1. Free motion

$$S(q) = \int_{t'}^{t''} d\tau \, \frac{1}{2} m \, \dot{q}^2(\tau) \qquad\qquad \hat{H} = \frac{\hat{p}^2}{2m}, \quad \hat{V} = 0$$

Integrating by parts:

$$\int_{t'}^{t''} d\tau \, \dot{q}(\tau)^2 = q(\tau)\,\dot{q}(\tau) \Big|_{t'}^{t''} - \int_{t'}^{t''} d\tau \, q(\tau) \frac{d^2}{d\tau^2} q(\tau)$$

$\hookrightarrow$ Gaussian integral w. $A = -m \dfrac{d^2}{d\tau^2}$

$q(t) = q_c(t) + r(t)$, with $q_c(t') = q'$, $q_c(t'') = q''$, $r(t') = r(t'') = 0$

$$S(q_c + r) = S(q_c) + S(r) + m \int_{t'}^{t''} d\tau \, \dot{q}_c(\tau)\,\dot{r}(\tau) .$$

and $\displaystyle\int_{t'}^{t''} d\tau \, \dot{q}_c(\tau)\,\dot{r}(\tau) = \cancel{\dot{q}_c(\tau)\,\dot{r}(\tau)} \Big|_{t'}^{t''} - \int_{t'}^{t''} d\tau \, \ddot{q}_c(\tau)\, r(\tau)$

$m\,\ddot{q}_c(\tau) = 0 \;\Rightarrow\;$ linear term in $r(t)$ vanishes.

Then: $\displaystyle q_c(\tau) = q' + \frac{\tau - t'}{t'' - t'}(q'' - q')$

$$\Rightarrow S(q_c) = \frac{m}{2} \frac{(q'' - q')^2}{t'' - t'}$$

$$\Rightarrow \langle q'' | e^{-i\hat{H}(t''-t')} | q \rangle = \mathcal{N} \exp\left\{ + i \frac{m}{2} \frac{(q'' - q')^2}{t'' - t'} \right\}$$

The normalization factor is given by:

$$\mathcal{N} = \int \mathcal{D}r \, \exp\left\{ i \frac{m}{2} \int_{t'}^{t''} d\tau \, \dot{r}(\tau)^2 \right\}.$$

↳ independent of q', q'' (i.e. independent of the b.c.).

2. <u>Harmonic oscillator</u>

$$\hat{H} = \frac{\hat{P}^2}{2} + \frac{1}{2}\omega^2 \hat{Q}^2 \qquad m = 1$$

$$S(q) = \int_{t'}^{t''} dt \left[\frac{1}{2} \dot{q}(t)^2 - \frac{1}{2}\omega^2 q(t)^2 \right]$$

$$\langle q'' t'' | q' t' \rangle = \int_{q'}^{q''} \mathcal{D}q \, \exp\{ i S(q) \}$$

Matrix A $\rightarrow$ diff. op. $\left(-\frac{d^2}{dt^2} - \omega^2 \right)$

In order to separate the dependence on q', q'' $\rightarrow$ change of vars.

$$q(t) = q_c(t) + r(t)$$

$$S(q_c + r) = S(q_c) + S(r) + \int_{t'}^{t''} d\tau \left[\dot{q}_c(\tau)\dot{r}(\tau) - \omega^2 q_c(\tau) r(\tau) \right]$$

$$= S(q_c) + S(r) + \int_{t'}^{t''} d\tau \left[-\ddot{q}_c(\tau) - \omega^2 q_c(\tau) \right] r(\tau)$$

Classical sol.n $\qquad \ddot{q}_c(\tau) = -\omega^2 q_c(\tau)$

$$\Rightarrow \quad S(q) = S(q_c) + S(r)$$

Hence: $\quad \langle q'' t'' | q' t' \rangle = \mathcal{N}(T,\omega) \exp\left\{ i S(q_c) \right\}.$

$$\mathcal{N}(\omega,T) = \int \mathcal{D}r \exp\left\{ i \int_{t'}^{t''} d\tau \left[\frac{1}{2}\dot{r}(t)^2 - \frac{1}{2}\omega^2 r(t)^2 \right] \right\}.$$

and $\quad r(t') = r(t'') = 0.$

$\rightsquigarrow$ divergences in the normalization factor $\mathcal{N}(T,\omega)$.

