NON-WIENER FUNCTIONAL INTEGRALS

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A study is made of Feynman path integrals and some similar integrals which are used to solve the initial-value problem for the Schrödinger equation. The S matrix and the partition function are found. The relationship between these integrals and operator symbols is found. In particular, it is shown that functional integrals of this kind depend strongly on the adopted approximations of finite multiplicity. The relation between the Feynman integral and the Wick formula is discussed.

In the present paper we consider functional integrals that are not related to measures in function spaces. Such integrals were first considered by Feynman [1, 2]. Recently the integral introduced by Feynman in [2], the so-called integral "over paths in phase space," has again attracted the attention of a number of authors who hope that the difficulties encountered in the quantization of Yang–Mills fields can be avoided by means of this integral [3-5]. In this connection, one occasionally comes across the almost mystical belief that the use of this integral is a new method of quantization that liberates one from the tortuous question of the order in which the operators $\hat{p}$ and $\hat{q}$ in a quantum expression that corresponds to a classical problem should occur.

A Feynman integral has the form

$$\langle x|e^{\frac{i\hat{H}t}{h}}|y\rangle = \int e^{\frac{i}{h}\int_0^t S(p(\tau),q(\tau))d\tau} \Pi\, dp(\tau)\, dq(\tau),$$

where $p(\tau)$ and $q(\tau)$ are paths in phase space with the boundary conditions $q(0) = x$, $q(t) = y$.

It is assumed that this integral is equal to the limit as $N \to \infty$ of the integrals of finite multiplicity obtained from (*) by replacing the integral in the argument of the exponential function by the integral sum $\sum_0^N S(p_k, q_k)\Delta_k$ with subsequent integration over $\Pi dp_k dq_k$ with weight $(2\pi h)^{-N}$. In this paper we show that the limit depends strongly on the choice of the points p_k and q_k in the integral sum. Choosing these points appropriately, one can arrange that the limit is a matrix element of the operator $e^{it\hat{H}/h}$ corresponding to the quantization in which the classical expression pq is associated with the operator $\hat{q}\hat{p}$, $\hat{p}\hat{q}$, or $(\hat{p}\hat{q} + \hat{q}\hat{p})/2$.

Thus, the integral (*) has meaning only if an additional specification is made concerning the finite-dimensional approximations of which the integral is the limit. This specification is as necessary for the integral as the specification of initial or boundary conditions for a differential equation.† As well as the Feynman integral (*) we shall also consider analogous integrals associated with second quantization. We find an expression in terms of functional integrals for the scattering operator and the partition function.

1. QUANTIZATION AND OPERATOR SYMBOLS

Notation. Let L be the phase space of a classical mechanical system with n degrees of freedom and $q = (q^1, \ldots, q^n)$ and $p = (p^1, \ldots, p^n)$ be canonical coordinates in L. The expressions qp, qx, p^2 etc., are

†The fact that a functional integral may depend on finite-dimensional approximations was previously known. This was pointed out as early as in [11]. A similar phenomenon is encountered in the theory of stochastic integrals.

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a shorthand for $qp = \sum q^i p^i$, $qx = \sum q^i x_i$, $p^2 = \sum (p^i)^2$ etc.; the expressions dp, dq, etc., are a shorthand for products of the differentials: $dp = dp^1 \ldots dp^n$, $dq = dq^1 \ldots dq^n$. By L_2 we denote the Hilbert space of functions $f(x)$, $(f, g) = \int f\bar{g}dx$. The operators $\hat{p}_k$ and $\hat{q}_k$ on L_2 have the form

$$(\hat{p}_k f)(x) = \frac{h}{i}\frac{\partial f}{\partial x_k},\ (\hat{q}_k f)(x) = x_k f(x).$$

The problem of quantization is to associate every classical observable, i.e., every real function $f(p, q)$, with a quantum observable, i.e., a selfadjoint operator $\hat{f}$ on a certain Hilbert space. The following conditions are assumed to be satisfied.

1) The correspondence $f \to \hat{f}$ depends on the parameter h (the Planck constant), the commutator satisfying $[\hat{f}, \hat{g}] = ih\hat{c} + o(h)$, where $\hat{c}$ is the operator corresponding to the function $c = [f, g]$, where $[f, g]$ are the Poisson brackets.

2) The function $f(p, q)$ is, in a certain sense, the limit of the operator $\hat{f}$ as $h \to 0$.

It is obvious that "quantization" is an operation with a considerable degree of ambiguity. In this paper we shall consider various versions of quantization. In what follows, we shall call the function $f(q, p)$ corresponding to the operator f the symbol of this operator.

$\hat{q}\hat{p}$ Quantization. With every polynomial

$$f(p, q) = \sum f_{m_1, \ldots, m_n, m_1', \ldots, m_n'} (p^1)^{m_1} \ldots (p^n)^{m_n} (q^1)^{m_1'} \ldots (q^n)^{m_n'} \tag{1}$$

we associate an operator on L_2:

$$\hat{f} = \sum f_{m_1, \ldots, m_n, m_1', \ldots, m_n'} \hat{q}_1^{m_1'} \ldots \hat{q}_n^{m_n'} \hat{p}_1^{m_1} \ldots \hat{p}_n^{m_n},$$

$\hat{f}$ is a differential operator with polynomial coefficients. We continue the correspondence between polynomials and polynomial differential operators to a correspondence between functions and operators of a more general form. To this end we note that if the polynomial $f(p, q)$ corresponds to $\hat{f}$, then

$$\begin{aligned} &\hat{p}_k \hat{f} \leftrightarrow \left(p^k - ih\frac{\partial}{\partial q}\right) f,\quad \hat{f}\hat{p}_k \leftrightarrow f p^k, \\ &\hat{q}_k \hat{f} \leftrightarrow q^k f,\quad \hat{f}\hat{q}_k \leftrightarrow \left(q^k - ih\frac{\partial}{\partial p^k}\right) f \end{aligned} \tag{2}$$

(Proof: note that if $m_k' > 0$, then

$$\hat{p}_k^1 \hat{q}_k^{m_k'} = (\hat{p}_k\hat{q}_k - \hat{q}_k\hat{p}_k)\,\hat{q}_k^{m_k'-1} + \hat{q}_k\hat{p}_k\hat{q}_k^{m_k'-1} = -ih\hat{q}_k^{m_k'-1} + \hat{q}_k\hat{p}_k\hat{q}_k^{m_k'-1} = -min\,\hat{q}_k^{m_k'-1} + \hat{q}_k^{m_k'}\hat{p}_k,$$

and thefore $\hat{p}_k\hat{f} \longleftrightarrow (p^k - ih\partial/\partial q^k)f$. Similarly, one can prove the fourth formula of (2); the second and third are obvious). We require that Eqs. (2) remain valid if f is not a polynomial. Suppose $\hat{f}$ is some operator on L_2 and $K(x, y) = \langle x|\hat{f}|y\rangle$ is its kernel [for operators of the form (1) $K(x, y)$ is a generalized function]. Note that if the kernel $K(x, y)$ corresponds to the operator f, then

$$\begin{aligned} &\hat{p}_k\hat{f} \leftrightarrow -ih\frac{\partial}{\partial x_k}K,\ \hat{f}\hat{p}_k \leftrightarrow ih\frac{\partial}{\partial y_k}K, \\ &\hat{q}_k\hat{f} \leftrightarrow x_k K,\quad \hat{f}\hat{q}_k \leftrightarrow y_k K. \end{aligned} \tag{3}$$

We shall seek a correspondence between functions and operators in the form

$$\begin{aligned} f(p, q) &= \int L(p, q\,|\,x, y) K(x, y)\,dx\,dy, \\ K(x, y) &= \int L^*(x, y\,|\,p, q) f(p, q)\,dp\,dq. \end{aligned} \tag{4}$$

