Problem Set 10

Advanced Logic
27 th November 2022
Due date: Friday, 2 December.

Note: the scores for these problems add up to 110%, so a perfect score corresponds to 10% extra credit.

1. Let M be the theory in the language of strings axiomatized by all of the following sentences (where c may be any constant of the language of strings other than "")

$$
\begin{array}{ll}
\text{M1} & \forall x(x = \text{""} \oplus x) \\
\text{M2} & \forall x(x = x \oplus \text{"}\prime) \\
\text{M3} & \forall x \forall y \forall z((x \oplus y) \oplus z = x \oplus (y \oplus z))
\end{array}
$$

Show the following:

(a) (20%) Show that for any length-one string a, if c is the constant that denotes a in the standard string structure $\mathbb{S}$, then $c = \langle a \rangle$ is a theorem of **M**.

(b) (30%) Show that for any strings s_1 and s_2, $\langle s_1 \rangle \oplus \langle s_2 \rangle = \langle s_1 \oplus s_2 \rangle$ is a theorem of M.

Hint: use induction on s_1.

(c) (30%) Using what you showed in parts (a) and (b), prove that for any closed term t in the language of strings, $t = \langle [\![t]\!]_S \rangle$ is a theorem of M.

Reminder: $[\![t]\!]_{\mathbb{S}}$ is the denotation of t in the standard string structure.

Hint: use induction on the construction of t.

(d) (15%) Let t_1 and t_2 be any closed terms in the language of strings. Using what you showed in (c), prove that if the sentence $t_1 = t_2$ is true in $\mathbb{S}$, then it is a theorem of M.

2. Let $M+$ be the result of adding to M :, for any two constants c and c' of the language of strings other than "", each of the following axioms:

$$
\begin{array}{ll}
\text{M4} & \forall x\,(c \oplus x \neq \text{"}\prime\prime) \\
\text{M5} & \forall x \forall y(c \oplus x = c \oplus y \rightarrow x = y) \\
\text{M6} & \forall x\,(c \oplus x \neq c' \oplus x)
\end{array}
$$

(a) (5%) Show that for any two distinct strings s and t, $\langle s \rangle \neq \langle t \rangle$ is a theorem of $\mathrm{M} + \mathrm{s}$ Hint: use induction on t.

(b) (5%) Using what you showed in part (a) and in part (c) of the previous exercise, conclude that for any closed terms t_1 and t_2 of the language of strings, if $t_1 \neq t_2$ is true in $\mathbb{S}$, it is a theorem of $M+$.

(c) (5%) Using what you showed in part (b) of this exercise and part (d) of the previous exercise, conclude that if P is any sentence of the language of strings that does not include any quantifiers or the predicate $\leq$ and is true in $\mathbb{S}, P$ is a theorem of M+.

Hint: Use induction on the construction of P, with the induction hypothesis whichever of P and $\neg P$ is true in $\sim$ is a theorem of $M+$.