Lecture 4

Scalar field correlators

4.1 Field correlators

As discussed in the previous lecture the field correlators are obtained from the partition function taking functional derivatives:

$$\begin{aligned} G^{(n)}\left(x_1, \ldots, x_n\right) &= \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle && (4.1)\\ &= \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\ldots\left(\frac{1}{i}\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}. && (4.2)\end{aligned}$$

The perturbative definition of the path integral has led to an expansion where we can classify terms according to the number of currents that appear. We denoted this number by E above. Clearly the only terms that contribute to an n-point function are the ones with $E=n$. For a given values of E there will be contibutions from all the values of V that satisfy

$$E = 2P - 3V\,,$$

where P in an integer. Therefore the calculation of correlators can be expressed as a perturbative expansion in powers of the coupling:

$$G^{(n)}\left(x_1, \ldots, x_n\right) = \sum_V g^V G^{(n,V)}\left(x_1, \ldots, x_n\right)\,. \tag{4.3}$$

Each functional derivative replaces a factor of J in the integrand with a Dirac delta. Performing the corresponding integration leads to replacing the argument at the end of the propagator that was connected to the current with the argument of the functional derivative:

$$\frac{\delta}{\delta J(x)}\int \ldots d^d y \ldots J(y)\Delta(y,\ldots)\ldots = \int \ldots \Delta(x,\ldots)\ldots\,. \tag{4.4}$$

4.1.1 Two-point correlator

The two-point function

$$\begin{aligned} G^{(2)}(x,y) &= \langle T\phi(x)\phi(y)\rangle && (4.5)\\ &= \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right) Z[J]\Big|_{J=0}. && (4.6)\end{aligned}$$

The only terms in Eq. (**??**) that contribute are the ones corresponding to $E=2$.

V=0 At the lowest order in g, *i.e.* for $V=0$, we have

$$\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet = \frac{i}{2}\int d^D z_1\, d^D z_2\, J(z_1)\Delta(z_1-z_2)J(z_2)\,. \tag{4.7}$$

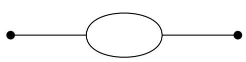
Taking the functional derivatives with respect to J we get a factor of two, from acting with each derivative on both $J(z_1)$ and $J(z_2)$:

$$G^{(2,0)}(x,y) = \frac{1}{i}\Delta(x-y)\,, \tag{4.8}$$

where the second index in the suffix indicates the order in the perturbative expansion as discussed above.
It is easy to verify that there are no contributions to $Z[J]$ with $E=2$ and $V=1$.

V=2 The next contributions to the two-point functions come from terms with $V=2$, and there are two distinct diagram topologies at this order.

a. The first diagram topology that contributes to $G^{(2)}$ is

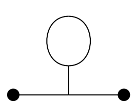


$$= \frac{1}{2^2}\int d^Dz_1\, d^Dz_2\, d^Dw_1\, d^Dw_2 \times$$
$$\times\, J(z_1)\Delta(z_1-w_1)\Delta(w_1-w_2)^2\Delta(w_2-z_2)J(z_2)\,. \tag{4.9}$$

Taking derivatives and inserting the appropriate factor of i yields a total contribution of

$$G_a^{(2,2)}(x,y) = -\frac{1}{2}\int d^Dw_1\, d^Dw_2\, \Delta(x-w_1)\Delta(w_1-w_2)^2\Delta(w_2-y)\,. \tag{4.10}$$

b. The other contribution comes from



$$= \frac{1}{2^2}\int d^Dz_1\, d^Dz_2\, d^Dw_1\, d^Dw_2 \times$$
$$\times\, J(z_1)\Delta(z_1-w_1)\Delta(w_1-z_2)\Delta(w_1-w_2)\Delta(0)J(z_2)\,. \tag{4.11}$$

