Completeness and its consequences

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Two theorems

Last week we proved the

Soundness Theorem

Whenever $\Gamma \vdash P$, $\Gamma \vDash P$.

Soundness Theorem (alternative form)

Every satisfiable set of formulae is consistent.

Today we'll cover its converse:

Completeness Theorem

Whenever $\Gamma \vDash P$, $\Gamma \vdash P$.

Completeness Theorem (alternative form)

Every consistent set of formulae is satisfiable.

The Completeness Theorem

The Completeness Theorem

If $\Gamma \vDash P$, then $\Gamma \vdash P$.

Three key notions for this proof

Definition

Γ is **negation-complete** $:=$ for each formula P, either $P \in \Gamma$ or $\neg P \in \Gamma$.

Definition

Γ is **closed** $:=$ for each formula P, if $\Gamma \vdash P$, then $P \in \Gamma$.

Fact

If Γ is consistent and negation-complete, Γ is closed.

Proof: suppose is negation-complete, $\Gamma \vdash P$, but $P \notin \gamma$. Then $\neg P \in \Gamma$, so $\Gamma \vdash \neg P$, so Γ is inconsistent.

Definition

Γ is **witness-complete** $:=$ for each formula P and variable v, either $\forall v \neg P \in \Gamma$ or there is a term t such that $P[t/v] \in \Gamma$.

Strategy

Step Zero: every **negation-complete**, **witness-complete**, consistent set of formulae in the **identity-free** language $\mathcal{L}_{\neg,\wedge,\vee,\to,\forall,\exists}(\Sigma)$ is satisfiable.

Step One: every negation-complete, witness-complete, consistent set of formulae in $\mathcal{L}(\Sigma)$ is satisfiable.

Step Two: every witness-complete, consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ is a subset of some negation-complete, witness-complete, consistent Γ^+, and is thus satisfiable by Step One.

Step Three: every consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ in which countably infinitely many variables don't occur free is a subset of some witness-complete, consistent Γ^+, and is thus satisfiable by Step Two.

Step Four: every consistent $\Gamma \subseteq \mathcal{L}(\Sigma)$ can be turned by a relettering of free variables into one in which countably many infinitely many variables don't occur free, and is thus staisfiable by Step Three.

Step Zero: the identity-free language

Suppose Γ is a consistent, negation-complete, and witness-complete set of identity-free formulae of a signature Σ. Consider, the following structure S and assignment g:

$$
\begin{aligned}
D &:= \text{Terms}(\Sigma) \\
I_c &:= c \text{ for each individual constant of } \Sigma. \\
I_f(t_1, \ldots, t_n) &:= f(t_1, \ldots, t_n) \text{ for each } n\text{-ary function symbol } f \text{ of } \Sigma. \\
I_F &:= \{\langle t_1, \ldots, t_n \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \text{ for each } n\text{-palce predicate } F \text{ of } \Sigma. \\
g(v) &:= v \text{ for each variable } v.
\end{aligned}
$$

We will prove that for all (identity-free) formulae P, $S, g \Vdash P$ iff $P \in \Gamma$.

Proof for Step Zero

First we need to show that $[\![t]\!]_S^g = t$ for every term t. This is a trivial induction.

Next, we show by induction on the construction of formulae that every formula P has the following property: for every formula Q that can be got from P by zero or more substitutions, $S, g \Vdash Q$ iff $Q \in \Gamma$.

(i) Atomic formulae: $S, g \Vdash F(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_S^g, \ldots, [\![t_n]\!]_S^g \rangle \in I_F$, iff $\langle t_1, \ldots, t_n \rangle \in I_F$, iff $F(t_1, \ldots, t_n) \in \Gamma$.

(ii) Negation. Suppose $S, g \Vdash Q$ iff $Q \in \Gamma$. Then $S, g \Vdash \neg Q$ iff $Q \notin \Gamma$. But since Γ is consistent and negation-complete, $Q \notin \Gamma$ iff $\neg Q \in \Gamma$.

(ii) Conjunction. Suppose $S, g \Vdash Q$ iff $Q \in \Gamma$ and $S, g \Vdash Q'$ iff $Q' \in \Gamma$. Then $S, g \Vdash Q \wedge Q'$ iff $Q \in \Gamma$ and $Q' \in \Gamma$. But if $P \in \Gamma$ and $Q \in \Gamma$ we must have $P \wedge Q \in \Gamma$ (by closure, using $\wedge$Intro), and if $P \wedge Q \in \Gamma$ we must have $P \in \Gamma$ and $Q \in \Gamma$ (by closure, using $\wedge$Elim).

