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# Solving A Class of Sum Power Minimization Problems by Generalized Water-Filling

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***Abstract*—Radio Resource Management (RRM) plays an important role in wireless communication systems, especially in more advanced systems with more constraint conditions. In this paper, we first propose a generalized water-filling approach to solve the power allocation problem of minimizing sum power while meeting the target sum rate constraint with weights. Based on this sum power objective function, we extend the proposed method to more complicated RRM problems with more stringent constrains. The proposed algorithms with this generalized approach possess several distinguished features. They provide exact optimal solutions based on non-derivative methods, as the implementation of the proposed algorithms invokes neither the derivative nor the gradient. With geometric interpretation, the proposed algorithms provide more insights to and intuitions of the problems and could be used to efficiently solve a family of the sum power minimization problems. Optimality of the proposed algorithms is strictly proved. Numerical results that illustrate the steps and demonstrate efficiency of the proposed algorithms are presented.**

**Keywords**

**Water-filling, minimum sum power, maximum sum data rate, channel capacity, optimal radio resource management (RRM), QoS, optimization methods.**

## I. Introduction

Transmit power and data rates are two fundamental radio resources. The required Quality of Service (QoS) can be satisfied by adjusting either power or rate or both in a complementary way. The optimal allocation of these scarce radio resources for different users/channels directly affects system performance. Therefore, for radio resource management (RRM) with different target and/or priorities, the problems or mathematical models may be formulated through maximizing the objective function of transmission throughput (or sum rate), subject to the sum power constraint; or through minimizing the objective function of sum power, subject to the throughput constraint. In this paper, we simply refer to the former problem as the original RRM problem, and the latter one as the (sum) power problem to distinguish itself from the original RRM problem.

### A. Background: Original Problem

The water-filling principle has been applied to efficiently compute the solutions of the original RRM problem. As an analogy, it is similar to the process of pouring a fixed amount of water into a tank, the bottom of which has the stair levels that are analogous to the inverse of the sub-channel gains. This principle can be extended to deal with an array of RRM problems [1] -[10]. Most of these references are focused on the throughput maximization problems, *i.e.*, the original RRM problem. In our recent paper [11], we proposed the generalized water-filling (GWF) algorithm to solve the original RRM problem, and further extended GWF to solve the more general RRM problems with more complicated constraints, such as individual peak power constraints (WFPP), group peak power constraints (WFGPP), and group bounded power constraints (WFGBP). This paper develops based on the concept of peak power constraints from [12] and its following works, where the upper bound of (average) powers is the constraints under investigation. The complication level increases with more constraints. These general RRA problems can find their applications in the advanced communication systems. For example, for WFGBP, the fact that the transmit powers of the groups of the mobile users are bounded leads to group bounded power constraints. For different WFPP problems, similar constraints are investigated in [13]-[15] with different approaches.

### B. Our Work for Sum Power Minimization Problem

In this paper, a set of new problems are formed by changing the throughput of objective function, in the original RRM problem, into the sum power; while by changing the sum power constraint, in the original RRM problem, into the throughput constraint. At the same time, we will use the operator of minimization for this set of new problems. Thus, this set of new problems are readily distinguishable from the original RRM problems mentioned above. Correspondingly, this set of problems, with the objective functions having (sum) power forms, are referred to as P-GWF, P-WFPP, P-WFGPP, and P-WFGBP respectively here. These problems have their practical applications. For example, when we consider different class of QoS services, the constraint of the different lower and upper bounds of the power allocation is reflected in our target problems. Without loss of clarity, we will use the same set of abbreviations to denote the problems and the algorithms that solve the corresponding problems.

The solution to the sum power problem has not been studied as well as the original problem for throughput maximization in the open literature. For the sum power problem, the constraint

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is to meet the sum rate, which is not a linear function of the allocated power. Normally, this sum power problem is more difficult to solve than the original RRM problem due to the difference in the problem structure. The sum power problem in its most basic form, or simple extension of it with equally weighted cases has been previously presented in [16] and [17] respectively. The solution can be obtained by finding the water level(s) through solving a non-linear system in the parameter of the water level(s). This non-linear system, in the simplest form, consists of an equation determined by the sum of non-linear composite expressions that includes a logarithmic operation and an inequality for the non-negativity of the water-level.

In this paper, we significantly extend beyond earlier proposed approach [11] to solve the sum power RRM problems, including the basic form (P-GWF) and its extended and generalized forms. The stated generalized approach has a distinguished feature that the proposed algorithms start from geometric interpretations of the target problems. These geometric interpretations and the relationships they formed provide more insights into the problems; and such insights assist us to efficiently solve the target problems with optimal solutions. The proposed algorithms possess simple procedures due to the fact that the proposed algorithms belong to the non-derivative methods (which have been defined in [18] and [19]) that use neither the derivative nor the gradient during theirs implementation. The proposed algorithm P-GWF for the basic sum power problem has two advantages: it provides the exact solution, and thus eliminates the iterative steps of finding the water level through solving the non-linear system. On the other hand, the machinery of the proposed approach enabled us to solve the more generalized RRM problems with more stringent constraints. In our numerical examples, it is shown that with optimal power allocation for the generalized RRM problems, the water levels are different for the different constraints of lower and upper bounds. The conventional approach of determining the water level(s) might not be able to solve this kind of generalized problems. The difference between our approach and those of others is summarized below.

*First*, for the simple case of P-GWF problem that can be solved through the conventional approach, it is generalized into a weighted case in this paper. Together, the corresponding algorithm with less computation is also proposed.

*Second*, for the more complicated P-WFPP, P-WFGPP, and P-WFGBP cases, their solutions cannot be computed exactly by the conventional approach, but these (optimal) solutions can be computed exactly by our approach in this paper.

*Third*, for each of these problems, our approach only takes a low degree polynomial computational complexity for the exact solution, unlike the popular primal-dual interior point method (PD-IPM) that only computes an $\epsilon$ solution, which is not an optimal solution and requires more computations (refer to [20], [21], and references therein).

*Fourth*, for the mentioned problems of P-WFPP, P-WFGPP, and P-WFGBP, similar results have not been reported in the open literature, to the best of the authors' knowledge. For example, [11] and [22] provided efficient algorithms for some RRM problems that have different structures, unlike the structure of these target problems in this paper. The approaches discussed in [11] and [22] cannot solve the target problems in this paper. The proposed algorithms are novel and efficient.

In the remaining of the paper, the problem statement, the conventional approach, and the preparation or the illustration for the proposed P-GWF are discussed in Section II. The extended and generalized sum power RRM problems with additional stringent constraints are further investigated in Section III. Numerical examples and complexity analysis are presented in Section IV. Section V concludes the paper. Appendices provide the strict optimality proofs for the extended and generalized algorithms to compute the minimum sum power problems.

## II. Generalized Water-Filling for Sum Power Problem

### A. Problem Statement and Conventional Approach

The *original basic RRM problem* can be described by the following: given $P > 0$, which is the total power or volume of the water; the allocated power and the propagation path gain of the $i$th channel, which are denoted by $s_i$ and $a_i$ respectively, $i = 1 \dots K$; and $K$ which is the total number of channels, letting $\{a_i\}_{i=1}^K$ be a sorted sequence with monotonically decreasing (the indexes can be arbitrarily renumbered to satisfy this condition), in which $a_i > 0, \forall i$, find a group of the powers $\{s_i\}$ to satisfy:

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^K} & \frac{1}{2}\sum_{i=1}^K \log_2(1+a_i s_i) \\ \text{subject to:} & \sum_{i=1}^K s_i = P; 0 \le s_i, \ \forall i. \end{array} \tag{1}$$

Extensive investigation to solve problem (1) has been reported in the open literature. Using a geometrical approach to solve this problem has been discussed in our earlier paper [11].

The *basic sum power RRM problem* can be stated as: given $B > 0$, which denotes the number of the target transmission bits (or sum rate of the system), find a group of the powers $\{s_i\}$ to satisfy:

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K s_i \\ \text{subject to:} & \sum_{i=1}^K \frac{1}{2}\log_2(1+a_i s_i) = B; 0 \le s_i, \ \forall i. \end{array} \tag{2}$$

Note that if only the first constraint is substituted with $\frac{1}{2}\sum_{i=1}^K \log_2(1+a_i s_i) \ge B$, the new problem, as a convex optimization problem, is equivalent to (2). Solving problem (2) is important, especially when saving energy/power is indeed the first priority of the system design.

Problem (1) is to solve the throughput maximization problem; while problem (2) is to solve the sum power minimization problem. Generally, the solution to (1) can not be directly applied to (2). This paper focuses on the investigation of the solution to (2) and its extended and more generalized forms.

To find the solution to problem (2), conventional approach usually starts from the equivalent form of the Karush-Kuhn-

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Tucker (KKT) conditions of problem (2). This equivalent form is:

$$\begin{cases} s_i = \left(\mu - \frac{1}{a_i}\right)^+, \text{ for } i = 1, \ldots, K; \\ \sum_{i \in \{l|\mu - \frac{1}{a_l} \geq 0,\ 1 \leq l \leq K\}} \log_2(1 + a_i s_i) = 2B; \mu \geq 0, \end{cases} \tag{3}$$

where $(x)^+ = \max\{0, x\}$. $\mu$ is the water level chosen to satisfy the sum rate constraint with equality ($\frac{1}{2}\sum_{i=1}^{K} \log_2(1 + a_i s_i) = B$).

Enumeration can be utilized to find the water level $\mu$ in (3). This statement means that solving the equation in $\mu$:

$$\sum_{i \in \{l|\mu - \frac{1}{a_l} \geq 0,\ 1 \leq l \leq k\}} \log_2\left(1 + a_i\left(\mu - \frac{1}{a_i}\right)\right) = 2B, \tag{4}$$

can find the water level $\mu$ in (3), where the index $k$ runs up from 1 to $K$. Further, after the index $k$ only runs some steps that are not greater than $K$, due to $\{\frac{1}{a_l}\}$ keeping monotonicity, the solution to (4) can be obtained. That is to say, we may use fewer steps to find $\mu$. This algorithm or solution to (3) is referred to as the conventional sum power water-filling algorithm or solution, denoted by P-CWF. The detail can be furthermore referred to in [16]. Since P-CWF results from the motivation to solve the system directly, it is a non-geometric approach.

### *B. Illustration of the Proposed Generalized Water-Filling Algorithm (P-GWF)*

In this section, we apply our proposed generalized water-filling methodology [11] to solve problem (2). Similar to GWF [11], Figs. 1 (a)-(c) illustrate the proposed P-GWF algorithm for the sum power problems. Suppose there are 4 steps/stairs ($K = 4$) with unit width inside a water tank. In the conventional approach, the dashed horizontal line, which is the water level $\mu$, needs to be determined first and then the powers (water volume above the step) are solved.

