Provability

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Defining provability

$\vdash$ is defined to be the smallest relation between sets of formulae and formulae meeting the following conditions.

$$\frac{P \in \mathcal{L}(\Sigma)}{P \vdash P}\ \text{Assumption} \qquad \frac{\Gamma \vdash P \qquad \Delta \subseteq \mathcal{L}(\Sigma)}{\Gamma, \Delta \vdash P}\ \text{Weakening}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ {\to}\text{Intro} \qquad \frac{\Gamma \vdash P \to Q \qquad \Gamma \vdash P}{\Gamma \vdash Q}\ {\to}\text{Elim}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ {\wedge}\text{Intro} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P}\ {\wedge}\text{Elim1} \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}\ {\wedge}\text{Elim2}$$

$$\frac{\Gamma \vdash P \qquad Q \in \mathcal{L}(\Sigma)}{\Gamma \vdash P \vee Q}\ {\vee}\text{Intro1} \qquad \frac{\Gamma \vdash Q \qquad P \in \mathcal{L}(\Sigma)}{\Gamma \vdash P \vee Q}\ {\vee}\text{Intro2}$$

$$\frac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma \vdash R}\ {\vee}\text{Elim}$$

Defining provability (contd.)

$$\frac{\Gamma, P \vdash Q \qquad \Gamma, P \vdash \neg Q}{\Gamma \vdash \neg P}\ \neg\text{Intro} \qquad \frac{\Gamma \vdash \neg\neg P}{\Gamma \vdash P}\ \text{DNE}$$

$$\frac{\Gamma \vdash P[u/v] \qquad u \notin FV(\Gamma, \forall v P)}{\Gamma \vdash \forall v P}\ \forall\text{Intro} \qquad \frac{\Gamma \vdash \forall v P \qquad t \in \mathrm{Terms}(\Sigma)}{\Gamma \vdash P[t/v]}\ \forall\text{Elim}$$

$$\frac{\Gamma \vdash P[t/v]}{\Gamma \vdash \exists v P}\ \exists\text{Intro} \qquad \frac{\Gamma \vdash \exists v P \qquad \Gamma, P[u/v] \vdash Q \qquad u \notin FV(\Gamma, Q, \exists v P)}{\Gamma \vdash Q}\ \exists\text{Elim}$$

$$\frac{t \in \mathrm{Terms}(\Sigma)}{\vdash t = t}\ {=}\text{Intro} \qquad \frac{\Gamma \vdash s = t \qquad \Gamma \vdash P[s/v]}{\Gamma \vdash P[t/v]}\ {=}\text{Elim}$$

A proof by induction: provability is compact

Compactness of provability

$\Gamma \vdash P$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash P$.

The right-to-left direction is just a matter of applying Weakening.

The left-to-right direction needs an induction. Let's say that a sequent $\Gamma \triangleright P$ is *compactable* iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash P$; we are trying to show that every provable sequent is compactable.

Assumption: if $\Gamma \triangleright P$ is an instance of Assumption, $\Gamma = \{P\}$ which is finite.

Weakening: if $\Gamma \triangleright P$ follows by Weakening from some provable compactable sequent, that sequent must be $\Delta \triangleright P$ for some $\Delta \subseteq \Gamma$. By the IH, there's a finite $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash P$; since $\Delta_0 \subseteq \Gamma$, this means $\Gamma \triangleright P$ is also compactable.

$\rightarrow$**Intro:** suppose $\Gamma \triangleright P$ follows by $\rightarrow$Intro from some provable compactable sequent. Then there must be some Q and R such that $P = Q \rightarrow R$ and that sequent is $\Gamma, Q \triangleright R$. By the IH, there's a finite subset Δ of $\Gamma \cup \{Q\}$ such that $\Delta \triangleright R$. Let $\Gamma_0 = \Delta \setminus \{Q\}$; note that Γ_0 is finite since Δ is. Then $\Gamma_0 \subseteq \Gamma$ and $\Delta \subseteq \Gamma_0 \cup \{Q\}$, so by Weakening $\Gamma_0, Q \vdash R$, so by $\rightarrow$Intro, $\Gamma_0 \vdash Q \rightarrow R$.

