Non-classical correlations in the language of Bayesian game theory

9th semester thesis project progress report

by

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ABSTRACT

An efficient way to study non-local correlations is from the perspective of equilibria in games with incomplete information, aka Bayesian games, drawing inspiration from concepts in both game theory and quantum physics. These equilibria can be categorized as general communication equilibria, belief-invariant equilibria, and correlated equilibria, all of which contain the well-known Nash equilibria in decreasing order of effectiveness. In the 1990s, the concept of belief-invariant equilibrium first surfaced in game theory. However, the class of non-signaling correlations related to belief-invariance naturally emerged in the 80s in quantum theory. The two theoretical roots of the concept are explained and combined in [1]. The aforementioned types of equilibria and quantum-correlated equilibria are studied using tools from quantum information but speaking the language of algorithmic game theory. This report reviews the broad framework of belief-invariant communication equilibria provided in [1] that includes particular examples of correlated and quantum correlated equilibria. The work also includes the Bells theorem and its violations as a result of non-locality, a subject of great interest in the theoretical underpinnings of quantum mechanics. This report also presents an intuitive way to approach the already familiar non-local games, such as the CHSH in Bayesian game-theoretic language, to demonstrate bell violation and Tsilerson bound. Ref.[2] reviews bell theorem extensively and studies the possibility of setting up Bell's inequality violating experiment in the context of cosmology and string theory. The framework we study here provides an abstract but comprehensive language to study bell violating scenarios. To view the framework with an eye on exploring profound implications in foundational physics experiments remains a deep quest for this project. Another profound quest that's presently on focus in context to game theory is "quantum advantage with separable states".

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Chapter 1

Introduction

Imagine that you are in a situation to resolve a dispute between your two angry friends or siblings. A smart way to settle the issue is by listening to the concerns causing the fight from each of them "privately" and giving back your advice (again, "privately"). But can you resolve their issue by being less evil? that is, by not extracting embarrassing private information from them? Well, quantum theory provides you with a way to do that!

The above-mentioned scenario, which everyone would have experienced in life, can be seen as a form of a game with incomplete information (also called a Bayesian game) with a trusted mediator. The mathematical proof that some quantum correlations, unlike all other correlations in the universe, cannot result from any local cause was made by Northern Irish physicist John Bell in 1964. The strong connection between Bell nonlocality and Bayesian games was earlier discussed by Nicolas Brunner, and Noah Linden in their work [3]. Before I go further, I want to emphasize here the following beautiful lines from their paper " More recently a theory of generalized nonlocal correlations has been developed [4, 3], which has a direct impact on fundamental questions in the foundations of quantum mechanics [5, 6]. In a completely different area, but only three years after Bell's ground-breaking discovery, Harsanyi [7] developed a framework for games with incomplete information, that is, games in which players have only partial information about the setting in which the game is played ". It's important to note that the notion in bayesian game theory "non-signaling strategies," had already naturally aroused in the foundations of quantum mechanics a decade before its formulation. This report reviews Ref.[1], which unifies the frameworks developed later since John Bell's discovery and the CHSH game.

Chapter 2

Preliminaries

This chapter gives a short summary of some preliminary concepts and notations.

2.1 Use of notations

2.1.1 Sets and tuples

In this section I will explain some use of notations in this report with an intention to make it easy for a reader while going through this report. Here I use capital alphabets to denote sets. For example, $S_1 = \{0,1\}$. If $n \in \mathbb{N}$, then $[n] := \{1, 2, \ldots, n\}$. A Cartesian product of sets $S_{i\in[n]}$ denotes $S := \times_{i\in[n]} S_i := S_1 \times S_2 \times \ldots S_n$. An element $s \in S$ is a tuple denoted as $s := (s_1, s_2, \ldots, s_n)$, where $s_i \in S_i$ for $i \in [n]$. We can also denote a tuple as $s = (s_i)_{i\in[n]}$. For example, $S_1 = \{0,1\}$ and $S_2 = \{0,1,2\}$, then $S := \times_{i\in[2]} S_i = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$.

If we select an element $i \in [n]$, then s_{-i} denotes a tuple of every other elements except i. That is $s_{-i} \in \times_{j\neq i}^{n} S_j$. We can also denote the full tuple as $s := (s_i, s_{-i})$. This notation is essential to keep in mind when describing unilateral deviation of a player i in game when defining its equilibrium. Also when speaking about a marginal of a joint distribution, which we'll discuss shortly.

2.1.2 Functions

Consider R_1 as an input set and S_1 as the output set. Then $S_1^{R_1} : \{f_\gamma : R_1 \to S_1 \mid \gamma = 1, 2, \ldots, |S_1|^{|R_1|}\}$ denote the set of all possible functions mapping from R_1 to S_1.

For example, if $R_1 = \{0,1\}$ and $S_1 = \{0,1\}$, then

$$S_1^{R_1} := \{f_1 : x \mapsto 0, f_2 : x \mapsto x, f_3 : x \mapsto x \oplus 1, f_4 : x \mapsto 1\}$$

.

2.2 Joint probability distribution

If we have a set $R = \times_{i\in[n]} R_i$, then $P(r) = \Pr\{\mathbf{R} = r\}$ denotes a joint distribution over R, where $\mathbf{R}$ is a random variable that takes in values $r \in R$.

2.2.1 Example

If $R = \{0,1\} \times \{0,1\} = \{(0,0),(0,1),(1,0),(1,1)\}$, it is always convenient and intuitive to write down probability distributions as probability vectors like so:

$$P_r = \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix} \tag{2.1}$$

Where, $\sum_{(r_1,r_2)\in R} p_{r_1 r_2} = 1,\ \ p_{r_1 r_2} \in [0,1]$.

2.2.2 Marginals

If $P(r)$ is a joint distribution for $r \in R$, then for an $i \in [n]$, the marginal of r_i is described by the distribution $P(r_i) = \sum_{r_{-i}} P(r) = \sum_{r_{-i}} P(r_i r_{-i})$, for $r_{-i} \in \times_{j\neq i}^{n} R_i$. We can also speak about the marginal of rest of the elements as r_{-i} by the distribution $P(r_{-i}) = \sum_{r_i \in R_i} P(r) = \sum_{r_i\in R_i} P(r_i r_{-i})$.

