Theories and labellings

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Theories

Definition

A **theory** in a signature Σ is a set T of sentences (closed formulae) of Σ such that whenever P is closed and $T \vdash P$, $T \in \Gamma$.

Note: given the Soundness and Completeness theorems, we could just as well write $\vDash$ rather than $\vdash$ in this definition.

Finite axiomatizability

Definition

When T is a theory and $A \subseteq T$, A **axiomatizes** T iff T is the set of all sentences P such that $A \vdash P$.

Definition

Theory T is **finitely axiomatizable** iff it is axiomatized by some finite set of sentences.

Many theories widely used in mathematics are finitely axiomatizable (e.g. NBG set theory). Others are not (e.g. Peano Arithemtic, ZFC set theory). Later we'll discuss a notion of **effective axiomatizability** that includes the latter.

Consistent and inconsistent theories

A theory T is a set of formulae, so it makes sense to ask whether it is consistent (proves every formula in its signature). Given Soundness and Competeness this comes to the same thing as being satisfiable (true in some structure).

Since every theory contains all the sentences it proves, and an inconsistent set proves every setnence, there is exactly one inconsistent theory in a signature Σ, namely the set of all sentences of Σ. It is maximal with respect to the inclusion relation on the set of theories in Σ.

There is also a minimal theory, namely the set of all valid sentences of Σ. All other Σ-theories extend this one.

Negation-complete theories

Definition

A **negation-complete theory** in Σ is a theory T such that for every sentence P of Σ, either $P \in T$ or $\neg P \in T$.

Note that this is a slightly different sense of "negation-complete" from the one we applied to sets of formulae. A theory can't be negation-complete in the old sense (containing every *formula* or its negation), since it can't contain open formulae.

Every consistent theory can be extended to a negation-complete, consistent theory. The only theory that extends a negation-complete, consistent theory is the inconsistent one.

A reminder of where we're headed

Gödel's first incompleteness theorem (first pass)

No theory is

1. consistent
2. negation-complete
3. finitely axiomatizable $\leftarrow$ we'll eventually strengthen to "effectively axiomatizable"
4. "sufficiently strong" $\leftarrow$ we still need to explain this

Theory of a structure

Definition

When S is a structure for a signature Σ, $\operatorname{Th} S$, the **theory of** S is the set of all sentences of Σ that are true in S.

Obviously the theory of any structure is consistent and negation-complete. Also, any consistent and negation-complete theory is the theory of some structure. (Since it's consistent it's true in some structure; since it's negation-complete, it can't leave out any sentence true in that structure.)

Definition

Structures S and S' are **elementarily equivalent** iff $\operatorname{Th} S = \operatorname{Th} S'$.

Isomorphism

Definition

When S and S' are structures for a signature Σ, an **embedding** from S to S' is an injection h from the domain of S to that of S' such that (i) for every n-ary predicate F of Σ, $I_F(x_1, \ldots, x_n)$ iff $I'_F(hx_1, \ldots, hx_n)$, and (ii) for every n-ary function symbol f of Σ, $I'_f(hx_1, \ldots, hx_n) = h(I_f(x_1, \ldots, x_n))$.

An **isomorphism** from S to S' is an embedding from S to S' that is surjective (and hence bijective).

S is **isomorphic** to S' iff there exists an isomorphism from S to S'.

An induction on terms establishes that if h is an embedding from S to S', $[\![t]\!]_{S'}^{h \circ g} = h([\![t]\!]_S^g)$ for every Σ-term t.

Isomorphic structures are elementarily equivalent

An induction on formulae establishes that if h is an isomorphism from S to S', $S, g \Vdash P$ iff $S', h \circ g \Vdash P$ for every Σ-formula P. Looking at the case where P is a sentence, it follows that isomorphic structures are always elementarily equivalent.

The converse of this claim is false.

An important example

Recall that Arith, the *signature of arithmetic* is the signature with individual constant $\mathtt{0}$, function symbols $\mathtt{suc}$, $\mathtt{+}$, $\mathtt{x}$, and predicate $\leq$.

The *standard model of arithmetic* is the structure for Arith with domain $\mathbb{N}$, where $I_{\mathtt{0}} = 0$, $I_{\mathtt{suc}} = suc$, $I_{\mathtt{+}} = +$, $I_{\mathtt{.}} = \times$, and $I_{\leq} = \leq$. We call this $\mathbb{N}_{0,\mathsf{suc},+,\times,\leq}$, or just $\mathbb{N}$ for short.

