Problem Set 4

Advanced Logic

26th September 2022

Throughout this problem set, A is some arbitrary set, and A^* is the set of lists over A; s, t, u are arbitrary members of A^*. $\oplus$ denotes the list concatenation operation on A^*, defined to obey the following recursion clauses:

$$[] \oplus t = t$$
$$(a : s) \oplus t = a : (s \oplus t)$$

Note that you can and should rely on earlier results in proving later ones.

1. (15%) Prove that if $s \oplus t = s$ then $t = []$.
   (a) $s \oplus t = s$
   (b) iff $(s : []) \oplus t = s$, since $[s] = (s : [])$
   (c) iff $s : ([] \oplus t) = s$, by clause 2
   (d) iff $s : t = s$, by clause 1
   (e) iff $t = []$
   (f) therefore, we reach our conclusion here.
2. (15%) Prove that if $s \oplus t = []$ then $s = t = []$.
   (a) by the injective property of Axiom of the List, we have that $[]$ is not in the range of any $cons_a$
   (b) $if s \oplus t = []$, then s and t are all $[]$
   (c) Thus, s = t = []
3. (15%) Prove that if $s \oplus t = (a : u)$ then either $s = []$ or $s = (a : s')$ for some s'.
   (a) Assume that a is just any arbitrary element in A, then $(a : s')$ is just the transformation of list s after taking out an element in A.
   (b) iff $s \oplus t = (a : s') \oplus t = a : (s' \oplus t)$, by clause 2
   (c) the case of $s = []$ is tritvial because a : t and a : u is the same as both u and t are all arbitrary elements
   (d) Moreover, for the second case, s' and t and u are all arbitrary, thus, it is the same.

For the following problems, we define a function final : $A^* \to \mathcal{P}(A^*)$ recursively as follows:

$$\text{final } [] = \{[]\}$$
$$\text{final}(a : s) = \text{ final } s \cup \{(a : s)\}$$

We say that s is a final sublist of t iff $s \in$ final t.

5. (15%) Prove that [] is a final sublist of every list.
    (a) Assume that X is the set that has all the list and $x \in X$, then for every final function, we have $x = a : s$ and $\text{final}(a : s) = \text{ final } s \cup \{(a : s)\}$, since x is monotonicly popping out elements, it will reach [] as always.
    (b) Thus [] will always be in the form of $a : s$ and then we can use clause 1 to get $\{[]\}$. Thus, we have the conclusion

6. (10%) Prove that every list is a final sublist of itself.
    (a) Assume that x is an arbitrary list of all list, then $\text{final}(a : s) = \text{ final } s \cup \{(a : s)\}$ by clause 2, and x is (a:s), thus, we have $final(s) \cup \{x\}$. Since x $\in finalx$, we say that x is a final sublist of itself. Therefore, as x is arbitrary, every list is a final sublist of itself.

7. (10%) Prove that t is a final sublist of $s \oplus t$.
    (a) t $\in final(s \oplus t)$ iff $final(t) \cup \{s \oplus t\}$
    (b) By the recursive structure, we have that t = a : u
    (c) $final(t) = final(a : u) = finalu \cup \{(a : u)\}$
    (d) $\{(a : u)\} = \{t\}$
    (e) Thus, t $\in final(s \oplus t)$

8. (10%) Prove that if s is a final sublist of t, then $t = u \oplus s$ for some u.
    (a) $s \in final(t)$
    (b) iff $final(a : u) = final(u) \cup \{(a : u)\}$ or $final(a : s) = final(s) \cup \{(a : s)\}$, as s and u are all arbitrary
    (c) $a : u = a : ([] \oplus u) = (a : []) \oplus u$,by clause 1 and 2 in q1
    (d) $(a : []) = [a]$ is just arbitrary list, thus, it can be substituted as s
    (e) thus, we have $t = s \oplus u$
    (f) since u and s are all arbitrary, we have $t = u \oplus s$

9. (10%) Prove that if $s \oplus s' = t \oplus t'$ then either s' is a final sublist of t' or t' is a final sublist of s'.

10. (10% extra credit) Play through the level 'Advanced Multiplication World' in the Lean Natural Numbers Game (https://www . ma. imperial . ac . uk/ buzzard/xena/natural_number_game/). To show that you've completed the levels, send us a screenshot of the last level of Advanced Multiplication World open on your computer screen, with your name showing somewhere in the screenshot (e.g. in a text editor window).