The Representability Theorem

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Min, the minimal theory of strings

$M1$	$\forall x \forall y (\neg (c \oplus x = ""))$ (for $c \neq ""$)
$M2$	$\forall x \forall y (c \oplus x = c \oplus y) \rightarrow x = y$
$M3$	$\forall x \neg (c_1 \oplus x = c_2 \oplus x)$
$M4$	$\forall x ("" \oplus x = x)$
$M5$	$\forall x \forall y ((c \oplus x) \oplus y = c \oplus (x \oplus y))$
$M6$	$c = c \oplus ""$
$M7$	$\forall x ("" \leq x)$
$M8$	$\forall x (x \leq "" \leftrightarrow x = "")$
$M9$	$\forall x \forall y ((c_1 \oplus x \leq c_2 \oplus y) \leftrightarrow x \leq y)$
$M10$	$\forall x \forall y (x \leq y \vee y \leq x)$
$M11$	$\forall x (x = "" \vee \exists y (x = c_1 \oplus y \vee \cdots \vee x = c_n \oplus y))$

Key facts about Min

1. Min can capture the labelling function.

2. Min can capture the substitution function.

3. For any axiom-set Ax, if Min can represent Ax, then Min can represent the relation *A is a proof of P from Ax*.

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And more boringly, we can add:

0. Min can represent any finite set of srtrings.

A generalization behind these facts

The previous facts can all be subsumed under the following generalizations:

4. Min can represent every *decidable* relation among strings.
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Intuitively: function f is *computable* if we can set up an ideal computer such that when it's run with strings $s_1, \ldots, s_n$ as input, it will stop and print out $f(s_1, \ldots, s_n)$ as output, and relation R among strings is *decidable* if we can set up an ideal computer such that when it's run with strings $s_1, \ldots, s_n$ as input, it'll stop and print out `yes` if $Rs_1 \ldots s_n$ and `no` otherwise.

We'll give precise versions of these informal definitions next week. But at our current informal level, it should seem obvious enough that the labelling and substitution functions are computable, and that the relation *A is a proof of P from Ax* is decidable whenever Ax is.

How we will prove these facts (preview)

We will prove a theorem to the following effect (I'll explain 'sufficiently simple' later):

Representability Theorem (rough statement)

Every relation that is definable in $\mathbb{S}$ *by a sufficiently simple formula* is represented in Min, and every function that is definable in $\mathbb{S}$ *by a sufficiently simple formula* is capturable in Min.

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We will prove a theorem to the following effect (I'll explain 'sufficiently simple' later):

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Given this, we can establish the facts on the previous slide by showing that:

1,2. The labelling and substitution functions have a sufficiently simple definitions in $\mathbb{S}$.

3. Whenever axiom-set Ax that has a sufficiently simple definition in $\mathbb{S}$, the relation *s is a proof of P from Ax* has a sufficiently simple definition in $\mathbb{S}$.

4. Every finite set has a sufficiently simple definition in $\mathbb{S}$.

5,6. Every decidable set and every computable function has a sufficiently simple definition in $\mathbb{S}$.

The Representability Theorem

Bounded formulae

Now it's time to get precise about what we were calling 'sufficiently simple formulae'.

There are actually two relevant notions of simplicity.

Definition

The set of **bounded formulae** of Str is the smallest set of formulae such that:

- Every atomic formula is bounded
- When P and Q are bounded, $\neg P$, $P \to Q$, $P \vee Q$, and $P \wedge Q$ are bounded.
- When P is bounded and t is any term, $\forall v(v \leq t \to P)$ and $\exists v(v \leq t \wedge P)$ are bounded.

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Definition

P is a Σ_1 **formula** of Str iff either P is a bounded formula of Str, or P is $\exists v Q$ for some bounded formula Q of Str.

Min, the minimal theory of strings

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Representation theorem

(i) If a set is defined in $\mathbb{S}$ by a bounded formula, it is representable in Min.

(ii) If a set is defined in $\mathbb{S}$ by a Σ_1 formula, it is semi-representable in Min.

(iii) If a partial function is defined in $\mathbb{S}$ by a bounded or Σ_1 term, it is capturable in Min.

Min includes all true quantifier-free sentences

Here's the first thing we'll prove on the way to showing this.

Quantifier-Free Omniscience

If A is a *quantifier-free* sentence of the language of strings, A is true in $\mathbb{S}$ iff $\mathsf{Min} \vDash A$

Note that this immediately immediately implies:

- If relation R is definable in $\mathbb{S}$ by a quantifier-free formula P, P represents R in Min.

