

Problem Set 4

Advanced Logic

26th September 2022

Throughout this problem set, A is some arbitrary set, and A^* is the set of lists over A ; s, t, u are arbitrary members of A^* . \oplus denotes the list concatenation operation on A^* , defined to obey the following recursion clauses:

$$\begin{aligned}\square \oplus t &= t \\ (a : s) \oplus t &= a : (s \oplus t)\end{aligned}$$

Note that you can and should rely on earlier results in proving later ones.

1. (15%) Prove that if $s \oplus t = s$ then $t = \square$.
 - (a) $s \oplus t = s$
 - (b) iff $(s : \square) \oplus t = s$, since $[s] = (s : \square)$
 - (c) iff $s : (\square \oplus t) = s$, by clause 2
 - (d) iff $s : t = s$, by clause 1
 - (e) iff $t = \square$
 - (f) therefore, we reach our conclusion here.
2. (15%) Prove that if $s \oplus t = \square$ then $s = t = \square$.
 - (a) by the injective property of Axiom of the List, we have that \square is not in the range of any $cons_a$
 - (b) if $s \oplus t = \square$, then $s = \square$
 - (c) if $s \oplus t = \square$, then $s = \square$
3. (15%) Prove that if $s \oplus t = (a : u)$ then either $s = \square$ or $s = (a : s')$ for some s' .

For the following problems, we define a function $\text{final} : A^* \rightarrow \mathcal{P}(A^*)$ recursively as follows:

$$\begin{aligned}\text{final } \square &= \{\square\} \\ \text{final}(a : s) &= \text{final } s \cup \{(a : s)\}\end{aligned}$$

We say that s is a final sublist of t iff $s \in \text{final } t$.

5. (15%) Prove that \square is a final sublist of every list.

6. (10%) Prove that every list is a final sublist of itself.
7. (10%) Prove that t is a final sublist of $s \oplus t$.
8. (10%) Prove that if s is a final sublist of t , then $t = u \oplus s$ for some u .
9. (10%) Prove that if $s \oplus s' = t \oplus t'$ then either s' is a final sublist of t' or t' is a final sublist of s' .
10. (10% extra credit) Play through the level 'Advanced Multiplication World' in the Lean Natural Numbers Game (https://www.ma.imperial.ac.uk/buzzard/xena/natural_number_game/). To show that you've completed the levels, send us a screenshot of the last level of Advanced Multiplication World open on your computer screen, with your name showing somewhere in the screenshot (e.g. in a text editor window).