1. Translation Ward id.

$$\begin{cases} x^\mu \mapsto x'^\mu = x^\mu + a^\mu \\ \\ \phi(x) \mapsto \phi'(x') = \phi(x) \end{cases}$$

$a^\mu = \varepsilon\, \delta^{\mu\rho}$

↳ translation in the ρ direction.

$$\phi'(x) = \phi(x-a) = \phi(x) - a^\mu \partial_\mu \phi(x) = \phi(x) - \varepsilon\, \partial_\rho \phi(x)$$

$$S[\phi'] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi'(x)\, \partial^\mu \phi'(x) - \frac{1}{2} m^2 \phi'(x)^2 \right]$$

$$= S[\phi] - \varepsilon \int d^D x \left[\partial_\rho \left(\partial_\mu \phi(x) \right) \partial^\mu \phi(x) - m^2 \left(\partial_\rho \phi(x) \right) \phi(x) \right]$$

$$= S[\phi] - \varepsilon \int d^D x\, \partial^\sigma \left\{ g_{\sigma\rho}\, \frac{1}{2} \left(\partial_\mu \phi(x)\, \partial^\mu \phi(x) - m^2 \phi(x)^2 \right) \right\}$$

$$\Rightarrow \quad \delta S = - \varepsilon \int d^D x\, \partial^\sigma J_\sigma(x) = 0$$

Check that the theory is invariant under translations.

In order to compute the Noether current J_μ, we promote ε to a <u>local</u> quantity, i.e. $\varepsilon = \varepsilon(x)$

We get an extra term in δS

$$\delta S \Big|_{\text{extra}} = - \int d^D x\, \left(\partial_\mu \varepsilon(x) \right) \left(\partial_\rho \phi(x) \right) \left(\partial^\mu \phi(x) \right)$$

Integrating by parts:

$$\delta S \Big|_{\text{extra}} = \int d^D x \; \varepsilon(x)\, \partial^\sigma \left\{ (\partial_\rho \phi(x))(\partial_\sigma \phi(x)) \right\} .$$

Hence:

$$\delta S = \int d^D x \; \varepsilon(x)\, \partial^\sigma \left\{ (\partial_\rho \phi(x))(\partial_\sigma \phi(x)) - g_{\rho\sigma} \mathcal{L}(\phi(x), \partial\phi(x)) \right\}$$

$$\Rightarrow \quad J_{\sigma,\rho} = T_{\sigma\rho} = (\partial_\rho \phi(x))(\partial_\sigma \phi(x)) - g_{\rho\sigma} \mathcal{L}(\phi(x), \partial\phi(x)) .$$

2. $\int d\psi_\alpha = 0 \,, \qquad \int d\psi_\alpha \, \psi_\beta = \delta_{\alpha\beta}$

$\rightsquigarrow$ see new lecture notes w. a dedicated Appendix.

$$\exp\left(\bar\psi_\alpha A_{\alpha\beta} \psi_\beta\right) = \sum_{k=0}^{N} \frac{1}{k!} \left(\bar\psi_\alpha A_{\alpha\beta} \psi_\beta\right)^k$$

Only the term w. $k = N$ yields a non-zero contribution to the integral. - because we need a factor of $\bar\psi_{\alpha_1} \cdots \bar\psi_{\alpha_N} \psi_{\beta_1} \cdots \psi_{\beta_N}$.

$k = N$ term: $\frac{1}{N!} A_{\alpha_1\beta_1} \cdots A_{\alpha_N\beta_N} \bar\psi_{\alpha_1}\psi_{\beta_1} \cdots \bar\psi_{\alpha_N}\psi_{\beta_N}$. $\quad : (*)$

Re-order the product of $\psi, \bar\psi$: $\bar\psi_{\alpha_1}\psi_{\beta_1} \cdots \bar\psi_{\alpha_N}\psi_{\beta_N} = (-)^{\#} \bar\psi_{\alpha_1} \cdots \bar\psi_{\alpha_N} \psi_{\beta_1} \cdots \psi_{\beta_N}$

$\#$: number of permutations

$$\# = \sum_{k=1}^{N-1} k = \frac{N(N-1)}{2}$$

$$(*) = \frac{1}{N!} A_{\alpha_1\beta_1} \cdots A_{\alpha_N\beta_N} (-)^{\#} \varepsilon_{\alpha_1\cdots\alpha_N} \varepsilon_{\beta_1\cdots\beta_N} \bar{\psi}_1 \cdots \bar{\psi}_N \psi_1 \cdots \psi_N$$

Consider $M_{\alpha_1\cdots\alpha_N} = A_{\alpha_1\beta_1} \cdots A_{\alpha_N\beta_N} \varepsilon_{\beta_1\cdots\beta_N}$

$$M_{\cdots\alpha_i\cdots\alpha_j\cdots} = \cdots A_{\alpha_i\beta_i} \cdots A_{\alpha_j\beta_j} \cdots \varepsilon_{\cdots\beta_i\cdots\beta_j\cdots}$$

$$M_{\cdots\alpha_j\cdots\alpha_i\cdots} = \cdots A_{\alpha_j\beta_i} \cdots A_{\alpha_i\beta_j} \cdots \varepsilon_{\cdots\beta_i\cdots\beta_j\cdots}$$

$$= (-)^{P_{ij}} \cdots A_{\alpha_j\beta_j} \cdots A_{\alpha_i\beta_i} \cdots \varepsilon_{\cdots\beta_i\cdots\beta_j\cdots}$$

$$\Rightarrow \quad \varepsilon_{\alpha_1\cdots\alpha_N} M_{\alpha_1\cdots\alpha_N} = N!\, M_{1\cdots N}$$

$$(*) = A_{1\beta_1} \cdots A_{N\beta_N} \varepsilon_{\beta_1\cdots\beta_N} \bar{\psi}_1 \cdots \bar{\psi}_N \psi_1 \cdots \psi_N \cdot (-)^{\#}$$

$$= (-)^{\#} \det A \, \bar{\psi}_1 \cdots \bar{\psi}_N \psi_1 \cdots \psi_N$$

Finally, we can perform the integrals according to the rules given in the text.

$$\int \prod_{\beta=1}^{N} d\psi_\beta \prod_{\alpha=1}^{N} d\bar{\psi}_\alpha \exp\left(\sum_{\alpha\beta} \bar{\psi}_\alpha A_{\alpha\beta} \psi_\beta\right) = (-)^{\#} \det A .$$

3. Dirac propagator

$$S(x-y) = \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(x-y)} \, i \, \frac{(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

$$(i\gamma_\mu \partial^\mu_x - m) \, S(x-y) = i \int \frac{d^D p}{(2\pi)^D} (\not{p}-m) \frac{(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

$$(\not{p}-m)(\not{p}+m) = \left(p_\mu p_\sigma \gamma^\mu \gamma^\sigma - m\not{p} + m\not{p} - m^2 \right) = p^2 - m^2$$

$$\Rightarrow \; (\square) = i \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot(x-y)} = i\,\delta(x-y)$$