Quantum Field Theory

Problem Sheet 7

1. *Feynman parameters*

Using the exponentiation
$$\frac{1}{A}=\int_0^\infty d\lambda\, e^{-\lambda A}\,,$$
and the definition of the Gamma function,
$$\Gamma(z)=\int_0^\infty dt\, t^{z-1}e^{-x}\,,$$
prove the Feynman parametrization formula:
$$\frac{1}{A_1^{\alpha_1}\ldots A_n^{\alpha_n}}=\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}\int_0^1 dx_1\ldots dx_n\,\delta(1-x_1-\ldots-x_n)\,\frac{x_1^{\alpha_1-1}\ldots x_n^{\alpha_n-1}}{[x_1A_1+\ldots+x_nA_n]^{\alpha_1+\ldots+\alpha_n}}\,.$$

2. *Cheng-Wu theorem*

Prove that the Feynman parametrization remains true if the integration
$$\int_0^1 dx_1\ldots dx_n\,\delta(1-x_1-\ldots-x_n)$$
is replaced by
$$\int_0^1\left(\prod_{l\in\nu}dx_l\right)\delta\left(1-\sum_{l\in\nu}x_l\right)\int_0^\infty\left(\prod_{k\notin\nu}dx_k\,.\right)$$

3. *Schwinger time representation*

Using the exponentiation formula
$$\frac{1}{(p^2+M^2)^a}=\frac{1}{\Gamma(a)}\int_0^\infty dT\,T^{a-1}e^{-T(p^2+M^2)}\,,$$
show that
$$\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{p^2+M^2}=\frac{1}{(4\pi)^{D/2}}\frac{1}{\Gamma(a)}\left(M^2\right)^{D/2-a}\int_0^\infty d\tau\tau^{a-D/2-1}e^{-\tau}\,,$$
where M^2 is a generic function that does not depend on p, and the integral has already been Wick rotated to Euclidean space.

Identify the divergence of the integral, and show that the UV divergence in the p integration translates into a divergence of the integral over τ near the lower end of the integration interval.

Regularize the integral by introducing a non vanishing lower integration limit

$$\int_{T_0}^{\infty} dT\, T^{a-D/2-1} e^{-TM^2}$$

and discuss the degree of divergence of the integral.

4. *Massless tadpoles*

   Consider the identity

$$\int \frac{d^Dp}{(2\pi)^D} \frac{m^2}{p^2(p^2+m^2)} = \int \frac{d^Dp}{(2\pi)^D} \left[\frac{1}{p^2} - \frac{1}{(p^2+m^2)}\right].$$

   Evaluate the LHS and the second term on the RHS using Schwinger's proper-time parametrization, and show that

$$\int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2} = 0\,.$$

   This result is known as *Veltman's formula.*

5. *Momentum cut-off and Pauli-Villars*

   Compute

$$\int_\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2}\,,$$
$$\int_\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{[(p-k)^2+m^2]\,(p^2+m^2)}\,,$$

   where the suffix indicates that the integral over the radial momentum coordinate is cut off at Λ.

   Remember that the solid angle in D dimensions is given by

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,.$$

   The Pauli-Villars regularizaition is defined by replacing each propagator by a more convergent expression:

$$\frac{1}{p^2+m^2} \longrightarrow \frac{1}{p^2+m^2} - \frac{1}{p^2+M^2}\,.$$

   Discuss what happens to the two integrals above when PV regularization is used.