Example

Consider the joint distribution of Eq.(2.1). The marginals of r_1 and r_2 are:

$$P_{r_1} = \begin{bmatrix} p_{00} + p_{01} \\ p_{10} + p_{11} \end{bmatrix}, \quad P_{r_2} = \begin{bmatrix} p_{00} + p_{10} \\ p_{01} + p_{11} \end{bmatrix} \tag{2.2}$$

2.2.3 Product distribution

$P(r)$ is a product distribution if $P(r) = \prod_{i\in[n]} P(r_i)$, where $P(r_i) = \sum_{r_{-i}} P(r)$ are marginals of each element $i \in [n]$.

Example

Consider the joint distribution of the form,

$$P_r = \begin{bmatrix} p_0q_0 \\ p_0q_1 \\ p_1q_0 \\ p_1q_1 \end{bmatrix} \tag{2.3}$$

where $\sum_{s_1\in S_1} p_{s_1} = 1, p_{s_1} \in [0,1]$ and $\sum_{s_2\in S_2} q_{s_2} = 1, q_{s_2} \in [0,1]$.

$$P_{r_1} = \begin{bmatrix} p_0q_0 + p_0q_1 \\ p_1q_0 + p_1q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \quad P_{r_2} = \begin{bmatrix} p_0q_0 + p_1q_0 \\ p_0q_1 + p_1q_1 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \tag{2.4}$$

And of course,

$$P_r = \begin{bmatrix} p_0q_0 \\ p_0q_1 \\ p_1q_0 \\ p_1q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \otimes \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \tag{2.5}$$

2.3 Joint conditional probability distribution (correlation)

Chapter 3

n-player Bayesian games with correlation device

3.1 The Game

Following the notations and definitions from [1], the n-player game with incomplete information/Bayesian game with correlation device is defined as follows,

$$G = (N, T_i, P, A_i, (Q, R_i, S_i), v_i)$$

.

where the notations are described as follows,

- $N = [n]$ is the set of n players.
- T_i is the set of types for player $i \in [n]$.
- A_i is the set of actions player i can take.
- $T =_{i \in N} T_i$ is the finite set of type profiles.
- $A =_{i \in N} A_i$ is the finite set of action profiles.
- $P(t)$ is the prior probability distribution over $t \in T$.
- $v_i : T_i \times A_i \to \mathbb{R}$ is the utility function for player $i \in N$.
- the general correlation device:
  - R_i is the set of inputs for player i that the player can feed to the correlation device.

- S_i is the set of outputs player i player can receive from the correlation device.
- Q is the joint conditional probability distribution over the output profiles $S =_i S_i$ given an input from the input profiles $T =_i T_i$.

The role of the correlation device will be explained in detail further in the section. I have created a depiction of the entire game Fig.[3.1] in order to have an easy intuition of the framework.

3.2 Strategies and equilibria in Bayesian games: in the absence of correlation device

In a Bayesian game setting, the players privately receive their types from an external agent (which can be a mediator or the environment) who draws the type profile $t \in T$ from a joint probability distribution over T. The players then have to choose their actions based on their respective type. In this section we'll see how a strategy can be defined in a Bayesian game.

3.2.1 Pure strategies

In absence of correlation device a pure strategy for a player is a function mapping from the type set to action set. So for a player i a pure strategy is:

$$g_i : T_i \to A_i$$

We can say $g_i \in A_i^{T_i}$, where $A_i^{T_i}$ is the set of all functions mapping T_i to A_i; which will have $|A_i|^{|T_i|}$ number of functions as elements. Thus a strategy set for a player i is $A_i^{T_i}$.