(iii) Universal quantification. Suppose as the induction hypothesis that for all Q that can be got from P by substitutions, $S, g \Vdash Q$ iff $Q \in \Gamma$; and suppose Q can be got from $\forall v P$ by substitutions. Then Q is $\forall v Q'$ for some Q' that can be got from P by substitutions.

Suppose that $S, g \Vdash \forall v Q'$. Then $S, g[v \mapsto d] \Vdash Q'$ for all d in the domain, so by the Substitution Lemma, $S, g \Vdash Q'[t/v]$ for all t, so by the induction hypothesis, $Q'[t/v] \in \Gamma$ for all t. Given negation-completeness means that there is no t for which $\neg P[t/v] \in \Gamma$. Since Γ is witness-complete, it follows that $\forall v P \in \Gamma$.

Conversely, suppose that $\forall v P \in \Gamma$. Then by closure and $\forall$Elim, $P[t/v] \in \Gamma$ for all terms t, so by the induction hypothesis, $S, g \Vdash P[t/v]$ for all terms t. But then by the Substitution Lemma, $S, g[v \mapsto t] \Vdash P$ for all terms t, so $S, g \Vdash \forall v P$ (since everything in the domain of S is a term).

I'll leave the steps for $\vee, \rightarrow$, and $\exists$ as exercises.

Step One: adding identity

Once we add identity to the language, the structure that worked in Step Zero no longer does the job. Every atomic sentence of the form $t_1 = t_2$ where t_1 and t_2 are distinct terms is false on S on g. But a consistent Γ can of course contain some such formulae! To solve this, let's make the following definition:

Definition

Where t is any Σ-term, let $[t]_\Gamma$ be the set $\{s \mid t = s \in \Gamma\}$

Our new structure S' will have as its domain $\{[t]_\Gamma \mid t \in \text{Terms}(\Sigma)\}$.

And our new assignment g' will map each variable v to $[v]_\Gamma$.

Thanks to the =Intro and =Elim rules, we can prove the following (for a negation-complete, consistent Γ)

(a) $s = t \in \Gamma$ iff $[s]_\Gamma = [t]_\Gamma$.

Proof: Left to right: suppose $s = t \in \Gamma$ and $s = s' \in \Gamma$; then $t = s' \in \Gamma$ by =Elim and closure. Right to left: suppose $[s]_\Gamma = [t]_\gamma$. By =Intro and closure, $t = t \in \Gamma$, so $t \in [t]_\Gamma$, so $t \in [s]_\Gamma$, so $s = t \in \Gamma$.

(b) If $s_1 \in [t_1]_\Gamma$, and $\dots s_n \in [t_n]_\Gamma$, then $[f(s_1, \dots, s_n)]_\Gamma = [f(t_1, \dots, t_n)]_\Gamma$

Proof: if the hypothesis is true, each $t_i = s_i \in \Gamma$. By =Intro and closure, $f(t_1, \dots, t_n) = f(t_1, \dots, t_n) \in \Gamma$. So by n applications of =Elim and closure, $f(s_1, \dots, s_n) = f(t_1, \dots, t_n) \in \Gamma$. The conclusion follows by part (a).

(c) If $s_1 \in [t_1]_\Gamma$, and $\dots s_n \in [t_n]_\Gamma$, and $F(t_1, \dots, t_n) \in \Gamma$, then $F(s_1, \dots, s_n) \in \Gamma$.

Proof: by closure and =Elim.

So, we can coherently stipulate that the interpretatin functions of our new structure S' work as follows:

$$\begin{aligned} I_c &:= [c]_\Gamma \text{ for each individual constant } c. \\ I_f([t_1]_\Gamma, \ldots, [t_n]_\Gamma) &:= [f(t_1, \ldots, t_n)]_\Gamma \\ I_F &:= \{\langle [t_1]_\Gamma, \ldots, [t_n]_\Gamma \rangle \mid F(t_1, \ldots, t_n) \in \Gamma\} \end{aligned}$$

Another straightforward induction then proves that for every t, $[\![t]\!]^{g'}_{S'} = [t]_\Gamma$.

We can then redo the step for atomic formulae in the Step Zero proof:

$$\begin{aligned} S', g' \Vdash F(t_1, \dots, t_n) &\text{ iff } \langle [\![t_1]\!]_{S'}^{g'}, \dots, [\![t_n]\!]_{S'}^{g'} \rangle \in I_F \\ &\text{ iff } \langle [t_1]_\Gamma, \dots, [t_n]_\Gamma \rangle \in I_F \\ &\text{ iff } F(t_1, \dots, t_n) \in \Gamma \end{aligned}$$

And we also have atomic identity formulae.

