Problem Set 10

Advanced Logic
15th November 2022
Due date: Friday, 18 November.

1. (50%) Show that the following are definable in the standard string structure.
   (a) (10%) the set of all strings whose length is an even number.
   (b) (10%) the function that maps every ordered pair of strings $\langle s, t\rangle$ to its first element.
   (c) (10%) the set of all ordered pairs of strings $\langle s, t\rangle$ such that s is a length-one substring of t.
   (d) (10%) the function that maps each string to an equally long string comprised entirely of spaces.
   (e) (10%) the relation that holds between two strings s and t when s is a *line* of t---i.e., s would appear as a line if we pasted t into a text editor. That is: s doesn't contain any newlines, and either s is t, or s is an initial substring of t that's followed by a newline, or s is a final substring of t that's preceded by a newline, or s is a substring of t that's both preceded and followed by a newline.

> *Reminder:* The way to show that a set/relation is definable in a structure is to find a formula P such that when when the structure is expanded with a definition
>
> $$\forall v_1 \ldots \forall v_n(\text{YourNewPredicate}(v_1, \ldots, v_n) \leftrightarrow P)$$
>
> the extension of YourNewPredicate will be the desired set/relation. For functions, you can instead consider definitions of the form
>
> $$\forall v_1 \ldots \forall v_{n+1}(v_{n+1} = \text{YourNewFunctionSymbol}(v_1, \ldots, v_n) \leftrightarrow P)$$
>
> You can built up your definitions in stages.
>
> For this exercise, it is enough if you just write down definitions that work---I won't expect you to give the proof that they work (which will always just be a routine exercise in unpacking the definition of truth in a structure).

2. (25%) Suppose set X and binary relations R and R' are definable in a certain structure S with domain D_S. Show that the following are also definable:
   (a) (5%) $D_S \setminus X$
   (b) (5%) R^{-1}
   (c) (5%) $R \cup R'$
   (d) (5%) $R \circ R'$
   (e) (5%) $\{d \mid Rdd\}$

3. (25%) Here is a list of expressions in the language of strings, specified using various shorthands that have been introduced. For each one, say

   (i) what string it is (write out in full, with no shorthands like omitting parentheses or infix notation)

   (ii) whether it is a term or a formula

   (iii) what its free variables are (if any)

   (iv) if it's a term, what it denotes in the standard string structure on an assignment function that maps the variable x to the string `cat`.

   (v) if it's a formula, whether it is true in the standard string structure on an assignment function that maps the variable x to the string `cat`.

   (vi) if it's a formula: whether it is valid, inconsistent, or neither.

   (a) $""$
   (b) $x = "x"$
   (c) $x \oplus "" \oplus x$
   (d) $"x" \leq "" \oplus x$
   (e) $\exists x(x = "x")$
   (f) $\forall x(x \leq "" \to x = "")$
   (g) $(x = "x")["x"/x]$
   (h) $\forall x(x = "x")["x"/x]$
   (i) $\langle x \rangle$
   (j) $\langle x \rangle = "x"$
   (k) $\langle x = "x" \rangle$
   (l) $\langle\langle x \rangle\rangle$

EXTRA CREDIT: up to 10% for any of the following.

1. Suppose that structure S is *explicit*: that is, for every element of the domain, there is a term with no free variables that denotes it in S. Show that every finite subset of S's is definable in S.

   > *Reminder:* given the definition definition of "finite subset", you can show that all finite subsets of a set have a certain property by showing that the empty set has the property and that if a set has it, so does any set derived from that set by adding one extra element.

2. Using the compactness theorem, prove that if a set of sentences (in any signature) is true in arbitrarily large finite structures, it is also true in some infinite structure.

> *Hint:* You can help yourself to the fact that for each n, there is a sentence I_n that is true exactly in thos structures whose domain has size at least n---e.g., I_3 is $\exists x \exists y \exists z(\neg(x = y) \wedge \neg(x = z) \wedge \neg(y = z))$. Any structure in which all these sentences are true must have an infinite domain.

3. When S is a nonstandard model of true arithemtic, and a and b are two elements of the domain of S, nonstandard model of arithmetic, say that $a \lceil_S b$ iff $x \lceil y$ is true in S on the assignment $[x \mapsto a, y \mapsto b]$. Say that a and b are “in the same block” iff for for some number n, either $x = y + \langle n \rangle$ or $x = y + \langle n \rangle$ is true in S on this assignment.

   Prove that if a is a nonstandard element of the domain, then (i) there is a nonstandard element b such that $a \lceil_S b$ and b is not in the same block as a, and (ii) a nonstandard element c such that $c \lceil_S a$ and c is not in the same block as a.

   > *Hint:* you could start by showing that when two numbers are both nonstandard, their sum is not in the same block as either of them; then consider $a + a$. For part (ii), note that while true arithmetic doesn't imply that every number can be divided evenly by two, it does imply that whenever a number can't be divided evenly by two, its successor can.