The net contribution from this diagram is

$$G_b^{(2,2)}(x,y) = -\frac{1}{2}\int d^Dw_1\, d^Dw_2\, \Delta(x-w_1)\Delta(w_1-y)\Delta(w_1-w_2)\Delta(0))\,. \tag{4.12}$$

4.1.2 Momentum space

It is useful to write this correlators in momentum space. We already discussed the representation of the free propagator in momentum space:

$$\Delta(x) = \int \frac{d^Dp}{(2\pi)^D}\, e^{-ip\cdot x}\frac{1}{p^2-m^2+i\epsilon}\,. \tag{4.13}$$

The Fourier transform of the two-point correlator is defined as

$$\tilde{G}^2\left(p,p'\right) = \int d^D x\, d^D y\, e^{ip\cdot x} e^{ip'\cdot y} G^{(2)}(x,y) \tag{4.14}$$

$$= \int d^D z\, d^D y\, e^{ip\cdot z} e^{i(p+p')\cdot y} G^{(2)}(z) \tag{4.15}$$

$$= (2\pi)^D\, \delta(p+p') \int d^D z\, e^{ip\cdot z} G^{(2)}(z) \tag{4.16}$$

$$= (2\pi)^D\, \delta(p+p')\, \frac{1}{i} \tilde{\Delta}_F(p)\,. \tag{4.17}$$

Note that translation invariance of $G^{(2)}(x,y)$ produces an overall Dirac delta that implements conservation of the total momentum. Again we can look at the contributions to $\tilde{G}^{(2)}(p,p')$ order by order in perturbation theory.

V=0 At lowest order in g, we simply have the Fourier transform of the free propagator:

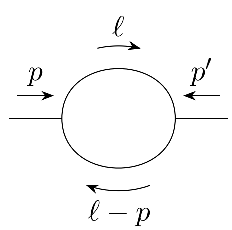
$$\tilde{G}^{(2,0)}(p,p') = (2\pi)^D\, \delta(p+p')\, \frac{1}{i} \tilde{\Delta}(p) \tag{4.18}$$

$$= (2\pi)^D\, \delta(p+p')\, \frac{1}{i} \frac{1}{p^2 - m^2 + i\epsilon}\,. \tag{4.19}$$

V=2 At order g^2, we obtain

$$\begin{aligned}\tilde{G}^{(2,2)}_a(p,p') = &-\frac{1}{2}\, (2\pi)^D\, \delta(p+p')\, \frac{1}{p^2-m^2+i\epsilon} \times \\ &\times \left\{ \int \frac{d^D \ell}{(2\pi)^D}\, \frac{1}{\ell^2 - m^2 + i\epsilon}\, \frac{1}{(\ell-p)^2 - m^2 + i\epsilon} \right\} \frac{1}{p^2 - m^2 + i\epsilon}\,.\end{aligned} \tag{4.20}$$

The above expression can be represented by a Feynman diagram in momentum space:



(4.21)

The Feynman rules in momentum space can be summarised as follows (adapted from Srednicki's book).

1. Draw n lines for a n-point correlator.

2. Leave one end of each external line free, and attach the other to the lines coming out of a vertex.

3. The i-th external line carries momentum p_i, which we assume to be incoming momentum, and represent with a line pointing towards the vertex.

4. Four-momenta flow along the arrows, and the total momentum is conserved at each vertex. For a diagram without loops, this fixes the momentum of *all* internal lines.

5. The value of the diagram is given by the product of a factor of $i/(p^2 - m^2 + i\epsilon)$ for each line with momentum p, a factor of $1/i$ for the external end of a line, and a factor $ig(1/i)^{\#}$ for each vertex, where # is the number of legs connecting at each vertex.

6. A diagram with L loops will have L internal momenta that are not fixed by momentum conservation. We integrate over those momenta, with measure $d^p/(2\pi)^D$.

