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It is shown that a correspondence exists between the Weyl-ordered Hamiltonian and the midpoint prescription in the discrete path integral for fermions. It is then proven that the Feynman rules obtained from the discrete and continuous path integral are equivalent.

I. INTRODUCTION

The correspondence between classical and quantum functions of the dynamical variables is an old problem in quantum theory. On one hand (the canonical one) the question arises on how to order noncommuting operators. On the other hand (the functional one) it is the action in the path integral that presents ambiguities.[1,2]

A well-known result in the bosonic case[3] is that the Weyl-ordered form of operators is equivalent to the midpoint prescription in the path integral provided that one uses $\langle b|b\rangle = \int dx\, b^* b$ for the scalar product. In other words,

$$\langle x_2|\exp[-i\epsilon\mathbf{H}_{WO}]|x_1\rangle = \int dp \exp\left\{i\epsilon\left[p\,\frac{x_2-x_1}{\epsilon} - H\left(p,\frac{x_1+x_2}{2}\right)\right]\right\}, \quad (1)$$

where $\mathbf{H}_{WO}$ is derived from the classical Hamiltonian $H(p,x)$ considered as a symmetrized function of the canonical operators $\mathbf{x}$ and $\mathbf{p}$. [Equation (1), as in the others that follow, is valid up to the order ϵ included.] In our language the properly stated Weyl ordering acts on the products of the operators $\mathbf{Q}_i$ by symmetrizing them, i.e.,

$$(\mathbf{Q}_1\cdots\mathbf{Q}_K)_{WO} = \frac{1}{K!}\sum_{\text{perm}} \mathbf{Q}_{i_1}\cdots\mathbf{Q}_{i_K}, \quad (2)$$

so that we should specify the canonical coordinate operators to be symmetrized. (An operator Weyl ordered in one coordinate system may not be such after a canonical transformation.)

The rhs of Eq. (1) is the infinitesimal form of the phase space path integral written on the time lattice, where the Hamiltonian H is evaluated at the midpoint. The corresponding finite path integral is a product of infinitesimal ones integrated over the intermediate coordinates; as shown by Sato,[4] in the limit of zero lattice spacing this reproduces the same Feynman rules as the continuous path integral. Since in the last decade the path integral has been used extensively as a tool for quantization, it would be useful to include the other half of the world, the fermions, in this picture; we do this in what follows.

II. A REPRESENTATION FOR FERMIONS

Let us now consider the quantum mechanics of fermions. We have operators obeying anticommutation rules:

$$\{\Psi^\mu,\Psi^\nu\} = \{\overline{\Psi}_\mu,\overline{\Psi}_\nu\} = 0; \qquad \{\Psi^\mu,\overline{\Psi}_\nu\} = \delta^\mu_\nu, \quad \mu,\nu = 1,\ldots,N. \quad (3)$$

In order to establish a correspondence between the canonical and functional forms of the propagator, let us introduce the eigenvectors of Ψ[5]:

$$\langle\psi|\Psi^\mu = \psi^\mu\langle\psi|, \quad \Psi^\mu|\psi\rangle = (-1)^N|\psi\rangle\psi^\mu, \quad (4)$$

with the completeness

$$\int|\psi\rangle \overleftarrow{d^N\psi}\langle\psi| = 1, \text{ where } \overleftarrow{d^N\psi} = \overleftarrow{d\psi^1}\cdots\overleftarrow{d\psi^N}. \quad (5)$$

