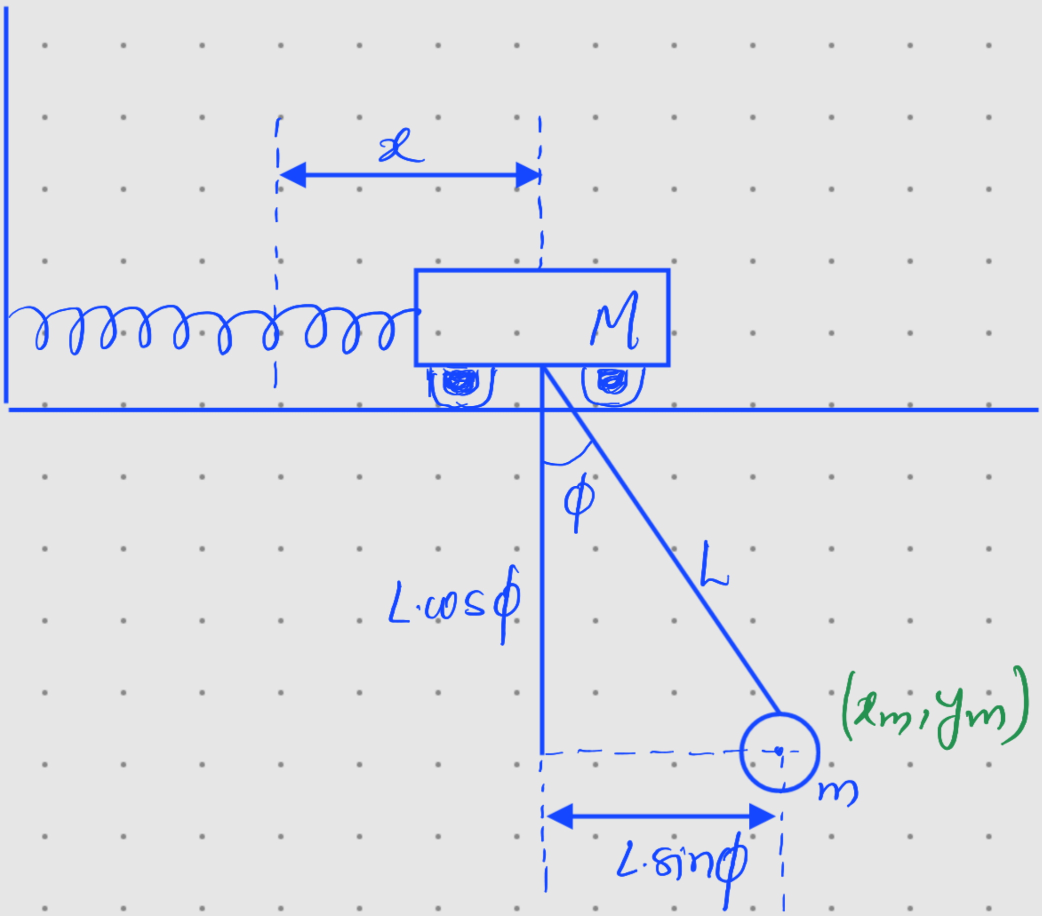
A Pendulum hanging from a Harmonic Oscillator



$$x_M = x$$

$$\left.\begin{aligned} x_m &= x + L\cdot\sin(\phi) \\ y_m &= -L\cdot\cos(\phi) \end{aligned}\right\} \quad \begin{aligned} \dot{x}_m &= \dot{x} + L\cdot\dot{\phi}\cdot\cos(\phi) \\ \dot{y}_m &= L\cdot\dot{\phi}\cdot\sin(\phi) \end{aligned}$$

$$\begin{aligned} v_m^2 = \dot{x}_m^2 + \dot{y}_m^2 &= \left\{\dot{x} + L\cdot\dot{\phi}\cdot\cos(\phi)\right\}^2 + L^2\cdot\dot{\phi}^2\cdot\sin^2(\phi) \\ &= \dot{x}^2 + 2\cdot\dot{x}\cdot L\cdot\dot{\phi}\cdot\cos(\phi) + L^2\cdot\dot{\phi}^2\cdot\cos^2(\phi) \\ &\quad + L^2\cdot\dot{\phi}^2\cdot\sin^2(\phi) \\ &= \dot{x}^2 + 2\cdot L\cdot\dot{x}\cdot\dot{\phi}\cdot\cos(\phi) + L^2\cdot\dot{\phi}^2 \end{aligned}$$

