Lecture 5

Fermion Fields

5.1 Fermion Path Integral

For the case of fermion fields, we want to define the path integral following the recipe we used for the scalar field. We will treat the fields ψ and $\bar{\psi}$ as Grassmann variables, *i.e.*

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0 \,. \tag{5.1}$$

In order to have a consistent implementation of the anticommuting properties of the fermion fields, the functional derivative with respect to a Grassmann variable must be a Grassmann variable itself. As a consequence

$$\frac{\delta^2 F}{\delta\psi_\alpha(x)\delta\psi_\beta(y)} = -\frac{\delta^2 F}{\delta\psi_\beta(y)\delta\psi_\alpha(x)} \,, \tag{5.2}$$

and

$$\frac{\delta^2 F}{\delta\psi_\alpha(x)\delta\psi_\alpha(x)} = 0 \,. \tag{5.3}$$

In the definition of the generating functional, we introduce independent sources for ψ and $\bar{\psi}$:

$$\int d^D y \, \left[\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)\right] \,, \tag{5.4}$$

such that

$$\frac{\delta}{\delta\eta(x)} \int d^D y \, \left[\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)\right] = -\bar{\psi}(x) \tag{5.5}$$

$$\frac{\delta}{\delta\bar{\eta}(x)} \int d^D y \, \left[\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)\right] = \psi(x) \,. \tag{5.6}$$

Free theory The action for the free Dirac field is

$$S_0\left[\psi, \bar{\psi}\right] = \int d^D x \, \bar{\psi}(x) \left(i\partial\!\!\!/ - m\right) \psi(x) \,. \tag{5.7}$$

Using the rules above for the functional derivative, we can find the classical equation of motion, *i.e.* Dirac's equation

$$\frac{\delta}{\delta\bar{\psi}(x)} S_0\left[\psi, \bar{\psi}\right] = 0 \quad \Longrightarrow \quad \left(i\partial\!\!\!/ - m\right) \psi(x) = 0 \,. \tag{5.8}$$

By analogy with the scalar case, we can write the generating functional for the free theory:

$$Z_0\left[\bar{\eta}, \eta\right] = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left\{i\left(S_0[\psi, \bar{\psi}] + \bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta\right)\right\} \tag{5.9}$$

$$= \exp\left[-\int d^D x \, d^D y \, \bar{\eta}(x) S(x-y) \eta(y)\right] \,. \tag{5.10}$$

The Feynman propagator for the Dirac field is

$$S(x-y) = \int_p e^{-ip\cdot(x-y)} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\,. \tag{5.11}$$

Note that, just like in the case of the scalar field, the propagator is the inverse of the quadratic term in the action. The propagator is a 4×4 matrix in spin space, which we can write explicitely:

$$\begin{aligned} &\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \\ &= S_{\alpha\beta}(x-y) = \int_p e^{ip\cdot(x-y)} \frac{i\left(p_\mu\left(\gamma^\mu\right)_{\alpha\beta} + m\,\delta_{\alpha\beta}\right)}{p^2-m^2+i\epsilon}\,. \end{aligned} \tag{5.12}$$

Because of the linear term in p in the propagator the fermionic propagator is not symmetric in its arguments, and will be denoted with an arrow pointing from one end to the other:

$$S(x-y) = \underset{x}{}\!\longrightarrow\!\underset{y}{}\,. \tag{5.13}$$

Correlators of fermion fields are computed by taking derivatives with respect to the source fields

$$\begin{aligned} &\langle 0|T\psi_{\alpha_1}(x_1)\ldots\bar{\psi}_{\beta_1}(y_1)\ldots|0\rangle_0 = \\ &= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}\ldots i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\; Z_0\left[\eta,\bar{\eta}\right]\big|_{\eta=\bar{\eta}=0}\,. \end{aligned} \tag{5.14}$$

Interacting theory If the interactions are specified by a potential $V\left(\psi,\bar{\psi}\right)$, the generating functional is defined as

$$Z\left[\eta,\bar{\eta}\right] \propto \exp\left[i\int d^Dx\, V\left(i\frac{\delta}{\delta\eta(x)}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]\; Z_0\left[\eta,\bar{\eta}\right]\,, \tag{5.15}$$

and the normalization is fixed by requiring

$$Z[0,0] = 1\,. \tag{5.16}$$

A double expansion in powers of the interaction, and in powers of the number of propagators defines the interacting path integral, as we did for the case of scalars.