7. Determine the symmetry factor associated to permutations of *internal* propagators and vertices.

Exercise 4.1.1 Use the Feynman rules in momentum space to compute $G_b^{(2,2)}$. Check that you get the same result by performing a Fourier transform of the result in position space.

4.2 Physical states

The eigenstates of the Hamiltonian form a complete set of states. They can be classified in three categories.

1. The vacuum state $|0\rangle$ is the lowest energy state, and corresponds to a state with no particles.

2. The one-particle states $|\mathbf{p}, \sigma\rangle$ are classified by their spatial momentum. Their energy is given by the relativistic dispersion relation

$$E_p = \sqrt{\mathbf{p}^2 + m_{\text{phys}}^2}\,, \tag{4.22}$$

where m_{phys} denotes the physical mass of the state. Note that the physical mass *does not* need to coincide with the mass that appears in the Lagrangian. We will discuss this point in detail later. Any other quantum number necessary to identify the particle is denoted here by σ. The states are normalised by imposing

$$\langle \mathbf{p}, \sigma | \mathbf{p}', \sigma' \rangle = \delta_{\sigma\sigma'}\, 2E_p\, (2\pi)^{D-1} \delta(\mathbf{p} - \mathbf{p}')\,. \tag{4.23}$$

3. The multiparticle states $|\mathbf{P}; n\rangle$ are classified by their total spatial momentum $\mathbf{P}$, plus other parameters such as the relative momenta between the particles, which we denote collectively by n. The energy of the mutiparticle states is $\sqrt{\mathbf{P}^2 + M^2}$, where M^2 is one of the parameters included in n. The threshold for producing multiparticle states is given by the energy of two particles at rest $M = 2m_{\text{phys}}$.

The completeness relation can be written as

$$\begin{aligned} |0\rangle\langle 0| + \sum_\sigma \int d\Omega_p |\mathbf{p}, \sigma\rangle \langle \mathbf{p}, \sigma| + \\ + \sum_n \int d\Omega_P |\mathbf{P}, n\rangle \langle \mathbf{P}, n| \; = 1\,, \end{aligned} \tag{4.24}$$

and we have introduced the Lorentz-invariant integration measure

$$d\Omega_p = \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{2E_p}\,. \tag{4.25}$$

Note that the 'sum' over n is a short-hand notation, which may involve integrations over continuum variables such as the relative momentum, or the invariant mass of the state.

4.3 Polology

In this section, we shall learn some features about the analytic structure of field correlators, and discuss their relevance in order to extract physical information from the correlators. Starting from an n-point correlator in momentum space,

$$\tilde{G}^{(n)}\left(p_1, \ldots, p_n\right) = \int d^D x_1 \ldots d^D x_n\, e^{-ip_1 \cdot x_1} \ldots e^{-ip_n \cdot x_n} \left\langle T\phi(x_1) \ldots \phi(x_n) \right\rangle, \tag{4.26}$$

we want to focus on the contribution coming from the sector where the values $x_1^0, \ldots, x_r^0$ are all larger than the values of $x_{r+1}^0, \ldots, x_n^0$, for some value of r between 1 and $n-1$. This contribution can be written as

$$\begin{aligned} \tilde{G}^{(n)}\left(p_1, \ldots, p_n\right) = \int d^D x_1 \ldots d^D x_n\, e^{-ip_1 \cdot x_1} \ldots e^{-ip_n \cdot x_n} \times \\ \times \theta\left(\min\{x_1^0 \ldots x_r^0\} - \max\{x_{r+1}^0 \ldots x_n^0\}\right) \\ \times \langle T\Big[\phi(x_1) \ldots \phi(x_r)\Big] T\Big[\phi(x_{r+1}) \ldots \phi(x_n)\Big] \rangle\,. \end{aligned} \tag{4.27}$$