Definition

True arithmetic is $\mathrm{Th}\,\mathbb{N}$ is the theory of the standard model of arithmetic.

Another important example

The *signature of string theory* Str is the signature with $\approx 120,000$ individual constants, `""`, `"a"`, `"b"`, `"c"`, ..., `quo`, `com`, `lpa`, `rpa`, `sem`, `new`; one binary function symbol $\oplus$, and one binary predicate $\leq$.

The *standard string structure* $\mathbb{S}$ is the structure for Str where the domain is the set of all strings (over the Unicode alphabet), where $I_{\texttt{""}} = []$, $I_{\texttt{"}c\texttt{"}} = c$ for every length-one string c except `"`, `,`, `(`, `)`, `;`, and newline; $I_{\texttt{quo}} = \texttt{"}$, $I_{\texttt{com}} = \texttt{,}$, ... Also, $I_{\oplus}$ is $\oplus$ (the concatenation function), and $I_{\leq} = \leq$ (the no-longer-than relation).

Definition

True string theory is $\mathrm{Th}\,\mathbb{S}$, the theory of the standard string structure.

The standard numeral function

Definition

The *standard numeral* function $\langle\cdot\rangle$ is the function from $\mathbb{N}$ to Terms(Arith) such that:

$$\langle 0\rangle = \mathtt{0}$$

$$\langle \mathrm{suc}\, n\rangle = \mathtt{suc(}\langle n\rangle\mathtt{)}$$

Fact

For every number n, $[\![\langle n\rangle]\!]_{\mathbb{N}} = n$.

Proof: a trivial induction on n.

Corollary: the standard model of arithmetic is **explicit**, where

Definition

Structure S is **explicit** iff for every element d of the domain, there is a term t with no free variables such that $[\![t]\!]_S = d$.

The string labelling function

Definition

let $\mathsf{Const} : U \to \mathsf{Terms}(\mathsf{Str})$ be the function such that $\mathsf{Const}(c) = \texttt{"} : (c : \texttt{"})$ for every character other than `"`, `,`, `(`, `)`, `;`, and newline; $\mathsf{Const}(\texttt{"}) = \texttt{quo}$, $\mathsf{Const}(\texttt{,}) = \texttt{com}$, $\mathsf{Const}(\texttt{(}) = \texttt{lpa}$, $\mathsf{Const}(\texttt{)}) = \texttt{rpa}$, $\mathsf{Const}(\texttt{;}) = \texttt{sem}$, and $\mathsf{Const}(\text{newline}) = \texttt{new}$.

Definition

The *string label* function $\langle\cdot\rangle$ is the function from U^* to $\mathsf{Terms}(\mathsf{Str})$ such that

$$\langle[]\rangle = \texttt{""}$$

$$\langle a : s\rangle = \texttt{⊕(}\,\mathsf{Const}(a)\texttt{,}\langle s\rangle\texttt{)}$$

Fact

For every string s, $[\![\langle s\rangle]\!]_{\mathbb{S}} = s$.

Proof: a trivial induction on s.

Baby arithmetic

Definition

Baby arithmetic is the theory in Arith axiomatized by the following sentences

$$\{\neg(\langle n\rangle = \langle m\rangle) \mid n \neq m \in \mathbb{N}\}$$

$$\{\langle n\rangle + \langle m\rangle = \langle n+m\rangle \mid n, m \in \mathbb{N}\}$$

$$\{\langle n\rangle \times \langle m\rangle = \langle n\times m\rangle \mid n, m \in \mathbb{N}\}$$

$$\{\langle n\rangle \leq \langle m\rangle \mid n \leq m \in \mathbb{N}\}$$

$$\{\neg(\langle n\rangle \leq \langle m\rangle) \mid m > n \in \mathbb{N}\}$$

Fact

If S is a model of baby arithemtic, the function $n \mapsto [\![\langle n\rangle]\!]_S$ is an embedding from the standard model of arithmetic to S.

Such structures may however contain all sorts of **non-standard** elements, i.e. those that are not $[\![\langle n\rangle]\!]_S$ for any n.

Baby string theory

Definition

Baby string theory is the theory in Arith axiomatized by the following sentences

$$\{\neg(\langle s\rangle = \langle t\rangle) \mid s \neq t \in U^*\}$$

$$\{\langle s\rangle \oplus \langle t\rangle = \langle s \oplus t\rangle \mid s, t \in^*\}$$

$$\{\langle s\rangle \leq \langle t\rangle \mid s \leq t \in U^*\}$$

$$\{\neg(\langle s\rangle \leq \langle t\rangle) \mid t > s \in U^*\}$$

Fact

If S is a model of baby string theory, the function $s \mapsto [\![\langle s\rangle]\!]_S$ is an embedding from the standard string structure to S.

