Nuclear Physics B163 (1980) 109–132

THE STRUCTURE OF YANG-MILLS THEORIES IN THE TEMPORAL GAUGE
(I). General formulation

G.C. ROSSI *
CEN-Saclay, Boîte Postale No. 2, 91190 Gif-sur-Yvette, France

M. TESTA **
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure 24, rue Lhomond 75231 Paris Cedex 05, France

Received 30 July 1979
(Revised 4 October 1979)

In this paper we discuss the $A_0 = 0$ gauge formulation of Yang-Mills theories by the aid of the Feynman propagation kernel. We show that Gauss' law is automatically implemented as a constraint on the states. The states not annihilated by Gauss' operator are shown to describe external non-abelian sources. The formula of the Feynman propagation kernel in the presence of external sources is explicitly given and applied to free quantum electrodynamics and the abelian Higgs model. The equivalence with the Coulomb-gauge formulation is proved.

1. Introduction

The elegance of the principle of local gauge invariance [1] has as a counterpart the necessity of introducing the non-canonical variable A_0 in the description of the gauge vector fields. In the calculation of physical quantities one is then confronted with the problem of its elimination, by the use of some gauge-fixing condition. The simplest way of doing it is clearly to set $A_0 = 0$.

Some nice features of the temporal gauge are the absence of ghosts and the fact that it provides the most suitable framework to study topological non-perturbative phenomena such as vacuum tunneling induced by instantons [2].

In the $A_0 = 0$ gauge, however, Gauss' law is lost and has to be implemented as a supplementary condition on the states [3].

In this paper we want to carefully study the formulation of gauge theories in the $A_0 = 0$ gauge by means of the Feynman propagation kernel [4], which gives the

* On leave of absence from Istituto di Fisica dell'Università di Roma, Rome, Italy.
** On leave of absence from INFN – Sezione di Roma, Rome, Italy.

amplitude for finding the field in the configuration $\boldsymbol{A}_2$ at time T_2, if it was in the configuration $\boldsymbol{A}_1$ at time T_1. The use of this kernel instead of the Green function generating functional has the advantage of treating all the states on the same ground without giving a special role to the vacuum.

We will show that within this formalism Gauss' law is automatically implemented as a constraint on the states [5] and that external sources can be introduced in a very natural and mathematically transparent way, despite the fact that A_0 is set equal to zero [6, 7]. The formula for the Feynman propagation kernel in the presence of any configuration of external sources is derived.

In this framework a simple perturbation expansion can be set up which will be the subject of a forthcoming publication.

The plan of the paper is as follows. In sect. 2 we define the Feynman propagation kernel, K, in the $A_0 = 0$ gauge and we show that it is obtained as a projection on the set of "physical" states (states which satisfy Gauss' law) of the kernel $\widetilde{K}$ corresponding to the theory in which Gauss' law is missing. In sect. 3 the state functionals contributing to $\widetilde{K}$ are shown to describe all possible distributions of external sources, in the sense that they belong to unitary finite-dimensional representations of the local gauge group. Sect. 4 is dedicated to a parenthetical discussion on the transformation properties of the states under gauge transformations with non-trivial winding number. We prove, on rather general grounds, that all "physical" states have the same θ-angle as the vacuum. In sect. 5 we make another brief digression showing that, although the color charges are not gauge-invariant operators, they nevertheless transform "physical" states into "physical" states. Using the results of sect. 3, in sect. 6 we obtain the expression for the Feynman propagation kernel describing an arbitrary external charge distribution coupled to the Yang-Mills field, as a group projection of $\widetilde{K}$. In sect. 7 we apply our formalism to two very simple examples: free electromagnetism and the abelian Higgs model in the tree approximation, showing how the interaction energy of an external source distribution can actually be computed. In sect. 8 we prove the equivalence of the Coulomb gauge to the $A_0 = 0$ gauge. Some technical points are discussed in the appendices.

2. Feynman propagation kernel in the $A_0 = 0$ gauge

We describe the dynamics of Yang-Mills field by the Feynman propagation kernel

$$K(\boldsymbol{A}_2, T_2; \boldsymbol{A}_1, T_1) = K(\boldsymbol{A}_2, \boldsymbol{A}_1; T_2 - T_1) = \int_{\substack{\boldsymbol{A}(\boldsymbol{x}, T_1) = \boldsymbol{A}_1(\boldsymbol{x}) \\ \boldsymbol{A}(\boldsymbol{x}, T_1) = \boldsymbol{A}_2(\boldsymbol{x})}} \delta A_\mu(\boldsymbol{x}, t)\, e^{iS_0}, \tag{2.1}$$

where *

$$S_0 = \int_{T_1}^{T_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \mathcal{L}_0, \quad \mathcal{L}_0 = -\frac{1}{2g^2}\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) , \tag{2.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] . \tag{2.3}$$

In (2.1) the A_0 integration is unrestricted. The reason for this is that we have chosen to describe the evolution of the system on the spacelike hypersurfaces t = constant. As a consequence, A_0 is not a canonical variable since the normal derivative of the normal component of A_μ ($\dot{A}_0$ in this case) does not appear in the action.

The integration over A_0 has only the function of guaranteeing the validity of Gauss' law and it must be extended over the whole time interval $T_1 \leqslant t \leqslant T_2$.

As is well-known, eq. (2.1) is meaningless unless, by a gauge-fixing procedure, we eliminate the infinite gauge volume due to the invariance of the lagrangian under the gauge transformations

$$A_\mu \to A_\mu^{U_w} = U_w^+ A_\mu U_w + iU_w^+ \partial_\mu U_w , \tag{2.4a}$$

$$(A_\mu^{U_w})^{U_v} \equiv (A_\mu)^{U_w U_v} , \tag{2.4b}$$

$$U_w(\boldsymbol{x}, t) = \exp(i\lambda_a w^a(\boldsymbol{x}, t)) . \tag{2.5}$$

For the reasons explained in sect. 1 and the insight that will be gained on the structure of the theory, we choose to work in the $A_0 = 0$ gauge. We will discuss in sect. 8 the Coulomb gauge $\partial^i A_i = 0$.

Following the standard Faddeev-Popov prescription [9], we introduce in (2.1) the identity

$$1 = \Delta \int \mathcal{D}w(\boldsymbol{x}, t)\, \delta(A_0^{U_w}) , \tag{2.6}$$

where the integration is extended over all points in space and over the whole time interval between T_1 and T_2, including the ends.

At each time, the integral in (2.6) runs over the group, $\mathcal{G}$ (local gauge group), of the differentiable gauge transformations going to the identity at spatial infinity. The reason for this restriction will be clear in a moment (see also sect. 8).

The gauge transformations belonging to $\mathcal{G}$ are known to fall in homotopy classes, characterized by an integer n (winding number). Since the final result will be shown

* We take SU(N) as the gauge group and use the notations

$$A_\mu = A_\mu^a \lambda^a, \quad [\lambda^a, \lambda^b] = if^{abc}\lambda^c, \quad \mathrm{Tr}(\lambda^a\lambda^b) = \tfrac{1}{2}\delta^{ab},$$

with A_μ real. The λ's are the SU(N) generators in the fundamental representation. In the functional integral we have simply indicated by $\delta h(\boldsymbol{x}, t)$ the cartesian measure $\Pi_{\boldsymbol{x}}\Pi_t\Pi_{a=1}^{N^2-1}\,\delta h^a(\boldsymbol{x}, t)$. $\mathcal{D}\, h(\boldsymbol{x}, t) = \mu(h)\, \delta h(\boldsymbol{x}, t)$ will be the corresponding normalized invariant measure over the gauge group. The index convention and the metrics of ref. [8] are used.

to be independent on the particular homotopy class chosen in (2.6), we can, without loss of generality, restrict the integration over the subgroup $\mathcal{G}_0$ of the homotopically trivial gauge transformations.

From the definition (2.6), one easily sees that Δ is independent on A_μ and can only depend exponentially on $T_2 - T_1$. Then it merely represents a shift in the zero of the energies [see eq. (2.13)] which will be dropped in the following.

Introducing (2.6) into (2.1) and making the change of variables $A_\mu \to A_\mu^{U_w}$, we obtain

$$K(A_2, A_1; T_2 - T_1) = \int_{\mathcal{G}_0} \mathcal{D}w(\boldsymbol{x}, t) \int_{\substack{A(\boldsymbol{x}, T_1) = A_1(\boldsymbol{x})^{U_w(\boldsymbol{x}, T_1)} \\ A(\boldsymbol{x}, T_2) = A_2(\boldsymbol{x})^{U_w(\boldsymbol{x}, T_2)}}} \delta A(\boldsymbol{x}, t) \mathrm{e}^{iS_0(A_0=0)} . \tag{2.7}$$

It is now possible to drop the gauge volume corresponding to the gauge integrations at all times except those at times T_1 and T_2. We thus get

$$K(A_2, A_1; T_2 - T_1) = \int_{\mathcal{G}_0} \mathcal{D}w_1(\boldsymbol{x}) \mathcal{D}w_2(\boldsymbol{x}) \int_{A_1^{U_{w_1}}, A_2^{U_{w_2}}} \delta A(\boldsymbol{x}, t)\, \mathrm{e}^{iS_0(A_0=0)} , \tag{2.8}$$

where we have set $w(\boldsymbol{x}, T_{1,2}) = w_{1,2}(\boldsymbol{x})$.

