Solutions to Problem Set 7

Advanced Logic

15th December 2022

1. (50%) Show that $\Gamma, P, Q \vdash R$ if and only if $\Gamma, P \wedge Q \vdash R$ (for any formulas P, Q, R and set of formulas Γ of some first-order language $\mathcal{L}(\Sigma)$).

> Left to right: suppose $\Gamma, P, Q \vdash R$. Then $\Gamma, P \vdash Q \to R$ and $\Gamma \vdash P \to (Q \to R)$ by $\to$Intro, and thus $\Gamma, P \wedge Q \vdash P \to (Q \to R)$ by Weakening. By Assumption $P \wedge Q \vdash P \wedge Q$, so by Weakening $\Gamma, P \wedge Q \vdash P \wedge Q$, so by $\wedge$Elim, $\Gamma, P \wedge Q \vdash P$ and $\Gamma, P \wedge Q \vdash Q$. Hence by $\to$Elim we have $\Gamma, P \wedge Q \vdash Q \to R$ and by a second appleal to $\to$Elim, $\Gamma, P \wedge Q \vdash R$.
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> Right to left: suppose $\Gamma, P \wedge Q \vdash R$. Then $\Gamma \vdash P \wedge Q \to R$ by $\to$Intro, and so $\Gamma, P, Q \vdash P \wedge Q \to R$ by Weakening. Also, $P \vdash P$ and $Q \vdash Q$ by Assumption, so $\Gamma, P, Q \vdash P$ and $\Gamma, P, Q \vdash Q$ by Weakening, and hence $\Gamma, P, Q \vdash P \wedge Q$ by $\wedge$Intro. Hence finally we can appeal to $\to$Elim to conclude that $\Gamma, P, Q \vdash R$.
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> *Note:* you could avoid having to bring in the $\to$ rules by appealing to the *Cut* principle according to which if $\Gamma, P \vdash Q$ and $\Delta \vdash P$, $\Gamma, \Delta \vdash Q$. We proved this in lecture.

2. (30%) Show that the following three conditions on a set of formulae Γ are equivalent:

   a. $\Gamma \vdash P$ and $\Gamma \vdash \neg P$ for some P
   b. $\Gamma \vdash Q$ for every formula Q
   c. $\Gamma \vdash \neg\forall x(x = x)$

> To show that (a) implies (b), suppose $\Gamma \vdash P$ and $\Gamma \vdash \neg P$, and let Q be an arbitrary formula. By Weakening, $\Gamma, \neg Q \vdash P$ and $\Gamma, \neg Q \vdash \neg P$, so by $\neg$Intro, $\Gamma \vdash \neg\neg Q$. Finally by DNE we can conclude that $\Gamma \vdash Q$.
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> It is obvious that (b) implies (c).
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> To show that (c) implies (a), it suffices to show that $\Gamma \vdash \forall x(x = x)$ for all Γ. By $=$Intro, $\vdash x = x$, so by $\forall$Intro, $\vdash \forall x(x = x)$ (this is legitimate since the variable x is not free in the empty set). Thus by Weakening, $\Gamma \vdash \forall x(x = x)$ for any Γ.

3. (10%) Show that for any terms t_1, t_2, t_3 and variable v:

   a. $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$
   b. $t_1 = t_2 \vdash t_2 = t_1$
   c. $t_1 = t_2 \vdash t_3[t_1/v] = t_3[t_2/v]$

(a.) Choose a variable v not free in any of $t_1 - -t_3$, and note that $t_2 = t_3$ is $(v = t_3)[t_2/v]$ while $t_1 = t_3$ is $(v = t_3)[t_1/v]$. By Assumption and Weakening, $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_2$ and $t_1 = t_2, t_2 = t_3 \vdash t_2 = t_3$, so by =Elim, $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$.