$$T = T_m + T_M = \frac{1}{2} m \cdot v_m^2 + \frac{1}{2} \cdot M \cdot v_M^2$$

$$= \frac{1}{2} \cdot m \cdot \left(\dot{x}^2 + 2 \cdot L \cdot \dot{x} \cdot \dot{\phi} \cdot \cos(\phi) + L^2 \cdot \dot{\phi}^2\right)$$

$$+ \frac{1}{2} \cdot M \cdot \dot{x}^2$$

$$= \frac{1}{2} \cdot m \cdot \dot{x}^2 + m \cdot L \cdot \dot{x} \cdot \dot{\phi} \cdot \cos(\phi) + \frac{1}{2} m \cdot L^2 \cdot \dot{\phi}^2$$

$$+ \frac{1}{2} \cdot M \cdot \dot{x}^2$$

$$V = \frac{1}{2} \cdot K \cdot x^2 - m \cdot g \cdot L \cdot \cos(\phi)$$

$$\Rightarrow \boxed{\begin{aligned} L = T - V &= \frac{1}{2} \cdot m \cdot \dot{x}^2 + m \cdot L \cdot \dot{x} \cdot \dot{\phi} \cdot \cos(\phi) + \frac{1}{2} \cdot m \cdot L^2 \cdot \dot{\phi}^2 \\ &+ \frac{1}{2} \cdot M \cdot \dot{x}^2 - \frac{1}{2} \cdot K \cdot x^2 + m g L \cdot \cos(\phi) \end{aligned}}$$

<u>Lagrangians</u>

$$\frac{\partial L}{\partial \dot{x}} = m \cdot \dot{x} + m \cdot L \cdot \dot{\phi} \cdot \cos(\phi) + M \cdot \dot{x}$$

$$= (m + M)\dot{x} + m \cdot L \cdot \dot{\phi} \cdot \cos(\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m + M)\ddot{x} + m \cdot L \cdot \ddot{\phi} \cdot \cos\phi - m \cdot L \cdot \dot{\phi}^2 \cdot \sin(\phi)$$

$$\frac{\partial L}{\partial \dot{\phi}} = m \cdot L \cdot \dot{x} \cdot \cos(\phi) + m \cdot L^2 \cdot \dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = m \cdot L \cdot \ddot{x} \cdot \cos(\phi) - m \cdot L \cdot \dot{x} \cdot \sin(\phi) \cdot \dot{\phi}$$

$$+ m \cdot L^2 \cdot \ddot{\phi}$$

$$\frac{\partial L}{\partial x} = -K\cdot x$$

$$\frac{\partial L}{\partial \phi} = -m\cdot L\cdot \dot{x}\cdot\dot{\phi}\cdot\sin(\phi) - m\cdot g\cdot L\cdot\sin(\phi)$$

Lagrangian eqns

for x:

$$(m+M)\ddot{x} + m\cdot L\cdot\ddot{\phi}\cdot\cos(\phi) - m\cdot L\cdot\dot{\phi}^2\cdot\sin(\phi) + Kx = 0$$

$$\Rightarrow \boxed{\ddot{x} = \frac{m}{m+M}\cdot L\cdot\dot{\phi}^2\cdot\sin(\phi) - \frac{m}{m+M}\cdot L\cdot\ddot{\phi}\cdot\cos(\phi) - \frac{K}{m+M}\cdot x}$$

for ϕ:

$$m\cdot L\cdot\ddot{x}\cdot\cos(\phi) \cancel{- m\cdot L\cdot\dot{x}\cdot\sin(\phi)\cdot\dot{\phi}} + m\cdot L^2\cdot\ddot{\phi}$$
$$\cancel{+ m\cdot L\cdot\dot{x}\,\dot{\phi}\cdot\sin(\phi)} + m\cdot g\cdot L\cdot\sin(\phi) = 0$$

$$\Rightarrow \cancel{m}\cdot L^{\cancel{2}}\cdot\ddot{\phi} + \cancel{m}\cdot g\cdot \cancel{L}\sin(\phi) + \cancel{m}\cdot\cancel{L}\cdot\ddot{x}\cdot\cos(\phi) = 0$$

$$\Rightarrow L\ddot{\phi} + g\cdot\sin(\phi) + \ddot{x}\cdot\cos(\phi) = 0$$

$$\Rightarrow \boxed{\ddot{\phi} = \frac{1}{L}\left(-g\cdot\sin(\phi) - \ddot{x}\cdot\cos(\phi)\right)}$$

for small angles: $\sin(\phi)\simeq\phi$ and $\cos(\phi)\simeq 1$

$$\Rightarrow Kx + m\cdot L[\ddot{\phi} - \dot{\phi}^2\cdot\phi] + (M+m)\ddot{x} = 0$$

$$\Rightarrow g\phi + \ddot{x} + L\ddot{\phi} = 0$$