We can introduce a complete set of states inbetween the two T-ordered products, and look at the result coming from the one-particle states. Defining new integration variables

$$x_i = x_1 + y_i\,, \quad \text{for } i = 2, \ldots, r\,, \tag{4.28}$$

we can write

$$\begin{aligned}\phi(x_1)\ldots\phi(x_r) &= \phi(x_1)\ldots\phi(x_1+y_i)\ldots\phi(x_1+y_r)\\ &= e^{iP\cdot x_1}\phi(0)e^{-iP\cdot x_1}\ldots e^{iP\cdot x_1}\phi(y_i)e^{-iP\cdot x_1}\ldots e^{iP\cdot x_1}\phi(y_r)e^{-iP\cdot x_1}\,,\end{aligned} \tag{4.29}$$

and hence

$$\langle 0|T\phi(x_1)\ldots\phi(x_r)|p,\sigma\rangle = e^{-ip\cdot x_1}\langle 0|T\phi(0)\ldots\phi(y_r)|p,\sigma\rangle\,. \tag{4.30}$$

A similar shift of the integration variables can be done using

$$x_i = x_{r+1} + y_i\,, \quad \text{for } i = r+2, \ldots, n\,, \tag{4.31}$$

The argument of the theta function can be rewritten as

$$\min\{x_1^0\ldots x_r^0\} - \max\{x_{r+1}^0\ldots x_n^0\} = x_1^0 - x_{r+1}^0 + \min\{0\ldots y_r^0\} - \max\{0\ldots y_n^0\}\,. \tag{4.32}$$

Using the integral representation of the theta,

$$\theta(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega\,\frac{e^{-i\omega\tau}}{\omega+i\epsilon}\,, \tag{4.33}$$

and performing the integrals over x_1 and x_{r+1}, yields

$$\begin{aligned}\tilde{G}^{(n)}(p_1,\ldots,p_n) = &\int d^Dy_2\ldots d^Dy_r\,d^Dy_{r+2}\ldots d^Dy_n\\ &\times e^{-ip_2\cdot y_2}\ldots e^{-ip_r\cdot y_r}\,e^{-ip_{r+2}\cdot y_{r+2}}\ldots e^{-ip_n\cdot y_n}\\ &\times \frac{-1}{2\pi i}\int\frac{d\omega}{\omega+i\epsilon}\,\exp\left\{-i\omega\left[\min\{0\ldots y_r^0\}-\max\{0\ldots y_n^0\}\right]\right\}\\ &\times \sum_\sigma\int d\Omega_p\langle 0|T\phi(0)\ldots\phi(y_r)|p,\sigma\rangle\langle p,\sigma|T\phi(0)\ldots\phi(y_n)|0\rangle\\ &\times (2\pi)^D\delta(\mathbf{p}-\mathbf{p}_1-\ldots-\mathbf{p}_r)\,\delta(E_p+\omega-p_1^0-\ldots-p_r^0)\\ &\times (2\pi)^D\delta(\mathbf{p}+\mathbf{p}_{r+1}+\ldots+\mathbf{p}_n)\,\delta(E_p+\omega+p_{r+1}^0+\ldots+p_n^0)\,.\end{aligned} \tag{4.34}$$

Performing the integrals over the spatial components of p and ω yields

$$\begin{aligned}&\delta(\mathbf{p}_1+\ldots+\mathbf{p}_n)\,\text{and}\,,\\ &\delta(p_1^0+\ldots+p_n^0)\frac{1}{q^0-E_p+i\epsilon}\,,\end{aligned} \tag{4.35}$$

respectively, where the four-momentum q is defined as

$$q = p_1 + \ldots + p_r = -p_{r+1} - \ldots - p_n \,. \tag{4.36}$$

Finally, if we are interested in the residue at the pole, we can rewrite

$$\frac{1}{q^0 - E_p + i\epsilon} \longrightarrow \frac{2E_p}{q^2 - m_{\text{phys}}^2 + i\epsilon} \tag{4.37}$$

