Models

Professor Cian Dorr

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New York University

First-order structures

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Definition

A *structure* for a first-order signature Σ is a tuple $\langle D, (I_s)_{s \in R_\Sigma}, (I_t)_{t \in F_\Sigma} \rangle$, where:

- D (called the "domain" of the structure) is not empty
- $I_s \subseteq D^n$ when $s \in R_\Sigma$ and $a_\Sigma s = n$
- $I_t \in D^{D^n}$ when $t \in F_\Sigma$ and $a_\Sigma s = n$.

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Definition

When S is a structure for Σ with domain D, an *assignment function for S* is a function from Var to D.

Truth in a structure on an assignment

We define a notion of a formula P being **true in** a given structure S on a given assignment g.

- $S, g \Vdash P$ means 'P is true in S on g'.
- We also write this as '$g \in [\![P]\!]_S$'.
- Or as '$[\![P]\!]_S^g = 1$'.

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In order to do this for a language with function symbols, we also need to define what it is for a given element d of a structure's domain to be the **denotation of** a term t **on** an assignment g.

- We write this as '$d = [\![t]\!]_S(g)$', or '$d = [\![t]\!]_S^g$'.

Definition of denotation on an assignment

Definition

Given a structure S for Σ, $[\![\cdot]\!]_S$ is the function from $\mathrm{Terms}(\Sigma)$ to $D^{D^{\mathrm{Var}}}$ such that

1. $[\![v]\!]_S^g = gv$ for any variable v.
2. $[\![c]\!]_S^g = I_c$ if c is an individual constant of Σ (for any g).
3. $[\![f(t_1, \ldots, t_n)]\!]_S^g = I_f([\![t_1]\!]_S^g, \ldots, [\![t_n]\!]_S^g)$

Definition of truth on an assignment

Given a structure S for Σ, the relation $S, g \Vdash P$ is the unique relation between assignment functions for S and formulae of Σ such that:

- $S, g \Vdash F(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]^g_S, \ldots, [\![t_n]\!]^g_S \rangle \in I_F$.
- $S, g \Vdash t_1 = t_2$ iff $[\![t_1]\!]^g_S = [\![t_2]\!]^g_S$.
- $S, g \Vdash \neg P$ iff $S, g \nVdash P$.
- $S, g \Vdash P \to Q$ iff $S, g \nVdash P$ or $S, g \Vdash Q$.
- $S, g \Vdash P \wedge Q$ iff $S, g \Vdash P$ and $S, g \Vdash Q$.
- $S, g \Vdash P \vee Q$ iff either $S, g \Vdash P$ or $S, g \Vdash Q$.
- $S, g \Vdash \forall v P$ iff $S, g[v \mapsto d] \Vdash P$ for all d in the domain of S.
- $S, g \Vdash \exists v P$ iff $S, g[v \mapsto d] \Vdash P$ for some d in the domain of S.

Defining logical notions

Definition

Formula P of $\mathcal{L}(\Sigma)$ is **valid** iff P is true in every Σ-structure on every assignment.

Definition

Sequent $\Gamma \triangleright P$ of $\mathcal{L}(\Sigma)$ is **valid** iff for every Σ-structure S and every assignment g for S, if every member of Γ is true in S on g, then P is true in S on g.

Another way of saying that the sequent $\Gamma \triangleright P$ is valid is to say that P is a **logical consequence** of Γ, or in symbols, $\Gamma \vDash P$.

Definition

Set Γ of formulae of $\mathcal{L}(\Sigma)$ is **satisfiable** (logically consistent) iff there is a Σ-structure S and an assignment g for S such that every member of Γ is true in S on g.

Fact

$\Gamma \vDash P$ iff $\Gamma \cup \{\neg P\}$ is not satisfiable; Γ is satisfiable iff $\Gamma \nvDash \bot$ [$:= \neg\forall x(x = x))$]

Irrelevance Lemma for terms

If $gv = hv$ for all $v \in FV(t)$, then $[\![t]\!]_S^g = [\![t]\!]_S$.

Irrelevance Lemma for formulae

If $gv = hv$ for all $v \in FV(P)$, then $h \in [\![P]\!]_S$ if $g \in [\![P]\!]_S$.

Irrelevance Lemma for terms

If $gv = hv$ for all $v \in FV(t)$, then $[\![t]\!]_S^g = [\![t]\!]_S$.

Irrelevance Lemma for formulae

If $gv = hv$ for all $v \in FV(P)$, then $h \in [\![P]\!]_S$ if $g \in [\![P]\!]_S$.

Substitution Lemma for terms

$[\![t[s/v]]\!]_S^g = [\![t]\!]_S^{g[v \mapsto [\![s]\!]_S^g]}$.

