Relations on a set

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Proving Cantor's Theorem (recap)

Cantor's Theorem

For any set A, $A \lnsim \mathcal{P}A$.

Proof: Clearly, $A \lesssim \mathcal{P}A$, since the function $[x \mapsto \{x\}] : A \to \mathcal{P}A$ is obviously injective. So it suffices to show that $A \not\approx \mathcal{P}A$, i.e. that *no function from A to $\mathcal{P}A$ is a bijection*. We'll actually show that no function from A to $\mathcal{P}A$ is even a *surjection*.

Suppose that f is a function from A to $\mathcal{P}A$. Define f's "diagonal set" as:

$$D_f := \{x \in A \mid x \notin fx\}$$

Suppose for contradiction that there's an element $y \in A$ such that $fy = D_f$.

Then $y \in fy$ iff $y \in D_f$ (since $fy = D_f$)

Also, $y \in D_f$ iff $y \notin fy$ (since for any $x \in A$, $x \in D_f$ iff $x \notin fx$).

So, $y \in fy$ iff $y \notin fy$: Contradiction!

A note on the terminology of ‘diagonal set’

When we have a relation R from A to A, $\operatorname{diag} R$ is the set $\{x \in A \mid Rxx\}$. It’s so-called because we can determine what’s in it by representing R as a grid of ticks and crosses, and looking down the diagonal of the grid.

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & \checkmark & \checkmark & \times & \checkmark \\
b & \times & \times & \times & \checkmark \\
c & \checkmark & \times & \checkmark & \checkmark \\
d & \checkmark & \times & \checkmark & \times
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
 \\
\boxed{\checkmark} \\
\boxtimes \\
\boxed{\checkmark} \\
\boxtimes
\end{array}
$$

But what does that use of ‘diagonal’ have to do with functions from a set to its powerset?

Currying and uncurrying

Well, it turns out there's a very close connection between (a) *relations from A to B* and (b) *functions from A to $\mathcal{P}B$*.

- Any relation R from A to B determines a function $\operatorname{curry} R$ from A to $\mathcal{P}B$, namely $[x \mapsto \{y \in B \mid Rxy\}]$.
- Any function f from A to $\mathcal{P}B$ determines a relation $\operatorname{uncurry} f$ from A to B, namely $\{\langle x, y\rangle \mid y \in fx\}$.

These operations are inverses:

$$\begin{aligned}
\operatorname{curry}(\operatorname{uncurry} f) &= [x \mapsto \{y \mid \langle x, y\rangle \in \operatorname{uncurry} f\}] \\
&= [x \mapsto \{y \mid y \in fx\}] = [x \mapsto fx] = f \\
\operatorname{uncurry}(\operatorname{curry} R) &= \{\langle x, y\rangle \mid y \in (\operatorname{curry} R)x\} \\
&= \{\langle x, y\rangle \mid y \in \{z \mid Rxz\}\} = \{\langle x, y\rangle \mid Rxy\} = R
\end{aligned}$$

Why we called D_f the "diagonal set"

Given our $f : A \to \mathcal{P}A$, we have:

$$\begin{aligned} \operatorname{curry} f &:= \{\langle x, y\rangle \in A \times A \mid y \in fx\} \\ \operatorname{diag}(\operatorname{curry} f) &:= \{x \in A \mid \langle x, x\rangle \in \operatorname{curry} f\} = \{x \in A \mid x \in fx\} \end{aligned}$$

and so

$$A \setminus \operatorname{diag}(\operatorname{curry} f) := \{x \in A \mid x \notin fx\} = D_f$$

Cantor's theorem and the real numbers (aside)

Cantor proved the above theorem on the way to proving that *the set of real numbers* ($\mathbb{R}$) *is bigger than the set of natural numbers* ($\mathbb{N}$).

To get this from our version, it suffices to show that $\mathcal{P}\mathbb{N} \lesssim \mathbb{R}$. Given this, if we had $\mathbb{N} \sim \mathbb{R}$ we would have $\mathcal{P}\mathbb{N} \lesssim \mathbb{N}$, which is ruled out by Cantor's theorem. Since obviously $\mathbb{N} \lesssim \mathbb{R}$, we get $\mathcal{P}\mathbb{N} \lnsim \mathbb{R}$.

