Non-classical correlations in the language of Bayesian game theory

This title would literally make sense by the end!

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Course: 9th semester thesis project

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My initial motivation

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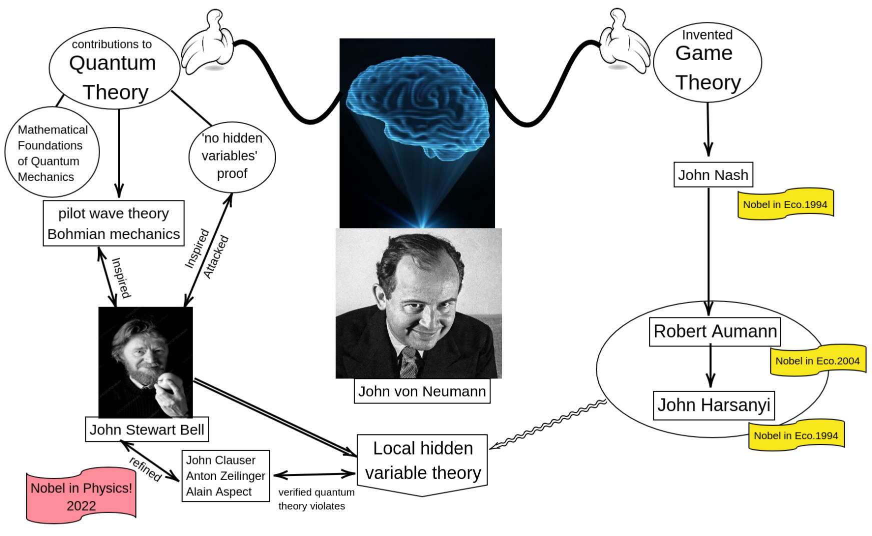


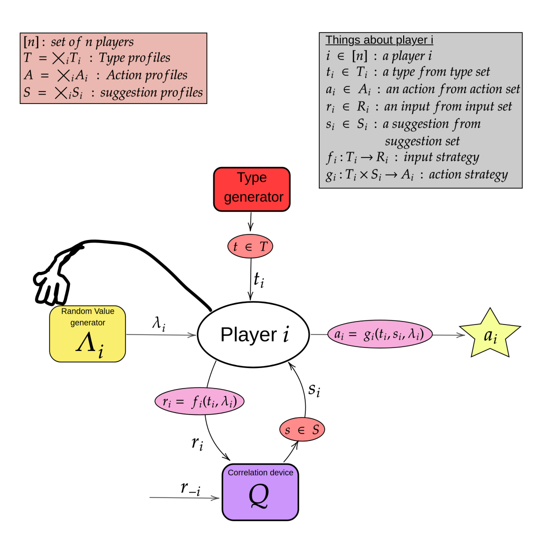
References for now and future

- The main reference until now (more work in progress):
  - Vincenzo Auletta, Diodato Ferraioli, Ashutosh Rai, Giannicola Scarpa, and Andreas Winter. Belief-invariant and quantum equilibria in games of incomplete information. Theoretical Computer Science, 895:151–177, dec 2021.
- An eye to keep on exploring profound implications in foundational physics(for the future):
  - Sayantan Choudhury, Sudhakar Panda, and Rajeev Singh. Bell violation in the sky. The European Physical Journal C, 77(2):1–181, 2017.

The general framework

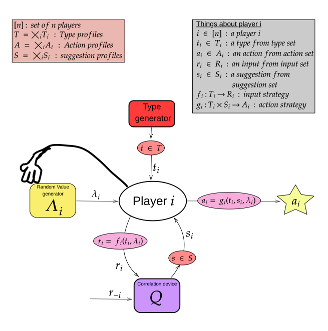
n-player Bayesian game

- A set of players: $[n] := \{1, 2, \dots, n\}$
- Action set for each player $i \in [n]$: A_i
  - Action profile $A = \times_i A_i$
- Type set for each player: T_i
  - Type profile $T = \times_i T_i$
  - A joint distribution over type profiles $P(t)$, where $t \in T$
- Utility function for each player i, $v_i : A \times T \to \mathbb{R}$
- A communication device Q:
  - Set of inputs it takes from player i: R_i
  - Set of outputs it gives to player i: S_i
  - The resulting correlation $Q(s \mid r)$, where $s \in S$ and $r \in R$