Substitution Lemma for formulae

$S, g \Vdash P[s/v]$ iff $S, g[v \mapsto [\![s]\!]_S^g] \Vdash P$.

Soundness and Completeness

Two theorem

Two reason for being interested in models comes from the following key theorem:

Soundness Theorem

Whenever $\Gamma \vdash P$, $\Gamma \vDash P$.

Completeness Theorem

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Say that Γ is *consistent* iff there is some P such that $\Gamma \nvdash P$. Then the above claims are equivalent to:

Soundness Theorem (alternative form)

Every satisfiable set of formulae is consistent.

Completeness Theorem (alternativbe form)

Every consistent set of formulae is satisfiable.

The usefulness of the Soundness Theorem

Up to now we have had great ways of showing that sequents are provable (e.g. constructing a proof of them), but no good ways of showing that they *aren't* provable. We can look at a bunch of attempts and say 'This isn't a proof, and this isn't, and this isn't. . . '; but this procedure never rules out that there's a proof we haven't considered yet.

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Thanks to the Soundness Theorem, we have a way of showing that $\Gamma \nvdash P$: we construct a structure S and assignment g such that every member of Γ is true in S on g, but P isn't true in S on g.

More on the philosophical significance of the Soundness Theorem

Some discussions suggest the Soundness Theorem is supposed to actually provide a justification for reasoning in accordance with the classical rules baked into $\vdash$. But this is a problematic thought for at least two separate reasons.

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Some discussions suggest the Soundness Theorem is supposed to actually provide a justification for reasoning in accordance with the classical rules baked into $\vdash$. But this is a problematic thought for at least two separate reasons.

1. Like every proof we've ever done, the proof of the Soundness Theorem requires using many of these very rules in the metalanguage. If one were really worried about those rules, one wouldn't accept the proof.

2. The imagined justification would require, e.g. going from 'Sentence P is valid' to actually asserting a certain sentence P—accepting it as *really* true. But since there isn't a set that contains everything, it is obscure how such a transition would be justified.

Proving the Soundness Theorem

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Weakening: Suppose $\Gamma \vDash P$ and Δ is a set of formulae. We need to show $\Gamma, \Delta \vdash P$: in other words, for every structure S and assignment g for S, if every member of $\Gamma \cup \Delta$ is true in S on g, P is true in S on g. But if every member of $\Gamma \cup \Delta$ is true in S on g, every member of Γ is; so by our induction hypothesis, P is.

$\vee$**Intro1:** Suppose $\Gamma \vDash P$, and suppose every member of Γ is true in S on g. Then by the induction hypothesis, P is true in S on g, so either P is true in S on g or Q is, so by the definition of $[\![P \vee Q]\!]$, $P \vee Q$ is true in S on g.

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$\vee$**Elim:** Suppose $\Gamma \vDash P \vee Q$, $\Gamma, P \vDash R$, and $\Gamma, Q \vDash R$. Suppose every member of Γ is true in S on g. Then by the IH, $P \vee Q$ is true in S on g, so by the definition of $[\![P \vee Q]\!]$, either P is true in S on g or Q is. In the former case, R is true in S on g by the second part of the IH; in the latter case, R is also true in S by the third part of the IH. So, R is true in S on g.

$\lor$**Intro1:** Suppose $\Gamma \vDash P$, and suppose every member of Γ is true in S on g. Then by the induction hypothesis, P is true in S on g, so either P is true in S on g or Q is, so by the definition of $[\![P \lor Q]\!]$, $P \lor Q$ is true in S on g.

$\lor$**Elim:** Suppose $\Gamma \vDash P \lor Q$, $\Gamma, P \vDash R$, and $\Gamma, Q \vDash R$. Suppose every member of Γ is true in S on g. Then by the IH, $P \lor Q$ is true in S on g, so by the definition of $[\![P \lor Q]\!]$, either P is true in S on g or Q is. In the former case, R is true in S on g by the second part of the IH; in the latter case, R is also true in S by the third part of the IH. So, R is true in S on g.

$\forall$**Intro:** Suppose $\Gamma \vDash P$ and v is a variable that isn't free in any member of Γ. Suppose every member of Γ is true in a certain S on a certain g. Let d be an arbitrary member of the domain of S. Since v isn't free in Γ, $g[v \mapsto d]$ agrees with g on $FV(\Gamma)$, so by the Irrelevance Lemma, every member of Γ is true in S on $g[v \mapsto d]$, so by the IH, P is true in S on $g[v \mapsto d]$. Since this holds for every d, we can conclude (looking at the definition of $[\![\forall v P]\!]$) that $\forall v P$ is true in S on g.

The Completeness Theorem

The Completeness Theorem

The proof of the Soundness Theorem is straightforward and unsurprising. Much more interesting is the proof of it's companion, which we'll discuss next week:

The Completeness Theorem

If $\Gamma \vDash P$, then $\Gamma \vdash P$.