Collecting all the terms, and ignoring a phase factor that reduces to one at the pole, we find that the one-particle state contribution to the integration over the specific sector that we considered above yields

$$\tilde{G}^{(n)}(p_1, \ldots, p_n) \longrightarrow \delta(p_1 + \ldots + p_n) \, \frac{1}{q^2 - m_{\text{phys}}^2 + i\epsilon} \sum_\sigma M_{0|q\sigma}(p_2, \ldots, p_r) M_{q\sigma|0}(p_{r+2}, \ldots, p_n) \,. \tag{4.38}$$

In the equation above we have defined

$$M_{0|q\sigma}(p_2, \ldots, p_r) = \int d^D y_2 \ldots d^D y_r \, e^{-ip_2 \cdot y_2} \ldots e^{-ip_r \cdot y_r} \langle 0 | T \phi(0) \ldots \phi(y_r) | q, \sigma \rangle \,, \tag{4.39}$$

$$M_{q\sigma|0}(p_{r+2}, \ldots, p_n) = \int d^D y_{r+2} \ldots d^D y_n \, e^{-ip_{r+2} \cdot y_{r+2}} \ldots e^{-ip_n \cdot y_n} \langle q, \sigma | T \phi(0) \ldots \phi(y_n) | 0 \rangle \,, \tag{4.40}$$

and we notice that this contribution appears multiplied by a Dirac delta that enforces the conservation of total momentum. The important result here is that the correlators in momentum space have a pole singularity whenever $q = p_1 + \ldots + p_r$ *goes on-shell*, *i.e.* $q^2 = m_{\text{phys}}^2$.

Note that these results are completely general, and in particular do not rely on the perturbative definition of the correlators.

4.4 Källen-Lehmann representation

Let us now come back to the 2-point function, and find a representation that allows us to extract some physical information about the scalar field. We define the full propagator as

$$\Delta_F(x - y) = i \langle 0 | T \phi(x) \phi(y) | 0 \rangle \,, \tag{4.41}$$

and define the field so that

$$\langle 0 | \phi(x) | 0 \rangle = 0 \,, \text{ and } \langle \mathbf{p} | \phi(0) | 0 \rangle = 1 \,, \tag{4.42}$$

where $|\mathbf{p}\rangle$ represents the physical one-particle state, and we have dropped the dependence on σ. As usual the full propagator in momentum space is defined by taking the Fourier transform:

$$\tilde{\Delta}_F(p) = \int d^D x \, e^{ip \cdot (x-y)} \Delta(x - y) \,. \tag{4.43}$$

Free theory With these conventions, the free theory result for the propagator is

$$\tilde{\Delta}(p) = \frac{1}{p^2 - m^2 + i\epsilon} \,. \tag{4.44}$$

Eq. (4.50) shows that $\tilde{\Delta}$ has a pole at $p^2 = m^2$. For the free particle we find a pole in the propagator, at a value which coincides with the parameter in the Lagrangian.

Interacting theory For the interacting theory, we can derive a general expression, which again does not rely on the perturbative definition of the two-point function. Let us first consider the case where $x^0 > y^0$:

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle =& \langle 0|\phi(x)\phi(y)|0\rangle \\ =& \langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle + \\ &+ \int d\Omega_p \,\langle 0|\phi(x)|\mathbf{p}\rangle\langle \mathbf{p}|\phi(y)|0\rangle + \\ &+ \sum_n \int d\Omega_P \,\langle 0|\phi(x)|\mathbf{P},n\rangle\langle \mathbf{P},n|\phi(y)|0\rangle \,.\end{aligned} \tag{4.45}$$

The conventions in Eq. (4.48) allow us to simplify the expression above.

