Universitat Autònoma de Barcelona

Grau's Final Treball in Physics

Quantum entanglement and game theory

Miquel Cerda Ramon

Director: Andreas Winter

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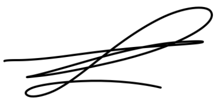

Resume

The objective of this work is to study how we can get advantages within game theory by introducing quantum mechanics. We will talk about the elements of game theory. We are going to present what Bell's non-locality is and quantum entanglement through which we are going to talk about the quantum strategies that we can apply within game theory. Finally we are going to explain a method of building games with quantum advantage.

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1. Introduction

In this work, we will make a brief explanation of the concepts of quantum entanglement, Bell non-locality and game theory in order to bring these concepts together and study quantum strategies and their benefits within incomplete information games.

In 1935 Albert Einstein, Boris Podolsky and Nathan Rosen published an article called ”Can Quantum Mechanical Description of Physical Reality Be Considered Complete?”[1]. In this article what they intend is to demonstrate that the quantum theory was incomplete. In short, they pose the following problem. Alice and Bob share an interlocked state which for example we will say is $|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle$. Where $\uparrow$ $\sqrt{\text{two}}$ corresponds to the spin of the particle up and $\downarrow$ down. As quantum mechanics says, if Alice now makes a measurement to find out where her spin is pointing, by looking at the result, she will immediately know where Bob's particle's spin is pointing. Alice's measuring action instantly influences Bob's particle and apparently this influence would travel faster than light. Einstein disagreed with this phenomenon, which he called "spooky action at a distance." Einstein claimed that there had to be some hidden variables that would explain this phenomenon and that would make this fit with local realism, which he defended.

This article divided the world of physics into two camps, for or against local realism. The theory of local realism is based on two hypotheses. The first is that all objects have a defined state. Against the defenders of the quantum theory who defend that they can be in indefinite states. And second, that the effects of local actions, such as measuring, cannot travel faster than light.

In 1964, John Bell published the influential article called "On the Einstein Podolsky Rosen Paradox" [2]. Bell showed in this article that there were experiments that would allow us to distinguish between quantum mechanics and any other theory that followed local realism. These experiments are based on the fact that local realism leads to requirements for certain phenomena that do not satisfy quantum mechanics. The problem was that the technology to carry out these experiments did not exist. The intention was to end the argument, either the quantum theory was incomplete or the local realism was wrong.

Experimentally, the violation of Bell's inequalities has already been confirmed. Therefore, the concept of local realism has been shown to be wrong. The non-locality present within quantum physics becomes a very powerful tool. Right now it is one of the fields where most research time is spent (within the field of physics).

Bell's inequalities can also be discussed within game theory [3]. Game theory is of great importance in economics, sociology, biology, etc. The games he poses can be used as models of situations that we observe in real life, therefore, his analyzes are of great importance. In games like CHSH or GHZ, we can see that by applying strategies with the help of quantum mechanics, the game's own Bell inequality can be violated and therefore more interesting equilibria can also be reached.

We will structure the work as follows. First of all we are going to introduce the mathematical tools and concepts of quantum mechanics necessary to be able to develop the other points of the work. In section 3, we are going to talk about entanglement and non-locality. In this section we will introduce several games that can be used as Bell tests. In section 4 we are going to introduce the incomplete information games and we are going to talk about the different correlations and kinds of equilibria. Finally, in section 5 we are going to explain what competitive games are and how we can modify collaborative games to turn them into competitive games in which there is a quantum advantage.

2. Mathematical preliminaries and postulates of quantum mechanics

Later we will see games where we apply a quantum correlation that involves a quantum balance. That is why we have to introduce the tools that we will use in these cases. We have based ourselves on the book [4], and on the notes [6].

2.1. mathematical preliminaries

Here we are going to introduce the mathematical objects that we will use the most in this work. We are not going to delve very deeply into each of them, only covering the necessary parts to develop the game theory that is the central part of the work. We will work inside a complex vector space, more specifically in the Hilbert space. Hilbert space is endowed with a linear scalar product in the second component and antilinear in the first. The elements of this vector space are the column vectors.

$$|\psi\rangle = \begin{pmatrix} \psi_{one} \\ \psi_{two} \\ \vdots \\ \psi_n \end{pmatrix} \tag{one}$$

where we will use Dirac notation, in which the column vector is denoted ket ($|\psi\rangle$). Since it is a complex space, $\psi_j \in$ c. $|\psi\rangle$ possesses all the characteristics so that C_n be a vector space. Normally we we will meet in space C_{two} which has the canonical basis:

$$|0\rangle = \begin{pmatrix} one \\ 0 \end{pmatrix}, \ |one\rangle = \begin{pmatrix} 0 \\ one \end{pmatrix} \tag{two}$$

2.1.1. Scalar product

In C_n the scalar product between two vectors is defined $|\psi\rangle$ Y $|\varphi\rangle$ What:

$$(|\psi\rangle, |\varphi\rangle) = (\psi^*_{one}, \psi^*_{two}, \cdots, \psi^*_n) \begin{pmatrix} \varphi_{one} \\ \varphi_{two} \\ \vdots \\ \varphi_n \end{pmatrix} = \sum_{j=1}^{n} \psi_j^* \varphi_j \tag{3}$$

In quantum mechanics, the most common notation when we talk about the scalar product between two vectors is $\langle\psi|\varphi\rangle$. We can check that $\langle\psi|\varphi\rangle = \langle\phi|\psi\rangle^*$

2.1.2. Linear operators

We will use linear operators A, in which A is a function that goes from $C_n \rightarrow C_n$. Yes $\{|v_j\rangle\}$ is a basis:

$$A|v_j\rangle = \sum_{k=1}^{n} A_{kj}|v_k\rangle \tag{4}$$

Therefore, if $|\varphi\rangle = \sum_i \lambda_i |v_i\rangle$:

$$A|\varphi\rangle = \sum_i \lambda_i A|v_i\rangle = \sum_i \lambda_i \sum_{k=1}^{n} A_{ki}|v_k\rangle \tag{5}$$

2.1.3. tensor product

vector space $C_n \otimes C_m$, is the vector space resulting from the tensor product between the spaces C_n Y C_m. The elements of $C_n \otimes C_m$, are linear combinations of $|\psi\rangle \otimes |\varphi\rangle$, where $|\psi\rangle \in C_n$ Y $|\varphi\rangle \in C_m$. Yes $\{|v_i\rangle\}$ is base of C_n Y $\{|or_i\rangle\}$ is base of C_m, the base of space $C_n \otimes C_m$ will be:

$$\{|v_i\rangle \otimes |or_j\rangle; \quad i = one, ..., n; \quad j = one, ..., m\} \tag{6}$$

We take this opportunity to say that we will change the notation later. For simplicity, we abbreviate from $|\varphi\rangle \otimes |\psi\rangle$ a $|\varphi\rangle |\psi\rangle$.

2.2. Quantum mechanical postulates

2.2.1. Postulate 1

For every physical system there is a Hilbert space H associated, called the state space. The system is fully described by a positive trace operator equal to 1 (Tr ρ=one). If the system is prepared with probability P_i in the state ρ_i, so:

$$\rho = \sum_i P_i \rho_i \tag{7}$$

Pure state is called the state that is p∑ you can write in the form ∑ma $\rho = |\psi\rangle\langle\psi|$. The mixed states can diagonalize and define in the form $\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i P_i = 1$. Therefore we can write the mixed states of the form (7).