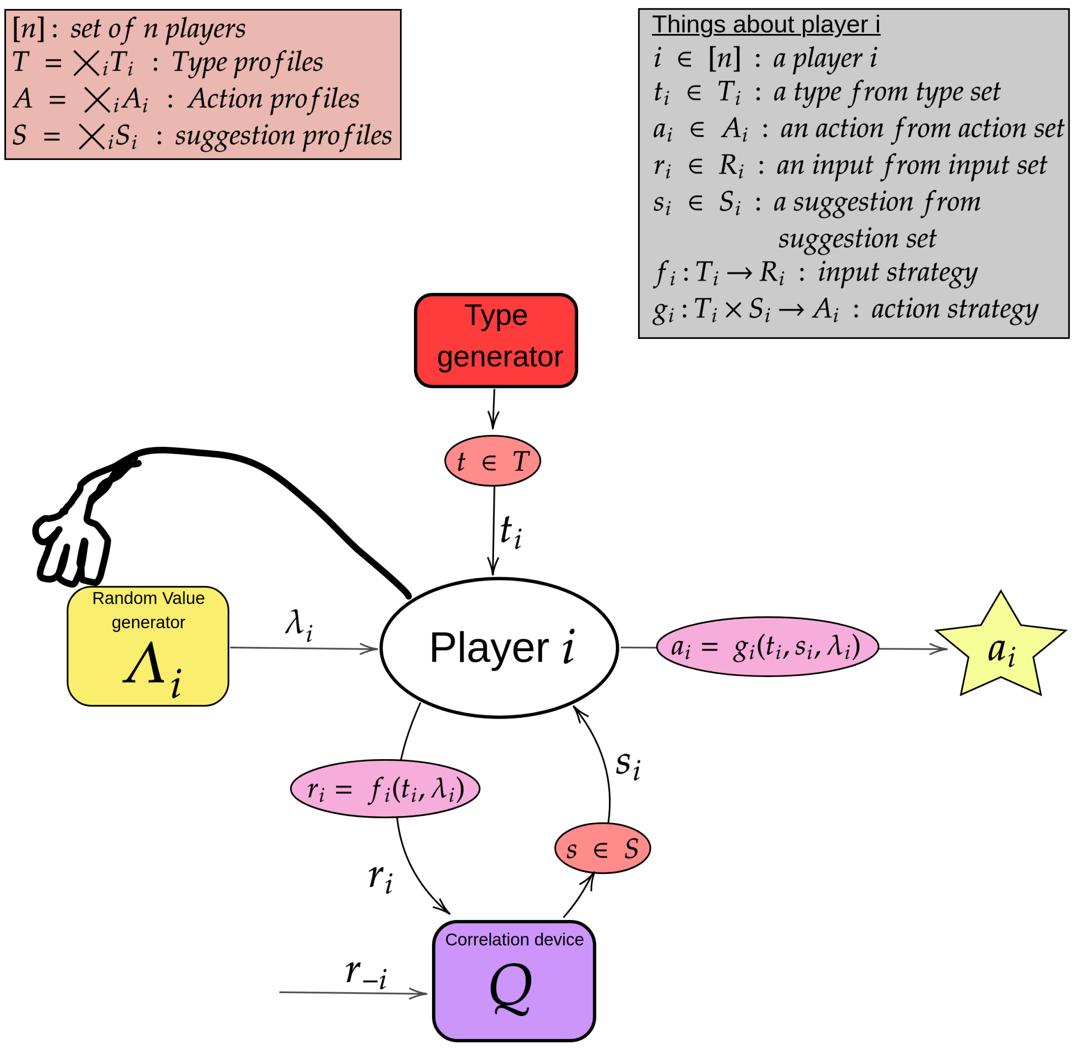

Figure 3.1: Depiction of the entire game. The reader can revisit this image as making progress through this report for better intuition.

Given the type profiles are drawn from a probability distribution P over T, the expected utility for a player i for strategy profile $g = (g_i)_{i\in[n]}$ [1] given the player received the type t_i is:

$$\langle v_i(t_i, g)\rangle = \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)v_i(t, g(t))$$

Where $t = (t_i)_{i\in[n]} \in T$ and $g(t) = (g_i(t_i))_{i\in[n]}$. And with the notation $T_{-i} =_{j\neq i}^{n} T_j$, $P(t_{-i}|t_i)$ implies the probability that the rest of the players type profile is $t_{-i} \in T_{-i}$ given player i's type is t_i.

The overall utility for player i is:

$$\langle v_i(g)\rangle = \sum_{t_i\in T_i} P(t_i)\langle v_i(t_i, g)\rangle = \sum_{t\in T} P(t)v_i(t, g(t))$$

3.2.2 Mixing strategies with random variable

In ref.[1], a mixed strategy is formulated by assigning a random variable λ_i as an argument to the pure strategy function g_i, so that $g_i(., \lambda_i)$ is now a random function. For better understanding, we can say all player $i \in [n]$ holds a random value generator that output a variable λ_i with probability $\Lambda_i(\lambda_i)$. And of course $\sum_{\lambda_i} \Lambda_i(\lambda_i) = 1$. The player i now just looks at the outcome λ_i emitted by the random value generator and then takes the action.

Now given the type t_i of the player, the player can emulate any mixed strategy $\tilde{g}_i \in (\Delta A_i)^{T_i}$ using a suitable random value generator Λ_i and taking a suitable decision based on its outcome. So given t_i, if player wants to decide a_i on seeing different λ_i, then

$$g_i(a_i|t_i) = \sum_{\lambda_i} g_i(a_i|t_i, \lambda_i)\Lambda_i(\lambda_i)$$

[1] The notation means $(g_i)_{i\in[n]} = (g_1, g_2, \ldots, g_n)$

To understand the procedure, here are some examples I made.

Examples:

$A_i = \{walk, run\}$, $T_i = \{lazy, testy\}$

1. $\Lambda_i \to$ "a coin".

   - Decision:
     - $(t_i = lazy)$: then if $head \to walk$, $tail \to walk$
     - $(t_i = testy)$: then if $head \to walk$, $tail \to run$
   - strategy:
     - $g_i(walk|lazy, head) = 1, g_i(walk|lazy, tail) = 1,$

       $g_i(run|lazy, head) = 0, g_i(run|lazy, tail) = 0$
     - $g_i(walk|testy, head) = 1, g_i(walk|testy, tail) = 0,$

       $g_i(run|testy, head) = 0, g_i(walk|testy, tail) = 1$

   $\implies g_i(walk|lazy) = 1, g_i(run|lazy) = 0, g_i(walk|testy) = 1/2,$

   $g_i(run|testy) = 1/2$

2. $\Lambda_i \to$ "a dice"..

   - Decision:
     - $(t_i = lazy)$: then if $\lambda_i = even \to walk$, $\lambda_i = odd \to walk$
     - $(t_i = testy)$: then if $\lambda_i = even \to walk$, $\lambda_i = odd \to run$
   - strategy:
     - $g_i(walk|lazy, \lambda_i = even) = 1, g_i(walk|lazy, \lambda_i = odd) = 1,$

       $g_i(run|lazy, \lambda_i = even) = 0, g_i(run|lazy, \lambda_i = odd) = 0$
     - $g_i(walk|testy, \lambda_i = even) = 1, g_i(walk|testy, \lambda_i = odd) = 0,$

       $g_i(run|testy, \lambda_i = even) = 0, g_i(walk|testy, \lambda_i = odd) = 1$

$$\implies g_i(walk|lazy) = 1, g_i(run|lazy) = 0, g_i(walk|testy) = 1/2,$$
$$g_i(run|testy) = 1/2$$

The above example also shows how random variables with a different number of outcomes can emulate the same mixed strategies (in this case, a coin and a dice).

3.2.3 Nash equilibrium

The strategy g is a nash equilibrium if for all i, t_i and a_i,

$$\sum_{t_{-i},\lambda} P\left(t_{-i} \mid t_i\right) \Lambda(\lambda) v_i\left(t, g_{-i}\left(t_{-i}, \lambda_{-i}\right) g_i\left(t_i, \lambda_i\right)\right)$$
$$\geq \sum_{t_{-i},\lambda} P\left(t_{-i} \mid t_i\right) \Lambda(\lambda) v_i\left(t, g_{-i}\left(t_{-i}, \lambda_{-i}\right) a_i\right)$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, and, by independence of $\lambda_i, \Lambda(\lambda) = \prod_{i=1}^{n} \Lambda_i\left(\lambda_i\right)$. By Nash's theorem [8], every game of incomplete information has an equilibrium.

