Definitions

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10th November 2022

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Definitions of predicates

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Where F is an n-ary predicate that is in Σ^+ but not in Σ, a **definition of F in Σ** is a sentence of the form $\forall v_1 \ldots \forall v_n(F(v_1, \ldots, v_n) \leftrightarrow P)$, where P is a formula of Σ and $v_1, \ldots, v_n$ are distinct variables.

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'Possible definition' might be less misleading than 'definition'.

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Where f is an n-ary function symbol that occurs in Σ^+ but not in Σ, a **definition of f in Σ** is a sentence of the form $\forall v_1 \ldots \forall v_{n+1}(v_{n+1} = f(v_1, \ldots, v_n) \leftrightarrow P)$, where P is a formula of Σ and $v_1, \ldots, v_{n+1}$ are distinct variables.

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$$\forall x \forall y(y = \text{tail}(x) \leftrightarrow ((x = \text{""} \wedge y = \text{""}) \vee \exists z(\text{LengthOne}(z) \wedge x = z \oplus y))$$

$$\forall x \forall y(y = \text{useless}(x) \to y \leq x)$$

Definitional extensions of a theory

Definition

Suppose D is a definition in Σ of a predicate F or function symbol f, and T is a theory in Σ. Then, D is **legitimate** in T iff either

(i) D is a definition of a predicate F, or

(ii) D is a definition $\forall v_1 \ldots \forall v_{n+1}(v_{n+1} = f(v_1, \ldots, v_n) \leftrightarrow P)$ of a function symbol f, such that $T \vdash \forall v_1 \ldots \forall v_n \exists! v_{n+1}(P)$.

- Here, $\exists! vQ$ is shorthand for $\exists v(Q \wedge \forall v'(Q \rightarrow v = v'))$.
- $\forall v_1 \ldots \forall v_n \exists! v_{n+1}(P)$ is called the **unique existence claim** for D.

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Example: $\forall x \forall y(y = \mathrm{pred}(x) \leftrightarrow x = \mathrm{suc}(y))$ is only legitimate in T if $T \vDash \forall x \exists y(x = \mathrm{suc}\, y)$. $\forall x \forall y(y = \mathrm{pred}(x) \leftrightarrow (x = \mathrm{suc}(y)) \wedge (x = 0 \wedge y = 0)$ is legitimate in any theory that includes $\forall x(x \neq 0 \leftrightarrow \exists y(x = \mathrm{suc}\, y))$.

Definitional extensions of a theory

Definition

Suppose T is a theory in Σ and T^+ is a theory in Σ^+. Then, T^+ is a **one-step definitional extension of** T iff there exactly one predicate or function symbol that's in Σ^+ but not Σ, and there is a definition D of this predicate or function symbol that is legitimate in T, such that T^+ is the set of all logical consequences of $T \cup D$ in Σ^+.

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Definition

T^+ is a **finite definitional extension** of T iff T^+ belongs to the closure of $\{T\}$ under the one-step definitional extension relation.

Or more intuitively: iff there's a list of theories $T, T_1, \ldots, T_n, T^+$ where each after the first is a one-step definitional extension of its predecessor.

Definitional extensions are conservative extensions

Definition

Theory T^+ in signature Σ^+ is a **conservative extension** of theory T in signature Σ iff T is the set of all Σ-sentences that are in T^+.

Note that this relation is a partial order (transitive, reflexive, antisymmetric).

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Fact

If T^+ is a finite definitional extension of T, T^+ is a conservative extension of T.

Idea of proof: first note that it's sufficient to show that it's true when T^+ is a one-step definitional extension of T.

To prove this, we take any structure S where T is true, and extend it into a structure S^+ for Σ^+ where the definition of the new predicate/function symbols are true. We show that T^+ is true in S^+. It follows that any Σ-sentence consistent with T is also consistent with T^+, so any Σ-sentence in T^+ must be a consequence of T.

More carefully:

Definition

Where S is a structure for Σ, a definition of a predicate or function symbol not in Σ is **legitimate in** S iff is legitimate in $\mathrm{Th}\, S$.

Equivalently: iff either it's a definition of a predicate, or it's a definition $\forall v_1 \dots \forall v_{n+1}(v_{n+1} = f(v_1, \dots, v_n) \leftrightarrow P))$ of a function symbol such that $S \Vdash \forall v_1 \dots v_n \exists! v_{n+1} P$.

