1. $$G_b^{(2,2)}(x,y) = -\frac{1}{2}\int d^Dw_1 d^Dw_2\, \Delta(x-w_1)\,\Delta(w_1-y)\,\Delta(w_1-w_2)\,\Delta(0)$$

In momentum space.

$$G_b^{(2,2)}(p,p') = \int d^Dx\, d^Dy\, e^{-ip\cdot x}\, e^{-ip'\cdot y}\left(-\frac{1}{2}\right)\times$$

$$\times\int d^Dw_1\, d^Dw_2 \int_{l_1 l_2 l_3 l_4} \frac{e^{il_1\cdot(x-w_1)}}{l_1^2-m^2+i\varepsilon}\;\frac{e^{il_2\cdot(w_1-y)}}{l_2^2-m^2+i\varepsilon}$$

$$\times\frac{e^{il_3\cdot(w_1-w_2)}}{l_3^2-m^2+i\varepsilon}\;\frac{1}{l_4^2-m^2+i\varepsilon} =$$

$$= \int_{l_1 l_2 l_3 l_4} (2\pi)^D\,\delta(l_1-p)\,(2\pi)^D\,\delta(l_2+p')\,(2\pi)\,\delta(l_1-l_2+l_3)\,(2\pi)^D\,\delta(l_3)$$

$$\times\left(-\frac{1}{2}\right)\prod_{k=1}^{4}\frac{1}{l_k^2-m^2+i\varepsilon} =$$

$$= -\frac{1}{2}\,\frac{1}{p^2-m^2+i\varepsilon}\,\frac{1}{p'^2-m^2+i\varepsilon}\,(2\pi)^D\,\delta(p+p')\,\frac{1}{m^2}\int\frac{d^Dl}{(2\pi)^D}\,\frac{1}{l^2-m^2+i\varepsilon}.$$

2. <u>$p_1 p_2 \to p_1' p_2'$ scattering</u>

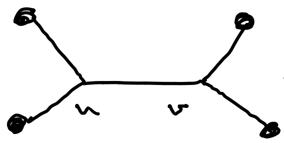
$$\tilde{G}^{(4)}(p_1,p_2;-p_1,-p_2) = \int d^Dx_1\, d^Dx_2\, d^Dx_1'\, d^Dx_2'\, e^{ip_1\cdot x_1}\, e^{ip_2\cdot x_2}\, e^{-ip_1'\cdot x_1'}\, e^{-ip_2'\cdot x_2'}$$

$$\times\, G^{(4)}(x_1,x_2,x_1',x_2').$$

$$G^{(4)}(x_1 x_2 x_3 x_4) = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J(x_4)}\right) Z[J]\,\Big|_{J=0}$$

We need to isolate the contributions to $Z[J]$ w. $E = 4$

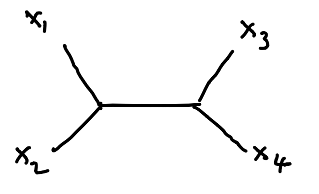
$$E = 2P - 3V \quad \Rightarrow \quad V = 2 \quad \text{and} \quad P = 5$$



$$= \frac{i}{2^3} \int d^Dz_1\, d^Dz_2\, d^Dz_3\, d^Dz_4\, d^Du\, d^Dv\; J(z_1)\, \Delta(z_1-u)$$

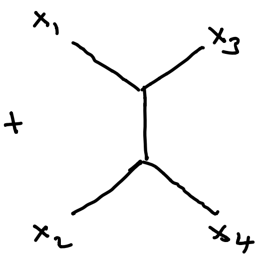
$$\times\, J(z_2)\, \Delta(z_2-u)\; \Delta(u-v)\; J(z_3)\; \Delta(z_3-v)\, J(z_4)\, \Delta(z_4-v) \, .$$

$G^{(4)}(x_1, x_2, x_3, x_4) =$



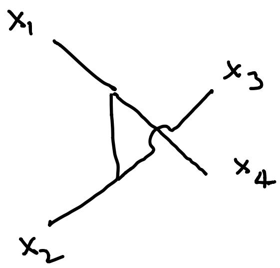
(a)

$+$



(b)

$+$



(c)

$$(a) = \int d^Du\, d^Dv\; \Delta(x_1-u)\, \Delta(x_2-u)\, \Delta(u-v)\, \Delta(x_3-v)\, \Delta(x_4-v)$$

$$\tilde{G}_a^{(4)}(p_1, p_2, -p_1', -p_2') = \int d^Dx_1\, d^Dx_2\, d^Dx_1'\, d^Dx_2'\, d^Du\, d^Dv \quad e^{ip_1\cdot x_1}\, e^{ip_2\cdot x_2}\, e^{-ip_1'\cdot x_1'}\, e^{-ip_2'\cdot x_2'}$$

$$\times \int_{l_1 l_2 l_3 l_4 l_5} e^{-il_1\cdot(x_1-u)}\, e^{-il_2\cdot(x_2-u)}\, e^{-il_3\cdot(u-v)}\, e^{-il_4\cdot(x_1'-v)}\, e^{-il_5\cdot(x_2'-v)}$$

$$\times \frac{1}{l_1^2-m^2}\; \frac{1}{l_2^2-m^2}\; \frac{1}{l_3^2-m^2}\; \frac{1}{l_4^2-m^2}\; \frac{1}{l_5^2-m^2} =$$

$$= \frac{1}{p_1^2-m^2}\; \frac{1}{p_2^2-m^2}\; \frac{1}{p_1'^2-m^2}\; \frac{1}{p_2'^2-m^2}\; \frac{1}{(p_1+p_2)^2-m^2}\; (2\pi)^D \delta(p_1+p_2-p_1'-p_2') \, .$$

