Solutions for Problem Set 1

Advanced Logic

21st September 2022

1. Prove that for any sets A and B, $A \subseteq B$ if and only if $A \cup B = B$.

Left to right: Suppose $A \subseteq B$. Then also $A \cup B \subseteq B$, since if $x \in A \cup B$, then either $x \in A$ or $x \in B$, and either way $x \in B$ since $A \subseteq B$. Also $B \subseteq A \cup B$ since obviously every member of B is either a member of A or a member of B. Hence, $A \cup B = B$ by the Axiom of Extensionalty.

Right to left: Suppose that $A \cup B = B$. Let x be an arbitrary member of A. Then x is either a member of A or a member of B, so $x \in A \cup B$, so $x \in B$. Since x was arbitrary we can conclude that $A \subseteq B$.

2. Prove that whenever R is a relation from A to B and S is a relation from B to C,

(a) If R and S are both serial, then $S \circ R$ is serial.

Suppose R and S are serial and consider $x \in A$. Since R is serial there is some $y \in B$ such that Rxy. Since S is serial, for any such y there is some $z \in C$ such that Syz. But in that case $(S \circ R)xz$, so x bears $S \circ R$ to something.

(b) If R and S are both functional, then $S \circ R$ is functional.

Suppose R and S are functional and consider some $x \in A$ and $z, z' \in C$ such that both $(S \circ R)xz$ and $(S \circ R)xz'$. By definition of $S \circ R$ there are $y, y' inB$ such that Rxy, Syz, Rxy', and $Sy'z'$. But then $y = y'$ since R is functional, and so $z = z'$ since S is functional.

3. Let $A = \{a, b\}$ and $B = \{c, d\}$ be two-membered sets. Then $A \times B$ is the four-mem- bered set $\{\langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle\}$, and there are thus 16 ($= 2^4$) relations from A to B. List all 16, for each one, specify whether it is serial, surjective, functional, and injective.

$\langle a, c\rangle$	$\langle a, d\rangle$	$\langle b, c\rangle$	$\langle b, d\rangle$	*Properties*
×	×	×	×	Functional, Injective
×	×	×	✓	Functional, Injective
×	×	✓	×	Functional, Injective
×	×	✓	✓	Surjective, Injective
×	✓	×	×	Functional, Injective
×	✓	×	✓	Serial, Functional,
×	✓	✓	×	Serial, Surjective, Functional, Injective
×	✓	✓	✓	Serial, Surjective,
✓	×	×	×	Functional, Injective
✓	×	×	✓	Serial, Surjective, Functional, Injective
✓	×	✓	×	Serial, Functional,
✓	×	✓	✓	Serial, Surjective,
✓	✓	×	×	Surjective, Injective
✓	✓	×	✓	Serial, Surjective,
✓	✓	✓	×	Serial, Surjective,
✓	✓	✓	✓	Serial, Surjective,

4. Prove that when R is a relation from A to B and S is a relation from B to C, $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Consider an arbitrary $x \in A$ and $z \in C$. Using the definitions of composition and converse, we have

$$\begin{aligned} \langle z, x\rangle \in (S \circ R)^{-1} &\text{ iff } \langle x, z\rangle \in S \circ R \\ &\text{ iff } \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S \text{ for some } y \in B \\ &\text{ iff } \langle y, x\rangle \in R^{-1} \text{ and } \langle z, y\rangle \in S^{-1} \text{ for some } y \in B \\ &\text{ iff } \langle z, x\rangle \in R^{-1} \circ S^{-1} \end{aligned}$$

So by the Axiom of Extensionality $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

5. Show that when R is a relation from A to B,

a. R is serial iff $\mathrm{id}_A \subseteq R^{-1} \circ R$

Left to right: Suppose R is serial. Every member of id_A is of the form $\langle x, x\rangle$ for some $x \in A$. For any such x, there is some $y \in B$ such that Rxy. In that case we also have $R^{-1}yx$ and hence $\langle x, x\rangle \in (R^{-1} \circ R)$.

Right to left: Suppose $\mathrm{id}_A \subseteq R^{-1} \circ R$ and $x \in A$. Then since $\langle x, x\rangle \in \mathrm{id}_A$, $\langle x, x\rangle \in R^{-1} \circ R$, i.e. there is some $y \in B$ such that Rxy and $R^{-1}yx$. We just need the first conjunct.

b. R is injective iff $R^{-1} \circ R \subseteq \mathrm{id}_A$.

Left to right: Suppose R is injective, and consider $\langle x, z\rangle \in R^{-1} \circ R$. Then there is some $y \in B$ such that Rxy and $R^{-1}yz$. But then Rzy, which implies $x = z$ since R is injective, and thus $\langle x, z\rangle \in \mathrm{id}_A$ as desired.

Right to left: Suppose $R^{-1} \circ R \subseteq \mathrm{id}_A$, and consider some $x, z \in A$ and $y \in B$ such that Rxy and Rzy. Then $R^{-1}yz$, so $(R^{-1} \circ R)xz$, so $\langle x, z\rangle \in \mathrm{id}_A$, i.e. $x = z$, thus establishing the injectivity of R.