Quantum Field Theory

Problem Sheet 1

1. The general Gaussian integral can be readily evaluated:

$$Z_A(b) = \int d^n x \, \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i \right) \tag{1}$$

$$= (2\pi)^{n/2} \left(\det A\right)^{-1/2} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) , \tag{2}$$

where $\Delta = A^{-1}$. The existence of Δ is guaranteed by the non-vanishing e.vals of A. Check the result in Eq. 2 by changing the integration variables in the integral:

$$x_i = y_i + \sum_{j=1}^{n} \Delta_{ij} b_j \, .$$

2. Let x and y be stochastic variables such that:

$$\langle x \rangle = \langle y \rangle = 0 \, , \langle x^2 \rangle = 5 \, , \langle xy \rangle = 3 \, , \langle y^2 \rangle = 2 \, ,$$

compute $\langle x^4 \rangle$, $\langle x^3 y \rangle$, $\langle x^2 y^2 \rangle$.

3. Compute the six-point function

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \rangle$$

using Wick's theorem.

4. Consider

$$\langle x_k F(x) \rangle_0 = \int d\mu_0(x) x_k F(x) \tag{3}$$

Show that

$$\langle x_k F(x) \rangle_0 = \sum_l \langle x_k x_l \rangle_0 \langle \frac{\partial F}{\partial x_l} \rangle_0 \, . \tag{4}$$

Hint:

$$-\sum_l \Delta_{kl} \frac{\partial}{\partial x_l} \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j \right) = x_k \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j \right) \, . \tag{5}$$

5. Compute the ratio

$$Z(\lambda)/Z(0) \tag{6}$$

for the potential

$$V(x) = \frac{1}{4!} \sum_{i=1}^{n} x_i^4 \, , \tag{7}$$

to second order in λ.

6. Show that all the vacuum contributions cancel when computing $\langle x_{i_1} x_{i_2} \rangle$. The final result is

$$\frac{1}{Z(\lambda)} \int d^n x \, e^{-S(x;\lambda)} x_{i_1} x_{i_2} = \left[\Delta_{i_1 i_2} - \lambda \left(\frac{1}{2} \sum_{i=1}^{n} \Delta_{ii_1} \Delta_{ii_2} \Delta_{ii} \right) + \right.$$
$$+ \lambda^2 \left(\frac{1}{4} \sum_{i,j=1}^{n} \Delta_{ii_1} \Delta_{ii_2} \Delta_{ij}^2 \Delta_{jj} + \frac{1}{6} \sum_{i,j=1}^{n} \Delta_{ii_1} \Delta_{ji_2} \Delta_{ij}^3 + \right.$$
$$\left. \left. + \frac{1}{4} \sum_{i,j=1}^{n} \Delta_{ii_1} \Delta_{ji_2} \Delta_{ij} \Delta_{ii} \Delta_{jj} \right) \right] . \tag{8}$$

Write a diagrammatic representation for the contributions $O(\lambda^2)$.