Advanced Logic: Problem Set 02

- Due date: Friday, September 16.
- Your proofs may appeal to any facts stated in the first three lectures.

1. (80%) Suppose we have a function $f : A \to B$. Let $f^* : \mathscr{P}B \to \mathscr{P}A$ be the function such that for any $Y \in \mathscr{P}B, f^*Y = \{x \in A \mid fx \in Y\}$.

   (a) Break down $f : A \to B$, this statement assumes that there is a function(a relation(Cartesian product of two elements from A and from B) that is both serial and functional) from set A to set B

   (b) Break down $f^* : \mathscr{P}B \to \mathscr{P}A$. This statement assumes that there exists a power set of B and a power set of A and assumes that there is a function from the one to the other

   (c) Combining the condition (a) and condition (b), the question asks us to prove that with all these condition above, we should have a causation structure that $Y \in \mathscr{P}B, f^*Y = \{x \in A \mid fx \in Y\}$, this is a single direction conditional statement.

   (d) Break down $Y \in \mathscr{P}B, f^*Y = \{x \in A \mid fx \in Y\}$, This Disgusting Notation represents that 2 assumptions. The first assumptions assumes that there is a set Y that is the element of power set of the set B, this $\mathscr{P}B$ is consistent with the above definition. The second assumption states that function f^* takes in a set called Y and spits out another set that is $\{x \in A \mid fx \in Y\}$ this structure itself is a biconditional structure, it states that(left) assumes that there is an element of set A, then function f can take this element and the result it spits out is an element of set Y. It (right) assumes that there is an result of taking x to function f and the result belongs to Y and x is an element of A. On the other direction of the equation, if there exists(we assumes) an element that can be take in by the function f^* and the result is an element of set Y, then this element also belongs to the set A.

   (e) Above are my understanding of the question and assumptions, and now I will start my proof. None of the above are my assumptions, thoses are assumptions coming from the question itself.

   (f) Now, I assumes that there exists an arbitrart element a in set A, namely any element(along) in set A but not an $\emptyset$. Let's start the journey of this little a. By the definition of power set, any arbitary elment in a set is also in its power set. Thus, a is also in $\mathscr{P}A$. Now, let's consider the result of function of f taking in a and what it's gonna spits out. $f(a)$ would be on set B by (a). We can use another label a_1, to represents the result of a after going through function f. Since a_1 is an element of set B, it is also an element of power set of B. Thus, a_1 is any arbitary element of power set of B. Therefore, a_1 is an element of power set of B.

(g) By observing the structure of (d), we found out that there is a label for any arbitary element in power set of B which is Y. Thus, we find out a_1 and Y are the same. Y is just another name for a_1. Therefore we prove the first assumption of (d). Now, we should using a_1 to prove the second assumption.

(h) The second assumption has two structure one is the equation sign and the other is the biconditional set builder.Let's consider the set builder first.

(i) (left to right) assumes that there is an arbitary element in set A and it is called x(left). From the assuption and (a) and conclusion(g), we can conclude that the result of input x into function f which is fx is an element of set Y. Thus, one direction is proved.

(j) (right to left) assumes that there exists an element x and it will be putted into function, and the result of it is an element of set Y. Since x is an arbitary element and fx is on B, thus x is also an element of set A because of (a). Thus, the second direction is proved.

(k) Combing (i) and (j), we showed the structure of the nasty set. And It is same for the left side of the equation sign. Here is why.

(l) Since Y is an input variable that we plug in to the function which is f^*Y, the result of that function is in $\mathscr{P}A$. So, the journey of a ends with power set of A by jumping from a to a_1 to Y to $\mathscr{P}A$. Thus, we proved the equation sign.

(m) Similary, that is same from the other direction of the equation sign. As we consider biconditional structure of the set builder which is the result of synthesizing (a) and (b) and (d) and either left($\{x \in A\}$) or right($\{fx \in Y\}$) of the set builder. To be clear, in case of (d), we treat fx as a element of set Y and element of power set of B. Thus, x also end its journey at power set of A. Thus, we proved the equation sign from the left side and the right side.

(n)

2. (50%) Show that f is injective iff f^* is surjective.

   (a) Since we already do the reasoning above, we can just use the conclusion of the reasoning and take it as granted.

(b) Assume that f^* is surjective which means that for every y in B there is some x in A. From (1)-(c), there exists an relation that $f^*Y = \{x \in A \mid fx \in Y\}$. To prove it is injective, we need to that when relation exists, and there is an arbitary element in set A which is x, then there is no other element in set A that can be putted into the relation and the result is the same. The existence of the relation is the thing we need to solve. Thankful, we already have the relation from the conclution of (1)-(c). Thus, we can use the relation to prove that f is injective. This x is an element of A.

3. (30%) Show that f is surjective iff f^* is injective.

