Problem Set 4

Advanced Logic

26th September 2022

Throughout this problem set, A is some arbitrary set, and A^* is the set of lists over A; s, t, u are arbitrary members of A^*. $\oplus$ denotes the list concatenation operation on A^*, defined to obey the following recursion clauses:

$$[] \oplus t = t$$
$$(a : s) \oplus t = a : (s \oplus t)$$

Note that you can and should rely on earlier results in proving later ones.

1. (15%) Prove that if $s \oplus t = s$ then $t = []$.
   (a) $s \oplus t = s$
   (b) iff $(s : []) \oplus t = s$, since $[s] = (s : [])$
   (c) iff
2. (15%) Prove that if $s \oplus t = []$ then $s = t = []$.
3. (15%) Prove that if $s \oplus t = (a : u)$ then either $s = []$ or $s = (a : s')$ for some s'.

For the following problems, we define a function final : $A^* \to \mathcal{P}(A^*)$ recursively as follows:

$$\text{final } [] = \{[]\}$$
$$\text{final}(a : s) = \text{ final } s \cup \{(a : s)\}$$

We say that s is a final sublist of t iff $s \in \text{final } t$.

5. (15%) Prove that [] is a final sublist of every list.
6. (10%) Prove that every list is a final sublist of itself.
7. (10%) Prove that t is a final sublist of $s \oplus t$.
8. (10%) Prove that if s is a final sublist of t, then $t = u \oplus s$ for some u.
9. (10%) Prove that if $s \oplus s' = t \oplus t'$ then either s' is a final sublist of t' or t' is a final sublist of s'.

10. (10% extra credit) Play through the level 'Advanced Multiplication World' in the Lean Natural Numbers Game (https://www . ma. imperial . ac . uk/ buzzard/xena/natural_number_game/). To show that you've completed the levels, send us a screenshot of the last level of Advanced Multiplication World open on your computer screen, with your name showing somewhere in the screenshot (e.g. in a text editor window).