(b.) Choose a variable v not free in t_1 or t_2 and note that $t_1 = t_1$ is $(v = t_1)[t_1/v]$, while $t_2 = t_1$ is $(v = t_1)[t_2/v]$. By =Intro we have $\vdash t_1 = t_1$ and by Weakening $t_1 = t_2 \vdash t_1 = t_1$; also by Assumption, $t_1 = t_2 \vdash t_1 = t_2$. Thus, we can apply =Elim to conclude that $t_1 = t_2 \vdash t_2 = t_1$.

4. (10%) Show that $\forall vP \dashv\vdash \neg\exists v\neg P$ for every formula P.

For this one I'm actually going to draw proof trees, and I'll appeal to the 'mixed premise' versions of the inference rules (see lecture 12). Right to left:

$$\dfrac{\dfrac{}{\forall vP, \exists v\neg P \vdash \forall vP}\,\mathrm{A}_w \qquad \dfrac{\dfrac{\dfrac{}{\exists v\neg P \vdash \exists v\neg P}\,\mathrm{A} \qquad \dfrac{\dfrac{\dfrac{}{\forall vP \vdash \forall vP}\,\mathrm{A}}{\forall vP \vdash P}\,\forall\mathrm{E} \qquad \dfrac{}{\neg P \vdash \neg P}\,\mathrm{A}}{\neg P \vdash \neg\forall vP}\,\neg\mathrm{I}_w}{\exists v\neg P \vdash \neg\forall vP}\,\exists\mathrm{E}_w}{\forall vP, \exists v\neg P \vdash \neg\forall vP}\,\mathrm{W}}{\forall vP \vdash \neg\exists v\neg P}\,\neg\mathrm{I}$$

And left to right:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P \vdash \neg P}\,\mathrm{A}}{\neg P \vdash \exists v\neg P}\,\exists\mathrm{I} \qquad \dfrac{}{\exists v\neg P \vdash \exists v\neg P}}{\neg\exists v\neg P \vdash \neg\neg P}\,\neg\mathrm{I}_w}{\neg\exists v\neg P \vdash P}\,\mathrm{DNE}}{\neg\exists v\neg P \vdash \forall vP}\,\forall\mathrm{I}$$

EXTRA CREDIT 10% for any of the following:

1. Show that for every formula P of $\mathcal{L}(\varnothing)$ (the first-order language with no non-logical constants at all), either $\forall x\forall y(x = y) \vdash P$ or $\forall x\forall y(x = y) \vdash \neg P$.

By induction on the construction of the formula P.

Base case: P is an atomic formula $u = v$. Then we have $\forall x\forall y(x = y) \vdash P$:

$$\dfrac{\dfrac{\dfrac{}{\forall x\forall y(x = y) \vdash \forall x\forall y(x = y)}\,\mathrm{A}}{\forall x\forall y(x = y) \vdash \forall y(u = y)}\,\forall\mathrm{E}}{\forall x\forall y(x = y) \vdash u = v}\,\forall\mathrm{E}$$

Induction steps:

(i) $\neg P$ has the property in question if P does, since if it's not the case that $\forall x\forall y(x = y) \vdash \neg P$, then $\forall x\forall y(x = y) \vdash P$ by the induction hypothesis, in which case $\forall x \forall y(x = y), \neg P \vdash P$ by Weakening and $\forall x \forall y(x = y), \neg P \vdash \neg P$ by Assmption and Weakening, so $\forall x \forall y(x = y) \vdash \neg\neg P$ by $\neg$Intro.

(ii) $P \wedge Q$ has the property if both P and Q does. For if $\forall x \forall y(x = y) \vdash P$ and $\forall x \forall y(x = y) \vdash Q$, $\forall x \forall y(x = y) \vdash P \wedge Q$ by $\wedge$Intro. Meanwhile, if $\forall x \forall y(x = y) \vdash \neg P$, then since $P \wedge Q \vdash P$ by Assumption and $\wedge$Elim1, $\forall x \forall y(x = y) \vdash \neg(P \wedge Q)$ by $\neg\text{Intro}_w$; analogous reasoning applies if $\forall x \forall y(x = y) \vdash \neg Q$.

