Midterm Review Sheet

Advanced Logic

19th October 2022

Here is a list of topics from the class so far that may come up on the midterm, and some sample problems related to these topics.

1 Basics of set theory

Axioms of Extensionality and Separation. Definitions: $\subseteq$, $\cup$, $\cap$, $\setminus$; relation; converse; serial, functional, surjective, injective; function, injection, surjection, bijection; reflexive, symmetric, transitive, antisymmetric relation on a set. Power set. Union and intersection of a set of sets.

Sample problem: Show that for any sets A, B, C, $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

Sample problem: Show that if R and S are reflexive and symmetric, $R \circ S$ is reflexive and symmetric. Give an example that shows that even if R and S are transitive (as well as being reflexive and symmetric), $R \circ S$ need not be transitive.

2 Sizes of sets

Equinumerous ($\sim$); 'at least as big as' ($\lesssim$); Dedekind-infinite set; finite set. Understand the statement of Cantor's theorem and its proof. Statement of Schröder-Bernstein and Cardinal Comparability theorems. Basic facts about finitude and countability (see list in lecture 8).

Sample problem: Show that when $A \sim B$ and $C \sim D$, $A \times C \sim B \times D$.

Sample problem: Suppose that R is a relation on A. Show that there exists a subset B of A such that there is no $x \in A$ such that for all y, Rxy iff $y \in B$.

Sample problem: Show, using the definitions of 'finite' and 'countably infinite', that the union of a finite set and a countably infinite set is countably infinite.

3 Closed sets

Set closed under a relation (or family of relations); closure of a set under a relation (or family of relations); proof by induction that every member of such a set has a certain property.

Sample problem: Suppose $B \subseteq A$, R is a transitive, reflexive relation on A, and C is the closure of B under R. Show that for every $y \in C$, there is some $x \in B$ such that Rxy.

4 Numbers and lists

The Axiom of Numbers and Axiom of Lists. Proving by induction that every member of $\mathbb{N}$ or A^* has some property. Format of a definition by recursion of a function whose argument is a number or list. Using such a definition to calculate the value of a function on a particular argument. Using such a definition in a proof by induction.

Sample problem: Give a recursive definition of a function that maps every string to the corresonding palindrome, e.g. mapping `cat` to `cattac`. (Your definition may use the string concatenation operation $\oplus$.) Prove that your function does in fact map `cat` to `cattac`.

Sample problem: Prove by induction that $1 + n = \mathrm{suc}\, n$ for every number n, using just the recursion clauses $n + 0 = n$ and $n + \mathrm{suc}\, m = \mathrm{suc}(n + m)$ and the definition $1 = \mathrm{suc}\, 0$.

Sample problem: Consider the functions $\mathbb{N} \to \mathcal{P}\mathbb{N}$ defined recursively as follows:

$$\begin{aligned} \mathrm{succs}\, 0 &\coloneqq \mathbb{N} \\ \mathrm{succs}(\mathrm{succ}\, n) &\coloneqq (\mathrm{succs}\, n) \setminus \{n\} \end{aligned}$$

Prove by induction that $n \in \mathrm{succs}\, n$ for all n.

5 First-order syntax

Relational signature; first-order signature. Terms versus formulae. Proving by induction that a certain property applies to all terms, or all formulae, of a certain signature Σ. Technique of recursive definition for terms and formulae. Intuitive understanding of the *occurs free in* relation between variables and terms/formulae, and of the operation of substituting a term for all free occurrences of a variable in a term or formula (I won't expect you to have memorized the official definitions of these concepts).

Sample problem: Give a recursive definition of a function `complexity' that maps every formula of a certain language $\mathcal{L}$ to the number of occurrences of the logical connectives `¬`, `∧`, `∨`, `→`, `∀`, `∃` in that formula. So, e.g., it should come out that $\mathrm{complexity}(x = y) = 0$, and $\mathrm{complexity}(\forall x(x = y \vee \neg(x = z))) = 3$. Prove that the function you defined actually does map $x = y$ to 0 and $\forall x(x = y \vee \neg(x = z))$ to 3.

Sample problem: Let Σ be a signature containing just one predicate F, of arity 1. Prove by induction that the last character of every formula in $\mathcal{L}(\Sigma)$ is `)`.

6 Provability

General concept of and nomenclature for provability (in classical logic). (I'll provide a list of all the 18 rules in the definition of $\vdash$ on the exam, so I don't need you to memorize these; I may also include a few other handy results like Cut and Contraposition that you'll be allowed to use.)

Sample problem: Show (using only the official rules, or the w-subscripted generalizations) that that for any formulas P and Q, the following conditions are equivalent: (i) $P \vdash \neg Q$; (ii) $Q \vdash \neg P$; (iii) $\vdash \neg(P \wedge Q)$.

Sample problem: Recall that $\vdash_M$ is the relation defined like $\vdash$ but without the DNE rule. Show that for any formula P, $\neg\neg\neg P \vdash_M \neg P$.