Definitions and structures

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Definition

Suppose S is a structure for Σ, and S^+ is a structure for Σ^+. Then, S^+ is a **definitional expansion** of S iff S^+ has the same domain as S, and for every predicate or function symbol that's in Σ^+ but not Σ, there is a definition of it in Σ that is legitimate in S and true in S^+.

Fact

If S is a structure for Σ and D is a definition that is legitimate in S, then there is a definitional expansion S^+ of S in which D is true.

Proof: if D is $\forall v_1 \ldots \forall v_n(F(v_1, \ldots, v_n) \leftrightarrow P)$, let the interpretation of F in S^+ be

$$\{\langle x_1, \ldots, x_n\rangle \mid S, [v_1 \mapsto x_1, \ldots, v_n \mapsto x_n] \Vdash P\}$$

And if D is $\forall v_1 \ldots \forall v_{n+1}(v_{n+1} = f(v_1, \ldots, v_n) \leftrightarrow P)$, let the interpretation of f in S^+ be

$$\{\langle x_1, \ldots, x_{n+1}\rangle \mid S, [v_1 \mapsto x_1, \ldots, v_{n+1} \mapsto x_{n+1}] \Vdash P\}$$

Note that this is guaranteed to be a *function*, since the legitimacy of the definition means that $\forall v_1 \ldots \forall v_n \exists! v_{n+1} P$ is true in S.

Eliminating definitions

Definitions can in principle be "eliminated". Given a definition D in Σ of a predicate F or function symbol f, we can define a function that maps every formula Q of Σ^+ to a Σ-formula Q^D such that $Q, D \vdash Q^D$ and $Q^D, D \vdash Q$.

When D is a definition $\forall v_1 \ldots \forall v_n(F(v_1, \ldots, v_n) \leftrightarrow P))$ of a predicate F, the definition of $\cdot^D$ is straightforward. Basically, we just go through the formula Q and every time we find an atomic formula $F(t_1, \ldots, t_n)$, we replace it with $P[t_1/v_1, \ldots, t_n/v_n]$.

- The only fiddly thing we need to do is handle the case where this substitution is ill-defined because of variable-capture; we deal with this by replacing the variables bound by quantifiers in P with fresh variables that don't occur in $t_1, \ldots, t_n$.

Eliminating definitions of function symbols

For definitions of function symbols, things are a bit more complicated, since it doesn't make sense to "replace" a function symbol with a formula. What we have to do is first turn our formula P into a logically equivalent "safe" formula.

Definition

Formula Q is safe iff

- The only occurrences in Q of terms other than variables are in identity-formulae, as the second argument of $=$.
- Whenever Q contains a term of the form $f(t_1, \ldots, t_n)$, all of $t_1, \ldots, t_n$ are variables.

Eliminating definitions of function symbols

Fact

Every formula is provably equivalent to a safe formula.

The intuition for why this is true is pretty clear:

Formula	Safe equivalent
$x + y \leq z$	$\exists x'(x' = x + y \wedge x' \leq z)$
$x + (y \times y') \leq z$	$\exists x' \exists x''(x'' = y \times y' \wedge x' = x + x'' \wedge x' \leq z)$

The formal proof is a bit intricate: it involves defining a function that maps each non-safe formula to a logically equivalent 'simpler' formula, and showing that if we apply this function enough times, the result will finally be a safe formula. I won't get into the details.

Eliminating definitions of function symbols

Once we have found a safe equivalent Q' of a function Q, we can define Q^D similarly to how we did it for function symbols: we just go through Q' and whenever we find an atomic formula of the form $u_{n+1} = f(u_1, \ldots, u_n)$, we replace it with $P[u_1/v_1, \ldots, u_{n+1}/v_{n+1}]$ (relettering bound variables in P to avoid capture).

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fact

If D is a definition of a predicate, then for any formulae $P_1, \dots, P_n, Q$ of the expanded language, $P_1, \dots, P_n \vdash Q$ iff $P_1^D, \dots, P_n^D \vdash Q^D$.

If D is a definition of a function symbol, then for any formulae $P_1, \dots, P_n, Q$ of the expanded language, $P_1, \dots, P_n \vdash Q$ iff $P_1^D, \dots, P_n^D, U \vdash Q^D$, where U is the unique existence claim corresponding to D.