(iii) $P \vee Q$ has the property if both P and Q do: similar to (ii), using $\vee$Intro and $\vee$Elim.

(iv) $P \to Q$ has the property if P and Q do. If $\forall x \forall y(x = y) \vdash Q$, then $\forall x \forall y(x = y) \vdash P \to Q$ by Weakening and $\to$Intro. If $\forall x \forall y(x = y) \vdash \neg P$, then $\forall x \forall y(x = y), P \vdash \neg P$ and $\forall x \forall y(x = y), P \vdash P$ by Weakening and Assumption, so $\forall x \forall y(x = y), P \vdash Q$ by Explosion (see lecture 11), so $\forall x \forall y(x = y) \vdash P \to Q$ by $\to$Intro. Finally, if $\forall x \forall y(x = y) \vdash P$ and $\forall x \forall y(x = y) \vdash \neg Q$, $\forall x \forall y(x = y) \vdash \neg(P \to Q)$ by $\neg$Intro, since we have $\forall x \forall y(x = y), P \to Q \vdash Q$ by $\to\text{Elim}_w$ and Assumption.

(v) Suppose P has the property. Then if $\forall x \forall y(x = y) \vdash P$, also $\forall x \forall y(x = y) \vdash \forall v P$ by $\forall$Intro. On the other hand, if $\forall x \forall y(x = y) \vdash \neg P$, then $\forall x \forall y(x = y), \forall v P \vdash \neg P$ by Weakening, but also $\forall v P \vdash P$ by $\forall$ELim, so $\forall x \forall y(x = y) \vdash \neg\forall v P$ by $\neg$Intro.

(v) Suppose P has the property. Then if $\forall x \forall y(x = y) \vdash P$, $\forall x \forall y(x = y) \vdash \exists v P$ by $\exists$*Intro*. On the other hand, if $\forall x \forall y(x = y) \vdash \neg P$, then $\forall x \forall y(x = y), P \vdash \neg P$ by Weakening, while $\exists v P \vdash \exists v P$ by Assumption, so $\forall x \forall y(x = y), \exists v P \vdash \neg P$ by $\exists\text{Elim}_w$ ******

2. Suppose F is a singulary predicate of Σ. Define a function $r_F : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma)$ as follows:

$$\begin{aligned}
r_F P &= P \quad \text{when } P \text{ is atomic} \\
r_F(\neg P) &= \neg r_F P \\
r_F(P \to Q) &= r_F P \to r_F Q \\
r_F(P \wedge Q) &= r_F P \wedge r_F Q \\
r_F(P \vee Q) &= r_F P \vee r_F Q \\
r_F(\forall v P) &= \forall v(Fv \to r_F P) \\
r_F(\exists v P) &= \exists v(Fv \wedge r_F P)
\end{aligned}$$

Show that $r_F[\Gamma], F(v_1), \ldots, F(v_n) \vdash r_F P$ whenever $\Gamma \vdash P$, where $v_1, \ldots, v_n$ are the free variables in Γ and P.

APOLOGY: I'm afraid I must admit that the thing I've asked you to prove here is not true! To see this, take $\Gamma = \emptyset$ and $P = \exists x(x = x)$, so that $r_F[\Gamma] = \emptyset$ and $r_F(P) = \exists x(F(x) \wedge x = x)$. There are no free variables in P, so for the claim in question to be true it would have to be the case that $\vdash r_F(P)$; but in fact there $\exists x(F(x) \wedge x = x)$ is not a theorem (as we can show by appealing to the soundness theorem).

When our signature contains function symbols we can also have a different kind of counterexample: although $\forall x G(x) \vdash G(f(x)$, $\forall x(F(x) \to G(x)), F(x) \nvdash G(f(x))$.

WHAT TO DO WITH THIS TRAIN WRECK? Here's a good fallback claim that IS true: if the signature doesn't contain any function symbols, then $\Gamma \vdash P$, $r_F[\Gamma], \Delta \vdash r_F P$, where $\Delta = \{F(v) \mid v \in \mathrm{Var}\}$.