So we just need to define an injection from $\mathcal{P}\mathbb{N}$ to $\mathbb{R}$. To do this, we map every set $X \subseteq N$ to the unique real number whose decimal expansion has a 1 in position n if $n \in X$ and a 0 in position n otherwise.

Properties of relations on a set

Relations on a set

So far, we have been looking at relations from a set A to a set B. But for lots of the relations we will be interested in, $A = B$.

A relation from A to A—i.e. a subset of $A^2 (= A \times A)$—is called a *relation on A*.

There are some important properties that apply to relations on a set, which we will cover in this section. (Warning: definitions incoming!)

The big four properties

Suppose R is a relation on A. Then:

Definition

R is *reflexive* $:=$ Rxx for all $x \in A$.

Definition

R is *transitive* $:=$ for all $x, y, z \in A$, if Rxy and Ryz, then Rxz.

Definition

R is *symmetric* $:=$ for all $x, y \in A$, if Rxy then Ryx.

Definition

R is *antisymmetric* $:=$ for all $x, y \in A$, if Rxy and Ryx, then $x = y$.

Combinations of the big four

Definition

R is a *preorder* $:=$ R is reflexive and transitive.

Definition

R is an *equivalence relation* $:=$ R is reflexive, transitive, and symmetric.

Definition

R is a *partial order* $:=$ R is reflexive, transitive, and antisymmetric.

Some examples

On the set of natural numbers $\mathbb{N}$:

- The relation *having at least as many digits in one's decimal expansion* is a preorder.
- The relation *having the same number digits in one's decimal expansion* is an equivalence relation.
- The greater-than-or-equal-to relation is a partial order.
- The relation *being a factor of* is also a partial order.

Some more terminology

When R is a relation on A:

Definition

R is *irreflexive* $:=$ for all $x \in A$, not Rxx.

Definition

R is *asymmetric* $:=$ for all $x, y \in A$, not both Rxy and Ryx.

Definition

R is *strongly connected* $:=$ for all $x, y \in A$, either Rxy or Ryx.

Definition

R is *connected* $:=$ for all $x, y \in A$, either Rxy or Ryx or $x = y$.

Closedness

Closed sets

Suppose R is a relation on A and $X \subseteq A$.

Definition

X is *closed under* R := for all $x, y \in A$, if $x \in X$ and Rxy, then $y \in X$.

Intuitively: if you start inside X, you can't "escape" by taking an R-step.

Examples

In $\mathbb{N}$, what does a set have to be like to be...

- Closed under $\{\langle x, y\rangle \mid y = x + 1\}$?
- Closed under $\{\langle x, y\rangle \mid x = y + 1\}$?
- Closed under $\{\langle x, y\rangle \mid y \geq x\}$?
- Closed under $\{\langle x, y\rangle \mid y = x + 1 \text{ or } x = y + 1\}$?

Closed sets, more generally

We extend the concept of closedness from *relations on* A to *relations from* $A^2 (= A \times A)$ *to* A:

Definition

Where R is a relation from $A \times A$ to A and $X \subseteq A$, X is *closed under* R iff for all $x, y, z \in A$, if $x \in X$ and $y \in X$ and $R\langle x, y\rangle z$, then $z \in X$.

A *function* $A \times A \to A$ is called a *binary operation on* A

- Examples: *addition* and *multiplication* are binary operations on $\mathbb{N}$
- the operation of *joining two strings to make a third* is a binary operation on the set of strings over a given alphabet...

We can extend this in the obvious way to relations from $A^3 (= (A \times A) \times A$ to A...

Closed under a family of relations

Sometimes we are interested in some family of relations $R_1, \ldots, R_n$, where each one is either from A to A or from A^2 to A or from A^3 to A. . . . We say that a set X is closed under this family of relations iff X is closed under every member of the family.