Set and Relations

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Sets

By a set we mean any collection M of determinate, distinct objects (called the elements of M) of our intuition or thought into a whole. (Cantor, 1895)

Talking about sets

We will not try to define 'A is a set' or 'x is an element of A'. We write '$x \in A$' for 'x is an element of A'; we also call elements 'members'.

- If we want to talk about a particular set, one thing we can do is to just list its members surrounded by curly brackets: e.g., $\{a, b, c\}$ means 'the set whose elements are a, b, c, and nothing else'.
- Alternatively, we can pick out the set by giving a condition such that its elements are all and only the things satisfying that condition, like so:

$$\{x \mid x \text{ is a student in this class who is a senior}\}$$

$$\{n \mid n \text{ is a prime number}\}$$

$$\{X \mid X \text{ is a set all of whose elements are prime numbers}\}$$

This is called *set builder notation*.

Subsets

Starting with the undefined notions *set* and *element*, we can define some important other things. Here's a crucial one:

Definition

'$A \subseteq B$' ('A is a subset of B') is defined to mean: A is a set, and B is a set, and every element of A is an element of B.

- Confusingly, 'x contains y' is sometimes used to mean '$y \in x$' and sometimes '$y \subseteq x$'. Look out for this!

A basic axiom about sets

Axiom of Extensionality

If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

In primitive terms: if A is a set, and B is a set, and every element of A is an element of B, and every element of B is an element of A, then $A = B$.

An inconsistent theory

To get anywhere, also need some axioms to us that there *exist* sets such as $\{n \mid n \text{ is a prime number}\}$ and so on. The use of the set-builder notation suggests the following:

Axiom Schema of Naïve Comprehension

There exists a set ($\{x \mid \phi(x)\}$) whose members are all and only those objects x such that $\phi(x)$.

This is an *axiom schema*: we get a particular *instance* of the schema by replacing '$\phi(x)$' with some formula like 'x is a prime number' or 'x is a student in this class' or 'x is either identical to a, identical to b, or identical to c', or what have you.

An inconsistent theory

Axiom Schema of Naïve Comprehension

There exists a set ($\{x \mid \phi(x)\}$) whose members are all and only those objects x such that $\phi(x)$.

Question: should we understand sets themselves count as 'objects' for the purposes of understanding this schema? It seems like we'd better, since we want to be able to talk about sets of sets as well as sets of non-sets.

But now things get weird. Suppose we plug in '$x = x$' for '$\phi(x)$'; then we get a set $\{x \mid x = x\}$ whose members are all and only those objects x such that $x = x$, i.e. all objects whatsoever *including that very set*! So it's an element of itself! Seems kind of bizarre.

An inconsistent theory

Actually Naïve Comprehension is not just bizarre, it's inconsistent. For consider what happens when we plug in '$x \notin x$' (i.e. 'it is not the case that $x \in x$') for $\phi(x)$. This gives a set $\{x \mid x \notin x\}$ whose elements are all and only those objects x (including sets!) such that $x \notin x$.

Now we ask: is it the case that $\{x \mid x \notin x\} \in \{x \mid x \notin x\}$?

- If $\{x \mid x \notin x\} \in \{x \mid x \notin x\}$, then $\{x \mid x \notin x\} \notin \{x \mid x \notin x\}$. (Because it contains *only* objects that aren't elements of themselves).
- If $\{x \mid x \notin x\} \notin \{x \mid x \notin x\}$, then $\{x \mid x \notin x\} \in \{x \mid x \notin x\}$. (Because it contains *all* objects that aren't elements of themselves.)

Like any pair of sentences of the forms 'If P then not-P' and 'If not-P then P', this implies a contradiction 'P and not-P'.

An inconsistent theory

1.	$\mid P$	Assumption
2.	$\mid P \to \neg P$	Premise
3.	$\mid \neg P$	1, 2, $\to$Elim
4.	$\neg P$	1–3, $\neg$Intro
5.	$\neg P \to P$	Premise
6.	P	4, 5, $\to$Elim
7.	$P \wedge \neg P$	4, 6, $\wedge$Intro

Avoiding the inconsistency

In practice, we'll always be considering sets of objects within some fixed "domain"—sets of numbers, sets of strings, sets of sets of numbers. . . .

