Problem Set 1 - second try Advanced Logic, Fall 2020

As I explained, I am making this available to allow for the possibility of a "do-over" if you are not satisfied with the answers you turned in for Problem Set 1 on Friday.

If you wish, you may turn solutions any of the following problems at any time before class on Tuesday September 13th. If you turn in a solution, it will replace whatever you may have turned in the first time around for the correspondingly numbered problem.

If you're turning in answers for this problem set, please help us keep track, by listing any cases where you want us to *keep* what you turned in first time around, and if possible, copy and paste the answer you want us to keep into the document containing your answers to these new problems.

1. (50%) Prove that for any sets A and B, $A \subseteq B$ if and only if $A \backslash B = \varnothing$.

   (a) Assume that $A \subseteq B$ and show that $A \backslash B = \varnothing$.

      i. Since $A \subseteq B$, for any element a in A, $a \in A$, $a \in B$.
      ii. By the definition of $\backslash$, $a \in A \backslash B$ if and only if $a \in A$ and $a \notin B$.
      iii. Therefore, if $a \in A \backslash B$, then $a \in A$ and $a \notin B$.
      iv. Combining (i) and (iii), we reach that $a \in B$ and $a \notin B$, thus $a \neq a$.(contradiction)
      v. By the definition of $\varnothing$, $\varnothing := \{x \mid x \neq x\}$
      vi. and substitue x for a by(iv), $a \neq a$, therefore, we satisfied the definition of empty set
      vii. for every element in $A \backslash B$, $a \in A \backslash B$, $a \in A$, $a \notin B$, $a \neq a$, $a \in \varnothing$ therefore, $A \backslash B \subseteq \varnothing$
      viii. By definition of $\varnothing$, $\varnothing$ is the subset of anyset, Thus, $\varnothing \subseteq A \backslash B$
      ix. Therefore, $A \backslash B = \varnothing$

   (b) Assume that $A \backslash B = \varnothing$ and show that $A \subseteq B$.

      i. To prove that A is a subset of B, we need to show that for any element a in A, $a \in B$.
      ii. With the assumption that $A \backslash B = \varnothing$, for any element a in $A \backslash B$, $a \in A$, $a \notin B$ and $a \in \varnothing$
      iii. Since a represents any element in A, $a \in A$ and $a \in \varnothing$, $A \subseteq \varnothing$

iv. Since $\varnothing$ is the subset of any set, $\varnothing \subseteq A$
v. Combining (iii) and (iv), we reach that $A = \varnothing$
vi. Thus, $A \subseteq B$ (empty set is the subset of any set)

2. (30%) Prove that whenever R is a relation from A to B and S is a relation from B to C,
   (a) If R and S are both surjective, then $S \circ R$ is surjective.
      i. By the definition of surjective, for every element in set C there is some element in set B such that $\langle b, c\rangle \in S$.
      ii. And for every element in set B there is some element in set A such that $\langle a, b\rangle \in R$.
      iii. Combining(i) and (ii), for every element in set C there is some element in set B and for every element in set B there is some element in A
      iv. By the definition of composition($S \circ R$), $\{\langle x, z\rangle \in A \times C \mid$ there exists $y \in B$ such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S\}$, from A to C
      v. Substitue x, y, z with a, b, c, respectively, we reach that $\{\langle a, c\rangle \in A \times C \mid$ there exists $b \in B$ such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S\}$
      vi. By the definition of ϕ(generic act), Axiom of Product Existence(I dont know how far I should go for the existence proof, it's a big rabbit hole), there exists a, b, c.
      vii. So, $S \circ R$ is surjective.
   (b) R and S are both injective, then $S \circ R$ is injective.
      i. By the definition of injective, R is injective iff whenever Rxy and $Rx'y, x = x'$.
      ii. By the given condition that R is injective, Rab and $Ra'b, a = a'$.
      iii. And S is injective, Sbc and $Sb'c, b = b'$.
      iv. By the definition of composition($S \circ R$), $\{\langle x, z\rangle \in A \times C \mid$ there exists $y \in B$ such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S\}$, from A to C
      v. Substitue x, y, z with a, b, c, respectively, we reach that $\{\langle a, c\rangle \in A \times C \mid$ there exists $b \in B$ such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S\}$
      vi. Similarly to (a), we reach that $S \circ R$ is injective.

4. (10%) Let A $= \{a, b\}$ be a 2-membered set B $= \{c, d, e\}$ be a three-membered set. List all the functions from A to B (there are $9 = 3^2$) and all the functions from B to A (there are $8 = 2^3$). For each of these 17 functions, specify whether it is injective and whether it is surjective.
   (a) A $= \{a, b\}$ be a 2-membered set B $= \{c, d, e\}$ be a three-membered set.
   (b) List all the functions from A to B (there are $9 = 3^2$)
      i. With the condition: $f(a) = c$

ii. $f_1 = \{\langle a, c\rangle, \langle b, d\rangle\}$, Injective
iii. $f_2 = \{\langle a, c\rangle, \langle b, e\rangle\}$, Injective
iv. $f_3 = \{\langle a, c\rangle, \langle b, c\rangle\}$, Surjective
v. With the condition: $f(a) = d$
vi. $f_4 = \{\langle a, d\rangle, \langle b, c\rangle\}$, Injective
vii. $f_5 = \{\langle a, d\rangle, \langle b, d\rangle\}$, Surjective
viii. $f_6 = \{\langle a, d\rangle, \langle b, e\rangle\}$, Injective
ix. With the condition: $f(a) = e$
x. $f_7 = \{\langle a, e\rangle, \langle b, c\rangle\}$, Injective
xi. $f_8 = \{\langle a, e\rangle, \langle b, d\rangle\}$, Injective
xii. $f_9 = \{\langle a, e\rangle, \langle b, e\rangle\}$, Surjective