3.3 Strategies and equilibria in Bayesian games: in the presence of correlation device

A more general solution concept than the well-known Nash equilibrium in games of complete information is correlated equilibrium. This powerful concept was first introduced by Robert Aumann in his work [9] in 1974. The idea is that players now look at private signals from a random value generator shared by all players to decide on their actions. This shared resource can be interpreted as a trusted mediator who draws an action profile a from a joint probability distribution over A and sends the corresponding action a_i privately to each player $i \in [n]$ as advice. If the joint probability distribution over A is such that no player wouldn't want to deviate from the advised action given rest of the players follow the advice, then the distribution is a correlated equilibrium.

Ever since Harsanyi[7] formulated games of incomplete information or Bayesian games, there have been quite some attempts to bring in the same concept of Aumann to the new framework. I leave the review of those pieces of literature for a future article. For now in this report, I discuss the equilibrium concepts described for Bayesian games in the presence of a correlation device from Ref.[1].

3.3.1 The general correlation device Q

Since a player's utility in Bayesian games depends on the inputs or types received by every player, including the player's own type, the player should get advice based on the type profile. A general correlation device is an object that privately takes inputs from every player, and based on the input profile, it implements a joint probability distribution over output profiles to generate advice for the players. The general correlation device can be dubbed a communication device if it directly takes private input communicated by the players. Firstly, for better intuition, let's see how it is different from the random value generator that is privately owned by the player to implement mixed strategies by the following points:

- The Correlation device Q is not something that the players privately own, unlike Λ_i; it's sort of a public property.
- However, players can only interact with it privately; meaning no player gets to know other player's business with it.
- Unlike the Λ_i, the Correlation device Q doesn't just spit out a random variable to a player. It takes private input from all the players before doing it.

Now let us see how the general correlation device works with the following points:

- All player $i \in [n]$ now has to decide the input r_i, based on their type t_i. Thus the player now has a new strategy $f_i : T_i \to R_i$.

  - In fact it can be a special case that $r_i = t_i$ when f_i is an identity function. The variable r_i is introduced in order to allow the player to fool the device by sending a false type.
  - For mixed strategy the player randomize the function by $f_i(t_i, \lambda_i)$ by using the random value generator Λ_i.

- Receives private inputs $r_i \in R_i$ from each player $i \in [n]$ and holds the input profile $r \in R$.

- Based on r it generates a joint conditional probability distribution $Q(s \mid r)$ over suggestion profiles S, with

$$\sum_{s \in S} Q(s|r) = 1 \quad \forall r \in R$$

- Outputs s based on $Q(s|r)$ and privately sends suggestion s_i to each player i.

- The only thing a player knows about Q is the joint probability distribution $Q(s \mid r)$ for all possible $r \in R$, but doesn't know the current r_{-i}. And obviously player knows what he sends in and what he receives out from it.

- Based on s_i and t_i, the player i decides on the action a_i by $g_i : S_i T_i \to A_i$.
  - For mixed strategy the player randomize the function by $g_i(t_i, s_i, \lambda_i)$ using the same random value λ_i from the random value generator Λ_i

Belief-invariant (aka non-signaling) correlations

If $All(S \mid R)$ denotes the set of all possible joint conditional probability distribution over S for every given input profiles R, there is a subset of correlations $BINV(S \mid R) \subset All(S \mid R)$ for which every joint conditional probability distribution $Q \in BINV(S \mid R)$

cannot give any information about other players inputs r_{-i} to each player i using his/her own input r_i and s_i generated by Q.

Formally, for a set $I \subset N$, let $R_I =_{i\in I} R_i$ and $S_I =_{i\in I} S_i$, a correlation $Q(s \mid r)$ is belief invariant if for all subsets $I \subset N$ and $J = N\backslash I$,

$$\sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r_J\right) = \sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r'_J\right) \quad \forall s_I \in S_I, r_I \in R_I, r_J, r'_J \in R_J$$

Expected utility given t_i

$$\langle v_i(t_i, f, g)\rangle = \sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f(t,\lambda)\big)v_i\big(t, g(t,s,\lambda)\big)$$

signaling and non-signaling communication equilibrium

(f, g, Q) is communication equilibrium if $\forall i \in [n]$, $\forall t_i \in T_i$, and for all random functions $f'_i \in R_i^{T_i}$ and $g'_i \in A_i^{T_i \times S_i}$:

$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f(t,\lambda)\big)v_i\big(t, g(t,s,\lambda)\big) \geq$$

$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f'_i(t_i,\lambda_i)f_{-i}(t_{-i},\lambda_{-i})\big)v_i\big(t, g'_i(t_i,s_i,\lambda_i)g_{-i}(t_{-i},s_{-i}\lambda_{-i})\big)$$

The communication equilibrium is called signaling if the conditional probability distribution Q is signaling. And the equilibrium is called belief-invariant or non-signaling communication equilibrium if Q is non-signaling.

Chapter 4

Quantum correlated strategies and equilibrium for Bayesian games

4.1 POVMs

In line with the fact that POVMs are a generalization of projection-valued measures (PVM), quantum measurements described by POVMs are also a generalization of quantum measurements described by PVMs (called projective measurements). A POVM can be compared to a PVM in a rough parallel to what a mixed state is to a pure one. A positive operator-valued measure (POVM) is a set $\{M_s\}_s$ of operators that satisfy non-negativity and completeness:

$$\forall s : M_s \geq 0, \quad \sum_s M_s = I.$$

The probability for obtaining outcome s with M_s acting on a density operator ρ (pure or mixed) is $Tr\{\rho M_s\}$. We also don't have to worry about post-measurement state for POVMs.

Ref.[1] gives the following form to illustrate classical probabilities in the quantum framework.

$$M_s = \sum_x \mu_s(x)|x\rangle\langle x|$$

where $\{|x\rangle\}$ is some fixed orthonormal basis, and $\mu_s \geq 0 \, \forall s \in S$ with $\sum_s \mu_s(x) = 1 \ \forall x$.

Completeness check:

$$\sum_s M_s = \sum_s \sum_x \mu_s(x)|x\rangle\langle x| = \sum_x \sum_s \mu_s(x)|x\rangle\langle x| = \sum_x |x\rangle\langle x| = 1$$

Examples: Here are some examples I came up with during the internship for better understanding.

