

# Non-classical correlations in the language of Bayesian game theory

*9th semester thesis project progress report*

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## ABSTRACT

An efficient way to study non-local correlations is from the perspective of equilibria in games with incomplete information, aka Bayesian games, drawing inspiration from concepts in both game theory and quantum physics. These equilibria can be categorized as general communication equilibria, belief-invariant equilibria, and correlated equilibria, all of which contain the well-known Nash equilibria in decreasing order of effectiveness. In the 1990s, the concept of belief-invariant equilibrium first surfaced in game theory. However, the class of non-signaling correlations related to belief-invariance naturally emerged in the 80s in quantum theory. The two theoretical roots of the concept are explained and combined in [1]. The aforementioned types of equilibria and quantum-correlated equilibria are studied using tools from quantum information but speaking the language of algorithmic game theory. This report reviews the broad framework of belief-invariant communication equilibria provided in [1] that includes particular examples of correlated and quantum correlated equilibria. The work also includes the Bells theorem and its violations as a result of non-locality, a subject of great interest in the theoretical underpinnings of quantum mechanics. This report also presents an intuitive way to approach the already familiar non-local games, such as the CHSH in Bayesian game-theoretic language, to demonstrate bell violation and Tsilerson bound. Ref.[2] reviews bell theorem extensively and studies the possibility of setting up Bell's inequality violating experiment in the context of cosmology and string theory. The framework we study here provides an abstract but comprehensive language to study bell violating scenarios. To view the framework with an eye on exploring profound implications in foundational physics experiments remains a deep quest for this project. Another profound quest that's presently on focus in context to game theory is "quantum advantage with separable states".

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# Chapter 1

## Introduction

Imagine that you are in a situation to resolve a dispute between your two angry friends or siblings. A smart way to settle the issue is by listening to the concerns causing the fight from each of them "privately" and giving back your advice (again, "privately"). But can you resolve their issue by being less evil? that is, by not extracting embarrassing private information from them? Well, quantum theory provides you with a way to do that!

The above-mentioned scenario, which everyone would have experienced in life, can be seen as a form of a game with incomplete information (also called a Bayesian game) with a trusted mediator. The mathematical proof that some quantum correlations, unlike all other correlations in the universe, cannot result from any local cause was made by Northern Irish physicist John Bell in 1964. The strong connection between Bell nonlocality and Bayesian games was earlier discussed by Nicolas Brunner, and Noah Linden in their work [3]. Before I go further, I want to emphasize here the following beautiful lines from their paper " More recently a theory of generalized nonlocal correlations has been developed [4, 3], which has a direct impact on fundamental questions in the foundations of quantum mechanics [5, 6]. In a completely different area, but only three years after Bell's ground-breaking discovery, Harsanyi [7] developed a framework for games with incomplete information, that is, games in which players have only partial information about the setting in which the game is played ". It's important to note that the notion in bayesian game theory "non-signaling strategies," had already naturally aroused in the foundations of quantum

mechanics a decade before its formulation. This report reviews Ref.[1], which unifies the frameworks developed later since John Bell's discovery and the CHSH game.

# Chapter 2

## Preliminaries

This chapter gives a short summary of some preliminary concepts and notations.

### 2.1 Use of notations

#### 2.1.1 Sets and tuples

In this section I will explain some use of notations in this report with an intention to make it easy for a reader while going through this report. Here I use capital alphabets to denote sets. For example,  $S_1 = \{0, 1\}$ . If  $n \in \mathbb{N}$ , then  $[n] := \{1, 2, \dots, n\}$ . A Cartesian product of sets  $S_{i \in [n]}$  denotes  $S := \times_{i \in [n]} S_i := S_1 \times S_2 \times \dots \times S_n$ . An element  $s \in S$  is a tuple denoted as  $s := (s_1, s_2, \dots, s_n)$ , where  $s_i \in S_i$  for  $i \in [n]$ . We can also denote a tuple as  $s = (s_i)_{i \in [n]}$ . For example,  $S_1 = \{0, 1\}$  and  $S_2 = \{0, 1, 2\}$ , then  $S := \times_{i \in [2]} S_i = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ .

If we select an element  $i \in [n]$ , then  $s_{-i}$  denotes a tuple of every other elements except  $i$ . That is  $s_{-i} \in \times_{j \neq i}^n S_j$ . We can also denote the full tuple as  $s := (s_i, s_{-i})$ . This notation is essential to keep in mind when describing unilateral deviation of a player  $i$  in game when defining its equilibrium. Also when speaking about a marginal of a joint distribution, which we'll discuss shortly.