$\rightarrow$**Elim:** suppose $\Gamma \triangleright P$ follows by $\rightarrow$Elim from two provable compactable sequents. Then there must be some Q such that one of those sequents is $\Gamma \triangleright Q \rightarrow P$ and the other is $\Gamma \triangleright Q$. By the IH, there are finite subsets Γ_1, Γ_2 of Γ such that $\Gamma_1 \vdash Q \rightarrow P$ and $\Gamma_2 \vdash Q$. Let $\Gamma_0 = \Gamma_1 \cup \Gamma_2$. By Weakening, $\Gamma_0 \vdash Q \rightarrow P$ and $\Gamma_0 \vdash Q$, so by $\rightarrow$Elim, $\Gamma_0 \vdash P$. Γ_0 is finite since it's the union of two finite sets.

Other rules similar.

Finite-sequent provability

We can shed more light on the last result by introducing a new relation $\vdash_{fin}$ between *finite* sets of formulae and formulae (a subset of $\mathcal{P}_{fin}(\mathcal{L}(\Sigma)) \times \mathcal{L}(\Sigma)$). It is defined just like $\vdash$, but the Weakening rule is changed to:

$$\frac{\Gamma \vdash_{fin} P \qquad Q \in \mathcal{L}(\Sigma)}{\Gamma, Q \vdash_{fin} P} \text{ One-formula Weakening}$$

Theorem

$\Gamma \vdash P$ iff for some $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash_{fin} P$

For the right-to-left direction, we first show by a trivial induction that $\Gamma_0 \vdash P$ whenever $\Gamma_0 \vdash_{fin} P$, and then appeal to Weakening to get that when $\Gamma_0 \vdash P$ and $\Gamma_0 \subseteq \Gamma$, $\Gamma \vdash P$.

For the left-to right direction, we first prove that $\vdash_{fin}$ is closed under the version of Weakening restricted to *finite* Δ (by induction on the size of Δ). Then the proof proceeds just like the one on the previous slide.

Provability and proofs

What about proofs?

When you're taught to *use* a formal system of logic, you're taught rules for writing down things called *proofs*. So far, we haven't even mentioned them!

But there's a sense in which a certain very abstract notion of “proof” is in play whenever one defines a set (or relation) as a closure.

Closures and derivations

Supose we have a family (R_i) of relations on a set A (which may be of different arities), such that C is the smallest subset of A closed under all the R_i. (Note that some of the R_i may be 1-ary, i.e. subsets of A, so C need not be $\varnothing$!)

Let a *derivation history* for (R_i) be a list $s \in A^*$ such that for each element y of s, there is some R_i (of arity n) and some $x_1, \ldots, x_{n-1}$ occurring earlier than y in s such that $R_i x_1 \ldots x_{n-1} y$. More carefully:

Definition

The set of derivation histories for (R_i) is the smallest set which contains $[]$ and is such that if it contains s, and $R_i x_1 \ldots x_{n-1} y$ for some $x_1 \ldots x_{n-1} \in \mathrm{elements}\, s$ (where n is the arity of R) then it contains $(y : s)$.

It is easy to show that s and t are derivation histories, $s \oplus t$ is. (Use induction on s.)

The existence of derivations

A *derivation of y* is a derivation history whose last element is y, i.e. which is $(y : s)$ for some $s \in A^*$.

Then we can show that for any $y \in A$, $y \in C$ iff there is a derivation of y.

Proof, left-to-right: When $y \in R_i$ for a singulary R_i, $[y]$ is a derivation of y. Otherwise, suppose $R_i x_1 \ldots x_{n-1} y$ for an n-ary R, where $x_1 \ldots x_{n-1} \in C$, and there is a derivation s_i of each x_i. Then $t := [y] \oplus s_1 \oplus \cdots \oplus s_{n-1}$ is a derivation of y.