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle =& \int d\Omega_p \, e^{-ip\cdot(x-y)} + \\ &+ \sum_n \int d\Omega_p \, e^{-ip\cdot(x-y)} \, |\langle \mathbf{p},n|\phi(0)|0\rangle|^2 \,.\end{aligned} \tag{4.46}$$

Because we are working with a scalar field, the matrix element $\langle \mathbf{p},n|\phi(0)|0\rangle$ is invariant under Lorentz transformations, and therefore can only depend on p via the invariant mass M^2. We can therefore introduce the *spectral density*

$$\rho(s) = \sum_n |\langle \mathbf{p},n|\phi(0)|0\rangle|^2 \, \delta(s - M^2) \,, \tag{4.47}$$

and write the two-point correlator as

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle =& \int d\Omega_p \, e^{-ip\cdot(x-y)} + \\ &+ \int_{4m^2}^{\infty} ds \, \rho(s) \int d\Omega_p \, e^{-ip\cdot(x-y)} \,.\end{aligned} \tag{4.48}$$

Similar manipulations for the case $y^0 > x^0$ yield

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int d\Omega_p\, e^{ip\cdot(x-y)} + \\ + \int_{4m^2}^{\infty} ds\, \rho(s) \int d\Omega_p\, e^{ip\cdot(x-y)} \,. \tag{4.49}$$

Collecting both contributions to the T-ordered product

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \,, \tag{4.50}$$

and using

$$\frac{1}{i}\int \frac{d^Dp}{(2\pi)^D}\frac{e^{-ip\cdot(x-y)}}{p^2-m_{\text{phys}}^2+i\epsilon} = \theta(x^0-y^0)\int d\Omega_p\, e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\int d\Omega_p\, e^{ip\cdot(x-y)} \,, \tag{4.51}$$

we finally obtain

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^Dp}{(2\pi)^D}\, e^{-ip\cdot(x-y)} \Big[\frac{1}{p^2-m_{\text{phys}}^2+i\epsilon} + \\ + \int_{4m_{\text{phys}}^2}^{\infty} ds\, \rho(s) \frac{1}{p^2-s+i\epsilon}\Big] \,. \tag{4.52}$$

Eq. (4.58) allows us to read the expression for the full propagator in momentum space:

$$\tilde{\Delta}_F(p) = \frac{1}{p^2-m_{\text{phys}}^2+i\epsilon} + \int_{4m_{\text{phys}}^2}^{\infty} ds\, \rho(s)\frac{1}{p^2-s+i\epsilon} \,. \tag{4.53}$$

We see that the two-point correlator of a field ϕ that satisfies the conditions in Eq. (4.48) has a pole for $p^2 = m_{\text{phys}}^2$, with residue exactly equal to one. Note that the field ϕ does not need to be the field that appears in the Lagrangian. Knowledge about the multiparticle states of the theory is encoded in the two-point function via the integral on the RHS side of Eq. (4.59).

4.5 S Matrix

In order to compute the quantum amplitude for a physical process involving arbitrary numbers of particles in the initial and final state, we need to compute the overlap of a state prepared in the distant past (the so-called *in* state), with the resulting final state in the distant future (the so-called *out* state). If we want to describe a $2 \to n$ process – like a pp collision at the LHC – we need to compute

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_n; \mathrm{out} | \mathbf{k}_1, \mathbf{k}_2; \mathrm{in} \rangle \,. \tag{4.54}$$

The S-matrix allows us to express this scalar product between in- and out-states in terms of states defined at any common reference time:

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_n; \mathrm{out} | \mathbf{k}_1, \mathbf{k}_2; \mathrm{in} \rangle = \langle \mathbf{p}_1, \ldots, \mathbf{p}_n | S | \mathbf{k}_1, \mathbf{k}_2 \rangle \,. \tag{4.55}$$