(c) all the functions from B to A (there are $8 = 2^3$)

i. With the condition: $f(c) = a$
ii. $g_1 = \{\langle c, a\rangle, \langle d, a\rangle\}$, Surjective
iii. $g_2 = \{\langle c, a\rangle, \langle d, b\rangle\}$, Injective
iv. $g_3 = \{\langle c, a\rangle, \langle e, a\rangle\}$, Surjective
v. $g_4 = \{\langle c, a\rangle, \langle e, b\rangle\}$, Injective
vi. With the condition: $f(c) = b$
vii. $g_5 = \{\langle c, b\rangle, \langle d, a\rangle\}$, Injective
viii. $g_6 = \{\langle c, b\rangle, \langle d, b\rangle\}$, Surjective
ix. $g_7 = \{\langle c, b\rangle, \langle e, a\rangle\}$, Injective
x. $g_8 = \{\langle c, b\rangle, \langle e, b\rangle\}$, Surjective

5. (10%) Prove that where R is a relation from A to B and S and T are relations from B to $C, (S \cup T) \circ R = (S \circ R) \cup (T \circ R)\cdot$

(a) To prove that $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$, we need to prove that

(b) Left Side: $(S \cup T) \circ R$ is a subset of $(S \circ R) \cup (T \circ R)$

i. Since binary relations can be defined as a set of ordered pairs which is a subset of Cartesian Product.
ii. It is a special case of set, thus, we can use set operations on it.
iii. S and T are some relations from B to C. Therefore, the Union of S and T is also a relation from B to C, we can call it K.
iv. $K \circ R$ means that the composition of R and K is the relation that $\{\langle x, z\rangle \in A \times C \mid \text{there exists } y \in B \text{ such that } \langle x, y\rangle \in R$ and $\langle y, z\rangle \in K\}$, from A to C
v. So, every ordered pair $\langle x, z\rangle$ in $K \circ R$ is a subset of $(S \cup T) \circ R$
vi. Next, I am gonna prove that every ordered pair $\langle x, z\rangle \subseteq (S \cup T) \circ R$ is a subset of $(S \circ R) \cup (T \circ R)$
vii. By the definition of composition, $S \circ R$ is the composition of R and S, which is the relation that $\{\langle x, z\rangle \in A \times C \mid \text{there exists}$ $y \in B$ such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S\}$, from A to C

viii. Thus, every ordered pair $\langle x, z\rangle \subseteq S \circ R$

ix. Similarly, $\langle x, z\rangle \subseteq T \circ R$

x. Thus, (vi) is proved.

xi. Combining (vi) and (v), we can conclude that $(S \cup T) \circ R$ is a subset of $(S \circ R) \cup (T \circ R)$

(c) Right Side: $(S \circ R) \cup (T \circ R)$ is a subset of $(S \cup T) \circ R$

i. Using the same method as the left side, but assuming that $(S \circ R) \cup (T \circ R)$, the ordered pair $\langle x, z\rangle$ is a subset of $(S \cup T) \circ R$

6. (10%) Show that where R is a relation from A to B,

(a) R is surjective iff id B $\subseteq R_\circ R^{-1}$

i. From left to right

A. Assume that R is a relation from A to B, it is surjective iff for every $b \in B$, there exists some $a \in A$ such that $\langle a, b\rangle \in R$

B. By the definition of converse, R^{-1} is the relation that $\{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$, from B to A

C. Since R is surjective, for every $b \in B$, there exists some $a \in A$ such that $\langle a, b\rangle \in R$

D. According to the assumption of set B exists, and the definition of id_B, for any set, $\langle b, b\rangle \in \text{id}_B$

E. Thus, besides, R is surjective from A to B, R is also surjective from R to R.

F. Since the composition of a relation and its converse is the identity relation, $R_\circ R^{-1}$ is the identity relation, which is the same as id_B

G. Thus, $\text{id}_B \subseteq R_\circ R^{-1}$

ii. From right to left

A. From right to left

B. It is assumed that B exists, therefore id_B also exists.

C. It is also assumed that R exists, thus, $R_\circ R^{-1}$ is just R itself by definition of composition

D. According to the PowerPoint(a few noteworthy fact), the composition of id_B and R is just R itself.

E. thus, $\text{id}_B \subseteq R$

F. thus, for every $b \in B$, there exists some $a \in A$ such that $\langle a, b\rangle \in R$ as R is defined by a relation from A to B

G. As a result, R is surjective.

(b) R is functional iff $R_\circ R^{-1} \subseteq \text{id}_\text{B}$