Strengthening the baby theories

There are many theories stronger than baby arithmetic/string theory that are true in the standard models of string theory. As we make the theories stronger we throw away more and more models by placing more constraints on the behaviour of their *non-standard* elements (if they have any).

Min, the minimal theory of strings

$S1$	$\forall x \forall y (\neg (c \oplus x = \texttt{""}))$	(for $c \neq \texttt{""}$)
$S2$	$\forall x \forall y (c \oplus x = c \oplus y) \to x = y$	
$S3$	$\forall x \neg (c_1 \oplus x = c_2 \oplus x)$	
$S4$	$\forall x (\texttt{""} \oplus x = x)$	
$S5$	$\forall x \forall y ((c \oplus x) \oplus y = c \oplus (x \oplus y))$	
$S6$	$c = c \oplus \texttt{""}$	
$S7$	$\forall x (\texttt{""} \leq x)$	
$S8$	$\forall x (x \leq \texttt{""} \leftrightarrow x = \texttt{""})$	
$S9$	$\forall x \forall y ((c_1 \oplus x \leq c_2 \oplus y) \leftrightarrow x \leq y)$	
$S10$	$\forall x \forall y (x \leq y \vee y \leq x)$	
$S11$	$\forall x (x = \texttt{""} \vee \exists y (x = c_1 \oplus y \vee \cdots \vee x = c_n \oplus y))$	

Min and Gödel's incompleteness theorem

This particular theory is going to play a starring role iin our initial version of Gödel's incompleteness theorem

Gödel's first incompleteness theorem (first pass)

No theory in Str is

1. consistent
2. negation-complete
3. finitely axiomatizable
4. "sufficiently strong" includes Min

If we make the theories strong enough, can we guarantee that their models won't have any non-standard elements at all?

No. It turns out that even *true arithmetic* and *true string theory* have non-standard models.

Non-standard models of arithmetic

Let Γ be:

$$\mathsf{Th}\,\mathbb{N} \cup \{\neg(x = \langle n \rangle) \mid n \in \mathbb{N}\}$$

Every finite subset of Γ is satisfiable. For any such finite subset there'll be a biggest n such that it contains $\neg(x = \langle n \rangle)$; so it'll be true on the assignment $[x \mapsto \langle \mathsf{suc}\, n \rangle]$.

So by the Compactness Theorem, Γ is satisfiable.

Any structure S in which Γ is true on some assignment must contain non-standard elements.

- In fact each such "non-standard model of arithmetic" has *many* non-standard elements. If gx is non-standard, so is $[\![\mathsf{suc}(x)]\!]^g$, since $\neg(\mathsf{suc}(x) = 0)$ and $\mathsf{suc}(x) = \mathsf{suc}(t) \to x = t$ must be true on every assignment (since they are logical consequences of true arithmetic).

Non-standard models of string theory

Similarly, we can use the Compactness Theorem to show that there are non-standard models of true string theory, looking now at the set

$$\mathrm{Th}\,\mathbb{S} \cup \{\neg(x = \langle s\rangle) \mid s \in U^*\}$$

How many countable non-standard models of true arithmetic are there?

Let P be the set of all prime numbers. For any $X \subseteq P$, let

$$T_X = \{\exists y(x = y \times \langle n \rangle) : n \in S\} \cup \{\neg\exists y(x = y \times \langle n \rangle) : n \in P \setminus S\}$$

For each S, every finite subset of $\mathrm{Th}\,\mathbb{N} \cup T_X$ is consistent, so by compactness and DLS $\mathrm{Th}\,\mathbb{N} \cup T_X$ has a countable model.

$T_X \cup T_{X'}$ is inconsistent unless $X = X'$. So for any countable model S of $\mathrm{Th}\,\mathbb{N}$, there are only countably many sets X of primes such that for some d in the domain, every member of T_X is true on an assignment where x is mapped to d.

If there were only countably many countable models of true arithmetic up to isomorphism, then only countably many sets X of prime numbers would be such that T_X has a countable model.

But there are uncountably many sets of prime numbers!

So: there are uncountably many non-isomorphic countable models of true arithmetic.