2.2.2. Postulate 2

The M Quantum measurements are characterized by a set $\{M_j\}$ of positive semidefinite matrices, which satisfy $\sum_j M_j = I$ The subscript j refers to the possible results of the measurement.

The probability of observing an outcome is determined by the call *born's rule*:

$$P(j|\rho) = \mathrm{Tr}\,\rho M_j \tag{8}$$

It is important to note that if we have a mixed state, when we make a measurement and observe the result of it, the mixed state will collapse to a pure state.

2.2.3. Postulate 3

The time evolution of an isolated quantum system is unitary. If the state in time *you* it is ρ and in time *you* it is ρ', there is a unitary *OR*, which depends on *you′−you*, such that:

$$\rho' = U\rho U^{\dagger} \tag{9}$$

2.2.4. Postulate 4

The state space of a composite system is the tensor product of the state spaces of each subsystem. If the subsystem i is ready in the state ρ_i, the total system will be in the state:

$$\rho = \rho_{\text{one}} \otimes \rho_{\text{two}} \otimes \cdots \otimes \rho_n \tag{10}$$

We can expand the *born's rule* for composite systems. The Hilbert space of the composite system will be $H = H_{\text{one}} \otimes \cdots \otimes H_n$. The joint probability that the player i observe the result j will be:

$$P(j_{\text{one}}, \ldots, j_n|\rho) = \mathrm{Tr}\,\rho(M^{\text{one}}_{j_{\text{one}}} \otimes \cdots \otimes M^{n}_{j_n}) \tag{eleven}$$

If we are given the state of a composite system, one can always determine the state of its subsystems. For simplicity let us assume a bipartite system, ρ_{AB}. The state of the subsystem belonging to Alice will be:

$$\rho_A = \mathrm{TR}_{B.}(\rho_{AB}) \tag{12}$$

Where $\mathrm{Tr}_{B.}$ is the partial trace over Bob's subsystem.

We are going to see another point in the case of a two-party system. Consider a system in a mixed state ρ. This state is∑ e can be seen as the partial state of a composite system and the composite system can be pure. Yes $\rho = \sum_i P_i|\psi_i\rangle\langle\psi_i|$, $\rho = \mathrm{Tr}_{B.}(|\psi\rangle\langle\psi|)$ with

$$|\psi\rangle = \sum_i \sqrt{P_i}|\psi_i\rangle \otimes |\beta_i\rangle$$

Where $\{|\beta_i\rangle\}$ is an orthonormal basis of the Hilbert space corresponding to Bob.

3. Intertwining and non-locality

3.1. interlacing

The concept of quantum entanglement has played a very special role in the development of quantum physics. For many years it was one of the main topics of discussion since the consequences of this quantum correlation, for many physicists, were not acceptable. In this section, we have been informed by the articles dealing with Bell non-locality [3] and [12], in which this discussion is discussed. We have also consulted [4] and [6]. For more depth on the subject, I recommend consulting [5].

In 1935, Einstein, with the collaboration of Podolsky and Rosen, published the article known as EPR [1], where the EPR paradox is stated. With this paradox, Einstein intended to strike a blow at quantum theory, of which he himself had been a participant in the development. With this paradox he accused quantum theory of being incomplete. Einstein explained that if two people shared a state that was entangled, just as quantum theory says, a local measurement in one part of the state would instantly affect the other part. Einstein based himself on this phenomenon to affirm that entanglement violated the concept of local realism, of which he himself was an advocate, and that therefore quantum theory was incomplete.

In 1964 John Bell raised what are called Bell's inequalities [2]. Local realism applied to certain phenomena led to these inequalities, and therefore, if experimentally, quantum entanglement violated these inequalities, local realism and Einstein's argument would be shown to be wrong. It took many years for the experiments to be carried out, but finally, quantum entanglement was shown to violate these inequalities and thus local realism was ruled out.

Quantum entanglement is a phenomenon, without classical equivalent, in which the quantum states of two or more objects must be described by a single state that involves all the objects in the system. That is, a set of entangled particles cannot be defined as individual particles with states defined. For example the state $|\psi\rangle = \frac{|0\rangle|one\rangle - |one\rangle|0\rangle}{\sqrt{two}}$ cannot be written in the form $|\psi\rangle = |\varphi\rangle|\phi\rangle$.

Demonstration

Be $|\varphi\rangle = \varphi_0|0\rangle + \varphi_{one}|one\rangle$ Y $|\phi\rangle = \phi_0|0\rangle + \phi_{one}|one\rangle$, Thus,

$$|\varphi\rangle|\phi\rangle = \varphi_0\phi_0|0\rangle|0\rangle + \varphi_0\phi_{one}|0\rangle|one\rangle + \varphi_{one}\phi_0|one\rangle|0\rangle + \varphi_{one}\phi_{one}|one\rangle|one\rangle \tag{13}$$

We want this state to match $|\psi\rangle = \frac{|0\rangle|one\rangle - |one\rangle|0\rangle}{\sqrt{two}}$

$$\varphi_0\phi_0=0, \qquad \varphi_0\phi_{one}=one/\sqrt{two}, \quad \varphi_{one}\phi_0=-one/\sqrt{two}, \qquad \varphi_{one}\phi_{one}=0$$

We clearly see that this system of equations has no solution. Therefore it is shown that the state $|\psi\rangle = \frac{|0\rangle|one\rangle - |one\rangle|0\rangle}{\sqrt{two}}$ cannot be written in the form $|\psi\rangle = |\varphi\rangle|\phi\rangle$ and we can assure that it is an interlocked state.

A pure state of a system composed of a $\bigotimes_{n=1}^{N} H_n$ it is called interlaced if it cannot be written as product:

$$|\psi\rangle \neq \bigotimes_{n=1}^{N} |\psi_n\rangle \tag{14}$$

A mixed state of a composite system is called entangled if there is no convex decomposition over the product states:

$$\rho \neq \sum_m p_m \left(\bigotimes_{n=1}^{N} \rho_{n,m} \right) \qquad \text{(fifteen)}$$

Quantum entanglement is the key in developing technologies such as quantum computing, quantum cryptography, in communication, etc., and therefore it is a very active research area.

3.2. not locality

In this section we build on the introduction to the book [3] to explain the concept and history of Bell's non-locality. For more depth on the subject, you can consult the article [12] or the same book [3].

One of the most famous statements in science is the one that has accompanied the quantum since the beginning: *There is indeterminacy in nature*. The beginnings of quantum theory did not leave any physicist of the time indifferent. There was a lot of discussion about the theory. Einstein was one of the biggest detractors of quantum theory. He thought that a theory without determinism and based on statistical laws could only be a temporary fix.

Even today, if you are asked for evidence of indeterminacy, many will answer with Heisenberg's uncertainty principle. This principle only works within the quantum field, not nature in general. In 1964 John Bell showed that there is indeterminacy in nature by observing the phenomenon called Bell's non-locality.

To demonstrate Bell's nonlocality, experiments specifically designed for this task had to be done. The work of Alain Aspect and collaborators in 1982 was the first proof of Bell's non-locality.