$[n]$: set of n players
$T = \times_i T_i$: Type profiles
$A = \times_i A_i$: Action profiles
$S = \times_i S_i$: suggestion profiles
Things about player i
$i \in [n]$: a player i
$t_i \in T_i$: a type from type set
$a_i \in A_i$: an action from action set
$r_i \in R_i$: an input from input set
$s_i \in S_i$: a suggestion from suggestion set
$f_i : T_i \to R_i$: input strategy
$g_i : T_i \times S_i \to A_i$: action strategy
Type generator
$t \in T$
t_i
Random Value generator
Λ_i
λ_i
Player i
$a_i = g_i(t_i, s_i, \lambda_i)$
a_i
$r_i = f_i(t_i, \lambda_i)$
s_i
$s \in S$
r_i
r_{-i}
Correlation device
Q

n-player Bayesian game >_ The easy depiction



Communication equilibrium: If $\forall i \in [n], \forall t_i \in T_i$

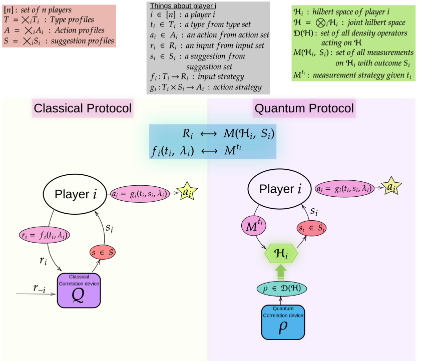
$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f(t,\lambda)\big)v_i\big(t, g(t,s,\lambda)\big) \geq$$

$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f_i'(t_i,\lambda_i)f_{-i}(t_{-i},\lambda_{-i})\big)v_i\big(t, g_i'(t_i,s_i,\lambda_i)g_{-i}(t_{-i},s_{-i}\lambda_{-i})\big)$$

Quantum formalism

Quantum formalism

- Adviser assigns a finite-dimensional Hilbert space $\mathscr{H}_i$ for each player $i \in [n]$
  - They can be any finite-level quantum system (a **qudit** register)
- Implements a state ρ on the joint Hilbert space $\mathscr{H} := \bigotimes_i \mathscr{H}_i := \mathscr{H}_1 \otimes \mathscr{H}_2 \cdots \otimes \mathscr{H}_n$.
  - i.e., $\rho \in \mathscr{D}(\mathscr{H})$
- Sends the assigned qudit register ($\mathscr{H}_i$) to each player $i \in [n]$ privately.
- Players then perform generalized measurement (POVM) $\{M_{a_i}^{t_i}\}_{s_i}$, to obtain the measurement outcome a_i, which would be their advice.
- The correlation formed will be then $Q(a \mid t) = \mathrm{Tr}\left(\rho(M_{a_1}^{t_1} \otimes M_{a_2}^{t_2} \otimes \cdots \otimes M_{a_n}^{t_n})\right)$



The canonical correlation formed:

$$Q(a \mid t) = \mathrm{Tr}\left(\rho(M_{a_1}^{t_1} \otimes M_{a_2}^{t_2} \otimes \cdots \otimes M_{a_n}^{t_n})\right)$$

$\sum_{s_J} \bigotimes_{j\in J} M_{s_j}^{r_j} = \bigotimes_{j\in J} \sum_{s_j} M_{s_j}^{r_j} = \bigotimes_{j\in J} \mathbb{I}$. So,

$$
\begin{aligned}
\sum_{s_J \in S_J} q\left(s_I, s_J \mid r_I, r_J\right) &= \sum_{s_J \in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r_j} \right) \\
&= \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} \mathbb{I} \right) \\
&= \sum_{s_J \in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r_j'} \right) \\
&= \sum_{s_J \in S_J} q\left(s_I, s_J \mid r_I, r_J'\right),
\end{aligned}
$$

