Quantum Field Theory

Problem Sheet 5

1. *Translation Ward identity*

   Find the variation of the action for the free scalar field under the field transformation
   $$\phi(x) \mapsto \phi'(x) = \phi(x) + a(x)\partial_\mu \phi(x)\,.$$
   Deduce the Ward identities generated by translation invariance.

2. *Grassmann integrals*

   Integrals over Grassmann variables are defined by specifying two *operational* rules:
   $$\int d\psi_\alpha = 0\,,$$
   $$\int d\psi_\alpha \psi_\beta = \delta_{\alpha\beta}\,.$$
   Briefly discuss why this is the case.

   Show that, for an $N \times N$ matrix $A_{\alpha\beta}$
   $$\int \prod_{\beta=1}^{N} d\psi_\beta \prod_{\alpha=1}^{N} d\bar{\psi}_\alpha \, \exp\left(\bar{\psi}_\alpha A_{\alpha\beta} \psi_\beta\right) = \det A$$

   Hint: it is useful to remember that
   $$\det A = \sum_{\beta_1 \dots \beta_N} \epsilon_{\beta_1 \dots \beta_N} A_{1\beta_1} \dots A_{N\beta_N}\,.$$

3. *Dirac propagator*

   Prove that the Dirac propagator is the inverse of the kinetic term in the action, *i.e.*
   $$\left(i\partial\!\!\!/_x - m\right)_{\alpha\beta} S_{\beta\gamma}(x-y) = i\delta(x-y)\delta_{\alpha\gamma}\,.$$

4. *LSZ reduction for fermions*

   For the case of fermions the operator $\psi(x)$ can be decomposed as
   $$\psi(x) = \int d\Omega_p \sum_{s=\pm 1/2} \left[e^{-ip\cdot x} a(\mathbf{p},s) u(\mathbf{p},s) + e^{ip\cdot x} b^\dagger(\mathbf{p},s) v(\mathbf{p},s)\right]\,.$$

This relation can be inverted, yielding:

$$a^\dagger(\mathbf{p},s) = \int d^3x\, e^{-ip\cdot x}\bar{\psi}(x)\gamma^0 u(\mathbf{p},s)\,,$$
$$b^\dagger(\mathbf{p},s) = \int d^3x\, e^{-ip\cdot x}\bar{v}(\mathbf{p},s)\gamma^0 \psi(x)\,.$$

Following the same reasoning that we used in the case of a scalar field, let us introduce in the interacting theory time-dependent creation/annihilation operators for fermions and antifermions according to the expressions above. Show that

$$a^\dagger(+\infty,\mathbf{p},s) - a^\dagger(-\infty,\mathbf{p},s) = \int d^4x\, e^{-ip\cdot x}\bar{\psi}(x)(i\overleftarrow{\partial\!\!\!/} + m)u(\mathbf{p},s)\,, \tag{1}$$

and similar relations for the other creation/annihilation operators.

The scattering amplitude for a $2 \longrightarrow 2$ process can be written as:

$$\begin{aligned} &\langle p_1', s'; p_2', r'; \mathrm{out}|p_1, s; p_2, r; \mathrm{in}\rangle = \\ &= \langle 0|T\left(b(+\infty,\mathbf{p}_2',r')a(+\infty,\mathbf{p}_1',s')a^\dagger(-\infty,\mathbf{p}_1,s)b^\dagger(-\infty,\mathbf{p}_2,r)\right)|0\rangle\,. \end{aligned} \tag{2}$$

Show that

$$\begin{aligned} \langle p_1', s'; p_2', r'; \mathrm{out}|p_1, s; p_2, r; \mathrm{in}\rangle = &\ (-i)^2(i)^2 \\ &\times \int d^4x_1\, e^{-ip_1\cdot x_1} \int d^4x_2\, e^{-ip_2\cdot x_2} \int d^4x_1'\, e^{ip_1'\cdot x_1'} \int d^4x_2'\, e^{ik_2'\cdot x_2'} \\ &\times \left[\bar{u}(\mathbf{p}_1',s')\left(i\overrightarrow{\partial\!\!\!/}_{x_1'} - m\right)\right]_{\alpha_1} \left[\bar{v}(\mathbf{p}_2,r)\left(i\overrightarrow{\partial\!\!\!/}_{x_2} - m\right)\right]_{\beta_2} \\ &\times \langle 0|T\left(\bar{\psi}_{\alpha_1}(x_1')\psi_{\alpha_1}(x_2')\bar{\psi}_{\beta_1}(x_1)\psi_{\beta_2}(x_2)\right)|0\rangle \\ &\times \left[\left(-i\overleftarrow{\partial\!\!\!/}_{x_1} - m\right)u(\mathbf{p}_1,s)\right]_{\beta_1} \left[\left(-i\overleftarrow{\partial\!\!\!/}_{x_2'} - m\right)v(\mathbf{p}_2',r')\right]_{\alpha_2}\,. \end{aligned} \tag{3}$$