Right-to-left: we show by induction that for every s, if s is a derivation history, then $\mathrm{elements}\, s \subseteq C$. Base case trivial since $\mathrm{elements}[] = \varnothing \subseteq C$. Induction step: suppose s is such that if it's a derivation history, then $\mathrm{elements}\, s \subseteq C$. Suppose $(y : s)$ is a derivation history. Then s is a derivation history and there is an n-ary R_i such that for some $x_1 \ldots x_{n-1} \in \mathrm{elements}\, s$, $R_i x_1 \ldots x_{n-1} y$. But by the induction hypothesis, $\mathrm{elements}\, s \subseteq C$. Since C is closed under R_i, it follows that $y \in C$.

The use of derivation histories

A common situation: we are interested in some finite number of (R_i) such that for each one, it is a mechanical matter to check whether $R_i x_1 \ldots x_n$ (when $x_1 \ldots x_n$ are "given" to us in some appropriate way, e.g. they are strings we have written down).

Figuring out whether a given y belongs to the smallest set closed under (R_i) may still be very hard!

But if we are presented (in the same canonical way) with a list of elements of A, it will be a mechanical matter to check whether it is a derivation of y. We just go through it step by step and check, for each step, whether it bears each R_i to an appropriate tuple of previous steps.

- It's easier if we have an ``annotated'' derivation history where for each element of the list we are told which R_i applies and where in the earlier list we are to find the relevant $x_1, \ldots, x_{n-1}$; but even without this information, there are only finitely many possibilities to search through.

Proofs

A derivation history for $\vdash$ is a list of sequents, where each one is either an instance of Assumption, an instance of $=$Intro, follows from some earlier sequent by Weakening, $\to$Intro, ...; follows from two earlier sequents by $\to$Elim, $\wedge$Intro,...; or follows from three earlier sequents by $\vee$Elim.

Such derivation histories may involve sequents $\Gamma \triangleright P$ where Γ is infinite. There is no clear sense in which one could *write down* such a derivation history.

By contrast, derivation histories for $\vdash_{fin}$ are lists of *finite* objects. These are more like what we'd expect a `proof' to be.

Proofs as strings

If we're treating formulas as strings, it's natural to think that proofs should be strings too. Derivation histories in $\vdash_{fin}$ aren't strings; they are lists of ordered pairs of a finite set of formulae and a formula.

But we can represent any such list unambiguously as a string.

1. First, convert each finite set of formulae into a string by listing them in alphabetical order and joining them with a character that never appears in formulae: `;`, say.
2. Second, convert each ordered pair of such a string and a formula into a single string by joining the two with another character that never appears in formulae: `>`, say.
3. Finally, convert the resulting list of strings into a single string by joining them all with some third character that never appears in formulae: newline, say.

Tree-style proofs

In constructing a proof in this sense of some given sequent, one typically ends up making a lot of arbitrary choices about how to order the lines. *Proof theorists* are interested in properties of proofs that don't depend on these arbitrary line-numbering choices: for these purposes, it's more useful to think of formal proofs as *trees* of formulae rather than lists of formulae. E.g. the following (cf. our earlier discussion of Explosion) would be an unambiguous visual representation of a unique proof:

$$\dfrac{\dfrac{\dfrac{\Gamma \triangleright P}{\Gamma, \neg Q \triangleright P}\text{ Weakening} \qquad \dfrac{\Gamma \triangleright \neg P}{\Gamma, \neg Q \triangleright \neg P}\text{ Weakening}}{\Gamma \triangleright \neg\neg Q}\ \neg\text{Intro}}{\Gamma \triangleright Q}\text{ DNE}$$

The theory of trees can be developed along similar lines to the theory of lists; but we haven't done this, so we'll stick with our less elegant linear conception of proof.

