**Problem Set 4**
Advanced Logic
6th October 2022

Throughout this problem set, $A$ is some arbitrary set, and $A^*$ is the set of lists over $A$; $s, t, u$ are arbitrary members of $A^*$. $\oplus$ denotes the list concatenation operation on $A^*$, defined to obey the following recursion clauses:

$$[] \oplus t = t$$
$$(a : s) \oplus t = a : (s \oplus t)$$

Note that you can and should rely on earlier results in proving later ones.

1. (15%) Prove that if $s \oplus t = s$ then $t = []$.

> By induction on $s$, generalizaing over $t$.
>
> Base case: since $[] \oplus t = t$ for all $t$, if $[] \oplus t = []$ then $t = []$.
>
> Induction step: suppose that for any $t$ such that $s \oplus t = s$, $t = []$. Let $a \in A$, and consider some $t$ such that $(a : s) \oplus t = (a : s)$. Then $(a : s \oplus t) = (a : s)$, so $s \oplus t = s$ by the Injective Property of lists, so $t = []$ by the induction hypothesis.

2. (15%) Prove that if $s \oplus t = []$ then $s = t = []$.

> If $s \oplus t = []$ then there can't be any $a \in A$ and $s' \in A^*$ such that $s = (a : s')$, since then we would have $[] = (a : s') \oplus t = (a : s \oplus t)$, which is ruled out by the Injective Property of lists. So the only remaining possibility is that $s = []$. And then since $[] \oplus t = []$ we have $t = []$ by the previous result.
>
> (This can also be formatted as an induction.)

3. (15%) Prove that if $s \oplus t = (a : u)$ then either $s = []$ or $s = (a : s')$ for some $s'$.

> Suppose $s \oplus t = (a : u)$. For every list $s$, either $s = []$ or there is some $b \in A$ and $s' \in A^*$ such that $s = (b : s')$. In the first case we are done, so let $s = (b : s')$. But then we have $s \oplus t = (b : s') \oplus t = b : (s \oplus t) = a : u$, which implies $b = a$ (as well as $s \oplus t = u$) by the Injective Property of lists.

For the following problems, we define a function final : $A^* \to \mathcal{P}(A^*)$ recursively as follows:

$$\text{final}\,[] = \{[]\}$$
$$\text{final}(a : s) = \text{final}\,s \cup \{(a : s)\}$$

We say that $s$ is a *final sublist* of $t$ iff $s \in \text{final}\,t$.

4. (15%) Prove that $[]$ is a final sublist of every list.

> By induction. Base case: $[] \in \text{final}[]$ by definition.
>
> Induction step: suppose $[] \in \text{final}\,s$, and consider $a \in A$. Then $\text{final}(a : s) = \text{final}\,s \cup \{(a : s)\}$, so

$[] \in \text{final}(a : s)$.

5. (10%) Prove that every list is a final sublist of itself.

By induction. Base case: [] is a final sublist of itself by definition.

Induction step: suppose $s \in \text{final}\, s$, and consider $a \in A$. Then $\text{final}(a : s) = \text{final}\, s \cup \{(a : s)\}$, so since $(a : s) \in \{(a : s)\}$, $(a : s) \in \text{final}(a : s)$.

6. (10%) Prove that $t$ is a final sublist of $s \oplus t$.

By induction on $s$, for a given arbtrary $t$. Base case: $t$ is a final sublist of $[] \oplus t = t$ by the previous problem.

Induction step: suppose $t \in \text{final}(s \oplus t)$ and $a \in A$. Then $\text{final}((a : s) \oplus t) = \text{final}(a : (s \oplus t)) = \text{final}(s \oplus t) \cup \{a : (s \oplus t)\}$, and so we also have $t \in \text{final}((a : s) \oplus t)$.

7. (10%) Prove that if $s$ is a final sublist of $t$, then $t = u \oplus s$ for some $u$.

By induction on $t$, generalizing over $s$.

Base case: if $s$ is a final sublist of [] then $s = []$ and since $[] = [] \oplus []$ we have $t = [] \oplus s$.

Induction step: suppose that whenever $s$ is a final sublist of $t$ then $t = u \oplus s$ for some $u$. Now suppose that $s$ is a final sublist of $(a : t)$. Then eiter $s$ is a final sublist of $t$ or $s = (a : t)$. In the first case, by the IH there exists $u$ such that $t = u \oplus s$ and hence $(a : t) = a : (u \oplus s) = (a : u) \oplus s$. In the second case, we have $(a : t) = [] \oplus s$.

8. (10%) Prove that if $s \oplus s' = t \oplus t'$ then either $s'$ is a final sublist of $t'$ or $t'$ is a final sublist of $s'$.

By induction on $s$, generalizing over $s', t, t'$.

*Base case:* $[] \oplus s' = s'$, so if $[] \oplus s' = t \oplus t'$ we have $s' = t \oplus t'$ and thus $t' \in \text{final}\, s'$ by problem 6 above.

*Induction step:* suppose for induction that for any $s', t, t'$ such that $s \oplus s' = t \oplus t'$, either $s'$ is a final sublist of $t'$ or $t'$ is a final sublist of $s'$. Fix $a \in A$, and further suppose that $(a : s) \oplus s' = t \oplus t'$. Then by problem 3 above, either $t = []$ or there is a $u$ such that $t = (a : u)$.

In the first case, we have $t' = [] \oplus t' = (a : s) \oplus s'$, so $s'$ is a final sublist of $t'$ by problem 6 above.

In the second case, we have $(a : s) \oplus s' = (a : u) \oplus t'$, hence $a : (s \oplus s') = a : (u \oplus t')$, hence $s \oplus s' = u \oplus t'$ (by the Injective Property of Lists), and thus either $s' \in \text{final}\, t'$ or $t' \in \text{final}\, s'$ by the IH.

9. (10% extra credit) Play through the level 'Advanced Multiplication World' in the Lean Natural Numbers Game (`https://www.ma.imperial.ac.uk/~buzzard/xena/natural_number_game/`). To show that you've completed the levels, send us a screenshot of the last level of Advanced Multiplication World open on your computer screen, with your name showing somewhere in the screenshot (e.g. in a text editor window).

See last week's problem set solutions for a link to a document with all the solutions to this game.