Solutions to Problem Set 11

Advanced Logic

6th December 2022

Note: the scores for these problems add up to 110%, so a perfect score corresponds to 10% extra credit.

1. Let M be the theory in the language of strings axiomatized by all of the following sentences (where c may be any constant of the language of strings other than "")

$$\begin{array}{ll} \text{M1} & \forall x(x = \texttt{""} \oplus x) \\ \text{M2} & \forall x(x = x \oplus \texttt{""}) \\ \text{M3} & \forall x \forall y \forall z((x \oplus y) \oplus z = x \oplus (y \oplus z)) \end{array}$$

Show the following:

(a) (20%) Show that for any length-one string a, if c is the constant that denotes a in the standard string structure $\mathbb{S}$, then $c = \langle a \rangle$ is a theorem of M.

> When a is a single character and c is the constant that denotes it, $\langle a \rangle$ is the term $c \oplus \texttt{""}$, so what we are trying to show is that $M \vDash c = c \oplus \texttt{""}$. This follows by ∀Elim from M2.

(b) (30%) Show that for any strings s_1 and s_2, $\langle s_1 \rangle \oplus \langle s_2 \rangle = \langle s_1 \oplus s_2 \rangle$ is a theorem of M.

> By induction on s_1, for a fixed s_2.
>
> *Base case:* s_1 is the empty string [], so $s_1 \oplus s_2 = s_2$ and $\langle s_1 \rangle = \texttt{""}$, so we just need to show that $M \vDash \texttt{""} \oplus \langle s_2 \rangle = \langle s_2 \rangle$. This follows by ∀Elim from M1.
>
> *Induction step:* suppose $M \vDash \langle s \rangle \oplus \langle s_2 \rangle = \langle s \oplus s_2 \rangle$; we show that $M \vDash \langle a : s \rangle \oplus \langle s_2 \rangle = \langle (a : s) \oplus s_2 \rangle$ for each character. $(a : s) \oplus s_2 = a : (s \oplus s_2)$, so $\langle (a : s) \oplus s_2 \rangle = c \oplus \langle s \oplus s_2 \rangle$ and $\langle a : s \rangle = c \oplus \langle s \rangle$ (where c is the constant that denotes the length-one string $[a]$ in $\mathbb{S}$), so what we want to show is that $M \vDash (c \oplus \langle s \rangle) \oplus \langle s_2 \rangle = c \oplus \langle s \oplus s_2 \rangle$. By M3, it suffices to show that $M \vDash c \oplus (\langle s \rangle \oplus \langle s_2 \rangle = c \oplus \langle s \oplus s_2 \rangle$. But this follows from the IH given that M is closed under =Intro and =Elim.

(c) (30%) Using what you showed in parts (a) and (b), prove that for any closed term t in the language of strings, $t = \langle [\![t]\!]_{\mathbb{S}} \rangle$ is a theorem of M.

> *Reminder:* $[\![t]\!]_{\mathbb{S}}$ is the denotation of t in the standard string structure.
>
> We prove this by induction on the construction of t.
>
> Base case: t is an individual constant: either "", in which case $\langle [\![t]\!]_{\mathbb{S}} \rangle = \langle [] \rangle = \texttt{""}$, so that the thing we are trying to show M proves is an instance of =Intro, or else it's the constant c denoting a one-character string $[a]$, in which case $\langle [\![t]\!]_{\mathbb{S}} \rangle = \langle [a] \rangle = c \oplus \langle [] \rangle = c \oplus \texttt{""}$, so that the thing we are trying to show M

proves is $c = c \oplus$ "", which follows from M2.

Induction step: t is $t_1 \oplus t_2$, where $M \vDash t_1 = \langle [\![t_1]\!]_{\mathbb{S}} \rangle$ and $M \vDash t_2 = \langle [\![t_2]\!]_{\mathbb{S}} \rangle$. Since $[\![t]\!]_{\mathbb{S}} = [\![t_1]\!]_{\mathbb{S}} \oplus [\![t_2]\!]_{\mathbb{S}}$, part b implies that $M \vDash \langle [\![t_1]\!]_{\mathbb{S}} \rangle \oplus \langle [\![t_2]\!]_{\mathbb{S}} \rangle = \langle [\![t]\!]_{\mathbb{S}} \rangle$. But then (by =Elim), $M \vDash t_1 \oplus t_2 = \langle [\![t]\!]_{\mathbb{S}} \rangle$, i.e. $M \vDash t = \langle [\![t]\!]_{\mathbb{S}} \rangle$.

(d) (15%) Let t_1 and t_2 be any closed terms in the language of strings. Using what you showed in (c), prove that if the sentence $t_1 = t_2$ is true in $\mathbb{S}$, then it is a theorem of M.

Suppose $t_1 = t_2$ is true in $\mathbb{S}$; then $[\![t_1]\!]_{\mathbb{S}} = [\![t_2]\!]_{\mathbb{S}}$. But by part (c), $M \vDash t_1 = \langle [\![t_1]\!]_{\mathbb{S}} \rangle$ and $M \vDash t_2 = \langle [\![t_2]\!]_{\mathbb{S}} \rangle$; so by the transitivity of identity, $M \vDash t_1 = t_2$.