This provides an alternative way of proving the fact about conservative extensions: we show that when T^+ is a one-step definitional extension of T with definition D,

Notational variants

Suppose Σ_1 and Σ_2 are signatures, and J is a bijection that maps every predicate of Σ_1 to one of Σ_2 with the same arity, and maps every function symbol of Σ_1 to one of Σ_2 with the same arity.

Then we can, obviously, use J to define mappings J^* and J^{**} from the terms and formulae of Σ_1 to those of Σ_2, via the following recursive definitions:

$$\begin{aligned}
J^*(v) &:= v \text{ for } v \text{ a variable} \\
J^*(f(s_1, \ldots, s_n)) &:= (Jf)(J^* s_1, \ldots, J^* s_n) \\
J^{**}(F(s_1, \ldots, s_n)) &:= (JF)(J^* s_1, \ldots, J^* s_n) \\
J^{**}(s_1 = s_2) &:= J^* s_1 = J^* s_2 \\
J^{**}(\neg P) &:= \neg J^{**} P \\
J^{**}(P \# Q) &:= J^{**} P \# J^{**} Q \text{ for } \# = \rightarrow, \wedge, \vee \\
J^{**}(\# v P) &:= \# v (J^{**} P) \text{ for } \# = \forall, \exists
\end{aligned}$$

Notational variants

Definition

Theory T_1 in signature Σ_1 is a **notational variant** of theory T_2 in signature Σ_2 iff there is a substitution function J from Σ_1 to Σ_2 such that $T_2 = \{J^{**}P \mid P \in T_1\}$.

Interpretation

Definition

Theory T_1 in signature Σ_1 **interprets** theory T_2 in signature Σ_2 iff T_1 has a definitional extension T_1^+ that extends some notational variant of T_2.

This finally gives us the notion of "at least as strong as" we are looking for—a "sufficiently strong" theory will be one that *interprets* Min.

Gödel's incompleteness theorem, again

Gödel's first incompleteness theorem (first pass)

No theory in Str is

1. consistent
2. negation-complete
3. finitely axiomatizable
4. includes Min

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Gödel's first incompleteness theorem (second pass)

No theory is

1. consistent
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Gödel's incompleteness theorem, again

To get from the first-pass version to the second version, suppose for contradiction that T is consistent, negation-complete, finitely axiomatizable, and has a definitional extension T^+ that extends some notational variant Min' of Min, in signature Str'.

We can assume that T^+ only adds finitely many definitions to T, since Str' is finite. So, T^+ is finitely axiomatizable. It is consistent since T is, and it's also negation-complete. Now consider the set of all sentences of Str' that belong to T^+. It's also consistent and negation-complete, and we can show that it's finitely axiomatizable too.

Definability in a structure

Definable sets and relations

Definition

Let S be a structure for Σ. An n-ary relation R of S's domain is **definable in** S iff there is a definitional expansion of S that includes a n-ary predicate F with R as its interpretation.

This is equivalent to the claim that there is a Σ-formula P with n free variables $v_1, \ldots, v_n$ such that for all $x_1, \ldots, x_n$ in the domain of S, $S, [v_1 \mapsto x_1, \ldots, v_n \mapsto x_n] \Vdash P$ iff $Rx_1 \ldots x_n$.

Definable functions

For an an n-ary function f on S's domain, we could also say that f is definable in S iff there is a definitional expansion of S that includes an n-ary function symbol g with f as its interpretation.

But since n-ary functions are $n+1$-ary relations, this actually comes to the same thing: for an n-ary function on S's domain, it's the interpretation of an $n+1$-ary predicate in a definitional expansion of S iff it's the interpretation of an n-ary function symbol in a definitional expansion of S.

The existence of undefinable sets

Since there are only countably many formulae in the language, the set of definable subsets of a structure's domain is always countable.

If the domain is infinite, then by Cantor's theorem it follows that not all subsets of the domain are definable.

Undefinable sets of strings

For the standard string structure (or indeed any other structure whose domain includes the formulae of its signature), we can actually give an example.

Recall that in proving Cantor's theorem, we considered an arbitrary function $f : A \to \mathcal{P}A$, and showed that f isn't a surjection by showing that $\{x \mid x \notin fx\}$ isn't in its range.

Similarly: when P is a formula with one free variable v, say that P is *true of x* in S iff $S, [v \mapsto x] \nVdash P$. Then by the same reasoning, the set of all formulae that are not true of themselves in $\mathbb{S}$ is not definable in $\mathbb{S}$.