$S', g' \Vdash s = t$ iff $[\![s]\!]_{S'}^{g'} = [\![t]\!]_{S'}^{g'}$, iff $[s]_\Gamma = [t]_\Gamma$, iff $s = t \in \Gamma$.

The rest of the Step Zero proof goes through just as before.

Step Two: sets that are witness complete but not negation complete

Extensibility Lemma

Every consistent Γ has a consistent, negation-complete extension (i.e. superset).

Note that if Γ is witness-complete, so are all of its extensions; so given Step One, this implies that every witness-complete consistent set is satisfiable.

Proving the Extensibility Lemma

There are countably infinitely many formulae; enumerate them as P_0, P_1, P_2. Define a countably infinite sequence of sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ recursively as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{P_n\} & \text{if this is consistent} \\ \Gamma_n \cup \{\neg P_n\} & \text{otherwise} \end{cases}$$

Finally let Γ^+ be $\bigcup_n \Gamma_n$.

Γ^+ is negation-complete.

Each Γ_n is consistent (induction on n, using $\neg$Intro and Cut to get that if $\Gamma_n \vdash \neg P_n$ and $\Gamma_n, \neg P_n \vdash \bot$, then $\Gamma_n \vdash \bot$.).

By the compactness of provability, this implies that Γ^+ is consistent.

Step Three: sets that aren't witness-complete

Say that Γ is *abstemious* iff there is a countably infinite set $v_1, v_2, \ldots$ of variables that aren't free in any element of Γ.

There are only countably many pairs $\langle P, u \rangle$ of a formula P and variable u. Enumerate them as $\langle P_1, u_1 \rangle, \langle P_2, u_2 \rangle, \ldots$. We define another sequence of extensions of Γ, as follows:

$$\Gamma^0 := \Gamma$$

$$\Gamma^{n+1} := \begin{cases} \Gamma^n \cup \{P_n[v_n/u_n]\} & \text{if this is consistent} \\ \Gamma^n \cup \{\forall u_n \neg P_n\} & \text{otherwise} \end{cases}$$

Define $\Gamma' = \bigcup_n \Gamma^n$.

Γ' is obviously witness-complete.

To show that it's consistent, we show that each Γ^n is consistent. But this follows from $\forall$Intro and Cut.

Step Four

Now we have that every abstemious, consistent set of formulae is satisfiable. What about the case of a non-abstemious set?

This is a little fiddly to work through, but what we do is to pick some function f from variables to variables whose range excludes countably infinitely many variables, and turn it into a function f^* on [sets of] formulae in the obvious way. We check that if $\Gamma \vdash P$ then $f^*[\Gamma] \vdash f^*P$, and conclude that if Γ is satsifiable, $f^*[\Gamma]$ is abstemious and satisfiable. So there's an S, g such that $S, g \Vdash f^*[\Gamma]$. Finally, if we let $g^*(v) = g(fv)$, it is straightforward to show that $S, g^* \Vdash P$ iff $S, g \Vdash f^*P$; thus $S, g^* \Vdash \Gamma$.

Consequences of the Completeness Theorem

The Compactness Theorem

If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Proof:

- We have already noted the *compactness of provability*: if $\Gamma \vdash P$, then $\Gamma_0 \vdash P$ for some finite $\Gamma_0 \subseteq \Gamma$.
- So, if there is a proof of a contradiction from Γ, there is a proof of a contradiction from some finite $\Gamma_0 \subseteq \Gamma$.
- So by the completeness theorem, if Γ is unsatisfiable, there is a proof of a contradiction from some finite $\Gamma_0 \subseteq \Gamma$.
- So by the soundness theorem, if Γ is unsatisfiable, some finite $\Gamma_0 \subseteq \Gamma$ is unsatisfiable.

An application

Definition

A **theory** in a signature Σ is a set T of sentences (closed formulae) of Σ such that whenever P is closed and $T \vdash P$, $T \in \Gamma$.

Definition

When S is a structure for a signature Σ, $\mathrm{Th}\, S$, the **theory of** S is the set of all sentences (closed formulae) of Σ that are true in S.

Definition

True arithmetic, $\mathrm{Th}\, \mathbb{N}$ is the theory of the standard model of arithmetic (in the signature $0, \mathrm{suc}, +, \times, \leq$).

