Solutions to Problem Set 5
Advanced Logic
23rd October 2022

Note: some of the following problems mention the function elements : $A^* \to \mathcal{P}A$. Recall that this is defined recursively, by the clauses

$$\text{elements}\,[] = \varnothing$$
$$\text{elements}\,(a : s) = \text{elements}\,s \cup \{a\}$$

1. (20%) Prove that for any $s, t \in A^*$, $\text{elements}(s \oplus t) = \text{elements}\,s \cup \text{elements}\,t$.

By induction on s, for a fixed arbitrary $t \in A^*$.

Base case:
$$\begin{aligned} \text{elements}([] \oplus t) &= \text{elements}(t) && \text{by definition of } \oplus \quad . \\ &= \varnothing \cup \text{elements}(t) \\ &= \text{elements}[] \cup \text{elements}\,s && \text{by definition of elements} \end{aligned}$$

Induction step: suppose $\text{elements}(s \oplus t) = \text{elements}\,s \cup \text{elements}\,t$, and let $a \in A$. Then,

$$\begin{aligned} \text{elements}((a : s) \oplus t) &= \text{elements}(a : (s \oplus t)) && \text{(by definition of } \oplus \\ &= \text{elements}(s \oplus t) \cup \{a\} && \text{by definition of elements} \\ &= \text{elements}\,s \cup \text{elements}\,t \cup \{a\} && \text{by IH} \\ &= (\text{elements}\,s \cup \{a\}) \cup \text{elements}\,t \\ &= \text{elements}(a : s) \cup \text{elements}\,t && \text{by definition of elements} \end{aligned}$$

2. (30%) Prove that $X \subseteq A$ is finite iff there exists $s \in A^*$ such that $\text{elements}\,s = X$.

> *Hint:* You'll need one inductive proof for each direction of this. Here and in all the following problems, you can use our official definition of finitude (Lecture 5) or the fact (Lecture 8) that a set is finite iff it is equinumerous with pred n for some n.

Left to right: we prove by induction (using our official definition of 'finite') that for every finite subset X of A, there exists an appropriate s.

Base case: $X = \varnothing$. Then we can set $s = []$, since $\text{elements}[] = \varnothing$.

Induction step: suppose X^+ is the result of adding one element to X, i.e. $X^+ = X \cup \{a\}$ for some $a \in A \setminus X$; and suppose for induction that $X = \text{elements}\,s$. Then $X^+ = \text{elements}(a : s)$.

We conclude that an appropriate s exists for every set in the closure of $\{\varnothing\}$ under adding one element, i.e. for every finite subset of A.

Right to left: we need to show that for all $s \in A^*$, $\text{elements}\,s$ is finite.

Base case: $\text{elements}[] = \varnothing$ which is finite.

Induction step: suppose $\text{elements}\,s$ is finite. Then $\text{elements}(a : s) = \text{elements}\,s \cup \{a\}$ which is also finite, since the set of finite subsets of A is closed under adding one element.

3. (30%) Prove that every subset of a finite set is finite.

> You'll need a proof by induction, for which the following fact might be useful: $B \subseteq A \cup \{x\}$ iff either $B \subseteq A$ or $B = B' \cup \{x\}$ for some $B' \subseteq A$.

We prove this by induction using the official definition of 'finite'.

Base case: every subset of $\varnothing$ is finite since $\varnothing$ is the only subset of $\varnothing$.

Induction step: suppose every subset of X is finite, and that $X^+ = X \cup \{a\}$ for some $a \in A \setminus X$. Then every subset of X^+ is either a subset of X, in which case it is finite by hypothesis, or of the form $Y \cup \{a\}$ where Y is a subset of X, in which case it is finite since the set of finite subsets of A is closed under adding one element.

4. (10%) Prove that whenever A and B are finite sets, so is $A \cup B$.

Fix a finite A and B; we prove by induction that $A \cup X$ is finite for all finite $X \subseteq B$, which implies that in particular $A \cup B$ is finite.