It is usual to separate the S matrix into the identity operator, corresponding to particles not interacting, plus a non-trivial part which is usually denoted T:

$$S = 1 + iT \,. \tag{4.56}$$

4.6 LSZ reduction

An important corollary of the result shown in section 4.3 is obtained by setting $r = 1$. In this case, the previous discussion allows us to conclude that the correlators in momentum space have a pole whenever the momentum of one of the fields is on-shell. Therefore an n-point correlation function has (at least) n poles, each corresponding to one of the momenta $p_i \to m_{\mathrm{phys}}^2$. The residue at this multiple pole yields the S-matrix for a scattering process involving n-particles:

$$\begin{aligned}
\langle p_1' \ldots p_{m'}'; \mathrm{out} | p_1 \ldots p_m; \mathrm{in} \rangle =& \langle p_1' \ldots p_{m'}' | S | p_1 \ldots p_m \rangle & (4.57)\\
=& \lim_{p_j^2, p_k'^2 \to m_{\mathrm{phys}}^2} \prod_{k=1}^{m'} (p_k'^2 - m_{\mathrm{phys}}^2 + i\epsilon) \prod_{j=1}^{m} (p_j^2 - m_{\mathrm{phys}}^2 + i\epsilon) & \\
& \times \tilde{G}^{(m+m')}(p_1, \ldots, p_m, -p_1', \ldots, -p_{m'}') \,, & (4.58)
\end{aligned}$$

where $n = m + m'$, and the fields are normalised so that

$$\langle \mathbf{p} | \phi(0) | 0 \rangle = 1 \,. \tag{4.59}$$

We will return to the question of the normalisation of the field later. The LSZ reduction formula provides an elegant way to represent quantum amplitudes using Feynman diagrams in momentum space. We adopt the same rules discussed above, with the following modifications.

1. We associate an outgoing momentum to the external lines that correspond to particles in the final state.

2. We multiply each external line by a factor of $-i(p^2 - m_{\mathrm{phys}}^2 + i\epsilon)$ – the correlators multiplied by these factors are called *truncated* (or *amputated*) correlators.

A heuristic derivation of the LSZ reduction formula is discussed in Problem Sheet 4.

Exercise 4.6.1 Compute the amplitude for the scattering process

$$p_1 p_2 \longrightarrow p_1' p_2'$$

at order g^2 in the ϕ^3 scalar theory. You can assume that $m_{\mathrm{phys}} = m$ in this calculation.

4.7 Optical Theorem

Physical constraints translate into relations between correlators. It is important to be able to derive these relations, and to understand their physical content. One example is provided by the unitarity of the S-matrix, *i.e.* by the conservation of probability in quantum mechanics. Unitarity is written as

$$S^\dagger S = 1\,. \tag{4.60}$$

Inserting the representation of S in terms of the transition matrix yields

$$-i\left(T - T^\dagger\right) = T^\dagger T\,. \tag{4.61}$$

Let us consider the matrix element of Eq. (4.61) between an initial state a and a final state b, and let us factor out a Dirac delta that corresponds to total momentum conservation,

$$\langle b|T|a\rangle = (2\pi)^D \delta(P_a - P_b)\,\mathcal{M}(a \to b)\,. \tag{4.62}$$

Some simple algebra yields on the LHS

$$-i(2\pi)^D \delta\left(P_a - P_b\right)\left[\mathcal{M}(a \to b) - \mathcal{M}(b \to a)^*\right]\,. \tag{4.63}$$

On the RHS we can insert a complete set of states, and rewrite it as

$$(2\pi)^D \delta\left(P_a - P_b\right) \sum_f \int d\Omega_f\, (2\pi)^D \delta\left(P_a - P_f\right) \mathcal{M}(b \to f)^* \mathcal{M}(a \to f)\,. \tag{4.64}$$

The unitarity condition simplifies for $a = b$,

$$2\mathrm{Im}\, \mathcal{M}(a \to a) = \sum_f \int d\Omega_f (2\pi)^D \delta\left(P_a - P_f\right) \left|\mathcal{M}(a \to f)\right|^2\,. \tag{4.65}$$

This is the so-called *optical theorem*, which relates the imaginary part of the forward $a \to a$ amplitude (LHS) to the total cross section $a \to f$ (RHS), summed over *all* final states f.

