The Recursion Theorem

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Relational structures

Relational structures, intuitively

Often in mathematics (including metalogic), we are interested in bundles comprising a set D together with one or more relations on D (which could be regular binary relations on D, i.e. subsets of $D \times D$, or ternary relations (subsets of $D^3 = (D \times D) \times D \ldots$, or. . . .

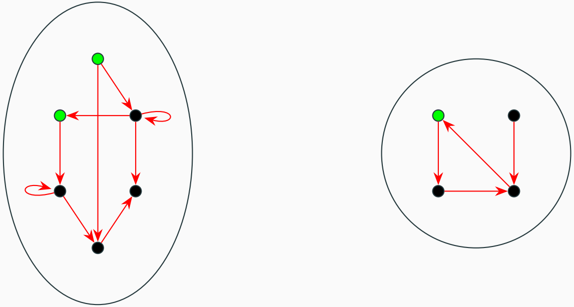
- When we talk of 'n-ary relations' on a set, we include the case of *singulary* (1-ary) relations on a set as a special case: 'singulary relation on A' is just another way of saying 'subset of A'. We'll identify the 'range' of a subset with the subset itself.

The most common way to make sense of such "bundles" is just to look at a tuple where the first co-ordinate is the set A and the remaining co-ordinates are the relations on A that we are interested in.

Example: DXR-bundles

For example, we could consider triples of the form $\langle D, X, R\rangle$ where D is a set, X is a subset of D, and R is a binary relation on D. Let's call such triples "DXR-bundles" as a mnemonic. (We won't use this terminology after today).

We can picture a DXR-bundle as a set where some elements may be are colored green (the ones in X) and where there may be arrows between some elements (the pairs in R).

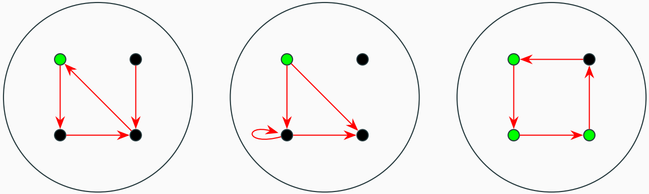


Some properties of DXR-bundles

Definition

A DXR-bundle $S = \langle D, X, R \rangle$ is *algebraic* iff (i) X is a singleton and (ii) R is a function.

The DXR-bundle pictured on the left is algebraic, while the ones on the right are not.



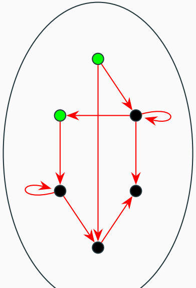
Some properties of DXR-bundles

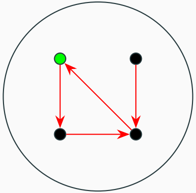
Definition

A DXR-bundle $S = \langle D, X, R \rangle$ has the *Inductive Property* iff D is the closure of X under R.

That is: for every $C \subseteq D$, if $x \in C$ whenever $x \in X$, and $y \in C$ whenever $x \in C$ and Rxy, then $C = D$.

The DXR-bundle on the left has the Inductive Property; the one on the right doesn't





Some properties of DXR-bundles

Definition

A DXR-bundle $S = \langle D, X, R \rangle$ has the *Injective Property* iff (i) R is injective, and (ii) no element of X is in the range of R.

Definition

Suppose $S = \langle D, X, R\rangle$ and $S' = \langle D', X', R'\rangle$ are DXR-bundles. Then a *homomorphism from* S *to* S' is any function $h : D \to D'$ such that

(i) whenever $x \in X$, $hx \in X'$

(ii) whenever Rxy, $R'(hx)(hy)$.

The Recursion Theorem for DXR-bundles

Using the special DXR-bundle terminology introduced on the last few slides, we can state the following theorem. (We won't prove it yet since it'll be more useful to prove a more general theorem of which it's a special case.)

