Lecture 7

Divergences

7.1 A first look at divergences

In this lecture we will aim to develop a self-consistent treatment of divergences in QFT. This is a vast topic, which can hardly be addressed exhaustively in the time that we have. Therefore we will follow the following steps.

1. Compute the scalar two-point function beyond the first order in perturbation theory. As we try to perform this calculation we will encounter our first divergent integral.
2. Discuss the regularization of divergencies; *i.e.* a procedure that allows us to manipulate well-defined mathematical expressions, and to identify the structure of the divergencies.
3. Discuss the renormalization of divergencies; *i.e.* the conditions that are necessary for a quantum field theory to produce finite, unambiguous predictions.

7.1.1 Two-point function in perturbation theory

Working in perturbation theory, we compute the two-point function

$$\tilde{G}^{(2)}\left(p,p'\right) = (2\pi)^D \delta\left(p+p'\right) \frac{1}{i} \tilde{\Delta}_F(p)\,, \tag{7.1}$$

as a Taylor expansion in powers of the coupling constant

$$\tilde{G}^{(2)}\left(p,p'\right) = \sum_k g^k \tilde{G}^{(2,k)}\left(p,p'\right)\,. \tag{7.2}$$

As discussed before, the delta function in Eq. 7.1 ensures momentum conservation. For all practical purposes, we should remember that it is there, and work on the perturbative expansion of the full propagator

$$\tilde{\Delta}_F(p) = \sum_k g^k \tilde{\Delta}_F^{(k)}(p)\,. \tag{7.3}$$

From our previous computations

$$\begin{aligned}\frac{1}{i}\tilde{\Delta}_F^{(2)}(p) = &-\frac{1}{2}\frac{1}{p^2-m^2+i\epsilon}\left(\int \frac{d^D\ell}{(2\pi)^D}\,\frac{1}{\ell^2-m^2+i\epsilon}\,\frac{1}{(\ell-p)^2-m^2+i\epsilon}\right)\times \\ &\times \frac{1}{p^2-m^2+i\epsilon}\,.,\end{aligned} \tag{7.4}$$

and therefore the $O(g^2)$ contribution to the correlator can be written as

$$\frac{1}{i}\tilde{\Delta}(p)\left(i\Pi(p^2)\right)\frac{1}{i}\tilde{\Delta}(p)\,, \tag{7.5}$$

where

$$i\Pi(p^2) = \frac{g^2}{2}\int \frac{d^D\ell}{(2\pi)^D}\,\frac{1}{\ell^2-m^2+i\epsilon}\,\frac{1}{(\ell-p)^2-m^2+i\epsilon}\,. \tag{7.6}$$

7.1.2 Evaluation of $\Pi(p^2)$

Feynman parameters The product of propagators in Eq. (7.6) can be rewritten using Feynman parameters. The general formula

$$\frac{1}{A_1^{\alpha_1}\dots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\dots+\alpha_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)} \int_0^1 dx_1\, x_1^{\alpha_1-1}\dots\int_0^1 dx_n\, x_n^{\alpha_n-1} \times \delta\left(1-x_1-\dots-x_n\right) \frac{1}{(x_1 A_1+\dots+x_n A_n)^{\alpha_1+\dots+\alpha_n}}\,, \tag{7.7}$$

can be applied to the integrand above, yielding

$$\frac{1}{\ell^2-m^2+i\epsilon}\frac{1}{(\ell-p)^2-m^2+i\epsilon} = \int_0^1 dx\, \frac{1}{\left(q^2-M^2+i\epsilon\right)^2}\,, \tag{7.8}$$

where $q=\ell-xp$, and $M^2(x,p)=m^2-x(1-x)p^2$. Hence, we have

$$i\Pi(p) = \frac{g^2}{2}\int_0^1 dx \int \frac{d^Dq}{(2\pi)^D}\, \frac{1}{\left(q^2-M^2+i\epsilon\right)^2}\,. \tag{7.9}$$

Wick rotation It is useful to introduce Euclidean momenta in order to perform the integration. Because of the location of the poles, we can rotate the integration contour clockwise by $\pi/2$ to run along the purely imaginary axis. Introducing

$$q^0 = iq^0_E\,, \quad \mathbf{q}=\mathbf{q}_E\,, \tag{7.10}$$

we can rewrite

$$\Pi(p^2) = \frac{g^2}{2}\int_0^1 dx \int \frac{d^Dq_E}{(2\pi)^D}\, \frac{1}{\left(q_E^2+M^2\right)^2}\,. \tag{7.11}$$

