Resource Allocation in OFDMA-based Wireless Networks- The 4^{th} Homework

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Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- Given a sub-channel allocation for a single user

$$P1.1: \min_{\boldsymbol{p}} \quad \sum_{k\in\mathcal{C}} p^k$$

$$\text{subject to.} \quad C1: \sum_{k\in\mathcal{C}} \log_2\Big(1+\boxed{\frac{a^k p^k h^k}{N^k}}\Big) \geq R^{\min},$$

$$C2: \; p^k \geq 0, \qquad \forall k \in \mathcal{C}.$$

- Problem P1.1 is a convex problem and same in UL and DL transmissions. This problem is optimally solved by water-filling method in [1]* and [2]†.

* J. Cioffi,“ Ee379c course reader, chapter 4,” http://www.stanford.edu/group/cioffi/ee379c, frequently updated

† P. He and L. Zhao,“Solving a class of sum-power minimization problems by generalized water-filling”, *IEEE Transactions on wireless Communication*, vol. 14, no. , pp. 6792-6804, 2015.

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- Water-filling method to address problem P1.1:
  - Problem P1.1 is convex and therefore its local solution is the unique optimal solution. Using Lagrangian method, the optimal solution to problem P1.1 can be obtained in polynomial time. The optimal solution is

$$p^k = \left[\lambda - \frac{1}{H^k}\right]^+, \tag{1}$$

    where $[x]^+ = \max(x, 0)$, $H^k = \dfrac{h^k}{N^k}$ is channel gain for user on sub-channel k and $\{H^k\}_{k=1}^{|\mathcal{C}|}$ is a sorted sequence with monotonically decreasing, and λ is the water-level chosen to satisfy the minimum target rate constraint C1 in P1.1.

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- Based on Lagrangian function, we have

$$\mathcal{L}(p^k,\mu) = \sum_{k=1}^{|\mathcal{C}|} p^k - \mu\Big(\sum_{k=1}^{|\mathcal{C}|} \log_2(1+p^kH^k) - R^{\min}\Big)$$

$$\bigtriangledown_{p^k}\mathcal{L}(p^k,\mu) = 1 - \mu\Big[\frac{H^k}{\ln 2(1+p^kH^k)}\Big] = 0$$

$$p^k = \frac{\frac{\mu H^k}{\ln 2} - 1}{H^k} \quad \longrightarrow \quad p^k = \Big[\lambda - \frac{1}{H^k}\Big]^+$$

where $\lambda = \dfrac{\mu}{\ln 2}$.

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- The value of λ is water level chosen to satisfy minimum target rate constraint with equality $\sum_{k=1}^{|\mathcal{C}|} \log_2(1+p^k H^k) = R^{\min}$.
- According to equation $p^k = (\lambda - \frac{1}{H^k})$, we have

$$R^{\min} = \sum_{k=1}^{|\mathcal{C}|} \log_2(1 + (\lambda - \frac{1}{H^k})H^k)$$

$$= \sum_{k=1}^{|\mathcal{C}|} \log_2(1 + (\lambda H^k - 1))$$

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- The rate constraint becomes

$$R^{\min} = \sum_{k=1}^{|\mathcal{C}|} \log_2(\lambda H^k) = \log_2\Big(\prod_{k=1}^{|\mathcal{C}|} \lambda H^k\Big) = \log_2\Big(\lambda^{|\mathcal{C}|} \prod_{k=1}^{|\mathcal{C}|} H^k\Big).$$

So, the value of λ is obtained by

$$\lambda = \left(\frac{2^{R^{\min}}}{\prod_{k=1}^{|\mathcal{C}|} H^k}\right)^{\frac{1}{|\mathcal{C}|}} \tag{2}$$

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- $p^k = \left[\lambda - \frac{1}{H^k}\right]^+$ implies that there exists k^* (called the optimal number of sub-channels with positive power) so that

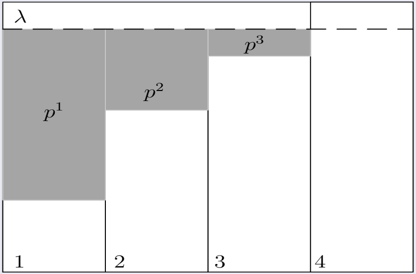
$$p^k = \begin{cases} \lambda - \frac{1}{H^k} & k \leq k^*, \\ 0 & k > k^*. \end{cases} \tag{3}$$