Because of the invariance of the action and of the functional integration measure, δA, under arbitrary time-independent gauge transformations, one of the two gauge integrations in (2.8) is always trivial, since the integrand only depends on $U_{w_1} U_{w_2}^+$. By the same argument it follows that kernels obtained in integrating (2.6) over different homotopy classes are equal.

The important observation to make about eq. (2.8) is that, in the $A_0 = 0$ gauge, Gauss' law appears to be automatically implemented as a constraint on the states.

In order to clarify this statement and the role of the integrations over the initial and final gauge transformations in (2.8), it is useful to introduce the auxiliary quantity

$$\tilde{K}(A_2, A_1; T_2 - T_1) = \int_{A_1, A_2} \delta A(\boldsymbol{x}, t)\, \mathrm{e}^{iS_0(A_0=0)} . \tag{2.9}$$

Physically $\tilde{K}$ describes the quantum theory of a Yang-Mills field without Gauss' law. If we introduce the eigenvectors of the field operator A at a time t_0:

$$A^a(\boldsymbol{x}, t_0) | A \rangle = A^a(\boldsymbol{x}) | A \rangle , \qquad a = 1, 2, ..., N^2 - 1 , \tag{2.10}$$

$\tilde{K}$ is given by

$$\tilde{K}(A_2, A_1; T_2 - T_1) = \langle A_2 | \mathrm{e}^{-iH(T_2 - T_1)} | A_1 \rangle , \tag{2.11}$$

where H is the hamiltonian of the system represented in functional formalism by the operator:

$$\mathcal{H} = \int \mathrm{d}\boldsymbol{x} \left[\tfrac{1}{2} g^2 \frac{\delta^2}{\delta A_i^a(\boldsymbol{x})\, \delta A_a^i(\boldsymbol{x})} + \frac{1}{4g^2} F_{ij}^a F_a^{ij} \right] . \tag{2.12}$$

In terms of the eigenfunctionals of $\mathcal{H}$, $\psi_n(A) \equiv \langle A | n \rangle$, $\widetilde{K}$ can be written symbolically as [4]

$$\widetilde{K}(A_2, A_1; T_2 - T_1) = \sum_n \mathrm{e}^{-iE_n(T_2 - T_1)} \psi_n(A_2)\, \psi_n^*(A_1) . \tag{2.13}$$

We see from this formula that the gauge integrations in (2.8) have simply the effect of picking up in $\widetilde{K}$ the set of states (which will be called "physical" states) that obey Gauss' law. They are of the form *

$$\psi_{\mathrm{ph}}(A) = \int_{\mathcal{G}_0} \mathcal{D}w\ \psi(A^{U_w}) \tag{2.14}$$

and are invariant under gauge transformations which go to I at spatial infinity. By taking the functional derivative of the equation

$$\psi_{\mathrm{ph}}(A^{U_h}) = \psi_{\mathrm{ph}}(A) \tag{2.15}$$

with respect to $h^a(\boldsymbol{x})$ at $h^a(\boldsymbol{x}) = 0$, we obtain Gauss' law:

$$-iD_{ab}^j \frac{\delta}{\delta A_b^j(\boldsymbol{x})} \psi_{\mathrm{ph}}(A)$$

$$\equiv -i(\partial^j \delta_{ab} + f_{acb} A_c^j(\boldsymbol{x})) \frac{\delta}{\delta A_b^j(\boldsymbol{x})} \psi_{\mathrm{ph}}(A) = 0 . \tag{2.16}$$

From eq. (2.14) it should be clear why we have restricted U_w in (2.6) to tend to the identity at spatial infinity. Had we allowed for more general gauge transformations, we would have got constraints on the states stronger than Gauss' law.

In order to illustrate this point we first recall that under a general transformation $A \to A^R$ of the fields a state functional behaves as

$$\psi(A^R) = \mathcal{U}_R \psi(A) , \tag{2.17}$$

where $\mathcal{U}_R$ is the unitary operator, which implements the transformation law R and

$$A^R = \mathcal{U}_R A \mathcal{U}_R^+ , \tag{2.18a}$$

$$\mathcal{U}_R \mathcal{U}_S \equiv \mathcal{U}_{RS} . \tag{2.18b}$$

For a constant gauge transformation (global color rotation), $V_r = \exp(+ir^a\lambda^a)$, we have

$$\mathcal{U}_{V_r} = \mathrm{e}^{+ir^a Q^a} , \tag{2.19}$$

* A similar way of implementing Gauss' law has been proposed in ref. [10], without, however, specifying the integration domain.

where

$$Q^a = f^{abc} \int \mathrm{d}\boldsymbol{x}\, \dot{A}_i^b(\boldsymbol{x})\, A^{ci}(\boldsymbol{x}) \tag{2.20}$$

are the global color charges *.

Suppose now for a moment that we allow the integration in (2.6), and hence in (2.8) and in (2.14), to run, say, over all gauge transformations. Denoting by $\phi(A)$ the resulting averaged state we would get

$$\mathrm{e}^{ir^a Q^a} \phi(A) = \mathrm{e}^{ir^a Q^a} \int \mathcal{D}v\, \psi(A^{U_v}) = \int \mathcal{D}v\, \psi(A^{V_r U_v}) = \phi(A)\,. \tag{2.21}$$

The change of integration variables which leads to the last equality in (2.21) has been possible by the enlargement of the integration domain in the definition of ϕ.

Eq. (2.21) would imply that all color charges are identically zero since they annihilate all physical states. This situation is in general too restrictive and physically unacceptable **.

3. Structure of the states

In this section we want to discuss the structure of the set of states of the theory described by the kernel $\tilde{K}$, defined in eq. (2.9).

The starting point of this analysis is the identity

$$\tilde{K}(A_2^U; A_1^U; T_2 - T_1) = \tilde{K}(A_2, A_1; T_2 - T_1)\,, \tag{3.1}$$

valid for any time-independent gauge transformation with arbitrary winding number. We stress again that (3.1) is only a consequence of the invariance of the action and of the measure δA, under such gauge transformations. This is the translation in functional language of the gauge invariance of the hamiltonian (2.12). In particular in the zero winding number sector, (3.1) is equivalent to the statement that Gauss' law commutes with the hamiltonian.

Eq. (3.1), in view of the formula (2.13), implies that energy-degenerate states transform among themselves according to

$$\mathcal{U}_{U_h} \psi_{n,s}^{(j)}(A) = \psi_{n,s}^{(j)}(A^{U_h}) = R_{ss'}^{(j)+}(U_h)\, \psi_{n,s'}\,, \tag{3.2}$$

where the $R^{(j)}$'s are unitary representations of the gauge group and the set of indices s labels the degenerate states belonging to the same representation (j).

In order to gain some insight into the physical significance of this degeneracy we first address ourselves to the problem of classifying the irreducible finite-

* The possibility of defining these global charges is discussed in sect. 5.

** An average such as in (2.21) would probably cause no harm in a theory in which the total charge of the "physical" states is shielded (color bleaching). See, for instance, the case of the Higgs model discussed in sect. 7.

dimensional unitary representations of $\mathcal{G}_0$ *. We will discuss the group $\mathcal{G}$ in sect. 4.

Consider the mapping

$$U_h(\boldsymbol{x}) \to R(U_h)\,, \tag{3.3}$$

where $U_h(x)$ is an element of the local gauge group $\mathcal{G}_0$ and $R(U_h)$ is a finite-dimensional unitary matrix, such that

$$U_h(\boldsymbol{x})\,U_{h'}(\boldsymbol{x}) \to R(U_h)R(U_{h'}) = R(U_h U_{h'})\,. \tag{3.4}$$

R is a functional of $U_h(\boldsymbol{x})$, which, as shown in appendix A, depends only on the values assumed by U_h in at most a finite number of points. This means that all the finite-dimensional irreducible unitary representations of $\mathcal{G}_0$ can be obtained by the following construction:

(i) choose p different points in space: $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_p$;

(ii) choose p irreducible finite-dimensional unitary representations of the global gauge group SU(N): $L^{j_1}, L^{j_2}, ..., L^{j_p}$;

(iii) consider $L^{j_1}[U_h(\boldsymbol{x}_1)], L^{j_2}[U_h(\boldsymbol{x}_2)], ..., L^{j_p}[U_h(\boldsymbol{x}_p)]$; denoting by M^j the generators of SU(N) in the representation (j), one has:

$$L^j[U_h(x_k)] = \exp(iM^j_a h^a(x_k))\;; \tag{3.5}$$

(iv) then **

$$R^{(j_p)}(U_h) = L^{j_1}[U_h(\boldsymbol{x}_1)] \times L^{j_2}[U_h(\boldsymbol{x}_2)] \times ... \times L^{j_p}[U_h(\boldsymbol{x}_p)]\,. \tag{3.6}$$

The physical meaning of the above construction can be seen by taking the functional derivative of eq. (3.2) with respect to $h^a(\boldsymbol{x})$ at $h^a(\boldsymbol{x}) = 0$. Using (3.5) and (3.6), we get, for instance, in the case $p = 2$ ***

$$-iD^l_{ab}\frac{\delta\psi^{(j_2)}_{n,s_1s_2}}{\delta A^l_b(\boldsymbol{x})} = -\delta(\boldsymbol{x}-\boldsymbol{x}_1)(M^{j_1}_a)_{s_1s_1'}\,\psi^{(j_2)}_{n,s_1's_2}(A)$$

$$-\,\delta(\boldsymbol{x}-\boldsymbol{x}_2)(M^{j_2}_a)_{s_2s_2'}\psi^{(j_2)}_{n,s_1s_2'}(A)\,. \tag{3.7}$$

Eq. (3.7) is Gauss' law for a state $\psi^{(j_2)}_{n,s_1s_2}$ describing the configuration of the gauge field in presence of two point-like (non-abelian) sources concentrated in $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ with color spin M^{j_1} and M^{j_2}. In general a state, belonging to the p-point representation (3.6), satisfies Gauss' law with p point-like sources.