We use left arrows as in $\int f(z)\overleftarrow{dz} = f(z)\overleftarrow{\partial}/\partial z$, corresponding to the right derivative, and we use the transposed relation (right arrows) for the left derivative; when unnecessary the arrows will be suppressed. The relationship (5) holds only if each component Ψ^μ represents one fermionic degree of freedom: The linear space generated by 1 and ψ^μ has dimension 2 indeed. The ψ's are generators of a Grassmann algebra and

$$\int\psi\,\overleftarrow{d\psi} = -\int\overleftarrow{d\psi}\,\psi = 1, \quad \int\overleftarrow{d\psi} = 0 \quad (6)$$

as usual. The following formulas hold:

$$\langle\psi|\psi'\rangle \equiv \delta(\psi-\psi') = (\psi'-\psi)^N\cdots(\psi'-\psi)^1, \quad (7)$$

$$\langle\psi|\overline{\Psi}_\mu = \frac{\vec{\partial}}{\partial\psi^\mu}\langle\psi|, \quad \overline{\Psi}_\mu|\psi\rangle = (-1)^N|\psi\rangle\frac{\overleftarrow{\partial}}{\partial\psi^\mu}. \quad (8)$$

For the eigenvectors of $\overline{\Psi}$ we have

$$\overline{\Psi}_\mu|\overline{\psi}\rangle = |\overline{\psi}\rangle\overline{\psi}_\mu, \quad \langle\overline{\psi}|\overline{\Psi}_\mu = (-1)^N\overline{\psi}_\mu\langle\overline{\psi}|. \quad (4')$$

As is readily seen Eqs. (4')[6] are the conjugate of the corresponding ones for Ψ. The same is true for the remaining equations; for instance,

$$\int|\overline{\psi}\rangle\overrightarrow{d^N\overline{\psi}}\langle\overline{\psi}| = 1, \quad \overrightarrow{d^N\overline{\psi}} = \overrightarrow{d\overline{\psi}_N}\cdots\overrightarrow{d\overline{\psi}_1} \quad (5')$$

and

$$\int\overrightarrow{d\overline{\psi}}\,\overline{\psi} = -\int\overline{\psi}\,\overrightarrow{d\overline{\psi}} = 1. \quad (6')$$

The connection between these bases is given by

$$\langle\overline{\psi}|\psi\rangle = Ce^{\overline{\psi}_\mu\psi^\mu}, \quad \langle\psi|\overline{\psi}\rangle = C^{-1}e^{\psi^\mu\overline{\psi}_\mu} \quad (9)$$

(without loss of generality, $C = 1$).

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III. WEYL ORDERING AND THE MIDPOINT RULE

The Weyl ordering is now defined ($\mathbf{Q}$ denotes Ψ or $\bar{\Psi}$) as

$$(\mathbf{Q}_1\cdots\mathbf{Q}_K)_{WO}=\frac{1}{K!}\sum_{\text{perm}}(-1)^{\sigma(\text{perm})}\mathbf{Q}_{i_1}\cdots\mathbf{Q}_{i_K}. \quad (10)$$

The sign of the permutation is due to the statistics.

A useful property of the Weyl ordering is

$$(\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_K)_{WO}=\tfrac{1}{2}(\mathbf{Q}_1(\mathbf{Q}_2\cdots\mathbf{Q}_K)_{WO}\pm(\mathbf{Q}_2\cdots\mathbf{Q}_K)_{WO}\mathbf{Q}_1), \quad (11)$$

valid for fermions and bosons, with the minus sign when $\mathbf{Q}_1$ and $(\mathbf{Q}_2\cdots\mathbf{Q}_K)_{WO}$ are fermions. [The proof relies on the fact that thanks to the symmetry of Weyl ordering, the nonvanishing (anti)commutators cancel.]

Now we can prove in general that to a Weyl-ordered fermionic Hamiltonian corresponds the midpoint prescription in the path integral, as sketched by several authors.[7] We proceed by induction.