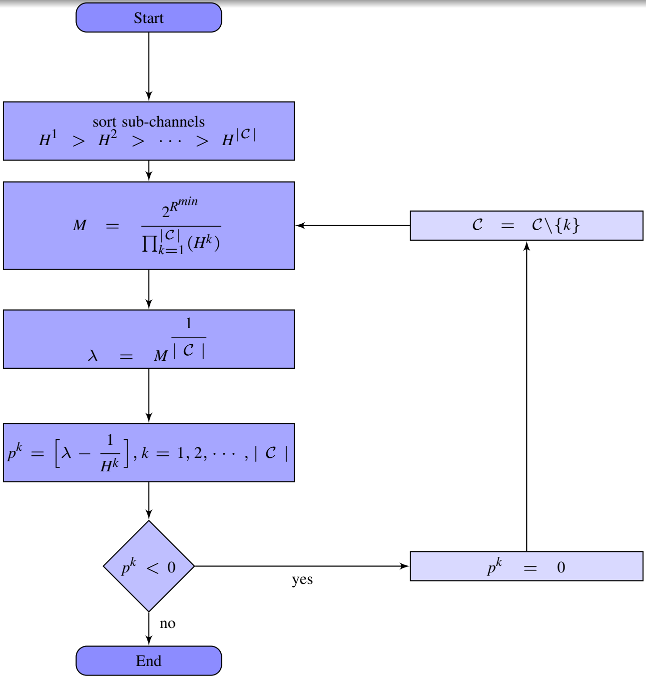
- For obtaining p^k in (3), we need to calculate the value of λ and k^*. Note that the method of obtaining these values affects the computational complexity.

Problem 1.1: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate Constraint for a Single-cell Single-user Network

Minimizing the aggregate power

- In water filling method, the dashed horizontal line, which is the water level λ, needs to be determined and then the powers levels at each sub-channel (water volume above the step) are obtained.




Start
sort sub-channels
$H^1 > H^2 > \cdots > H^{|\mathcal{C}|}$
$M = \frac{2^{R^{min}}}{\prod_{k=1}^{|\mathcal{C}|}(H^k)}$
$\mathcal{C} = \mathcal{C} \setminus \{k\}$
$\lambda = M^{\frac{1}{|\mathcal{C}|}}$
$p^k = \left[\lambda - \frac{1}{H^k}\right], k = 1, 2, \cdots, |\mathcal{C}|$
$p^k < 0$
yes
$p^k = 0$
no
End

Problem 1.2: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate and Mask Power Constraints for a Single-cell Single-user Network

Minimizing the aggregate power

- Given a sub-channel allocation

$$\begin{aligned} P1.2: \min_{\boldsymbol{P}} \quad & \sum_{k\in\mathcal{C}} p^k \\ \text{subject to.} \quad & C1: \sum_{k\in\mathcal{C}} \log_2\Big(1+\boxed{\frac{a^k p^k h^k}{\sigma^2}}\Big) \geq R^{min}, \\ & C2: \; 0 \leq p^k \leq p^{\max,k}, \quad \forall k \in \mathcal{C}. \end{aligned}$$

- The problem P1.2 is optimally solved by water-filling method in [2] * and [3] †.

* P. He and L. Zhao, "Solving a class of sum-power minimization problem by generalized water-filling," *IEEE Transactions on wireless Communication*, vol. 14, no. , pp. 6792-6804, 2015.

† S. Khakurel, C. Leung, and T. Le-Ngoc, "A generalized water-filling algorithm with linear complexity and finite convergence time," *IEEE Wireless Communications Letters*, vol. 3, no. 2, 2014.