* We have not investigated the existence and the structure of possible infinite-dimensional representations.

** Representations corresponding to different choices of the p points, but to the same choice of $L^{j_1}, L^{j_2}, ..., L^{j_p}$ are equivalent. This fact allows one to implement any possible space-time symmetry of the hamiltonian.

*** The minus sign on the r.h.s. of eq. (3.7) merely reflects the fact that non-abelian charges naturally couple as negative charges do in electrodynamics.

As a matter of principle this terminology is incorrect because at this stage the indices (s) in eq. (3.7) do not represent any physical degree of freedom, but only label the degeneracy of the hamiltonian (2.12). In other words the M_a's are not quantum observables but are to be thought as numerical matrices telling us how the degenerate states are transformed under infinitesimal gauge transformations. In sect. 6 we will show how to construct out of $\tilde{K}$ the propagation kernel for a system in which the indices (s) represent genuine degrees of freedom: those of non-abelian sources.

The construction leading to eq. (3.6), contributes evidence that, due to the compactness of the gauge group, only external sources with strength 1 in units of the coupling constant can possibly be coupled to the gauge field.

In the case of QED the gauge group is abelian and all the irreducible unitary representations are unidimensional. This together with the non-compactness of the gauge group allows one to have external sources of arbitrary strength. A state describing a continuous charge distribution $\rho(\boldsymbol{x})$, is characterized by the transformation law

$$\psi^{(\rho)}(A^{U_\lambda}) = \exp\left[i\int \rho(\boldsymbol{x})\,\lambda(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right]\psi^{(\rho)}(A)\,, \tag{3.8}$$

$$A_i^{U_\lambda}(\boldsymbol{x}) = A_i(\boldsymbol{x}) - \partial_i\lambda(\boldsymbol{x})\,, \tag{3.9}$$

for all gauge functions, such that

$$\lambda(\boldsymbol{x}) \underset{|\boldsymbol{x}|\to\infty}{\to} 0\,. \tag{3.10}$$

We conclude with a remark on the parametrization of the wave functional of a state containing a certain number of external sources in the representation $R^{(j)}$. Such a functional can always be written in the form

$$\psi_s^{(j)}(A) = F_{ss'}^{(j)}(A)\,\phi_{s'}(A)\,, \tag{3.11}$$

where $F_{ss'}^{(j)}(A)$ are arbitrary functionals of A satisfying

$$F_{ss'}^{(j)}(A^{U_w}) = R_{ss''}^{(j)+}(U_w)F_{s''s'}^{(j)}(A), \qquad U_w \in \mathcal{G}_0\,, \tag{3.12}$$

and the ϕ_s's are $d(j)$ gauge-invariant functionals, $d(j)$ being the dimensionality of $R^{(j)}$.

This means that, for instance, to completely describe a quark-antiquark-like pair of sources, one has to deal with N^2 gauge-invariant functionals. A state of the form

$$\psi_{rs}^{(q\bar{q})}(A) = \left[\mathrm{P}\exp(i\int_{\mathcal{P}} A^i(x)\,\mathrm{d}x_i\right]_{rs}\phi(A)\,, \tag{3.13}$$

where P denotes path-ordering along the path $\mathcal{P}$ and $\phi(A)$ is gauge invariant, though having the correct gauge transformation properties, is certainly not the most general state functional for the system.

A systematic way of constructing F is the following. Choose an arbitrary non-gauge-invariant functional $f(A) = \lambda^a f^a(A)$ (for instance $\partial^i A_i$). Assign to any field configuration A a gauge transformation with parameters $h_A^a(\boldsymbol{x})$ solutions of the equation *

$$f(A^{U_h^+}) = 0 . \tag{3.14}$$

Then

$$F_{ss'}^{(j)}(A) = R_{ss'}^{(j)+}(U_{h_A}) . \tag{3.15}$$

From (3.14) and (3.15) it is clear that F depends on the functional $f(A)$ one is choosing. The meaning of the above construction is the following. Inserting eqs. (3.11) and (3.15) in the eigenvalue equation for the hamiltonian (2.12), one obtains the eigenvalue equation for the hamiltonian of the Yang-Mills field coupled to the external sources described by $R^{(j)}$ in the gauge $f(A) = 0$ [12]. This is a reformulation of the procedure proposed in ref. [7].

4. θ-dependence of state functionals

Using the invariance of $\widetilde{K}(A_2, A_1; T)$ under gauge transformations with non-trivial winding number, one can generalize eq. (3.2) [2], showing that

$$\psi_{n,s}^{(j)}(A^{U_h^{(l)}}) = e^{il\theta_n} R_{ss'}^{(j)+}(U_h^{(l)})\, \psi_{n,s'}^{(j)}(A) , \tag{4.1}$$

where $U_h^{(l)}$ is a winding number-l gauge transformation.

In eq. (4.1) a different angle θ appears for each (degenerate) level of the theory. However states belonging to the "physical" sector [states which satisfy Gauss' law (2.16)] that are localized, i.e., that differ from the vacuum only inside a finite volume, all have the same angle.

In fact all localized "physical" states, $\psi_{\text{ph}}^{\text{v}}$, can be obtained by applying to the vacuum functional, ψ_Ω, a functional $C_{\text{v}}(A, -i\,\delta/\delta A)$ depending only on the values assumed by A and $-i\,\delta/\delta A$ inside a finite volume v and invariant under zero winding number gauge transformations:

$$\psi_{\text{ph}}^{\text{v}}(A) = C_{\text{v}}\left(A, -i\frac{\delta}{\delta A}\right) \psi_\Omega(A) , \tag{4.2}$$

with

$$C_{\text{v}}\left(A, -i\frac{\delta}{\delta A}\right) = C_{\text{v}}\left(A^U, -i\frac{\delta}{\delta A^U}\right) \equiv C_{\text{v}}^U . \tag{4.3}$$

* Suitable boundary conditions are to be associated to eq. (3.14) in order to have (apart from possible Gribov-like ambiguities [11]) a unique solution.

Under a $U_h^{(1)}$ gauge transformation we have

$$\psi_{\mathrm{ph}}^{\mathrm{v}}(A^{U_h^{(1)}}) = C_{\mathrm{v}}^{U_h^{(1)}}\, \mathrm{e}^{i\theta_\Omega}\psi_\Omega(A)\,, \tag{4.4}$$

where θ_Ω is the angle of the vacuum.

The gauge invariance of C_{v} entails

$$C_{\mathrm{v}}^{U_h^{(1)}[U_g^{(1)}]^{-1}} = C_{\mathrm{v}}\,, \tag{4.5}$$

and hence that $C_{\mathrm{v}}^{U_h^{(1)}}$ is independent on the particular representative of winding number-1 gauge transformation we have chosen. Taking, in (4.4), a representative which is equal to the identity in the region v, we then obtain

$$\psi_{\mathrm{ph}}^{\mathrm{v}}(A^{U_h^{(1)}}) = \mathrm{e}^{i\theta_\Omega}\psi_{\mathrm{ph}}^{\mathrm{v}}(A)\,. \tag{4.6}$$

It has to be noted that, if the vacuum has zero global color charge, the same is ture for the states (4.2). This follows from the fact that C_{v}, being localized in the volume v, is not able to distinguish between a constant gauge transformation (global color rotation) and one which has the same constant value in v and goes to the identity outside.

The physical meaning of this result is that, since a state carrying a non-zero color charge polarises the vacuum up to infinity (through Gauss' law), it cannot differ from ψ_Ω only in a limited region [13].

We have thus shown that all physical states with zero global color charge have the same θ-angle. If color is confined or bleached (as in presence of a complete Higgs mechanism), these are all the physical states of the theory. But also if color is not confined the θ-angle in any charged sector has to be the same, since all charge sectors can be obtained from the neutral one by using cluster properties.

For the "non-physical" states (states with external sources) we have not been able to prove correspondingly simple results. Obviously θ is the same for all the states related by the symmetry operations of the hamiltonian.

Eq. (4.1) shows that the phase factor $\exp(il\theta_n)$ is a representation of the homotopy group $\pi_{D-1}(\mathrm{G})$, where G is the global gauge group and D is the number of space-time dimensions. This fact may put restrictions on the possible values of θ. For pure Yang-Mills in 3 + 1 dimensions, where $\pi_3(\mathrm{SU}(N)/\mathrm{Z}_N) = \mathrm{Z}$, θ is unrestricted; but, for example, in QCD in 1 + 1 dimensions coupled with fields belonging to a representation of $\mathrm{SU}(N)/\mathrm{Z}_N$, where the relevant homotopy group is $\pi_1(\mathrm{SU}(N)/\mathrm{Z}_N) = \mathrm{Z}_N$, θ can only assume the N discrete values 0, $2\pi/N$, ..., $2\pi(N-1)/N$ [14]. More generally, if Z_N has a non-trivial subgroup, $\mathrm{Z}_{N'}$ $(1 < N' < N)$, and the gluons are coupled to fields which belong to a representation of $\mathrm{SU}(N)/\mathrm{Z}_{N'}$, then only the N' values of θ, given by $\mathrm{e}^{iN'\theta} = 1$, are allowed.