From Eqs. (2) and (3) for L and L* we obtain the equations

$$\begin{aligned} &\left(p^k - ih\frac{\partial}{\partial q^k}\right) L = ih\frac{\partial L}{\partial x_k},\ p^k L = -ih\frac{\partial L}{\partial y_k}, \\ &q^k L = x_k L,\ \left(q^k - ih\frac{\partial}{\partial p^k}\right) L = y_k L; \end{aligned} \tag{5}$$

$$-ih\frac{\partial L^*}{\partial x_k}=\left(p^k+ih\frac{\partial}{\partial q^k}\right)L^*,\ ih\frac{\partial L^*}{\partial y_k}=p^kL^*,$$
$$x_kL^*=q^kL^*,\ y_kL^*=\left(q^k+ih\frac{\partial}{\partial p^k}\right)L^*. \tag{5'}$$

All eight equations are obtained in exactly the same way; we shall therefore derive only the first of Eqs. (5). It follows from (2) and (3) that the symbol $(p^k - ih\partial/\partial q_k)f$ and the kernel $-ih\partial K/\partial x_k$ correspond to the operator $\hat{p}_k\hat{f}$. Hence, using (4), we obtain the identities

$$\left(p^k-ih\frac{\partial}{\partial q^k}\right)f=\int\left(p^k-ih\frac{\partial}{\partial q^k}\right)LKdx\,dy=\int L\left(-ih\frac{\partial}{\partial x_k}K\right)dx\,dy=ih\int\left(\frac{\partial}{\partial x_k}L\right)K\,dx\,dy$$

(the last is obtained by integration by parts). Thus,

$$\int\left(p^k-ih\frac{\partial}{\partial q^k}\right)LK\,dx\,dy=ih\int\frac{\partial L}{\partial x_k}K\,dx\,dy.$$

This equation must hold for arbitrary K. It follows that the first equation of (5) must hold. The unique solutions (to within a factor) of Eqs. (5) and (5') are the functions

$$L=\delta(q-x)\,e^{\frac{i}{h}p(y-q)},\ L^*=(2\pi h)^{-n}\delta(q-x)\,e^{-\frac{i}{h}p(y-q)}. \tag{6}$$

The factors can be determined from the condition that the identity operator corresponds, on the one hand, to the kernel $\delta(x-y)$ and, on the other hand, to the symbol $f\equiv 1$.

From (6) we obtain finally the connection between the symbols and kernels

$$f(p,q)=\int K(q,y)\,e^{\frac{i}{h}p(y-q)}dy, \tag{7}$$

$$K(x,y)=(2\pi h)^{-n}\int f(p,x)\,e^{-\frac{i}{h}p(y-x)}dp. \tag{7'}$$

Suppose $\varphi(x)\in L_2$. It follows from (7') that the function $(\hat{f}\varphi)(x)$ is given by

$$(\hat{f}\varphi)(x)=\int f(p,x)\,\tilde{\varphi}(p)e^{\frac{i}{h}px}dp,$$
$$\tilde{\varphi}(p)=(2\pi h)^{-n}\int\varphi(y)\,e^{-\frac{i}{h}py}dy. \tag{8}$$

Equations (8) serve as the foundation of the theory of pseudodifferential operators [6].†

It follows from Eqs. (7) and (7') that if $\hat{f}=\hat{f}_1\hat{f}_2$, then the symbols of the operators $\hat{f}$, $\hat{f}_1$, and $\hat{f}_2$ are related by the equations

$$f(p,q)=\frac{1}{(2\pi h)^n}\int f_1(p_1,q)\,f_2(p,q_1)\,e^{-\frac{i}{h}(p_1-p)(q_1-q)}dp_1\,dq_1. \tag{9}$$

Equation (9) is the starting point for the determination of the Feynman integral over paths in phase space.

Apart from this equation we should like to point out two remarkable equations, which we shall not however use in what follows:

$$\mathrm{Sp}\,\hat{f}=\int K(x,x)\,dx=\frac{1}{(2\pi h)^n}\int f(p,q)\,dp\,dq, \tag{10}$$

$$\mathrm{Sp}\,\hat{f}_1\hat{f}_2^*=\int K_1(x,y)\,\overline{K_2(x,y)}\,dx\,dy=\frac{1}{(2\pi h)^n}\int f_1(p,q)\,\overline{f_2(p,q)}\,dp\,dq. \tag{11}$$

†If the reader should feel that the above arguments are unconvincing, he may verify directly by means of (8) that every polynomial (1) is associated with an operator $\hat{f}$

$$(\hat{f}\varphi)(x)=\sum f_{m_1,\ldots,m_n'}\,x_1^{m_1'}\ldots x_n^{m_n'}\left(\frac{h}{i}\right)^{m_1+\ldots+m_n}\frac{\partial^{m_1+\ldots+m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\varphi.$$

Equation (11) plays the role of the Plancherel theorem for the transformation (7).†

$\hat{p}\hat{q}$ **Quantization.** With the polynomial (1) we now associate the following operator $\hat{f}$ on L_2:

$$\hat{f}=\sum f_{m_1,\ldots,m_n,m_1',\ldots,m_n'}\hat{p}_1^{m_1}\ldots\hat{p}_n^{m_n}\hat{q}_1^{m_1'}\ldots\hat{q}_n^{m_n'}. \tag{12}$$

As in the foregoing case, f is a polynomial differential operator on L_2.

The continuation of the correspondence between polynomial differential operators and polynomials to more general functions and operators is based on the same arguments as in the case of $\hat{q}\hat{p}$ quantization. Formulas hold that are the analogs of (2):

$$\begin{gathered}\hat{p}_k\hat{f}\leftrightarrow p^k f,\quad \hat{f}\hat{p}_k\leftrightarrow\left(p^k+ih\frac{\partial}{\partial q^k}\right)f,\\ \hat{q}_k\hat{f}\leftrightarrow\left(q^k+ih\frac{\partial}{\partial p^k}\right)f,\quad \hat{f}\hat{q}_k\leftrightarrow fq^k\end{gathered} \tag{13}$$

[the proof is the same as for Eqs. (2)]. The correspondence between the symbols and the kernels can be found in the form (4). The relationships (3) and (13) yield equations for L and L* that are similar to (5) and (5'):

$$\begin{gathered}p^kL=ih\frac{\partial L}{\partial x_k},\ \left(p^k+ih\frac{\partial}{\partial q^k}\right)L=-ih\frac{\partial L}{\partial y_k},\\ \left(q^k+ih\frac{\partial}{\partial p^k}\right)L+x_kL,\ q^kL=y_kL,\end{gathered} \tag{14}$$

$$\begin{gathered}-ih\frac{\partial L^*}{\partial x_k}=p^kL^*,\ ih\frac{\partial L^*}{\partial y_k}=\left(p^k-ih\frac{\partial}{\partial q^k}\right)L^*,\\ x_kL^*=\left(q^k-ih\frac{\partial}{\partial p^k}\right)L^*,\ y_kL^*=q^kL^*.\end{gathered} \tag{14'}$$

From (14) and (14') we can find L and L* to within a factor; the factor can be found from the condition that the kernel K(x, y) = δ(x−y) and the symbol f ≡ 1 correspond to the identity operator. Finally, we obtain

$$L=\delta(q-y)\,e^{-\frac{i}{h}p(x-q)},\quad L^*=(2\pi h)^{-n}\delta(q-y)\,e^{\frac{i}{h}p(x-q)}. \tag{15}$$