#### 2.1.2 Functions

Consider  $R_1$  as an input set and  $S_1$  as the output set. Then  $S_1^{R_1} : \{f_\gamma : R_1 \rightarrow S_1 \mid \gamma = 1, 2, \dots, |S_1|^{|R_1|}\}$  denote the set of all possible functions mapping from  $R_1$  to  $S_1$ .



For example, if  $R_1 = \{0, 1\}$  and  $S_1 = \{0, 1\}$ , then

$$S_1^{R_1} := \{f_1 : x \mapsto 0, f_2 : x \mapsto x, f_3 : x \mapsto x \oplus 1, f_4 : x \mapsto 1\}$$

.

## 2.2 Joint probability distribution

If we have a set  $R = \times_{i \in [n]} R_i$ , then  $P(r) = \Pr \{\mathbf{R} = r\}$  denotes a joint distribution over  $R$ , where  $\mathbf{R}$  is a random variable that takes in values  $r \in R$ .

### 2.2.1 Example

If  $R = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , it is always convenient and intuitive to write down probability distributions as probability vectors like so:

$$P_r = \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix} \quad (2.1)$$

Where,  $\sum_{(r_1, r_2) \in R} p_{r_1 r_2} = 1$ ,  $p_{r_1 r_2} \in [0, 1]$ .

### 2.2.2 Marginals

If  $P(r)$  is a joint distribution for  $r \in R$ , then for an  $i \in [n]$ , the marginal of  $r_i$  is described by the distribution  $P(r_i) = \sum_{r_{-i}} P(r) = \sum_{r_{-i}} P(r_i r_{-i})$ , for  $r_{-i} \in \times_{j \neq i}^n R_j$ .

We can also speak about the marginal of rest of the elements as  $r_{-i}$  by the distribution  $P(r_{-i}) = \sum_{r_i \in R_i} P(r) = \sum_{r_i \in R_i} P(r_i r_{-i})$ .

#### Example

Consider the joint distribution of Eq.(2.1). The marginals of  $r_1$  and  $r_2$  are:

$$P_{r_1} = \begin{bmatrix} p_{00} + p_{01} \\ p_{10} + p_{11} \end{bmatrix}, \quad P_{r_2} = \begin{bmatrix} p_{00} + p_{10} \\ p_{01} + p_{11} \end{bmatrix} \quad (2.2)$$

### 2.2.3 Product distribution

$P(r)$  is a product distribution if  $P(r) = \prod_{i \in [n]} P(r_i)$ , where  $P(r_i) = \sum_{r_{-i}} P(r)$  are marginals of each element  $i \in [n]$ .

#### Example

Consider the joint distribution of the form,

$$P_r = \begin{bmatrix} p_0 q_0 \\ p_0 q_1 \\ p_1 q_0 \\ p_1 q_1 \end{bmatrix} \quad (2.3)$$

where  $\sum_{s_1 \in S_1} p_{s_1} = 1, p_{s_1} \in [0, 1]$  and  $\sum_{s_2 \in S_2} q_{s_2} = 1, q_{s_2} \in [0, 1]$ .

$$P_{r_1} = \begin{bmatrix} p_0 q_0 + p_0 q_1 \\ p_1 q_0 + p_1 q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \quad P_{r_2} = \begin{bmatrix} p_0 q_0 + p_1 q_0 \\ p_0 q_1 + p_1 q_1 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \quad (2.4)$$

And of course,

$$P_r = \begin{bmatrix} p_0 q_0 \\ p_0 q_1 \\ p_1 q_0 \\ p_1 q_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \otimes \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \quad (2.5)$$

## 2.3 Joint conditional probability distribution (correlation)

Consider we have sets of input and output tuples  $R$  and  $S$ . Then the joint conditional probability distribution for the occurrence of  $s \in S$  given in  $r \in R$  has occurred is denoted  $Q(s \mid r)$ , with the condition  $\sum_s Q(s \mid r) = 1$ .

### 2.3.1 Product

The conditional probability distribution is a product distribution if  $Q(s \mid r) = \prod_{i \in [n]} Q(s_i \mid r_i)$

### 2.3.2 Local correlation

$Q(s \mid r)$  is locally correlated if it can be decomposed into a convex combination of the product distributions:

$$Q(s \mid r) = \sum_{\gamma} \prod_{i \in [n]} Q(s_i \mid r_i, \gamma_i) \Gamma(\gamma)$$

Where  $\Gamma(\gamma)$  is a joint probability distribution.