   (a) Same Framework

   (b) left side: assume that f is surjective, then we can conclude that for every element y in set B, there is some x in set A such that $f(x) = y$. Injective property states that there is one exact x that has relation with y. The definition of injective property needs two condition and one result. if Rxy and Rx'y, x = x'. The functional property assumes the single property(exact one) of y. and for every y we have some x, and for every x there is some y(Rxy). Since every y has some x and y can only be in a form of exact one. Thus, x can only be in a form of exact one x. Thus, Rxy, Rx'y, x = x'. Thus, it is injective.

   (c) right side: assumes that f^* has serial, functional, and injective property by the given information of the question. To prove it is surjective, we need to show for every y, there is some x. Now, we start the proof. Serial property proves that for every x there is some y. Serial property states that every x there is some y. Functional property states that it is the exact y. Injective property requires that we have some x from every y. Therefore, by combing all 3 property, for every y(we get some y, at least, by serial property, we have B is at least as big as A), By the uniqueness of x and y(functional and injective), for every y, there is some x. Thus, it is proved.

4. (a) (10%) Using the Axiom of Separation to show that there is no set that contains all sets. (Hint: adapt the reasoning in Russell's Paradox.)

      i. first of all, I am confused with the question that the "Axiom of Seperation" and the definition in the power point "Axiom Schema of Seperation". Are Axiom and Axiom of Seperation the same thing in this class? For the following proof, I will assume this is the definition in the power point.

      ii. Axiom Schema of Separation: For every set A, there is a set $\{x \in A \mid \phi(x)\}$ whose elements are all and only those objects x such that $x \in A$ and $\phi(x)$

      iii. Assumes that there exists a set that contains all sets. Let's call it A. According to the question, A is the set that contains all set. Therefore, A contains itself. However, according to the Axiom

Schema of Seperation and the definition of the notation of ϕ, it is saying that x is blank.(by discussing with professor). Thus, either the deinition of ϕ is wrong or the definition of axiom schema of seperation is wrong or the assumption that set contains all set is wrong. Thus, the assumption is wrong. Thus, this set does not exists.

iv.

(b) (10%) Show that for any set A, there is no injective function from $\mathscr{P}A$ to A.

i. The injective property of a function requires an exact x from B to A. The relation $\mathscr{P}A$ and A is that A is a subset(include itself) of $\mathscr{P}A$. Assumes that A is a set that contains all the set in the universe including it self. Then, apply a similar reasoning of Russell paradox, we can conclude that this injective property is not satisfied because it reference to it self therefore it is not the exact x. By giving an counterexample, I showed there is no injective function from $\mathscr{P}A$ to A

(c) (10%) Suppose that $f : A \to B$ and $g : B \to A$ are functions such that for any $x \in A, x = g(fx)$, and for any $y \in B, y = f(gy)$. Show that $g = f - 1$.

i. $f : A \to B$

ii. $g : B \to A$

iii. above are causation, and the below is the effect.

iv. $x \in A, x = g(fx)$

v. any $y \in B, y = f(gy)$

vi. above is the causation, and the below is the final effect we need to prove.

vii. $g = f - 1$

viii. The causation chain of the question has three levels, the second level has two parts. All the above are assumptions given by the question, and now I will start my proof.

ix. first level:

x. Since $f : A \to B$ and $g : B \to A$, we can assume that there is an element a that is in set A, and another element b in set B, and it has an relation Rab that is bijection by definition of bijection.

xi. By (iv) and (v), we have that fx is in set B, g(fx) is in set A. so it is jumping back and forth. On the other side, gy is in set A, and f(gy) is in set B. Thus, it is also jumping back and forth. Together, they have a bijection relation. The assumption of the question is consistent with my assumptions(I know we are safe, yeah!).

xii. To show g = f - 1, we need to show that g is a subset of f - 1 and f - 1 is a subet of g.

xiii. From left to right: Since we have a bijection relation. By the definition of equinumerious, $A \sim B$. Then B is at least as big as A and A is at least as big as B.($A \precsim B$ and $B \precsim A$), thus. (suspisious)Assume that A is DK-infinite, then B is also DK-infinite, (equinumerious), By Cardinal Comparability Theorem, which we have have not proved yet because it need (AOC),

$$\text{For any sets } A \text{ and } B, \text{ either } A \precsim B \text{ or } B \precsim A.$$

I give the above sentence a whole sentence space because it deserves it.

xiv. This correponds with my suspisious assumption that A is DK-infite by using the law of excluded the middle. Since B is at least as big as A, A is equinumerious with some subset of B, and there is an injective function from A to B. Thus, f-1 is a subset of g.

xv. We proved one direction of the equation sign, the other side is the same structure by Cardinal Comparability Theorem and Axiom of Choice.

xvi. However, the above reasoning and proof is flawed. Because we have not proved the Axiom of Choice yet.