5. A remark on color charges

The global color charges (2.20) are non-gauge-invariant and, therefore, apparently non-physical operators. We want to show that they can nevertheless be used for the classification of the states. Let us in fact consider the identity

$$\mathcal{U}_{U_h}\mathcal{U}_{V_h} = \mathcal{U}_{V_r}\mathcal{U}_{V_r^+ U_h V_r} \tag{5.1}$$

and notice that $V_r^+ U_h V_r$ goes to I at spatial infinity. Applying (5.1) to a state $\psi_s^{(j)}(A)$ we get [see eq. (3.2)]

$$\mathcal{U}_{U_h}\mathcal{U}_{V_r}\psi_s^{(j)}(A) = R_{ss'}^{(j)+}(V_r^+ U_h V_r)\,\mathcal{U}_{V_r}\psi_{s'}^{(j)}(A)\,, \tag{5.2}$$

if

$$\mathcal{U}_{U_h}\psi_s^{(j)}(A) = R_{ss'}^{(j)+}(U_h)\psi_{s'}^{(j)}(A)\,. \tag{5.3}$$

Eqs. (5.2) and (5.3) show that the states $\psi^{(j)}(A)$ and $\mathcal{U}_{V_r}\psi^{(j)}(A)$ belong to equivalent representations of $\mathcal{G}_0$. However $\mathcal{U}_{V_r}\psi^{(j)}(A)$ describes a situation in which each spin M^{jk} appearing in $R^{(j)}$ is rotated by $L^{jk}(V_r^+) = \exp(-iM_a^{jk}r^a)$. In particular if $\psi(A)$ is a "physical" state (R = I), $\mathcal{U}_{V_r}\psi(A)$ is also a "physical" state.

One can show explicitly that the state $Q^a\psi_{\rm ph}$ is a "physical" state. In fact under a gauge transformation $U_h \in \mathcal{G}_0$, one has

$$\begin{aligned}\mathcal{U}_{U_h}Q^a\psi_{\rm ph} &= \mathcal{U}_{U_h}Q^a\mathcal{U}_{U_h}^+\psi_{\rm ph}\\ &= \Big(\int_{S_\infty} \dot{A}_i^a\,{\rm d}s^i - \int \sigma_{ab}(h)D_i^{bc}\dot{A}^{ic}(\boldsymbol{x})\,{\rm d}\boldsymbol{x}\Big)\psi_{\rm ph}\,,\end{aligned} \tag{5.4}$$

where S_∞ is the surface at spatial infinity and $\sigma_{ab}(h)$ is defined in eq. (C.1? of appendix C. The first term in eq. (5.4) represents the total "electrix" flux through the surface at infinity. On a "physical" state it is simply equal to the total charge of the state, while the second term vanishes.

6. Propagation kernel in presence of external sources

Unlike the abelian case, a non-abelian external source can never be static, because its color spin has to be coupled to the gauge field.

We now want to study the quantum mechanics of a given system of external sources in the presence of the gauge field by constructing its Feynman propagation kernel. We will show in appendix B that our construction corresponds to the effective lagrangian

$$\mathcal{L} = -\frac{1}{2g^2}\,{\rm Tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_j m_a^j(t)A_0^a(\boldsymbol{x}_j, t)\,, \tag{6.1}$$

where $m_a^{(j)}(t)$ evolves according to

$$D_0^{ab} m_b^j(t) \equiv \dot{m}_a^j(t) + f^{acb} A_0^c(\boldsymbol{x}_j, t) m_b^j(t) = 0 \,. \tag{6.2}$$

Eq. (6.2) simply expresses the conservation of the color current.

In order to have consistency between eq. (6.2) and the quantum evolution equation of m_a^j, the following commutation relations must hold

$$[m_a^j(t), m_b^j(t)] = i f^{abc} m_c^j(t) \,. \tag{6.3}$$

Eqs. (6.3) imply that $m_a^j(t)$ is unitarily equivalent to M_a^j [eq. (3.5)].

Let us consider first for simplicity the case of only one point-like source located at $\boldsymbol{x}_j$, belonging to the representation (j) of SU(N).

We define such a system by the propagation kernel

$$K_{s_2 s_1}^j(A_2, A_1; T_2 - T_1) = \sum_n \mathrm{e}^{-iE_n(T_2 - T_1)} \psi_{n,s_2}^j(A_2) [\psi_{n,s_1}^j(A_1)]^* \,, \tag{6.4}$$

where the sum is to be extended over all the states appearing in eq. (2.13) which transform as

$$\psi_{n,s}^j(A^{U_w}) = L^{j+}[U_w(\boldsymbol{x}_j)]_{sr} \psi_{n,r}^j(A) , \qquad U_w \in \mathcal{G}_0 \,. \tag{6.5}$$

The kernel $K_{s_2 s_1}^j(A_2, A_1; T_2 - T_1)$ describes a quantum system whose physical degrees of freedom are the Yang-Mills fields A and the set of indices s. It represents the amplitude to find at time T_2 the gauge field in the configuration A_2 and the color spin of the source with component s_2, if at time T_1 the field configuration was A_1 and the color spin component of the source s_1. Correspondingly the wave functional describing a state of such system is of the form $\psi_s^j(A)$ and represents the amplitude for finding the Yang-Mills field in the configuration A and the source in the configuration s. Eq. (6.4) implies that the energy eigenfunctionals of this system are not degenerate. On them the Gauss operator $iD_k^{ab}\, \delta/\delta A_k^b(\boldsymbol{x})$ is identically equal to the matrix M_a^j [see eq. (6.5)] which now represents a physical operator acting on physical degrees of freedom.

Apparently, setting $A_0 = 0$ in eq. (6.2) seems to imply that m_a^j does not evolve in time in contradiction with eq. (6.4) where the color spin component of the source is not conserved. The point is that one cannot set $A_0 = 0$ in the equations of motion with given arbitrary initial and final field configurations, as required in the quantum case, since, if $A_0 = 0$, Gauss' law becomes a constraint on the values of $\dot{A}$. In the classical theory one can set $A_0 = 0$, provided the initial values of A and $\dot{A}$ are given in such a way as to satisfy Gauss' law at the initial time. Gauss' law will then be true at any other time.

The relation between K_{s_2, s_1}^j and $\tilde{K}$ [eq. (2.9)] is very simple and follows from the orthogonality relation

$$\int_{\mathcal{G}_0} \mathcal{D}w(\boldsymbol{x}) L^i[U_w(\boldsymbol{x}_i)]_{rs} L^j[U_w(\boldsymbol{x}_j)]_{pq}^* = \delta_{ij} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \delta_{rp} \delta_{sq} \,, \tag{6.6}$$

which generalizes the usual orthogonality relation of group representations.

From eqs. (6.4)–(6.6) we have

$$K^j_{s_2 s_1}(A_2, A_1; T_2 - T_1) = \int_{\mathcal{G}_0} \mathcal{D}w_1(\boldsymbol{x})\, \mathcal{D}w_2(\boldsymbol{x}) \tilde{K}(A_2^{U_{w_2}}, A_1^{U_{w_1}}; T_2 - T_1)$$

$$\times L^j[U_{w_2}(\boldsymbol{x}_j)]_{s_2 r} L^j[U_{w_1}(\boldsymbol{x}_j)]^*_{s_1 r}$$

$$= \int_{\mathcal{G}_0} \mathcal{D}w(\boldsymbol{x}) \tilde{K}(A_2^{U_w}, A_1; T_2 - T_1) L^j[U_w(\boldsymbol{x}_j)]_{s_2 s_1}\,. \tag{6.7}$$

In the last equality we have used the gauge invariance of $\tilde{K}$ [eq. (3.1)] in order to eliminate one of the two gauge integrations.

Eq. (6.7) is immediately generalized to any number of external sources by using eq. (3.6). Furthermore the previous discussion has been carried out for simplicity in the pure Yang-Mills case. If other fields ϕ (gauge-invariantly coupled) are present, formula (6.7) becomes

$$K^{(j)}_{s_2 s_1}(A_2, \phi_2; A_1, \phi_1; T_2 - T_1)$$

$$= \int_{\mathcal{G}_0} \mathcal{D}w(x) \tilde{K}(A_2^{U_w}, \phi_2^{U_w}; A_1, \phi_1; T_2 - T_1) R^{(j)}_{s_2 s_1}(U_w)\,, \tag{6.8}$$

where $\phi_2^{U_w}$ denotes the gauge transformed ϕ_2 configuration.

Eqs. (6.7) and (6.8) are formal but can be used to compute perturbatively the energy of a given external source configuration. This will be explicitly shown in a forthcoming publication.

7. Two examples: QED and abelian Higgs model

In this section we want to clarify with two examples the foregoing formal considerations, by discussing free QED and the abelian Higgs model.