Recursion Theorem for DXR-bundles

Suppose $S = \langle D, X, R \rangle$ is a DXR-bundle with the Inductive and Injective Properties and $S' = \langle D', X', R' \rangle$ is an algebraic DXR-bundles. Then there is a unique homomorphism from S to S'.

The Recursion Theorems for DXR-bundles and Numbers

We can consider the natural numbers as a DXR-bundle $\langle \mathbb{N}, \{0\}, \mathrm{suc} \rangle$. It is algebraic, and has the Inductive and Injective properties. So as a special case of the Recursion Theorem for DXR-bundles, we have that there is a unique homomorphism from $\langle \mathbb{N}, \{0\}, \mathrm{suc} \rangle$ to any algebraic DXR-bundle.

In other words:

Recursion Theorem for Numbers

Suppose D is a set, $z \in D$, and $s : D \to D$. Then there is a unique function $f : \mathbb{N} \to D$ such that $f0 = z$ and for all $n \in \mathbb{N}$, $f(\mathrm{suc}\, n) = s(fn)$.

The supposition can be restated as '$\langle D, \{z\}, s \rangle$ is an algebraic DXR-bundle'. And the property singling out f is equivalent to '$fx \in \{z\}$ whenever $x \in \{0\}$, and whenever $\langle n, m \rangle \in \mathrm{suc}$, $\langle fn, fm \rangle \in s$', in other words '$f$ is a homomorphism from $\langle \mathbb{N}, \{0\}, \mathrm{suc} \rangle$ to $\langle D, \{z\}, s \rangle$'.

Proving the Recursion Theorem for DXR-bundles

Let's prove the Recursion Theorem for DXR-bundles, since the idea of the proof is exactly the same as that of the more general Recursion Theorem we'll have later.

Proof: Let $X^+ = X \times X'$.

Let R^+ be the binary relation on $D \times D'$ such that $R^+\langle x, x'\rangle\langle y, y'\rangle$ iff Rxy and $R'x'y'$.

Let F be the closure of X^+ under R^+.

We will first prove that F is a function from D to D', and then verify that it meets the further conditions required to be a homomorphism from S to S'. Finally we'll show that there is at most one homomorphism from S to S'.

Proving the Recursion Theorem

(i) F is serial, i.e. for all $x \in D'$ there exists $x' \in D'$ such that Fxx'. By induction.

Base case: if $x \in X$, then since X' is nonempty, there is some $x' \in D'$ such that $\langle x, x' \rangle \in X^+ \subseteq F$.

Induction step: Suppose Fxx' and Rxy. Since R' is serial, there exists y' such that $R'x'y'$. But then $R^+\langle x, x'\rangle\langle y, y'\rangle$, so Fyy' since F is closed under R^+.

Proving the Recursion Theorem

(ii) F is functional, i.e. for any $x \in D$, if Fxx' and Fxx'' then $x' = x''$. By induction.

Base case: suppose $x \in X$. Then since x isn't in the range of R, no ordered pair $\langle x, x' \rangle$ is in the range of any R^+. So the only way $\langle x, x' \rangle$ and $\langle x, x'' \rangle$ could both be in F is if both are in $X^+ = X \times X'$. But since X' is a singleton, it follows that $x' = x''$.

Induction step: suppose that x is F-related to a unique x', Rxy, and Fyy' and Fyy''; we will show that $y' = y''$. Since $y \notin X$, neither $\langle y, y' \rangle$ nor $\langle y, y'' \rangle$ is in X^+, so both must be in the range of R^+: i.e. we have $R^+\langle z, z' \rangle\langle y, y' \rangle$ and $R^+\langle u, u' \rangle\langle y, y'' \rangle$ for some $\langle z, z' \rangle$ and $\langle u, u' \rangle$ in F. But then Rzy and Ruy, so $z = u = x$ since R is injective. So by the induction hypothesis, $z' = u' = x'$. Thus we have $R'x'y'$ and $R'x'y''$, hence $y = y''$ by the fact that R' is functional.

Proving the Recursion Theorem

(iii) F is a homomorphism from S to S'.