Let us use $d_i$ to denote the "step depth" of the $i$th stair which is the height of the $i$th step to the bottom of the tank, and is given below:

$$d_i = \frac{1}{a_i}, \text{ for } i = 1, 2, \ldots, K. \tag{5}$$

Since the sequence $\{a_i\}$ is sorted with monotonically decreasing, the step depth of the stairs indexed by $\{1, \cdots, K\}$ is monotonically increasing.

Instead of trying to determine the water level $\mu$, which is a real nonnegative number, we aim to determine the highest (shallowest) water level step under water, which is an integer number between 1 and $K$, and denoted it by $k^*$. Based on the result of $k^*$, we can write out the solutions for power allocation in problem (2) instantly.

Fig. 1(a) illustrates the concept of $k^*$. Since the third level is the highest level under water, we have $k^* = 3$. The shaded area denotes the allocated power for the third step by $s_3^*$.

*We define the achieved data rate using power below step $k$ by $R(k)$ that can be expressed by*

$$\begin{aligned} R(k) &= \frac{1}{2}\sum_{i=1}^{k-1} \log_2\left(1 + a_i \overline{s}(k)_i\right) \\ &= \frac{1}{2}\sum_{i=1}^{k-1} \log_2\left[1 + a_i\left(\frac{1}{a_k} - \frac{1}{a_i}\right)\right] \\ &= \frac{1}{2}\sum_{i=1}^{k-1} \log_2\left[\frac{a_i}{a_k}\right] = \frac{1}{2}\log_2\left[\Pi_{i=1}^{k-1}\left(\frac{a_i}{a_k}\right)\right], \end{aligned} \tag{6}$$

where $\overline{s}_i(k) = \frac{1}{a_k} - \frac{1}{a_i}$ in $R(k)$. To include the case of $k = 1$ in (6), we define the two special cases as follows: $\sum_{i=m}^{n} b_i = 0$, for $m > n$, and

$$\Pi_{i=m}^{n} b_i = 1, \text{ for } m > n, \tag{7}$$

where $\{b_i\}$ is assumed to be a general number sequence.

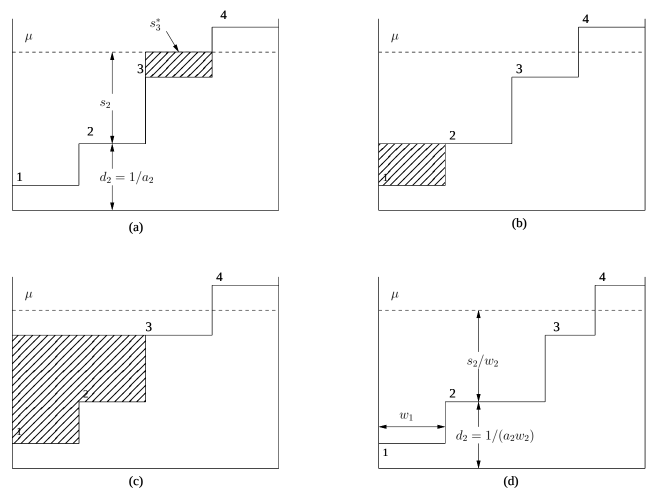

Fig. 1. Illustration for the proposed generalized Water-Filling Algorithm for the sum power problems (P-GWF). (a) Water level step $k^* = 3$, allocated power for the third step $s_3^*$, and step/stair depth $d_i = 1/a_i$. (b) $ER(k)$ (which is determined by the shadowed area, representing the total water/power, up to, but excluding step $k$) when $k = 2$. (c) $ER(k)$ when $k = 3$. (d) The weighted case.

*Let $ER(k)$ denote the Exponential Rate function achieved with the power below step $k$, which can be written by*

$$ER(k) = 2^{2R(k)} = \Pi_{i=1}^{k-1}\left(\frac{a_i}{a_k}\right), \quad \text{for} \quad k = 1, \ldots, K, \tag{8}$$

where the factor "2" in the exponent comes from the fractional coefficient before the sum of logarithm functions in (6). Further the exponential rate target given is defined by $\eta$: $\eta = 2^{2B}$.

In the following, we explain how to find the water level step $k^*$ without the knowledge of the water level $\mu$. Fig. 1(b) and Fig. 1(c) illustrate the concepts of $ER(k)$ achieved by the power from the shadowed area for the cases of $k = 2$ and $k = 3$ respectively. As an example of Fig.1(c), the water volume under step 3 can be expressed as the sum of the two terms: (i) the step depth difference between the 3rd and the 1st step, $(1/a_3 - 1/a_1)$, and (ii) the step depth difference between

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the 3rd and the 2nd step, $(1/a_3 - 1/a_2)$. Thus, the achieved data rate using power under the 3rd step can be written as

$$\begin{aligned} R(k=3) &= \frac{1}{2}\log_2\left[1 + a_1\left(\frac{1}{a_3} - \frac{1}{a_1}\right)\right] \\ &+ \frac{1}{2}\log_2\left[1 + a_2\left(\frac{1}{a_3} - \frac{1}{a_2}\right)\right] \\ &= \frac{1}{2}\log_2\left[\frac{a_1 a_2}{a_3^2}\right]. \end{aligned} \tag{9}$$

Therefore, the corresponding $ER(k=3)$ is given by

$$ER(k=3) = 2^{2\cdot R(k=3)} = \frac{a_1 a_2}{a_3^2}, \tag{10}$$

which is an expansion of the composite form of (8).

To clearly understand the procedures of the proposed algorithm, the line of the methodology is briefly summarized by the following: using $\{\frac{1}{a_k} - \frac{1}{a_i}\}_{i=1}^{k-1}$, for $k = 1, \ldots, K$, to define the achieved data rate or the exponential rate sequence in $k$; using the exponential rate sequence to determine the highest step $k^*$; and then this $k^*$ is used to compute the optimal solution to the target problem. *Thus, $\{\frac{1}{a_k} - \frac{1}{a_i}\}_{i=1}^{k-1}, \forall k$, is not guaranteed to be the optimal solution. It is only utilized for computing the optimal solution in this proposed algorithm.*

The explicit solution, on the other hand, is optimal, and its optimality proof to (2) will be introduced, as an instance of the generalized case in following subsection.

### C. Extend to Weighted Case

For the weighted or generalized case, an extended problem can be stated as: given the weighted coefficients $w_i > 0, \forall i$, associated with $\{a_i w_i\}_{i=1}^K$ which are assumed to be in decreasing order (similar to the case in Subsection II-A, the indexes can be arbitrarily renumbered to satisfy this condition), find a group of the powers $\{s_i\}$ which are the solutions to the following problem,

$$\begin{aligned} &\min_{\{s_i\}_{i=1}^K} \quad \sum_{i=1}^K s_i \\ &\text{subject to:} \quad \tfrac{1}{2}\sum_{i=1}^K w_i \log_2(1 + a_i s_i) = B; 0 \le s_i,\ \forall i. \end{aligned} \tag{11}$$

In Fig. 1(d), the width of the $i$th stair/step is denoted by $w_i$. The term $s_i$ denotes the allocated power represented by the area above step $i$ under water. The value of $1/a_i$ denotes the area, under the $i$th step to the bottom of the tank. Hence, the step depth of the $i$th step is given by

$$d_i = 1/(a_i w_i), \quad i = 1, \cdots, K. \tag{12}$$

Then $R(k)$ can be expressed by $R(k) =$

$$\begin{aligned} &= \frac{1}{2}\sum_{i=1}^{k-1} w_i \log_2\left[1 + a_i w_i\left(\frac{1}{a_k w_k} - \frac{1}{a_i w_i}\right)\right] \\ &= \frac{1}{2}\sum_{i=1}^{k-1} w_i \log_2\left[\frac{a_i w_i}{a_k w_k}\right] \\ &= \frac{1}{2}\sum_{i=1}^{k-1} \log_2\left[\frac{a_i w_i}{a_k w_k}\right]^{w_i}. \end{aligned} \tag{13}$$

The corresponding exponential rate function is

$$ER(k) = 2^{2R(k)} = \Pi_{i=1}^{k-1}\left(\frac{a_i w_i}{a_k w_k}\right)^{w_i}, \text{ for } k = 1, \ldots, K. \tag{14}$$

Based on these extended definitions, we have the following proposition to compute the solution to (11).

**Proposition 2.1.** The explicit solution, by finite amounts of computation, to (11) is:

$$\begin{aligned} s_i &= \left[\frac{s_{k^*}}{w_{k^*}} + (d_{k^*} - d_i)\right] w_i, && \text{for } 1 \le i \le k^*; \\ \text{and } s_i &= 0, && \text{for } k^* < i \le K, \end{aligned} \tag{15}$$

where

$$k^* = \max\left\{k \middle| ER(k) < \eta,\ \ 1 \le k \le K\right\} \tag{16}$$

with $ER(k)$ defined by (14), the power level for this step is

$$s_{k^*} = \frac{1}{a_{k^*}}\left[\left(\frac{\eta}{ER(k^*)}\right)^{\frac{1}{\sum_{i=1}^{k^*} w_i}} - 1\right] \tag{17}$$

and the optimal sum power allocated in (11) is:

$$P^* = \sum_{i=1}^{k^*}\left[\frac{1}{a_{k^*} w_{k^*}} \cdot \left(\frac{\eta}{ER(k^*)}\right)^{\frac{1}{\sum_{i=1}^{k^*} w_i}} - \frac{1}{a_i w_i}\right] w_i. \tag{18}$$

Prior to the formal proof, we first show how $s_{k^*}$ in (17) is obtained, and whether $\{s_i\}$ in (15), including $s_{k^*}$, is a feasible solution to (11). Using (15) and (16), the first constraint, *i.e.*, the rate constraint, of (11) leads to the following equation in $s_{k^*}$:

$$\begin{aligned} B &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2(1 + a_i s_i) \\ &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2[1 + a_i w_i(\tfrac{s_{k^*}}{w_{k^*}} + \tfrac{1}{a_{k^*} w_{k^*}} - \tfrac{1}{a_i w_i})] \\ &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2[a_i w_i(\tfrac{s_{k^*}}{w_{k^*}} + \tfrac{1}{a_{k^*} w_{k^*}})] \\ &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2\{\tfrac{a_i w_i}{a_{k^*} w_{k^*}}[a_{k^*} w_{k^*}(\tfrac{s_{k^*}}{w_{k^*}} + \tfrac{1}{a_{k^*} w_{k^*}})]\} \\ &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2(\tfrac{a_i w_i}{a_{k^*} w_{k^*}}) \\ &+ \tfrac{1}{2}[\log_2(1 + a_{k^*} s_{k^*})]\sum_{i=1}^{k^*} w_i. \end{aligned} \tag{19}$$

So, this equation in $s_{k^*}$ is further simplified as below:

$$\begin{aligned} B &= \tfrac{1}{2}\sum_{i=1}^{k^*} w_i \log_2(\tfrac{a_i w_i}{a_{k^*} w_{k^*}}) \\ &+ \tfrac{1}{2}[\log_2(1 + a_{k^*} s_{k^*})]\sum_{i=1}^{k^*} w_i. \end{aligned} \tag{20}$$

Finally, (17) is obtained by solving this equation: (20), and then $\{s_i\}$ in (15) is the feasible solution.

