Exercise 1

$$Z_A = \int d^n x \, \exp\left(-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j\right).$$

$$= (2\pi)^{n/2} (\det A)^{-1/2}.$$

$$Z_A(b) = \int d^n x \, \exp\left(-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j + \sum_i b_i x_i\right)$$

$$x_i = y_i + \sum_k \Delta_{ik} b_k \, , \qquad d^n x = d^n y$$

$$(\cdots) = -\frac{1}{2}\left[\sum_{ij} y_i A_{ij} y_j + \sum_{ijk} y_i A_{ij} \Delta_{jk} b_k + \sum_{ijk} \Delta_{ik} b_k A_{ij} y_j +\right.$$

$$\left. + \sum_{ijkk'} \Delta_{ik} b_k A_{ij} \Delta_{jk'} b_{k'}\right] + \sum_i b_i y_i + \sum_{ik} b_i \Delta_{ik} b_k =$$

$$= -\frac{1}{2}\left[\sum_{ij} y_i A_{ij} y_j + \cancel{\sum_i y_i b_i} + \cancel{\sum_j b_j y_j} + \sum_{kk'} b_k \Delta_{kk'} b_{k'}\right]$$

$$+ \cancel{\sum_j b_j y_j} + \sum_{kk'} b_k \Delta_{kk'} b_{k'} =$$

$$= -\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \frac{1}{2}\sum_{ij} b_i \Delta_{ij} b_j$$

$$\Rightarrow Z_A(b) = \underbrace{\int d^n y \, \exp\left(-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j\right)} \cdot \exp\left(\frac{1}{2}\sum_{ij} b_i \Delta_{ij} b_j\right).$$

$$= (2\pi)^{n/2} (\det A)^{-1/2} \cdot \exp\left(\frac{1}{2}\sum_{ij} b_i \Delta_{ij} b_j\right).$$

Exercise 2

x, y stochastic variables

$$\langle x \rangle = \langle y \rangle = 0 \qquad \langle x^2 \rangle = 5 \qquad \langle xy \rangle = 3 \qquad \langle y^2 \rangle = 2$$

then :

$$\langle x^4 \rangle = 3 \langle x^2 \rangle = 3 \times 25 = 75$$

$$\langle x^3 y \rangle = \langle x x x y \rangle = x x x y + x x x y + x x x y$$

$$= 3 \langle x^2 \rangle \langle xy \rangle = 3 \times 3 \times 5 = 45$$

$$\langle x^2 y^2 \rangle = \langle x^2 \rangle \langle y^2 \rangle + 2 \langle xy \rangle^2 = 2 \times 5 + 2 \times 3^2 = 28 .$$

Exercise 3

We want to compute $\langle x_{i_1} x_{i_2} \cdots x_{i_6} \rangle$

Possible pairings & respective contributions.

1. $(i_1 i_2)(i_3 i_4)(i_5 i_6) \longrightarrow \Delta_{i_1 i_2} \Delta_{i_3 i_4} \Delta_{i_5 i_6}$
2. $(i_1 i_2)(i_3 i_5)(i_4 i_6)$
3. $(i_1 i_2)(i_3 i_6)(i_4 i_5)$
4. $(i_1 i_3)(i_2 i_4)(i_5 i_6)$
5. $(i_1 i_3)(i_2 i_5)(i_4 i_6)$
6. $(i_1 i_3)(i_2 i_6)(i_4 i_5)$

7. $(i_1 i_4)(i_2 i_3)(i_5 i_6)$

8. $(i_1 i_4)(i_2 i_5)(i_3 i_6)$

9. $(i_1 i_4)(i_2 i_6)(i_3 i_5)$

10. $(i_1 i_5)(i_2 i_3)(i_4 i_6)$

11. $(i_1 i_5)(i_2 i_4)(i_3 i_6)$

12. $(i_1 i_5)(i_2 i_6)(i_3 i_4)$

13. $(i_1 i_6)(i_2 i_3)(i_4 i_5)$

14. $(i_1 i_6)(i_2 i_4)(i_3 i_5)$

15. $(i_1 i_6)(i_2 i_5)(i_3 i_4)$

Exercice 4

$$\langle x_k F(x) \rangle_0 = \int d^n x \, \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right) x_k \, F(x)$$

$$= \int d^n x \, (-) \sum_{\ell} \Delta_{k\ell} \left[\frac{\partial}{\partial x_\ell} \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right)\right] F(x)$$

$$= - \sum_{\ell} \Delta_{k\ell} \int d^n x \, (-) \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right) \frac{\partial}{\partial x_\ell} F(x)$$

$$= \sum_{\ell} \langle x_k x_\ell \rangle_0 \left\langle \frac{\partial F}{\partial x_\ell} \right\rangle_0 .$$

Exercice 5

$$\frac{Z(\lambda)}{Z(0)} = \langle e^{-\lambda V(x)} \rangle_0 = \exp\left[-\lambda \, V\left(\frac{\partial}{\partial b}\right)\right] \exp\left(\frac{1}{2} \sum_{ij} b_i \Delta_{ij} b_j\right)\bigg|_{b=0}$$

$$V(x) = \frac{1}{4!} \sum_i x_i^4$$

$$\Rightarrow \quad \exp\left[-\lambda V(x)\right] = 1 - \lambda V(x) + \frac{\lambda^2}{2!} V(x)^2 + \dots$$

$$= 1 - \frac{\lambda}{4!} \sum_i x_i^4 + \frac{\lambda^2}{2} \frac{1}{(4!)^2} \sum_{ij} x_i^4 x_j^4 + \dots$$

$$\frac{Z(\lambda)}{Z(0)} = 1 - \frac{\lambda}{4!} \sum_i \langle x_i^4 \rangle_0 + \frac{\lambda^2}{2} \frac{1}{(4!)^2} \sum_{ij} \langle x_i^4 x_j^4 \rangle_0 + O(\lambda^3)$$

Using Wick's thm: $\quad \langle x_i^4 \rangle_0 = 3\, \Delta_{ii}^2$

$$\langle x_i^4 x_j^4 \rangle_0 = x_i x_i x_i x_i x_j x_j x_j x_j \longrightarrow 9\, \Delta_{ii}^2\, \Delta_{jj}^2$$

$$= x_i x_i x_i x_i x_j x_j x_j x_j \longrightarrow \frac{4!}{2 \cdot 2} \frac{4!}{2 \cdot 2} \cdot 2 \quad \Delta_{ii}\, \Delta_{jj}\, \Delta_{ij}^2$$

$$= x_i x_i x_i x_i x_j x_j x_j x_j \longrightarrow 4 \cdot 3 \cdot 2 \quad \Delta_{ij}^4 .$$

# of pairing : $7 \times 5 \times 3 = 105$

we found: $9 + 24 + 72 = 105$ ✓