Problem 1.2: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate and Mask Power Constraints for a Single-cell Single-user Network

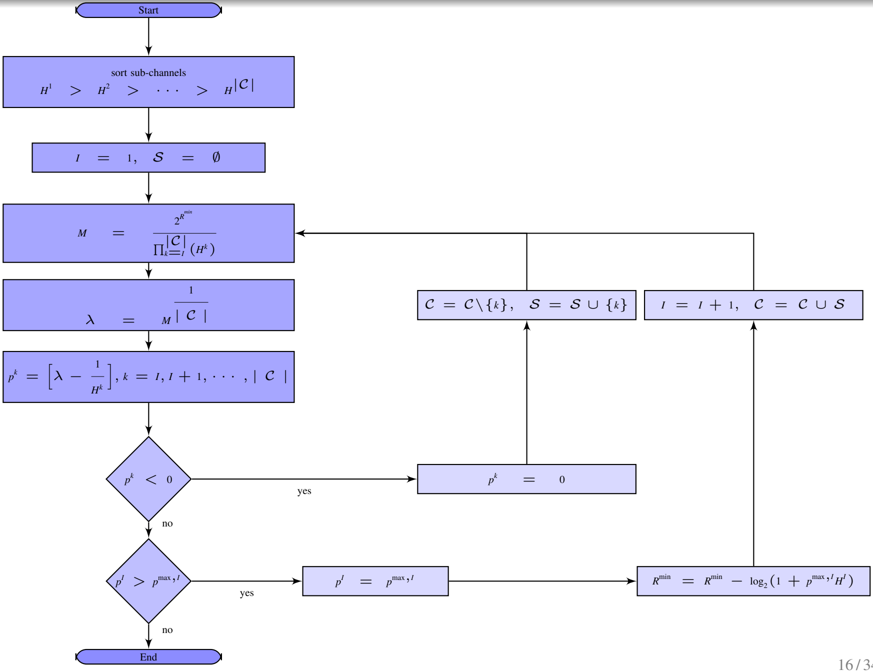
Minimizing the aggregate power

- In the water-filling method, transmit power on each sub-channel k is obtained by
$$p^k = \left[\lambda - \frac{1}{H^k}\right]^+_{p^{max,k}} \tag{4}$$
where $[x]^+ = \max(x, 0)$, $[x]_y = \min(x, y)$, and λ is given by (**??**).
- Obtaining transmit power on each sub-channel k is same as with solution of problem P1.1, but to satisfy the mask power constraint on each sub-channel k, following algorithm is employed.

Problem 1.2: Minimizing the Aggregate Transmit Power Subject to Minimum Data Rate and Mask Power Constraints for a Single-cell Single-user Network

Minimizing the aggregate power

1: **Initialization**:
2: $\{H^k\}_{k=1}^{|\mathcal{C}|}$, for $k = 1, 2, \cdots, |\mathcal{C}|$, $p^{\max,k}$, and $\mathcal{S} = \emptyset$.
3: Compute λ and p^k using flowchart on page 10 .
4: Add sub-channels with negative power level into $\mathcal{S}$ i.e., $\mathcal{S} = \{\forall k \in \mathcal{C} \mid p^k < 0\}$.
5: If $p^i > p^{\max,i}$, $p^i = p^{\max,i}$, $\mathcal{C} = \mathcal{C} \backslash \{i\}$.
6: $R^i = \log_2(1 + p^{\max,i} H^i)$, $R^{\min} = R^{\min} - R^i$.
7: $\mathcal{C} = \mathcal{C} \cup \mathcal{S}$, Then return to step 3.


Start
sort sub-channels
$H^1 > H^2 > \cdots > H^{|\mathcal{C}|}$
$I = 1, \ \mathcal{S} = \emptyset$
$M = \frac{2^{R^{min}}}{\prod_{k=I}^{|\mathcal{C}|}(H^k)}$
$\lambda = \frac{1}{M^{|\mathcal{C}|}}$
$p^k = \left[\lambda - \frac{1}{H^k}\right], k = I, I+1, \cdots, |\mathcal{C}|$
$p^k < 0$
yes
$p^k = 0$
$\mathcal{C} = \mathcal{C} \setminus \{k\}, \ \mathcal{S} = \mathcal{S} \cup \{k\}$
no
$p^I > p^{\max, I}$
yes
$p^I = p^{\max, I}$
$R^{\min} = R^{\min} - \log_2(1 + p^{\max, I} H^I)$
$I = I + 1, \ \mathcal{C} = \mathcal{C} \cup \mathcal{S}$
no
End

Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- Given a sub-channel allocation

$$\begin{aligned} P2.1: \max_{\boldsymbol{P}} \quad & \sum_{k\in\mathcal{C}} \log_2\left(1+\boxed{\frac{a^k p^k h^k}{N^k}}\right) \\ \text{subject to.} \quad & C1: \sum_{k\in\mathcal{C}} a^k p^k \leq p^{\max}, \\ & C2: \ p^k \geq 0, \qquad \forall k \in \mathcal{C}. \end{aligned}$$

- Problem P2.1 is a convex problem and same in UL and DL transmissions. This problem is optimally solved by water-filling method in [2]* and [4]†.

* P. He, L. Zhao, S. Zhou, and Z. Niu, "Water-Filling: A geometric approach and its application to solve generalized radio resource allocation problems," *IEEE Transaction on Wireless Communication.*, vol. 12, no. 7, pp. 3637-3647, 2013

† D. P. Palomar and J. R. Fonollosa, “Practical algorithms for a family of waterfilling Solutions,” *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 686-695, Feb 2005

Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- Water-filling method to address problem P2.1:
  - Problem P2.1 is convex and therefore its local solution is the unique optimal solution. Using Lagrangian method, the optimal solution to problem P2.1 can be obtained in polynomial time. The optimal solution is

    $$p^k = [\lambda - \frac{1}{H^k}]^+, \tag{5}$$

    where $[x]^+ = \max(x, 0)$, $H^k = \dfrac{h^k}{N^k}$ is channel gain for user on sub-channel k and $\{H^k\}_{k=1}^{|\mathcal{C}|}$ is a sorted sequence with monotonically decreasing, and λ is the water-level chosen to satisfy the maximum transmit power constraint C1 in P2.1.

Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- Based on Lagrangian function, we have

$$\mathcal{L}(p^k, \mu) = (\sum_{k=1}^{|\mathcal{C}|} \log_2(1 + p^k H^k)) + \mu(p^{\max} - \sum_{k=1}^{|\mathcal{C}|} p^k)$$

$$\bigtriangledown_{p^k} \mathcal{L}(p^k, \mu) = (\frac{1}{\ln 2})(\frac{H^k}{1 + p^k H^k}) - \mu = 0$$

$$p^k = (\frac{1}{\ln 2})(\frac{1}{\mu}) - \frac{1}{H^k} \quad \longrightarrow \quad p^k = \left[\lambda - \frac{1}{H^k}\right]^+$$

where $\lambda = \dfrac{1}{\mu \ln 2}$.

Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- The value of λ is water level chosen to satisfy maximum transmit power constraint with equality $\sum_{k=1}^{|\mathcal{C}|} p^k = p^{\max}$.
- According to equation $p^k = (\lambda - \frac{1}{H^k})$, we have

$$p^{\max} = \sum_{k=1}^{|\mathcal{C}|} (\lambda - \frac{1}{H^k})$$

$$= | \mathcal{C} | \lambda - \sum_{k=1}^{|\mathcal{C}|} \frac{1}{H^k}$$

Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- The maximum transmit power constraint becomes

$$p^{\max} = |\mathcal{C}|\lambda - \sum_{k=1}^{|\mathcal{C}|} \frac{1}{H^k}.$$

So, the value of λ is obtained by

$$\lambda = \frac{1}{|\mathcal{C}|}\left(p^{\max} + \sum_{k=1}^{|\mathcal{C}|} \frac{1}{H^k}\right). \tag{6}$$