**Proof of Proposition 2.1.** The formal proof is stated as follows. System (15) implies that

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} = \frac{w_i}{\frac{1}{a_i} + s_i}, \text{ for } 1 \le i \le k^*. \tag{21}$$

Let

$$\lambda = \left(\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}}\right)^{-1}. \tag{22}$$

From geometric view, $\lambda$ itself is the water level $\mu$. According to the definitions of $k^*$ and $s_{k^*}$, for $k^* < i \le K$, $\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} \ge \frac{w_i}{\frac{1}{a_i} + s_i}$ and $s_i = 0$. This statement can be explained as follows:

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$s_i = 0$ first comes from (15). To show the inequality, assume to the contrary that, for $k^* < i \le K$, $\frac{w_{k^*}}{\frac{1}{a_{k^*}}+s_{k^*}} < \frac{w_i}{\frac{1}{a_i}+s_i}$ holds. That is to say, $\frac{1}{a_{k^*}w_{k^*}} + \frac{s_{k^*}}{w_{k^*}} > \frac{1}{a_{k^*+1}w_{k^*+1}}$. Then,

$$\begin{aligned} & 1 + a_i w_i \left(\frac{1}{a_{k^*}w_{k^*}} + \frac{s_{k^*}}{w_{k^*}} - \frac{1}{a_i w_i}\right) \\ > \quad & 1 + a_i w_i \left(\frac{1}{a_{k^*+1}w_{k^*+1}} - \frac{1}{a_i w_i}\right), \forall i. \end{aligned} \tag{23}$$

The logarithm operation is applied to both sides of the inequality; and then the summation operation with the weights is applied with the index running from 1 to $k^*$. Finally put the exponentiation on the last result of both sides. As a result, $\eta > ER(k^*+1)$. However, a contradiction of the maximum $k^*$ not being maximum is acquired. Therefore, for $k^* < i \le K$, $\frac{w_{k^*}}{\frac{1}{a_{k^*}}+s_{k^*}} \ge \frac{w_i}{\frac{1}{a_i}+s_i}$.

Let $\sigma_i = 1 - \lambda \frac{w_i}{\frac{1}{a_i}+s_i}$. Then

$$\sigma_i \ge 0, \text{ for } k^* < i \le K; \sigma_i = 0, \text{ for } 1 \le i \le k^*. \tag{24}$$

Therefore, the following system holds:

$$\begin{cases} 1 - \frac{\lambda w_i}{\frac{1}{a_i}+s_i} - \sigma_i = 0, & \text{for } 1 \le i \le K; \\ s_i \ge 0, \sigma_i s_i = 0, \sigma_i \ge 0, & \forall i; \\ \sum_{i=1}^K w_i \log_2(1 + a_i s_i) \ge 2B, & \\ \lambda \left[\sum_{i=1}^K w_i \log_2(1 + a_i s_i) - 2B\right] = 0, & \\ \lambda \ge 0. & \end{cases} \tag{25}$$

By observation, the equation and inequality set above is a set of the KKT conditions of the problem in (11) and the water level $\mu$ is equal to the Lagrange multiplier $\lambda$ mentioned above. Note that the Lagrange function of the problem (11) is

$$\begin{aligned} & L(\{s_i\}, \lambda, \{\sigma_i\}) \\ = \quad & \textstyle\sum_{i=1}^K s_i - \lambda \left[\sum_{i=1}^K w_i \log_2(1 + a_i s_i) - 2B\right] \\ - \quad & \textstyle\sum_{i=1}^K \sigma_i s_i. \end{aligned} \tag{26}$$

Since problem (11) is, in essence, a differentiable convex optimization problem, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. We observe that the General Constraint Qualification (refer to (3.71) of Theorem 3.8 in [19]) of the problem holds. This (3.71) is often abbreviated as GCQ, for which, it is seen that Slater's condition [21] is a special case and implies GCQ. Proposition 2.1 hence is proved.

**Remark 2.1.** Proposition 2.1, at the first line in the formal proof, stated that $\{s_k\}_{k=1}^K$ in (15) implies (21), then the Lagrange multipliers are constructed by (22) and (24), and it is seen that for $\{s_k\}_{k=1}^K$ in (15), there exists the group of Lagrange multipliers constructed above to satisfy the KKT conditions of the problem (11). Therefore, according to optimization theory, $\{s_k\}_{k=1}^K$ in (15) is the optimal solution to problem (11). Within the statement of Proposition 2.1, it is worth mentioning that (14) determines $ER(k)$, then obtains $k^*$ by (16), and further $s_{k^*}$ by (17).

Thus, the first step is to calculate $ER(k)$, then find the water level step, $k^*$ from (16), which is the maximal index of $ER(k) < \eta$. The corresponding power level for this step, $s_{k^*}$, can be obtained by applying (17). Then for those steps with index higher than $k^*$, the power level is assigned with zero. For those steps below $k^*$, the power level is assigned by the first expression in (15). The first term $(s_{k^*}/w_{k^*})$ inside the square bracket denotes the depth of the $k^*$th step to the water level. The second term inside the square bracket denotes the step depth difference between the $k^*$th step and the $i$th step. Therefore, the sum inside the square bracket means the depth of the $i$th step to the water level. When this quantity is multiplied by the width of this step, the volume of the water above this step (allocated power) can be obtained.

When the weighting factors are set to ones, a corollary of Proposition 2.1 is stated as follows.

**Corollary 2.1.** The explicit solution to (2) is:

$$s_i = \begin{cases} s_{k^*} + (d_{k^*} - d_i) & 1 \le i \le k^* \\ 0, & k^* < i \le K, \end{cases} \tag{27}$$

where the water level step $k^*$ is given by

$$k^* = \max\left\{k \middle| ER(k) < \eta, \ \ 1 \le k \le K\right\} \tag{28}$$

with $ER(k)$ defined by (8), the power level for this step is

$$s_{k^*} = \frac{1}{a_{k^*}}\left[\left(\frac{\eta}{ER(k^*)}\right)^{\frac{1}{k^*}} - 1\right] \tag{29}$$

and the optimal sum power allocated in (2) is:

$$P^* = \sum_{i=1}^K s_i = \sum_{i=1}^{k^*}\left[\frac{1}{a_{k^*}} \cdot \left(\frac{\eta}{ER(k^*)}\right)^{\frac{1}{k^*}} - \frac{1}{a_i}\right]. \tag{30}$$

Note that the solution $\{s_i\}$ expressed in (27) has an identical geometric form to that in [11]. However it is solved differently, since it is stemming from (28) and (29).

Similar to the weighed case, the first step of the proposed approach is to find the water level step $k^*$ based on (28). Then the power $s_{k^*}$ at this step can be determined based on (29). For those steps with index higher than $k^*$, no power is assigned. For those steps with index lower than $k^*$, their power levels are obtained by adding $s_{k^*}$ to the corresponding level depth difference, between the $k^*$th step and the $i$th step, which are shown in (27).

In the following descriptions of algorithmic implementation, only weighted case is provided.

From Proposition 2.1, when $k^*$ is obtained, $ER(k^*)$ is known. Then it is memorized to compute $s_{k^*}$. Thus, how to search $k^*$ is a key point for the proposed P-GWF. The procedure of P-GWF approach is stated below:

**Algorithm P-GWF**:

1) Let $\eta = 2^{2B}$. Initialize $W_s = 0; ER_M = ER^* = 1; i = 1$. If $K = 1$, output the optimal solution $s_1^* = \frac{1}{a_1}[\eta^{\frac{1}{w_1}} - 1]$; else go to 2).
2) Compute $W_s \leftarrow W_s + w_i; ER^* \leftarrow ER^* \cdot (\frac{d_{i+1}}{d_i})^{W_s}$. Then $i \leftarrow i + 1$, where the symbol "$\leftarrow$" represents the assignment operation.
3) If $ER^* < \eta$ and $i < K$, $ER_M = ER^*$, and repeat the step 2); else, for $ER^* \ge \eta$, output $k^* = i-1, W_s = W_s -$

$w_i$; for $i = K$ and $ER^* < \eta, k^* = i, W_s = W_s + w_i$ and $ER_M = ER^*$. Finally, let $s_{k^*} = \frac{1}{a_{k^*}}\left[(\frac{\eta}{ER_M})^{\frac{1}{W_s}} - 1\right]$.

We can observe that $\frac{s_{k^*}}{w_{k^*}} + d_{k^*}$ is the water level due to $\frac{s_{k^*}}{w_{k^*}} + d_{k^*} = \frac{s_i}{w_i} + d_i$, for $1 \le i \le k^*$. In addition, $W_s$ and $ER_M$ are used, in each of the iteration, for a factor of less computation.

## III. Solving Generalized RRM Problem Using P-GWF

In this section, we generalize the basic sum power RRM problem (2) to the sum power problems of WFPP, WFGPP, and WFGBP. The last case, P-WFGBP, is the most generalized RRM problem which will strip down to the other forms when applied to special values of lower and upper bounds and the number of its groups. To the best of the authors' knowledge, there is no existing algorithm reported in the open literatures to compute the exact solution for the generalized sum power problem P-WFGBP.

### A. Weighted Water-Filling with Individual Peak Power Constraints (P-WFPP) for Sum Power Problems

Let $P_i$ denote the peak power restriction of the $i$th channel. The weighted P-WFPP problem is stated by

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K s_i \\ \text{subject to:} & \sum_{i=1}^K w_i \log_2(1+a_i s_i) \ge 2B; 0 \le s_i \le P_i, \ \forall i. \end{array} \tag{31}$$

Comparing the problem (31) with (11), the constraint of $0 \le s_i$ is replaced with $0 \le s_i \le P_i$, *i.e.*, adding additional individual peak power constraint, and $\sum_{i=1}^K w_i \log_2(1 + a_i s_i) = 2B$ is replaced with $\sum_{i=1}^K w_i \log_2(1 + a_i s_i) \ge 2B$. In fact, by properly further reducing some allocated power(s), we can reach the equality constraint of the transmitted bits. Thus, problem (31) is reasonably assumed here.