Fitch-style proof formatting

In your introductory Logic class, you probably learnt a way of writing down proofs that look something like this:

1 -	A	Assumption
2 -	B	Assumption
3 -	A	Reiteration, 1
4 -	$B \to A$	$\to$-Intro 2--3
5 -	$A \to (B \to A)$	$\to$-Intro 1--4

Any such proof can be understood as a proof in our sense (in a system with some more rules, which are derived rules for $\vdash$) where the vertical lines on the left correspond to the formulae on the left hand side of $\triangleright$.

The Fitch-style proof on the previous line corresponds in this sense to the following list of sequents:

$$\begin{aligned} A &\rhd A \\ A, B &\rhd B \\ A, B &\rhd A \\ A &\rhd B \to A \\ &\rhd A \to (B \to A) \end{aligned}$$

This is in fact a proof in our sense.

Fitch proofs enforce certain choices about the ordering of lines, so not every proof in our sense corresponds to a Fitch proof. It turns out that every provable sequent *does* have a Fitch proof, but we won't show this since it would require a rigorous definition of ``Fitch proof'' (which is fiddly!).

Sometimes, the lists of sequents corresponding to valid Fitch-style proofs are not valid proofs in our sense, e.g.:

1	A	Assumption	
2	$\quad B$	Assumption	
3	$\quad A \wedge B$	1,2 $\wedge$I	

$$\rightsquigarrow \quad \begin{array}{l} A \rhd A \\ A, B \rhd B \\ A, B \rhd A \wedge B \end{array}$$

This isn't a proof in our sense, but would become one if we added

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge\mathsf{Intro}_w$$

This can easily be seen to be a *derived* rule for $\vdash$:

$$\frac{\dfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A} \qquad \dfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\ \mathsf{Weakening}}{\Gamma, \Delta \vdash A \wedge B}\ \wedge\mathsf{Intro}$$

Our other two-premise and three-premise rules have 'mixed antecedent' generalizations that are derivable in the same way.

$$\frac{\Gamma \vdash P \to Q \qquad \Delta \vdash P}{\Gamma, \Delta \vdash Q} \to\text{Elim}_w \qquad\qquad \frac{\Gamma, P \vdash Q \qquad \Delta, P \vdash \neg Q}{\Gamma, \Delta \vdash \neg P} \neg\text{Intro}_w$$

$$\frac{\Gamma \vdash P \vee Q \qquad \Delta, P \vdash R \qquad \Delta', Q \vdash R}{\Gamma, \Delta, \Delta' \vdash R} \vee\text{Elim}_w$$

$$\frac{\Gamma \vdash \exists v P \qquad \Delta, P[u/v] \vdash Q \qquad u \notin FV(\Gamma, \Delta, Q, \exists v P)}{\Gamma, \Delta \vdash Q} \exists\text{Elim}_w$$

$$\frac{\Gamma \vdash s = t \qquad \Delta \vdash P[s/v]}{\Gamma, \Delta \vdash P[t/v]} =\text{Elim}_w$$

We would have got the very same relation $\vdash$ if we used these clauses rather than the original ones. We'd get a different notion of 'proof', although there's a straightforward recipe for turning proofs-in-the-new-sense into proofs-in-the-old-sense.

Derived rules

There are many lots of other useful derived rules. For example:

$$\dfrac{\Gamma \vdash P \lor Q \qquad \Gamma \vdash \neg P}{\Gamma \vdash Q}\ \text{Disjunctive Syllogism}$$

This is also derivable for $\vdash$:

$$\dfrac{\dfrac{\Gamma \vdash P \lor Q \qquad \dfrac{\Gamma \vdash \neg P \qquad \dfrac{}{P \vdash P}\,\mathsf{A}}{\Gamma, P \vdash \neg\neg Q}\,\neg\mathsf{I}_w \qquad \dfrac{\dfrac{\dfrac{}{Q \vdash Q}\,\mathsf{A}}{\Gamma, Q \vdash Q}\,\mathsf{W} \qquad \dfrac{}{\neg Q \vdash \neg Q}\,\mathsf{A}}{\Gamma, Q \vdash \neg\neg Q}\,\neg\mathsf{I}_w}{\Gamma \vdash \neg\neg Q}\,\lor\mathsf{E}}{\Gamma \vdash Q}\,\mathsf{DNE}$$

