Quantum Field Theory

Problem Sheet 2

1. *Path Integral for free motion*

   The Hamiltonian for the free particle is

   $$H = P^2/(2m)\,.$$

   Using the following change of variables:

   $$q(t) = q_c(t) + r(t)\,,$$

   with boundary conditions $q_c(t') = q', q_c(t'') = q''$, and imposing that q_c satisfies the classical equations of motion, show that

   $$\langle q'', t'' | q', t' \rangle = N \exp\left[i \frac{m}{2} \frac{(q'' - q')^2}{t'' - t'} \right]\,.$$

   Write an expression for the normalization factor N.

2. *Path Integral for the harmonic oscillator*

   Same question for the harmonic oscillator, with

   $$H = \frac{1}{2}P^2 + \frac{1}{2}m\omega^2 Q^2\,.$$

3. *Weyl ordering*

   Find the Weyl ordered product $[A^2 B]$.

4. *Schrödinger equation from path integrals*

   The quantum mechanical amplitude $\langle q'', t'' | q', t' \rangle$ for a system described by the lagrangian

   $$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$$

   is given by the path integral

   $$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}x \, e^{iS[x]}\,.$$

   Discretise the interval $T = t'' - t'$, and write an expression for the path integral (see lecture notes).

Consider the last time slice, closest to q'', and show that

$$\langle q'', t'' | q', t' \rangle = \int \frac{dx_{n-1}}{2\pi i \epsilon} \exp i \left[\frac{(q'' - x_{n-1})^2}{2\epsilon} - \epsilon V \left(\frac{q'' + x_{n-1}}{2} \right) \right] \langle x_{n-1}, t'' - \epsilon | q', t' \rangle \,.$$

For $\epsilon \to 0$ the kinetic term in the exponential suppresses contributions for large values of $\delta = (x_{n-1} - q'')$. Expanding the RHS in powers of δ, derive the Schrödinger equation for the transition amplitude:

$$i \frac{\partial}{\partial t''} \langle q'', t'' | q', t' \rangle = \left[-\frac{1}{2} \frac{\partial^2}{\partial q''^2} + V(q'') \right] \langle q'', t'' | q', t' \rangle \,.$$