1. $\Big\{M_1 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{4}|2\rangle\langle 2|, \quad M_2 = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|, \quad M_3 = \frac{1}{4}|2\rangle\langle 2|\Big\}$
2. $\Big\{M_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \quad M_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\Big\}$

Here are some other examples of POVMs that are not of the above form:
Consider two arbitrary non-orthonormal states $|a\rangle$ and $|b\rangle$ so that $\langle a|b\rangle \neq 0$. Since $\langle\psi|a\rangle\langle a|\psi\rangle \geq 0$ for any $|\psi\rangle \in \mathcal{H}$, the following can be a POVM:

$$M_1 = \alpha|a\rangle\langle a|, \quad M_2 = \beta|b\rangle\langle b|, \quad M_3 = I - M_1 - M_3$$

.

Considering normalized states, $\alpha, \beta > 0$ just has to satisfy,

$$\alpha + \beta|\langle a|b\rangle|^2, \quad \alpha|\langle a|b\rangle|^2 + \beta, \quad \beta|\langle a^\perp|b\rangle|^2, \quad \alpha|\langle a|b^\perp\rangle|^2 < 1$$

Then the following is a POVM,

$M_1 = \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}|a\rangle\langle a|, \quad M_2 = \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}|b\rangle\langle b|, \quad M_3 = I - \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}\big(|a\rangle\langle a| + |b\rangle\langle b|\big)$.

4.2 Quantum formalism

The procedure of the Bayesian game setup with the quantum correlated device can be understood with the following points:

1. The quantum correlation device assigns Hilbert spaces $\mathcal{H}_i$ (a qudit register) to each player i and implements a density operator ρ on the joint Hilbert space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$.

- The state ρ can be pure or mixed. In fact, the ensemble of the composite pure states can be either formed from a joint or disjoint distribution.

2. Sends the respective qudit registers to its assigned players so they can do the local measurement M^{t_i} based on their type to get their suggestions s_i.

3. So in this setup, the information about the player's type stays with the player itself as its not communicated to the device (or the mediator). And the device implements the state ρ independent of the players type profile.

    - This is the quantum analog of classical correlated strategy where the correlation device implements joint probability distribution independent of the player's type profile. The reason why the equilibrium that arises of it is called quantum correlated equilibrium.

So the formulation now is,

$$Q(s \mid r) = Tr(\rho M_s^r)$$

Where $M_s^r = \bigotimes_i M_{s_i}^{r_i}$. Again, for better intuition I created the following depiction Fig.[4.1].

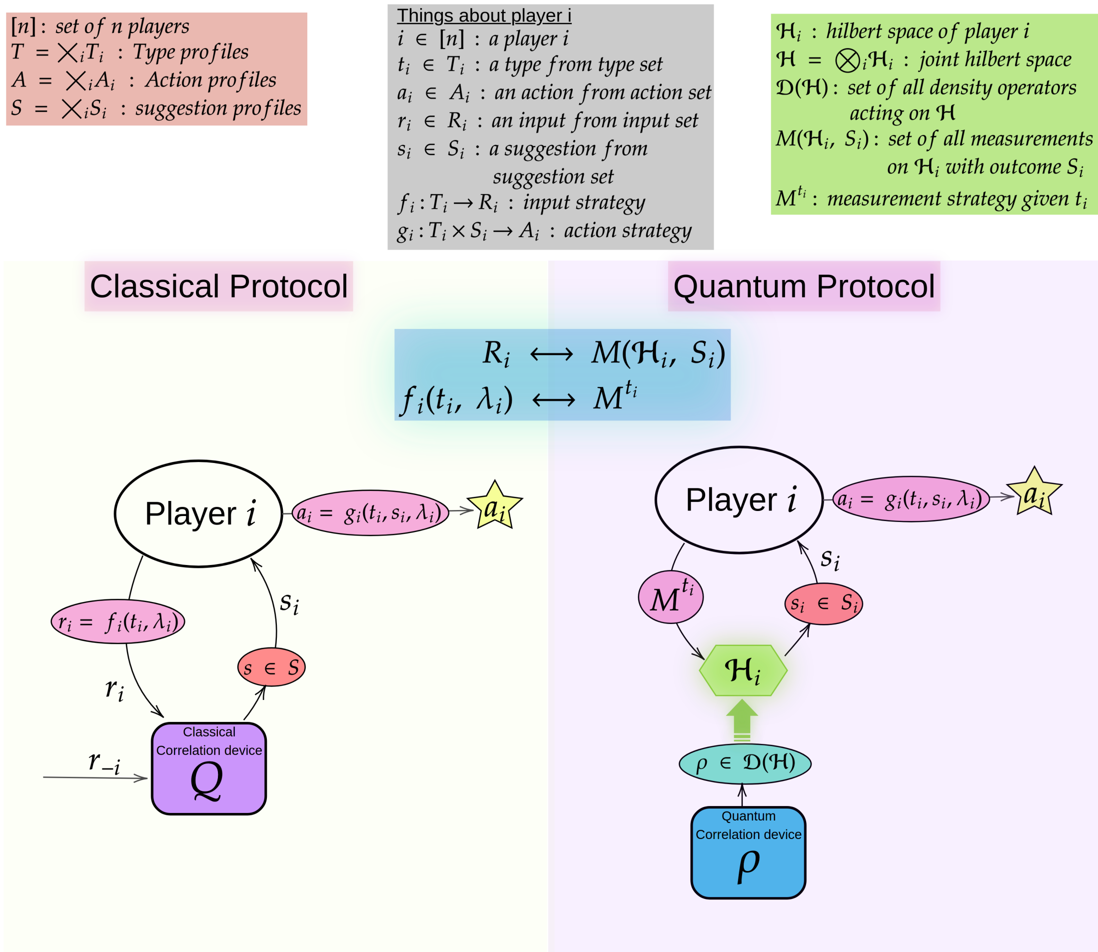

Figure 4.1: Easy depiction of the game setup with quantum correlation device and its comparison with the classical setup. Assume the type generator part in Fig.[3.1] is already there. Its been removed here just to give emphasis to the part of the image where the distinction is made.