We can prove this by induction on provable sequents.

(a) for any instance of Assumption $P \triangleright P$, $r_F P \triangleright r_F P$ is also an instance of Assumption and hence provable, and hence $r_F P, \Delta \vdash r_F$ by Weakening.

(b) Every instance $\triangleright t = t$ of =Intro has the property in question, since $\Delta \triangleright t = t$ by =Intro and Weakening.

(c) Suppose for induction that provable sequents $\Gamma \triangleright P$ and $\Gamma \triangleright Q$ are such that $r_F[\Gamma], \Delta, \vdash r_F(P)$ and $r_F[\Gamma], \Delta \vdash r_F(Q)$. Then $r_F[\Gamma], \Delta \vdash r_F(P) \wedge r_F(Q)$ by $\wedge$Intro. But $r_F(P) \wedge r_F(Q) = r_F(P \wedge Q)$ by definition of r_F, so the seqent $\Gamma \triangleright P \wedge Q$ has the property in question as well.

(d) Suppose the induction hypothesis holds for a provable sequent $\Gamma \triangleright P$, where variable v is not free in Γ. That is: $\Gamma, \Delta \vdash r_F(P)$. Let $\Delta^- = \Delta \setminus Fv$; then $\Gamma, \Delta^- \vdash F(v) \to r_F(P)$ by $\to$Intro, and thus $\Gamma, \Delta^- \vdash \forall v(F(v) \to r_F(P))$ by $\forall$Intro (which is legitimate since v isn't free in Γ or Δ^-). It follows by Weakening that $\Gamma, \Delta \vdash \forall v(F(v) \to r_F(P))$ Since $r_F(\forall v P) = \forall v(F(v) \to r_F(P))$, we have established that the sequent $\Gamma \triangleright \forall v P$ (which follows by $\forall$I from the one we started with) has the property in question.

(e) Suppose that the induction hyptothesis holds for a provable sequent $\Gamma \triangleright \forall v P$, i.e. that $r_F[\Gamma], \Delta \vdash \forall v(F(v) \to r_F(P))$; let u be any variable (noting that there are no terms in the language other than variables). Then, $r_F[\Gamma], \Delta \vdash F(u) \to r_F(P[u/v])$ by $\forall$Elim; since $F(u) \in \Delta$, we also have $r_F[\Gamma], \Delta \vdash F(u)$ by Assumption and Weakening, so by $\to$Elim$_w$, $f_F[\Gamma], \Delta \vdash r_F(P[u/v])$. Thus the sequent $\Gamma \triangleright P[u/v]$, which follows by $\forall$Elim from the one we started with, has the property in question.

(f) Suppose the induction hypothesis holds for a provable sequent $\Gamma \triangleright P[u/v]$, i.e. that $r_F[\Gamma], \Delta \vdash r_F(P[u/v])$. Since $F(u) \in \Delta$, we also have $r_F[\Gamma], \Delta \vdash F(u)$ by Assumption and Weakening, and thus $r_F[\Gamma], \Delta \vdash F(u) \wedge r_F(P[u/v])$ by $\wedge$Intro. hen by $\exists$Intro, $r_F[\Gamma], \Delta \vdash \exists v(F(v) \wedge r_F(P))$. But since $r_F(\exists v P) = \exists v(F(v) \wedge r_F(P))$, this means that the sequent $\Gamma \triangleright \exists v P$, which follows by $\exists$Intro from the one we started with, has the property in question.

