Advanced Logic: Introduction

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What is metalogic?

In formal logic, we consider *formal languages* (artificial languages, generally designed to be much simpler than any natural language) and *formal proof systems* which provide an exact specification of what counts as a correct proof (in a given formal language).

One thing you can do with a formal language and proof system is *use* the language to say things, and *use* the proof system to back up what you're saying with reasoning.

In an introductory logic class, the focus is on using certain formal languages (those of propositional and first-order logic) and formal proof systems (for classical propositional and first order logic).

This class is a *metalogic* class, which means it's concerned with reasoning *about* (as opposed to *with*) formal languages and proof systems.

Sample questions of metalogic

Can this sentence be proved in this system of logic, or not?

Is this formal system *inconsistent* in the sense that it can prove every sentence (in the relevant formal language)?

Is this formal system *negation-complete* in the sense that it can prove the negation of a sentence whenever it can't prove that sentence?

Is there a way of "simulating" or "interpreting" system A within system B?

Is this formal system *decidable*, in the sense that you could we in principle program a computer to tell you whether an arbitrary sentence is a theorem of the system?

These are questions of *mathematics* (as well as questions of logic). We'll answer them by giving ordinary mathematical proofs, reasoning about things like formal sentences and formal sentences as mathematical objects analogous to numbers, sets, etc.

The proofs we'll be giving will be written according to standard mathematical conventions, in English supplemented with various defined expressions and special symbols. Just like proofs in any other branch of mathematics, it would be *possible* to fully formalize these proofs as well. But we won't do that, because (a) it's a ton of work, and (b) we would have to be extra careful not to get confused between the formal sentences and proofs we are *talking about* with the formal sentences and proofs we are *using*.

This means that one important skill you'll have to pick up as part of the course is the skill of writing good *informal* mathematical proofs.

Why metalogic is philosophically interesting

If you're planning on using a formal system, reasoning about that system is going to be useful. For example, if you can show that a certain sentence can't be proved, you can give up on trying to prove it.

But on top of this, it so happens that the central body of results in metalogic (done between roughly 1870 and 1930) includes some philosophically deep results about the limitations of all possible formal systems, and by extension (arguably) about the limitations inherent in our nature as finite beings.

Where we are going

Gödel's first incompleteness theorem (first pass)

Any consistent theory that is reasonably simple and sufficiently strong leaves some questions in its own language unanswered.

Gödel's first incompleteness theorem (second pass)

Any consistent theory in first-order classical logic that is effectively axiomatizable and interprets minimal arithmetic is negation-incomplete.

Consequences for philosophy of mathematics

Since Euclid, mathematicians have started with *axioms* and proved theorems based on them.

But sometimes, one may have reason to think—one may even *prove*—that there is no way of proving a positive or negative answer to some question based on the existing stock of axioms. Sometimes, this has spurred mathematicians to formulate *new* axioms.

Could this process be completed? Could we find some finite list of axioms which are (in principle) sufficient to prove the answer to every mathematical question?

The first incompleteness theorem says no. If we have a finite list of axioms, the set of all theorems provable from those axioms will be "reasonably simple". And the axioms we already have are already "sufficiently strong". So any extended set of axioms will either be inconsistent, or still leave some questions unsettled.

Consequences for philosophy more generally

This isn't specific to mathematics! On any topic where one can formulate rigorous theories at all, one may be tempted to think one has formulated a consistent theory that (correctly) settles every question about the topic.

But in a wide range of cases, the first incompleteness theorem implies that one would be wrong to think this. For if one has managed to formulate the theory, it is "reasonably simple". And it turns out that even rather weak theories are often "sufficiently strong"—there are all sorts of ways in which minimal arithmetic can be simulated within, e.g., physics.

[I]t must be brought about that every fallacy becomes nothing other than a calculating error, and every sophism expressed in this new type of notation becomes in fact nothing other than a grammatical or linguistic error, easily proved to be such by the very laws of this philosophical grammar. Once this has been achieved, when controversies arise, there will be no more need for a disputation between two philosophers than there would be between two accountants [computistas]. It would be enough for them to pick up their pens and sit at their abacuses, and say to each other (perhaps having summoned a mutual friend): 'Let us calculate.'

Gottfried Wilhelm Leibniz [GP.VII]

Consequences for philosophy more generally

A further destination

The proof of the first incompleteness theorem actually provides a recipe for taking a reasonably simple, moderately strong theory T and finding a sentence G_T such that T doesn't prove G_T or its negation unless it's inconsistent. But the relevant G_T does not look very interesting apart from the fact that it witnesses the fact that T isn't negation-complete.

Gödel proved a *second* incompleteness theorem which (for theories that satisfy a slightly higher standard of strength) provides a much more interesting witness of negation-incompleteness.

Gödel's Second Incompleteness Theorem

Any consistent theory T that is reasonably simple and moderately strong does not prove its own consistency.

Gödel's Second Incompleteness Theorem

Any consistent theory in first-order classical logic T that is effectively axiomatizable and interprets Peano arithmetic does not prove the sentence in its language that formalizes 'There is no proof of a contradiction from T'.

Why is this interesting?

A natural thought: you shouldn't follow some method of reasoning unless you can show it won't lead you into contradiction.

- Distinguish this from the stronger thought that you shouldn't follow some method unless you are show *on independent grounds* that it won't lead you into a contradiction. This leads to a regress worry: what method are you supposed to use to acquire your *first* belief of the form 'Such-and-such rules won't lead me into a contradiction'? By contrast, the weaker thought allows me to use my method in showing that it won't lead to a contradiction.
- The second incompleteness theorem suggests that even the weaker thought is bad. Consider the totality of all methods available to me. Since I'm a finite being, it seems to be "reasonably simple" in the relevant sense, as well as "moderately strong". So it can't be used to show that it won't lead me into contradiction!