A comment on divergencies The integral in Eq. (7.11) is clearly divergent for $D\geq 4$. In particular, it is quadratically divergent in the UV for the case $D=6$, which is the one we will be interested in. Before developing more sophisticated tool, we can make a simple, but rather deep, observation. If we take the derivative of $\Pi(p^2)$ with respect to p^2:

$$\Pi'(p^2) = -g^2\int_0^1 dx\, x(x-1) \int \frac{d^Dq_E}{(2\pi)^D}\, \frac{1}{\left(q_E^2+m^2\right)^3}\,, \tag{7.12}$$

which is still divergent, but only for $D\geq 6$. Similarly

$$\Pi''(p^2) = 3g^2\int_0^1 dx\, x^2(x-1)^2 \int \frac{d^Dq_E}{(2\pi)^D}\, \frac{1}{\left(q_E^2+m^2\right)^4} \tag{7.13}$$

is divergent only for $D \geq 8$, and in particular is finite for $D = 6$. Therefore the function $\Pi(p^2)$ can be reconstructed by integrating its second derivative twice,

$$\Pi(p^2) = \Pi(\mu_1^2) + \Pi'(\mu_2^2)\left(p^2 - \mu_1^2\right) + \int_{\mu_1^2}^{p^2} ds' \int_{\mu_2^2}^{s'} ds\, \Pi''(s)\,. \tag{7.14}$$

Eq. (7.14) shows clearly that $\Pi(p^2)$ in $D = 6$ is well defined for all values of p^2 provided we fix the values of $\Pi(\mu_1^2)$, and $\Pi'(\mu_2^2)$, *i.e.* the values of the function and its derivative at two *arbitrary* values of the scale.

7.2 Regularization

In order to make progress in our understanding of these divergencies, we need to first regulate the theory, *i.e.* we need to choose a prescription that makes the loop integrals mathematically well defined. This is clearly a necessary condition in order to be able to manipulate these expressions, and eventually define a predictive theory that yields finite results for physical quantities.

There are several ways of regularizing the theory, here we list some of the most common procedures, some of which we will explore in tutorials.

1. Sharp cutoff in Euclidean momenta, $q_E^2 \leq \Lambda^2$.

2. Pauli-Villars regulator. The propagators are modifed in order to have a less divergent behaviour at large values of the momenta:
$$\frac{1}{p^2 - m^2 + i\epsilon} \mapsto \frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - M^2 + i\epsilon}\,, \tag{7.15}$$
where M is a mass scale that plays the role of the UV cutoff.

3. Schwinger-time regularization – see PS7.

4. Work in generic dimension D, and define the divergent integrals by analytical continuation.

7.2.1 Dimensional Regularization

In dimensional regularization (DimReg), loop integrals are computed in a generic dimension D, where the intregration is actually convergent, and then defined for arbitrary values of D by analytical continuation. Let us look at the integral that we encountered above for the two-point function. After Wick rotation, we are interested in

$$I_D = \int \frac{d^D q_E}{(2\pi)^D} \frac{1}{\left(q_E^2 + m^2\right)^2}\,, \tag{7.16}$$

which is easily computed in spherical coordinates:

$$I_D = \int \frac{d\Omega_D}{(2\pi)^D} \frac{1}{2} \int_0^\infty dq_E^2 \, (q_E^2)^{D/2-1} \frac{1}{\left(q_E^2 + m^2\right)^2} \tag{7.17}$$

$$= \frac{1}{(2\pi)^D} \frac{2\pi^{D/2}}{\Gamma(D/2)} \frac{1}{2} \left(\frac{1}{M^2}\right)^{2-D/2} \int_0^1 d\xi \, \xi^{1-D/2} (1-\xi)^{D/2-1} \tag{7.18}$$

$$= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)} \left(\frac{1}{M^2}\right)^{2-D/2} . \tag{7.19}$$

Mathematical aside In deriving the above result we have made use of a number of useful properties/tricks, which we summarise here.

