Solutions for Problem Set 2

Advanced Logic

21st September 2022

1. Suppose we have a function $f : A \to B$. Let $f^* : \mathcal{P}B \to \mathcal{P}A$ be the function such that for any $Y \in \mathcal{P}B$, $f^*Y = \{x \in A \mid fx \in Y\}$.

a. Show that f is injective iff f^* is surjective.

> *Left-to-right:* suppose f is injective and $X \in PA$. Let $Y = \{fx \mid x \in X\}$. Note that for any $x \in A$, if $fx \in Y$, then $fx = fx'$ for some $x' \in X$, so $x \in X$ since f is injective. So $f^*Y = \{x \in A \mid fx \in Y\} = \{x \in A \mid x \in X\} = X$. Thus X is in the range of f^*.
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> *Right-to-left:* Suppose f^* is surjective, and that $x, z \in A$ are such that $fx = fz$. Then for every $Y \subseteq B$, $x \in f^*Y$ iff $z \in f^*Y$. Now consider some Y such that $f^*Y = \{x\}$: such a Y must exist, since f^* is surjective. Then $z \in \{x\}$ iff $x \in \{x\}$, so $z \in \{x\}$, so $x = z$.

b. Show that f is surjective iff f^* is injective.

> *Left-to-right:* suppose f is surjective, and that $Y, Z \subseteq B$ are such that $f^*Y = f^*Z$. We will first show that $Y \subseteq Z$. Consider some arbitrary $y \in Y$. Since f is surjective, $y = fx$ for some $x \in A$. Since $fx \in Y$, $x \in f^*Y = f^*Z$; hence $fx \in Z$, i.e. $y \in Z$. Since y was arbitrary we can conclude that $Y \subseteq Z$. By parallele reasoning we can show $Z \subseteq Y$. So we can conclude that $Y = Z$, establishing that f^* is one-to-one.
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> *Right-to-left:* Suppose f^* is injective. Then for every $y \in B$, $f^*\{y\} \neq f^*\varnothing$, i.e. $f^*\{y\} \neq \varnothing$, so there is some $x \in A$ such that $x \in f^*\{y\}$. But in that case $fx = y$, so y is in the range of f.

2. (10%) Using the Axiom of Separation to show that there is no set that contains all sets. (*Hint:* adapt the reasoning in Russell's Paradox.)

> Suppose V was a set containing every set. Then by the Axiom of Separation, there is a set R ($= \{x \in V \mid x \notin x\}$) such that for any $x \in V$, $x \in R$ iff $x \notin x$. Since R is a set, $R \in V$, so it follows that $R \in R$ iff $R \notin R$: a contradiction.

3. (10%) Show that for any set A, there is no injective function from $\mathcal{P}A$ to A.

> One approach to this is to adapt the proof of Cantor's Theorem.
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> Suppose for contradiction that $f : \mathcal{P}A \to A$ is injective; then f can be considered as a bijection from $\mathcal{P}A$ to some $B \subseteq A$, and f^{-1} is a bijection $B \to \mathcal{P}A$. But consider the set $D := \{x \in B \mid x \notin f^{-1}x\}$. Since f^{-1} is supposedly a bijection, D is $f^{-1}y$ for some $y \in B$, so we have both $y \in D$ iff $y \in f^{-1}y$ and $y \in D$ iff $y \notin f^{-1}y$, and hence $y \in f^{-1}y$ iff $y \notin f^{-1}y$: a contradiction.
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> Alternatively, we could just appeal to Cantor's theorem, which says that $\mathcal{A} \lesssim \mathcal{P}A$, i.e. $A \lesssim \mathcal{P}A$ and not $A \sim \mathcal{P}A$. If there was an injection $\mathcal{P}A \to A$, that would mean $\mathcal{P}A \lesssim A$; by the Schröder-Bernstein theorem, the combination of this with $\mathcal{A} \lesssim \mathcal{P}A$ implies $A \sim \mathcal{P}A$, so we have a contradiction.

4. Suppose that $f : A \to B$ and $g : B \to A$ are functions such that for any $x \in A$, $x = g(fx)$, and for any $y \in B$, $y = f(gy)$. Show that $g = f^{-1}$.

(Note that we can't just assume that f^{-1} is a *function*, though this will follow from what we're being asked to prove.)

It suffices (by the Axiom of Extensionality) to show that (i) $\langle y, x\rangle \in f^{-1}$ for all $\langle y, x\rangle \in g$, and (ii) $\langle y, x\rangle \in g$ for all $\langle y, x\rangle \in f^{-1}$.

(i) Suppose $\langle y, x\rangle \in g$, i.e. $x = gy$; then $y = fx$, i.e. $\langle x, y\rangle \in f$, so $\langle y, x\rangle \in f^{-1}$.

(ii) Suppose $\langle y, x\rangle \in f^{-1}$; then $\langle x, y\rangle \in f$, i.e. $y = fx$, so $x = gy$, i.e. $\langle y, x\rangle \in g$.