Thus,

$$f(p,q)=\int K(x,q)\,e^{-\frac{i}{h}p(x-q)}\,dx, \tag{16}$$

$$K(x,y)=(2\pi h)^{-n}\int f(p,y)\,e^{\frac{i}{h}p(x-y)}\,dp. \tag{16'}$$

From (16) and (16') we obtain a multiplication formula: if $\hat{f}=\hat{f}_1\hat{f}_2$, then

$$f(p,q)=\frac{1}{(2\pi h)^n}\int f_1(p,q_1)\,f_2(p_1,q)\,e^{\frac{i}{h}(q-q_1)(p-p_1)}\,dq_1\,dp_1. \tag{17}$$

The formula for the trace and the Plancherel formula have the previous form (10) and (11).‡

We now introduce the following provisional notation. By $f_{\hat{q}\hat{p}}$ we denote the symbol corresponding to the operator $\hat{f}$ in $\hat{q}\hat{p}$ quantization ($\hat{q}\hat{p}$ symbol); by $f_{\hat{p}\hat{q}}$ we denote the symbol corresponding to the same operator in $\hat{p}\hat{q}$ quantization ($\hat{p}\hat{q}$ symbol). Equations (7) and (16) yield a relationship between $f_{\hat{p}\hat{q}}$ and $f_{\hat{q}\hat{p}}$

$$f_{\hat{p}\hat{q}}(p,q)=\frac{1}{(2\pi h)^n}\int f_{\hat{q}\hat{p}}(p',q')\,e^{\frac{i}{h}(p'-p)(q'-q)}\,dp'\,dq', \tag{18}$$

$$f_{\hat{q}\hat{p}}(p,q)=\frac{1}{(2\pi h)^n}\int f_{\hat{p}\hat{q}}(p',q')\,e^{-\frac{i}{h}(p'-p)(q'-q)}\,dp'\,dq'. \tag{18'}$$

† If the function K(x, y) is smooth and decreases rapidly, Eqs. (7') and (11) can be deduced directly from (7) after the manner of the formula for the inverse Fourier transformation and the Plancherel theorem for the Fourier transform. Thus, the possibility is opened up of approaching Eqs. (7) and (7') for the case when K(x, y) is a generalized function [corresponding to the polynomial f(p, q)] from the same point of view as is usually adopted when one considers the Fourier transform of generalized functions [7].

‡ As regards the derivation of Eqs. (16), (16'), and (17), the same arguments hold as in the case of $\hat{q}\hat{p}$ quantization.

Equation (11) yields the Plancherel formula for the transformations (18) and (18'):

$$\int f_{\hat{p}\hat{q}}(p,q)\overline{g_{\hat{p}\hat{q}}(p,q)}\,dp\,dq=\int f_{\hat{q}\hat{p}}(p,q)\overline{g_{\hat{q}\hat{p}}(p,q)}\,dp\,dq. \tag{19}$$

Finally, let us consider the expression for the symbol of the adjoint operator in terms of the symbol of the original operator. Suppose

$$\hat{f}=\sum f_{m_1,\ldots,m_n,m_1',\ldots,m_n'}\hat{p}_1^{m_1}\ldots\hat{p}_n^{m_n}\hat{q}_1^{m_1'}\ldots\hat{q}_n^{m_n'},$$

Then $\hat{f}^*=\sum \bar{f}_{m_1,\ldots,m_{n'}}\hat{q}_n^{m_n'}\ldots\hat{p}_1^{m_1}$. We denote the $\hat{p}\hat{q}$ and $\hat{q}\hat{p}$ symbols of the operator $\hat{f}^*$ by $f^*_{\hat{p}\hat{q}}$ and $f^*_{\hat{q}\hat{p}}$, respectively. Obviously,

$$f^*_{\hat{q}\hat{p}}(p,q)=\overline{f_{\hat{p}\hat{q}}}(p,q)=\frac{1}{(2\pi h)^n}\int \bar{f}_{\hat{q}\hat{p}}(p',q')e^{-\frac{i}{h}(p-p')(q-q')}dp'\,dq'. \tag{20}$$

Similarly

$$f^*_{\hat{p}\hat{q}}(p,q)=\bar{f}_{\hat{q}\hat{p}}(p,q)=\frac{1}{(2\pi h)^n}\int \bar{f}_{\hat{p}\hat{q}}(p',q')e^{-\frac{i}{h}(p-p')(q-q')}dp'\,dq'. \tag{20'}$$

Symmetric or Weyl Quantization. Let A and B be noncommuting operators. We consider the operator $(\alpha A+\beta B)^n$ and expand it in powers of α and β:

$$(\alpha A+\beta B)^n=\sum\frac{n!}{k!l!}\alpha^k\beta^l(A^kB^l). \tag{21}$$

We shall call the operator (A^kB^l) defined by this equation the symmetric product of A^k and B^l.

Examples: $(AB)=(AB+BA)/2$, $A^2B=(A^2B+ABA+BA^2)/3$.

We associate the polynomial (1) with an operator on L_2:

$$\hat{f}=\sum f_{m_1,\ldots,m_n,m_1',\ldots,m_n'}(\hat{p}_1^{m_1}\hat{q}_1^{m_1'})\ldots(\hat{p}_n^{m_n}q_n^{m_n'}). \tag{22}$$

Using induction on the degree of the polynomial f, one can readily verify that the correspondence obtained in this manner between the polynomials and the polynomial differential operators is one to one [8].

As in the foregoing cases the main role in the further extension of the correspondence between the polynomials and the polynomial differential operators is played by the formulas that express the symbols of the operators $\hat{p}_k\hat{f}$, $\hat{f}\hat{p}_k$, $\hat{q}_k\hat{f}$, $\hat{q}_k\hat{f}$, and $\hat{f}\hat{q}_k$ in terms of the symbol of the operator f:

$$\begin{gathered}\hat{p}_k\hat{f}\leftrightarrow\left(p^k-\frac{ih}{2}\frac{\partial}{\partial q^k}\right)f,\ \hat{f}\hat{p}_k\leftrightarrow\left(p^k+\frac{ih}{2}\frac{\partial}{\partial q^k}\right)f,\\ \hat{q}_k\hat{f}\leftrightarrow\left(q^k+\frac{ih}{2}\frac{\partial}{\partial q^k}\right)f,\ \hat{f}\hat{q}_k\leftrightarrow\left(q^k-\frac{ih}{2}\frac{\partial}{\partial p^k}\right)f.\end{gathered} \tag{23}$$

Formulas (23) are proved in [8]. The proof is more complicated than that of formulas (2) and (13).

From (3) and (23) we find equations for the functions L and L* that realize the relationships between the symbols and the kernels of the operators:

$$\begin{gathered}\left(p^k-\frac{ih}{2}\frac{\partial}{\partial q^k}\right)L=ih\frac{\partial L}{\partial x_k},\ \left(p^k+\frac{ih}{2}\frac{\partial}{\partial q^k}\right)L=-ih\frac{\partial L}{\partial y_k},\\ \left(q^k+\frac{ih}{2}\frac{\partial}{\partial p^k}\right)L=x_kL,\ \left(q^k-\frac{ih}{2}\frac{\partial}{\partial p^k}\right)L=y_kL,\\ -ih\frac{\partial L^*}{\partial x_k}=\left(p^k+\frac{ih}{2}\frac{\partial}{\partial q^k}\right)L^*,\ ih\frac{\partial L^*}{\partial y_k}=\left(p^k-\frac{ih}{2}\frac{\partial}{\partial q^k}\right)L^*,\\ x_kL^*=\left(q^k-\frac{ih}{2}\frac{\partial}{\partial p^k}\right)L^*,\ \ y_kL^*=\left(q^k+\frac{ih}{2}\frac{\partial}{\partial p^k}\right)L^*.\end{gathered} \tag{24}$$