Two key notions for this proof

Definition

Γ is **negation-complete** $:=$ for each formula P, either $P \in \Gamma$ or $\neg P \in \Gamma$.

Definition

Γ is **witness-complete** $:=$ for each formula P and variable v, either $\forall v \neg P \in \Gamma$ or there is a term t such that $P[t/v] \in \Gamma$.

(Note: the Russell book uses "witness-complete" for a slightly different notion.)

The proof will work by first showing that consistent sets Γ that have these further properties are satisfiable, and then showing that other consistent sets Γ have consistent supersets Γ^+ that have these properties.

Strategy

Step One: every **negation-complete**, **witness-complete**, consistent set of formulae in the **identity-free** language $\mathcal{L}_{\neg,\wedge,\vee,\to,\forall,\exists}(\Sigma)$ is satisfiable.

Step Two: every negation-complete, witness-complete, consistent set of formulae in $\mathcal{L}(\Sigma)$ is satisfiable.

Step Three: every witness-complete, consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ is a subset of some negation-complete, witness-complete, consistent Γ^+, and is thus satisfiable by Step Two.

Step Four: every consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ in which countably infinitely many variables don't occur free is a subset of some witness-complete, consistent Γ^+, and is thus satisfiable by Step Three.

Step Five: every consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ can be turned by a relettering of free variables into one in which countably many infinitely many variables don't occur free, and is thus staisfiable by Step Four

Step One: the identity-free language

Suppose Γ is a consistent, negation-complete, and witness-complete set of identity-free formulae of a signature Σ. Consider, the following structure S and assignment g:

$$
\begin{aligned}
D &:= \text{Terms}(\Sigma) \\
I_c &:= c \text{ for each individual constant of } \Sigma. \\
I_f(t_1, \ldots, t_n) &:= f(t_1, \ldots, t_n) \text{ for each } n\text{-ary function symbol } f \text{ of } \Sigma. \\
I_F &:= \{\langle t_1, \ldots, t_n \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \text{ for each } n\text{-palce predicate } F \text{ of } \Sigma. \\
g(v) &:= v \text{ for each variable } v.
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We will prove that for all (identity-free) formulae P, $S, g \Vdash P$ iff $P \in \Gamma$.

Proof for Step One

First we need to show that $[\![t]\!]_S^g = t$ for every term t. This is a trivial induction.

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(i) Atomic formulae: $S, g \Vdash F(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]^g_S, \ldots, [\![t_n]\!]^g_S \rangle \in I_F$, iff $\langle t_1, \ldots, t_n \rangle \in I_F$, iff $F(t_1, \ldots, t_n) \in \Gamma$.

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(ii) Negation. Suppose $S, g \Vdash P$ iff $P \in \Gamma$. Then $S, g \Vdash \neg P$ iff $P \notin \Gamma$. But since Γ is consistent and negation-complete, $P \notin \Gamma$ iff $\neg P \in \Gamma$.

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(ii) Conjunction. Suppose $S, g \Vdash P$ iff $P \in \Gamma$ and $S, g \Vdash Q$ iff $Q \in \Gamma$. Then $S, g \Vdash P \wedge Q$ iff $P \in \Gamma$ and $Q \in \Gamma$.

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(iii) Universal quantification. Suppose as the induction hypothesis that t, $S, g \Vdash P[t/v]$ iff $P[t/v] \in \Gamma$. Suppose that $S, g \Vdash \forall vP$. Then by closure and $\forall$Elim, $S, g \Vdash P[t/v]$ for all t, so by the induction hypothesis, $P[t/v] \in \Gamma$ for all t. But since Γ is witness-complete, there is a t such that $\exists v \neg P \rightarrow \neg P[t/v] \in \Gamma$. Since Γ is consistent and closed under *modus tollens*, it follows that $\neg\exists v \neg P \in \Gamma$, and hence that $\forall vP \in \Gamma$ (using the quantifier rules).

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Conversely, suppose that $\forall v P \in \Gamma$. Then by closure and $\forall$Elim, $P[t/v] \in \Gamma$ for all terms t, so by the induction hypothesis, $S, g \Vdash P[t/v]$ for all terms t. But then $S, g[v \mapsto t] \Vdash P$ for all terms t by the Substitution Lemma, so $S, g \Vdash \forall v P$.

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I'll leave the steps for $\vee, \to$, and $\exists$ as exercises.

Step Two: adding identity

Once we add identity to the language, the result no longer goes through. Every atomic sentence of the form $t_1 = t_2$ where t_1 and t_2 are distinct terms is false on S on g. But a consistent Γ can of course contain some such formulae!

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Our new structure S' will have as its domain $\{[t]_\Gamma \mid t \in \mathrm{Terms}(\Sigma)\}$.