Proposition 2.1 in subsection II-C provides an explicit exact solution using the proposed approach. P-WFPP problem can be obtained with some modifications to P-GWF. For convenience, the expression (14) can be extended into the expression:

$$ER(i_k) = \Pi_{t=1}^{k-1}\left(\frac{d_{i_k}}{d_{i_t}}\right)^{w_{i_t}}, \text{ for } k = 1, \ldots, |E|,$$

where $E$ is a subsequence of the sequence $\{1, 2, \ldots, K\}$, $|E|$ is the cardinality of the set $E$, so $E$ can be expressed through $\{i_1, i_2, \ldots, i_{|E|}\}$. Especially, if $E$ is taken as the sequence $\{1, 2, \ldots, K\}$, then the extended expression is regressed into the original expression (14). Similarly, some corresponding changes in (15)-(17) are also made (*i.e.*, the subscripts of sequence are replaced with those of the subsequence). For avoiding notation-wise tediousness, these extended expressions are still labelled by (15)-(17) in the following algorithm descriptions.

**Algorithm P-WFPP**:

**Input:** arrays $\{a_i, w_i, P_i\}$ for $i = 1, 2, \ldots, K$, the set $E = \{1, 2, \ldots, K\}$, and $\eta = 2^{2B}$.

1) Utilize (15)-(17) to compute $\{s_i\}$.
2) The set $\Lambda$ is defined by the set $\{i|s_i > P_i, i \in E\}$. If $\Lambda$ is the empty set, output $\{s_i\}_{i=1}^K$; else, $s_i = P_i$, for $i \in \Lambda$.
3) Update $E$ with $E \setminus \Lambda$ and $\eta$ with $\eta / \left[\Pi_{t\in\Lambda}(1 + a_t P_t)^{w_t}\right]$. Then return to 1) of the P-WFPP.

**Remark 3.1.1.** 3) in P-WFPP is a dynamic power distribution process. The state of this process is the difference between the individual peak power sequence and the current power distribution sequence obtained by P-GWF. The control of this process is to use (15)-(17) of Algorithm P-GWF based on the state mentioned above. And, a new state appears for next time stage. Therefore, a dynamic power distribution process, P-WFPP, with the state feedback is formed. Since the finite set $E$ is getting smaller and smaller until the set $\Lambda$ is empty, P-WFPP carries out $K$ loops to compute the optimal solution, at most. In detail, updating $E$ with $E \setminus \Lambda$ is to remove the set $\{i|s_i > P_i\}$, when $s_i \leftarrow P_i, \forall i$ in the set. Then over the updated set $E$, the exponent rate $\eta$ is updated with $\eta / \left[\Pi_{t\in\Lambda}(1 + a_t P_t)^{w_t}\right]$ correspondingly. Further, the process of updating $E$ and $\eta$ is a middle process, from the current state to form the current control, based on system theory.

For Algorithm P-WFPP, we can obtain the following results:

**Proposition 3.1**: P-WFPP can provide the exact optimal solution to the problem (31) by finite amounts of computation.

Its proof is placed in Appendix A.

### B. Weighted Water-Filling with Group Peak Power Constraints (P-WFGPP) for Sum Power Problems

Let $\{\chi_i\}_{i=1}^T$ be a partition of the index set: $\{1, \ldots, K\}$. For convenience, the elements of $\chi_i$ can be listed, monotonically increasing, *i.e.*, $i_1 < i_2 < \ldots < i_{|\chi_i|}$. Let $\overline{P}_i(> 0)$ denote the upper or peak power bound of the power constraint for the $i$th group channels, $\forall i$. The weighted P-WFGPP problem can be written by

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K s_i \\ \text{subject to:} & \sum_{i=1}^K w_i \log_2(1+a_i s_i) \ge 2B; 0 \le s_k, \ \forall k; \\ & \sum_{k=1}^K s_k \le P; \\ & \sum_{k\in\chi_i} s_k \le \overline{P}_i, \quad i = 1, \ldots, T. \end{array} \tag{32}$$

Comparing the problem (32) with (31), we know that the constraints of $0 \le s_i \le P_i, \forall i$, are extended to $0 \le \sum_{k\in\chi_i} s_k \le \overline{P}_i$, *i.e.*, if every $\chi_i$ is taken as a singleton, the problem (32) is regressed into the problem (31).

To solve the problem (32), let us recall the original preliminary RRM problem and its solution (GWF) reported in our earlier work [11], for preparation:

$$\begin{array}{ll} \max_{\{s_k\}_{k=1}^{K'}} & \sum_{k=1}^{K'} w_k \log_2(1+a_k s_k) \\ \text{subject to:} & \sum_{k=1}^{K'} s_k \le P_{\text{total}}; 0 \le s_k, \ \forall k. \end{array} \tag{33}$$

GWF gives the following solution to (33) by [11]:

$$\begin{array}{ll} s_m = \left[\dfrac{s_{k'^*}}{w_{k'^*}} + (d_{k'^*} - d_m)\right] w_m, & \text{for } 1 \le m \le k'^*; \\ s_m = 0, & \text{for } k'^* < m \le K', \end{array} \tag{34}$$

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where

$$k'^{*} = \max\left\{k \middle| P_2(k) > 0,\ \ 1 \leq k \leq K'\right\} \tag{35}$$

and the power level for this step is

$$s_{k'^{*}} = \frac{w_{k'^{*}}}{\sum_{m=1}^{k'^{*}} w_m} P_2(k'^{*}), \tag{36}$$

where

$$\begin{aligned} P_2(k) = \ & \left[P_{\text{total}} - \textstyle\sum_{m=1}^{k-1}\left(d_k - d_m\right) w_m\right]^{+}, \\ & \text{for } k = 1, \ldots, K', \end{aligned} \tag{37}$$

and

$$d_m = 1/(a_m w_m), \quad m = 1, \cdots, K'. \tag{38}$$

The following statement presents an algorithm which is a combination and modification of the GWF and P-GWF. This algorithm is termed as the P-WFGPP.

Similarly, for convenience, the expression (14) can be extended into the expression:

$$ER(i_k) = \Pi_{t=1}^{k-1}\left(\frac{d_{i_k}}{d_{i_t}}\right)^{w_{i_t}},\ \text{for } k = 1, \ldots, |E|,$$

where $E$ is a subsequence of the sequence $\{1, 2, \cdots, K\}$.

**Algorithm P-WFGPP**:

**Input:** Sets $\{a_k, w_k\}_{k=1}^{K}$, $\{\overline{P}_i\}_{1=1}^{T}$, $E = \{1, 2, \ldots, T\}$, and $\eta = 2^{2B}$.

1) Let $n = 1$ and $\Lambda = \emptyset$ (empty set). Utilize (15)-(17) to compute $\{s_k\}_{k=1}^{K}$.
2) The set $\Lambda_n$ is assigned by the set $\{i | \sum_{k\in\chi_i} s_k > \overline{P}_i, i \in E\}$. If $\Lambda_n$ is the empty set, output $\{s_i\}_{i=1}^{K}$; else, $P_{\text{total}} \leftarrow \overline{P}_i, K' \leftarrow |\chi_i|, \chi_i$ is renamed into the set $\{i_1, \ldots, i_{K'}\}$, and then utilize (34)-(36) from GWF, for $i \in \Lambda_n$.
3) Update $E$ with $E \setminus \Lambda_n$ and $\eta$ with $\eta / \left[\Pi_{i\in\Lambda_n}\Pi_{t\in\chi_i}\left(1 + a_t s_t\right)^{w_t}\right]$. Then $n \leftarrow n+1, K \leftarrow K - \sum_{i\in\Lambda_n}|\chi_i|$ and return to the second statement in 1) of the P-WFGPP.

**Proposition 3.2**: P-WFGPP can provide the exact optimal solution to the problem (32) by finite amounts of computation.

Its proof is presented in Appendix B.

*C. Weighted Water-Filling with Group Bounded Power Constraints for Sum Power Problem (P-WFGBP)*

Let $\{\chi_i\}_{i=1}^{T}$ be a partition of the index set: $\{1, \ldots, K\}$. Assume that $0 \leq \underline{P}_i \leq \overline{P}_i$, and $\underline{P}_i$ and $\overline{P}_i$ denote the lower bound and the upper bound of the power constraint for the $i$th group channels, $\forall i$. The weighted P-WFGBP problem is stated by

$$\begin{aligned} \min_{\{s_i\}_{i=1}^{K}} \quad & \textstyle\sum_{i=1}^{K} s_i \\ \text{subject to:} \quad & \textstyle\sum_{i=1}^{K} w_i \log_2(1 + a_i s_i) \geq 2B; 0 \leq s_k,\ \forall k; \\ & \underline{P}_i \leq \textstyle\sum_{k\in\chi_i} s_k \leq \overline{P}_i, i = 1, \ldots, T. \end{aligned} \tag{39}$$

Comparing the problem (39) with (32), it is seen that the constraints of $0 \leq \sum_{k\in\chi_i} s_i \leq \overline{P}_i, \forall i$, are generalized to $\underline{P}_i \leq \sum_{k\in\chi_i} s_k \leq \overline{P}_i$, *i.e.*, adding additional group lower bound power constraints. The lower bound of the additional constraint can be used to guarantee the fair transmitted rate from the $i$th group transmission, whereas the upper bound of the additional constraint can be used to limit the total interference from the $i$th group. The problem (39) is thus referred to as power (weighted) water-filling with group bounded power constraints (P-WFGBP).

Similarly, due to the explicit solution using generalized view approach that is provided in Proposition 2.1, the proposed GWF and P-WFGPP can be applied to the P-WFGBP problem with some modifications. The following statement presents a generalized algorithm, which is based on a meaningful combination and modification of the GWF and P-WFGPP.

Note, for the problem:

$$\begin{aligned} \max_{\{s_k\}_{k\in\chi_i}} \quad & \textstyle\sum_{k\in\chi_i} w_k \log_2(1 + a_k s_k) \\ \text{subject to:} \quad & \textstyle\sum_{k\in\chi_i} s_k = \overline{P}_i; 0 \leq s_k,\ \forall k, \end{aligned} \tag{40}$$

its optimal value is denoted by $\overline{V}_i$, for $i = 1, \ldots, T$. It can be observed that there does not exist any solution to problem (39), if $\sum_{i=1}^{T} \overline{V}_i < 2B$. Further, if $\sum_{i=1}^{T} \overline{V}_i = 2B$, the optimal solution to (40) denoted by $\{s_k^*\}_{k\in\chi_i}$, for $i = 1, \ldots, T$, can determine the optimal solution, $\{\{s_k^*\}_{k\in\chi_i}\}_{i=1}^{T}$, to (39). Hence, $\sum_{i=1}^{T} \overline{V}_i > 2B$ is assumed in the following. Further, $\overline{P}_i$, in the problem mentioned above, is replaced with $\underline{P}_i, \forall i$, and the corresponding optimal value is denoted by $\underline{V}_i, \forall i$. If $\sum_{i=1}^{T} \underline{V}_i \geq 2B$, the optimal solutions to the problems undergo a similar process to that mentioned above. For $i = 1, \ldots, T$, this constitutes an optimal solution to problem (39) and the optimal value is $\sum_{i=1}^{T} \underline{P}_i$ with practical meaning. Therefore, we only consider the cases under $\sum_{i=1}^{T} \overline{V}_i > 2B$, which has been assumed before, together with $\sum_{i=1}^{T} \underline{V}_i < 2B$.