From (24) and the normalization condition

$$f\equiv 1\leftrightarrow K(x,y)=\delta(x-y)$$

we find that

$$L = \delta\left(q - \frac{x+y}{2}\right) e^{-\frac{1}{ih} p(y-x)},\quad L^* = \frac{1}{(2\pi h)^n}\,\delta\left(q - \frac{x+y}{2}\right) e^{\frac{1}{ih} p(y-x)}, \tag{25}$$

and hence

$$\begin{gathered} f(p, q) = \int K(q - \xi/2, q + \xi/2)\, e^{p\xi/ih} d\xi, \\ K(x, y) = \frac{1}{(2\pi h)^n} \int f\left(p, \frac{x+y}{2}\right) e^{\frac{1}{ih} p(y-x)} dp. \end{gathered} \tag{26}$$

Equations (26) yield a multiplication formula ($q_i = q_i^1, \ldots, q_i^n$; $p_i = p_i^1, \ldots, p_i^n$):

$$f(p, q) = \frac{1}{(\pi h)^{2n}} \int f_1(p_1, q_1) f_2(p_2, q_2) \exp\left\{ -\frac{2}{ih} \begin{vmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q \\ p_1 & p_2 & p \end{vmatrix} \right\} dp_1\, dp_2\, dq_1\, dq_2. \tag{27}$$

As before, the trace formula (10) and the Plancherel formula (11) hold.

In contrast to $\hat{q}\hat{p}$ and $\hat{p}\hat{q}$ quantization there is a very simple relationship between the symbols f and f* of the operator $\hat{f}$ and the adjoint $\hat{f}^*$ in the case of Weyl quantization. Using the relationship between the corresponding kernels $K^*(x, y) = K(y, x)$, we find from the first equation of (26) that

$$f^*(p, q) = \overline{f(p, q)}. \tag{28}$$

In particular, a real symbol corresponds to a self-adjoint operator.

Second Quantization in the Bose Case. This is based on the operators of creation and annihilation $\hat{a}_k^* = 1/\sqrt{2}\,(\hat{q}_k - i\hat{p}_k)$ and $\hat{a}_k = 1/\sqrt{2}(\hat{q}_k + i\hat{p}_k)$. There are different possible correspondences between functions and operators. We shall restrict ourselves to the so-called Wick normal form. We rewrite the polynomial f(p, q) in terms of the variables

$$\begin{gathered} a(k) = \frac{1}{\sqrt{2}}(q_k + ip_k),\quad a^*(k) = \frac{1}{\sqrt{2}}(q_k - ip_k): \\ f = \sum_{k, k'} \sum_{m_i, m_i'} \varphi_{m_1, \ldots, m_k | m_1', \ldots, m_{k'}'}\, a^*(m_1) \ldots a^*(m_k)\, a(m_1') \ldots a(m_{k'}'). \end{gathered} \tag{29}$$

The polynomial f is associated with the operator

$$\hat{f} = \sum_{k, k'} \sum_{m_i, m_i'} \varphi_{m_1, \ldots, m_k | m_1', \ldots, m_{k'}'}\, \hat{a}^*(m_1) \ldots \hat{a}^*(m_k)\, \hat{a}(m_1') \ldots \hat{a}(m_{k'}'). \tag{30}$$

The coefficients $\varphi_{m_1, \ldots, m_k | m_1', \ldots, m_{k'}}$ are assumed to be symmetric individually with respect to the first and second group of subscripts. The further extension of this correspondence and the derivation of the principal formulas of the operator calculus can be made in accordance with a general plan.†

The principal formulas [9]‡: if $\hat{f}_i \longleftrightarrow f_i(a^*, a)$, then

$$\hat{f} = \hat{f}_1 \hat{f}_2 \leftrightarrow \int f_1(a^*, \alpha) f_2(\alpha^*, a)\, e^{-\frac{1}{h}(a^* - \alpha^*)(\alpha - a)} \Pi\, d\alpha^*\, d\alpha; \tag{31}$$

if $\hat{g} = \hat{f}^*$, then

$$g(a^*, a) = f^*(a^*, a), \tag{32}$$

where f* is the complex conjugate of the function f [a^* and a are complex conjugate variables, $(a^*)^* = a$]

$$\mathrm{Sp}\,\hat{f} = \int f(a^*, a)\, \Pi\, da^*\, da. \tag{33}$$

†Second quantization has been much better studied than the previously described quantizations. In particular, it is known that to every bounded operator there corresponds a symbol which is an entire function of 2n complex variables $a(k)$ and $a^*(k)$ (regarded as independent). This result and also the formulas that follow below can be obtained most conveniently by considering the realization of the operators $\hat{a}_k$ and $\hat{a}_k^*$ in the Fok space without appealing to the expression of the operators by means of kernels on L_2 (see [9]).
‡In [9] only the case h = 1 is considered.

In Eq. (33)

$$\Pi da^* da = (2\pi h)^{-n} dp_1 dq_1 \dots dp_n dq_n.$$

In (31) $\Pi\alpha^* d\alpha$ has a similar meaning (the normalization factor is defined by the condition $\int e^{-1/h a^* a} \cdot \Pi da^* da = 1$. Recall that $a(k) = (q_k + ip_k)/\sqrt{2}$, $a^*(k) = (q_k - ip_k)/\sqrt{2}$, where q_k and p_k are real variables).

Second Quantization in the Fermi Case. Let G be an algebra with 2n anticommuting generators $a(1), \dots, a(n); a^*(1), \dots, a^*(n)$:

$$a(i)a(j) + a(j)a(i) = a(i)a^*(j) + a^*(j)a(i) = a^*(i)a^*(j) + a^*(j)a^*(i) = 0.$$

The elements of G can be uniquely expressed in the form (29), the coefficients $\varphi_{m_1,\dots,m_k|m_1',\dots,m_k'}$ being antisymmetric individually with respect to the first and second group of subscripts. With an element of G of the form (29) we associate an operator (30) on the Fok space of the Fermi system. Thus, the symbol of the operator (30) in the Fermi case is not represented by a function of 2n complex variables, as before, but by an element of a Grassmann algebra. Equation (31) remains in force.† Equation (32) also remains in force but note that f^* is now not the complex conjugate function, which would be meaningless, but an element of G of the form

$$f^* = \sum \bar{\varphi}_{m_1,\dots,m_k|m_1',\dots,m_{k'}'} a^*(m_{k'}') \dots a^*(m_1') a(m_k) \dots a(m_1).$$

Equation (33) ceases to hold and is replaced by

$$\mathrm{Sp}\,\hat{f} = \int f(a^*, a)\, e^{\frac{2}{h} a^* a} \Pi\, da\, da^*. \tag{34}$$

In (31) and (34) $\Pi da^* da$ includes the normalization factor h^n:

$$\Pi da^* da = h^n da_1^* da_1 \dots da_n^* da_n.$$

The integrals in (31) and (34) are the so-called integrals with respect to anticommuting variables.‡

In view of the considerable analogy between the formal properties of functions and elements of a Grassmann algebra G, the elements of this algebra can be naturally called functions with anticommuting variables. Quantization by means of the Wick normal form (in both variants) is remarkable in that it can be well transferred to the case of an infinite number of degrees of freedom. At the same time (29) is transformed into an infinite series (convergent for all a_k and a_k^* that are square-summable†† in the Bose case and formal in the Fermi case). Equations (31), (33), and (34) remain in force; the integrals in these equations become functional integrals and they are to be understood as the limit of integrals of finite multiplicity [9].

2. FUNCTIONAL INTEGRAL FOR $(it\hat{H}/h)$

General Comments. Let $\hat{H}$ be some Hamiltonian and H(p, q) be its symbol. Let us find the symbol G(p, q|t) of the operator $e^{it\hat{H}/h}$. We note first that

$$e^{i\frac{t\hat{H}}{h}} = 1 + \frac{it\hat{H}}{h} + \frac{t^2}{h^2}\hat{R},$$

where $\hat{R}$ is an operator that can be expanded in integral powers of t. Turning to the symbols, we find that

$$G(p, q\,|\,t) = 1 + \frac{itH}{h} + \frac{t^2}{h^2} R = e^{\frac{itH}{h}} + \frac{t^2}{h^2} r(t).$$

†In the Fermi case the parameter h does not have such a clear physical meaning as in the Bose case.

