# Final Exam Review Sheet

Advanced Logic
8th December 2022

This is a list of topics from the class that may come up on the final exam. The concepts and techniques from before the midterm is still relevant (e.g. you might find yourself needing to prove by induction that every number, or string, or term, or formula has some property); however with the exception of the question on provability there will not be questions that only draw on pre-midterm content.

The sample problems are included here because I believe that working on them would be a helpful way of preparing for the exam. I make no claim that the problems are similar to the ones on the exam in terms of difficulty---making up problems that are easy enough to put on an in-class exam but not trivial turns out to be quite hard!

## 1 Provability

Understand what a sequent is, and what a provable sequent is; be comfortable working with the 18 rules to show that sequents are provable. (You don't have to memorize the rules: I'll include all the ones you need in a supplement to exam.) Know how to show by induction that all provable sequents have a certain property.

*Sample problem:* Show that for any set of formulae $\Gamma$ and formulae $P, Q, R$. (a) $\Gamma, P \wedge Q \vdash R$ if and only if $\Gamma, P, Q \vdash R$, and (b) $\Gamma, P \vee Q \vdash R$ if and only if $\Gamma, P \vdash R$ and $\Gamma, Q \vdash R$.

*Sample problem:* Prove that if $\Gamma \vdash P$ and $t$ is substitutable for $v$ in $P$ and every member of $\Gamma$, then $\Gamma[t/v] \vdash P[t/v]$ for any term $t$ (where $\Gamma[t/v] \coloneqq \{Q[t/v] \mid Q \in \Gamma\}$). (You don't have to cover all 18 rules: it's enough if you do the cases for Assumption, Weakening, and the introduction and elimination rules for one of $\rightarrow, \wedge, \vee$ and one of $\forall, \exists$.

*Sample problem:* Prove that $\forall v P \dashv\vdash \forall u(P[u/v])$ and $\exists v P \dashv\vdash \exists u(P[u/v])$ so long as $P[u/v]$ is defined and $u$ is not free in $P$.

## 2 Structures and truth in a structure

Understand what a signature is and what a structure for a signature is, including the standard model of arithmetic and the standard string structure. Understand the definition of the denotation of a term in a structure relative to an assignment. Understand what it is for a formula to be true in a structure on an assignment, and how the notions of validity, logical consequence, and satisfiability are defined in terms of this.

*Sample problem:* Show that if $\Gamma \vDash P[u/v]$ and $u$ is any variable that is not free in any member of $\Gamma$, or in $\forall v P$, then $\Gamma \vDash \forall v P$. Given examples to show that this can fail when (i) $u$ is free in some member of $\Gamma$, or (ii) $u$ is free in $\forall v P$ though not in any member of $\Gamma$.

## 3 The soundness and completeness theorems

Understand the statement of both theorems. Know how to prove the soundness of various rules. Have some sense of the strategy used for proving the completeness theorem, and understand important concepts involved in this proof such as negation-completeness and witness-completeness. Understand the statements of the Downward Löwenheim-Skolem Theorem and the Compactness Theorem; know how these follow from the Completeness Theorem, and be able to apply them appropriately.

*Sample problems:* (a) Suppose that $\Gamma$ is a set of sentences and $S$ is a structure in which every member of $\Gamma$ is true. Show that there exists a negation-complete and witness-complete set of formulae $\Gamma^+$ extending $\Gamma$ such that every member of $\Gamma^+$ is true on $g$, for some assignment $g$ for $S$.

(b) Suppose that $\Gamma$ is a negation-complete, witness-complete set of formulae, $S$ is a structure with domain $D$, and $g$ is an assignment for $S$ such that $S, g \Vdash P$ for every $P \in \Gamma$. Let $S'$ be the structure with domain $D' = \{[\![t]\!]_S^g \mid t \text{ a term of } \Sigma\}$, where the extension of every predicate and function symbol is just the restriction to $D'$ its interpretation in $S$. Show that $S', g \Vdash P$ for every $P \in \Gamma$.

(c) Explain why (a) and (b) above jointly imply that whenever a theory is true in a structure, it is true in some countable substructure of that structure.

## 4 Definability

Understand what it is for formula to be true of an $n$-tuple of elements in a structure's domain, to define a set of $n$-tuples, and for such a set to be *definable* in the structure. Know the statement of Tarski's Undefinability Theorem (the set of sentences true in $\mathbb{S}$ is not definable in $\mathbb{S}$), and grasp the main idea of how it's proved.