Demonstrating Bell violation and Tsirelson bound with CHSH game in Bayesian game-theoretic language

The CHSH game

Objective:

$$\mathrm{LOC}(S \mid R) \subset \mathrm{Q}(S \mid R) \subset \mathrm{BINV}(S \mid R)$$

The CHSH game

- $N = \{1, 2\}$, $T_i = \{0, 1\}$, $A_i = \{0, 1\}$, $P(t) = \frac{1}{4}\ \forall t \in T$
- the utility function is described as:

$$v_{i=1,2}(t_1 t_2, a_1 a_2) = \begin{cases} 0 & \text{if } t_1 \cdot t_2 \neq a_1 \oplus a_2 \\ 1 & \text{if } t_1 \cdot t_2 = a_1 \oplus a_2 \end{cases}$$

- spread it out:

$V_a^t =$

a \ t	00	01	10	11
00	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$
01	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
10	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
11	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$

- $N = \{1,2\}, T_i = \{0,1\}, A_i = \{0,1\}, P(t) = \frac{1}{4}\ \forall t \in T$
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$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

- The strategy set of this "Bayesian game" is then:

$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\}$$

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} \text{a} \backslash \text{t} & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

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$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\}$$

- The Nash equlibria of this game are:

$$(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$$

- payoff tensor:

$V_a^t =$

a \ t	00	01	10	11
00	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$
01	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
10	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
11	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$

- The Nash equilibria of this game are(all are product distribution(deterministic), not a Local correlation!):

$$(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$$

- All gives the payoff profile $(3/4, 3/4)$:
  - Although the number is correct, conceptually, this doesn't define the actual classical bound! (when it comes to conflicting interest games)

- payoff tensor:

$V_a^t =$

a \ t	00	01	10	11
00	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$
01	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
10	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
11	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$

- The **convex hull** of Nash equilibria forms a specific class of correlated equilibria.
- Convex combination by maximal distribution:

$$\begin{aligned} Q(a \mid t) = {} & \frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^1(t_2)} + \frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^1(t_2)} \\ & + \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^4(t_2)} \\ & + \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^4(t_2)} \end{aligned} \tag{1}$$

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

- spreading out the conditional probability distribution:

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \tag{2}$$

- This is the Local correlation!
- Conceptually, when speaking about Aumann's correlated advice of functions (strategies) in bayesian games, this is it!
- $$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \tag{3}$$
- Of course, the maximum payoff profile we get with this correlation is $(3/4, 3/4)$

$$N = \{1,2\},\ T_i = \{0,1\},\ A_i = \{0,1\},\ P(t) = \frac{1}{4}\ \ \forall t \in T$$

Consider the quantum strategy (ρ, M^{t_1}, M^{t_2})

- $\rho = |\phi^+\rangle\langle\phi^+|$ where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$
- POVMs:

$$\begin{aligned} M^0_{a_1} &= |\phi_{a_1}(\theta^0_1)\rangle\langle\phi_{a_1}(\theta^0_1)|, \quad & M^1_{a_1} &= |\phi_{a_1}(\theta^1_1)\rangle\langle\phi_{a_1}(\theta^1_1)| \\ M^0_{a_2} &= |\phi_{a_2}(\theta^0_2)\rangle\langle\phi_{a_2}(\theta^0_2)|, \quad & M^1_{a_2} &= |\phi_{a_2}(\theta^1_2)\rangle\langle\phi_{a_2}(\theta^1_2)| \end{aligned}$$

where $\{|\phi_0(\theta^{t_i}_i)\rangle = \cos\theta^{t_i}_i|0\rangle + \sin\theta^{t_i}_i|1\rangle, \quad |\phi_1(\theta^{t_i}_i)\rangle = -\sin\theta^{t_i}_i|0\rangle + \cos\theta^{t_i}_i|1\rangle\}$.

