3. *Dimension reduction*. A common belief is that most realistic data sets can be approximated by a *manifold*, or higher dimensional curved surface, in an even higher dimensional space. Investigate this for the MNIST data set of hand-drawn digits.

   A first cut at this is to perform an SVD on the data set. (You will first need to flatten the images into row or column vectors before performing the SVD.) How quickly do the singular values decrease?

   But translating even a single image over a part of the plane will result in a very high-dimensional vector space of images, without giving significantly more information. If we have a translation vector $\boldsymbol{b}$, so that an image is shifted like this $h(\boldsymbol{x}) \mapsto h(\boldsymbol{x}-\boldsymbol{b})$, you can treat $\boldsymbol{b}$ as a two-parameter vector that helps to describe a given image, resulting in an extra two dimensions for the manifold. Translate all the MNIST images so that their center of mass (or brightness) is at the center of the image frame. Re-try the SVD on this set if centered images. How quickly do the singular values decrease now? Can other transformations be used to reduce the apparent dimension of the data set? What happens if you separate out a single digit (such as "9" or "3")? Would nonlinear transformations help represent these hand-drawn digits?