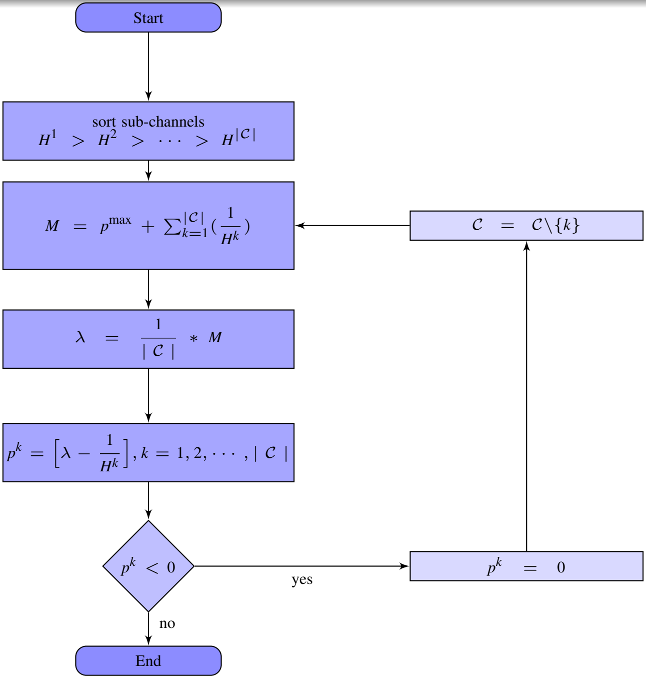
Problem 2.1: Maximizing the Aggregate Data Rate Subject to Maximum Power Constraint for a Single-cell Single-user Network

Maximizing the aggregate data rate

- Using water-filling method, in each iteration, the water-level λ is obtained by

$$\lambda = \frac{1}{\mid \mathcal{C} \mid}\Big(p^{\max} + \sum_{k=1}^{|\mathcal{C}|} \frac{1}{H^k}\Big), \tag{7}$$

- Based on λ, the power level on every sub-channel k is calculated by $p^k{=}\lambda{-}\dfrac{1}{H^k}$.
- If this obtained power violates the constraint C2 in P2.1 for a given sub-channel k, this sub-channel is removed from the set $\mathcal{C}$ i.e., $\mathcal{C}{=}\mathcal{C}\backslash\{k\}$.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sort sub-channels
$H^1 > H^2 > \cdots > H^{|\mathcal{C}|}$
$M = p^{\max} + \sum_{k=1}^{|\mathcal{C}|}(\frac{1}{H^k})$
$\mathcal{C} = \mathcal{C} \backslash \{k\}$
$\lambda = \frac{1}{|\mathcal{C}|} * M$
$p^k = \left[\lambda - \frac{1}{H^k}\right], k = 1, 2, \cdots, |\mathcal{C}|$
$p^k < 0$
yes
$p^k = 0$
no
End

Problem 2.2: Maximizing the Aggregate Data Rate Subject to Maximum Power and Mask Power Constraints for a Single-cell Single-user Network

Maximizing the aggregate data rate

- Given a sub-channel allocation

$$
\begin{aligned}
P2.2:\ \max_{\boldsymbol{P}} \quad & \sum_{k\in\mathcal{C}} \log_2\left(1+\boxed{\frac{a^k p^k h^k}{N^k}}\right) \\
\text{subject to.} \quad & C1:\ \sum_{k\in\mathcal{C}} a^k p^k \le p^{\max}, \\
& C2:\ 0 \le p^k \le p^{\max,k}, \qquad \forall k \in \mathcal{C}.
\end{aligned}
$$

- The problem P2.2 is optimally solved by water-filling method in [2]*.

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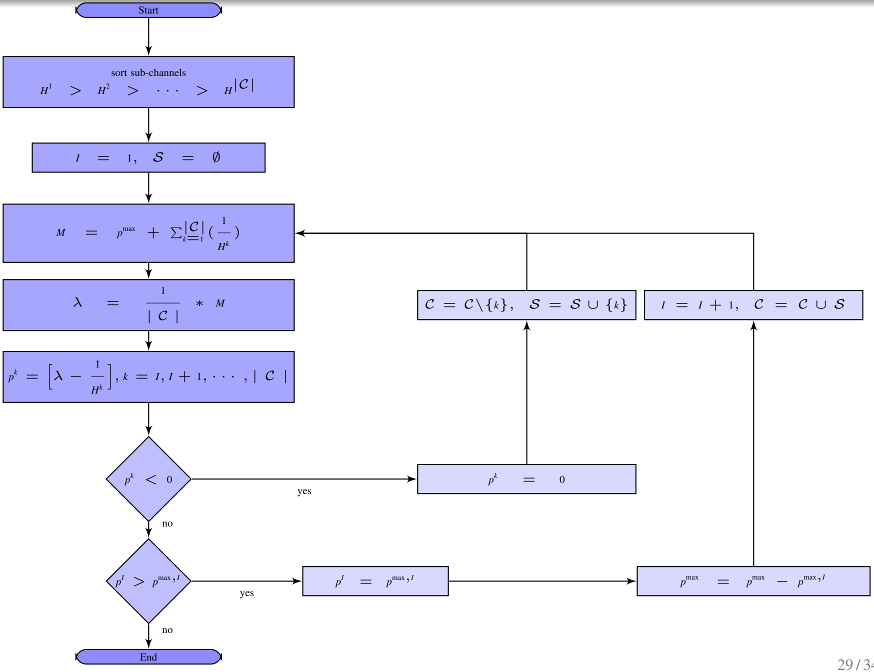
where $[x]^+ = \max(x, 0)$, $[x]_y = \min(x, y)$, and λ is given by (7).