It is seen that if $\underline{P}_i = 0, \forall i$, then problem P-WFGBP (39) is reduced into problem P-WFGPP (32); and if $\chi_i$ is regressed to a singleton and $\underline{P}_i = 0, \forall i$, then problem P-WFGBP (39) is reduced into problem P-WFPP (31). Thus, problem P-WFGBP (39) is the most general form of the RRA problems. It is called the generalized problem in this paper. The corresponding algorithm is described below.

**Algorithm P-WFGBP**:

**Input:** the channel gains $\{a_k\}_{k=1}^{K}$, the weights $\{w_k\}_{k=1}^{K}$, the group lower and upper power bounds $\{\underline{P}_i, \overline{P}_i\}_{i=1}^{T}$, the partition $\{\chi_i\}_{i=1}^{T}$ and the (weighted) sum-rate constraint $B$.

1) $P_{\text{total}} \leftarrow \underline{P}_i, K' \leftarrow |\chi_i|, \chi_i$ is written into the set $\{i_1, \ldots, i_{K'}\}$, and then utilize (34)-(36) from GWF, for $i = 1, \ldots, T$. Hence, the solutions $\{\{s'_{i_t}\}_{t\in\chi_i}\}_{i=1}^{T}$ are obtained.
2) Update $B$ with $B - \frac{1}{2}\sum_{k=1}^{K} w_k \log_2(1 + a_k s'_k)$, $\frac{1}{a_k}$ with $\frac{1}{a_k} + s'_k, \forall k$, and $\overline{P}_i$ with $\overline{P}_i - \underline{P}_i, \forall i$.
3) Utilize P-WFGPP to compute $\{s_k\}_{k=1}^{K}$ as the optimal solution to (32) under the updated parameters.
4) Output the optimal solution $\{s_k\} \leftarrow \{s_k + s'_k\}$ to the problem (39).

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**Remark 3.3.1.** Due to its definition mentioned above, P-WFGBP carries out $T$ loops to compute the exact optimal solution, at most.

For optimality of the proposed P-WFGBP, we have the following conclusion:

**Proposition 3.3**: P-WFGBP can provide the exact optimal solution to the problem (39) via finite amounts of computation.

Its proof is presented in Appendix C.

**Remark 3.3.2.** If we chose an approach, similar to P-CWF to directly solve the class of problems (39), a non-linear system with non-linear equations and inequalities in multiple dual variables (as below) would have had to be solved in a difficult manner:

$$\begin{cases} \sum_{i=1}^{T}\sum_{j\in\chi_i} w_j \log_2[1+a_j\left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+] \geq 2B; \\ \lambda\{\sum_{i=1}^{T}\sum_{j\in\chi_i} w_j \log_2[1+a_j\left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+] - 2B\} \\ = 0; \\ \underline{P}_i \leq \sum_{j\in\chi_i}\left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+ \leq \overline{P}_i, \\ \text{ for } i=1,2,\ldots,T; \\ \underline{\sigma}_i[\sum_{j\in\chi_i}\left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+ - \underline{P}_i] = 0, \\ \text{ for } i=1,2,\ldots,T; \\ \overline{\sigma}_i[\sum_{j\in\chi_i}\left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+ - \overline{P}_i] = 0, \\ \text{ for } i=1,2,\ldots,T; \\ \lambda\geq 0; \underline{\sigma}_i \geq 0, \overline{\sigma}_i\geq 0, \text{ for } i=1,2,\ldots,T. \end{cases} \tag{41}$$

For example, the system of (3) and (4) in [22] is defined, which was claimed to find a very general multiple water level multiple constrained water filling result. However, it cannot be used for the exact solution to the mentioned problem (39). The reason is stated as follows. It is seen that $s_j = \left(\frac{w_j\lambda}{1+\overline{\sigma}_i-\underline{\sigma}_i}-\frac{1}{a_j}\right)^+$, for $j\in\chi_i, i=1,2,\ldots,T$, where $\{s_j\}$ is the solution to problem (39). According to (3) in [22], the water levels should be taken as $\mu_k = \frac{\lambda}{1+\overline{\sigma}_k-\underline{\sigma}_k}$, for $k=1,2,\ldots,T$. Since $\{s_j\}$ is the solution, it should also satisfy the second, and the fourth to the sixth constraints in the system (41). Thus, a further developed version from (4) in [22] cannot include only the water-levels $\{\mu_k\}$. It also should include other dual variables, as a prerequisite for solving the problem, although (4) in [22] does not need the dual variables which correspond to non-negativeness constraints of the solution $\{s_j\}$ due to using the function $(\cdot)^+$. This point for the assumed form (4) in [22] results in [22] not being able to be used to solve the target problem (39).

The algorithms proposed in [11] are to compute the solutions to the maximum throughput problems. The minimum sum power problems discussed in this paper are different. In addition, KKT conditions for these two cases are different. As a result, algorithms of [11] cannot directly be used for the problems discussed here.

## IV. Numerical Results and Complexity Analysis

A few numerical examples are presented in this section to illustrate the steps of the proposed algorithms.

**Example 1.** Instantiate a case of P-WFPP problem by

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^2} & \sum_{i=1}^2 s_i \\ \text{subject to:} & \sum_{i=1}^2 \log(1+a_i s_i) \geq 3; 0\leq s_1\leq 1; \\ & 0\leq s_2\leq 8, \end{array} \tag{42}$$

where $a_1 = 1$ and $a_2 = 0.5$. Utilizing the proposed P-WFPP, the optimal solution is $\{s_1 = 1, s_2 = 6\}$ that is shown in Fig. 2(a).

**Example 2.** Instantiate another case of P-WFPP problem with multiple channels:

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^8} & \sum_{i=1}^8 s_i \\ \text{subject to:} & \sum_{i=1}^8 \log(1+a_i s_i) \geq 7; 0\leq s_i\leq i, \forall i, \end{array} \tag{43}$$

where $a_i = 1/i, \forall i$. That is to say, the step depth monotonically increases from 1 to 8, as shown in Fig. 3 with the solution.

Using the proposed P-WFPP, the result of the first iteration is $s_6 = s_{k^*} = 6[(\frac{128}{64.8})^{\frac{1}{6}} - 1](\doteq 0.72)$ and then $\{s_i = i\}_{i=1}^3 \cup \{s_i = s_6 + 6 - i\}_{i=4}^6 \cup \{s_i = 0\}_{i=7}^8 \doteq \{s_i = i\}_{i=1}^3 \cup \{s_i = 6.72 - i\}_{i=4}^6 \cup \{s_i = 0\}_{i=7}^8$. The result of the second iteration is $s_8 = s_{k^*} = 8[(\frac{105}{32})^{\frac{1}{5}} - 1](\doteq 2.146)$ and then $\{s_i = i\}_{i=1}^5 \cup \{s_i = s_8 + 8 - i\}_{i=6}^8 (\doteq \{s_i = i\}_{i=1}^5 \cup \{s_i = 10.146 - i\}_{i=6}^8)$. The result of the third iteration is $s_8 = s_{k^*} = 8[(\frac{21}{8})^{\frac{1}{3}} - 1](\doteq 3.036)$ and then $\{s_i = i\}_{i=1}^5 \cup \{s_i = s_8 + 8 - i\}_{i=6}^8 (\doteq \{1, 2, 3, 4, 5, 5.036, 4.036, 3.036\})$.

According to the algorithm and Proposition 3.1, the result of the third iteration is indeed the optimal solution. These results are illustrated in Figs. 3(a)-(c).

**Example 3.** Instantiate a case of weighted P-WFPP problem by

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^2} & \sum_{i=1}^2 s_i \\ \text{subject to:} & \sum_{i=1}^2 w_i\log(1+a_i s_i) \geq 3; 0\leq s_i\leq 12, \forall i, \end{array} \tag{44}$$

where $a_1 = 1, a_2 = 0.5, w_1 = 0.4$ and $w_2 = 0.6$. Utilizing the proposed P-WFPP, the optimal result is $\{s_1 = \frac{64}{49}\sqrt{56} - 1, s_2 = 12\}$, which is shown in Fig. 2(b).

**Example 4.** As the last example, we instantiate a case of the sum power weighted water-filling with group bounded power constraints (P-WFGBP) problem by:

$$\begin{array}{ll} \min_{\{s_i\}_{i=1}^3} & \sum_{i=1}^3 s_i \\ \text{subject to:} & \sum_{i=1}^3 w_i\log(1+a_i s_i) \geq 3; 0\leq s_i, \forall i; \\ & 1\leq s_1+s_2\leq 12; s_3\leq 12, \end{array} \tag{45}$$

where $a_1 = a_2 = a_3 = 1, w_1 = 0.3, w_2 = 0.2$ and $w_3 = 0.5$.

Utilizing the proposed Algorithm: P-WFGBP with inputting: $\chi_1 = \{1, 2\}, \chi_2 = \{3\}$ and $B \leftarrow 1.5$. The optimal solution is $\{s_1 = \frac{2^{2.8}\times 3^{0.7}}{5^{0.5}} - 1, s_2 = \frac{2^{3.8}}{5^{0.5}\times 3^{0.3}} - 1, s_3 = \frac{2^{2.8}\times 5^{0.5}}{3^{0.3}} - 1\}$ which is shown in Fig. 2(c), where the stair width for the three channels are 0.3, 0.2, 0.5 respectively specified by their weighting factors. The step depth is calculated using $1/(a_i w_i)$, leading to the step depth values of 10/3, 5, and 2 respectively for the three channels.

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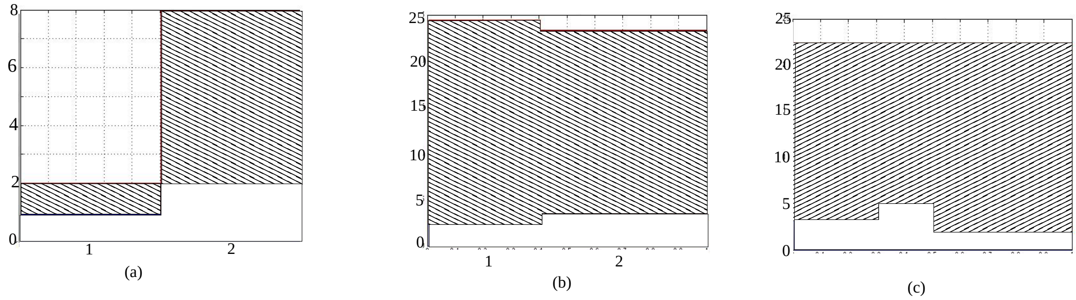


Fig. 2. Illustration for Examples 1, 3, 4 respectively.