Let's see what are the three most important roles of Bell's non-locality. The first is to evidence indeterminism. The second is that Bell's non-locality is used in the creation of quantum devices since it provides the most convincing certificates. These devices are intended for the areas of quantum computing and cryptography. This field is in continuous development and is one of the areas in which more work and research is being done lately. Finally, it is the contribution of Bell nonlocality with new ideas for new physical theories.

Bell nonlocality tests are usually experiments in laboratories. To make it easier to understand the phenomenon of non-locality, examples of games like television contests are usually given. These types of examples, more familiar, help to better understand the concept of Bell's non-locality.

In these games, the players, named alphabetically Alice, Bob, Charlie, etc., are on the same team. In each round of the game each player is asked a question (input) and each of them has to give an answer (output). Players know the rules and the list of possible questions in advance. Before starting, the players can decide what strategy they are going to follow to give the answers in each round. Players can use different resources to get their answers and to coordinate them with their partners. For example, if the players can communicate with each other during the round by, for example, a telephone, they could agree with each other and easily win the game. The telephone resource would therefore be a very powerful resource and is called a signaling resource. Non-signaling resources are more interesting. An example of a non-communication resource would be for example that each player had a list of answers for each question. As we can see if Alice, for example, did something to her list, the other players would not notice. Not being able to send any messages to the other players by manipulating his roster makes him a non-communication resource. We are going to propose three types of games and strategies that players can follow in each of them: An example of a non-communication resource would be for example that each player had a list of answers for each question. As we can see if Alice, for example, did something to her list, the other players would not notice. Not being able to send any messages to the other players by manipulating his roster makes him a non-communication resource. We are going to propose three types of games and strategies that players can follow in each of them: An example of a non-communication resource would be for example that each player had a list of answers for each question. As we can see if Alice, for example, did something to her list, the other players would not notice. Not being able to send any messages to the other players by manipulating his roster makes him a non-communication resource. We are going to propose three types of games and strategies that players can follow in each of them:

(i) The first is a game with the rule that players have to give the same answer if they are asked the same question. This game is easily won by agreeing in advance to always give the same answer.

(ii) Second, the players have to give the same answer only if they receive the same question. If they get different answers, they have to answer differently. This game is won by previously agreeing on a list of answers to each question.

(iii) The third game for example may be that they have to give different answers if they are asked the question number one, but the same answer if they ask another question that is not one. This game cannot be won using a predetermined strategy.

We refer to *bell town* to the processes by which the player i generates its response without player j's input being important in this process, with $i \neq j$. If we call λ the strategy they follow, the probability that the player i give an answer (output) "a" To the question x it is $P_\lambda(a/x)$. For two players it would be as follows.

$$P(a, b/x, y) = \int d\lambda Q(\lambda) P_\lambda(a/x) P_\lambda(b/y) \tag{16}$$

Where $Q(\lambda)$ is the probability distribution that defines the strategy. The statistics will be *local* if they can be written in the above way. will be *not local* if they cannot be written in the above way. A *Bell's test* is a game in which the strategy follows non-local statistics.

We could think that the non-local resources are the communication devices. Quantum obliges us to extend this definition. Let's look at the following example. If two players share a physical system that we quantumly describe as ρ_{AB}. We assume that the strategy that gives the outputs to the players is make local measurements to the shared state. Assuming that Π_x^a is the positive operator corresponding to an input x and an output a for Alice, equivalently for Bob. We will have the following statistics:

$$P(a, b/x, y) = TR(\Pi_x^a \otimes \Pi_Y^b \rho_{AB}) \tag{17}$$

In general, these statistics cannot be written in the form (16). So we conclude that some shared quantum states are non-local resources.

3.3. Bell tests

3.3.1. Clauser-Horne-Shimony-Holt (CHSH)

The CHSH game consists of the following: We have two players. The question asked to the players can be 0 or 1, x *and* $y \in \{0,\text{one}\}$ and the answers they can give $a, b \in \{0,\text{one}\}$. The distribution of the questions is uniform and therefore with probability $\frac{\text{one}}{4}$ each pair of questions. When the players give their answers, if they meet the conditions, the players are given a payment and we will say that they have won. We will see the conditions to win and the payments in the following tables:

	0	one
0	1.1	0.0
one	0.0	1.1

Table 1: You pay in the event that xy=0

	0	one
0	0.0	1.1
one	1.1	0.0

Table 2: You pay in the event that xy=one

As we see in the tables, the players win by giving the same answer when the questions are (x *and* y) = (0,0), (0, eleven,0), and giving different answers when (x *and* y) = (1,one).

To check that this game is a Bell test, we need to see what is the probability that the players have of winning the game with a local resource. Local resource is the one that is used in a local strategy that will carry the local statistics that we have seen before.

In this case, a local resource may be to agree on a response before the game begins. Whether the two players agree to answer 0 or 1, you get the maximum chance of winning the game with local resources. The maximum probability they have of winning by agreeing on the same answer is equal to $\frac{3}{4}$.

$$P_{rl}(\text{gain}) \leq \frac{3}{4} \tag{18}$$

Where P_{rl}(win) is the probability of winning using a local resource.

Therefore, if we see that they win with a probability greater than 3/4, we can ensure that they are using non-local resources. The game can be won with probability 1 when both players use non-local resources. With non-local resources you can win with probability 1 with what are called PR-*box* that exist in mathematics but do not exist in nature. With quantum entanglement, this limit can also be overcome by making measurements to a shared entangled state between the players. The interleaved state in question is:

$$|\phi\rangle = \frac{\text{one}}{\sqrt{\text{two}}}(|0\rangle|0\rangle + |\text{one}\rangle|\text{one}\rangle) \tag{19}$$

With this state, Alice has to make the X and Z measurements for questions 0 and 1, respectively. Bob has to make the measurements of $\frac{X+Z}{\sqrt{\text{two}}}$ $\frac{X-Z}{\sqrt{\text{two}}}$ for questions 0 and 1 respectively. Where X corresponds to the Pauli matrix σ_x and Z to the σ_z. By following this strategy, it can be shown that they manage to win the game with probability $cos^{\text{two}}(\pi/8)$ ~0,85.

3.3.2. Greenberger-Horne-Zeilinger

A nonlocality test done by Greenberger, Horne, and Zeilinger in 1989. We consider the three-player game. We will discuss this game in detail later. The three players can receive two types of questions, X *and* $Z \in \{0,\text{one}\}$. The questions can be the following, (X *and* Z) = (1,one,one), (one,0,0), (0,one,0 and (0,0,1), uniformly distributed, that is, with probability $\frac{\text{one}}{4}$ each. The responses they can be, $a\ b\ c \in \{0,\text{one}\}$. The answers they have to give to win are, A=0 module 2, for question (1,1,1) and A=1 modulo 2 for questions (1,0,0), (0,1,0) and (0,0,1), where $A=a+b+c$. Suppose that whenever they win, each player receives a payoff equal to 1. If they don't win, the payoff is 0.

As in the case of the CHSH game, we are going to find out what is the maximum probability of winning with a local strategy. As before, they can agree on a response or they can even agree on an order of response. They could agree, for example, to give the answers (1,1,1), (1,0,0), (0,1,0) and (0,0,1) cyclically. Both with this strategy, and by fixing one of those 4 possible answers, they will win the game with probability 3/4. This probability is equivalent to the maximum that can be reached with a local strategy.