Let us start with free QED. The propagation kernel $\tilde{K}$ [eq. (2.9)] can be explicitly computed because the action is a quadratic functional of the gauge fields and in the euclidean region is given by

$$\tilde{K}(A_2, A_1; T) = f(T) \exp[W_{\mathrm{G}}(A_2, A_1; T) + W_{\mathrm{NG}}(A_2, A_1; T)]\,, \tag{7.1}$$

$$W_{\mathrm{G}}(A_2, A_1; T) = -\tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, [A_1^l(\boldsymbol{x})\, \Gamma^{(\alpha)}_{lm}(\boldsymbol{x}-\boldsymbol{y}; T) A_1^m(\boldsymbol{y}) + (1 \to 2)$$

$$+ A_1^l(\boldsymbol{x})\, \Gamma^{(\beta)}_{lm}(\boldsymbol{x}-\boldsymbol{y}; T) A_2^m(\boldsymbol{y}) + (1 \to 2)]\,, \tag{7.2a}$$

$$W_{\mathrm{NG}}(A_2, A_1; T) = -\tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, (A_2^l(\boldsymbol{x}) - A_1^l(\boldsymbol{x}))$$

$$\times\, \Gamma^{(\gamma)}_{lm}(\boldsymbol{x}-\boldsymbol{y}; T)(A_2^m(\boldsymbol{y}) - A_1^m(\boldsymbol{y}))\,, \tag{7.2b}$$

$$\Gamma^{(\alpha)}_{lm}(z;T) = -\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{p}\boldsymbol{z}} \sqrt{\boldsymbol{p}^2}\, \coth(\sqrt{\boldsymbol{p}^2}\,T) \left(g_{lm} + \frac{p_l p_m}{\boldsymbol{p}^2}\right), \tag{7.3a}$$

$$\Gamma^{(\beta)}_{lm}(z;T) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{p}\boldsymbol{z}} \sqrt{\boldsymbol{p}^2}\, \frac{1}{\sinh(\sqrt{\boldsymbol{p}^2}\,T)} \left(g_{lm} + \frac{p_l p_m}{\boldsymbol{p}^2}\right), \tag{7.3b}$$

$$\Gamma^{(\gamma)}_{lm}(z;T) = \frac{1}{T}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\, \mathrm{e}^{i\boldsymbol{p}\boldsymbol{z}} \frac{p_l p_m}{\boldsymbol{p}^2}. \tag{7.3c}$$

$f(T)$ is a function of time only, which in the limit $T \to +\infty$ behaves as

$$f(T) \underset{T\to+\infty}{\to} \mathrm{e}^{-E_\Omega T}, \tag{7.4}$$

where E_Ω is the zero-point energy of the vacuum.

In eq. (7.1) we have explicitly separated a gauge-invariant part W_{G} and a non-gauge-invariant one W_{NG}.

Eqs. (7.1)–(7.3) are obtained by first solving the classical equations of motion for $\boldsymbol{A}_{\mathrm{cl}}$ in the $A_0 = 0$ gauge with the boundary conditions

$$\begin{aligned} \boldsymbol{A}_{\mathrm{cl}}(\boldsymbol{x}, T_1) &= \boldsymbol{A}_1(\boldsymbol{x}), \\ \boldsymbol{A}_{\mathrm{cl}}(\boldsymbol{x}, T_2) &= \boldsymbol{A}_2(\boldsymbol{x}), \end{aligned} \qquad T = T_2 - T_1, \tag{7.5}$$

and then making the shift $\boldsymbol{A} \to \boldsymbol{A}_{\mathrm{cl}} + \hat{\boldsymbol{A}}$. The integration over the quantum fluctuation variables $\hat{\boldsymbol{A}}$ is performed with homogeneous boundary conditions and gives a function only depending on T which we have indicated by $f(T)$ in eq. (7.1).

In the limit $T \to +\infty$ only the first two terms survive in eq. (7.2a) and one gets [see eq. (2.13) with $iT \to T$]

$$\tilde{K}(A_2, A_1; T) \underset{T\to+\infty}{\to} \mathrm{e}^{-E_\Omega T} \psi_\Omega(A_2)\, \psi^*_\Omega(A_1), \tag{7.6}$$

$$\psi_\Omega(A) = \exp\left[\tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\, A^l(\boldsymbol{x}) \sqrt{\partial_i \partial^i} \left(g_{lm} - \partial_l \frac{1}{\partial^i \partial_i} \partial_m\right) A^m(\boldsymbol{x})\right]. \tag{7.7}$$

The state functional (7.7) is the lowest energy eigenfunctional of the system and satisfies Gauss' law, being invariant under time-independent gauge transformations $A_i \to A_i - \partial_i \lambda$ with $\lambda(\boldsymbol{x}) \xrightarrow[|\boldsymbol{x}|\to\infty]{} 0$. Ψ_Ω then represents the vacuum wave functional of the system [15]. In the following we will take $E_\Omega = 0$.

The last two terms in eq. (7.2a) are also gauge invariant and represent the contribution to the kernel due to sourceless excited states (many-photon states).

The term W_{NG} [eq. (7.2b)] is not gauge invariant and describes the whole set of states with every possible external charge distribution.

We can now study QED in presence of a given external charge distribution, $\rho(\boldsymbol{x})$, using eq. (6.7) specialized to the abelian case. We have

$$K^{(\rho)}(A_2, A_1; T) = \int \delta\lambda(\boldsymbol{x})\, \tilde{K}(A_2^{U_\lambda}, A_1; T) \exp[-i \int \rho(\boldsymbol{x})\, \lambda(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}]$$

$$= \exp[W_G(A_2, A_1; T)] \int \delta\lambda(x) \exp\left[-\frac{1}{2T}\int \mathrm{d}\boldsymbol{x}\, (\partial_l A_2^l - \partial_l A_1^l - \partial^l\partial_l\lambda)\right.$$

$$\left.\times \frac{1}{\partial^l\partial_l} (\partial_l A_2^l - \partial_l A_1^l - \partial_l\partial^l\lambda)\right] \exp[-i\int \rho(x)\lambda(x)\,\mathrm{d}x]\,. \tag{7.8}$$

We perform the gaussian functional integration over $\lambda(\boldsymbol{x})$, getting

$$K^{(\rho)}(A_2, A_1; T) = \exp\left[W_G - i\int \frac{\partial_l(A_2^l(\boldsymbol{x}) - A_1^l(\boldsymbol{x}))\,\rho(\boldsymbol{x}')}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\right.$$

$$\left. - T\int \frac{\rho(\boldsymbol{x})\,\rho(\boldsymbol{x}')}{8\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\right]. \tag{7.9}$$

As before, the lowest energy state in this sector is selected by taking in (7.9) the limit $T \to +\infty$. Its wave functional and energy can be immediately read off from eq. (7.9) and are given by

$$\psi^{(\rho)}(A) = \exp\left(-i\int \frac{\partial_l A^l(\boldsymbol{x})\,\rho(\boldsymbol{x}')}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\right)\psi_\Omega(A)\,, \tag{7.10}$$

$$E^{(\rho)} = \int \frac{\rho(\boldsymbol{x})\,\rho(\boldsymbol{x}')}{8\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\,. \tag{7.11}$$

As expected, $\psi^{(\rho)}$ satisfies Gauss' law with a charge distribution $\rho(\boldsymbol{x})$ and $E^{(\rho)}$ is the Coulomb self-energy.

A similar technique can be used to discuss in the tree approximation the presence of sources in the more interesting situation of the abelian Higgs model described by the lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - V(\phi^*\phi)\,, \tag{7.12}$$

$$D_\mu = \partial_\mu + igA_\mu\,, \tag{7.13}$$

$$V(\phi^*\phi) = \tfrac{1}{4}\beta(\phi^*\phi - \eta^2)^2\,, \tag{7.14}$$

As usual we write the complex field ϕ in the polar representation

$$\phi(\boldsymbol{x}) = (\eta + \sqrt{\tfrac{1}{2}}\,\sigma(\boldsymbol{x}))\,\mathrm{e}^{ig\chi(\boldsymbol{x})}\,. \tag{7.15}$$

Keeping only the bilinear terms in the fields A, σ and χ, we obtain in the $A_0 = 0$ gauge the hamiltonian

$$\mathcal{H} = \int \mathrm{d}\boldsymbol{x}\left[\frac{1}{2}\,\frac{\delta^2}{\delta A^i(\boldsymbol{x})\,\delta A_i(\boldsymbol{x})} - \frac{1}{2}\,\frac{\delta^2}{\delta\sigma(\boldsymbol{x})} - \frac{1}{2\mu^2}\,\frac{\delta^2}{\delta\chi(\boldsymbol{x})^2} + \tfrac{1}{4}F_{ij}F^{ij}\right.$$

$$\left. - \tfrac{1}{2}\mu^2 A^i A_i - \tfrac{1}{2}\partial^i\sigma\,\partial_i\sigma + \tfrac{1}{2}m^2\sigma^2 - \tfrac{1}{2}\mu^2\partial^i\chi\partial_i\chi - \mu^2 A^i\partial_i\chi\right], \tag{7.16}$$

where $\mu = \sqrt{2}\,g\eta$ is the mass the photon acquires with the Higgs mechanism and

$m = \eta\sqrt{\beta}$ is the mass of the σ field. The physical states satisfy Gauss' law:

$$-i\left(\partial^j \frac{\delta}{\delta A^j(\boldsymbol{x})} + \frac{\delta}{\delta\chi(\boldsymbol{x})}\right)\psi_{\mathrm{ph}} = 0\,. \tag{7.17}$$