2. Let M+ be the result of adding to M:, for any two distinct constants c and c' of the language of strings other than "", each of the following axioms:

$$\begin{array}{ll} \text{M4} & \forall x(c \oplus x \neq \text{""}) \\ \text{M5} & \forall x \forall y(c \oplus x = c \oplus y \rightarrow x = y) \\ \text{M6} & \forall x(c \oplus x \neq c' \oplus x) \end{array}$$

THERE WAS A TYPO IN THE STATEMENT OF THIS PROBLEM! A prime symbol was missing from the statement of M5, which should have been

$$\text{M5'} \quad \forall x \forall y(c \oplus x = c' \oplus y \rightarrow x = y)$$

My apologies. Those who struggled to solve the problem as written will receive generous credit.

(a) (5%) Show that for any two distinct strings s and t, $\langle s \rangle \neq \langle t \rangle$ is a theorem of M+.

We show by induction that every string t has the following property: for all $s \neq t$, $M \vDash \langle s \rangle \neq \langle t \rangle$.

Base case: $t = []$, so $\langle t \rangle =$ "". Then if $s \neq t$, $s = (a : s')$ for some character a and string s', so $\langle s \rangle = c \oplus \langle s' \rangle$ (where c is the constant that denotes $[a]$). Thus, $\langle s \rangle \neq \langle t \rangle$ follows from M4.

Induction step: $t = (a : t')$, where t' has the property in question; then $\langle t \rangle = c \oplus \langle t' \rangle$, where c is the constant for a. Now suppose $s \neq t$. There are three possible cases. (i) $s = []$; (ii) $s = (b : s')$ where $s' \neq t'$; (iii) $s = (b : t')$ where $b \neq a$.

In case (i), $\langle s \rangle =$ ""; then $\langle s \rangle \neq \langle t \rangle$ follows from M4.

In case (ii), we have $M \vDash \langle s' \rangle \neq \langle t' \rangle$ by the induction hypothesis. Substituting $\langle s' \rangle$ and $\langle t' \rangle$ for x and y in M5$'$, and contraposing, we have that $M \vdash c \oplus \langle s' \rangle \neq c' \oplus \langle t' \rangle$, where c' is the constant that denotes b in the standard string structure.

In case (iii), we have that $\langle s \rangle = c' \oplus \langle t' \rangle$, where c' is the constant for b. So by axiom M6, we have $M \vDash \langle s \rangle \neq \langle t \rangle$.

(b) (5%) Using what you showed in part (a) and in part (c) of the previous exercise, conclude that for any closed terms t_1 and t_2 of the language of strings, if $t_1 \neq t_2$ is true in $\mathbb{S}$, it is a theorem of $M+$.

> Suppose t_1 and t_2 are terms such that $t_1 \neq t_2$ is true in $\mathbb{S}$. Then $[\![t_1]\!]_{\mathbb{S}} \neq [\![t_2]\!]_{\mathbb{S}}$, so by part a, M+ proves $\langle [\![t_1]\!]_{\mathbb{S}} \rangle \neq \langle [\![t_2]\!]_{\mathbb{S}} \rangle$. By part (c) of the previous exercise, M+ also proves $t_1 = \langle [\![t_1]\!]_{\mathbb{S}} \rangle$ and $t_2 = \langle [\![t_2]\!]_{\mathbb{S}} \rangle$. So by two applications of =Elim, we can conclude that M+ proves $t_1 \neq t_2$.

(c) (5%) Using what you showed in part (b) of this exercise and part (d) of the previous exercise, conclude that if P is any sentence of the language of strings that does not include any quantifiers or the predicate $\leq$ and is true in $\mathbb{S}$, P is a theorem of M+.

> We show by induction that every quantifier and $\leq$-free formula P has the following property: *whichever of P and $\neg P$ is true in $\mathbb{S}$ is a theorem of M+.*
>
> Base case: if P is an atomic formula and doesn't include $\leq$ or any free variables, it's of the form $t_1 = t_2$ for some closed terms t_1 and t_2. Part (d) of problem 1 says that this is a theorem of M (and hence of M+) if it's true in $\mathbb{S}$, and part (b) of the present problem says that its negation is a theorem of M+ if the negation is true in $\mathbb{S}$ (i.e., if the identity formula is false in $\mathbb{S}$).
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> Induction steps:
>
> (i) Suppose P has the property. Then $\neg P$ also has it. If it's true in $\mathbb{S}$, P is false, so by the IH $M+ \vDash \neg P$; if it's false in $\mathbb{S}$, P is true, so by the IH $M+ \vDash P$, in which case we also have $M+ \vDash \neg\neg P$ (since $P \vDash \neg\neg P$).
>
> (ii) Suppose P and Q have the property. Then $P \wedge Q$ also has it: if it's true in $\mathbb{S}$, P and Q are too, so $M+ \vDash P$ and $M+ \vDash Q$, so $M+ \vDash P \wedge Q$ by $\wedge$Intro; if it's false in $\mathbb{S}$, either P or Q is false, so either $M+ \vDash \neg P$ or $M+ \vDash \neg Q$, and in either case $M+ \vDash \neg(P \wedge Q)$ (since $\neg P \vDash \neg(P \wedge Q)$ and $\neg Q \vDash \neg(P \wedge Q)$.
>
> (iii) Similarly, if P and Q has the property, $P \rightarrow Q$ has it: if it's true in $\mathbb{S}$, then either P is false in $\mathbb{S}$ (in which case $M+ \vDash \neg P$ and so $M+ \vDash P \rightarrow Q$), or Q is true in $\mathbb{S}$ (in which case $M+ \vDash Q$ and so again $M+ \vDash P \rightarrow Q$). If it's false in $\mathbb{S}$, then P is true and Q is false, so $M+ \vDash P$ and $M+ \vDash \neg Q$, so $M+ \vDash \neg(P \rightarrow Q)$.
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> (iv) The case of disjunction is similar.