Any Hamiltonian is a polynomial in the fermionic variables, so it is sufficient to prove the theorem for a product of Ψ's and $\bar{\Psi}$'s. For $\mathbf{H}=\bar{\Psi}_1\cdots\bar{\Psi}_K$, which is Weyl ordered, we have

$$\begin{aligned}\langle\psi_2|\exp(-i\epsilon\mathbf{H})|\psi_1\rangle&=\int\langle\psi_2|(1-i\epsilon\mathbf{H})|\bar{\psi}\rangle d^N\bar{\psi}\langle\bar{\psi}|\psi_1\rangle\\&=\int\exp\left\{i\epsilon\left[i\bar{\psi}_\mu\,\frac{\psi_2^\mu-\psi_1^\mu}{\epsilon}-H(\bar{\psi})\right]\right\}d^N\bar{\psi}.\end{aligned} \quad (12)$$

It is easy to recognize the exponential of the action under the integral. Here the theorem holds: Since the Hamiltonian depends only on $\bar{\Psi}$ we have no ordering ambiguities and no midpoint at all.

Now let us suppose the correspondence is valid for a Weyl-ordered operator $(\mathbf{A})_{WO}$ corresponding to the classical function $A(\psi,\bar{\psi})$ and consider $B=\psi A$; we have

$$\mathbf{B}_{WO}\equiv(\Psi\mathbf{A})_{WO}=\tfrac{1}{2}(\Psi\mathbf{A}_{WO}\pm\mathbf{A}_{WO}\Psi), \quad (13)$$

having used Eq. (11). Finally,

$$\begin{aligned}\langle\psi_2|\exp(-i\epsilon\mathbf{B}_{WO})|\psi_1\rangle&\equiv\left\langle\psi_2\left|\exp\left[\frac{-i\epsilon(\Psi\mathbf{A}_{WO}\pm\mathbf{A}_{WO}\Psi)}{2}\right]\right|\psi_1\right\rangle\\&=\int\exp\left\{i\epsilon\left[i\bar{\psi}_\mu\,\frac{\psi_2^\mu-\psi_1^\mu}{\epsilon}-\frac{(\psi_2A(\bar{\psi},\psi_{MP})\pm A(\bar{\psi},\psi_{MP})\psi_1)}{2}\right]\right\}d^N\bar{\psi}\\&=\int\exp\left\{i\epsilon\left[i\bar{\psi}_\mu\,\frac{\psi_2^\mu-\psi_1^\mu}{\epsilon}-B(\bar{\psi},\psi_{MP})\right]\right\}d^N\bar{\psi},\quad\text{with }\psi_{MP}=\frac{\psi_1+\psi_2}{2}.\end{aligned} \quad (14)$$

Now any product of Ψ's and $\bar{\Psi}$'s can be generated by iteration of (13), starting from $\mathbf{A}$ equal to a product of $\bar{\Psi}$'s; thus for a general Hamiltonian H we have

$$\langle\psi_2|\exp(-i\epsilon\mathbf{H}_{WO}|\psi_1\rangle=\int d^N\bar{\psi}\exp\left\{i\epsilon\left[i\bar{\psi}_\mu\frac{\psi_2^\mu-\psi_1^\mu}{\epsilon}-H(\bar{\psi},\psi_{MP})\right]\right\}. \quad (15)$$

Equation (15) proves our first statement. In Sec. IV we show how the corresponding finite form of the discrete path integral gives rise to the naive Feynman rules which we can obtain from the more usual continuous path integral: Since the former is well defined we may gain more insight of the latter.

IV. EQUIVALENCE BETWEEN THE MIDPOINT AND CONTINUOUS PATH INTEGRALS

Let us recall a few points about the quantization of fermions via path integrals.

Given a Hamiltonian $H=H_0+H_I$, where $H_0=\omega\bar{\psi}\psi$, we have, for the action integral,

$$S=\int dt\,\{i\bar{\psi}\dot{\psi}-\omega\bar{\psi}\psi-H_I(\bar{\psi},\psi)\}. \quad (16)$$