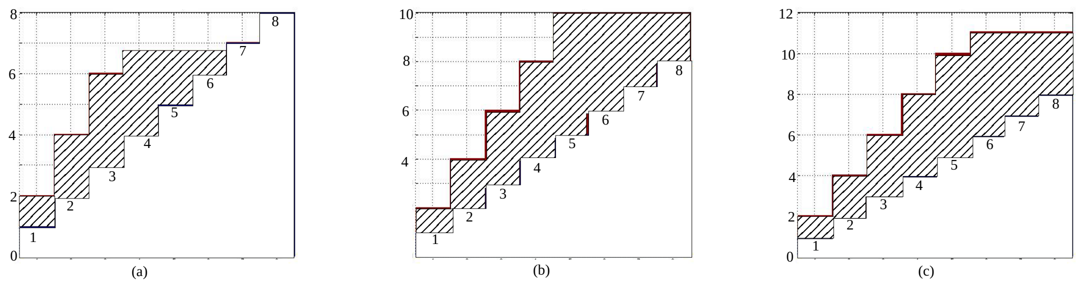


Fig. 3. Illustration for Example 2, results for the first, second and third iterations respectively.

### A. Complexity Analysis

For the non-weighted basic sum power water-filling problem (2), according to the expressions (9) and (10) in [22, Section 3], the conventional (sum power) water-filling algorithm had an exponential worst-case complexity [22, Section 3] of $2^K$, where $K$ is the number of the channels, even though the channel gains had been sorted in decreasing order. Pointing to this case, [22] and [16] proposed an improved algorithm with worst-case complexity of $K$ iterations. Since each iteration consists of the multiple basic elementary function evaluations, the arithmetic operations, and the logical operations, the proposed P-GWF is measured on these operations more accurately than the iterations. P-GWF uses $K$ iterations, each of which includes 8 operations: 1 basic elementary function evaluation (BE), 5 arithmetic operations (AOs), and 2 logical operations (LOs). For an algorithm, a total of these numbers right down to such operation(s) can measure the complexity level of the algorithm [23, Chapter 8].

The conventional approach [16, p. 310] requires a total of $O(K^2)$ operations which consist of $\frac{(K+1)K}{2}+1$ BEs, $\frac{(K+1)K}{2}+4$ AOs and $K$ LOs, under the $K+4$ memory unit requirement with a worst-case complexity of $K$ iterations.

As mentioned above, the proposed P-GWF uses $8K$ operations, which consist of $K$ BEs, $5K$ AOs, and $2K$ LOs under the $K+4$ memory unit requirement with a worst-case complexity of $K$ iterations.

For P-WFPP, it needs $K$ loops to compute the optimal solution, at most. The required number of operations is, at worst, $\sum_{i=1}^{K} 8i = 4K^2 + 4K$ fundamental basic elementary function evaluations, arithmetical and logical operations.

For P-WFGBP, it needs $T$ loops to compute the optimal solution, at most, where $T \leq K$. The required number of operations, at worst, is $T \times O(K^2)$ fundamental basic elementary function evaluations, arithmetical and logical operations. However, it is known from the prior works mentioned above that PD-IPM needs the computational complexity of $O(K^{3.5} log(1/\epsilon))$, to compute an $\epsilon$ solution.

In this complexity analysis, we didn't take sorting procedure into consideration. It is stated in [22] that the channel gain sequences come from the eigenvalues of a matrix. There are many algorithms to compute the eigenvalues and eigenvectors, with the eigenvalues sorted.

## V. Conclusion

In this paper, we extended the approach proposed in [11] to solve the basic RRM problem with an objective to minimize the sum power and constrained by a target sum data rate and the others. This is referred to as the sum power problem, as a contrast to the original sum data rate problem in [11]. We then extended the method to solve more complicated and more generalized sum power minimization RRM problems, such as, the one with individual peak power constraints, the one with the group peak power constraints, and the one with the group lower and upper bounded power constraints successively.

This class of sum power minimization problems has not been investigated so well as that of original throughput maximization problems in the open literature due to its more complicated structure. The proposed algorithms solve the RRM problems which take the sum power as the objective functions, with moderate complexity, and provide valuable references for

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engineering design and/or system optimization. The algorithms proposed in [11] cannot solve the problems here, as mentioned in Remark 3.3.2 and also reflected in the complexity of the proof for Proposition 3.2. For future direction, the proposed algorithms will be extended to efficiently and iteratively solve those more challenging and meaningful RRM problems.

## Appendix A
## Proof of Proposition 3.1

If the final set $E$ in P-WFPP is empty, it implies that $\frac{1}{2}\sum_{i=1}^{K} w_i \log_2(1+a_i P_i) < B$. Then no optimal solution exists.

If it is non-empty, it implies, based on the definition of P-WFPP, that

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}}+s_{k^*}} = \frac{w_i}{\frac{1}{a_i}+s_i}, \text{ for } \{i,k^*\} \subset E \text{ and } 0 < s_i \le P_i. \tag{46}$$

Let

$$\lambda = \left(\frac{w_{k^*}}{\frac{1}{a_{k^*}}+s_{k^*}}\right)^{-1}, \tag{47}$$

it is seen that

$$\underline{\sigma}_i = 1 - \lambda \frac{w_i}{\frac{1}{a_i}+s_i} \ge 0, \tag{48}$$

and let $\overline{\sigma}_i = 0$, for $i \in E$.

If $i \notin E$, then $s_i = P_i$. According to the definitions of $k^*$ and $s_{k^*}$, we have the following relationship:

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}}+s_{k^*}} < \frac{w_i}{\frac{1}{a_i}+s_i}, \text{ for } i \notin E, i.e., s_i = P_i. \tag{49}$$

A geometric interpretation for the derivation above is: the water level at the $k^*$th step of the stairs is higher than that at the $i$th step of the stairs. It is seen that

$$\overline{\sigma}_i = \lambda \frac{w_i}{\frac{1}{a_i}+s_i} - 1 > 0 \tag{50}$$

and let $\underline{\sigma}_i = 0$, for $i \notin E$. Then,

$$\begin{cases} \underline{\sigma}_i = 0 \text{ and } \overline{\sigma}_i > 0, \text{ for } i \notin E \\ \underline{\sigma}_i > 0 \text{ and } \overline{\sigma}_i = 0, \text{ for } i \in E \text{ and } s_i = 0 \\ \underline{\sigma}_i = \overline{\sigma}_i = 0, \text{ for } i \in E \text{ and } 0 < s_i \le P_i. \end{cases} \tag{51}$$

Therefore, the following system holds:

$$\begin{cases} 1 - \lambda \frac{w_i}{\frac{1}{a_i}+s_i} - \underline{\sigma}_i + \overline{\sigma}_i = 0, \text{for } 1 \le i \le K; \\ s_i \ge 0, \underline{\sigma}_i s_i = 0, \ \underline{\sigma}_i \ge 0, \forall i; \\ s_i \le P_i, \overline{\sigma}_i (s_i - P_i) = 0, \ \overline{\sigma}_i \ge 0, \forall i; \\ \sum_{i=1}^{K} w_i \log_2(1+a_i s_i) \ge 2B, \\ \lambda \left[\sum_{i=1}^{K} w_i \log_2(1+a_i s_i) - 2B\right] = 0, \lambda \ge 0. \end{cases} \tag{52}$$

By observation, the system above is a set of the KKT conditions of the problem in (31) and the water level at the $k^*$th step of the stairs is equal to the Lagrange multiplier $\lambda$ mentioned above. Note that the Lagrange function of the problem in (31) is

$$\begin{aligned} & L(\{s_i\}, \lambda, \{\underline{\sigma}_i\}, \{\overline{\sigma}_i\}) \\ = \ & \textstyle\sum_{i=1}^{K} s_i - \lambda \left[\sum_{i=1}^{K} w_i \log_2(1+a_i s_i) - 2B\right] \\ - \ & \textstyle\sum_{i=1}^{K} \underline{\sigma}_i s_i \\ + \ & \textstyle\sum_{i=1}^{K} \overline{\sigma}_i (s_i - P_i). \end{aligned} \tag{53}$$

Since problem (31) is a differentiable convex optimization problem, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. Note that GCQ of the problem mentioned above holds. Proposition 3.1 is hence proved.

## Appendix B
## Proof of Proposition 3.2

If the final set $E$ in P-WFGPP is empty, it implies that if the sum-rate achieves $B$ bits, the required sum power of the problem (32) is strictly greater than $\sum_{i=1}^{T} \overline{P}_i$. For this case, then there is no optimal solution to the problem (32).

If the final set $E$ is non-empty, it implies, according to the definition of P-WFGPP, we can take an $i_0 \in \cup_{i\in E}\chi_i$ such that

$$\begin{aligned} & \tfrac{w_{i_0}}{\frac{1}{a_{i_0}}+s_{i_0}} = \tfrac{w_{i_{k^*}}}{\frac{1}{a_{i_{k^*}}}+s_{i_{k^*}}} = \tfrac{w_{i_t}}{\frac{1}{a_{i_t}}+s_{i_t}}, \\ & \text{where } i_t \in \{i_1 < \ldots < i_{k^*}\} \subset \chi_i, \text{ and} \\ & \textstyle\sum_{k\in\chi_i} s_k \le \overline{P}_i, \text{ for } i \in E. \end{aligned} \tag{54}$$

Let

$$\lambda = \left(\frac{w_{i_{k^*}}}{\frac{1}{a_{i_{k^*}}}+s_{i_{k^*}}}\right)^{-1}, \tag{55}$$

it is seen that

$$\begin{aligned} & \mu_{i_t} = 0, \text{ for } t = 1, \ldots, k^*; \\ & \mu_{i_t} = 1 - \lambda \tfrac{w_{i_t}}{\frac{1}{a_{i_t}}+s_{i_t}} \ge 0, \text{ for } t = k^*+1, \ldots, |\chi_i|; \end{aligned} \tag{56}$$

and let $\sigma_i = 0$, for $i \in E$.

