\text{"} \leq \text{""} \oplus x$

   (e) $\exists x(x = \text{"}x\text{"})$

   (f) $\forall x(x \leq \text{""} \to x = \text{""})$

   (g) $(x = \text{"}x\text{"})[\text{"}x\text{"}/x]$

   (h) $\forall x(x = \text{"}x\text{"})[\text{"}x\text{"}/x]$

   (i) $\langle x \rangle$

   (j) $\langle x \rangle = \text{"}x\text{"}$

   (k) $\langle x = \text{"}x\text{"} \rangle$

   (l) $\langle \langle x \rangle \rangle$

	string	kind	FV	denot/TV	status
$""$	`""`	term	none	[]	
$x = "x"$	`=(x,"x")`	formula	x	F	neither
$x \oplus "" \oplus x$	`⊕(x,+("", x))`	term	x	`catcat`	
$"x" \leq "" \oplus x$	`≤("x",+("",x))`	formula	x	T	neither
$\exists x(x = "x")$	`∃x =(x,"x")`	formula	none	T	valid
$\forall x(x \leq "" \to x = "")$	`∀x (≤(x,"")→=(x,""))`	formula	none	T	neither
$(x = "x")["x"/x]$	`=("x","x")`	formula	none	T	valid
$\forall x(x = "x")["x"/x]$	`∀x =(x,"x")`	formula	none	F	neither
$\langle x \rangle$	`⊕("x","")`	term	none	`x`	
$\langle x \rangle = "x"$	`=(⊕("x",""),"x")`	formula	none	T	neither
$\langle x = "x" \rangle$	see (1) below	term	none	`=(x,"x")`	
$\langle\langle x \rangle\rangle$	see (2) below	term	none	`⊕("x","")`	

(1): `⊕("=",⊕(lpa,⊕("x",⊕(com,⊕(quo,⊕("x",⊕(quo,⊕(rpa,""))))))))`

(2): `⊕("⊕",⊕(lpa,⊕(quo,⊕("x",⊕(quo,⊕(com,⊕(quo,⊕(quo,⊕(rpa,"")))))))))`

EXTRA CREDIT: up to 10% for any of the following.

1. Suppose that structure S is *explicit*: that is, for every element of the domain, there is a term with no free variables that denotes it in S. Show that every finite subset of S's is definable in S.

   Given the definition of "finite subset", we can all finite subsets of a set are definable by showing that the empty set is and that if a set is, so does any set derived from that set by adding one extra element.

   Base case: the empty set is defined by the formula $x \neq x$.

   Induction step: suppose a set X is definable, and let F be a predicate (in a definitional extension of S) that defines it. Suppose $a \notin X$. Since S is explicit, there is some closed term t such that $[\![t]\!]_S = a$. Then $X \cup \{a\}$ is also definable, since if we further definitionally extend S with a preedicate H defined by

   $$\forall x(H(x) \leftrightarrow (G(x) \vee x = t))$$

   the extension of H will be $X \cup \{a\}$.

2. Using the compactness theorem, prove that if a set of sentences (in any signature) is true in arbitrarily large finite structures, it is also true in some infinite structure.

   We help yourself to the fact that for each n, there is a sentence I_n that is true exactly in thos structures whose domain has size at least n---e.g., I_3 is $\exists x \exists y \exists z(\neg(x = y) \wedge \neg(x = z) \wedge \neg(y = z))$.

   Suppose Γ is a set of sentences that is true in arbitrarily large finite structures, and let $\Gamma^+ = \Gamma \cup \{I_n \mid n \in \mathbb{N}\}$. Every finite subset of Γ is consistent. For suppose X is such a subset: then there is a number n such that the subset does not contain

I_n for any $n \geq m$, and since Γ has arbitrarily large finite models there is a finite model of Γ of size greater than n: every member of X that's in Γ is true in this model, and every member of X that's of the form I_n is true. By the compactness theorem, it follows that Γ^+ is consistent. Any model of Γ^+ must have an infinite domain, since if it had a finite domain of size m, I_{m+1} wouldn't be true in it. Since $\Gamma \subseteq \Gamma^+$, we can conclude that Γ has an infinite model.

3. When S is a nonstandard model of true arithemtic, and a and b are two elements of the domain of S, nonstandard model of arithmetic, say that $a \leq_S b$ iff $x \leq y$ is true in S on the assignment $[x \mapsto a, y \mapsto b]$. Say that a and b are "in the same block" iff for for some number n, either $x = y + \langle n \rangle$ or $x = y + \langle n \rangle$ is true in S on this assignment.

   Prove that if a is a nonstandard element of the domain, then (i) there is a nonstandard element b such that $a \leq_S b$ and b is not in the same block as a, and (ii) a nonstandard element c such that $c \leq_S a$ and c is not in the same block as a.

(i) First, we show that when a and b are both nonstandard, $a +_S b$ (i.e., $[\![x + y]\!]_S^g$, where $g = [x \mapsto a, y \mapsto b]$) is not in the same block as a. Suppose otherwise; then there's an $n \in \mathbb{N}$ such that either $x + y = x + \langle n \rangle$ is true on g, or $x = (x + y) + \langle n \rangle$ is true on g. The latter case can be ruled out, since then $x = x + (y + \langle n \rangle)$ and hence $y + \langle n \rangle = 0$, and hence $y = 0$ would have to be true on g, which can't be if b is nonstandard. (Here we appeal to the fact that S is a model of true arithmetic, which means that $\forall x \forall y \forall z((x + y) + z = x + (y + z))$, $\forall x \forall y(x = x + y \rightarrow y = 0$, and $\forall x \forall y(x + y = 0 \rightarrow x = 0)$ are all true in S.) The former case similarly can be ruled, since then $y = \langle n \rangle$ would have to be true on g which also can't happen if b is nonstandard.

Now, suppose a is nonstandard. Then $a +_S a$ is not in the same block as a by what we just showed. Also, $a +_S a$ is nonstandard, since if it were the denotation of $\langle n \rangle$, $x + x = \langle n \rangle$ would be true on g, which can only happen if n is even and $x = \langle n/2 \rangle$ is true on g. And finally, $a \leq_S a + a$, since $x \leq x + x$ is true on every assignment.

(ii) The following is a theorem of true arithmetic: $\forall x \exists y(x = y + y \vee \operatorname{suc} x = y + y)$. Thus, if a a nonstandard element, there is either an element such that $a = b +_S b$ or an element such that $\operatorname{suc}_S a = b +_S b$. In the former case, b must be nonstandard with $b \lceil_S a$ and in a different block from a by what we showed in (i). In the latter case, b must be nonstandard with $b \lceil_S \operatorname{suc}_S a$ and in a different block from $\operatorname{suc}_S a$ by the same reasoning, and $\operatorname{suc}_S a$ must be nonstandard, since it could only be the denotation of $\langle n \rangle$ if a were the denotation of $\langle n - 1 \rangle$.