### 2.3.3 Non-signalling

For a subset  $I \subset [n]$ , let  $R_I = \times_{i \in I} R_i$  and  $S_I = \times_{i \in I} S_i$ , a correlation  $Q(s \mid r)$  is belief invariant if for all subsets  $I \subset [n]$  and  $J = [n] \setminus I$ ,

$$\sum_{s_J \in S_J} Q(s_I, s_J \mid r_I, r_J) = \sum_{s_J \in S_J} Q(s_I, s_J \mid r_I, r'_J) \quad \forall s_I \in S_I, r_I \in R_I, r_J, r'_J \in R_J$$

# Chapter 3

## n-player Bayesian games with correlation device

### 3.1 The Game

Following the notations and definitions from [1], the n-player game with incomplete information/Bayesian game with correlation device is defined as follows,

$$G = (N, T_i, P, A_i, (Q, R_i, S_i), v_i)$$

where the notations are described as follows,

- $N = [n]$  is the set of  $n$  players.
- $T_i$  is the set of types for player  $i \in [n]$ .
- $A_i$  is the set of actions player  $i$  can take.
- $T = \times_{i \in N} T_i$  is the finite set of type profiles.
- $A = \times_{i \in N} A_i$  is the finite set of action profiles.
- $P(t)$  is the prior probability distribution over  $t \in T$ .
- $v_i : T_i \times A_i \rightarrow \mathbb{R}$  is the utility function for player  $i \in N$ .
- the general correlation device:
  - $R_i$  is the set of inputs for player  $i$  that the player can feed to the correlation device.

- $S_i$  is the set of outputs player  $i$  player can receive from the correlation device.
- $Q$  is the joint conditional probability distribution over the output profiles  $S = \times_i S_i$  given an input from the input profiles  $T = \times_i T_i$ .

The role of the correlation device will be explained in detail further in the section. I have created a depiction of the entire game Fig.[3.1] in order to have an easy intuition of the framework.

## 3.2 Strategies and equilibria in Bayesian games: in the absence of correlation device

In a Bayesian game setting, the players privately receive their types from an external agent (which can be a mediator or the environment) who draws the type profile  $t \in T$  from a joint probability distribution over  $T$ . The players then have to choose their actions based on their respective type. In this section we'll see how a strategy can be defined in a Bayesian game.

### 3.2.1 Pure strategies

In absence of correlation device a pure strategy for a player is a function mapping from the type set to action set. So for a player  $i$  a pure strategy is:

$$g_i : T_i \rightarrow A_i$$

We can say  $g_i \in A_i^{T_i}$ , where  $A_i^{T_i}$  is the set of all functions mapping  $T_i$  to  $A_i$ ; which will have  $|A_i|^{|T_i|}$  number of functions as elements. Thus a strategy set for a player  $i$  is  $A_i^{T_i}$ .