1. The solid angle in D dimension is

$$\Omega_D = \int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} . \tag{7.20}$$

2. We made a change of integration variable:

$$\xi = \frac{M^2}{q_E^2 + M^2} . \tag{7.21}$$

3. The Euler gamma function is defined as

$$\Gamma(z) = \int_0^\infty dt \, t^{z-1} e^{-t} . \tag{7.22}$$

4. It can be readily shown that

$$\Gamma(z+1) = z\Gamma(z) . \tag{7.23}$$

5. The beta function (Euler integral of the first kind) is defined as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 d\xi \, \xi^{\alpha-1} (1-\xi)^{\beta-1} . \tag{7.24}$$

6. For $n \geq 0$, and small ϵ, we have

$$\Gamma(n+1) = n! , \tag{7.25}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{n! 2^{(2n)}} \sqrt{\pi} , \tag{7.26}$$

$$\Gamma(-n+\epsilon) = \frac{(-)^n}{n!} \left[\frac{1}{\epsilon} - \gamma + \sum_{k=1}^n \frac{1}{k} + O(\epsilon)\right] , \tag{7.27}$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant.

Divergences of I_D Eq. (7.19) provides an expression for I_D which can be extended by analytical continuation to arbitrary values of D. It is interesting to note that the divergences that we identified in $D = 4$ and $D = 6$ appear in the regularized version as poles of the gamma function for negative integer values of its argument.

General formula It is useful to generalise the result in Eq. (7.19):

$$\boxed{\int \frac{d^D q_E}{(2\pi)^D} \frac{\left(q_E^2\right)^a}{\left(q_E^2 + M^2\right)^b} = \frac{\Gamma(b-a-D/2)\Gamma(a+D/2)}{(4\pi)^{D/2}\Gamma(b)\Gamma(D/2)} \left(M^2\right)^{-(b-a-D/2)}\,.} \tag{7.28}$$

Manipulations similar to the ones above allow you to derive the general formula.
It is useful to factor out the angular integral in the above expression:

$$\int \frac{d^D q_E}{(2\pi)^D} \frac{\left(q_E^2\right)^a}{\left(q_E^2 + M^2\right)^b} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \frac{1}{(2\pi)^D} \int_0^\infty dq \frac{q^{2a+D-1}}{(q^2+M^2)^b} \tag{7.29}$$

$$= \frac{2}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dq\, \frac{q^{2a+D-1}}{(q^2+M^2)^b}\,. \tag{7.30}$$

Comparing Eq. (7.28) with Eq. (7.30) we see that

$$\int_0^\infty dq \frac{q^{2a+D-1}}{(q^2+M^2)^b} = \frac{\Gamma(b-a-D/2)\Gamma(a+D/2)}{2\Gamma(b)} (M^2)^{-(b-a-D/2)}\,, \tag{7.31}$$

which can be rewritten as

$$\boxed{\int_0^\infty dq \frac{q^{2\alpha}}{(q^2+M^2)^\beta} = \frac{\Gamma(\beta-\alpha-1/2)\Gamma(\alpha+1/2)}{2\Gamma(\beta)} (M^2)^{-(\beta-\alpha-1/2)}\,.} \tag{7.32}$$

7.2.2 Structure of divergences

Let us now consider the case $D = 6$, and work out in more detail the structure of the divergences that appear in $\Pi(p^2)$. As discussed above, for $D = 6$ the integral I_D is logarithmically divergent, and therefore it is convergent as soon as $D < 6$. We will therefore use dimensional regularization, and define the integral in $D = 6 - 2\epsilon$.

Before we start manipulating the integral, we need to do some dimensional analysis first. For $D = 6$, the scalar field has mass dimensions

$$[\phi] = \frac{D-2}{2} = 2\,,$$

and therefore the coupling constant g is dimensionless. This is a useful property that we want to preserve; as we continue our expressions to $D = 6 - 2\epsilon$ we replace g in the action by $g\tilde{\mu}^\epsilon$, where $\tilde{\mu}$ is an arbitrary scale, and g remains a dimensionless coupling. This seemingly harmful rescaling has deep consequences: the regularization procedure – in this case changing the number of space-time dimensions – has automatically introduced a new scale in the problem.

With these choices for the regulator, we find

$$\Pi(p^2) = \frac{1}{2}\alpha\Gamma(\epsilon - 1)\int_0^1 dx\, M^2(x,p^2)\left(\frac{4\pi\tilde{\mu}^2}{M^2(x,p^2)}\right)^\epsilon . \tag{7.33}$$

ϵ dependence We can now compute explicitly the dependence on ϵ:

$$\Gamma(\epsilon - 1) = -\left[\frac{1}{\epsilon} - \gamma + 1 + O(\epsilon)\right] , \tag{7.34}$$

$$\left(\frac{4\pi\tilde{\mu}^2}{M^2(x,p^2)}\right)^\epsilon = 1 + \epsilon\log\left(\frac{4\pi\tilde{\mu}^2}{M^2(x,p^2)}\right) + O(\epsilon^2)\, , \tag{7.35}$$

and hence

$$\Gamma(\epsilon - 1)\left(\frac{4\pi\tilde{\mu}^2}{M^2(x,p^2)}\right)^\epsilon = -\left[\frac{1}{\epsilon} - \gamma + 1 + \log\left(\frac{4\pi\tilde{\mu}^2}{M^2(x,p^2)}\right) + O(\epsilon)\right] . \tag{7.36}$$