- Obtaining transmit power on each sub-channel k is same as with solution of problem P2.1, but to satisfy the mask power constraint on each sub-channel k, following algorithm is employed.

Problem 2.2: Maximizing the Aggregate Data Rate Subject to Maximum Power and Mask Power Constraints for a Single-cell Single-user Network

Maximizing the aggregate data rate

1: **Initialization**:
2: $\{H^k\}_{k=1}^{|\mathcal{C}|}$, for $k = 1, 2, \cdots, |\mathcal{C}|$, $p^{\max,k}$, and $\mathcal{S} = \emptyset$.
3: Compute λ and p^k using flowchart on page 24 .
4: Add sub-channels with negative power level into $\mathcal{S}$ i.e., $\mathcal{S} = \{\forall k \in \mathcal{C} \mid p^k < 0\}$.
5: If $p^i > p^{\max,i}$, $p^i = p^{\max,i}$, $\mathcal{C} = \mathcal{C} \backslash \{i\}$.
6: $P^{\max} = P^{\max} - p^{\max,i}$.
7: $\mathcal{C} = \mathcal{C} \cup \mathcal{S}$, Then return to step 3.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$\mathcal{C} = \mathcal{C} \setminus \{k\}, \ \mathcal{S} = \mathcal{S} \cup \{k\}$
no
$p^I > p^{\max, I}$
yes
$p^I = p^{\max, I}$
$p^{\max} = p^{\max} - p^{\max, I}$
$I = I + 1, \ \mathcal{C} = \mathcal{C} \cup \mathcal{S}$
no
End

Homework Illustration

- In this homework, apply water-filling method to minimize aggregate transmit power and maximize aggregate throughput (data rate) in a simple wireless network with 1 user.
- This homework has two sections namely applying water-filling method to minimize aggregate transmit power and maximize aggregate throughput (data rate) in a simple wireless network.
- The following parameters are fixed for both cases:
  - Cell coverage area = 500 m $\times$ 500 m
  - Background noise power $\sigma^2 = 10^{-10}$ W
  - The channel path gains are based on the Simplified path loss model and Rayleigh fading model i.e., $h^k = x^k d^{-\alpha}$ *, where
    - d is the distance from the user to the BS,
    - x^k is the exponential random parameters with mean of 1, and
    - $\alpha = 3$ is the path loss exponent

- Generating x^k in matlab:

$$x = \text{exprnd}(\text{mean}, [1, |\ \mathcal{C}\ |])$$

example: $x = \text{exprnd}(1, [1, 5])$

$$x = [\ 0.3807\ ,\ 0.3509\ ,\ 0.8158\ ,\ 3.9334\ ,\ 1.1061\ ]$$

* For more information, please refer to A. Goldsmith, "Wireless Comunications," *Cambridge University Press*, 2005.

Applying Water-filling Method to Minimize Aggregate Transmit Power

Part. I

Q.1 Solve the following problem manually:

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{k=1}^{10} p^k \\ \text{subject to.} \quad & C1: \sum_{k \in \mathcal{C}} \log_2(1 + \frac{a^k p^k h^k}{N^k}) \geq 5 \text{ bps}\backslash\text{Hz}, \\ & C2: \; p^k \geq 0, \qquad k = 1, 2, \cdots, 10, \end{aligned} \tag{9}$$

where, $\boldsymbol{h} = \{h^k\}_{k=1}^{10} = [1.5, 10, 5, 2, 4, 8, 6.5, 3.5, 7, 6] * 10^{-10}$.
Determine the transmit power on each specific sub-channel.