As we have already said, if we see that they win the game with a probability greater than 3/4, we can affirm that they are using a non-local resource.

Using a quantum strategy, the GHZ game can be won with probability 1. The quantum strategy is that the players share an entangled state. With this state, depending on the strategy to follow, they will make a series of measures. In the case of the GHZ game the interlaced state is:

$$|GHZ\rangle = \frac{\text{one}}{\sqrt{\text{two}}}(|0\rangle|0\rangle|0\rangle + |\text{one}\rangle|\text{one}\rangle|\text{one}\rangle) \tag{twenty}$$

The strategy to follow is the following:

- If you have received type 0, you measure $Y = \sigma_Y$
- If you have received type 1, you measure $X = \sigma_X$

After making these measurements, the players will have the result of one of the two eigenvalues of the operator in question (+1 or -1). With this eigenvalue, doing the calculation of:

$$(-\text{one})^{a_i} = r_i \tag{twenty-one}$$

We will be able to discover what action we should take. Where r is the result of the measurement. Let's see how the course of a round would be:

Since we know that the inputs are uniformly distributed, let us assume that the question (1,1,1) has come up. The players already know what strategy to follow, everyone will do the operator's measure X. We know that immediately after making the measurement, the state changes. Let's see the possible states in which it can result. A priori they are the following: $|+x\rangle|+x\rangle|+x\rangle$, $|+x\rangle|+x\rangle|-x\rangle$, $|+x\rangle|-x\rangle|+x\rangle$, $|-x\rangle|+x\rangle|+x\rangle$, $|+x\rangle|-x\rangle|-x\rangle$, $|-x\rangle|+x\rangle|-x\rangle$, $|-x\rangle|-x\rangle|+x\rangle$ Y $|-x\rangle|-x\rangle|-x\rangle$.

To calculate the probability that each state has to leave we will do the calculation:

$$P(|r_{one}\rangle|r_{two}\rangle|r_3\rangle) = |\langle r_{one}|\langle r_{two}|\langle r_3|GHZ\rangle|^{two} \tag{22}$$

If we do the calculations we will see that there are two groups:

- With probability 0, the states: $|-x\rangle|-x\rangle|-x\rangle$, $|+x\rangle|+x\rangle|-x\rangle$, $|+x\rangle|-x\rangle|+x\rangle$ Y $|-x\rangle|+x\rangle|+x\rangle$.
- with probability $\frac{one}{4}$, the states: $|+x\rangle|+x\rangle|+x\rangle$, $|+x\rangle|-x\rangle|-x\rangle$, $|-x\rangle|+x\rangle|-x\rangle$ Y $|-x\rangle|-x\rangle|+x\rangle$.

Suppose you have exited the state $|+x\rangle|-x\rangle|-x\rangle$. Now the players have to do the calculation to know what action (response) they have to take. Alice does the following calculation:

$$(-one)^{a_i} = +one \tag{23}$$

We see that Alice has to take action $a_{one}=0$.

Bob and Charlie have to do the same calculation:

$$(-one)^{a_i} = -one \tag{24}$$

We see how Bob and Charlie have to do the actions $a_{two}=a_3=$one.

The final combination is (a_{one}, a_{two}, a_3) = (0,one,1), we see that this group of actions does win the game. If we make the respective calculations of each state resulting from the measurement, we will see that all the actions (a_{one}, a_{two}, a_3) calculated, will guarantee us to win the game. Similarly we will reach the same conclusion with the other inputs (questions). Therefore, we see that the quantum strategy wins the game with probability 1.

3.3.3. Mermin-Peres, magic square game

There are two players who are separated at the beginning of the game and have no way of communicating with each other. The game consists of filling a 3x3 matrix with the values 1 and -1. Alice fills in the rows and Bob fills in the columns. Alice has to get all three numbers in each row to have a negative product, and Bob has to get all three numbers in his columns to have a negative product. Each round they are assigned a random row and column. Therefore, they win if the number they share is the same.

If, for example, Alice is assigned row 1 and Bob is assigned column 2, they will win if they agree on the number m_{12}.

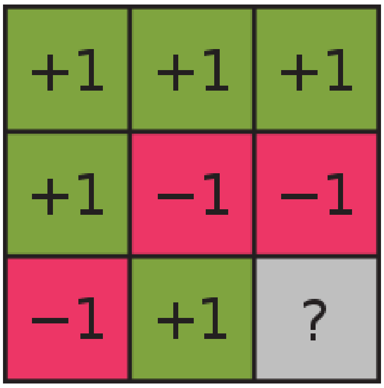

Figure 1: Example of how the matrix can look.

One strategy would be for the players to have an agreed matrix before starting the game. This would give them a chance of winning the game 8 times out of 9 since there is one square that it is impossible for the two players to match if they want to satisfy their two individual win conditions.

The two players could win with probability 1 if they could communicate after knowing which column and row they have been assigned. There is another way to be able to win this game with probability 1 and without communicating, using what is known as pseudo quantum telepathy. They do this by sharing two pairs of particles with entangled states. Knowing which row or column they have been assigned, they use that information to select the measurement they have to make to their particles.

$I \otimes yes_z$	$yes_z \otimes I$	$yes_z \otimes yes_z$
$yes_x \otimes I$	$I \otimes yes_x$	$yes_x \otimes yes_x$
$-S_x \otimes yes_z$	$-S_z \otimes yes_x$	$yes_Y \otimes yes_Y$

Table 3: Measurements they have to make to their particles for each cell.

The state they share is the following:

$$|\phi\rangle = \frac{\text{one}}{\sqrt{\text{two}}}(|+\rangle_a \otimes |+\rangle_b + |-\rangle_a \otimes |-\rangle_b) \otimes \frac{\text{one}}{\sqrt{\text{two}}}(|+\rangle_c \otimes |+\rangle_d + |-\rangle_c \otimes |-\rangle_d) \tag{25}$$

Where $|+\rangle$ Y $|-\rangle$ correspond to the eigenvectors of the Pauli matrix σ_x of values +1 and -1 respectively-mind.

$$|+\rangle = \frac{\text{one}}{\sqrt{\text{two}}}\begin{pmatrix}\text{one}\\ \text{one}\end{pmatrix} \quad |-\rangle = \frac{\text{one}}{\sqrt{\text{two}}}\begin{pmatrix}\text{one}\\ -\text{one}\end{pmatrix} \tag{26}$$

4. Games of incomplete information

In this section, we are going to explain the correlations and equilibria that we can find within game theory based on the article [7]. We are not going to explain all the characteristics of these equilibria and correlations, therefore, for more information, I recommend consulting the article [7].

In this work we want to study quantum entanglement, and how, thanks to it, there are games in which when a quantum strategy is applied, an advantage is obtained over classical strategies. We are going to focus on games with incomplete information. To do this, we are going to review game theory, from its origin, to the explanation of different types of games.