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} {}_{a}\backslash^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

-

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\backslash^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \\ 01 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 10 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 11 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \end{array} \tag{4}$$

-
$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \\ 01 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 10 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 11 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \end{array} \tag{5}$$

-
$$|\theta_1^0 - \theta_2^0| = |\theta_1^0 - \theta_2^1| = |\theta_1^1 - \theta_2^0| = \frac{\pi}{2} - |\theta_1^1 - \theta_2^1| \tag{6}$$

- solution that gives maximum $(\theta_1^0, \theta_1^1) = (0, \frac{\pi}{4})$ and $(\theta_2^0, \theta_2^1) = (\frac{\pi}{8}, -\frac{\pi}{8})$

- $$Q(a \mid t) = \frac{1}{2}\begin{bmatrix} \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \end{bmatrix} \quad (7)$$

- $$Q(a \mid t) = \quad (8)$$

a \ t	00	01	10	11
00	0.43	0.43	0.43	0.07
01	0.07	0.07	0.07	0.43
10	0.07	0.07	0.07	0.43
11	0.43	0.43	0.43	0.07

- And the maximum payoff profile is: $(0.85, 0.85)$

The CHSH game can be won with an average payoff profile (1,1) with the following non-signaling strategy:

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & 0.5 & 0.5 & 0.5 & 0 \\ 01 & 0 & 0 & 0 & 0.5 \\ 10 & 0 & 0 & 0 & 0.5 \\ 11 & 0.5 & 0.5 & 0.5 & 0 \end{array} \tag{9}$$

- This correlation can only be achieved if the types are communicated to the advisor.

Conclusion

Conclusion

- We had a brief look at the general framework for games with incomplete information and quantum formalism.
- We saw a comprehensive demonstration of Bell violation and Tsirelson's bound using the popular CHSH game, but speaking in the language of Bayesian game theory and Aumman's concept.
- Although the CHSH game is simple and already well analyzed, my attempt was to propose a unique conceptual and comprehensive methodology for approaching any sophisticated games with incomplete information of **conflicting interest**.

A new idea!

- Games of complete information apparently have no actual quantum advantage.
- However Games of incomplete information can be somehow converted to games of complete information (induced normal form and agent normal form).
- What are the games with complete information that is actually in its induced or agent normal form of a game with incomplete information that must have a quantum advantage of non-local correlation?

A quest on focus:

- Quantum advantage of separable states in conflicting interest games?

A quest on exploring(perhaps for the future):

- What will be the implication of the framework we are studying here for foundational physics experiments?

Thank You!

Games!

Games!

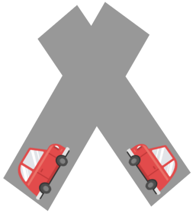
What is a game?

What is a game?

- An abstract object that has a finite number of "**players.**"
- Each player has a set of "**actions**" to take.
- A player will have a "**consequence.**" But what does that consequence depend on?
  1. The **action** that the **player** decides to take.
  2. The **actions** the "**rest of the players**" take.

Example:

$$
\begin{aligned}
(\text{stay}, \text{stay}) &\implies (\text{mad}, \text{mad}) \\
(\text{stay}, \text{cross}) &\implies (\text{mad}, \text{happy}) \\
(\text{cross}, \text{stay}) &\implies (\text{happy}, \text{mad}) \\
(\text{cross}, \text{cross}) &\implies (\text{crash}, \text{crash})
\end{aligned}
$$



Games!

What is Aumann’s concept then?

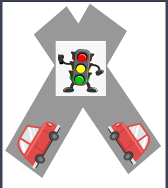
What is this thing about Aumann's concept then?

An adviser comes in...

- Of course, to advise the players
  - on what actions to take.
- But advice each players **privately**
  - For the betterment of all of them.
- Of course, then every player's actions will be **correlated** since they are listening to the same advisor.
  - So the advisor implements the advice in a way that any player disobeying the advice faces a bad consequence (**Aummann's correlated equilibrium!**).

Example:

Half of the time, advises:
(stay, cross) $\implies$ (mad, happy)
Half of the time, advises:
(cross, stay) $\implies$ (happy, mad)



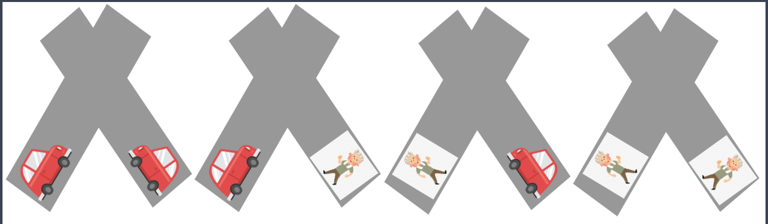
Games!