If $i \notin E$, then $\sum_{k\in\chi_i} s_k = \overline{P}_i$. According to the definitions of $i_{k^*}$ and $s_{i_{k^*}}$, we have the following relationship:

$$\begin{aligned} & \tfrac{w_{i_{k^*}}}{\frac{1}{a_{i_{k^*}}}+s_{i_{k^*}}} = \tfrac{w_{i_t}}{\frac{1}{a_{i_t}}+s_{i_t}} > \tfrac{w_{i_0}}{\frac{1}{a_{i_0}}+s_{i_0}} = \tfrac{1}{\lambda}, \\ & \text{where } i_t \in \{i_1 < \ldots < i_{k^*}\} \subset \chi_i \text{ and} \\ & \textstyle\sum_{k\in\chi_i} s_k = \overline{P}_i, \text{ for } i \notin E. \end{aligned} \tag{57}$$

It is seen that

$$\sigma_i = \lambda \frac{w_{i_{k^*}}}{\frac{1}{a_{i_{k^*}}}+s_{i_{k^*}}} - 1 > 0 \tag{58}$$

and let $\mu_{i_t} = 0$, for $1 \le i_t \le i_{k^*}$. since $i_{k^*+1} \le i_t \le i_{|\chi_i|}$,

$$1 + \sigma_i - \lambda \frac{w_{i_t}}{\frac{1}{a_{i_t}}+s_{i_t}} > 0 \text{ and } s_{i_t} = 0. \tag{59}$$

Thus, let $\mu_{i_t} = 1 + \sigma_i - \lambda \frac{w_{i_t}}{\frac{1}{a_{i_t}}+s_{i_t}} > 0$, for $i_{k^*+1} \le i_t \le i_{|\chi_i|}$. Note that $i \notin E$ has been assumed at the paragraph above.

The following details are expressed to obtain (57). We use the mathematical induction on the number of iterations, to prove truth of two statements. Some notations are introduced before the proof, for clarity. They are: the power distribution

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before accepting (15)-(17) at 1) of P-WFGPP is denoted by $\{\hat{s}_i\}$ at current iteration, and $\{\hat{\hat{s}}_i\}$ at the next iteration; after 1) of P-WFGPP and before 2) of P-WFGPP, the power distribution is denoted by $\{\overline{s}_i\}$ at the current iteration, and $\{\overline{\overline{s}}_i\}$ at the next iteration; and after 2) of P-WFGPP, the power distribution is denoted by $\{s_i\}$ at the current iteration. The first statement is that the water level over $E$ between 1) of P-WFGPP and 2) of P-WFGPP is higher than the water level over the complement of $E$ after 3) of P-WFGPP. The second statement is that as the number of iteration, $n$, increases, then $E$ gets less and less, and the water level over $E$ just after 1) of P-WFGPP becomes higher and higher. If the two statements are proved to be true, they imply truth of (57). Let us begin the formal proof. As $n = 1$ and $i_0 \in \cup_{i\in E}\chi_i$, (54)-(56) holds, with $\overline{s}_{i_0} > 0$. During 2) of P-WFGPP, $\Lambda_n$ is obtained. $\forall i \in \Lambda_n$, up to 3) of P-WFGPP, $i \in \Lambda_n$ means $i \notin E$. Since $P_i < \sum_{k\in\chi_i} \overline{s}_k$ and $P_{\text{total}} \leftarrow P_i$, $P_{\text{total}} < \sum_{k\in\chi_i} \overline{s}_k$. Thus, (37) implies

$$P_2(k) < \left[\sum_{k\in\chi_i} \overline{s}_k - \sum_{m=1}^{k-1} (d_k - d_m)\, w_m\right]^+, \forall k. \tag{60}$$

Further, (60) leads to $s_k \leq \overline{s}_k, \forall k \in \chi_i$ and $i \in \Lambda_n$. If

$$\left[\sum_{k\in\chi_i} \overline{s}_k - \sum_{m=1}^{k-1} (d_k - d_m)\, w_m\right]^+ > 0, k \in \chi_i \text{ and } i \in \Lambda_n, \text{ then } s_k < \overline{s}_k. \tag{61}$$

Therefore, $\frac{w_{i_0}}{\frac{1}{a_{i_0}} + \overline{s}_{i_0}} = \frac{w_j}{\frac{1}{a_j} + \overline{s}_j} = \frac{1}{\lambda}$, as $s_j > 0, j \in \chi_i$ and $i \in E$; while $\frac{1}{\lambda}(= \frac{w_{i_0}}{\frac{1}{a_{i_0}} + \overline{s}_{i_0}}) = \frac{w_j}{\frac{1}{a_j} + \overline{s}_j} < \frac{w_j}{\frac{1}{a_j} + s_j}$, due to $s_j < \overline{s}_j$, for $s_j > 0$, $j \in \chi_i$ and $i \in \Lambda_n$. Note $E = \{1, \ldots, T\}$ and the order of $s_j$ and $\overline{s}_j$ appearing, for $n = 1$ and before 3) of P-WFGPP. Hence, according to definition of the water level mentioned before, the first statement is true. At the same time,

$$\begin{array}{ll} & \eta / \left[\Pi_{i\in\Lambda_n} \Pi_{t\in\chi_i} (1 + a_t s_t)^{w_t}\right] \\ > & \eta / \left[\Pi_{i\in\Lambda_n} \Pi_{t\in\chi_i} (1 + a_t \overline{s}_t)^{w_t}\right], \end{array} \tag{62}$$

due to $s_t < \overline{s}_t$, as $t \in \chi_i$ and $i \in \Lambda_n$. Similarly, 3) of P-WFGPP and (17) at 1) of P-WFGPP result in $\overline{\overline{s}}_{i_0} > \overline{s}_{i_0}$. This similarity comes from the minuend of (37) and the dividend, $\eta$, of (17) becoming less by newly assigned ones, respectively. Further, it can obtained that $\overline{\overline{s}}_j > \overline{s}_j$, for $j \in \chi_i, i \in E$. This point means $\frac{w_j}{\frac{1}{a_j} + \overline{\overline{s}}_j} \leq \frac{w_j}{\frac{1}{a_j} + \overline{s}_j}$, for $j \in \cup_{i\in E}\chi_i$. Thus, the second statement is true. Therefore, if $n = 1$, the two statements are all true. Assume that if the number of iteration is $n$, the conclusion holds. The following is to prove by induction that if the number of iteration is $n + 1$, the conclusion holds. Similarly to proving the second statement for $n = 1$, the water level obtained by 1) of P-WFGPP is greater than the water level obtained by the $n$th iteration. Thus, such water levels, obtained by the $n + 1$ iterations, determine a monotonically increasing sequence. Hence, the second statement is true, as the number of iteration is $n + 1$. Further, $\frac{w_j}{\frac{1}{a_j} + \overline{s}_j}$, where $\overline{s}_j > 0$, $j \in \chi_i$ and $i \in \Lambda_{n+1}$, is strictly greater than the reciprocal of the water level, over $E$, obtained by the $(n+1)$th iteration, from the similar derivation to proving the the first statement for $n = 1$. For $\frac{w_j}{\frac{1}{a_j} + \overline{s}_j}$, where $\overline{s}_j > 0$, $j \in \chi_i, i \in \Lambda_k$ and $1 \leq k \leq n$, it is strictly greater than the reciprocal of the water level, over $E$, obtained by the $k$th iteration, from the assumption of the mathematical induction. Thus, the first statement is true, as the number of iteration is $n + 1$. Therefore, the two statements are all true, up to obtaining the final $E$. Due to D-WFF and GWF being used by 1) of P-WFGPP and 2) of P-WFGPP, together with consideration of the two statements, it is seen that (57) is true. This proof uses P-GWF and GWF, in turn, for each of the loops. Similar details for the other proofs can be obtained, by the approach mentioned above. The details are shown above for this proposition due to the pivot role of this proposition among the proposed propositions.

Then,

$$\begin{cases} \text{as } i \in E, & \sigma_i = 0 \text{ and } \mu_{i_t} = 0, \text{ for } s_{i_t} > 0; \\ & \sigma_i = 0 \text{ and } \mu_{i_t} \geq 0, \text{ for } s_{i_t} = 0. \\ \text{as } i \notin E, & \sigma_i \geq 0 \text{ and } \mu_{i_t} = 0, \text{ for } s_{i_t} > 0; \\ & \sigma_i \geq 0 \text{ and } \mu_{i_t} \geq 0, \text{ for } s_{i_t} = 0. \end{cases} \tag{63}$$

Therefore, the following system holds:

$$\begin{cases} 1 - \lambda \frac{w_{i_t}}{\frac{1}{a_{i_t}} + s_{i_t}} + \sigma_i - \mu_{i_t} = 0, \text{for } 1 \leq i \leq T, i_t \in \chi_i; \\ s_{i_t} \geq 0, \mu_{i_t} s_{i_t} = 0, \ \mu_{i_t} \geq 0, \forall i, i_t; \\ \sum_{k\in\chi_i} s_k \leq \overline{P}_i, \sigma_i(\sum_{k\in\chi_i} s_k - \overline{P}_i) = 0, \sigma_i \geq 0, \forall i; \\ \sum_{i=1}^K w_i \log_2(1 + a_i s_i) \geq 2B, \\ \lambda \left[\sum_{i=1}^K w_i \log_2(1 + a_i s_i) - 2B\right] = 0, \lambda \geq 0. \end{cases} \tag{64}$$

Note that, in the second line of the system just listed above, for any $i$, it determines the set of $\chi_i$. Successively, we take any $i_t \in \chi_i$. By observation, the system above is a set of the KKT conditions of the problem in (32) and the water level at the $k^*$th step of the stairs is equal to the Lagrange multiplier $\lambda$ mentioned above. Note that the Lagrange function of the problem in (32) is

$$\begin{array}{ll} & L(\{\{s_{i_t}\}_{i_t\in\chi_i}\}_{i=1}^T, \lambda, \{\sigma_i\}_{i=1}^T, \{\{\mu_{i_t}\}_{i_t\in\chi_i}\}_{i=1}^T) \\ = & \sum_{i=1}^T \sum_{i_t\in\chi_i} s_{i_t} \\ - & \lambda \left[\sum_{i=1}^T \sum_{i_t\in\chi_i} w_{i_t} \log_2 (1 + a_{i_t} s_{i_t}) - 2B\right] \\ - & \sum_{i=1}^T \sum_{i_t\in\chi_i} \mu_{i_t} s_{i_t} + \sum_{i=1}^T \sigma_{i_t} \left[\sum_{i_t\in\chi_i} s_{i_t} - \overline{P}_i\right]. \end{array} \tag{65}$$

Similarly, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. Also, GCQ of the problem (32) holds. Noting the fact that the final set $E$ can be constructed by finite amounts of computation (at most through $T$ loops), Proposition 3.2 is hence proved.