The vacuum functional turns out to be

$$\psi_\Omega(A, \sigma, \chi) = \exp[-\tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,(A^l(\boldsymbol{x}) + \partial^l\chi(\boldsymbol{x}))\Gamma_{lm}(x-y)(A^m(\boldsymbol{y}) + \partial^m\chi(\boldsymbol{y})) - \tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\sigma(\boldsymbol{x})\,\Gamma(x-y)\,\sigma(\boldsymbol{y})]\,, \tag{7.18}$$

$$\Gamma_{lm}(\boldsymbol{z}) = -\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{i\boldsymbol{p}\boldsymbol{z}}\left[\left(g_{lm} + \frac{p_l p_m}{\boldsymbol{p}^2}\right)\sqrt{\boldsymbol{p}^2 + \mu^2} - \frac{p_l p_m}{\boldsymbol{p}^2}\,\frac{\mu^2}{\sqrt{\boldsymbol{p}^2+\mu^2}}\right], \tag{7.19}$$

$$\Gamma(\boldsymbol{z}) = +\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{i\boldsymbol{p}\boldsymbol{z}}\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{7.20}$$

ψ_Ω is gauge invariant and satisfies Gauss' law (7.17). As expected ψ_Ω is the product of the vacuum functional of a free vector field of mass μ times the vacuum functional of a free scalar field of mass m. In order to find the expression of a state with a charge distribution $\rho(\boldsymbol{x})$, one could use the same method which led us to eqs. (7.10) and (7.11) in QED or alternatively directly solve the pair of equations

$$\mathcal{H}\psi^{(\rho)} = E^{(\rho)}\psi^{(\rho)}, \tag{7.21}$$

$$-i\left(\partial^j \frac{\delta}{\delta A^j(\boldsymbol{x})} + \frac{\delta}{\delta\chi(\boldsymbol{x})}\right)\psi^{(\rho)} = \rho(\boldsymbol{x})\,\psi^{(\rho)}. \tag{7.22}$$

The result for the state $\psi^{(\rho)}$ is

$$\psi^{(\rho)}(A, \sigma, \chi) = \exp[-i\int \mathrm{d}\boldsymbol{x}\,F(\boldsymbol{x})(\partial^k A_k(\boldsymbol{x}) - \mu^2\chi(\boldsymbol{x}))]\,\psi_\Omega(A, \sigma, \chi), \tag{7.23}$$

where

$$F(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{i\boldsymbol{p}\boldsymbol{x}}\frac{\rho(\boldsymbol{p})}{\mu^2 + \boldsymbol{p}^2}\,, \tag{7.24}$$

$$\rho(\boldsymbol{p}) = \int \mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-i\boldsymbol{p}\boldsymbol{x}}\rho(\boldsymbol{x})\,. \tag{7.25}$$

For the energy one finds

$$E^{(\rho)} = \int \frac{\mathrm{e}^{-\mu|\boldsymbol{x}-\boldsymbol{x}'|}}{8\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\rho(\boldsymbol{x})\,\rho(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'\,. \tag{7.26}$$

The total charge of $\psi^{(\rho)}$ carried by the quantized fields can be found by applying to the state (7.23) the total charge operator

$$Q = \int J_0(x)\, \mathrm{d}\boldsymbol{x} = i \int \frac{\delta}{\delta\chi(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x} \,. \tag{7.27}$$

It is easily seen that its value is $-\int \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$. This means that the external charge $\int \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ is completely screened (bleaching).

8. The Coulomb gauge

In this section we want to briefly discuss the form of the Feynman propagation kernel in the Coulomb gauge ($\partial^i A_i = 0$) and its relation to the one in the temporal gauge.

Because of the (formal) gauge invariance of the kernel defined in eq. (1.1), we can always take the boundary values of the gauge field satisfying the chosen gauge condition ($\partial_i A_1^i = \partial_i A_2^i = 0$).

The Coulomb gauge is formulated by introducing in (1.1) the identity

$$1 = \Delta(A) \int_{t \neq T_1, T_2} \mathcal{D} w(\boldsymbol{x}, t)\, \delta(\partial^i A_i^{U_w^+}) \,, \tag{8.1}$$

where the group integration is extended over all gauge transformations which go to the identity at spatial infinity. This restriction comes from the requirement of having a unique solution for the equation [11]

$$\partial^i (U_w \partial_i U_w^+) = 0 \,. \tag{8.2}$$

This is the analog of imposing, in the abelian case, boundary conditions that ensure uniqueness for the solution of the Laplace equation.

As shown by Gribov [11], however, the condition $U_w \xrightarrow[|x| \to \infty]{} \mathrm{I}$ is not enough to have a unique solution for the equation $\partial^i A_i^{U_w^+} = 0$ in the non-abelian case.

In eq. (8.1) it is understood that $\Delta(A)$ is computed taking into account all these possible solutions. In this way $\Delta(A)$ is gauge invariant [16].

Inserting (8.1) into (1.1) and extracting the infinite gauge volume, we get

$$K_\mathrm{c}(A_2, A_1; T) = \int_{A_1, A_2} \delta A\, \delta A_0\, \Delta(A) \delta(\partial^i A_i)\, \mathrm{e}^{iS} \,. \tag{8.3}$$

Performing the A_0 gaussian integration, one would get the usual non-local Coulomb interaction term. Here we want to show that, doing the A_0 integration in a different way, K_c is actually identical to the kernel $\bar{K}$ of eq. (2.8).

To this end we make in (8.3) the change of variables

$$\begin{aligned} A_k &\to A_k' = V^+ A_k V + i V^+ \partial_k V \,, \\ A_0 &\to A_0' = A_0 \,, \end{aligned} \tag{8.4}$$

with V such that

$$V^{+}A_0V+iV^{+}\dot{V}=0\,. \tag{8.5}$$

The general solution of (8.5) is

$$V(\boldsymbol{x},t)=[\mathrm{T}\exp(i\int_{T_1}^{t}A_0(\boldsymbol{x},\tau)\,\mathrm{d}\tau]\;V(\boldsymbol{x},T_1)\,. \tag{8.6}$$

With (8.4), A_0 disappears from the action and K_c becomes

$$K_c(A_2,A_1;T)=\int\delta A_0\int_{\substack{A_1^{V(\boldsymbol{x},T_1)}\\A_2^{V(\boldsymbol{x},T_2)}}}\delta A\;\Delta(A)\,\delta(\partial^i A_i^{V^+})\;\mathrm{e}^{iS(A_0=0)}\,. \tag{8.7}$$

We now perform the A_0 integration with the aid of eq. (8.1) itself. This is accomplished by means of the further change of variables $A_0^a(\boldsymbol{x},t)\to w^b(\boldsymbol{x},t)$, implicitly defined by

$$[\mathrm{T}\exp(i\int_{T_1}^{t}A_0(\boldsymbol{x},\tau)\,\mathrm{d}\tau)]\;V(\boldsymbol{x},T_1)=\mathrm{e}^{i\lambda^a w^a(\boldsymbol{x},t)}\equiv U_{\mathrm{w}}(\boldsymbol{x},t)\,. \tag{8.8}$$

The Jacobian of this transformation, J, is computed in appendix C and it turns out to be just the invariant measure over the gauge group, so that

$$\delta A_0=J\delta w=\mu(w)\,\delta w=\mathcal{D}w\,. \tag{8.9}$$

In this way, using (8.1), all gauge integrations $\mathcal{D}\,w(\boldsymbol{x},t)$, except those at times T_1 and T_2, can be performed and give 1. At the boundaries we are left with the integrations over $U_w(\boldsymbol{x},T_1)\equiv U_{w_1}(\boldsymbol{x})$ and $U_w(\boldsymbol{x},T_2)\equiv U_{w_2}(\boldsymbol{x})$ and we get back eq. (2.8). The argument can readily be generalized to the case in which matter fields are gauge invariantly coupled to the Yang-Mills field.

Strictly speaking the equivalence that we have established between the Coulomb and the $A_0=0$ gauge is formal in the sense that we have not taken into account complications arising in the Coulomb gauge from the existence of non-trivial winding number sectors in the $A_0=0$ gauge [17].

The authors would like to acknowledge the hospitality of the Service de Physique Théorique de l'Ecole Normale Supérieure in Paris and C. Itzykson for reading the manuscript.

Appendix A

In this appendix we want to prove the following theorem: Any irreducible finite-dimensional unitary representation of the local gauge group, $\mathcal{G}_0$,

$$U_h(\boldsymbol{x})\to R(U_h), \tag{A.1}$$

$$U_h(x)\,U_{h'}(x) \to R(U_h)R(U_{h'}) = R(U_h U_{h'}) , \tag{A.2}$$

is given by a functional R of $U_h(\boldsymbol{x})$ which only depends on the values assumed by U_h in a finite number of points.

Since every $U_h(\boldsymbol{x}) \in \mathcal{G}_0$ goes to I at spatial infinity we can compactify the 3-dimensional space, which then becomes a 3-dimensional sphere, S_3. The group under study is therefore the group of all differentiable gauge transformations $U_h(\boldsymbol{x})$, $\boldsymbol{x} \in S_3$, that are equal to I, say, at the north pole.

