Problem Set 8

Advanced Logic
30th October 2022
Due date: Friday, 4 November.

1. (50%) Prove the *Substitution Lemma for Terms*: for any signature Σ, any terms s and t of Σ, any variable v, any structure S for Σ, and any assignment g for S,
$$[\![t[s/v]]\!]_S^g = [\![t]\!]_S^{g[v \mapsto [\![s]\!]_S^g]}$$

> *Hint:* prove this by induction on the construction of terms (should it be t or s?)
>
> *Reminder:*
> $$g[v \mapsto [\![s]\!]_S^g](u) = \begin{cases} [\![s]\!]_S^g & \text{if } u = v \\ gu & \text{otherwise} \end{cases}$$

2. (30%) Using the result from part 1, prove the *Substitution Lemma for Formulae*: for any signature Σ, any term s of Σ, any formula P of Σ, any variable v, any structure S for Σ, and any assignment g for S,
$$S, g \Vdash P[s/v] \text{ iff } S, g[v \mapsto [\![s]\!]_S^g] \Vdash P$$

> *Hint:* prove this by induction on the construction of formulae . In the induction step for quantifiers, you will need to separately consider the case of formulae that begin with $\forall v$ (or $\exists v$) and formulae that begin with $\forall u$ (or $\exists u$) for some other variable u. You may if you wish rely on the "Irrelevance Lemma" according to which if $g(u) = h(u)$ for all $u \in FV(Q)$, $S, g \Vdash Q$ iff $S, h \Vdash Q$.

3. (10%) Using the result from part 2, prove the steps in the proof of the Soundness Theorem corresponding to the $\forall$Elim and $\exists$Intro rules. That is: show that if $\Gamma \vDash \forall v P$ then $\Gamma \vDash P[s/v]$ for every term s, and that if $\Gamma \vDash P[s/v]$, $\Gamma \vDash \exists v P$.

4. (5%) Using the result from part 2, prove the step in the proof of the Soundness Theorem corresponding to the =Elim rule. That is: show that if $\Gamma \vDash P[s/v]$ and $\Gamma \vDash s = t$, then $\Gamma \vDash P[t/v]$.

5. (5%) Using the result from part 2, prove the step in the proof of the Soundness Theorem corresponding to the $\exists$Elim rule. That is: show that if $\Gamma \vDash \exists v P$ and $\Gamma, P[u/v] \vDash Q$, then $\Gamma \vDash Q$, provided that u is not free in Γ, Q, or $\exists v P$.

   Give examples to show that this can fail when (i) u is free in Γ though not in Q or $\exists v P$; (ii) u is free in Q though not in Γ or $\exists v P$; (iii) u is free in $\exists v P$ though not in Γ or Q.

EXTRA CREDIT (2.5% each, up to a maximum of 10%) Prove the remaining facts about logical consequence required to complete the proof of the Soundness Theorem, namely:

1. If $\Gamma \vDash P$ and $\Gamma \vDash Q$ then $\Gamma \vDash P \wedge Q$ ($\wedge$Intro)
2. If $\Gamma \vDash P \wedge Q$ then $\Gamma \vDash P$ and $\Gamma \vDash Q$ ($\wedge$Elim1 and $\wedge$Elim2).
3. If $\Gamma, P \vDash Q$ then $\Gamma \vDash P \to Q$ ($\to$Intro).
4. If $\Gamma \vDash P \to Q$ and $\Gamma \vDash P$ then $\Gamma \vDash Q$ ($\to$Elim).
5. If $\Gamma, P \vDash Q$ and $\Gamma, P \vDash \neg Q$ then $\Gamma \vDash \neg P$ ($\neg$Intro).
6. If $\Gamma \vDash \neg\neg P$ then $\Gamma \vDash P$ (DNE).
7. $\vDash t = t$ for every term t (=Intro).