Collecting all contributions yields

$$\Pi(p^2) = -\frac{\alpha}{2}\int_0^1 dx\, M^2(x,p^2)\left[\frac{1}{\epsilon} + 1 + \log\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma M^2(x,p^2)}\right) + O(\epsilon)\right] \tag{7.37}$$

$$= \frac{\alpha}{2}\left[\left(\frac{1}{\epsilon} + 1\right)\left(\frac{1}{6}p^2 - m^2\right) - \int_0^1 dx\, M^2(x,p^2)\log\left(\frac{\mu^2}{M^2}\right)\right] + O(\epsilon)\, , \tag{7.38}$$

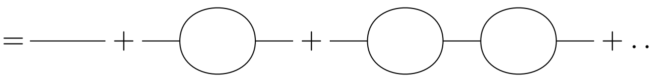
where

$$\alpha = \frac{g^2}{(4\pi)^3}\, , \quad \mu^2 = \frac{4\pi}{e^\gamma}\tilde{\mu}^2 . \tag{7.39}$$

Eq. (7.38) shows explicitly the structure of the divergences in the loop integral. They are given by the two terms proportional to $1/\epsilon$, and are proportional to p^2 and m^2. Understanding the structure of the divergent terms is the first step to be able to understand how to treat them. For the time being, we note that the divergent terms look like the contribution to the propagator that one would obtain from interaction vertices in the lagrangian that contain only two fields, *i.e.* vertices like $\partial_\mu\phi\partial^\mu\phi$ and ϕ^2. These vertices are usually called *counter terms*.

A geometric series Having evaluated $\Pi(p^2)$, we can easily resum an entire class of contributions:

$$\frac{1}{i}\tilde{\Delta}_F(p^2) = \tag{7.40}$$



$$\begin{aligned}
&= \frac{1}{i}\tilde{\Delta}(p^2) + \frac{1}{i}\tilde{\Delta}(p^2)\left[\left(i\Pi(p^2)\right)\left(\frac{1}{i}\tilde{\Delta}(p^2)\right)\right] + \\
&\quad + \frac{1}{i}\tilde{\Delta}(p^2)\left[\left(i\Pi(p^2)\right)\left(\frac{1}{i}\tilde{\Delta}(p^2)\right)\right]^2 + \ldots \qquad (7.41)\\
&= \frac{1}{i}\tilde{\Delta}(p^2)\sum_{k=0}^{\infty}\left[\Pi(p^2)\tilde{\Delta}(p^2)\right]^k \qquad (7.42)\\
&= \frac{1}{i}\tilde{\Delta}(p^2)\,\frac{1}{1-\Pi(p^2)\tilde{\Delta}(p^2)}\,. \qquad (7.43)
\end{aligned}$$

Hence the net result of the sum yields

$$\tilde{\Delta}_F(p^2) = \frac{1}{p^2 - m^2 - \Pi(p^2) + i\epsilon}\,, \tag{7.44}$$

where we have reintroduced the $i\epsilon$ term in the denominator (not to be confused with the parameter ϵ of DimReg).

Comparison with K-L According to the Källen-Lehmann representation of the propagator, we expect to find a pole for $p^2 = m_{\text{phys}}^2$, with the corresponding residue being equal to one. Two observations are in order.

1. In order to have such a pole we need
$$m_{\text{phys}}^2 - m^2 - \Pi(m_{\text{phys}}^2) = 0 \tag{7.45}$$
to hold, which clearly shows that $m_{\text{phys}}^2 \neq m^2$.

2. Similarly we can compute the residue of the propagator at $p^2 = m_{\text{phys}}^2$ by expanding the denominator in p^2 around m_{phys}^2,
$$p^2 - m^2 - \Pi(p^2) = \left(p^2 - m_{\text{phys}}^2\right)\left[1 - \Pi'(m_{\text{phys}}^2)\right] + O\left(\left(p^2 - m_{\text{phys}}^2\right)^2\right)\,, \tag{7.46}$$
which in turn yields
$$\begin{aligned}
\text{Res}_{m_{\text{phys}}^2}\tilde{\Delta}(p^2) &= \lim_{p^2\to m_{\text{phys}}^2}\left(p^2 - m_{\text{phys}}^2\right)\tilde{\Delta}(p^2) \qquad (7.47)\\
&= \frac{1}{1-\Pi'(m_{\text{phys}}^2)}\,. \qquad (7.48)
\end{aligned}$$

The latter equation shows that the field ϕ that appears in the Lagrangian does not have the normalization required to agree with the K-L representation.