4.3 Belief invariance of quantum correlated strategy

It can be easily showed that the quantum correlated strategy is non-signalling as follows: $\sum_{s_J} \bigotimes_{j\in J} M_{s_j}^{r_j} = \bigotimes_{j\in J} \sum_{s_j} M_{s_j}^{r_j} = \bigotimes_{j\in J} \mathbb{I}$. So,

$$\begin{aligned}
\sum_{s_J\in S_J} q\left(s_I, s_J \mid r_I, r_J\right) &= \sum_{s_J\in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r_j}\right) \\
&= \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} \mathbb{I}\right) \\
&= \sum_{s_J\in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r'_j}\right) \\
&= \sum_{s_J\in S_J} q\left(s_I, s_J \mid r_I, r'_J\right),
\end{aligned}$$

This fact naturally arises from non-signalling and non-cloning theorems of quantum theory.

4.4 Emulating classical correlations.

1. Local correlation: $Q(\mathbf{s} \mid \mathbf{r}) = \sum_\gamma V(\gamma) L_1\left(s_1 \mid r_1\gamma_1\right) \cdots L_n\left(s_n \mid r_n\gamma_n\right)$

$$\rho = \sum_\gamma V(\gamma) \left|\gamma_1\right\rangle \left\langle\gamma_1\right| \otimes \cdots \otimes \left|\gamma_n\right\rangle \left\langle\gamma_n\right|, \quad M_{s_i}^{r_i} = \sum L_i\left(s_i \mid r_i\gamma_i\right) \left|\gamma_i\right\rangle \left\langle\gamma_i\right|$$

So,

$$\mathrm{LOC}(S \mid R) \subset \mathrm{Q}(S \mid R) \subset \mathrm{BINV}(S \mid R)$$

This relation will be demonstrated in the following section 5.1.

2. canonical correlation:

$$\rho = \sum_{\mathbf{s}} Q(\mathbf{s}) \left|s_1\right\rangle \left\langle s_1\right| \otimes \cdots \otimes \left|s_n\right\rangle \left\langle s_n\right|, \quad M_{a_i}^{t_i} = \sum_{s_i} \delta_{g_i(t_i,s_i),a_i} \left|s_i\right\rangle \left\langle s_i\right|$$

So classical correlations can be emulated with mixed density operator of pure orthonormal states and suitable POVMs.

4.5 Quantum correlated equilibrium

Before we move on, lets introduce the following notation that I came up with for unilateral deviation of measurement strategy:

$$[X^{r_i}_{s_i}]M^{-r_i}_{-s_i} = \bigotimes_{j<i} M^{r_j}_{s_j} \otimes X^{r_i}_{s_i} \otimes \bigotimes_{j>i} M^{r_j}_{s_j}$$

4.5.1 what f_i is now on this setup?

Since the quantum correlation device does not take any inputs there is no point of R_i here. So players can choose any measurements they want from the infinite set $M(\mathcal{H}_i, S_i)$ based on their type. So the prior input strategy f_i transforms to POVM strategy in the quantum setup as follows:

$$\{f_i : T_i \to R_i\} \longleftrightarrow \{M_i : T_i \to M(\mathcal{H}_i, S_i)\}$$

So now M^{r_i} is re-denoted as $M_i(t_i) = M^{t_i}$.

4.5.2 Canonical correlation and solution

If we have $M^{t_i}_{a_i} = \sum_{s_i} \delta_{g_i(t_i,s_i),a_i} |s_i\rangle \langle s_i|$ or directly the observables as actions A_i, then we can write the representaion of cannonical correlation:

$$Q(a \mid t) = Tr(\rho M^t_a)$$

We now have $(M^{\mathbf{t}}, \rho)$ as equilibrium for all i, t_i and $X^{t_i}_{a_i} \in M(\mathcal{H}_i, A_i)\}$ if,

$$\left\langle v_{i,t_i}\left(M^{\mathbf{t}}, \rho\right)\right\rangle =$$
$$\sum_{t_{-i},\mathbf{a}} P\left(t_{-i} \mid t_i\right) \mathrm{Tr}\left(\rho M^t_a\right) v_i(t,a) \geq \sum_{\mathbf{t}_{-i},\mathbf{a}} P\left(t_{-i} \mid t_i\right) \mathrm{Tr}\left(\rho [X^{t_i}_{a_i}] M^{-t_i}_{-a_i}\right) v_i(t,a)$$

Chapter 5

Non-local games and their conversion to conflicting interest games

5.1 Demonstrating bell violation and Tsilerson bound with CHSH game in Bayesian game-theoretic language

The CHSH game in Bayesian game-theoretic language as in 3.1 can be described as follows:

$N = \{1, 2\}$, $T_i = \{0, 1\}$, $A_i = \{0, 1\}$, $P(t) = \frac{1}{4}$ $\forall t \in T$, and the utility function is described as:

$$v_{i=1,2}(t_1 t_2, a_1 a_2) = \begin{cases} 0 & \text{if } t_1 \cdot t_2 \neq a_1 \oplus a_2 \\ 1 & \text{if } t_1 \cdot t_2 = a_1 \oplus a_2 \end{cases} \tag{5.1}$$

Remember, in Bayesian game theory, the information about a player's type and the action they are going to execute, stays within the player itself. And there is no communication between the players. The CHSH game is a simple game to analyze. However, with this example, I present a much more comprehensive method to approach the Bayesian game, which would come in handy while analyzing more sophisticated conflicting interest games. I start by expanding the payoff function Eq.(5.1), to a payoff tensor:

$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array} \tag{5.2}$$

To clarify the notation, $v^{t_1t_2}_{a_1a_2} = (v_1(t_1t_2, a_1a_2), v_2(t_1t_2, a_1a_2))$. Where v_1 is the payoff for player 1, and v_2 is the payoff function for player 2.

Now, for the next step, I would write out every possible strategies for the player. Remember, strategy in a Bayesian is a "function" mapping from types to action. So the strategy set to both players $i \in \{1, 2\}]$ is:

$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\} \tag{5.3}$$

To show you an example, a strategy profile $g = (g_1^\gamma, g_2^\gamma)$ would give the payoff profile $V^{t_1t_2}_{g_1^\gamma(t_1)g_2^\gamma(t_2)} = (v_1(t_1t_2, g_1^\gamma(t_1)g_2^\gamma(t_2)), v_2(t_1t_2, g_1^\gamma(t_1)g_2^\gamma(t_2)))$. For example, the strategy profile $g = (g_1^1, g_2^3)$, the type profile $t = (0, 1)$ would give the payoff profile, $V^{t_1t_2}_{g_1^1(t_1)g_2^3(t_2)} = V^{01}_{g_1^1(0)g_2^3(1)} = V^{01}_{00} = (1, 1)$.