And our new assignment g' will map each variable v to $[v]_\Gamma$.

Thanks to the =Intro and =Elim rules, we can prove the following:

(a) $\Gamma \vdash s = t$ iff $[s]_\Gamma = [t]_\Gamma$.

(b) If $s_1 \in [t_1]_\Gamma$, and ... and $s_n \in [t_n]_\Gamma$, then $[f(s_1, \ldots, s_n)]_\Gamma = [f(t_1, \ldots, t_n)]_\Gamma]$

(c) If $s_1 \in [t_1]_\Gamma$, and ... $s_n \in [t_n]_\Gamma$, and $\Gamma \vdash F(t_1, \ldots, t_n)$, then $\Gamma \vdash F(s_1, \ldots, s_n)$.

So, we can consistently stipulate that the structure S' works as follows:

$$\begin{aligned} I_c &:= [c]_\Gamma \text{ for each individual constant } c. \\ I_f([t_1]_\Gamma, \ldots, [t_n]_\Gamma) &:= [f(t_1, \ldots, t_n)]_\Gamma \\ I_F &:= \{\langle [t_1]_\Gamma, \ldots, [t_n]_\Gamma \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \end{aligned}$$

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Another straightforward induction then proves that for every t, $[\![t]\!]_{S'}^{g'} = [t]_\Gamma$.

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$$\begin{aligned} I_c &\coloneqq [c]_\Gamma \text{ for each individual constant } c. \\ I_f([t_1]_\Gamma, \ldots, [t_n]_\Gamma) &\coloneqq [f(t_1, \ldots, t_n)]_\Gamma \\ I_F &\coloneqq \{\langle [t_1]_\Gamma, \ldots, [t_n]_\Gamma \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \end{aligned}$$

Another straightforward induction then proves that for every t, $[\![t]\!]^{g'}_{S'} = [t]_\Gamma$.

We can then redo the step for atomic formulae in the old proof. $S', g' \Vdash F(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]^{g'}_{S'}, \ldots, [\![t_n]\!]^{g'}_{S'} \rangle \in I_F$, iff $\langle [t_1]_\Gamma, \ldots, [t_n]_\Gamma \rangle \in I_F$, iff $F(t_1, \ldots, t_n) \in \Gamma$.

So, we can consistently stipulate that the structure S' works as follows:

$$\begin{aligned} I_c &\coloneqq [c]_\Gamma \text{ for each individual constant } c. \\ I_f([t_1]_\Gamma, \ldots, [t_n]_\Gamma) &\coloneqq [f(t_1, \ldots, t_n)]_\Gamma \\ I_F &\coloneqq \{\langle [t_1]_\Gamma, \ldots, [t_n]_\Gamma \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \end{aligned}$$

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And we also have atomic identity formulae.

$S', g' \Vdash s = t$ iff $\langle s \rangle^{g'}_{S'} = \langle t \rangle^{g'}_{S'}$, iff $[s]_\Gamma = [t]_\Gamma$, iff $s = t \in \Gamma$.

Step Three: sets that are witness complete but not negation complete

Extensibility Lemma

Every consistent Γ has a consistent, negation-complete supserset.

Note that if Γ is witness-complete, so are all of its supersets.

Proving the Extensibility Lemma

There are only countably many formulae P_0, P_1, P_2. Define a sequence of sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ recursively as follows:

$$\Gamma_0 = \Gamma$$
$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{P_n\} & \text{if this is consistent} \\ \Gamma_n \cup \{\neg P_n\} & \text{otherwise} \end{cases}$$

Finally let Γ^+ be $\bigcup_n \Gamma_n$. Γ^+ is negation-complete. Each Γ_n is consistent (induction on n). By the compactness of provability, this implies that Γ^+ is consistent.

Step Four: sets that aren't witness-complete

Say that Γ is *safe* iff there is a countably infinite set $v_1, v_2, \ldots$ of variables that aren't free in any element of Γ.

There are only countably many pairs $\langle P, v\rangle$ of a formula P and variable u. Enumerate them as $\langle P_1, u_1\rangle, \langle P_2, v_2\rangle, \ldots$. We define another sequence of extensions of Γ, as follows:

$$\Gamma^0 := \Gamma$$

$$\Gamma^{n+1} := \begin{cases} \Gamma^n \cup \{P_n[v_n/u_n]\} & \text{if this is consistent} \\ \Gamma^n \cup \{\forall u_n \neg P_n\} & \text{otherwise} \end{cases}$$

Define $\Gamma' = \bigcup_n \Gamma^n$.

Γ' is obviously witness-complete.

To show that it's consistent, we show that each Γ^n is consistent. But this follows from $\forall$Intro.

Step Five

Now we have that every safe, consistent set of formulae is satisfiable.