(g) Suppose the induction hypothesis holds for provable sequents $\Gamma \triangleright \exists v P$ and $\Gamma, P[u/v] \triangleright Q$, where u isn't free in Γ, Q, or $\exists v P$. That is, $r_F[\Gamma], \Delta \vdash \exists v(F(v) \wedge r_F(P))$ and $r_F[\Gamma], \Delta, r_F(P)[u/v] \vdash r_F(Q)$. By Cut, the second of these claims implies that $r_F[\Gamma], \Delta, F(u) \wedge r_F(P)[u/v] \vdash r_F(Q)$ (since $F(u) \wedge r_F(P)[u/v] \vdash r_F(P)[u/v]$ by Assumption and $\wedge$Elim). So by $\exists$Elim, we can conclude that $r_F[\Gamma], \Delta \vdash r_F(Q)$, i.e. that the sequent $\Gamma \triangleright Q$, which follows by $\exists$Elim from the one we started with, has the property in question.

We skip the cases for $\to$Intro, $\to$Elim, $\wedge$Elim $\vee$Intro, $\vee$Elim, $\neg$Intro, and DNE, which are all similar to the case of $\wedge$Intro.

3. Show, using the result of problem 4 above, that if $\Gamma \vdash P$, $f[\Gamma] \vdash_{\rightarrow,\vee,\wedge,\neg,\exists,=} fP$, where $f : \mathcal{L}(\Sigma) \rightarrow \mathcal{L}_{\rightarrow,\vee,\wedge,\neg,\exists,=}(\Sigma)$ is defined as follows:

$$\begin{aligned} fP &= P \quad \text{when } P \text{ is atomic} \\ f(\neg P) &= \neg fP \\ f(P \rightarrow Q) &= fP \rightarrow fQ \\ f(P \wedge Q) &= fP \wedge fQ \\ f(P \vee Q) &= fP \vee fQ \\ f(\forall vP) &= \neg\exists v \neg fP \\ f(\exists vP) &= \exists v(cP) \end{aligned}$$

> By induction on provable sequents.
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> Assumption: For any instance $P \triangleright P$ of Assumption, $fP \triangleright fP$ is also an instance of Assumption.
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> Weakening: if $\Gamma, \Delta \triangleright P$ follows from $\Gamma \triangleright P$ by Weakening, $f[\Gamma], f[\Delta] \triangleright f[P]$ follows from $f[\Gamma] \triangleright fP$ by Weakening.
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> $\wedge$Intro: if $\Gamma \triangleright P \wedge Q$ follows from $\Gamma \triangleright P$ and $\Gamma \triangleright Q$ by $\wedge$Intro, then $f[\Gamma] \triangleright fP \wedge fQ$ follows from $f[\Gamma] \triangleright fP$ and $f[\Gamma] \triangleright fQ$ by $\wedge$Intro; but $fP \wedge fQ$ is $f(P \wedge Q)$ by definition of f.
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> All the other steps are equally trivial except for:
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> $\forall$Intro: consider a case of $\forall$Intro, with input sequent $\Gamma \triangleright P[u/v]$ and output sequent $\Gamma \triangleright \forall vP$, where u isn't free in Γ or in $\forall vP$. Suppose for induction that $f[\Gamma] \vdash f(P[u/v])$; clearly f commutes with substitution, so $f(P[u/v])$ is $(fP)[u/v]$, and f doesn't change the free variables of a formula, so u is also not free in $f[\Gamma]$ or in $\forall v(fP)$. So by $\forall$Intro, $f[\Gamma] \vdash \forall v(fP)$. But then by the result of problem 4, $f[\Gamma] \vdash \neg\exists v \neg(fP)$, i.e. $f[\Gamma] \vdash f(\forall vP)$.
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> $\forall$Elim: consider a case of $\forall$Elim, with input sequent $\Gamma \triangleright \forall vP$ and output sequent $\Gamma \triangleright P[t/v]$, and suppose for induction that $f[\Gamma] \vdash f(\forall vP)$, i.e. $f[\Gamma] \vdash \neg\exists v \neg(fP)$. Then by the result of probelm 4, $f[\Gamma] \vdash \forall v(fP)$, so by $\forall$Elim, $f[\Gamma] \vdash (fP)[t/v]$; appealing again to the obvious fact that f commutes with substiution that means that $f[\Gamma] \vdash f(P[t/v])$ which is what we need.