‡Definition: $\int a_k da_k = \int a_k^* da_k^* = 1$, $\int da_k = \int da_k^* = 0$; the multiple integral is understood as a repeated integral; the differentials da_k and da_k^* anticommute with each other and with a_k and a_k^* (see [9]).

††I.e., $\sum |a_k^*|^2 < \infty$, $\sum |a_k|^2 < \infty$; a_k and a_k^* are assumed to be independent and in no way related variables.

We shall denote by $\hat{U}(t)$ and $\hat{r}(t)$ the operators whose symbols are represented by $e^{it\hat{H}/h}$ and $r(t)$, respectively. Note that the following identity holds:

$$e^{\frac{it\hat{H}}{h}} = (e^{\frac{it\hat{H}}{Nh}})^N = \left[\hat{U}\left(\frac{t}{N}\right) + \frac{t^2}{N^2h^2}\hat{r}\left(\frac{t}{N}\right)\right]^N.$$

In the limit $N \to \infty$ the second term ceases to play a role and therefore

$$e^{\frac{it\hat{H}}{h}} = \lim_{N\to\infty}\left[\hat{U}\left(\frac{t}{N}\right)\right]^N. \tag{35}$$

We shall denote the symbol of the operator $\hat{U}(t/N)^N$ by $G_N(p, q|t)$.

$\hat{q}\hat{p}$ Quantization. Using Eq. (9), we find that

$$G_N(p, q|t) = \frac{1}{(2\pi h)^{n(N-1)}}\int \exp\left\{\frac{it}{Nh}\sum_1^n H(p_k, q_{k-1}) - \frac{i}{h}\sum_1^n p_k(q_k - q_{k-1})\right\}\prod_1^{N-1} dp_\alpha dq_\alpha, \tag{36}$$

$$p_0 = p_N = p, \quad q_0 = q_N = q$$

(p_0 does not participate at all in the integrand. It is added to achieve symmetry between p and q). We set

$$p_k = p(t_k), \quad q_k = q(t_k), \quad t_k = kt/N, \quad \Delta = t/N.$$

It can be seen that the argument of the exponential function in (36) is an integral sum for the action integral

$$S = \int_0^t \left[H(p(\tau), q(\tau)) - p\frac{dq}{d\tau}\right]d\tau, \tag{37}$$

where $p(\tau)$, $q(\tau)$ is a closed path passing through the point p, q: $q(0) = q(t) = q$, $p(0) = p(t) = p$. Taking the factor $(1/2\pi h)^{n(N-1)}$ into the normalization of the differentials, we obtain

$$G(p, q|t) = \int e^{\frac{i}{h}S}\prod dp\,dq. \tag{38}$$

The integral is taken over all closed paths passing through the point p, q.

Applying Eq. (7') to the symbol $G_N(p, q|t)$, we find the kernel $K_N(x, y)$ of the operator $[\hat{U}(t/N)]^N$. It is given before, by the integral (36); the only difference is in the preintegral factor, which is now equal to $(2\pi h)^{-nN}$, and in the conditions on the path, which are now: $q_0 = x$, $q_N = y$ (p_N is a variable of integration). Making the passage to the limit $N \to \infty$ and including the factor $(2\pi h)^{-nN}$ in the normalization of the differentials, we obtain the Feynman formula

$$\langle x|e^{\frac{i\hat{H}t}{h}}|y\rangle = \int e^{\frac{i}{h}S}\prod dp\,dq. \tag{39}$$

The integral is taken over paths that satisfy the conditions $q(0) = x$, $q(t) = y$.

$\hat{p}\hat{q}$ Quantization. Using Eq. (17), we obtain

$$G_N(p, q|t) = \left(\frac{1}{2\pi h}\right)^{n(N-1)}\int \exp\left\{\frac{it}{hN}\sum_1^N H(p_{k-1}q_k) - \frac{i}{h}\sum_1^N p_{k-1}(q_k - q_{k-1})\right\}\prod_1^{N-1} dp_\alpha dq_\alpha, \tag{40}$$

$$p_0 = p_N = p, \quad q_0 = q_N = q.$$

Using Eq. (16'), we find that the kernel of the operator $[\hat{U}(t/N)]^N$ is defined by an equation that differs from (40) by the boundary conditions $q_0 = x$, $q_N = y$ and the preintegral factor, which is equal to $(1/2\pi h)^{nN}$. The formal expressions for the symbol and the kernel of the operator $e^{it\hat{H}/h}$ have the previous form (28) and (40).

Weyl Quantization. Using formula (27), we find that†

$$G_N(p,q\,|\,t)=\left(\frac{1}{\pi h}\right)^{2n(N-1)}\int\exp\left\{\frac{it}{hN}\sum_0^{N-1}H(p_k,q_k)\right.$$
$$\left.-\frac{2i}{h}\sum_1^{N-1}[(p_k-\xi_k)(\eta_{kH}-\eta_k)-(q_k-\eta_k)(\xi_{k-H}-\xi_k)]\right\}\prod_1^{N-1}dp_\alpha\,dq_\alpha\,d\xi_\alpha\,d\eta_\alpha, \tag{41}$$

$$\xi_1=p_0,\quad \eta_1=q_0,\quad \xi_N=p,\quad \eta_N=q.$$

Making the formal passage to the limit $N\to\infty$, we obtain

$$G(p,q\,|\,t)=\int\exp\left\{\frac{i}{h}\int_0^t\left[H(p(\tau),q(\tau))+2(p(\tau)-\xi(\tau))\frac{d\eta}{d\tau}-2(q(\tau)-\eta(\tau))\frac{d\xi}{d\tau}\right]d\tau\right\}\Pi\,dp\,dq\,d\xi\,d\eta, \tag{42}$$
$$\xi(0)=p(0),\quad \eta(0)=q(0),\quad \xi(t)=p,\quad \eta(t)=q.$$

Note that from a naive point of view the integral (42) is identical with (38). In the argument of the exponential function we replace the integral by an appropriate integral sum and denote the resulting expression by S_N:

$$S_N=\frac{i}{hN}\sum_0^{N-1}H(p_k,q_k)-\frac{2i}{h}\left[\sum_2^{N=1}(p_k-\xi_k)(\eta_{k+1}-\eta_k)\right.$$
$$\left.-\sum_1^{N-1}(q_{k+1}-\eta_{k+1})(\xi_{k+1}-\xi_k)-(q_2-\eta_2)\xi_1-(q_N-\eta_N)\xi_N\right].$$

Integrating with respect to ξ_k, $k = 2,\dots,N$, we obtain

$$\left(\frac{1}{\pi h}\right)^{2n(N-1)}\int e^{S_N}\Pi\,dp\,dq\,d\xi\,d\eta=\frac{1}{(2\pi h)^{n(N-1)}}\int\exp\left\{\frac{i}{hN}\sum_0^{N-1}H(p_k,q_k)-\frac{i}{h}\sum_2^{N-1}p_k(q_{k+1}-q_k)\right\}\Pi\,dp\,dq,$$
$$p_0=p_N=p,\quad q_0=q_N=q.$$