Suppose $x \in X$ and Fxx'. Then since x is not in the range of R, $\langle x, x' \rangle$ is not in the range of R^+, so it must be in X^+, so $x \in X'$.

Suppose Rxy, Fxx', and Fyy'. Then since y is not in X, $\langle y, y' \rangle$ is not in X^+, so there must be some $\langle z, z' \rangle \in F$ such that $R^+\langle z, z' \rangle\langle y, y' \rangle$, i.e. Rzy and $R'z'y'$. But since R is injective we must have $z = x$, and thus since F is functional we must have $z' = x'$, so $R'x'y'$.

Proving the Recursion Theorem

(iv) There is at most one homomorphism from S to S'.

Suppose g and h are homomorphisms from S to S': we prove by induction that $gx = hx$ for all $x \in D$.

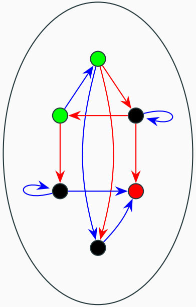
Base case: $x \in X$. Then gx and hx must both be in X', so $gx = hx$ since X' is a singleton.

Induction step: suppose $gx = hx$ and Rxy. Then $R'(gx)(gy)$ and $R'(gx)(hy)$; but then $gy = hy$ since R' is functional.

Example: DXYRS-bundles

As another example, we could consider quintuples of the form $\langle D, X, Y, R, S\rangle$, where D is a set, X and Y are subsets of D, and R and S are binary relations on D: let's call these 'DXYRS-bundles'.

We can picture a DXYRS-bundle as a set where some elements may be colored green (X) or red (Y) or both, and where any pair of elements may be connected by a red (R) or blue (S) arrow or both:



Example: DXYRS-bundles

All the definitions that we had for DXR-bundles carry over to DXYRS-bundles. For a DXYRS-bundle $B = \langle D, X, Y, R, S \rangle$:

- B is algebraic iff X and Y are singletons and R and S are functions.
- B has the Inductive Property iff D is the closure of $X \cup Y$ under R and S.
- B has the Injective Property iff R and S are injective and no two of X, Y, range R, and range S overlap.

For two DXYRS bundles $B = \langle D, X, Y, R, S \rangle$ and $B' = \langle D', X', Y', R', S' \rangle$:

- A homomorphism from B to B' is a function $h : D \to D'$ such that $hx \in X'$ whenever $x \in X$, $hx \in Y'$ whenever $x \in Y$, $R'(hx)(hy)$ whenver Rxy, and $S'(hx)(hy)$ whenever Sxy.

And we have a Recursion Theorem for DXYRS-bundles too: if B has the Inductive and Injective properties and B' is algebraic, there is a unique homomorphism from B to B'.

Generalizing

We need a flexible way of generalizing these concepts to "bundles" that build in any number of relations of any given "arity". Here's how we'll do it.

Definition

A *relational signature* Σ is an ordered pair $\langle I_\Sigma, a_\Sigma \rangle$, where I_Σ is a set and a_Σ is a function from $I \to \mathbb{N}^+$ such that $a_\Sigma(i) \neq 0$ for all $i \in I$.

The idea is that each such Σ will correspond to a particular "type" of relational structure, which contains as many n-ary relations are there are elements of I_Σ mapped to the number n by a_Σ. The elements of I are just arbitrary labels.

Notation: Sometimes we'll give a function a name like $(a_i)_{i \in I}$ or just (a_i), where I is the function's domain. When we pick a name like for a function, we get to write 'a_i' (with a subscript) to denote the value of the funciton on a particular argument $i \in I$. This makes things easier to read, and is convenient when we are just using the set I as an arbitrary collection of labels rather than caring about it for its own sake.

Relational structures, carefully

So, officially:

Definition

Where $\Sigma = \langle I_\Sigma, a_\Sigma \rangle$ is a relational signature, a *relational structure for* Σ is a pair $S = \langle D, (R_i) \rangle$, where for each $i \in I_\Sigma$, $R_i \subseteq D^{a_\Sigma i}$.