Let us take on S_3 a system of curvilinear coordinates (ξ, η, ζ) varying between 0 and 1, such that $\xi = 0$ corresponds to the north pole.

Let P_t be the region of S_3 defined by $\xi \leqslant t$ and $\bar{P}_t$ the region $\xi \geqslant t$. Denote by $\mathcal{G}_0(t)$ ($\bar{\mathcal{G}}_0(t)$) the subgroup of $\mathcal{G}_0$ of all differentiable gauge transformations equal to I in P_t ($\bar{P}_t$) and by $U^{[t_1 t_2]}$ ($t_1 < t_2$) the gauge transformations of $\mathcal{G}_0$ which are I outside the coordinate surfaces $\xi = t_1$ and $\xi = t_2$. With these notations

$$P_0 = \text{north pole}, \qquad P_1 = S_3 ,$$

$$\mathcal{G}_0(0) = \mathcal{G}_0, \qquad \mathcal{G}_0(1) = \mathrm{I} ,$$

$$U^{[0,t]} \in \bar{\mathcal{G}}_0(t) , \qquad U^{[t,1]} \in \mathcal{G}_0(t) , \tag{A.3}$$

$$[R(U^{[0,t]}), R(U^{[t,1]})] = 0 . \tag{A.4}$$

We start by showing that R depends on the values assumed by U in a finite number of coordinate surfaces $\xi = \text{const}$.

Let us divide S_3 in two parts by taking a t such that $R(U^{[0,t]}) \neq 1$. If such a t does not exist, R can at most depend on the value assumed by U at the south pole ($\xi = 1$) and the theorem is proven. If this is not the case, we want to first prove that R can depend on the values assumed by U in only a finite number of coordinate surfaces with $\xi \geqslant t$ (besides of course the values assumed by U in P_t). Eq. (A.4) implies that both $R(U^{[0,t]})$ and $R(U^{[t,1]})$ are reducible. This means that the representation space Σ of R can be decomposed into a direct sum of two orthogonal subspaces

$$\Sigma = \Sigma_1 \oplus \Sigma_2 \tag{A.5}$$

such that $R(U^{[0,t]})$ acts on Σ_1 in a non-trivial way. Because of (A.4) $R(U^{[t,1]})$ acts trivially on Σ_1, and can act non-trivially only on Σ_2.

Now let us take a $t' > t$; if $R(U^{[t,t']}) = 1$, R does not depend on the values assumed by U in the strip between t and t'. If $R(U^{[t,t']}) \neq 1$, the equation

$$[R(U^{[t,t']}), R(U^{[0,t]})] = 0 \tag{A.6}$$

implies that $R(U^{[t,t']})$ must act in a non-trivial way on a subspace $\Sigma_2' \subseteq \Sigma_2$.

Furthermore, since

$$[R(U^{[t,t']}), R(U^{[t',1]})] = 0 , \tag{A.7}$$

$R(U^{[t',1]})$ can act non-trivially only on the orthogonal complement of Σ_2' in Σ_2.

This argument shows that when $R(U^{[t,t']})$ is non-trivial, $R(U^{[t',1]})$ acts in a non-trivial way in a proper subspace of the one in which $R(U^{[t,1]})$ acts non-trivially. Varying t', this reduction can happen only a finite number of times (due to the finite dimensionality of the representation) and then $R(U)$ may depend non-trivially on the values assumed by U in only a finite number of coordinate surfaces with $\xi > t$. By a similar argument one can also show that R may depend on the values assumed by U in only a finite number of coordinate surfaces with $\xi < t$.

In a 1-dimensional space (S_1) the theorem would be demonstrated. For S_3 one can repeat a similar argument in a new system of coordinates (ξ', η', ζ') with a different north pole * and then R may depend only on the values assumed by U on the intersections of the ξ, ξ' coordinate surfaces.

Repeating this construction once more we conclude the demonstration of the theorem.

Appendix B

In this appendix we want to prove that the effective lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = -\frac{1}{2g^2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + m_a^j(t)A_0^a(\boldsymbol{x}_j, t) \tag{B.1}$$

leads to the Feynman propagation kernel (6.7). The $m_a^j(t)$'s are quantum operators with equal time commutation relations:

$$[m_a^j(t), m_b^j(t)] = if^{abc}m_c^j(t)\,. \tag{B.2}$$

They represent the color degrees of freedom of a point-like source sitting at $\boldsymbol{x}_j$ belonging to the representation j of SU(N), and satisfy the equations of motion

$$\dot{m}_a^j(t) + f^{acb}A_0^c(\boldsymbol{x}_j, t)\,m_b^j(t) = 0\,. \tag{B.3}$$

Because of (B.2), they are unitarily equivalent to the infinitesimal generator M^j [eq. (3.5)].

Following Feynman [18], in order to compute the kernel of the coupled system, we first have to find the propagation kernel for the spin system described by (B.2) and (B.3), as if A_0 were an external field. This is simply given by

$$K^j(s_2, T_2; s_1, T_1) = [\mathrm{T}\exp i\int_{T_1}^{T_2} M_a^j A_0^a(\boldsymbol{x}_j, t)\,\mathrm{d}t]_{s_2 s_1}\,. \tag{B.4}$$

* Of course the point in which U must be equal to 1 is kept fixed.

The kernel for the total system is then

$$K^j_{s_2 s_1}(A_2, A_1; T_2 - T_1) = \int_{A_1, A_2} \delta A_\mu(\boldsymbol{x}, t) \exp[i \int_{T_1}^{T_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\, \mathcal{L}_0]$$
$$\times K^j(s_2, T_2; s_1, T_1)\,. \tag{B.5}$$

We now go to the $A_0 = 0$ gauge, inserting eq. (2.6) in (B.5). Making the change of variables $A_\mu \to A_\mu^{U_w}$, we can trivially perform the A_0 integration using the δ-function. Dropping the infinite gauge volume, as in sect. 2, we get $(w(\boldsymbol{x}_1, T_{1,2}) \equiv w_{1,2}(\boldsymbol{x}))$

$$K^j_{s_2 s_1}(A_2, A_1; T_2 - T_1) = \int \mathcal{D}w_1(\boldsymbol{x})\, \mathcal{D}w_2(\boldsymbol{x})$$
$$\times \int_{A_1^{U_{w_1}}, A_2^{U_{w_2}}} \delta \boldsymbol{A}(\boldsymbol{x}, t)\, \mathrm{e}^{iS_0(A_0=0)} K^j_{A_0=0}(s_2, T_2; s_1, T_1)\,, \tag{B.6}$$

where

$$K^j_{A_0=0}(s_2, T_2; s_1, T_1)$$
$$= [\mathrm{T} \exp(i)^2 \int_{T_1}^{T_2} M^j_a \operatorname{Tr}(\lambda_a U_w(\boldsymbol{x}, \tau)\, \dot{U}^+_w(\boldsymbol{x}, \tau))\, \mathrm{d}\tau]_{s_2 s_1}\,. \tag{B.7}$$

Using eqs. (C.10) and (C.18) of appendix C, one easily concludes that the equality

$$\operatorname{Tr}\left(\lambda_a \mathrm{e}^{i\lambda_b w^b} \frac{\partial}{\partial t} \mathrm{e}^{-i\lambda_b w^b}\right) = \operatorname{Tr}\left(M^j_a\, \mathrm{e}^{iM^j_b w^b} \frac{\partial}{\partial t} \mathrm{e}^{-iM^j_b w^b}\right) \tag{B.8}$$

holds for any representation j. We can then write

$$M^j_a \operatorname{Tr}(\lambda_a U_w(\boldsymbol{x}_j, \tau)\, \dot{U}^+_w(\boldsymbol{x}_j, \tau)) = \mathrm{e}^{iM^j_b w^b(\boldsymbol{x}_j, \tau)} \frac{\partial}{\partial \tau} \mathrm{e}^{-iM^j_b w^b(\boldsymbol{x}_j, \tau)} \tag{B.9}$$

and hence

$$K^j_{A_0=0}(s_2, T_2; s_1, T_1) = [\mathrm{e}^{iM^j_b w^b(\boldsymbol{x}_j, T_2)}\, \mathrm{e}^{-iM^j_b w^b(\boldsymbol{x}_j, T_1)}]_{s_2 s_1}$$
$$= L^j[U_{w_2}(\boldsymbol{x}_j)\, U^+_{w_1}(\boldsymbol{x}_j)]_{s_2 s_1}\,. \tag{B.10}$$

Inserting (B.10) in (B.6), we finally get eq. (6.7).

Appendix C

Let us consider the functional change of variables $A^a_0(\boldsymbol{x}, t) \to w^b(\boldsymbol{x}, t)$, implicitly defined by

$$\mathrm{T}\exp(i\int_{T_1}^{t} A_0(\boldsymbol{x},\tau)\,\mathrm{d}\tau)\,\mathrm{e}^{i\lambda_a w^a(\boldsymbol{x},T_1)} = \mathrm{e}^{i\lambda_a w^a(\boldsymbol{x},t)} \equiv U_w(\boldsymbol{x},t)\,. \tag{C.1}$$

Since eq. (C.1) is local in $\boldsymbol{x}$, at each point $\boldsymbol{x}$ the Jacobian is

$$J(\boldsymbol{x}) = \left|\det_{\substack{a\ b\\ t\ t'}}\left(\frac{\delta A_0^a(\boldsymbol{x},t)}{\delta w^b(\boldsymbol{x},t')}\right)\right|\ . \tag{C.2}$$

From (C.1) it immediately follows that $\delta w^b(\boldsymbol{x},t')/\delta A_0^a(\boldsymbol{x},t)$ is a triangular matrix in t, t', because it is proportional to $\theta(t'-t)$. Its determinant $(J(\boldsymbol{x})^{-1})$ is then the product of the determinants of the diagonal $(t=t')$ blocks.