Appendix

5.A Differentiation in Grassmann variables

Grassmann algebra A Grassmann algebra $\mathcal{A}$, over $\mathbb{R}$ or $\mathbb{C}$, is constructed from a set of generators θ_i satisfying

$$\theta_i \theta_j + \theta_j \theta_i = 0\,. \tag{5.17}$$

Note that

1. all elements are first degree polynomials in each generator;
2. if the number of generators is finite and equal to n, the algebra is vector space of dimension 2^n.

Grassmannian parity Parity is defined as an automorphism on $\mathcal{A}$ is defined by

$$P(\theta_i) = -\theta_i\,. \tag{5.18}$$

The action of P on a monomial is

$$P(\theta_{i_1} \dots \theta_{i_p}) = (-)^p \theta_{i_1} \dots \theta_{i_p}\,. \tag{5.19}$$

The reflection defines two eigenspaces containing the even and odd elements:

$$P(\mathcal{A}^\pm) = \pm \mathcal{A}^\pm\,. \tag{5.20}$$

Grassmann differentiation Differentiation is defined as a linear mapping

$$D : \mathcal{A} \to \mathcal{A}\,, \tag{5.21}$$

which satisfies

$$D(A_1 A_2) = P(A_1) D(A_2) + D(A_1) A_2\,, \tag{5.22}$$

which guarantees that

$$DP + PD = 0\,. \tag{5.23}$$

Note that the image of $\mathcal{A}^{\pm}$ belongs to $\mathcal{A}^{\mp}$, *i.e.* derivation changes the parity of product of Grassmann variables.

We can introduce the nilpotent differential operators $\partial/\partial\theta_i$ by

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}\,. \tag{5.24}$$

The differential operators and the generators can be considered as operators acting on the elements of $\mathcal{A}$ from the left. They satisfy the anticommutation relations:

$$\theta_i\theta_j + \theta_j\theta_i = 0\,, \tag{5.25}$$

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0\,, \tag{5.26}$$

$$\theta_i\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\theta_i = 0\,. \tag{5.27}$$

Chain rule If $\sigma(\theta) \in \mathcal{A}^-$, $x(\theta) \in \mathcal{A}^+$, then

$$\frac{\partial}{\partial\theta}f(\sigma, x) = \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x}\,. \tag{5.28}$$

5.B Integration in Grassmann variables

To a given differential operator D we associate an integral operator I. The idea is to generalise the concept of *definite* integral to the case of Grassmann variables. I is defined by requiring a number of properties that are *expected* to hold for an integral.

1. I is linear
$$I(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 I(A_1) + \lambda_2 I(A_2)\,; \tag{5.29}$$
2. $ID = DI = 0\,;$
3. $D(A) = 0 \implies I(BA) = I(B)A\,;$
4. $PI + IP = 0$.

Note that a nilpotent differentiation operator D satisfies all these conditions. We shall therefore define the integration operation to be identical to differentiation:

$$\int d\theta_i A \equiv \frac{\partial}{\partial\theta_i}A\,. \tag{5.30}$$

Show that

$$\int d\theta f(\theta) = a^{-1}\int d\theta' f(a\theta' + b)\,. \tag{5.31}$$

Note that the Jacobian for this change of variables is a^{-1}, *i.e.* the inverse of the usual Jacobian for commuting variables. You can prove the generic result

$$\int d\theta_1 \dots d\theta_n = \int d\theta'_1 \dots d\theta'_n J(\theta')\,, \tag{5.32}$$

where

$$J^{-1} = \det \frac{\partial \theta_i}{\partial \theta'_j}\,. \tag{5.33}$$