These two observations suggest a general line of thinking: the fields and couplings that appear in the Lagrangian, the so-called *bare fields* and *bare coupings* are not physical. The physical variables need to be defined according to some well-specified prescription. The process of specifying these quantities is known as *renormalization* of the theory.

7.3 Renormalization

7.3.1 Renormalized perturbation theory

Renormalization of the field As suggested by the comparison with K-L, we define a *renormalized field*

$$\phi(x) = Z^{1/2}\phi_R(x)\,, \tag{7.49}$$

where Z is the renormalization constant of the field. In terms of the new field the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m^2 Z\phi_R^2 + \frac{g}{3!}Z^{3/2}\phi_R^3 + \frac{1}{2}\delta_Z\partial_\mu\phi_R\partial^\mu\phi_R\,, \tag{7.50}$$

where we have introduced $\delta_Z = Z - 1$.

Renormalization of mass and coupling We can also introduce a renormalized mass and a renormalized coupling

$$m^2 Z = Z_m m_R^2\,, \quad gZ^{3/2} = Z_g g_R \tag{7.51}$$

where we have introduced two new renormalization constants Z_m and Z_g.

Renormalized perturbation theory We can rewrite the Lagrangian one more time as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 + \frac{Z_g g_R}{3!}\phi_R^3 + \frac{1}{2}\delta_Z\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}\delta_m\phi_R^2\,, \tag{7.52}$$

where $\delta_m = Z_m - 1$. The expression above for the Lagrangian looks identical to the bare one, except that fields and couplings are now renormalized, and there two counter terms proportional to δ_Z and δ_m respectively. We can therefore define the path integral in terms of renormalized quantities; this will lead to *renormalized perturbation theory.*

We can now compute $\Pi(p^2)$ in renormalized perturbation theory, the result is identical to the one obtained before, plus the contribution of the counter terms:

$$\Pi(p^2) = \frac{\alpha}{2}\left[\left(\frac{1}{\epsilon}+1\right)\left(\frac{1}{6}p^2 - m_R^2\right) + \int_0^1 dx\, M^2\log\left(\frac{M^2}{\mu^2}\right)\right] + \delta_Z p^2 - \delta_m m_R^2 + O(\alpha^2)\,. \tag{7.53}$$

Is it useful to rewrite the equation above as:

$$\Pi(p^2) = \frac{\alpha}{2}\int_0^1 dx\, M^2 \log\left(\frac{M^2}{m_R^2}\right) + \left[\frac{\alpha}{6}\left(\frac{1}{2\epsilon} + \log(\mu/m_R) + \frac{1}{2}\right) + \delta_Z\right] p^2 - \\ - \left[\alpha\left(\frac{1}{2\epsilon} + \log(\mu/m_R) + \frac{1}{2}\right) + \delta_m\right] m_R^2 + O\left(\alpha^2\right) . \tag{7.54}$$

Note that at this stage we have not yet defined the renormalization constants, and therefore δ_Z and δ_m are free parameters that need to be fixed. Note also that by choosing

$$\delta_z = -\frac{\alpha}{6}\left(\frac{1}{2\epsilon} + \log(\mu/m_R) + \frac{1}{2} + \kappa_Z\right) + O\left(\alpha^2\right) , \tag{7.55}$$

$$\delta_m = -\alpha\left(\frac{1}{2\epsilon} + \log(\mu/m_R) + \frac{1}{2} + \kappa_m\right) + O\left(\alpha^2\right) , \tag{7.56}$$

where κ_Z and κ_m are finite constants, we obtain a value for $\Pi(p^2)$ which is finite when $\epsilon \to 0$, and independent of the arbitrary scale μ.

Renormalization conditions In order to fully determine the renormalization constants, we need to specify a so-called *renormalization scheme*. The renormalization scheme is defined by imposing a number of conditions that are sufficient to determine all the renormalization constants. Clearly these conditions are necessary in order to have a predictive framework.

7.3.2 Renormalization scheme

MOM scheme, MSbar scheme