Base case: $A \cup \varnothing = A$ which is finite by hypothesis.

Induction step: Suppose $A \cup X$ is finite, and let $X^+ = X \cup \{b\}$ for some $b \in B \setminus X$. Then $A \cup X^+ = A \cup X \cup \{b\}$, which is finite since the result of adding one element to a finite set is finite.

5. (10%) Prove that whenever A and B are finite sets, so is $A \times B$.

> *Hint:* Once again, you'll need an induction. The following facts may be useful: (i) $A \times \{x\} \sim A$ (for any A, x); (ii) $(A \cup B) \times C = (A \times C) \cup (A \times C)$.

Fix a finite A; we prove inductively that $A \times X$ is finite for every finite $X \subseteq B$; it follows that $A \times B$ is finite.

Base case: $A \times \varnothing = \varnothing$ which is finite.

Induction step: Suppose $A \times X$ is finite, and $X^+ = X \cup \{b\}$ for some $b \in B \setminus X$. Then $A \times X^+ = (A \times X) \cup (A \times \{b\}$. But $A \times X$ is finite by hypothesis, and $A \times \{b\}$ is also finite since it is equinumerous with A which is finite. So by the result of problem 4, $(A \times X) \cup (A \times \{b\}$ is finite.

For the two extra-credit problems, suppose that $R \subseteq (D \times D) \times D$, $B \subseteq D$, and C is the closure of B under R. We recursively define a subset V of D^* (the "derivations") as follows:

- $[] \in V$
- $(x : s) \in V$ iff $s \in V$ and either $x \in B$ or there exist $y, z \in \text{elements}\, s$ such that $Ryzx$.

5. (5%) Prove that if $s \in V$ and $t \in V$, $s \oplus t \in V$.

Fix $t \in V$; we prove by induction that $s \oplus t \in V$ for all $s \in V$.

Base case: $s = []$. Then $s \oplus t = t$, so $s \oplus t \in V$.

Induction step: suppose $s \in V$, $(x : s) \in V$, and $s \oplus t \in V$; we need to show that $(x : s) \oplus t \in V$, i.e. $(x : s \oplus t) \in V$. If $x \in B$, then $(x : s \oplus t) \in V$. Otherwise, there exist $y, z \in \text{elements}\, s$ such that $Ryzx$. But $\text{elements}\, s \subseteq \text{elements}(s \oplus t)$ by what we proved in problem 1, so in that case too $x : (s \oplus t)$ meets the test for inclusion in V.

6. (5%) Prove that that for any $a \in D$, $a \in C$ iff there exists $s \in D^*$ such that $(a : s) \in V$.

Left to right: by induction on C.

Base case: if $a \in B$, then $a : [] \in V$.

Induction step: Suppose $Ryza$, where $(y : s') \in V$ and $(z : s'') \in V$. Then $(y : s') \oplus (z : s'') \in V$ by the previous problem. Also, $y \in \text{elements}(y : s')$ by definition of elements, so $y \in \text{elements}((y : s') \oplus (z : s''))$ by problem 1; $z \in \text{elements}((y : s') \oplus (z : s''))$ by the same reasoning. So, by the recursion clause in the definition of V, $(a : (y : s') \oplus (z : s'')) \in V$.

Right to left: we prove by induction that if $s \in V$, then elements $s \subseteq C$. Since $a \in \text{elements}(a : s)$, this implies that if $(a : s) \in V$, $a \in C$.

Base step: $\text{elements}[] = \varnothing \subseteq C$, trivially.

Induction step: suppose $s \in V$ and $(x : s) \in V$ and elements $s \subseteq C$; we will show that $x \in C$, from which it follows given the IH that $\text{elements}(x : s) = \text{elements}\, s \cup \{x\} \subseteq C$. If $x \in B$, then $x \in C$ since $B \subseteq C$. Otherwise, there must be $y, z \in \text{elements}\, s$ such that $Ryzx$. By hypothesis $y \in C$ and $z \in C$; so $x \in C$ since C is closed under R.