- The elements of I_Σ can be anything we like, but it might be helpful to think of them as sets of *colors*. Then we can diagram a relational structure for Σ by a collection of dots where each dot can have one or more of the colors i for which $a_\Sigma i = 1$; each pair of dots can be connected by an arrow with any of the colors i for which $a_\Sigma i = 2$; each triple of dots can be connected by a "double tailed arrow" with any of the colors in i for which $a_\Sigma i = 3$; and so on.
- It'll often be convenient to use sets of numbers or sets of strings.

DXR-structures and DXYRS-structures as relational structures

If I and J are one-element sets $\{a\}$ and $\{b\}$, there is a natural one-to-one correspondence between relational structures for I, J, $\varnothing$ and DXY-structures: we just set $Q_a = X$ and $S_b = R$.

If I and J are two-element sets $\{a, b\}$ and $\{c, d\}$, we can get a one-to-one correspondence between relational structures for I, J, $\varnothing$ and DXYRS-structures: to get a DXYRS structure out of such a relational structure, we have to pick one element from each of I and J as the 'first', say a and c respectively, and then set $Q_a = X$, $Q_b = Y$, $S_c = R$, $S_d = S$.

Examples

To make any of our examples into an official relational structure, we'll need to pick some arbitrary index sets.

- For the natural numbers with 0 and suc, we could set $I = J = \{0\}$ and $K = \varnothing$, and take $Q_0 = \{0\}$ and $S_0 = \mathrm{suc}$. (Or we could have J be $\{\mathrm{suc}\}$ and $S_{\mathrm{suc}} = \mathrm{suc}$.)
- For the natural numbers with 0, suc, $+$, $\times$, we could instead take $K = \{1, 2\}$ and set $t_1 = +$ and $t_2 = \times$.
- For A^*, we could set $I = \{0\}$, $Q_0 = \{[]\}$, $J = A$, $S_a = \mathrm{cons}_a$.

Definitions for relational structure

Definition

A relational structure $\langle D, (R_i)\rangle$ for Σ is an *algebraic structure for* iff R_i is a function whenever $a_\Sigma i > 1$ and R_i is a singleton whenever $a_\Sigma i = 1$.

- Why is it natural to group the two parts of the definition together? Well, an *n*-ary relation R on D is an $n-1$-ary function iff for all $x_1, \ldots, x_{n-1} \in D$ there is exactly one x_n such that $Rx_1 \ldots x_{n-1}x_n$.
- In the case of $n = 1$, 'for all $x_1, \ldots, x_{n-1} \in D$' isn't adding anything; so we can identify *0-ary functions on D* with *singleton subsets of D*.

The Inductive Property

Definition

A relational structure $S = \langle D, (R_i) \rangle$ has the *inductive property* iff D is the only subset of D closed under each (R_i).

- When R is an n-ary relation on D, $C \subseteq D$ is closed under R iff whenever $x_1, \ldots, x_{n-1} \in C$ and $Rx_1 \ldots x_{n-1}x_n$, $x_n \in C$.
- For $n = 1$, this just means $R \subseteq C$!
- So, another way to put this definition is: D is the closure of $\bigcup\{R_i \mid a_\Sigma i = 1\}$ under all of the R_i for which $a_\Sigma i > 1$.

The Inductive Property

Suppose we have a relational structure for Σ, $\langle D, (R_i) \rangle$, that has the Inductive Property, and we want to show that every element of D has a certain property ϕ. Then we can do so by a proof by induction.

In this kind of proof by induction, there is just one step: we consider R_i (where $a_\Sigma i = n$) and suppose that $\phi(x_1), \ldots, \phi(x_{n-1})$ and $Rx_1 \ldots x_n$. Then we have to prove that $\phi(x_n)$. If we can do this for all i, we can conclude that every element of D is ϕ.