*Sample question:* (a) Where $s$ is a string in which the only character is the symbol `•`, let a *multiplication-derivation for* $s$ be a string $m$ in which (a) the first line is just the character `;`, and (b) when $l$ and $l'$ are two successive lines of $m$, $l' =$ `•` $\oplus\, l \oplus s$. Show that the set of ordered pairs $\{\langle s, m\rangle \mid m \text{ is a multiplication-derivation for } s\}$ is definable in the standard string structure $\mathbb{S}$. (b) Using this result, conclude that the set of ordered triples $\{\langle s, t, u\rangle \mid s, t, u$ all consist entirely of `•`s, and the length of $u$ is the product of the lengths of $s$ and $t\}$ is definable in $\mathbb{S}$.

> Note: for problems like (a) it'll pay to break up the task into several small definitions with mnemonic names like LineOf$(x, y)$; that way, if one of your definitions is broken we'll be able to tell what it was supposed to do and the whole answer won't come crashing down.

## 5 Representability

Understand what it is for a set of (or relation among) strings to be representable or semi-representable, and what it is for a function from strings to strings to be capturable, in a given theory (in a signature including the language of strings). Grasp the basic logical relationships between these notions. Know the statement of Tarski's Unrepresentability Theorem (no consistent theory that extends Min represents itself).

*Sample problem:* Suppose that theory $T$ represents a certain set of strings $X$ and captures a certain function $f$ from strings to strings. (a) Show that $T$ represents the set $\{s \mid fs \in X\}$. (b) Give an example that shows that $T$ need not represent the set $\{fs \mid s \in X\}$.

> For part (b), let $T = \text{Min}$, and consider the function $f$ that takes every string that's a proof of some sentence from the axioms of Min to that sentence and every other string to the empty string.

## 6 The Incompleteness Theorems

Understand the statement of the Diagonal Lemma and be able to apply the lemma in new contexts. Understand the statement of the First Incompleteness Theorem, and have a general sense for how it is proved, and in particular an understanding of the relevant facts about the theory Min mentioned in our version of the theorem. Understand the statement of the Second Incompleteness Theorem and follow the five-premise proof of it given in lecture (though you need not follow my very sketchy description of why being "moderately strong" suffices to derive Internal NEC and Internal MP).

*Sample question:* Gödel's First Incompleteness Theorem states that no theory (i) is consistent, (ii) is negation-complete, (iii) extends or interprets Min, and (iv) is effectively axiomatizable. For each combination of three of these four properties, give an example of a theory that has those three properties, and explain in a sentence or two why it has those three.

*Sample question:* Suppose we have a theory $T$ that extends Min and includes a formula $A(x)$ of one free variable such that

(i) $T \vDash A\langle P\rangle \to P$ for every sentence $P$

(ii) $T \vDash A\langle P\rangle \lor A\langle \neg P\rangle$ for every sentence $P$.

Show that $T$ is inconsistent. (*Hint:* use the Diagonal Lemma.)

## 7 Computability

Even though we didn't get to this until the last two lectures, there will be some questions about this on the exam, though there will be enough choice that you won't need to answer any questions mentioning the relevant concepts if you prefer.

Understand at an intuitive level the concepts *computable partial function*, *decidable set*, and *semi-decidable set*, and how they fit together. Know what the Church-Turing thesis says. Be able to give examples of sets that are decidable, semi-decidable but not decidable, and not semi-decidable. Know that the decidable sets are exactly the ones representable in Min, and the semi-decidable sets are exactly the ones semi-representable in Min. Know that every decidably axiomatizable theory is semi-decidable.

*Sample question:* Suppose that $X$ is a decidable set of newline-free strings and $R$ is a decidable binary relation between newline-free strings. Show that the set of strings in which every line is either a member of $X$ or bears $R$ to some previous line is decidable, and that the closure of $X$ under $R$ is semi-decidable.

> A maximally rigorous answer to a question like this would involve actually writing out programs Py (built up from the given programs that decide $X$ and $R$ respectively). But on the exam, I'll be happy if you can write out in English an unambiguous description of an appropriate mechanistic computational procedure (appealing to the Church-Turing thesis to justify the assumption that it could in principle be implemented in Py, or by a Turing machine, or in any other adequate model of computation).

*Sample question:* (a) Show that when $R$ and $S$ are decidable binary relations among strings, $S \circ R$ is semi-decidable. (b) Give an example where $S \circ R$ is not decidable.

> *Hint:* there are lots of possibilities here, but it might help to note that if $S = R^{-1}$, $\langle x, x \rangle \in S \circ R$ iff there is a $y$ such that $Rxy$, so any case where $R$ is decidable but $\{x \mid Rxy \text{ for some } y\}$ can be turned into an example of what you're looking for.