There are some useful derived rules where it's not so clear how one would add them to a Fitch-style proof system. Particularly interesting are so-called *structural* rules, which don't involve any particular connective, such as

$$\frac{\Gamma, P \vdash Q \qquad \Delta \vdash P}{\Gamma, \Delta \vdash Q}\,\text{Cut}$$

This is straightforward to derive using the $\to$ rules:

$$\frac{\dfrac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\,{\to}\text{Intro} \qquad \Delta \vdash P}{\Gamma, \Delta \vdash Q}\,{\to}\text{Elim}_w$$

An important special case of Cut is Transitivity: if $P \vdash Q$ and $R \vdash P$, then $R \vdash Q$. (Take $\Gamma = \varnothing$ and $\Delta = \{R\}$.)

A useful generalization of Cut is "Transitivity$^+$": if $\Gamma \vdash P$ for all $P \in \Delta$ and $\Gamma, \Delta \vdash Q$, then $\Gamma \vdash Q$.

Two more examples:

$$\frac{\Gamma, P \vdash Q}{\Gamma, \neg Q \vdash \neg P}\ \text{Contraposition} \qquad \textit{Proof:}\ \frac{\Gamma, P \vdash Q \qquad \dfrac{\dfrac{}{\neg Q \vdash \neg Q}\ \text{Assumption}}{P, \neg Q \vdash \neg Q}\ \text{Weakening}}{\Gamma, \neg Q \vdash \neg P}\ \neg\text{Intro}_w$$

$$\frac{P \in \Gamma}{\Gamma \vdash P}\ \text{Assumption}_w \qquad \textit{Proof:}\ \frac{\dfrac{}{P \vdash P}\ \text{Assumption}}{\Gamma \vdash P}\ \text{Weakening}$$

Provable Equivalence

Provable equivalence

Definition

$P \dashv\vdash Q$ (P and Q are provably equivalent) iff $P \vdash Q$ and $Q \vdash P$.

Fact

$P \dashv\vdash Q$ iff for all Γ, $\Gamma \vdash P$ iff $\Gamma \vdash Q$.

Proof, right to left: by Assumption, $P \vdash P$ and $Q \vdash Q$.

Left-to-right: suppose $\Gamma \vdash P$. Since $P \vdash Q$, $\Gamma, P \vdash Q$ by Weakening, so $\Gamma \vdash P \to Q$ by $\to$Intro; hence $\Gamma \vdash Q$ by $\to$Elim.

Provable equivalence

Provable equivalence is a congruence

Suppose $P \dashv\vdash Q$ and $P' \dashv\vdash Q'$. Then $\neg P \dashv\vdash \neg Q$, $P \wedge P' \dashv\vdash Q \wedge Q'$, $P \vee P' \dashv\vdash Q \vee Q'$, $P \to P' \dashv\vdash Q \to Q'$, and for any variable v, $\forall v P \dashv\vdash \forall v Q$ and $\exists v P \dashv\vdash \exists v Q$.

Proof for $\wedge$: by $\wedge$Elim, $P \wedge P' \vdash P$ and $P \wedge P' \vdash P'$. But then by the hypothesis, $P \wedge P' \vdash Q$ and $P \wedge P' \vdash Q'$. So by $\wedge$Intro, $P \wedge P' \vdash Q \wedge Q'$.

Proof for $\forall$: $\forall v P \vdash P$ by $\forall$Elim (since $P = P[v/v]$. So by the hypothesis, $\forall v P \vdash Q$. Since v isn't free in $\forall v P$ or $\forall v Q$, we can apply $\forall$Intro to conclude that $\forall v P \vdash \forall v Q$.

Other cases similar.