3. Solution of Schrödinger eq. $\quad$ [IGNORE THIS!]

$$\hat{H} = \frac{\hat{p}^2}{2m} \qquad |p\rangle = \int dq \, e^{iqp} |q\rangle$$

$$\langle p | p' \rangle = (2\pi)\, \delta(p-p')$$

Time evolution operator $\quad U(t,t')$

$$i \frac{\partial}{\partial t} U(t,t') = \hat{H}\, U(t,t')$$

$$\Rightarrow \quad i \frac{\partial}{\partial t} \langle p | U(t,t') | q \rangle = \langle p | \hat{H}\, U(t,t') | q \rangle$$

$$= \frac{p^2}{2m} \langle p | U(t,t') | q \rangle$$

b.c. : $\langle p | U(t', t') | q \rangle = \langle p | q \rangle = e^{-ipq}$

$$\Rightarrow \quad \langle p | U(t, t') | q \rangle = \exp\left[-i q \cdot p - i (t - t') \frac{p^2}{2m} \right]$$

$$\langle q | U(t, t') | q' \rangle = \int \frac{dp}{2\pi} \langle q | p \rangle \langle p | U(t, t') | q' \rangle$$

$$= \int \frac{dp}{2\pi} \exp\left[-i (q - q') \cdot p - i (t - t') \frac{p^2}{2m} \right]$$

$$\propto \exp\left[i \frac{m (q - q')^2}{2 (t - t')} \right].$$

3. Weyl ordering

In order to compute $[A^2 B]$, consider $(\alpha A + \beta B)^3$ and expand.

$$(\alpha A + \beta B)(\alpha A + \beta B) = \alpha^2 A^2 + \alpha\beta (AB + BA) + \beta^2 B^2$$

$$\Rightarrow (\alpha A + \beta B)^3 = \dots + \alpha^2 \beta \left(A^2 B + ABA + BA^2 \right) + \dots$$

$$[A^2 B] = \left(A^2 B + ABA + BA^2 \right) \frac{2}{3!} \qquad 1/3$$

4. Schrödinger eq. again

$$\langle q'', t'' | q', t' \rangle = U(q', q'', T)$$

$$T = t'' - t'$$

$$\langle q'' t'' | q' t' \rangle = \lim_{n \to \infty} \frac{1}{(2\pi i \varepsilon)^{1/2}} \int \prod_{k=1}^{n-1} \frac{dx_k}{(2\pi i \varepsilon)^{1/2}} \times$$

$$\times \exp\left\{ i \varepsilon \sum_{m=1}^{n} \left[\frac{1}{2} \left(\frac{x_m - x_{m-1}}{\varepsilon} \right)^2 - V\left(\frac{x_m + x_{m-1}}{2} \right) \right] \right\}.$$

$$= \int \frac{dx_{n-1}}{(2\pi i\varepsilon)^{1/2}} \exp\left\{ i\varepsilon\left[\frac{1}{2}\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)^2 - V\left(\frac{x_n + x_{n-1}}{2}\right)\right]\right\} \times$$

$$\times \frac{1}{(2\pi i\varepsilon)^{1/2}} \int \prod_{k=1}^{n-2} \frac{dx_k}{(2\pi i\varepsilon)^{1/2}} \exp\left\{ i\varepsilon \sum_{j=1}^{n-1} [\ldots] \right\}.$$

$$= \int \frac{dx_{n-1}}{(2\pi i\varepsilon)^{1/2}} \exp\left\{ i\varepsilon\left[\frac{1}{2}\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)^2 - V\left(\frac{x_n + x_{n-1}}{2}\right)\right]\right\} U(q', x_{n-1}, T-\varepsilon)$$

Expand the integrand around $x_{n-1} = x_n = q''$

$$U(q', x_{n-1}, T-\varepsilon) = U(q', q'', T-\varepsilon) + z \frac{\partial}{\partial x} U(q', x, T-\varepsilon)\Big|_{x=q''} +$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial x^2} U(q', x, T-\varepsilon)\Big|_{x=q''} z^2 + \ldots$$

$z = x_{n-1} - q''$, $\quad dz = dx_{n-1}$

$$\langle q'' t'' | q' t' \rangle = e^{-i\varepsilon V(q'')} \int \frac{dz}{(2\pi i\varepsilon)^{1/2}} \exp\left\{ -\frac{1}{2\varepsilon i} z^2 \right\} \Big\{ U(q', q'', T-\varepsilon) +$$

$$+ z \frac{\partial}{\partial x} U(q', x, T-\varepsilon)\Big|_{x=q''} + \frac{1}{2} \frac{\partial^2}{\partial x^2} U(q', x, T-\varepsilon)\Big|_{x=q''} z^2 + \ldots \Big\}$$

Gaussian integral: $\displaystyle \int dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

$$\int dx\, x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \langle q'' t'' | q' t \rangle = e^{-i\varepsilon V(q'')} \frac{1}{(2\pi i\varepsilon)^{1/2}} \left\{ (2\pi i\varepsilon)^{1/2} + (2\pi i\varepsilon)^{1/2} \frac{i\varepsilon}{2} \frac{\partial^2}{\partial x^2} \right\} \times$$

$$\times U(q', x, T-\varepsilon) \Big|$$

$\Big|_{x=q''}$

Expand in ε:

$$i \frac{\partial}{\partial T} U(q', q'', T) = \left[-\frac{\partial^2}{\partial x^2} + V(x) \right] U(q', x, T) \Big|_{x = q''}$$