This doesn't yet give us what we need, since the terms that define the functions we're interested in, e.g. substitution and labelling, are not quantifier-free. But it's a start.

First batch of facts to prove

For any closed terms t_1, t_2 in the language of strings:

Fact 1 If $t_1 = t_2$ is true in $\mathbb{S}$, $\mathsf{Min} \vDash t_1 = t_2$.

Fact 2 If $t_1 = t_2$ is false in $\mathbb{S}$, $\mathsf{Min} \vDash t_1 \neq t_2$.

Fact 3 If $t_1 \leq t_2$ is true in $\mathbb{S}$, $\mathsf{Min} \vDash t_1 \leq t_2$.

Fact 4 If $t_1 \leq t_2$ is false in $\mathbb{S}$, $\mathsf{Min} \vDash \neg t_1 \leq t_2$.

These straightforwardly entail Quantifier-Free Omniscience, by induction on the construction of formulae.

The proof of Facts 1–4 turns on:

Label Lemma

For any closed term t in the language of strings, and any string s, if $t = \langle s \rangle$ is true in $\mathbb{S}$, it's a theorem of Min.

Problem Set 11 walks you through the proof of this, and the proof of facts 1–4 from it.

The next thing we'll do is strengthen Quantifier-Free Omniscience to:

Bounded Omniscience

If P is a bounded sentence of the language of strings, P is true in $\mathbb{S}$ iff $\mathsf{Min} \vDash P$.

Given that the negation of a bounded formula is bounded, this immediately implies the first part of the Representation Theorem:

- If R is definable in $\mathbb{S}$ by a bounded formula P, P represents Y in Min.

Proving Bounded Omniscience

Listing Lemma

When s is any string, and $s_1, \ldots, s_n$ are *all* the strings which are no longer than s, Min $\vDash \forall x(x \leq \langle s \rangle \leftrightarrow (x = \langle s_1 \rangle \vee \cdots \vee x = \langle s_n \rangle))$.

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Equivalence Lemma

For every bounded formula P and assignment g, there is a quantifier-free sentence P' such that Min $\vDash P[v \mapsto \langle gv \rangle] \leftrightarrow P'$.

Proof: by induction on the complexity of P.

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Bounded Omniscience follows immediately from Quantifier-Free Omniscience and the Equivalence Lemma.

Positive Σ_1 Omniscience

Given Bounded Omniscience, we can immediately infer the stronger

Σ_1 Omniscience

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If A is a Σ_1 sentence of the language of strings, A is true in $\mathbb{S}$ iff $\mathsf{Min} \vDash A$.

Proof: The right-to-left direction follows from the truth of Min in $\mathbb{S}$. For the left-to-right direction, let P be $\exists v Q$ where Q is bounded. If P is true in $\mathbb{S}$, then there must be some string s such that Q is true in $\mathbb{S}$ on the assignment $[v \mapsto s]$; then the sentence $Q[v \mapsto \langle s \rangle]$ is true in $\mathbb{S}$, so by Bounded Omniscience it is a theorem of Min, so by Existential Generalization $\exists v Q$ is also a theorem of Min.

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Σ_1 Omniscience implies the second part of the Representation Theorem: if a Σ_1 formula defines a set in $\mathbb{S}$, it semi-represents it in Min. But it need not *represent* the set, since the negation of a Σ_1 formula is no Σ_1, and generally not even Σ_1-equivalent.

Σ_1-equivalence

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A Σ_1-**equivalent** formula P is one such that for some Σ_1-formula Q, Min $\models P \leftrightarrow Q$

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Also, if P is Σ_1-equivalent, so is $\exists v P$.

Capturing functions

We still have to show the third part of the representation theorem: that if a function f is defined in $\mathbb{S}$ by a Σ_1 formula, it is capturable in Min.

Let's just focus on 1-ary functions (the general case is similar). Suppose f is defined in $\mathbb{S}$ by the Σ_1 formula $Q(x, y) := \exists x\, P(x, y, z)$, where $P(x, y, z)$ is a bounded formula.

Then by Σ_1-omniscience, we have $\text{Min} \vDash \exists z P(\langle s \rangle, \langle fs \rangle, z)$ for all s. Unfortunately, Min may not prove the *unique existence claim*, $\forall x \exists! y \exists x (P(x, y, z))$, and if it doesn't, $Q(x, y)$ doesn't capture f in Min.