Part. I

Q.2 Solve the above problem with considering peak power constraint on each SC k, i.e., $p^{\max,k} = 85$ mw

Part. I

Q.3 Compare and discuss the achieved aggregate transmit power in Q.1 and Q.2.

Applying Water-filling Method to Minimize Aggregate Transmit Power

Part. I

Q.4 Simulate water-filling algorithm to solve problem P1.1 in course slides where the minimum target rate is set to 50 bps\Hz (i.e., $R^{\min} = 50$ bps\Hz) with respect to varying number of sub-channels from 50 to 250 (i.e., $|\mathcal{C}| = \{50, 100, 150, 200, 250\}$).

Part. I

Q.5 Simulate water-filling algorithm to address problem P2.1 in course slides where the minimum target rate is set to 50 bps\Hz (i.e., $R^{\min} = 50$ bps\Hz) and peak power (mask power) on each sub-channel is set to 0.5 mW (i.e., $p^{\max,k} = 0.5$ mW) with respect to varying number of sub-channels from 50 to 250 (i.e., $|\mathcal{C}| = \{50, 100, 150, 200, 250\}$).

In Q.5 and Q.6, please achieve your results by averaging from at least 15 independent snapshots with a different location for the user in each snapshot.

Part. I

Q.6 Compare and discuss the results in Q.5 and Q.6. Explain the impacts of considering the mask power at each sub-channel on obtained aggregate transmit power.

Applying Water-filling Method to Maximize Aggregate Data Rate

Part. II

Q.1 Solve the following problem:

$$\begin{aligned} \max_{\boldsymbol{p}} \quad & \sum_{k=1}^{10} \log_2\left(1+\frac{a^k p^k h^k}{N^k}\right) \\ \text{subject to.} \quad & C1: \sum_{k=1}^{10} a^k p^k \leq 25 \text{ mW}, \\ & C2: p^k \geq 0, \qquad k=1,2,\cdots,10, \end{aligned} \tag{10}$$

where, $\boldsymbol{h} = \{h^k\}_{k=1}^{10} = [1.5, 10, 5, 2, 4, 8, 6.5, 3.5, 7, 6] * 10^{-10}$.
Determine the transmit power on each specific sub-channel.

Part. II

Q.2 Solve the above problem with considering peak power constraint on each SC k, $p^{\max,k} = 10$ mw

Part. II

Q.3 Compare and discuss the achieved aggregate data rate in Q.1 and Q.2.

Applying Water-filling Method to Maximize Aggregate Data Rate

Part. II

Q.4 Simulate water-filling algorithm to solve problem P2.1 in course slides where the number of sub-channels is set to 200 (i.e., $|\mathcal{C}| = 200$) with respect to varying maximum transmit power budget from 25 mW to 50 mW (i.e., $P^{\max} = \{25\text{ mW}, 30\text{ mW}, 35\text{ mW}, 40\text{ mW}, 45\text{ mW}, 50\text{ mW}\}$).

Part. II

Q.5 Simulate water-filling algorithm to address problem P2.2 in course slides where the number of sub-channels is set to 200 (i.e., $|\mathcal{C}| = 200$) and peak power on each sub-channel is set to 0.5 mW (i.e., $p^{\max,k} = 0.5$ mW) with respect to varying maximum transmit power budget from 25 mW to 50 mW (i.e., $P^{\max} = \{25\text{ mW}, 30\text{ mW}, 35\text{ mW}, 40\text{ mW}, 45\text{ mW}, 50\text{ mW}\}$).

In Q.5 and Q.6, please achieve your results by averaging from at least 15 independent snapshots with a different location for the user in each snapshot.

Part. II

Q.6 Compare and discuss the results in Q.5 and Q.6. Explain the impacts of considering the mask power at each sub-channel on obtained aggregate data rate.