- Here we have a 'base case' for every singulary R_i and an 'induction step' for every n-ary R_i where $n > 1$. But if we're proving something about relational structures, for an arbitrary signature Σ, we won't be listing these separately in our proof.

The Injective Property

Definition

A relational structure $S = \langle D, (R_i) \rangle$ has the Injective Property iff

(i) R_i is injective whenever $a_\Sigma i > 1$.

(ii) Whenever $i \neq j$, the ranges of R_i and R_j do not overlap.

- We identify the "range" of a set considered as a singulary relation with the set itself.

A homomorphism from one relational structure to another will be a function from the carrier set of the first to the carrier set of the second that preserves all the relations. That is:

Definition

Suppose $S = \langle D, (R_i) \rangle$ and $S' = \langle D', (R'_i) \rangle$ are two relational structures for Σ. Then a *homomorphism from* S *to* S' is a function $h : D \to D'$ such that for each i, whenever $R_i x_1 \ldots x_n$, $R'_i (hx_1) \ldots (hx_n)$.

When the relational structures we are interested in are both algebraic structures, the condition for a homorphism can be restated as follows:

Fact

Suppose $S = \langle D, (f_k) \rangle$ and $S' = \langle D', (f'_k) \rangle$ are two algebraic structures for Σ. Then h is a homomorphism from S to S' iff $h : D \to D'$ and for each i, $h(f_k(x_1, \ldots, x_{n-1})) = f'_k(hx_1, \ldots, hx_{n-1})$.

Examples of homomorphisms

* length

* dots

* naturals to $\{0, 1\}$

* interpretation function

Easy facts about homomorphisms

Fact

If h is a homomorphism from S to S' and g is a homomorphism from S' to S'', $g \circ h$ is a homomorphism from S to S''.

Fact

The identity function on the domain of a structure is a homomorphism from that structure to itself.

One more definition

Definition

h is an *isomorphism* from S to S' iff h is a homomorphism from S to S', h is a bijection, and h^{-1} is a homomorphism from S' to S.

Fact

When S and S' are algebraic structures, h is an isomorphism from S to S' iff h is a homomorphism from S to S' and h is a bijection.

Proof: Let $S = \langle D, (f_i) \rangle$ and $S' = \langle D', (g_i) \rangle$, and h a bijective homomorphism from S to S' It suffices to show that h^{-1} is a homomorphism from S' to S. Fix i and $y_1, \ldots, y_n \in D'$ (where $n = a_\Sigma i$). Since h is a homomorphism we have

$$\begin{aligned} h(f_k(h^{-1}y_1, \ldots, h^{-1}y_n)) &= g_k(h(h^{-1}y_1), \ldots, h(h^{-1}y_n)) \\ &= g_k(y_1, \ldots, y_n) \end{aligned}$$

and hence

$$f_k(h^{-1}y_1, \ldots, h^{-1}y_n) = h^{-1}(g_k(y_1, \ldots, y_n))$$

The Recursion Theorem

The Recursion Theorem

The Recursion Theorem

Suppose that $S = \langle D, (R_i) \rangle$ and $S' = \langle D', (R'_i) \rangle$ are relational structures for Σ such that S has the Injective and Inductive properties and S' is an algebraic structure. Then there is a unique homomorphism from S to S'.

Proving the Recursion Theorem

Proof: For each i (where $a_\Sigma i = n$), we define R_i^+ to be an n-ary relation on $D \times D'$:

$$R_i^+ := \{\langle\langle x_1, x_1'\rangle, \ldots, \langle x_n, x_n'\rangle\rangle \mid Rx_1 \ldots x_n \text{ and } R'x_1' \ldots x_n'\}.$$

Let F be the smallest subset of $D \times D'$ closed under each (R_i^+).

We will first prove that F is a function from D to D', and then verify that it meets the further conditions required to be a homomorphism from S to S', and that it is the only such homomorphism.

Proving the Recursion Theorem

(i) F is serial, i.e. for all $x \in D'$ there exists $x' \in D'$ such that Fxx'. By induction (using S's Inductive Property).