Games with incomplete information

Games with incomplete information

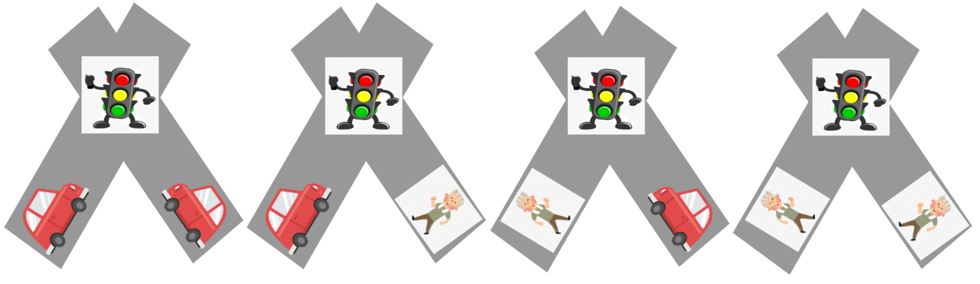
What is this ”**incomplete information**”?

- Now, each player may have a ”**type**”.
- The **consequence** to a player now depends on:
  - the **type** of the player as well.
  - the **types** of **other players** as well.
- But a player does not know what type of players they are facing. But however, they may have a belief about other players’ types (the incomplete information).



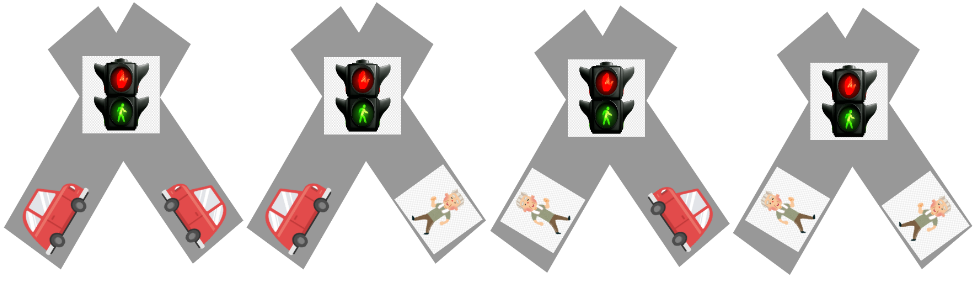
How would we bring in Aumann's concept here?

Would this be right?



How would we bring in Aumann's concept here?

Or this?



Correlations

Correlations

So what is a ”**correlation**” here?

- Say Player 1 is ”blue,” and Player 2 is ”red.”
- Players can be of **types**: {driver, walker}
- Players **actions** to choose: {stay, cross}

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 0 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1 \end{array}$$

- Something that a player can learn about the other player's types
  - by observing the advice they receive multiple times on repeated events

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 0 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1 \end{array}$$

$$Q(a_1 \mid t_1 \, \text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1 \, \text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

Signalling!

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 0 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1 \end{array}$$

$$Q(a_1 \mid t_1\,\text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1\,\text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

$$Q(a_2 \mid t_2\,\text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_2 \mid t_2\,\text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

Non-signalling!

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 1/2 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1/2 \end{array}$$

$$Q(a_1 \mid t_1\,\text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1\,\text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}$$

$$Q(a_2 \mid t_2\,\text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_2 \mid t_2\,\text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}$$

General :

For a subset $I \subset [n]$, let $R_I = \times_{i \in I} R_i$ and $S_I = \times_{i \in I} S_i$, a correlation $Q(s \mid r)$ is belief invariant if for all subsets $I \subset [n]$ and $J = [n] \backslash I$,

$$\sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r_J\right) = \sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r'_J\right) \quad \forall s_I \in S_I, r_I \in R_I, r_J, r'_J \in R_J$$