The fundamental object is the generator of the Green's functions (in the following we take $N=1$)

$$\begin{aligned}Z(\bar{\chi},\chi)&=\int D\psi\,D\bar{\psi}\exp\left\{i\left[S+\int dt(\bar{\chi}\psi+\bar{\psi}\chi)\right]\right\}\\&=\exp\left[-i\int dt\,H_I\left(\frac{\overleftarrow{\delta}}{i\delta\chi},\frac{\overrightarrow{\delta}}{i\delta\bar{\chi}}\right)\right]Z_0(\bar{\chi},\chi),\end{aligned} \quad (17)$$

where

$$Z_0(\bar{\chi},\chi)=\int D\psi\,D\bar{\psi}\exp\left[i\int dt(i\bar{\psi}\dot{\psi}-\omega\bar{\psi}\psi+\bar{\chi}\psi+\bar{\psi}\chi)\right]. \quad (18)$$

The interaction Hamiltonian defines the vertices, while the derivatives of Z_0 give the free propagators:

$$\langle\psi(t)\bar{\psi}(t')\rangle_0=\frac{\overrightarrow{\delta}}{i\delta\bar{\chi}(t)}\log Z_0\frac{\overleftarrow{\delta}}{i\delta\chi(t')}\bigg|_{\bar{\chi},\chi=0}. \quad (19)$$

Eventually this machinery generates the usual Feynman rules of the perturbation expansion.

To compute the propagators we complete the square in Z_0 to obtain

$$Z_0(\bar{\chi},\chi)=\det\left[\frac{d}{dt}+i\omega\right]\exp\left[-\int dt\,\bar{\chi}\left(\frac{d}{dt}+i\omega\right)^{-1}\chi\right]. \quad (18')$$

Finally,

$$\begin{aligned}\langle\psi(t)\bar{\psi}(t')\rangle_0&=\left(\frac{d}{dt}-i\omega\right)\Delta_F(t-t')\\&=\theta(t-t')\exp[-i\omega(t-t')],\end{aligned} \quad (20)$$

where

$$\Delta_F(t) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{e^{i\omega' t}}{\omega^2 - \omega'^2 - i\epsilon}$$

is the Feynman propagator.

Now we turn our attention to the discrete formalism. On the time lattice, where $t = t_0 + k\epsilon$, the generator of the exact Green's functions takes the form

$$\begin{aligned} Z^{\text{lat}}(\bar\chi,\chi) = \lim_{\substack{T\to+\infty \\ t_0\to-\infty}} \int \langle 0|\psi(T)\rangle \prod_{k=1}^{T} d\psi(k)d\bar\psi(k) \\ \times \exp\Big\{ i\epsilon \sum_{k=1}^{T} \Big[i\bar\psi(k)\frac{\psi(k)-\psi(k-1)}{\epsilon} \\ - H(\bar\psi(k),\psi(k)) + \bar\chi(k)\psi(k) \\ + \bar\psi(k)\chi(k)\Big]\Big\} d\psi(0)\langle\psi(0)|0\rangle. \end{aligned} \tag{21}$$

Now $\langle 0|\psi\rangle \propto 1$ and $\langle\psi|0\rangle \propto \psi$ (see Ref. 6), so that in the exponential we can consider $\psi(0) = 0$ because of the presence of the last factor; also, the integral over $\psi(0)$ is nothing but the normalization of the vacuum state: $\langle 0|0\rangle = 1$. Thus we have

$$\begin{aligned} Z^{\text{lat}}(\bar\chi,\chi) = \lim_{\substack{T\to+\infty \\ t_0\to-\infty}} \exp\Big[-i\sum_{k=1}^{T} H_I\Big(\frac{\overleftarrow{\partial}}{i\epsilon\partial\chi},\frac{\overrightarrow{\partial}}{i\epsilon\partial\bar\chi}\Big)\epsilon\Big] \\ \times Z_0^{\text{lat}}(\bar\chi,\chi) \end{aligned} \tag{22}$$

and

$$\begin{aligned} Z_0^{\text{lat}}(\bar\chi,\chi) = \lim_{\substack{T\to+\infty \\ t_0\to-\infty}} \det A \\ \times \exp\Big[-\epsilon^2 \sum \bar\chi(i) A^{-1}(i,j)\chi(j)\Big], \end{aligned} \tag{23}$$

