Problem Set 8

Advanced Logic
30th October 2022
Due date: Friday, 4 November.

1. (50%) Prove the Substitution Lemma for Terms: for any signature Σ, any terms s and t of Σ, any variable v, any structure S for Σ, and any assignment g for S,

$$[\![t[s/v]]\!]_S^g = [\![t]\!]_S^{g\left[v \mapsto [\![s]\!]_S^g\right]}$$

Hint: prove this by induction on the construction of terms (should it be t or s?)

Reminder:

$$g\left[v \mapsto [\![s]\!]_S^g\right](u) = \begin{cases} [\![s]\!]_S^g & \text{if } u = v \\ gu & \text{otherwise} \end{cases}$$

Answer for q1

1. costant case, variable case, one place function case, 2 place function case and so on
2. Induction Hypothesis: The construction of t(assume the hint). because $t[s/v]$ is substitute the occurence of variable v with term s in the term t.
3. there are two cases for the construction of t, either it is gu or gs.
4. gs is the case that const case, n-place function case, and ordinary variable besides u = v
5. To prove the lemma, we can prove from two direction prove each direction is a subset of the other
6. left to right:
7. $[\![t[s/v]]\!]_S^g \subseteq [\![t]\!]_S^{g\left[v \mapsto [\![s]\!]_S^g\right]}$
8. if t is a constant, then $[\![c[s/v]]\!]_S^g$ = c
9. as the construction said, $[\![t]\!]_S^{g\left[v \mapsto [\![s]\!]_S^g\right]}$, $v \mapsto [\![s]\!]_S^g$, has two cases, gu or gs, however, since t is constant, the right side is also costant.

10. gu = c, gs = c

11. $c \subseteq c$, it is proved, constant is the base case

12. for the variable case, we uses the IH. there are two cases for IH, either u = v or u != v, if u = v, then we use gs. then t contains s. if u != v, then we have gu. Since u does not occur, it is gu = v. In both cases, the $\subseteq$ holds as t contains s and v by $t[u/v]$,

13. for the n-place function cases,

14. for a n-place function, case, it is the individual of the each single term

15. for each single term, the reasoning is similar, and the $\subseteq$ is hold.

16. Thus, the left to right direction is proved

17. For the right to left direction, the reasoning is the same

2. (30%) Using the result from part 1, prove the Substitution Lemma for Formulae: for any signature Σ, any term s of Σ, any formula P of Σ, any variable v, any structure S for Σ, and any assignment g for S,

$$S, g \Vdash P[s/v] \text{ iff } S, g\,[v \mapsto [\![s]\!]_S^g] \Vdash P$$

Hint: prove this by induction on the construction of formulae. In the induction step for quantifiers, you will need to separately consider the case of formulae that begin with $\forall v$ (or $\exists v$) and formulae that begin with $\forall u$ (or $\exists u$) for some other variable u. You may if you wish rely on the "Irrelevance Lemma" according to which if $g(u) = h(u)$ for all $u \in FV(Q), S, g \Vdash Q$ iff $S, h \Vdash Q$.

1. begin the proof by the construction of $\vdash$

2. in the base case, formulas contains only empty set. Thus, it is trivial that the target is valid

3. IH: $\forall v$, $\exists v$

4. from right to left (it looks like this direction is easier)

5. In this case, $\vdash$ has all the formulas. By substitution lemma, we can substitute any variable with another term if this term is in t. Since t is in one of the formula, P is a logical consequence of S and s which are the structure and the term.

6. Since it's a logical sequent, the ordered pair $\vdash P$ is valid.

7. left to right

8. By expanding the definition of the assignment function which convert variables to certain terms on the domain of the formula transformation, we have the denotation exposed. The denotion is the assignment function after the transformation with respect to the structure.

9. this is the substitution lemma in the other direction.

10. IH: $\forall u$, $\exists u$

11. This induction hypothese is similar, but it is the case that if u = v. Thus, we use the same reasoning of using the substitution lemma but we use the other special case of substituting the variable with a new one.

4. (10%) Using the result from part 2, prove the steps in the proof of the Soundness Theorem corresponding to the $\forall$ Elim and $\exists$ Intro rules. That is: show that if $\Gamma \vDash \forall v P$ then $\Gamma \vDash P[s/v]$ for every term s, and that if $\Gamma \vDash P[s/v], \Gamma \vDash \exists v P$.

5. (5%) Using the result from part 2, prove the step in the proof of the Soundness Theorem corresponding to the $=$ Elim rule. That is: show that if $\Gamma \vDash P[s/v]$ and $\Gamma \vDash s = t$, then $\Gamma \vDash P[t/v]$

6. (5%) Using the result from part 2, prove the step in the proof of the Soundness Theorem corresponding to the $\exists$ Elim rule. That is: show that if $\Gamma \vDash \exists v P$ and $\Gamma, P[u/v] \vDash Q$, then $\Gamma \vDash Q$, provided that u is not free in Γ, Q, or $\exists v P$.

Give examples to show that this can fail when (i) u is free in Γ though not in Q or $\exists v P$; (ii) u is free in Q though not in Γ or $\exists v P$; (iii) u is free in $\exists v P$ though not in Γ or Q EXTRA CREDIT (2.5% each, up to a maximum of 10%) Prove the remaining facts about logical consequence required to complete the proof of the Soundness Theorem, namely:

1. If $\Gamma \vDash P$ and $\Gamma \vDash Q$ then $\Gamma \vDash P \wedge Q (\wedge \text{Intro})$

2. If $\Gamma \vDash P \wedge Q$ then $\Gamma \vDash P$ and $\Gamma \vDash Q (\wedge$ Elim1 and $\wedge$ Elim2).

3. If $\Gamma, P \vDash Q$ then $\Gamma \vDash P \rightarrow Q (\rightarrow \text{Intro})$

4. If $\Gamma \vDash P \rightarrow Q$ and $\Gamma \vDash P$ then $\Gamma \vDash Q (\rightarrow \text{Elim})$

5. If $\Gamma, P \vDash Q$ and $\Gamma, P \vDash \neg Q$ then $\Gamma \vDash \neg P (\neg \text{Intro})$

6. If $\Gamma \vDash \neg\neg P$ then $\Gamma \vDash P$(DNE).

7. $\vDash t = t$ for every term $t (= \text{Intro})$.