Problem Set 7

Advanced Logic
18th October 2022
Due date: Friday, 28th October.

1. (50%) Show that $\Gamma, P, Q \vdash R$ if and only if $\Gamma, P \wedge Q \vdash R$ (for any formulas P, Q, R and set of formulas Γ of some first-order language $\mathcal{L}(\Sigma)$).
   Answer

   (a) Γ is the set of formulas, PQR is formula. We need to show in 3 cases. holds from left to right and right to left

   (b) left to right: if $\Gamma, P, Q \Vdash R$ then, $\Gamma, P \wedge Q \Vdash R$

   (c) 2 parts, $\Gamma and P, Q$. Let A proves $P \vdash R$, B proves $Q \vdash R$ and we want to show that $A \wedge B \vdash R$

   (d) This step is trivial by the definition of $\vdash, \wedge Intro$

   (e) Now, we want to show that $\Gamma \vdash R$

   (f) By definition of $\vdash$, you can add any premises, as long as it is in the first order language $\mathcal{L}(\Sigma)$ although it would Weakening the proof.

   (g) Γ is therefore legally added but it weakens the proof.

   (h) Thus, the proof is made.

   (i) right to left:

   (j) Similar to left to right, but instead we use $\wedge Elim$

   (k) the reverse of the weakening is the still the weakening itself.

2. (30%) Show that the following three conditions on a set of formulae

Γ are equivalent:
a. $\Gamma \vdash P$ and $\Gamma \vdash \neg P$ for some P

1. With the premises of Γ, we can deduce that either not P or P.

2. This implies that $\Gamma \vdash Q$ for every formula Q

3. Since, Γ is the set of all formulas, it is obvious that it contains the formula P and formula not P.

b. $\Gamma \vdash Q$ for every formula Q

1. This implies that $\Gamma \vdash \neg\forall x(x = x)$
2. a formula can be broken down to terms and arrangement of terms(symbols, predicates, quantifer).
3. For every formula Q, Q can be broken to variables such as x, symbols suchas "=", and quantifer such as "for all".
4. Thus, $\Gamma \vdash Q$ for every formula Q

c. $\Gamma \vdash \neg\forall x(x = x)$

1. This implies a. $\Gamma \vdash P$ and $\Gamma \vdash \neg P$ for some P
2. This says that not every formula is identical with the premises that Γ.
3. The case of the substitution instance which is used to achieve capture-free substituion fits into the case of x is not equal x.
4. Thus, it implies (a)

[Hint: show that (a) implies (b), (b) implies (c), and (c) implies (a)]

3. (10%) Show that for any terms t_1, t_2, t_3 and variable v :
   a. $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$
   b. $t_1 = t_2 \vdash t_2 = t_1$
   c. $t_1 = t_2 \vdash t_3\,[t_1/v] = t_3\,[t_2/v]$
4. (10%) Show that $\forall vP \dashv\vdash \neg\exists\neg P$ for every formula P.

EXTRA CREDIT 10% for any of the following:

1. Show that for every formula P of $\mathcal{L}(\varnothing)$ (the first-order language with no non-logical constants at all), either $\forall x\forall y(x = y) \vdash P$ or $\forall x\forall y(x = y) \vdash \neg P$.

[Hint: this will require an induction on the construction of the formula P.]

3. Suppose F is a singulary predicate of Σ. Define a function $r_F : \mathcal{L}(\Sigma) \to \mathcal{L}(\Sigma)$ as follows:

$$\begin{aligned}
r_F P &= P \quad \text{when } P \text{ is atomic} \\
r_F(\neg P) &= \neg r_F P \\
r_F(P \to Q) &= r_F P \to r_F Q \\
r_F(P \wedge Q) &= r_F P \wedge r_F Q \\
r_F(P \vee Q) &= r_F P \vee r_F Q \\
r_F(\forall v P) &= \forall v\,(Fv \to r_F P) \\
r_F(\exists v P) &= \exists v\,(Fv \wedge r_F P)
\end{aligned}$$

Show that $r_F[\Gamma], F\left(v_1\right), \ldots, F\left(v_n\right) \vdash r_F P$ whenever $\Gamma \vdash P$, where $v_1, \ldots, v_n$ are the free variables in Γ and P. [Hint: This will require an induction on provable sequents. It'll be enough to do the]

4. Show, using the result of problem 4 above, that $\Gamma \vdash P, f[\Gamma] \vdash \rightarrow, v, \wedge, \neg, \exists, = fP$, where $f : \mathcal{L}(\Sigma) \rightarrow \mathcal{L}_{\rightarrow,\vee,\wedge,\neg,\exists,=}(\Sigma)$ is defined as follows:

$$
\begin{aligned}
fP &= P \quad \text{when } P \text{ is atomic} \\
f(\neg P) &= \neg fP \\
f(P \rightarrow Q) &= fP \rightarrow fQ \\
f(P \wedge Q) &= fP \wedge fQ \\
f(P \vee Q) &= fP \vee fQ \\
f(\forall vP) &= \neg\exists\neg v fP \\
f(\exists vP) &= \exists v(cP)
\end{aligned}
$$

[Hint: This will require an induction on provable sequents. It'll be enough to do the steps for Assumption, Weakening, $\forall Intro, \forall$ Elim, and one or two other rules.]