with

$$A(i,j) = [(1+i\epsilon\omega)\delta_{ij} - \delta_{i,j+1}]. \tag{24}$$

The inverse of this matrix is the free propagator

$$\begin{aligned} &\langle\psi(j+m)\bar\psi(j)\rangle_0^{\text{lat}} \\ &= \frac{\overrightarrow{\partial}}{i\epsilon\partial\bar\chi(j+m)} \log Z_0^{\text{lat}} \frac{\overleftarrow{\partial}}{i\epsilon\partial\chi(j)}\Big|_{\bar\chi,\chi=0} \\ &= A^{-1}(j+m,j). \end{aligned} \tag{25}$$

After some algebra we obtain

$$A^{-1}(j+m,j) = \begin{cases} 0, & m<0, \\ (1+i\epsilon\omega)^{-(m+1)}, & m\geqslant 0, \end{cases} \tag{25'}$$

and in the limit $\epsilon\to 0$ we have

$$\langle\psi(j+m)\bar\psi(j)\rangle_0^{\text{lat}} = \begin{cases} 0, & m<0, \\ \exp(-i\epsilon\omega m), & m\geqslant 0. \end{cases} \tag{26}$$

We see that the "discrete" propagator (26) is equivalent to the "continuous" one (20) when $m\epsilon = t - t' \neq 0$. For $t - t' = 0$ we have $\langle\psi(t)\bar\psi(t')\rangle_0 = (d/dt - i\omega)\Delta_F(0) = \frac{1}{2}$, although it is not clear to what it corresponds on the time lattice formalism. Let us clarify this point.

Until now we have ignored the ambiguity in the form of the action integral on the time lattice. This means that we can evaluate the Hamiltonian in Eq. (21) at any point between $\psi(k-1)$ and $\psi(k)$. Actually, no choice modifies the discrete free propagator (26); however, in order to have full equivalence between the discrete and continuous formalisms we must cure the $t - t' = 0$ disease. This is what the midpoint choice does; as a matter of fact, when we let $\psi(k)\to\psi_{MP}(k) = [\psi(k-1)+\psi(k)]/2$ in the interaction Hamiltonian H_I we have, for the equal-time propagator actually appearing in the perturbation expansion,

$$\langle\psi_{MP}(j)\bar\psi(j)\rangle_0^{\text{lat}} = \tfrac{1}{2}(0+1) = \tfrac{1}{2} = \langle\psi(t)\bar\psi(t)\rangle_0$$

and so we have succeeded in achieving the aim.

V. CONCLUSIONS

In this paper we have shown how the Weyl-ordered form of a fermionic Hamiltonian corresponds to the midpoint discrete path integral and in turn the latter corresponds, in the perturbation scheme, to the naive Feynman rules of the continuous path integral. This result may serve, for instance, as a useful tool in the study of the behavior of the quantum theory under coordinate transformation. In fact, to this purpose the operator or discrete path integral formalisms are the most reliable ways to work,[2] while the continuous path integral is the most direct approach. For instance, if an extra fermionic potential contribution is needed to make the Weyl Hamiltonian covariant, it must be added to the continuous fermionic Lagrangian as well in order to obtain the covariant Feynman rules.

In addition, we want to point out that the proof follows the same lines as in the bosonic case; this stresses the symmetry between bosons and fermions.

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[6]In this notation the bra $\langle\psi|$ is *not* the conjugate of $|\psi\rangle$: $|\psi\rangle^\dagger \neq \langle\psi| = |\bar\psi\rangle^\dagger$ up to a phase factor. For instance, we have $|\psi\rangle = -|0\rangle + |1\rangle\psi$ and $\langle\psi| = \psi\langle 0| + \langle 1|$, where $|0\rangle$ and $|1\rangle$ are the usual occupation number eigenvectors.

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