4.8 Ward identities

The final example of relations between correlators that we are going to discuss are the so-called *Ward identities*. Ward identities are equalities between field correlators that are obtained as a consequence of symmetries of the system. In classical mechanics, symmetries of the action translate into conserved currents according to Noether's theorem. As we will show in this section, the analogue of current conservation in quantum field theory is precisely the Ward identity.

In order to derive the identities, let us start by considering a symmetry transformation of the field, *i.e.* a transformation

$$\phi(x) \mapsto \phi'(x) = \phi(x) + \epsilon\delta\phi(x)\,, \tag{4.66}$$

such that for constant ϵ the action is unchanged. If we introduce a dependence on the space-time coordinate, $\epsilon(x)$, then the variation of the action can be written

$$\delta S = \int d^Dx\, \frac{\delta S}{\delta\phi(x)}\epsilon(x)\delta\phi(x) \tag{4.67}$$

$$= -\int d^Dx\, \epsilon(x)\partial_\mu j^\mu(x)\,, \tag{4.68}$$

where $j^\mu(x)$ is precisely the Noether current that is conserved in the classical theory.

In order to derive the Ward identities, we use Eq. (4.66) to perform a change of integration variables in the functional integral

$$\int \mathcal{D}\phi\, e^{iS[\phi]}O(\phi) = \int \mathcal{D}\phi'\, e^{iS[\phi']}O(\phi')\,, \tag{4.69}$$

and then expand the RHS to first order in ϵ:

$$\int \mathcal{D}\phi\, e^{iS[\phi]}O(\phi) = \int \mathcal{D}\phi\, e^{iS[\phi]}\, \left[1 + i\delta S[\phi]\right]\, \left[O(\phi) + \delta O\right]\,, \tag{4.70}$$

where O is a generic function of the field ϕ. We can now substitute the expressions for δS and δO:

$$\int \mathcal{D}\phi\, e^{iS[\phi]} \left\{-i\int d^Dx\, \epsilon(x)\partial_\mu j^\mu(x) O(\phi) + \int d^Dx\, \frac{\delta O(\phi)}{\delta\phi(x)}\epsilon(x)\delta\phi(x)\right\} = 0\,. \tag{4.71}$$

Rearranging the terms above allows us to write the identity in a way that makes its physical content more obvious:

$$\int d^Dx\, \epsilon(x) \left\{-i\langle\partial_\mu j^\mu(x) O(\phi)\rangle + \langle\frac{\delta O(\phi)}{\delta\phi(x)}\delta\phi(x)\rangle\right\} = 0\,. \tag{4.72}$$

Eq. (4.72) is sometimes referred to as an *integrated Ward identity*. Since it has to be satisfied for every function $\epsilon(x)$, we can derive the *Ward identity*:

$$-i\langle\partial_\mu j^\mu(x) O(\phi)\rangle + \langle\frac{\delta O(\phi)}{\delta\phi(x)}\delta\phi(x)\rangle = 0\,. \tag{4.73}$$

There are two important physical results encoded in Eq. (4.73).

1. Symmetry in QFT translates into a relation between correlators. This is true beyond perturbation theory and is used in defining the renormalization conditions in QFT.

2. Current conservation in QFT is realised at the level of the insertion of $\partial_\mu j^\mu(x)$ in field correlators, up to the terms that come from the variation of O. If O is a product of local fields, this variations is localised in space-time, *i.e.* the contributions are all proportional to Dirac deltas. These terms are called *contact terms*.

Note that in deriving the Ward identity above we have assumed that the integration measure $\mathcal{D}\phi$ is invariant, *i.e.* $\mathcal{D}\phi = \mathcal{D}\phi'$. There are examples where the measure is *not* invariant, which lead to extra terms in the Ward identities. In these cases the Ward identities are called *anomalous*.