Similar manipulations f. (b) + (c) yield:

$$\tilde{G}^{(4)}_b(p_1,p_2;-p_1',p_2') = \frac{1}{p_1^2-m^2}\,\frac{1}{p_2^2-m^2}\,\frac{1}{p_1'^2-m^2}\,\frac{1}{p_2'^2-m^2}\,\frac{1}{(p_1-p_1')^2-m^2}(2\pi)^D\,\delta(\underline{P})$$

$$\tilde{G}^{(4)}_c(p_1,p_2;-p_1',-p_2') = \frac{1}{p_1^2-m^2}\,\frac{1}{p_2^2-m^2}\,\frac{1}{p_1'^2-m^2}\,\frac{1}{p_2'^2-m^2}\,\frac{1}{(k_1-k_2')^2-m^2}(2\pi)^D\,\delta(\underline{P})$$

$$\underline{P} = P_{tot} = p_1+p_2-p_1'-p_2'$$

$$\Rightarrow \mathcal{M}(p_1p_2\to p_1'p_2') = (2\pi)\,\delta(p_1+p_2-p_1'-p_2')\;g^2\left\{\frac{1}{(p_1+p_2)^2-m^2} + \right.$$

$$\left. + \frac{1}{(p_1-p_1')^2-m^2} + \frac{1}{(p_1-p_2')^2-m^2}\right\}.$$

- You should try to write directly $\tilde{G}^{(4)}(p\ldots)$ using Feynman rules in momentum space.
- You should sort out the factors of "i".

3. LSZ reduction

Wave packet : $\quad a_1^+(t) = \int d^3k\; f_1(\vec{k})\; a^+(k,t)$

$$|in\rangle = \lim_{t\to-\infty} a_1^+(t)\,a_2^+(t)\,|0\rangle$$

$$|out\rangle = \lim_{t\to+\infty} a_{1'}^+(t)\,a_{2'}^+(t)\,|0\rangle$$

Amplitude : $\quad \langle out|in\rangle = \langle 0|\,a_{1'}(+\infty)\,a_{2'}(+\infty)\,a_1^+(-\infty)\,a_2^+(-\infty)\,|0\rangle$

$\hookrightarrow \langle out|in\rangle = \langle 0|\,T\,a_{1'}(+\infty)\,a_{2'}(+\infty)\,a_1^+(-\infty)\,a_2^+(-\infty)\,|0\rangle$

Write $a_i^+(-\infty)$ as a fn. of $a_i^+(+\infty)$, and likewise $a_i(+\infty)$ as a fn. of $a_i(-\infty)$. Then the T-ordering will "swap" a_i and a_i^+.

$$a^+(k,t) = -i\int d^3x\, e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)$$ (see Richard's notes)

Then: $a_1^+(+\infty) - a_1^+(-\infty) = \int dt\, \partial_0 a_1^+(t) =$

$$= -i\int d^3k\, f_1(\vec{k})\int d^4x\, \partial_0\left(e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)\right) . \qquad : (*)$$

$$\left(e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)\right) = \left(i k_0\, e^{-ik\cdot x}\phi(x) + e^{-ik\cdot x}\partial_0\phi(x)\right)$$

$$\partial_0\left(e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)\right) = k_0^2\, e^{-ik\cdot x}\phi(x) + i k_0\, \cancel{e^{-ik\cdot x}}\,\partial_0\phi(x) -$$

$$- i k_0\, \cancel{e^{-ik\cdot x}}\,\partial_0\phi(x) + e^{-ik\cdot x}\partial_0^2\phi(x) = e^{-ik\cdot x}\left(\partial_0^2 + E_k^2\right)\phi(x)$$

__NB__: $k_0 = E_k = \sqrt{\vec{k}^2 + m^2}$

$$(*) = -i\int d^3k\, f_1(\vec{k})\int d^4x\, e^{-ik\cdot x}\left(\partial_0^2 + \vec{k}^2 + m^2\right)\phi(x)$$

$$= -i\int d^3k\, f_1(\vec{k})\int d^4x\, e^{-ik\cdot x}\left(\partial_0^2 - \nabla^2 + m^2\right)\phi(x)$$

So finally: $$a_1^+(-\infty) = a_1^+(+\infty) + i\int d^3k\, f_1(\vec{k})\int d^4x\, e^{-ik\cdot x}$$

$$\times\left(\partial^2 + m^2\right)\phi(x) .$$

Similarly,

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ik\cdot x}(\partial^2 + m^2)\,\phi(x)$$