A key point was the definition of Nash equilibrium. Nash showed that in any game of incomplete information, there is an equilibrium. From then on, the subject was investigated much more and the characteristics and properties of these equilibria were studied. Motivated by all this, one of the key points was how to motivate or help players to reach a balance that would benefit them. This in game theory is called advice. To continue we are going to explain how an incomplete information game works. In a game of incomplete information or also called Bayesian games, each player is given what we call a type. This type can be anything. It can be, for example, personal characteristics of each player (intuitive, determined, rich, poor, etc.) or it could also be the secret interests of each player, for example, interest in investing in certain types of companies. In such a game a solution may be a communication equilibrium. In a communication equilibrium, the players communicate their type to a trusted mediator who, through a correlation, advises them to take an action. The communication balance is not always the best option since players are not always willing to reveal their type to a mediator, since as we have said, the type information is often sensitive. the players communicate their type to a trusted mediator who, by means of a correlation, advises them to take an action. The communication balance is not always the best option since players are not always willing to reveal their type to a mediator, since as we have said, the type information is often sensitive. the players communicate their type to a trusted mediator who, by means of a correlation, advises them to take an action. The communication balance is not always the best option since players are not always willing to reveal their type to a mediator, since as we have said, the type information is often sensitive.

An incomplete information set can be defined with the following objects:

- A finite set of players we call N, where $N = [n]$;
- A finite set of player profiles we call *type* $T := T_{\text{one}} \times \cdots \times T_n$;
- A finite set of actions $A := A_{\text{one}} \times \cdots \times A_n$;
- A probability distribution of the types $P(you)$.
- For each player, a reward function $v_i : T \times A \to$ − R

As we have already said, the game consists of the following:
The types are distributed according to the probability distribution P. Each player learns his type and carries out the strategy g_i, where $g_i(a_i | t_i)$ is a probability distribution of a_i about the guys you_i. this strategy g_i will advise you what action to take. A pure strategy is a map that leads from T_i a A_i, this means that the player decides an action based only on its type information. A mixed strategy is a probability distribution over pure strategies, therefore, the function $g_i : T_i \to A_i$ becomes a random variable. To make this distribution explicit, we introduce the independent and local random variables λ_i, with probability $\Lambda_i(\lambda_i)$.

Once an action is taken, the player gets paid according to his reward function. v_i. This reward function usually also depends on the other players, both their actions and their types. The average profit of the player i it is:

$$\langle v_i \rangle = \text{AND}_{\text{you,g}} v_i(\text{t}, g_{\text{one}}(you_{\text{one}}), \ldots, g_n(you_n)) = \sum_{\text{you,a}} P(\text{t}) v_i(\text{t,to}) \prod_{i=1}^{n} g_i(a_i | t_i) \tag{27}$$

Where t $= (you_{\text{one}}, \ldots, t_n)$, equal forgYa.A game solution is a set of strategies g $= (g_{\text{one}}, \ldots, g_n)$. We can say that the players look out for their own interest, since if we fix all the strategies except the player's i, he will choose the strategy that maximizes his average profit. The solution will be a balance

(Nash equilibrium) if neither player has an incentive to change strategy. Mathematically it can be explained as follows for any i, you_i, Y a_i:

$$\sum_{you_{-i},\lambda} P(t)\Lambda(\lambda) v_i(t, g_{-i}(t_{-i},\lambda_{-i}) g_i(you_i, \lambda_i)) \geq \sum_{you_{-i},\lambda} P(t)\Lambda(\lambda) v_i(t, g_{-i}(t_{-i},\lambda_{-i}) a_i) \tag{28}$$

This inequality is summarized in that, if the strategy is an equilibrium, any action a_i can only match the player's expected profit i if it coincides with the action recommended by the strategy g_i. Lastly, the expected social benefit is the sum of the expected benefits of each player. $SW(g) = \sum_i \langle v_i \rangle$.

4.1. Correlations

To define the different types of equilibria, we have to first define the correlations. As we have seen in the previous section, to have a well-defined expected benefit we need to have defined the conditional probability of the action regarding the type t.

For this purpose, we will define the correlation as a joint conditional probability distribution:

$$Q(s_{one}, \ldots, yes_n / r_{one}, \ldots r_n) \geq 0 \; \forall s_i \in yes_i,\; r_j \in R_j. \tag{29}$$

Where r_i are inputs and s_i are the outputs. Obviously you have to respect the following condition:

$$\sum_s Q(s/r) = one \; \forall r \in R \tag{30}$$

4.1.1. no communication

Also known in English as belief invariant. Q is belief invariant if the distribution of outputs s_i given the inputs r_i, does not give any additional information on r_j. This would not be the case of a correlation where for example s_i be equal to r_j. Formally it is expressed as follows. for a set $I \subset N, J = NEITHER, R_I = \times_{i\in I} R_i, yes_I = \times_{i\in I} yes_i$:

$$\sum_{s_J \in yes_J} Q(s_I, s_J / r_I, r_J) = \sum_{s_J \in yes_J} Q(s_I, s_J / r_I, r'_J) \; \forall s_I \in yes_I,\; r_I \in R_I,\; r_J, r'_J \in R_J \tag{31}$$

Later we will see examples of belief invariant correlations.

4.1.2. local

Q is local, if a player i looking at its part of a random variable (independent of r) $y = (y_{one}, \ldots, y_n)$, with distribution $P(y)$, you can simulate the mapping locally by only doing operations that depend on r_i Y y_i. Formally:

$$Q(s/r) = \sum_y V(y) L_{one}(s_{one} / r_{one} y_{one}) \cdots L_n(s_n / r_n y_n) \tag{32}$$

Any local correlation is also belief invariant, since the condition of equation (24) is fulfilled.

4.1.3. quantum

In the case of quantum strategies, the strategies consist of a set of local measures that we have to do with an interlaced state. The local measurements are $M_{you_i} = (M_{you_i}{}^{a_i} ; a_i \in A_i)$, for the game i with type you_i, and the state, ρ. Now the correlation $Q(a, t)$ we can write it as follows.

$$Q(a, t) = Tr \rho (M_{you_{one}}^{a_{one}} \otimes \cdots \otimes M_{you_n}^{a_n}) \tag{33}$$

Finally I want to comment that all local correlations can be quantum correlations but not all quantum correlations can be belief invariant correlations.

$$\text{Locals}(G) \subset \text{Quantum}(G) \subset \text{BI}(G) \tag{3. 4}$$

4.2. balance classes

Finally, we are going to talk about the different kinds of equilibria. We are going to talk about the characteristics, their advantages and disadvantages. At the end of this section we will see an example where we will see the repercussions of some of these equilibria.

We will study the equilibria with communication and with access to a correlation device that works with the inputs given by each player through a private channel of each player with the device. The inputs that each player gives them obviously depends on the type they get at the start of the game. Therefore, the player i, through a function F_i you can get the input you give to the device. $F_i: T_i \rightarrow R_i$. After each player submits their input r_i, the device gives them the output s_i, with each player will get the action that the device recommends them to do through a function $g_i: T_i \times yes_i \rightarrow A_i$.

For a mixed strategy, the player's expected payoff i it is:

$$\langle v_i \rangle = \sum_{t,s,\lambda} P(t)\Lambda(\lambda)Q(s\,/\,f_{one}(you_{one}, \lambda_{one}), \ldots, f_n(you_n, \lambda_n))v_i(t, g_{one}(you_{one}, yes_{one}, \lambda_{one}), \ldots, g_n(you_n, yes_n, \lambda_n)) \quad (35)$$

4.2.1. communication balance

It is the most general balance. Here the correlation Q does not have any restrictions. To define the balance well we have to define the correlation Q of the correlation device and the strategy followed by the players, that is, the functions $\{F_i\}$ Y $\{g_i\}$.