Axiom Schema of Separation

For every set A, there is a set $\{x \in A \mid \phi(x)\}$ whose elements are all and only those objects x such that $x \in A$ and $\phi(x)$.

This doesn't tell us that there *exists* e.g. a set $\mathbb{N}$ of all natural numbers. But once we have $\mathbb{N}$, we can use Separation to secure the existence of $\{x \in \mathbb{N} \mid x \text{ is prime}\}$, and so on.

More notation

Definition

$A \cap B := \{x \mid x \in A \text{ and } x \in B\}$

Definition

$A \cup B := \{x \mid x \in A \text{ or } x \in B\}$

Definition

$A \backslash B := \{x \mid x \in A \text{ and } x \notin B\}$

Definition

$\varnothing := \{x \mid x \neq x\}$

Ordered pairs

Sets correspond to conditions specified by a sentence with *one* variable: ‘x is prime’; ‘x is prime and x is odd’; ‘x is in this class and x is a senior’,. . . .

But for many purposes it’ll be important to reason in a similar way about conditions specified by sentences with *two* variables: ‘x is taller than y’; ‘every factor of x is a factor of y’; ‘x is prime and y is a set of natural numbers and $x \in y$’;. . .

In set theory the standard way to do this is in terms of *sets of ordered pairs*, a.k.a. *relations*.

Ordered pairs

New primitive notation: $\langle x, y \rangle$, meaning 'the ordered pair whose first co-ordinate is x and whose second co-ordinate is y'.

It's governed by the following axiom:

Axiom of Pair Uniqueness

If $\langle x, y \rangle = \langle x', y' \rangle$, then $x = x'$ and $y = y'$.

Products and relations

Axiom of Product Existence

For any sets A and B, there is a set $A \times B$ whose members are all and only those ordered pairs $\langle x, y \rangle$ such that $x \in A$ and $y \in B$.

Definition

R is a *relation from* A *to* $B := R \subseteq A \times B$.

When we are dealing with a relation R from A to B, we'll often write Rxy to abbreviate $\langle x, y \rangle \in R$.

Product Existence and Separation jointly imply that for any sets A and B condition $\phi(x, y)$, there is a relation R from A to B such that Rxy holds exactly when $\phi(x, y)$: namely, $\{\langle x, y \rangle \in A \times B \mid \phi(x, y)\}$.

Notation for relations

Definition

Suppose R is a relation from A to B. Then R^{-1}—the *converse* of R—is the relation $\{\langle y, x\rangle \in B \times A \mid \langle x, y\rangle \in R\}$ from B to A.

Definition

Suppose R is a relation from A to B and S is a relation from B to C. Then $S \circ R$—the *composition* of R and S—is the relation $\{\langle x, z\rangle \in A \times C \mid \text{there exists } y \in B \text{ such that } \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S\}$, from A to C.

Definition

For any set A, id_A is the relation $\{\langle x, y\rangle \in A \times A \mid x = y\}$ from A to A.

A few noteworthy facts

When R is a relation from A to B and S is a relation from B to C:

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

$$\mathrm{id}_B \circ R = R = R \circ \mathrm{id}_A$$

Four properties of relations

When R is a relation from A to B:

Definition

R is *serial* iff for every $x \in A$, there is some $y \in B$ such that Rxy.

Definition

R is *surjective* iff for every $y \in B$, there is some $x \in A$ such that Rxy.

Definition

R is *functional* iff whenever Rxy and Rxy', $y = y'$.

Definition

R is *injective* iff whenever Rxy and $Rx'y$, $x = x'$.

How to write informal proofs: an example

Theorem. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. By the Axiom of Extensionality, it suffices to show that (a) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and (b) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

(a) Suppose $x \in A \cap (B \cup C)$. Then $x \in A$, and either $x \in B$ or $x \in C$. In the first case, $x \in A \cap B$; in the second case, $x \in A \cap C$, so either way, $x \in (A \cap B) \cup (A \cap C)$.

(b) Suppose $x \in (A \cap B) \cup (A \cap C)$. Then there are two cases: either $x \in A \cap B$, or $x \in A \cap C$. In the first case, $x \in A$, and also $x \in B \cup C$ since $x \in B$, hence $x \in A \cap (B \cup C)$. In the second case, $x \in A$, and also $x \in B \cup C$ since $x \in C$, hence $x \in A \cap (B \cup C)$. So either way, we have $x \in A \cap (B \cup C)$.