The clever move

Now we invoke the trick. Consider the formula

$$Q'(x,y) \coloneqq \exists z(P(x,y,z) \wedge (\forall y' \leq y)(\forall z' \leq z)(P(x,y',z') \rightarrow y' = y)) \\ \vee(\neg\exists y \exists z\, P(x,y,z) \wedge y = \text{""})$$

- $Q'(x,y)$ defines f in $\mathbb{S}$ given that $Q(x,y)$ does: if for all s, Q is true of $\langle s, fs\rangle$ and not of $\langle s,t\rangle$ for any other t, then for all s, the first disjunct of $Q'(x,y)$ is true of $\langle s, fs\rangle$ and neither disjunct is true of $\langle s,t\rangle$ for any other t.
- For any string s, Min proves $Q'(\langle s\rangle, \langle fs\rangle)$. Since $Q(x,y)$ defines f, there's a string t such that $P(\langle s\rangle, \langle fs\rangle, \langle t\rangle)$ is true in $\mathbb{S}$, while $P(\langle s\rangle, \langle t'\rangle, \langle t''\rangle)$ is false in $\mathbb{S}$ for any t'' no longer than t and $t' \neq fs$. By Bounded Omniscience, Min proves these sentences. So by the Listings Lemma, Min proves the first disjunct of $Q'(\langle s\rangle, \langle fs\rangle)$.

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Now we invoke the trick. Consider the formula

$$Q'(x,y) := \exists z(P(x,y,z) \wedge (\forall y' \leq y)(\forall z' \leq^* z)(P(x,y',z') \rightarrow y' = y)) \\ \vee(\neg\exists y \exists z\, P(x,y,z) \wedge y = \text{""})$$

- Thanks to axiom M10, Min proves the unique existence claim for $Q'(x,y)$. Sketch of formal proof: first, note that $\forall x \exists y\, Q'(x,y)$ is a logical truth. Now suppose for contradiction that $Q'(x,y)$, $Q'(x,y')$, and $y \neq y'$. We can't have $\neg\exists y \exists z\, P(x,y,z)$, since then we'd have have $y = \text{""}$ and $y' = \text{""}$ and hence $y = y'$. So, there must exist z and z' such that both

  $$P(x,y,z) \wedge (\forall y'' \leq y)(\forall z'' \leq z)(P(x,y'',z'') \rightarrow y'' = y)$$
  $$P(x,y',z') \wedge (\forall y'' \leq y')(\forall z'' \leq z')(P(x,y'',z'') \rightarrow y'' = y')$$

  By axiom M10, we have that $y \leq y'$ or $y' \leq y$. Suppose it's the former. Then by the second of the above claims, it must be false that $z \leq z'$, so by M10, $z' \leq z$

Finish the proof

This means that for any string d:

$$\mathsf{Min}\langle fd\rangle \ \texttt{= the y } \exists\texttt{z (}B(\langle d\rangle, \texttt{y}, \texttt{z}) \ \wedge \ (\forall\texttt{yy})(\forall\texttt{zz})(B(\langle d\rangle, \texttt{y}, \texttt{z}) \ \rightarrow \ \texttt{y=y}))$$

We can show this by showing two things. First:

(i) $\mathsf{Min}\ \exists\texttt{z (}B(\langle d\rangle, \langle fd\rangle, \texttt{z}) \ \wedge \ (\forall\texttt{y}\langle fd\rangle)(\forall\texttt{zz})(B(\langle d\rangle, \texttt{y}, \texttt{z}) \ \rightarrow \ \texttt{y=}\langle fd\rangle))$

This part follows from Σ_1 Omniscience. Second:

(ii) $\mathsf{Min}\ \forall\texttt{y (}\exists\texttt{z (}B(\langle d\rangle, \texttt{y}, \texttt{z}) \ \wedge \ (\forall\texttt{yy})(\forall\texttt{zz})(B(\langle d\rangle, \texttt{y}, \texttt{z}) \ \rightarrow \ \texttt{y=y})) \ \rightarrow \ \texttt{y = }\langle fd\rangle)$

To show that this we'll need to rely on the last unused axiom of S, S10. This lets us reduce the universal quantification to two weaker things, one restricted by the formula $\texttt{y}$ $\langle fd\rangle$ and the other restricted by the formula $\langle fd\rangle$ $\texttt{y}$. Both parts follow from (i).