Simplifying the language

Considered the function from $\mathcal{L}(\Sigma)$ to $\mathcal{L}_{\neg,\wedge,\forall,=}(\Sigma)$ given by

$$\begin{aligned} fP &= P \text{ when } P \text{ is atomic} \\ f(\neg P) &= \neg fP \\ f(P \wedge Q) &= fP \wedge fQ \\ f(P \vee Q) &= \neg(\neg fP \wedge \neg fQ) \\ f(P \to Q) &= \neg(fP \wedge \neg fQ) \\ f(\forall v P) &= \forall v fP \\ f(\exists v P) &= \neg \forall v \neg fP \end{aligned}$$

One thing we can prove about this is that for all P, $P \dashv\vdash fP$. This will require an induction on the construction of the formula P, but first we'll need some initial lemmas.

Lemmas

Facts

For any formulae P and Q and variable v:

$$P \vee Q \dashv\vdash \neg(\neg P \wedge \neg Q)$$

$$P \to Q \dashv\vdash \neg(P \wedge \neg Q)$$

$$\exists v P \dashv\vdash \neg \forall v \neg P$$

These are pretty straightforward exercises in proof construction similar to what you're used to from Logic. I'll do the first pair on the next slide.

1 -	$P \vee Q$	As.
2 -	$(\neg P \wedge \neg Q)$	As.
3 -	$\neg P$	2, $\wedge$E
4 -	Q	1, 3, D.S.
5 -	$\neg Q$	2, $\wedge$E
6 -	$\neg(\neg P \wedge \neg Q)$	$\neg$I, 2--5

1 -	$\neg(\neg P \wedge \neg Q)$	As.
2 -	$\neg(P \vee Q)$	As.
3 -	P	As.
4 -	$P \vee Q$	$\vee$I
5 -	$\neg P$	$\neg$I, 2-3
6 -	Q	As.
7 -	$P \vee Q$	$\vee$I
8 -	$\neg Q$	$\neg$I, 2-3
9 -	$\neg P \wedge \neg Q$	$\wedge$I, 5, 8
10 -	$\neg\neg(P \vee Q)$	$\neg$I,

Now we are ready to prove that $P \dashv\vdash fP$ for all P, by induction on the construction of P.

Base case: if P is atomic, $fP = P$, so we have $P \dashv\vdash fP$ by Assumption.

Induction step for $\neg$: suppose $P = \neg Q$ where $Q \dashv\vdash fQ$. Then $fP = \neg fQ$, so we have $P \dashv\vdash fP$ by the fact that $\dashv\vdash$ is a congruence.

Induction steps for $\wedge$ and $\forall$: similar.

Induction step for $\vee$: suppose $P = Q \vee R$, where $Q \dashv\vdash fQ$ and $R \dashv\vdash fR$. Then $fP = \neg(\neg fQ \wedge \neg fR)$, so by the lemma, $fP \dashv\vdash fQ \vee fR$. But $P \dashv\vdash fQ \vee fR$ by the induction hypothesis and the fact that $\dashv\vdash$ is a congruence.

Induction step for $\to$: similar, but using the $\to$ part of the lemma.

Induction step for $\exists$: similar, but using the $\exists$ part.

We can also prove something stronger. Recall that $\vdash_{\neg,\wedge,\forall,=}$ is the relation between sets of formulae and formulae of $\mathcal{L}_{\neg,\wedge,\forall}(\Sigma)$ defined just like $\vdash$ but without the clauses for $\rightarrow$Intro, $\rightarrow$Elim, $\vee$Intro1, $\vee$Intro2, $\vee$Elim, $\exists$Intro, or Elim. Call it $\vdash'$ for short. Then we can show the following:

Fact

$\Gamma \vdash P$ iff $f[\Gamma] \vdash' fP$.

Right-to-left: Suppose $f[\Gamma] \vdash' fP$. Then $f[\Gamma] \vdash fP$ (since $\vdash' \subseteq \vdash$). But by the previous result, every member of $f[\Gamma]$ is provably equivalent to some member of Γ, and hence proved by Γ, so by Transitivity$^+$, $\Gamma \vdash fP$; then by Cut, $\Gamma \vdash P$.

The left-to-right direction requires an induction on the derivation of $\Gamma \triangleright P$ (see next slide).