If we have $x = (x_{one}, \ldots, x_n)$, $x_{-i} = (x_{one}, \ldots, x_{i-one}, x_{i+1}, \ldots, x_n)$. With this, we define that (F,g,Q) is an equilibrium of the game G, yes for each player i and for all functions $\varphi: T_i \rightarrow R_i$ Y $\chi: yes_i \rightarrow A_i$,

$$\sum_{you_{-i},s,\lambda} P(t_{-i}/t_i)\Lambda(\lambda)Q(s\,/\,f_i(you_i, \lambda_i)F_{-i}(t_{-i},\lambda_{-i}))v_i(t, g_i(you_i, yes_i, \lambda_i)g_{-i}(t_{-i},s_{-i},\lambda_{-i}))$$
$$\geq \sum_{you_{-i},s,\lambda} P(t_{-i}/t_i)\Lambda(\lambda)Q(s\,/\,\varphi_i(you_i, \lambda_i)F_{-i}(t_{-i},\lambda_{-i}))v_i(t, \chi(you_i, yes_i, \lambda_i)g_{-i}(t_{-i},s_{-i},\lambda_{-i}))$$

Here we can see the main concept of Nash equilibrium, where players have no incentive to change strategy.

4.2.2. Balance belief invariant

We obtain a belief invariant equilibrium if the correlation we use is belief invariant, as defined in the correlations section (4.1.1).

4.2.3. correlated equilibrium

Set (F,g,Q) is a correlated equilibrium if the distribution of the outputs of Q is independent of the inputs. $Q(s\,/r) = Q(s)$.

The set (g,Q) is a correlated equilibrium if and only if for any i, you_i, s_i Y a_i:

$$\sum_{you_{-i},s_{-i}} P(t_{-i}/t_i)Q(s)v_i(t,g(t,y)) \geq \sum_{you_{-i},s_{-i}} P(t_{-i}/t_i)Q(s)v_i(t, a_i g_{-i}(t_{-i},s_{-i})) \quad (36)$$

4.2.4. quantum balance

As we know, the solution of a game is the set (f,g,Q). In the case of quantum equilibrium, the solution consists of a set of local measures, $M_{you_i} = (M_{you_i}^{a_i}; a_i \in A_i)$, and the state ρ. It will be a quantum balance if the correlation we use is quantum, defined as (26).

We can define player payouts as:

$$\langle v_i \rangle = \sum_{you_{-i},a} P(you_{-i}|t_i)) Tr\rho(M_{you_{one}a_{one}} \otimes \cdots \otimes M_{you_n a_n}) v_i(you, a) \tag{37}$$

Using quantum correlations, the mediator does not need to know the type of the players. The mediator only has to send the players their fair share of the state ρ and advice on making measurements $\{M_{you_i} : you_i \in T_i\}$. Therefore the quantum solution will be (M_{you}, ρ). The quantum solution (M_{you}, ρ) is an equilibrium if and only if:

$$\begin{aligned}\langle v_i \rangle = & \sum_{you_{-i},a} P(you_{-i}|t_i)) Tr\rho(M_{you_{one}a_{one}} \otimes \cdots \otimes M_{you_n a_n}) v_i(you, a) \\ \geq & \sum_{you_{-i},a} P(you_{-i}|t_i)) Tr\rho(M_{you_{one}a_{one}} \otimes \cdots \otimes M_{you_{i-one} a_{i-one}} \otimes N_{you_i a_i} \otimes M_{you_{i+1} a_{i+1}} \otimes \cdots \otimes M_{you_n a_n}) v_i(you)\end{aligned} \tag{38}$$

for any player i, you_i Y $N_{you_i} = (N_{you_i a_i} : a_i \in A_i)$.

4.3. CHSH example

We are going to see the case of the CHSH game and see the belief invariant, correlated and quantum equilibria, with their respective social welfares (social benefits). As we have already seen in section (3.3.1), the CHSH game consists of the following: We have two players. The type of the players can be 0 or 1, and the actions they can take a_i they can also be either 0 or 1. The distribution of the types is uniform and therefore with probability $\frac{one}{4}$ each. This game is a collaborative game in the sense that the payouts are the same for the two players. Payments are distributed according to tables 1 and 2 of section (3.3.1). If they collaborate, both win the same, but if they don't, both lose.

We first consider the belief invariant equilibrium. With the following correlation, $Q(00/t_{one}you_{two}=0) = Q(\text{eleven}/t_{one}you_{two}=0) = \frac{one}{two}$, $Q(01/t_{one}you_{two}=1) = Q(10/t_{one}you_{two}=1) = \frac{one}{two}$. The two players communicate their type to the mediator (device of correlation) and this recommends them to take action 00 or 11 randomly in the event that $you_{one}you_{two}$ =0. Same for the case $you_{one}you_{two}$=1 that recommends actions 10 i 01 randomly, where 10 means that the first player must take action 1 and the second action 0. We see that each player has the probability of doing the action $a_i \in A_i$ with probability $\frac{one}{two}$ without the type of the other player influencing. For Therefore, no extra information can be gained after the mediator's recommendation. We also see that the probability of winning the game is 1, so they always get paid, and since the payout is always 1, each player gets an expected profit v_i=1 and therefore, an expected social benefit (social welfare) SW=one.

The correlated equilibrium is one in which a correlation is used where the outputs are independent of the inputs. The belief invariant equilibrium that we have just seen would not be correlated since it does not comply with this rule. Since when the device recommends, for example, the action 00 or 11, it is because the product of the types is equal to 0, therefore there is a relationship between inputs and outputs. A correlated equilibrium for the GHSH game would be using the correlation $Q(00) = Q(11) = \frac{one}{two}$. We see how the actions recommended the device are independent of the inputs. This equilibrium wins the game with a probability of $\frac{3}{4}$ what corresponds to the probability that $you_{one}you_{two}$=0. We see that the players are also not interested in ignoring the recommendation since it is impossible to have a greater benefit. With this correlation the expected social benefit will be $SW = \frac{3}{4}$.

As we have seen in section (3.3.1), there is a quantum equilibrium that wins the game with probability $cos^{two}(\frac{\pi}{8})$. To this equilibrium there corresponds an expected social benefit equal to $SW = cos^{two}(\frac{\pi}{8}) \sim 0{,}85$. We see as the SW of this equilibrium is less than that of the belief invariant equilibrium, but a point in favor of this strategy is that we have not had the need to reveal our type to the mediator.

5. Competitive games

This section is based on what has been learned in articles [7], [8], [9], [10] and [11].

Competitive games, also known as non-cooperative games or conflict of interest games, can be variations of collaborative games. The difference between competitive and collaborative games is that players have preference when choosing which move to make.

In a competitive game, there may be several Nash equilibria. They are Nash equilibria since even if the player receives advice to take a less appetizing action, in the sense of benefits, the player cannot increase his benefits by not following the strategy (advice).

Let's look at the case of the CHSH game. We have seen this collaborative game before, now we modify it to see how strategies and payoffs would change if there is a conflict of interest.