Let T be:

$$\mathrm{Th}\,\mathbb{N} \cup \{\neg(x = 0), \neg(x = suc(0)), \neg(x = suc(suc(0))), \ldots\}$$

Obviously every finite subset of T is satisfiable: just choose an assignment in the standard model of arithmetic that maps x to a big enough number.

So by the Compactness Theorem, T is satisfiable.

Any structure S in which T is true on some assignment must contain *non-standard* elements, that can't be reached from I_0 by any chain of applications of I_{suc}.

- In fact each such "non-standard model of arithmetic" has *many* non-standard elements: if gx is non-standard, so is $[\![\mathrm{suc}(x)]\!]^g$, since $\neg(\mathrm{suc}(x) = 0)$ and $\mathrm{suc}(x) = \mathrm{suc}(t) \to x = t$ must be true on every assignment (since they are logical consequences of $\mathrm{Th}\,\mathbb{N}$).

Isomorphism and elementary equivalence

Definition

When S and S' are structures for a signature Σ, an **isomorphism** from S to S' is a a bijection h from the domain of S to that of S' such that (i) for every n-ary predicate F of Σ, $I_F(x_1, \ldots, x_n)$ iff $I'_F(hx_1, \ldots, hx_n)$, and (ii) for every n-ary function symbol f of Σ, $I'_f(hx_1, \ldots, hx_n) = h(I_f(x_1, \ldots, x_n))$.

S is **isomorphic** to S' iff there exists an isomorphism from S to S'.

Definition

Two structures S and S' for a signature Σ are **elementarily equivalent** iff exactly the same sentences (closed formulae) of Σ are true in S as are true in S'.

It is easy to see that isomorphic models are always elementarily equivalent (show that if h is an isomorphism from S to S', then $S, g \Vdash P$ iff $S', h \circ g \Vdash P$). The converse is false.

Categorical theory

Definition

Theory T is **categorical** iff any two structures in which T is true are isomorphic.

The existence of non-standard models of arithmetic implies that $\mathrm{Th}\,\mathbb{N}$ is not categorical. It follows that no theory true in the standard model of arithmetic is categorical.

By contrast, it turns out that every structure elementarily equivalent to a *finite* structure is isomorphic to it. So the theory of a finite structure is categorical.

The Downward Löwenheim-Skolem Theorem

Notice that the structure constructed in our proof of the Completeness Theorem has a countable domain (since the set of terms, and hence any set of non-overlapping sets of terms, is countable). So it actually establishes the stronger fact that every consistent set of formulae has a *countable* model. Combining this with the Soundness Theorem, we get

Downward Löwenheim-Skolem Theorem

Every satisfiable set of formulae is satisfiable in a countable structure.

This implies that if ZFC is satisfiable, ZFC has a countable model.

Is there a paradox ("Skolem's paradox") here, given that ZFC proves the formalisation of "there are uncountable sets"?

Compare: the sentence $\exists x\, \mathrm{Red}(x)$ has a model whose domain is a set of non-red things. Does that mean that we can't use it to mean that something is red?

Full-strength Löwenheim-Skolem

The following are also true, though we won't prove them:

Downward Löwenheim-Skolem Theorem (strong form)

Every set of formulae that is satisfiable in some structure is satisfiable in some countable substructure of that structure.

Upward Löwenheim-Skolem Theorem

Every set of formulae that is satisfiable in an infinite structure is satisfiable in structures of arbitrary infinite cardinality.

How many countable non-standard models of true arithmetic are there?

Let P be the set of all prime numbers. For any $X \subseteq P$, let

$$T_X = \{\exists y(x = y \times \langle n \rangle) : n \in S\} \cup \{\neg \exists y(x = y \times \langle n \rangle) : n \in P \setminus S\}$$

For each S, every finite subset of $\mathrm{Th}\,\mathbb{N} \cup T_X$ is consistent, so by compactness and DLS $\mathrm{Th}\,\mathbb{N} \cup T_X$ has a countable model.

$T_X \cup T_{X'}$ is inconsistent unless $X = X'$. So for any countable model S of $\mathrm{Th}\,\mathbb{N}$, there are only countably many sets X of primes such that for some d in the domain, every member of T_X is true on an assignment where x is mapped to d.

If there were only countably many countable models of true arithmetic up to isomorphism, then only countably many sets X of prime numbers would be such that T_X has a countable model.

But there are uncountably many sets of prime numbers!

So: there are uncountably many non-isomorphic countable models of $\mathrm{Th}\,\mathbb{N}$.