In the argument of the exponential function on the left we have the integral sum for the action (37). Thus, Eq. (38) evens out the difference between $\hat{q}\hat{p}$, $\hat{p}\hat{q}$, and the Weyl symbols. One can readily find an example when all three kinds of symbol for the operator $\hat{H}$ are identical but different for the operator $\exp(it\hat{H}/h)$:

$$\hat{H}=\frac{1}{2}(\hat{p}^2+\hat{q}^2),\ H(p,q)=\frac{1}{2}(p^2+q^2)\ \text{ in all three cases,}$$

$$G(p,q\,|\,t)=\frac{1}{\cos ht}\begin{cases}\exp\left\{\dfrac{p^2+q^2}{2h}\operatorname{tg}ht+ipq\dfrac{1-\cos ht}{h\cos th}\right\} & \text{in the qp case,}\\ \exp\left\{\dfrac{p^2+q^2}{2h}\operatorname{tg}ht-ipq\dfrac{1-\cos ht}{h\cos ht}\right\} & \text{in the pq case,}\\ \exp\left\{\dfrac{p^2+q^2}{2h}\operatorname{tg}ht\right\} & \text{in the Weyl case.}\end{cases} \tag{43}$$

Because of the especial simplicity of $\hat{H}$, Eqs. (43) can be readily verified directly. They can also be obtained by means of the equation $G(p,q|t) = \lim G_N(p,q|t)$, where G_N has the form (36), (40), or (41). Going over from the symbols to the kernels in accordance with Eq. (27), one can obtain an expression for the kernel $K_N(x, y)$ of the operator $[\hat{U}(t/N)]^N$. However, it does not have such a lucid form as in the foregoing cases and we therefore omit it.

Second Quantization. Using Eq. (31), we find that the symbol $G_N(a^*,a|t)$ of the operator $[\hat{U}(t/N)]^N$ is

$$G_N(a^*,a\,|\,t)=\int\exp\left\{\frac{it}{hN}\sum_0^{N-1}H(a_k^*,a_{k+1})+\frac{1}{h}\sum_0^{N-1}(a_k^*-a_{k+1}^*)a_{k+1}\right\}\prod_1^{N-1}da_k^*\,da_k,$$
$$a_0^*=a_N^*=a^*,\quad a_0=a_N=a. \tag{44}$$

Equation (44) is equally valid in the Bose and Fermi cases. The formal passage to the limit $N\to\infty$ yields

†In the derivation of (41) we use the identity

$$\begin{vmatrix}1 & 1 & 1\\ \eta_k & q_k & \eta_{k+1}\\ \xi_k & p_k & \xi_{k+1}\end{vmatrix}=-(p_k-\xi_k)(\eta_{k+1}-\eta_k)+(q_k-\eta_k)(\xi_{k+1}-\xi_k).$$

the following equation for G:

$$G(a^*, a \mid t) = \int \exp\left\{\frac{i}{h}\int_0^t H(a^*(\tau), a(\tau))\, d\tau + \frac{1}{h}\int_0^t \frac{da^*}{d\tau} a(\tau)\, d\tau\right\} \prod da^*\, da. \tag{45}$$

3. FUNCTIONAL INTEGRALS FOR THE S MATRIX

Suppose $\hat{S}(t_2, t_1) = e^{it_2\hat{H}_0/h} e^{-i(t_2-t_1)\hat{H}/h} e^{-it_1\hat{H}_0/h}$. In the first order in $\tau = t_1 - t_2$ let us find the symbol of the operator $\hat{S}(t_1, t_2)$. We shall use the integral equation

$$\hat{S}(t_2, t_1) = 1 - \frac{i}{h}\int_{t_1}^{t_2} \hat{V}(t)\, S(t, t_1)\, dt, \qquad \hat{V}(t) = e^{\frac{it}{h}\hat{H}_0}\hat{V}e^{-\frac{it}{h}\hat{H}}, \hat{V} = \hat{H} - \hat{H}_0. \tag{46}$$

Expanding $\hat{S}$ in a perturbation theory series, we obtain

$$\hat{S}(t_2, t_1) = 1 - \frac{i}{h}\int_1^{t_2} \hat{V}(t)\, dt + (t_1 - t_2)^2 \hat{R}, \tag{47}$$

where $\hat{R}$ has a finite limit as $t_2 \to t_1$. Going over from the operators to the symbols in (47), we obtain

$$S(t_2, t_1) = 1 - \frac{i}{h}\int_{t_1}^{t_2} V(t)\, dt + (t_1 - t_2)^2 R = e^{-\frac{i}{h}\int_{t_1}^{t_2} V(t)\, dt} + (t_1 - t_2)^2 r.$$

Reverting to the operators, we obtain

$$\hat{S}(t_2, t_1) = \hat{U}(t_2, t_1) + (t_1 - t_2)^2 \hat{r}.$$

Note that the definition of $\hat{S}(t_2, t_1)$ yields the identity

$$\hat{S}(t_2, t_1) = \hat{S}(t_2, t_2 - \tau)\hat{S}(t_2 - \tau, t_2 - 2\tau) \ldots \hat{S}(t_1 + \tau, t_1), \quad \tau = (t_2 - t_1)/N. \tag{48}$$

Consider the operator

$$\hat{U}_N(t_2, t_1) = \hat{U}(t_2, t_2 - \tau)\hat{U}(t_2 - \tau, t_2 - 2\tau) \ldots \hat{U}(t_1 + \tau, t_1).$$

Since $\hat{U}(t, t - \tau)$ differs from $\hat{S}(t, t - \tau)$ by a quantity of order $\tau^2 = (t_2 - t_1/N)^2$, we have $\lim_{N\to\infty} \hat{U}_N(t_2, t_1) = \hat{S}(t_2, t_1)$. Let the $\hat{q}\hat{p}$ symbol of the operator $\hat{V}(t)$ be equal to $V(t|p, q)$. For the symbol $U_N(t_2, t_1|p, q)$ of the operator $\hat{U}_N(t_2, t_1)$ the following equation holds:

$$U_N(t_2, t_1 \mid p, q) = \left(\frac{1}{2\pi h}\right)^{h(N-1)} \int \exp\left\{-\frac{i}{h}\sum_0^{N-1}\int_{t_1+k\tau}^{t_1+(k+1)\tau} V(s \mid p_{k+1}, q_k) - \frac{i}{h}\sum_0^{N-1} p_{k+1}(q_{k+1} - q_k)\right\} \prod_0^{N-1} dp_k\, dq_k, \tag{49}$$

$$p_0 = p_N = p, \quad q_0 = q_N = q.$$

Similarly, one can obtain an expression for the $\hat{p}\hat{q}$ symbol and for the Weyl symbol of the operator $\hat{U}_N$. We shall not write down the corresponding equations,† but merely point out that in all three cases the argument of the exponential function is an integral sum for the integral

$$K = -\frac{i}{h}\int_{t_1}^{t_2}\left(V(s \mid p(s), q(s)) + p(s)\frac{dq}{ds}\right) ds.$$

In the case of second quantization the function U_N is defined by the integral

$$U_N(t_2, t_1 \mid a^*, a) = \int \exp\left\{-\frac{i}{h}\sum_0^{N-1}\int_{t_1+k\tau}^{t_1+(k+1)\tau} V(s \mid a_k^*, a_{k+1})\, ds + \frac{1}{h}\sum_0^{N-1}(a_k^* - a_{k+1}^*)\, a_{k+1}\right\}, \tag{50}$$

$$a_0 = a_N = a, \quad a_0^* = a_N^* = a^*.$$

†A difference is that the second term in the exponential function in (49) is replaced by the second term of the exponential functions of the integrands of (40), (41), or (42) and the points p_{k+1} and q_k in the first term are replaced by (p_k, q_{k+1}) or (p_k, q_k).