So now we would further expand the payoff tensor of Eq.(5.2) as follows:

$$V_a^t = \begin{array}{c|cccc}
g \backslash t & 00 & 01 & 10 & 11 \\
\hline
g_1^1g_2^1 & (1,1) & (1,1) & (1,1) & (0,0) \\
g_1^1g_2^2 & (1,1) & (0,0) & (1,1) & (1,1) \\
g_1^1g_2^3 & (0,0) & (1,1) & (0,0) & (0,0) \\
g_1^1g_2^4 & (0,0) & (0,0) & (0,0) & (1,1) \\
g_1^2g_2^1 & (1,1) & (1,1) & (0,0) & (1,1) \\
g_1^2g_2^2 & (1,1) & (0,0) & (0,0) & (0,0) \\
g_1^2g_2^3 & (0,0) & (1,1) & (1,1) & (1,1) \\
g_1^2g_2^4 & (0,0) & (0,0) & (1,1) & (0,0) \\
g_1^3g_2^1 & (0,0) & (0,0) & (1,1) & (0,0) \\
g_1^3g_2^2 & (0,0) & (1,1) & (1,1) & (1,1) \\
g_1^3g_2^3 & (1,1) & (0,0) & (0,0) & (0,0) \\
g_1^3g_2^4 & (1,1) & (1,1) & (0,0) & (1,1) \\
g_1^4g_2^1 & (0,0) & (0,0) & (0,0) & (1,1) \\
g_1^4g_2^2 & (0,0) & (1,1) & (0,0) & (0,0) \\
g_1^4g_2^3 & (1,1) & (0,0) & (1,1) & (1,1) \\
g_1^4g_2^4 & (1,1) & (1,1) & (1,1) & (0,0)
\end{array} \tag{5.4}$$

Now we can take average over the distribution of type profiles to obtain the payoff of each strategy profiles.

$$V_a^t = \begin{array}{c|c} g & \sum_t V_g^t P(t) \\ \hline g_1^1 g_2^1 & (3/4, 3/4) \\ g_1^1 g_2^2 & (3/4, 3/4) \\ g_1^1 g_2^3 & (1/4, 1/4) \\ g_1^1 g_2^4 & (1/4, 1/4) \\ g_1^2 g_2^1 & (3/4, 3/4) \\ g_1^2 g_2^2 & (1/4, 1/4) \\ g_1^2 g_2^3 & (3/4, 3/4) \\ g_1^2 g_2^4 & (1/4, 1/4) \\ g_1^3 g_2^1 & (1/4, 1/4) \\ g_1^3 g_2^2 & (3/4, 3/4) \\ g_1^3 g_2^3 & (1/4, 1/4) \\ g_1^3 g_2^4 & (3/4, 3/4) \\ g_1^4 g_2^1 & (1/4, 1/4) \\ g_1^4 g_2^2 & (1/4, 1/4) \\ g_1^4 g_2^3 & (3/4, 3/4) \\ g_1^4 g_2^4 & (3/4, 3/4) \end{array} \tag{5.5}$$

5.1.1 Nash equilibrium

It is obvious that the Nash equilibrium for common interest games (where players always get equal payoffs) is the strategy profile that gives the highest payoff for each player. However, we further our analysis as this section emphasizes the methodology for approaching a Bayesian game, which should incredibly help in analyzing far more sophisticated games of conflicting interest.

For demonstration, we can now convert Eq.(5.5) to the good old payoff matrix:

$$V_a^t = \begin{array}{c|cccc} \text{pl 1} \backslash \text{pl 2} & g_2^1 & g_2^2 & g_2^3 & g_2^4 \\ \hline g_1^1 & (3/4, 3/4) & (3/4, 3/4) & (1/4, 1/4) & (1/4, 1/4) \\ g_1^2 & (3/4, 3/4) & (1/4, 1/4) & (3/4, 3/4) & (1/4, 1/4) \\ g_1^3 & (1/4, 1/4) & (3/4, 3/4) & (1/4, 1/4) & (3/4, 3/4) \\ g_1^4 & (1/4, 1/4) & (1/4, 1/4) & (3/4, 3/4) & (3/4, 3/4) \end{array} \tag{5.6}$$

It is now easy to see which all strategy profiles are Bayesian Nash equilibrium. They are $(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$.

With the above payoff matrix, we now have kind of converted a game of incomplete information (bayesian game) to complete information. As a matter of fact, one can actually convert a bayesian game to its induced normal form [10] and also agent normal form [11], which we will discuss in the final thesis. Games of complete information don't necessarily have any actual quantum advantage [12]. However, I find it's a great idea to look into certain games of complete information that are in their induced normal form or agent form of a particular bayesian game that would have an actual quantum advantage.

5.1.2 Local correlation: Aummann's correlated equilibrium

Most of us may have heard the maximum classical winning probability is well known to be $\frac{3}{4}$. This is, of course, apparent from the previous subsection. Even though same number, it is not true that the classical boundary of local realism is set by what we have discussed in the previous section. Nash equilibria are a result of independent decision-making and are never locally correlated. The concept of Amman's correlated equilibria is a far more general and profound concept than that of Nash.