Assumption: $fP \triangleright fP$ is an instance of Assumption, so $fP \vdash' fP$.

Weakening: if $f[\Gamma] \vdash_{\neg,\wedge,\forall,=} fP$ then $f[\Gamma], f[\Delta] \vdash\vdash' fP$ by Weakening for $\vdash'$.

$\vee$**Intro1:** Suppose $f[\Gamma] \vdash' fP$. We have that $\neg fP \wedge \neg fQ \vdash' \neg fP$ (by Assumption and $\wedge$Elim1$_w$ for $\vdash'$). So by $\neg$Intro$_w$ for $\vdash'$, $f[\Gamma] \vdash \neg(\neg fP \wedge fQ)$, which is $f(P \vee Q)$.

$\vee$**Intro2:** Similar.

$\vee$**Elim:** Suppose $f[\Gamma] \vdash' f(P \vee Q)$, i.e. $\neg(\neg fP \wedge \neg fQ)$, and moreover $f[\Gamma], fP \vdash' fR$ and $f[\Gamma], fQ \vdash' fR$. By Contraposition for $\vdash'$ (established in the same way as for $\vdash$) we have $f[\Gamma], \neg fR \vdash' \neg fP$ and $f[\Gamma], \neg fR \vdash' \neg fQ$, so by $\wedge$Intro $f[\Gamma], \neg fR \vdash' \neg fP \wedge \neg fQ$. So by $\neg$Intro$_w$ for $\vdash'$, $f[\Gamma] \vdash' \neg\neg fR$, and so $f[\Gamma] \vdash' f[R]$by DNE.

$\to$**Intro:** Suppose $f[\Gamma], fP \vdash' fQ$. We have $fP \wedge \neg fQ \vdash' \neg fQ$ by Assumption$_w$ and $\wedge$Elim, and so, $f[\Gamma], fP \wedge \neg fQ \vdash' \neg fP$ by $\neg$Intro$_w$. But also $fP \wedge \neg fQ \vdash' fP$ by Assumption and $\wedge$Elim. So by $\neg$Intro$_w$, $f[\Gamma] \vdash' \neg(fP \wedge \neg fQ)$, which is $f(P \to Q)$.

$\to$**Elim:** Suppose $f[\Gamma] \vdash' f(P \to Q)$ (i.e. $\neg(fP \wedge \neg fQ)$) and also $f[\Gamma] \vdash' fP$. By Assumption $\neg fQ \vdash' \neg fQ$, so by $\wedge$Intro$_w$, $f[\Gamma], \neg fQ \vdash' fP \wedge \neg fQ$. So by $\neg$Intro$_w$, $f[\Gamma] \vdash' \neg\neg fQ$, and by DNE, $f[\Gamma] \vdash' fQ$.

$\exists$**Intro:** Suppos $f[\Gamma] \vdash' f(P[t/v]])$. By $\forall$Elim for $\vdash$, $\forall v \neg fP \vdash' \neg fP[t/v])$. But $\neg fP[t/v]$ is $\neg((fP)[t/v])$ (induction!), so by $\neg$Intro$_w$, $f[\Gamma] \vdash' \neg\forall v \neg fP$, which is $f(\exists vP)$.

$\exists$**Elim:** Suppose $f[\Gamma] \vdash' f(\exists vP)$ $(= \neg\forall v \neg P)$ and $f[\Gamma], f(P[u/v]) \vdash' fQ$, where v is not free in Γ, $\exists vP$, or Q. It is easy to show (induction!) that $f(P[u/v]) = fP[u/v]$. By Contraposition, $f[\Gamma], \neg fQ \vdash' \neg fP[u/v]$, which is $(\neg fP)[u/v]$. So by $\forall$-intro, $f[\Gamma], \neg fQ \vdash' \forall v \neg fP$. So by $\neg$Intro$_w$, $f[\Gamma] \vdash \neg\neg fQ$, and by DNE $f[\Gamma] \vdash fQ$.