Problem Set 10

Advanced Logic

27th November 2022

Due date: Friday, 2 December.

Note: the scores for these problems add up to 110%, so a perfect score corresponds to 10% extra credit.

1. Let M be the theory in the language of strings axiomatized by all of the following sentences (where c may be any constant of the language of strings other than "")

$$\begin{array}{ll} \text{M1} & \forall x(x = \texttt{""} \oplus x) \\ \text{M2} & \forall x(x = x \oplus \texttt{""}) \\ \text{M3} & \forall x \forall y \forall z((x \oplus y) \oplus z = x \oplus (y \oplus z)) \end{array}$$

Show the following:

(a) (20%) Show that for any length-one string a, if c is the constant that denotes a in the standard string structure $\mathbb{S}$, then $c = \langle a \rangle$ is a theorem of M.

(b) (30%) Show that for any strings s_1 and s_2, $\langle s_1 \rangle \oplus \langle s_2 \rangle = \langle s_1 \oplus s_2 \rangle$ is a theorem of M.

> *Hint:* use induction on s_1.

(c) (30%) Using what you showed in parts (a) and (b), prove that for any closed term t in the language of strings, $t = \langle [\![t]\!]_{\mathbb{S}} \rangle$ is a theorem of M.

> *Reminder:* $[\![t]\!]_{\mathbb{S}}$ is the denotation of t in the standard string structure.
> *Hint:* use induction on the construction of t.

(d) (15%) Let t_1 and t_2 be any closed terms in the language of strings. Using what you showed in (c), prove that if the sentence $t_1 = t_2$ is true in $\mathbb{S}$, then it is a theorem of M.

2. Let M+ be the result of adding to M:, for any two constants c and c' of the language of strings other than "", each of the following axioms:

$$\begin{array}{ll} \text{M4} & \forall x(c \oplus x \neq \texttt{""}) \\ \text{M5} & \forall x \forall y(c \oplus x = c \oplus y \to x = y) \\ \text{M6} & \forall x(c \oplus x \neq c' \oplus x) \end{array}$$

(a) (5%) Show that for any two distinct strings s and t, $\langle s \rangle \neq \langle t \rangle$ is a theorem of M+.

> *Hint:* use induction on t.

(b) (5%) Using what you showed in part (a) and in part (c) of the previous exercise, conclude that for any closed terms t_1 and t_2 of the language of strings, if $t_1 \neq t_2$ is true in $\mathbb{S}$, it is a theorem of $M+$.

(c) (5%) Using what you showed in part (b) of this exercise and part (d) of the previous exercise, conclude that if P is any sentence of the language of strings that does not include any quantifiers or the predicate $\leq$ and is true in $\mathbb{S}$, P is a theorem of M+.

> *Hint:* Use induction on the construction of P, with the induction hypothesis *whichever of* P *and* $\neg P$ *is true in* $\sim$ *is a theorem of* $M+$.