Suppose that $Fx_1x'_1 \ldots$ and $Fx_{n-1}x'_{n-1}$ and $R_ix_1 \ldots x_n$. Since R'_i is serial, there exists x'_n such that $R'_ix'_1 \ldots x'_{n-1}x'_n$. But then $R_i^+\langle x_1, x'_1\rangle \ldots \langle x_n, x'_n\rangle$, so Fyy' since F is closed under R_i^+.

Proving the Recursion Theorem

(ii) F is functional, i.e. for any $x \in D$, if Fxx' and Fxx'' then $x' = x''$. By induction (using S's Inductive Property). Suppose for induction that x_1 is F-related to a unique x_1',..., and x_{n-1} is F-related to a unique x_{n-1}', $Rx_1 \dots x_n$, Fx_nx_n', and Fx_nx_n''. We will show $x_n' = x_n''$.

By S's Injective Property, x_n isn't in the range of any R_j for $j \neq i$; thus neither $\langle x_n, x_n' \rangle$ nor $\langle x_n, x_n'' \rangle$ is in the range of any R^+j for $j \neq i$. So both are in the range of R_i^+ restricted to F—i.e. there exist $\langle y_1, y_1' \rangle, \dots, \langle y_{n-1}, y_{n-1}' \rangle, \langle z_1, z_1' \rangle, \dots, \langle z_{n-1}, z_{n-1}' \rangle \in F$ such that

$$R_i^+ \langle y_1, y_1' \rangle \dots \langle y_{n-1}, y_{n-1}' \rangle \langle x_n, x_n' \rangle \quad \text{and}$$
$$R_i^+ \langle z_1, z_1' \rangle \dots \langle z_{n-1}, z_{n-1}' \rangle \langle x_n, x_n'' \rangle$$

But then we have $R_i y_1 \dots y_{n-1} x_n$ and $R_i z_1 \dots z_{n-1} x_n$; so by S's Injective Property, each $y_k = z_k = x_k$. But then by the IH, each $y_k' = z_k' = x_k'$. So we have $R_i' x_1' \dots x_{n-1}' x_n'$ and $R_i' x_1' \dots x_{n-1}' x_n''$, hence $x_n' = x_n''$ since R_i' is functional.

Proving the Recursion Theorem

(iii) F is a homomorphism.

Suppose Fx_1x_1', ..., Fx_nx_n', and $R_ix_1 \ldots x_n$. Since x_n isn't in the range of R_j for any $j \neq i$, $\langle x_n, x_n' \rangle$ isn't in the range of R_j^+ for any $j \neq i$, so there must be $\langle y_1, y_1' \rangle, \ldots, \langle y_{n-1}, y_{n-1}' \rangle \in F$ such that

$$R_i^+ \langle y_1, y_1' \rangle \ldots \langle y_{n-1}, y_{n-1}' \rangle \langle x_n, x_n' \rangle$$

But then $R^i y_1 \ldots y_{n-1} x_n$, so each $y_k = x_k$ because R^i is injective, and so each $y_k' = x_k'$ because F is functional. Thus we have

$$R_i^+ \langle x_1, x_1' \rangle \ldots \langle x_n, x_n' \rangle$$

and hence

$$R_i' x_1' \ldots x_n'.$$

Proving the Recursion Theorem

(iv) There is at most one homomorphism from S to S'.

Suppose g and h are both homomorphisms from S to S'. Using S's Inductive Property, we'll prove by induction that $gx = hx$ for every $x \in D$.

Suppose for induction that $gx_1 = hx_1, \ldots,$ and $gx_{n-1} = hx_{n-1}$, and $Rx_1 \ldots x_{n-1}x_n$.

Then $R'(gx_1) \ldots (gx_{n-1})(gx_n)$ and $R'(gx_1) \ldots (gx_{n-1})(hx_n)$ since g and h are homomorphisms.

Since R' is functional, it follows that $gx_n = hx_n$.