If the Hamiltonian $\hat{H}_0$ is quadratic, $\hat{H}_0 = \sum \omega(p)\hat{a}^*(p)\hat{a}(p)$, then the operator $\hat{V}(s)$ and its corresponding symbol can be readily calculated. It follows from the equations

$$e^{\frac{it\hat{H}_0}{h}}\hat{a}(p)e^{-\frac{itH_0}{h}} = e^{-\frac{it\omega(p)}{h}}\hat{a}(p),$$
$$e^{\frac{it\hat{H}_0}{h}}a^*(p)e^{-\frac{it\hat{H}_0}{h}} = e^{\frac{it\omega(p)}{h}}\hat{a}^*(p)$$

that

$$V(s\,|\,a^*, a) = V(\tilde{a}^*, \tilde{a}),\quad \tilde{a}^*(p) = e^{\frac{is\omega(p)}{h}}a^*(p), a(p) = e^{-\frac{is\omega(p)}{h}}a(p), \tag{51}$$

where $V(a^*, a)$ is the symbol of the interaction operator $\hat{V}$.

Equations (50) and (51) are equally valid in the Bose and Fermi cases. The argument of the exponential function in (50) is an integral sum for the integral

$$K = -\frac{i}{h}\int_{t_1}^{t_2} V(s\,|\,a^*(s), a(s))\,ds + \frac{1}{h}\int_{t_1}^{t_2}\frac{da^*(s)}{ds}a(s)\,ds.$$

The symbol S of the scattering operator can be obtained from (49) and (50) by the successive passage to the limit†

$$S = \lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\ \lim_{N\to\infty} U_N.$$

4. EXPRESSION FOR THE PARTITION FUNCTION

If we replace the parameter t by $i\beta$ in Eqs. (36), (40), (41), (42), and (44), we obtain the symbol G_N of the operator $[\hat{U}(i\beta/N)]^N$. Applying Eqs. (10), (33), or (34), we obtain an expression for $\Xi_N(\beta) = \mathrm{Sp}[\hat{U}(i\beta/N)]^N$ in the form of a multiple integral whose limit as $N \to \infty$ is the partition function $\Xi(\beta/h) = \lim \Xi_N(\beta/h) = \mathrm{Sp}e^{-\beta\hat{H}/h}$. We give the final formula for the case of second quantization, which is the most important:

$$\Xi_N(\beta/h) = \int \exp\left\{-\frac{\beta}{Nh}\sum_0^{N-1} H(a_k^*, a_{k+1}) + \frac{1}{h}\sum_0^{N-1}(a_k^* - \dot{a}_{k+1})a_{k+1}\right\}\prod_2^N da_k^*\,da_k,$$

$$\text{where}\quad \begin{cases} a^*(N) = -a^*(0) & a(N) = -a(0)\ \text{in the Fermi case.}\\ a^*(N) = a^*(0) & a(N) = a(0)\ \text{in the Bose case.}\end{cases}$$

The difference between the Fermi and Bose cases is due to the difference between the trace formulas (33) and (34).

5. WICK FORMULA

In this section we obtain the expression (38) for the symbol of the operator $e^{it\hat{H}/h}$ by means of the Wick formula.‡ The arguments developed here are conditional to the same extent as Eq. (38) itself. We shall restrict ourselves to the case of $\hat{q}\hat{p}$ quantization and a single degree of freedom.

In accordance with the Wick formula the $\hat{q}\hat{p}$ symbol of the operator $e^{it\hat{H}/h}$ is

$$G(p, q\,|\,t) = \exp\left\{\int_0^t\int_0^t \frac{\delta}{\delta q(t_1)}\Delta(t_1 - t_2)\frac{\delta}{\delta p(t_2)}dt_1dt_2\right\}\exp\left\{\frac{i}{h}\int_0^t H(p(\tau), q(\tau))\,d\tau\right\}. \tag{52}$$

Equation (52) is to be understood as follows. First, arbitrary functions $p(\tau)$ and $q(\tau)$ are to be substituted as the arguments p and q into the symbol $H(p, q)$. To the functional

$$\exp\frac{i}{h}\int_0^t H(p(\tau), q(\tau))\,d\tau$$

†In the case of second quantization $S = 0$ for the Hamiltonian $\hat{H}_0$ considered above. A nonvanishing limit is possible for the Hamiltonian

$$H_0 = \int \omega(p)a^*(p)a(p)dp,$$

which is characteristic for a system with an infinite number of degrees of freedom.

‡For an understanding of this section, the most convenient derivation of the Wick formula can be found in [9]. To obtain Eqs. (52) and (53) one must apply the arguments of [9] to the operators $\hat{p}$ and $\hat{q}$ instead of $\hat{a}$ and $\hat{a}^*$ and set $\hat{H} = 0$.

one then applies the operator

$$\exp \int_0^t \int_0^t \frac{\delta}{\delta q(t_1)} \Delta(t_1 - t_2) \frac{\delta}{\delta p(t_2)} dt_1 dt_2.$$

The result is again a functional of the functions $p(\tau)$ and $q(\tau)$. The function $G(p, q|t)$ serves as the value of this functional on functions $p(\tau) = p$ and $q(\tau) = q$ that do not depend on τ.

The function Δ is the difference between the T product and the normal form. In our case

$$T(\hat{p}(t_1), \hat{q}(t_2)) = \theta(t_2 - t_1)\hat{p}(t_1)\hat{q}(t_2) + \theta(t_1 - t_2)\hat{q}(t_2)\hat{p}(t_1) = -ih\theta(t_2 - t_1) + \hat{q}(t_2)\hat{p}(t).$$

Thus,

$$\Delta(t_1, t_2) = -ih\theta(t_2 - t_1). \tag{53}$$

We replace all integrals in (52) by integral sums and set $q(t_k) = q_k$ and $p(t_k) = p_k$. Then G is replaced by the function $\tilde{G} = e^R F$, where

$$R = \sum \frac{\partial}{\partial q_k} \Delta(t_k - t_j) \frac{\partial}{\partial p_j} \alpha_i \alpha_j$$

is a differential operator with constant coefficients, and

$$F = \exp\left\{\frac{i}{h} \sum \alpha_k H(p_k, q_k)\right\}.$$

Since R is an operator with constant coefficients, e^R is an operator of convolution with the function Φ, which is the Fourier transform of the function

$$r = \exp\left\{-\sum u_k \Delta_{kj} v_j\right\}, \quad \Delta_{kj} = \Delta(t_k - t_j)\alpha_k \alpha_j.$$

One cannot directly calculate the Fourier transform of r since the quadratic form $\Sigma u_k \Delta_{kj} v_j$ is not positive definite. Therefore, as in all such cases, we replace r by the function

$$r_\varepsilon = \exp\left\{-(u, v) \; A_\varepsilon^{(N)} \begin{pmatrix} u \\ v \end{pmatrix}\right\},$$

where $A_\varepsilon^{(N)}$ is a positive definite matrix and

$$\lim_{\varepsilon \to 0} (u, v) A_\varepsilon^{(N)} \begin{pmatrix} u \\ v \end{pmatrix} = \sum u_k \Delta_{kj} v_j.$$

We then calculate the Fourier transform of the functions r_ε and set $\varepsilon = 0$. The Fourier transform of the functions r_ε is

$$\tilde{r}_\varepsilon = \frac{1}{\sqrt{\det(2\pi A_\varepsilon)}} e^{-\frac{1}{4}(q,p) A_\varepsilon^{(N)-1} \begin{pmatrix} q \\ p \end{pmatrix}}.$$