The Jacobian of (C.1) becomes

$$J = \prod_{\boldsymbol{x}}\prod_{t}\left|\det_{a\ b}\frac{\delta A_0^a(\boldsymbol{x},t)}{\delta w^b(\boldsymbol{x},t)}\right|\ . \tag{C.3}$$

We want to show that (up to infinite constants) J coincides with the weight $\mu(w)$ which appears in the functional group-invariant measure $\mathcal{D}w = \mu(w)\,\delta w$. $\mu(w)$ is a product of local measures $[\mu(w)]_{\boldsymbol{x},t}$:

$$\mu(w) = \prod_{\boldsymbol{x}}\prod_{t}\ [\mu(w)]_{\boldsymbol{x},t}\,, \tag{C.4}$$

and $[\mu(w)]_{\boldsymbol{x},t}$ is given by

$$[\mu(w)]_{\boldsymbol{x},t} = [\mu(0)]_{\boldsymbol{x},t}\left[\left|\det_{a\ b}\left(\left.\frac{\mathrm{d}u^a(\boldsymbol{x},t)}{\mathrm{d}w^b(\boldsymbol{x},t)}\right|_{v=0}\right)\right|\right]^{-1}\,, \tag{C.5}$$

where the relation between $u(\boldsymbol{x},t)$, $v(\boldsymbol{x},t)$ and $w(\boldsymbol{x},t)$ is implicitly given by

$$U_u(\boldsymbol{x},t) = U_w(\boldsymbol{x},t)\,U_v(\boldsymbol{x},t)\,. \tag{C.6}$$

From standard textbooks on group theory [19], one has

$$[\mu(w)]_{\boldsymbol{x},t} = \det_{a\ b}\left|\left(\frac{\mathrm{e}^{i\gamma}-1}{i\gamma}\right)\right|, \tag{C.7}$$

$$\gamma_{ab}(\boldsymbol{x},t) = iw^c(\boldsymbol{x},t)f^{cab}\,. \tag{C.8}$$

We will show that (up to constants)

$$\left(\frac{\mathrm{e}^{i\gamma}-1}{i\gamma}\right)_{ab} = \frac{\delta A_0^a(\boldsymbol{x},t)}{\delta w^b(\boldsymbol{x},t)}\ , \tag{C.9}$$

from which the equality $J = \mu(w)$ follows.

To this end we recall the formula [18]:

$$\frac{\delta U_w(\boldsymbol{x}, t)}{\delta w^a(\boldsymbol{x}, t')} U_w^+(x, t) = i\delta(t - t') \int_0^1 \mathrm{d}s\, U_{sw}(\boldsymbol{x}, t)\, \lambda^a U_{sw}^+(\boldsymbol{x}, t)$$

$$\equiv i\delta(t - t') \int_0^1 \mathrm{d}s\, \Sigma_a(sw)\,, \tag{C.10}$$

where, for brevity, we have put

$$U_{sw}(\boldsymbol{x}, t) = \exp(isw(\boldsymbol{x}, t)), \qquad w(\boldsymbol{x}, t) = \lambda_a w^a(\boldsymbol{x}, t)\,, \tag{C.11}$$

$$\Sigma_a(sw) = \sigma_{ab}(sw)\, \lambda_b\ , \tag{C.12}$$

$$\sigma_{ab}(sw) = 2\, \mathrm{Tr}(U_{sw}(\boldsymbol{x}, t)\, \lambda_a U_{sw}^+(\boldsymbol{x}, t)\, \lambda_b)\,. \tag{C.13}$$

Note the two properties of $\sigma_{ab}(sw)$:

$$\text{(i)} \quad \sigma_{ab}(sw) = \sigma_{ba}(-sw) = \sigma_{ab}^*(sw)\ ; \tag{C.14}$$

$$\text{(ii)} \quad \sigma_{ab}(sw) = (\mathrm{e}^{is\gamma})_{ab}\ . \tag{C.15}$$

Eq. (C.14) is an immediate consequence of the definition (C.13). Eq. (C.15) follows from the "equations of motion"

$$\frac{\mathrm{d}\Sigma_a(sw)}{\mathrm{d}s} = i[w, \Sigma_a(sw)]\ , \tag{C.16}$$

$$\Sigma_a(0) = \lambda_a\ ,$$

which imply by projecting on λ^b

$$\frac{\mathrm{d}\sigma_{ab}(sw)}{\mathrm{d}s} = i\sigma_{ad}(sw)\, \gamma_{db}\ ,$$

$$\sigma_{ab}(0) = \delta_{ab}\ . \tag{C.17}$$

The solution of eqs. (C.17) is clearly (C.15).

Since (C.15) only depends on the structure constant of SU(N), then

$$\sigma_{ab}(sw) = 2\, \mathrm{Tr}(L^j[U_{sw}] M_a^j L^{j+}[U_{sw}] M_b^j)\,, \tag{C.18}$$

$$L^j[U_{sw}] = \exp(isM_c^j w^c(\boldsymbol{x}, t))\,, \tag{C.19}$$

independently of the representation j to which the SU(N) generators M^j belong. In other words the mapping

$$U_w \to \sigma_{ab}(w) \tag{C.20}$$

defines the adjoint representation of SU(N).

Let us now take the functional derivative of eq. (C.1) with respect to $A_0^a(\boldsymbol{x}, t)$.

We get

$$\int_{T_1}^{t} \mathrm{d}t'' \frac{\delta U_w(\boldsymbol{x}, t)}{\delta w^c(\boldsymbol{x}, t'')} \frac{\delta w^c(\boldsymbol{x}, t'')}{\delta A_0^a(\boldsymbol{x}, t')} = i\theta(t-t')[\mathrm{T}\exp(i\int_{t'}^{t} A_0(\boldsymbol{x}, \tau)\,\mathrm{d}\tau)]\,\lambda^a$$
$$\times\ [\mathrm{T}\exp(i\int_{T_1}^{t'} A_0(\boldsymbol{x}, \tau)\,\mathrm{d}\tau)]\,\mathrm{e}^{i\lambda_a w^a(\boldsymbol{x}, T_1)}\ . \tag{C.21}$$

Using (C.10) and then setting $t = t'$, we obtain

$$\frac{\delta w^c(\boldsymbol{x}, t)}{\delta A_0^a(\boldsymbol{x}, t)} \int_0^1 \mathrm{d}s\, \Sigma_c(sw) = \theta(0)\,\lambda_a\ , \tag{C.22}$$

which after projection on λ_b, becomes

$$\frac{\delta w^c(\boldsymbol{x}, t)}{\delta A_0^a(\boldsymbol{x}, t)} \int_0^1 \mathrm{d}s\, \sigma_{cb}(sw) = \theta(0)\,\delta_{ab}\ . \tag{C.23}$$

In view of (C.15), the solution of (C.23) is (C.9).

References

[1] C.M. Yang and R.L. Mills, Phys. Rev. 96 (1954) 191.
[2] C. Callan, R. Dashen and D. Gross, Phys. Lett. 63B (1976) 334;
R.Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.
[3] H. Weyl, The theory of groups and quantum mechanics (Dover, 1950) pp. 256–261;
H.G. Loos, Phys. Rev. 188 (1969) 2342; D10 (1974) 4032.
[4] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965).
[5] A.M. Polyakov, Phys. Lett. 72B (1978) 477; Lectures given at the International Center for Theoretical Physics (Nov, 1977) notes taken by E. Gava, Internal report IC/78/4.
[6] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.
[7] J.L. Gervais and B. Sakita, Phys. Rev. D18 (1978) 453.
[8] J.D. Bjorken and S.D. Drell, Relativistic quantum mechanics and Relativistic quantum fields (McGraw-Hill, New York, 1964).
[9] L.D. Faddeev and V.N. Popov, Phys. Lett. 25B (1967) 29.
[10] P. Senjanovic, Phys. Lett. 72B (1978) 329.
[11] V.N. Gribov, Materials for 12th Leningrad Nuclear Physics Institute Winter School 1977, vol. 1, p. 147; Nucl. Phys. B139 (1978) 1.
[12] M. Creutz, I.J. Muzinich and T.N. Tudron, Brookhaven National Laboratory preprint BNL-24982 (1978).
[13] F. Strocchi, Scuola Normale Superiore Pisa, S.N.S. preprint 6/1979.
[14] E. Witten, Haward University preprint HUTP 78/A058 (1978).
[15] J.A. Wheeler, Geometrodynamics (Academic Press, New York, 1962) p. 74;
A. Katz, Nuovo Cim. 37 (1965) 342.
[16] C. Bender, T. Eguchi and H. Pagels, Phys. Rev. D17 (1978) 1086.
[17] J. Jackiw, I. Muzinich and C. Rebbi, Phys. Rev. D17 (1978) 1576.
[18] R.P. Feynman, Phys. Rev. 84 (1951) 108.
[19] A.A. Kirillov, Elements of the theory of representations (Springer-Verlag, Berlin, 1976).