Problem Set 9

Advanced Logic

9th November 2022 (apologies for late release!)

Due date: Friday, 11 November.

1. Suppose that h is a homomorphism from S to S', where S and S' are structures for some signature Σ.

   (a) (30%) Show that $[\![t]\!]_{S'}^{h \circ g} = h([\![t]\!]_S^g)$ for every term t of Σ and assignment g for S.

   (b) (30%) Use the previous result to show that if h is an *embedding*, then for every *quantifier-free* formula P of Σ and assignment g for S, $S', h \circ g \Vdash P$ iff $S, g \Vdash P$.

   (c) (20%) Show that if h is an *isomorphism*, the claim in part (b) is true for all formulae (including those with quantifiers).

   (d) (5%) Explain why (c) implies that isomorphic structures are elementarily equivalent.

   > *Definitions:* If S has domain D and interpretation function I and S' has domain D' and interpretation function I', a *homomorphism* from S to S' is a function $h : D \to D'$ such that for every n-ary function symbol f of Σ, $h(I_f(x_1, \ldots, x_n) = I'_f(hx_1, \ldots, x_n)$ and for every n-ary predicate F of Σ, $\langle hx_1, \ldots, hx_n \rangle \in I'_F$ whenever $\langle x_1, \ldots, x_n \rangle \in I_F$.
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   > An *embedding* is a homomorphism h with the further properties that (i) h is injective and (ii) $\langle hx_1, \ldots, hx_n \rangle \notin I'_F$ whenever $\langle x_1, \ldots, x_n \rangle \notin I_F$.
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   > An *isomorphism* is an embedding that is surjective.
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   > Two structures are *elementarily equivalent* iff the same sentences are true in them.

2. (15%) Prove the following (thus filling in one of the steps in the proof of the Completeness Theorem that was left as an exercise): if a set of formulae Γ is consistent and negation-complete, then for any formulae P and Q, $P \vee Q \in \Gamma$ iff either $P \in \Gamma$ or $Q \in \Gamma$.

EXTRA CREDIT

1. (5%) Prove the following (thus filling in another missing step in the proof of the Completeness Theorem): if a set of formulae Γ is consistent and negation-complete, then for any formulae P and Q, $P \to Q \in \Gamma$ iff either $P \notin \Gamma$ or $Q \in \Gamma$.

2. (5%) Prove the following (thus filling in the last missing step in the proof of the Completeness Theorem): if a set of formulae Γ is consistent, negation-complete, and witness-complete then for any formula P and variable v, $\exists v P \in \Gamma$ iff there is some term t such that $P[t/v] \in \Gamma$.

3. (10% --- more of a challenge) Show that if a structure S has a finite domain, then there is a sentence P true in S such that any other structure for the same signature in which P is true is isomorphic to S.

Hint: Try a sentence that begins with $\exists x_1 \ldots \exists x_n$, where n is the size of S's domain. What goes inside the existential quantifiers will be a big conjunction that, intuitively, says that $x_1, \ldots, x_n$ are the only things there are and that they are all distinct, and fully characterises how they are related by the function symbols and predicates of the signature.