We include the preexponential factor in the normalization of the differentials $\Pi dpdq$ and ignore it in what follows. We choose the matrix $A_\varepsilon^{(N)}$ in such a way that it becomes the operator

$$A_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varepsilon u(\tau) - \frac{ih}{2} \int_0^t \theta(\tau - \tau') v(\tau') d\tau' \\ -\frac{ih}{2} \int_0^t \theta(\tau' - \tau) u(\tau') d\tau' + \varepsilon v(\tau) \end{pmatrix} \tag{54}$$

in the limit $N \to \infty$. In what follows we are interested in the limit as $N \to \infty$; instead of the matrix $A_\varepsilon^{(N)}$ we shall therefore immediately invert the operator (54). To this end, we must solve the equation

$$A_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}.$$

Differentiating, we obtain

$$q'(\tau) = \varepsilon u'(\tau) - \frac{ih}{2} v(\tau), \; p'(\tau) = \varepsilon v'(\tau) + \frac{ih}{2} u(\tau). \tag{55}$$

In addition, from the equation itself it follows that

$$q(0)=\varepsilon u(0), \quad p(t)=\varepsilon v(t). \tag{56}$$

Setting $\varepsilon = 0$ in (55) and (56), we obtain

$$v(\tau)=\frac{2i}{h}q'(\tau),\ u(\tau)=-\frac{2i}{h}p(\tau),\ q(0)=p(t)=0. \tag{57}$$

Therefore, in the limit $\varepsilon \to 0$ the exponent in (53) is

$$-\frac{1}{4}(q,p)\begin{pmatrix}-\frac{2i}{h}p' \\ \frac{2i}{h}q'\end{pmatrix}=-\frac{i}{2h}\int_0^t\left(p\frac{dq}{d\tau}-q\frac{dp}{d\tau}\right)d\tau=-\frac{i}{h}\int_0^t p\frac{dq}{d\tau}d\tau \tag{58}$$

[in deriving the last equation we have used the boundary conditions (57)].

Note that the last integral in (58) is the limit of integral sums of the form

$$p_1q_1+p_2(q_2-q_1)+\ldots+p_N(q_N-q_{N-1}).$$

Therefore, the integral

$$\int_0^t (p(\tau)-p)\frac{d}{d\tau}(q(\tau)-q)\,d\tau$$

is the limit of the integral sums

$$\begin{aligned}&(p_1-p)(q_1-q)+(p_2-p)(q_2-q_1)+\ldots+(p_N-p)(q_N-q_{N-1})\\ &=p_1(q_1-q)+p_2(q_2-q_1)+\ldots+p_N(q_N-q_{N-1})+p(q-q_N).\end{aligned} \tag{59}$$

The right side of (59) is identical with the second term in the argument of the exponential function in Eq. (36). Thus, we again return to (37).

CONCLUSIONS

We begin by giving some criteria for an operator to be bounded.

A. If $f(p, q)$ is the qp symbol of the operator A, then

$$(A\xi,\eta)=\int f(p,x)\,\tilde{\xi}(p)\,\bar{\eta}(x)\,e^{\frac{i}{h}px}\,dp\,dx=(B\tilde{\xi},\eta),$$

where B is an operator with kernel equal to $f(x, y)e^{ixy/h}$,

$$\tilde{\xi}(p)=(2\pi h)^{-n}\int\xi(y)\,e^{-\frac{i}{h}py}\,dy,$$

$$\int|\tilde{\xi}(p)|^2dp=(2\pi h)^{-n}\int|\xi(x)|^2dx.$$

If ξ and η run through the unit sphere in L_2 independently, then $(2\pi h)^{n/2}\tilde{\xi}$ and η possess the same property. Therefore,

$$\|A\|=\sup_{\|\xi\|=\|\eta\|=1}|(A\xi,\eta)|=(2\pi h)^{-n/2}\|B\|.$$

B. If $f(p, q)$ is the symbol of the operator A, then the same arguments show that

$$\|A\|=(2\pi h)^{-n/2}\|B\|,$$

where B is the operator with kernel $f(x, y)e^{-ix/h}$.

C. If $f(p, q)$ is the Weyl symbol of the operator A, then

$$(A\xi,\eta)=(2\pi h)^{-n}\int f(p,q)\,e^{\frac{ps}{ih}}\,\xi\left(q-\frac{s}{2}\right)\bar{\eta}\left(q+\frac{s}{2}\right)dp\,dq\,ds.$$

By the Cauchy–Schwarz inequality

$$\left|\int \xi\left(q-\frac{s}{2}\right)\bar{\eta}\left(q+\frac{s}{2}\right)ds\right| \leqslant 2^n\left(\int|\xi(x)|^2dx\int|\eta(x)|^2dx\right)^{1/2}.$$

Therefore,

$$|(A\xi,\eta)|\leqslant(\pi h)^{-n}\int|f(p,q)|\,dpdq\,\|\xi\|\|\eta\|,$$
$$\|A\|\leqslant(\pi h)^{-n}\int|f(p,q)|\,dpdq.$$

We shall now list some of the questions whose solutions would illuminate the mathematical nature of the problems touched on in this paper.

1. The assertions made in the present paper that an integral G_N of finite multiplicity has the symbol of the operator $e^{it\hat{H}/h}$ or the scattering operator as its limit as $N \to \infty$ are none other than the probability hypotheses. † The proof of these hypotheses and also the elucidation of the extent to which the final result is independent of the details of the finite-dimensional approximation is an important unresolved problem. ‡ In this connection we would like to point out that an integral in a Wiener measure is much more stable against a change in the finite-dimensional approximations than the Feynman integrals considered in this paper.

2. With the exception of the Weyl and Wick symbols, which are studied in detail in [8, 9, 10], no theorems exist for other symbols on the approximation of an operator defined by an arbitrary symbol by operators with polynomial symbols.

3. The symbols considered in this paper behave very differently on the transition to an infinite number of degrees of freedom. It is shown in [8] that in the case of an infinite number of degrees of freedom one cannot use Weyl quantization to associate any symbol with a Hilbert – Schmidt operator (or, modifying somewhat the definition of quantization, one can associate nuclear operators but not a unique operator). On the other hand, it is known [9] that, using Wick quantization, one can associate a symbol with any bounded operator. It is not known what situation obtains in the case of $\hat{p}\hat{q}$ and $\hat{q}\hat{p}$ quantization.

4. The correspondence between operators and symbols is completely determined by the formulas that express the symbols of the operators $\hat{p}\hat{A}$, $\hat{A}\hat{p}$, $\hat{q}\hat{A}$, and $\hat{A}\hat{q}$ in terms of the symbol of $\hat{A}$. We shall say that a linear quantization is defined if these formulas have the form

$$\hat{p}_i\hat{A}\leftrightarrow L^1_{p_i}A,\quad \hat{A}\hat{p}_i\leftrightarrow L^2_{p_i}A,\quad \hat{q}_i\hat{A}\leftrightarrow L^1_{q_i}A,\quad \hat{A}\hat{q}_i\leftrightarrow L^2_{q_i}A,$$

where $L_p{}^1$ etc. are first-order differential operators for which the coefficients of the derivatives are constants and the free terms are linear. For example,

$$L^1_{p_i}A=\sum\left(\alpha^{(1)}_{ij}p_j+\beta^{(1)}_{ij}q_j+\gamma^{(1)}_{ij}\frac{\partial}{\partial p_j}+\delta^{(1)}_{ij}\frac{\partial}{\partial q_j}\right)A$$

(the matrices $\alpha^{(1)}_{ij}$ etc. are not arbitrary: they satisfy relations that follow from the commutation rules $[\hat{p}_i, \hat{q}_j] = ih\delta_{ij}$). It would be very interesting to construct a general theory of linear quantization and, for example, obtain criteria for an operator to belong to a certain class, (i.e., the criterion for boundedness, nuclearity, etc.) and learn how to solve the questions 1–3 in the general case.

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† The arguments adduced in this connection in Sec. 2 seem to be convincing but they do not have the strength of a mathematical proof.

‡ In particular, it would be very interesting to make the derivation of the Feynman integral from the Wick formula (as indicated in Sec. 3) completely correct [at least for Hamiltonians with polynomial symbols H(p, q)].

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