## APPENDIX C
## PROOF OF PROPOSITION 3.3

Let $\{s'_k\}_{k\in\chi_i}$ be the optimal solution to the problem:

$$\begin{array}{ll} \max_{\{s_k\}} & \sum_{k=1}^K w_k \log_2(1 + a_k s_k) \\ \text{subject to:} & 0 \leq s_k, \ \forall k; \sum_{k\in\chi_i} s_k = \underline{P}_i, i = 1, \ldots, T. \end{array} \tag{66}$$

Then, $\{s'_k\}$ satisfies the KKT conditions to the problem above. That is to say, there exist the non-negative numbers or the set of non-negative numbers, $\lambda_{0,i}, \sigma_{0,i}, \{\mu_{0,i_t}\}_{i_t\in\chi_i}$, $i$ of which

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determines the set of $\chi_i$ and then $i_t$ of which denotes any member of $\chi_i$, such that

$$\begin{cases} \frac{1}{\frac{1}{a_{i_t}w_{i_t}}+\frac{s'_{i_t}}{w_{i_t}}} = \lambda_{0,i} - \mu_{0,i_t}; \\ \lambda_{0,i} \geq 0, \sum_{k\in\chi_i} s'_k \leq \overline{P}_i, \lambda_{0,i}(\sum_{k\in\chi_i} s'_k - \underline{P}_i) = 0; \\ \mu_{0,i_t} \geq 0, s'_{i_t} \geq 0, \mu_{0,i_t}s'_{i_t} = 0, \forall i_t \in \chi_i, i = 1, \dots, T. \end{cases} \tag{67}$$

Since the set of $\cup_{i=1}^{T}\{s_{i_t}\}_{i_t\in\chi_i}$, $i_t$ of which has the same meaning as the one just mentioned, has been obtained by 2) and 3) of P-WFGBP, it satisfies another set of KKT conditions, mentioned blow.

There exist a non-negative number, and a group of sets consisting of non-negative numbers, $\lambda, \{\underline{\sigma}_i, \overline{\sigma}_i = 0\}, \{\mu_{i_t}\}_{i_t\in\chi_i}$, such that

$$\begin{cases} 1 - \frac{\lambda}{\frac{1}{a_{i_t}w_{i_t}}+\frac{s'_{i_t}+s_{i_t}}{w_{i_t}}} - \underline{\sigma}_i + \overline{\sigma}_i - \mu_{i_t} = 0; \\ \lambda \geq 0, \sum_{k=1}^{K} w_k \log_2[1 + a_k(s'_k + s_k)] \geq 2B, \\ \lambda\{\sum_{k=1}^{K} w_k \log_2[1 + a_k(s'_k + s_k)] - 2B\} = 0; \\ \underline{\sigma}_i \geq 0, \sum_{k\in\chi_i}(s'_k + s_k) \geq \underline{P}_i, \\ \underline{\sigma}_i[\sum_{k\in\chi_i}(s'_k + s_k) - \underline{P}_i] = 0; \\ \overline{\sigma}_i \geq 0, \sum_{k\in\chi_i}(s'_k + s_k) \leq \overline{P}_i, \\ \overline{\sigma}_i[\sum_{k\in\chi_i}(s'_k + s_k) - \overline{P}_i] = 0; \\ \mu_{i_t} \geq 0, s_{i_t} \geq 0, \mu_{i_t}s_{i_t} = 0, \forall i_t \in \chi_i, i = 1, \dots, T. \end{cases} \tag{68}$$

Further, partially update the Lagrange multipliers as below, with respect to the others keeping unchanged. If $s_{i_t} = 0, s'_{i_t} > 0$ and there exists $i$ such that $i_t \in \chi_i$, it is easy to see that $\frac{1}{\lambda_{0,i}} \geq \lambda$. This point stems from the following logical development. $s'_{i_t} > 0$ leads to $\mu_{0,i_t} = 0$ at line 1 of (67). Then, $\frac{1}{\lambda_{0,i}} = \frac{1}{a_{i_t}w_{i_t}} + \frac{s'_{i_t}}{w_{i_t}}$. At the same time, together with the mentioned $\overline{\sigma}_i = 0$, line 1 of (68) implies $\frac{\lambda}{\frac{1}{a_{i_t}w_{i_t}}+\frac{s'_{i_t}+s_{i_t}}{w_{i_t}}} = 1 - \underline{\sigma}_i - \mu_{i_t} \leq 1$, and then $\lambda \leq \frac{1}{a_{i_t}w_{i_t}} + \frac{s'_{i_t}+s_{i_t}}{w_{i_t}}$. Due to $s_{i_t} = 0$, $\lambda \leq \frac{1}{a_{i_t}w_{i_t}} + \frac{s'_{i_t}}{w_{i_t}} = \frac{1}{\lambda_{0,i}}$. Then let $\mu_{i_t} = 0$. If $\sum_{k\in\chi_i}(s'_k + s_k) = \underline{P}_i$, $\underline{\sigma}_i = \frac{\frac{1}{\lambda_{0,i}}-\lambda}{\frac{1}{a_{i_t}w_{i_t}}+\frac{s'_{i_t}}{w_{i_t}}} \geq 0$, and $\overline{\sigma}_i = 0$; if $\sum_{k\in\chi_i}(s'_k + s_k) = \overline{P}_i$, $\frac{1}{\lambda_{0,i}} \leq \lambda$ and $\overline{\sigma}_i = \frac{\lambda-\frac{1}{\lambda_{0,i}}}{\frac{1}{a_{i_t}w_{i_t}}+\frac{s'_{i_t}}{w_{i_t}}} \geq 0$, and $\underline{\sigma}_i = 0$.

Therefore, for the solution $\{s_k\}$ obtained by 4) of P-WFGBP, the Lagrange multipliers $\lambda, \{\underline{\sigma}_i, \overline{\sigma}_i\}_{i=1}^{T}$ and $\{\mu_k\}_{k=1}^{K}$ obtained above satisfy the KKT conditions of the problem (39), and the corresponding Lagrange function is:

$$\begin{aligned} & L(\{s_k\}, \lambda, \{\overline{\sigma}_i\}, \{\underline{\sigma}_i\}, \{\mu_k\}) \\ = \; & \sum_{k=1}^{K} s_k - \lambda[\sum_{k=1}^{K} w_k \log_2(1 + a_k s_k) - 2B] \\ + \; & \sum_{i=1}^{T} \overline{\sigma}_i[\sum_{j\in\chi_i} s_j - \overline{P}_i] \\ - \; & \sum_{i=1}^{T} \underline{\sigma}_i(\sum_{j\in\chi_i} s_j - \underline{P}_i) - \sum_{k=1}^{K} \mu_k s_k. \end{aligned} \tag{69}$$

Since the problem (39) is a differentiable convex optimization problem, similarly to the previous reasons, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. It is observed that GCQ of the problem (39) holds. Proposition 3.3 is thus proved.

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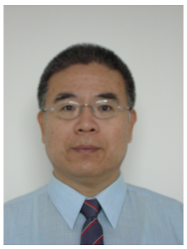
## REFERENCES

[1] S. Ko and S. Kim, "Block waterfilling with power borrowing for multicarrier communications," *in Proc. IEEE Vehicular Technology Conf.*, 2008.
[2] A. Kalakech, J. Louveaux, and L. Vandendorpe, "Application of gradient algorithm for optimizing power allocation in DSL systems," *Acoustics Speech and Signal Processing (ICASSP), 2010 IEEE International Conference on*, pp. 3142–3145, 2010.
[3] J. Peng, G. Wei, and J. Zhu, "Power allocation method for OFDM system with both total and per-antenna power constraints," *IEEE Communication Letters*, vol. 12, pp. 621–623, 2008.
[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory, 2nd Edition*, Wiley-Interscience, 2006.
[5] D. Zhao, Z. Fei, S. Li, and J. Kuang, "Improved iterative water-filling algorithm in MU MIMO system," *Wireless, Mobile and Multimedia Networks (ICWMNN 2010), IET 3rd International Conference on*, pp. 173–178, 2010.
[6] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. on Telecomm.*, vol. 10, pp. 585–596, 1999.
[7] T. Starr, J. Cioffi, and P. Silverman, *Understanding Digital Subscriber Line Technology*, Prentice Hall, 1999.
[8] N. Amitay and J. Salz, "Linear equalization theory in digital data transmission over dually polarized fading radio channels," *Bell Labs. Tech. J.*, vol. 63, pp. 2215–2259, 1984.
[9] D. Dash, "Paranoid secondary: Waterfilling in a cognitive interference channel with partial knowledge," *IEEE Trans. Wireless Communications*, vol. 39, pp. 1045–1055, 2012.
[10] M. Shinwari, A. Youssef, and W. Hamouda, "A water-filling based scheduling algorithm for the smart grid," *IEEE Transactions on Smart Grid*, vol. 3, pp. 710–719, 2012.
[11] P. He, L. Zhao, S. Zhou, and Z. Niu, "Water-filling: A geometric approach and its application to solve generalized radio resource allocation problems," *IEEE Trans. Wireless Communications*, vol. 12, no. 7, pp. 3637–3647, 2013.
[12] L. Zhang, Y.-C. Liang, and Y. Xin, "Joint beamforming and power allocation for multiple access channels in cognitive radio networks," *IEEE J. Select. Areas Communications*, vol. 26, pp. 38–51, 2008.
[13] R. Zhang, Y.-C. Liang, and S. Cui, "Dynamic resource allocation in cognitive radio networks," *IEEE Signal Processing Magazine*, vol. 27, pp. 102–114, 2010.
[14] C. Hsu, H. Su, and P. Lin, "Joint subcarrier pairing and power allocation for OFDM transmission with decode-and-forward relaying," *IEEE Trans. Inform. Theory*, vol. 59, pp. 399–414, 2011.
[15] E. Jorswieck, "Uplink throughput maximization with combined sum and individual power constraints," *IEEE Communication Letters*, vol. 10, pp. 816 –818, 2006.
[16] J. Cioffi, "Ee379c course reader, chapter 4," http://www.stanford.edu/group/cioffi/ee379c, frequently updated.
[17] P. Wang, X. Chen, X. Zhong, L. Xiao, S. Zhou, and J. Wang, *Cognitive Radio Systems*, chapter Power Allocation in OFDM-Based Cognitive Radio Systems, InTech, 2009.
[18] D. P. Bertsekas, *Nonlinear Programming, 2nd Edition*, Athena Scientific, Nashua, 1999.
[19] M. Avriel, *Nonlinear Programming: Analysis and Methods*, Prentice-Hall, Inc., 1976.
[20] L. Zhang, Y. Xin, Y.-C. Liang, and H. V. Poor, "Cognitive multiple access channels: Optimal power allocation for weighted sum rate maximization," *IEEE Trans. Communications*, vol. 57, pp. 2754–2762, 2009.
[21] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.

[22] D. Palomar, "Practical algorithms for a family of waterfilling solutions," *IEEE Trans. Signal Processing*, vol. 53, pp. 686–695, 2005.
[23] C. H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity, Unabridged edition*, Dover Publications, Mineola, 1998.



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