	0	one
0	one, $\frac{\text{one}}{\text{two}}$	0.0
one	0.0	$\frac{\text{one}}{\text{two}}$, one

Table 4: You pay in the event that $you_{\text{one}} you_{\text{two}}$=0

	0	one
0	0.0	one, $\frac{\text{one}}{\text{two}}$
one	$\frac{\text{one}}{\text{two}}$, one	0.0

Table 5: You pay in the event that $you_{\text{one}} you_{\text{two}}$=one

We see how in the case that $you_{\text{one}} you_{\text{two}}$=0, Alice prefers to do action 0, and Bob prefers action 1. We have a similar conflict in the case $you_{\text{one}} you_{\text{two}}$=1, in this case both prefer action 0.

If there is no communication, the pure strategies (0,0) and (1,1) lead to unequal benefits. If (0,0), Alice would have an average profit equal to $\frac{3}{4}$, instead, Bob would have an average profit equal to $\frac{3}{8}$. The same thing happens in the case of the pure strategy (1,1), which this time in favor of Bob.

Introducing communication into the game, using correlation $Q(00) = Q(11) = \frac{\text{one}}{\text{two}}$, you can get to a fair balance for both. In this case the two players would have equivalent expected payoffs a $\frac{9}{16}$ ~0,56. The expected social benefit of this last strategy is the same as that of the strategies with unequal benefits, the difference is that in this the two players would have the same benefits.

Finally, let us analyze the case of a belief invariant equilibrium. It is in which the following mapping is used.

$$\text{yes } you_{\text{one}} you_{\text{two}}=0, \qquad Q(00) = Q(11) = \frac{\text{one}}{\text{two}}$$

$$\text{yes } you_{\text{one}} you_{\text{two}}=\text{one}, \qquad Q(01) = Q(10) = \frac{\text{one}}{\text{two}}$$

The two players would have an expected payoff of $\frac{3}{4}$ and therefore an expected social benefit $SW = \frac{3}{4}$.

5.1. GHZ Modification

We are going to see the case of the GHZ game for n=3. We have already studied the collaborative game in section (3.3.2). We are going to study the belief invariant equilibrium before moving on to the conflict of interest game.

With communication, there are belief invariant equilibria that win the game with probability 1. Let's see the following correlation:

$$Q(111/\tau{=}0) = Q(100/\tau{=}0) = Q(010/\tau{=}0) = Q(001/\tau{=}0) = \frac{\text{one}}{4} \quad (39)$$

$$Q(000/\tau{=}1) = Q(110/\tau{=}1) = Q(101/\tau{=}1) = Q(011/\tau{=}1) = \frac{\text{one}}{4}$$

With this correlation, the players, in exchange for revealing their type to a mediator (correlation device), always win, which entails an expected social benefit, SW=1, which is the maximum that can be achieved. We see that we win the game with the same probability as in the quantum equilibrium explained in (3.3.2). The difference is that in the belief invariant strategy we have revealed our type to a mediator.

Now we are going to modify the game to be competitive. We do it by changing the payments. We define the concept of pass as what in the collaborative game was to win. Player payouts are now not 1 regardless of the action you have taken, instead each player's payout now depends on the action taken. If the player i do action 1 and generate a pass, win v(one). If they generate a pass and you have taken action 0, you win $v(0)$. We define the paid function F_i:

$$F_i(you) = \begin{cases} 0 & \text{Yes } V(you) = 0, \\ v(a_i) & \text{if } V(you) = 1. \end{cases} \quad (40)$$

where the function $V(you)$ represents whether to create a pass or not.

We assume that v(one) $> v(0)$. This creates a tension between the players, since everyone prefers to do the action a_i=1, but this can lead them to not generate a pass on some occasions. Now the best local strategy would Q(one, one, 1) = 1 since it corresponds to an expected social benefit $SW = \frac{3}{4} v$(one).

Both the classical belief invariant equilibrium and the quantum belief invariant equilibrium result in a profit. social medium of $SW = \frac{v(1)+v(0)}{\text{two}}$. We already know the difference between one and the other, the classic has to reveal its type and the quantum doesn't. We are going to verify that the belief invariant strategy with the correlation $Q(s/t)$ that wins the original (collaborative) pseudo-telepathic game, is also a belief invariant equilibrium for the modified (competitive) game.

The GHZ game is a special game, since it is won in a quantum way with probability 1. These games that win with probability 1 are called pseudo-telepathic. Apart from this, for each input you and for each player i, given a list of responses from the other players a_{-i}, there is only one response from the player i that generates a pass, that is, $V(you)$ = 1. This turns the game into what is called a unique pseudo-telepathic game.

To calculate the expected benefit of a player, we only need to know with what probability he will take each action. We can calculate this only knowing the distribution of the inputs and the strategy that follows. knowing $P(you)$ Y $Q(s/t)$ we can calculate AND a_i. Knowing this, we can calculate the player's expected profit i: AND $F_i(you) =: F_i$.

$$\text{AND}\, F_i(you) = \text{AND}_{a_i} v(a_i) \quad (41)$$

Let's check the following theorem:

Theorem Yes $Q(a/t)$ is a belief invariant strategy to win the unique pseudo-telepathic game, it will be a belief invariant equilibrium for the modified game. What's more, yes $Q(a/t)$ is a quantum strategy, it will be a quantum equilibrium for the modified game.

Demonstration We have to check that if all the players except the player i follow the strategy, the player i you cannot increase your profits by following another strategy. That is, you cannot get a better profit than $F_i = \text{AND}_{A_i} v(A_i)$.

All players follow the original strategy except the player i. This makes us have a strategy $Q(a/t)$ with the same marginal as the original strategy.

$$Q(a_{-i}/t_{-i}) = Q(a_{-i}/t_{-i}) \quad (42)$$

With random variables TY$A = A_{-i} A'_i$, randomly distributed with respect to $P.Q.'$, we can build an A_i With which $\nu(T, A_{-i} A_i) = 1$ with probability 1. Note that if the strategy were the original one, $Q(a/t)$, we we would find that $A_i = A_i$ with probability 1, due to the uniqueness and the pseudo telepathy of the game. We now make these two observations. The first, for any type you_n, A_i has the same distribution what A_i, Thus, F_i = AND$_{A_i}$ $\nu(A_i)$. Second, players will generate a pass only when $A'_i = A_i$. In this case the player's benefit i it is $F_i(T, A') = \nu(A'_i) = \nu(A_i)$, otherwise it would be 0. Therefore:

$$\begin{aligned} \text{AND} F_i(T, A) &= \text{AND}_{A'_i A_i} \{A = A_i\} \nu(A'_i) \\ &= \text{AND}_{A_i A'_i} \{\hat{A}_i = A'_i\} \nu(A_i) \\ &\leq \text{AND}_{A_i} \nu(A_i) \\ &= F_i \end{aligned} \tag{43}$$

Where $\{\Pi\}$ is the following function:

$$\{\Pi\} = \begin{cases} \text{one} & \text{if } \Pi \text{ is true} \\ 0 & \text{if } \Pi \text{ is a lie} \end{cases} \tag{44}$$

In this way, it is shown that $Q(a/t)$ is also a quantum strategy for the modified game.