Now coming back to our analysis, if a game has multiple Nash equilibria, then a convex hull of all of it forms a specific set of correlated equilibria (I will discuss these theorems in a separate chapter for Aumman's concept and as well for Harsanyi's framework in the final thesis). Since there are eight Nash equilibria, let us consider the convex combination by the maximal distribution,

$$g = \frac{1}{8}(g_1^1 \otimes g_2^1) + \frac{1}{8}(g_1^1 \otimes g_2^2) + \frac{1}{8}(g_1^2 \otimes g_2^1) + \frac{1}{8}(g_1^2 \otimes g_2^3) + \frac{1}{8}(g_1^3 \otimes g_2^2) + \frac{1}{8}(g_1^3 \otimes g_2^4) + \frac{1}{8}(g_1^4 \otimes g_2^3) + \frac{1}{8}(g_1^4 \otimes g_2^4) \tag{5.7}$$

The interpretation is that an adviser draws these strategy profiles from a uniform distribution and recommends the strategy ("the whole function") corresponding to each player privately. Now, writing out the expanded payoff tensor is a painful process for this case. Thus, a better and more efficient approach is to form the correlation $Q(a \mid t)$. To do so, we represent each function as deterministic channels $g_i^{\gamma}(a_i \mid t_i) = \delta_{a_i, g_i^{\gamma}(t_i)}$. So the correlation by Eq.(5.7) is,

$$\begin{aligned} Q(a \mid t) = &\frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^1(t_2)} + \frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^1(t_2)} \\ &+ \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^4(t_2)} \\ &+ \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^4(t_2)} \end{aligned} \tag{5.8}$$

Expanding the above conditional probability distribution strategy $Q(a \mid t)$ can a correlation matrix for each input and output:

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \tag{5.9}$$

Taking the marginals:

$$Q(a_1 \mid t_1(0)) = \begin{array}{c|cc} a_1 \backslash t_1 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}, \; Q(a_1 \mid t_1(1)) = \begin{array}{c|cc} a_1 \backslash t_1 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}$$

$$Q(a_2 \mid t_2(1)) = \begin{array}{c|cc} a_2 \backslash t_2 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}, \; Q(a_2 \mid t_2(1)) = \begin{array}{c|cc} a_2 \backslash t_2 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}$$

Thus, $Q(a \mid t)$ in Eq.(5.9) is clearly not a product distribution as $Q(a \mid t) \neq Q(a_1 \mid t_1)Q(a_2 \mid t_2)$, and hence this is not a product correlation formed by the independent choice of mixed strategy. It's a local correlation formed by correlated advice of full function strategy. And of the expected payoff profile is $\sum_t Q(a \mid t)v(a,t)P(t) = (3/4, 3/4)$. Hence, when speaking about the classical bound of a bayesian game, we can say it by the maximum payoff obtainable by Ammann's correlated equilibrium.

5.1.3 Non-local correlation

The quantum strategy (ρ, M^{t_1}, M^{t_2}) that gives the maximum winning probability is as follows:

$$\rho = |\phi^+\rangle\langle\phi^+|$$

where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is the maximally entangled state. And the measurement strategy is,

$$M^0_{a_1} = |\phi_{a_1}(0)\rangle \left\langle \phi_{a_1}(0) \right|, \quad M^1_{a_1} = \left| \phi_{a_1}\left(\frac{\pi}{4}\right)\right\rangle \left\langle \phi_{a_1}\left(\frac{\pi}{4}\right)\right|$$
$$M^0_{a_2} = \left|\phi_{a_2}\left(\frac{\pi}{8}\right)\right\rangle \left\langle \phi_{a_2}\left(\frac{\pi}{8}\right) \right|, \quad M^0_{a_2} = \left| \phi_{a_2}\left(-\frac{\pi}{8}\right)\right\rangle \left\langle \phi_{a_2}\left(-\frac{\pi}{8}\right)\right|$$

where $\phi_0(\theta) = \cos\theta|0\rangle + \sin\theta|1\rangle$ and $\phi_0(\theta) = -\sin\theta|0\rangle + \cos\theta|1\rangle$.

The corresponding stochastic matrix using the relation $Q(a \mid t) = \operatorname{Tr} \rho M^{t_1}_{a_1} \otimes M^{t_2}_{a_2}$ is,

$$[Q]^t_a = \frac{1}{2}\begin{bmatrix} \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \end{bmatrix} = \begin{bmatrix} 0.43 & 0.43 & 0.43 & 0.07 \\ 0.07 & 0.07 & 0.07 & 0.43 \\ 0.07 & 0.07 & 0.07 & 0.43 \\ 0.43 & 0.43 & 0.43 & 0.07 \end{bmatrix} \tag{5.10}$$

We can now evidently witness the bell violation in the conditional probability distribution of Eq.(5.12), as it significantly crosses the limit of local correlation Eq.(**??**). And the maximum winning probability with this quantum advice is 0.85!

Thus the non-local/quantum correlated strategies of this game can be described as follows:

$$[Q]^t_a = \begin{bmatrix} \frac{3}{8} < x^{00}_{00} \leq 0.43 & \frac{3}{8} < x^{01}_{00} \leq 0.43 & \frac{3}{8} < x^{10}_{00} \leq 0.43 & \frac{1}{8} > x^{11}_{00} \geq 0.07 \\ \frac{1}{8} > x^{00}_{01} \geq 0.07 & \frac{1}{8} > x^{01}_{01} \geq 0.07 & \frac{1}{8} > x^{10}_{01} \geq 0.07 & \frac{3}{8} < x^{11}_{01} \leq 0.43 \\ \frac{1}{8} > x^{00}_{10} \geq 0.07 & \frac{1}{8} > x^{01}_{10} \geq 0.07 & \frac{1}{8} > x^{10}_{10} \geq 0.07 & \frac{3}{8} < x^{11}_{10} \leq 0.43 \\ \frac{3}{8} < x^{00}_{11} \leq 0.43 & \frac{3}{8} < x^{01}_{11} \leq 0.43 & \frac{3}{8} < x^{10}_{11} \leq 0.43 & \frac{1}{8} > x^{11}_{11} \geq 0.07 \end{bmatrix} \tag{5.11}$$

As we can see, in-order to classically emulate the above non-local correlation, the players have to communicate their type to the correlation device or the trusted mediator. And finally, there you have the most "moral" solution to settle the dispute between your friends (or siblings)!. By implementing a quantum resource, you can not only avoid extracting private information from them, but you can also avoid directly giving them "manipulative" advice to settle the irritating fight.

5.1.4 Super Quantum correlation

The CHSH game can be won with probability 1 with the following non-signalling strategy:

$$[Q]_a^t = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix} \tag{5.12}$$

We can see the above correlation significantly crosses the limits of non-local quantum correlation. Such belief invariant correlations that cannot be emulated by quantum mechanics, but require the players to communicate with the device is called super quantum correlation. The upper limit 0.43 and lower limit 0.07 for corresponding x_a^ts of Eq.(5.11) is the Tsilerson bound of this game.

Chapter 6

Conclusion

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