5.2. Quantum Advantage Games

To verify that there is a quantum advantage, we will see how there are parameters for $\nu(a)$ such that the expected social benefit for the (quantum) belief invariant equilibrium is larger than the largest expected social benefit of any classical correlated equilibrium.

We describe the social benefit as $\sigma = \frac{\text{one}}{n}(F_{\text{one}} + F_{\text{two}} + \ldots + F_n)$. We also define $\bar{v} = max_a \nu(a)$.
Theorem

$$\max_{\text{DC Equil.}} \text{AND} \sigma(T, A) \leq \max_{\text{str. local}} \text{AND} \sigma(T, A) \leq \omega_c \bar{v}$$

Where ω_c is the maximum probability of winning the game using local strategies.

Demonstration The proof is clear. Any classical correlated equilibrium is a local strategy, therefore the first inequality is clear. For the second, we have to realize that σ can take values within the interval $[0, \bar{v}]$, but any local strategy makes it take the value of 0 with a probability equal to $1 - \omega_c$.

With Theorem 1 and 2 they imply that for any pseudo-telepathic game, there exists a competitive game derived from the original that has a quantum equilibrium with an expected social payoff unattainable for local strategies and in particular, unattainable for any classical correlated equilibrium, if you pay them ($\nu(a)$) are close enough.

For any game, we can define the player's expected payoff i, which follows a quantum strategy $Q(a/t)$, as follows:

$$\text{AND} F_i(T, A) = \text{AND} \nu(A_i) = \sum_{a_i} P_i(a_i) \nu(a_i) \tag{Four. Five}$$

The expected social benefit is:

$$SW = \sum_a P(a) \nu(a) \tag{46}$$

Where $P(a) = \frac{\text{one}}{n} \sum_{i=1}^{n} P_i(a)$.

For the classical correlated equilibrium, we know that it has an upper bound equal to $\omega_c \bar{v}$.

It is easy to see that if $v(a)$ meets the condition

$$v(a) > \omega_c \bar{v} \tag{47}$$

quantum equilibrium will always lead to an expected social benefit greater than that corresponding to the classical correlated equilibrium. This condition is valid for any type of game, but it does not mean that there are no values of $v(a)$, smaller than $\omega_c \bar{v}$, for which the condition is also satisfied

$$SW_{quantum} > SW_{correlated}. \tag{48}$$

5.2.1. GHZ modified

Let's go back to the GHZ game. In this case, the expected social benefit following the quantum strategy, which we have described above, is:

$$SW_{quantum} = \frac{v(1) + v(0)}{\text{two}} \tag{49}$$

The expected social benefit for classical correlated equilibria we already know that it has to be less than $\omega_c \bar{v}$. In the case of GHZ $v(\text{one}) > v(0)$, therefore:

$$SW_{correlated} \leq \omega_c \bar{v} = \bar{\omega}_c v(\text{one}) \tag{fifty}$$

In GHZ the value of ω_c is equal to $\frac{3}{4}$. Therefore $SW_{correlated} \leq \frac{3}{4} v(\text{one})$.

According to the condition that we have defined in the previous section, if $v(0)$ is greater than $\frac{3}{4} v(1)$, we will find what $SW_{quantum} > SW_{correlated}$. In this case, the SW quantum and correlated is $\frac{7}{8} v(1)$ and $\frac{6}{8} v(1)$ respectively. We see how if the condition is met $SW_{quantum} > SW_{correlated}$.

As we have said, each game gives us the option of trying to find a larger range of values for $v(a)$ that enforces condition (48). In this case, if we look for the limit $SW_{quantum} = SW_{correlated}$:

$$\frac{v(1) + v(0)}{\text{two}} = \omega_c v(\text{one}) \tag{51}$$

It follows that $v(0) = (2\omega_c - \text{one}) v(\text{one})$. If we do the calculations with the value $\omega_c = \frac{3}{4}$, we will find that $v(0) = \frac{v(\text{one})}{\text{two}}$.

We conclude that in the case of the game GHZ, whenever the value of $v(0)$ is greater than $\frac{v(\text{one})}{\text{two}}$, for strategy quantum, we will have an expected social benefit higher than the classical one.

6. Conclusions

In this work, we have studied the concept of quantum entanglement and how important it was, together with Bell's inequalities, to end, definitively, the division that had been in favor or against quantum mechanics since the publication of the article.

We have seen the different types of balances that we can achieve. We have verified that there are situations where the belief invariant equilibrium can overcome the correlated equilibrium in terms of social benefit. The belief invariant equilibrium is interesting since it does not allow players to find out information about other players. This is very useful in the application of games in sectors such as the economy, since often the information behind what we call player types is sensitive information and is not wanted to be revealed to other players.

With the introduction of quantum equilibria, we see that we can win games with probability 1 in what are called pseudo-telepathic games. The importance of these equilibria is that you can win the game with probability 1, but unlike belief invariant equilibria, you don't have to reveal your type to anyone.

We have described a method by which we can create conflict-of-interest games with quantum advantage. More specifically whenever $v(a) > \omega_c \bar{v}$, we will be able to construct a game where the expected social benefit corresponding to the quantum equilibrium is greater than that corresponding to the classical correlated equilibrium.

Previously, there had been examples of games with quantum advantage in competitive games, which were much more complex to analyze. The example that we have presented is new, but inspired by the article [8].

References

[1] A. Einstein, B. Podolsky, N. Rosen. Can quantum-mechanical description of physical reality be considered complete? Phys.Rev.,47,77-780.

[2] JS Bell. On the Einstein-Podolsky-Rosen paradox. Physics, 1(3):195-200, 1964.

[3] V. Scarani.*Bell Nonlocality*, Oxford University Press, 2019.

[4] JJ Sakurai, J. Napolitano.*Modern Quantum Mechanics*, Cambridge University Press, 2017.

[5] MB Plenio, S. Virmani. An introduction to entanglement measures. Quantum Information Compu-Station. Vol 7, No 1, 2007.

[6] E. Bagan.*Notes of Information and Quantum Computing*. Preliminary version 2.0. UAB/IFAE, 2007.

[7] V. Auletta, D. Ferraioli, A. Rai, G. Scarpa, A. Winter. Belief-Invariant and Quantum Equilibria in Games of Incomplete Information. arXiv:1605.07896

[8] B. Groisman, MM Gettrick, M. Mhalla, M. Pawlowski. How Quantum Information can improve Social Welfare. https://doi.org/10.14760/OWP-2020-13

[9] A. Pappa, N. Kumar, T. Lawson, M. Santha, S. Zhang, E. Diamanti, I. Kerenidis. Nonlocality and conflicting interest games. Phys. Rev. Lett. Vol.114, Iss. 2, 020401. 2015. https://link.aps.org/doi/10.1103/PhysRevLett.114.020401

[10] K. Bolonek-Lason. Three-player conflicting interest games and nonlocality. Quantum Inf Process 16, 186 (2017). https://doi.org/10.1007/s11128-017-1635-6

[11] S. Zhang. Quantum Strategic Game Theory. Proceedings of the 3rd Innovations in Theoretical Computer Science Conference. Pages 39-59. 2012. https://doi.org/10.1145/2090236.2090241

[12] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, S. Wehner. Bell nonlocality. Rev. Mod. Phys. Vol. 86, Iss. 2. 2014. 10.1103/RevModPhys.86.419