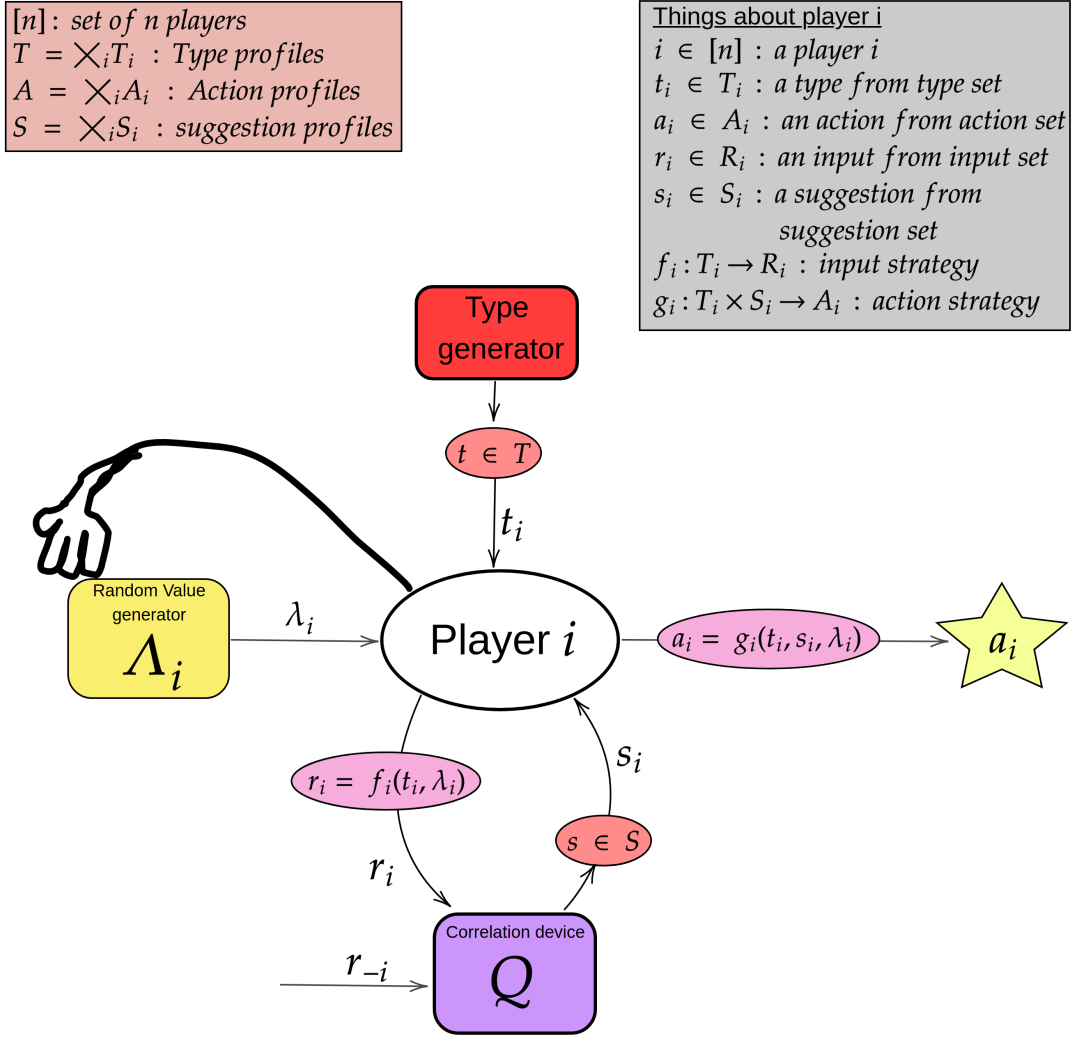


Figure 3.1: Depiction of the entire game. The reader can revisit this image as making progress through this report for better intuition.

Given the type profiles are drawn from a probability distribution  $P$  over  $T$ , the expected utility for a player  $i$  for strategy profile  $g = (g_i)_{i \in [n]}$ <sup>1</sup> given the player received the type  $t_i$  is:

$$\langle v_i(t_i, g) \rangle = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) v_i(t, g(t))$$

Where  $t = (t_i)_{i \in [n]} \in T$  and  $g(t) = (g_i(t_i))_{i \in [n]}$ . And with the notation  $T_{-i} = \times_{j \neq i}^n T_j$ ,  $P(t_{-i}|t_i)$  implies the probability that the rest of the players type profile is  $t_{-i} \in T_{-i}$  given player  $i$ 's type is  $t_i$ .

The overall utility for player  $i$  is:

$$\langle v_i(g) \rangle = \sum_{t_i \in T_i} P(t_i) \langle v_i(t_i, g) \rangle = \sum_{t \in T} P(t) v_i(t, g(t))$$

### 3.2.2 Mixing strategies with random variable

In ref.[1], a mixed strategy is formulated by assigning a random variable  $\lambda_i$  as an argument to the pure strategy function  $g_i$ , so that  $g_i(\cdot, \lambda_i)$  is now a random function. For better understanding, we can say all player  $i \in [n]$  holds a random value generator that output a variable  $\lambda_i$  with probability  $\Lambda_i(\lambda_i)$ . And of course  $\sum_{\lambda_i} \Lambda_i(\lambda_i) = 1$ . The player  $i$  now just looks at the outcome  $\lambda_i$  emitted by the random value generator and then takes the action.

Now given the type  $t_i$  of the player, the player can emulate any mixed strategy  $\tilde{g}_i \in (\Delta A_i)^{T_i}$  using a suitable random value generator  $\Lambda_i$  and taking a suitable decision based on its outcome. So given  $t_i$ , if player wants to decide  $a_i$  on seeing different  $\lambda_i$ , then

$$g_i(a_i|t_i) = \sum_{\lambda_i} g_i(a_i|t_i, \lambda_i) \Lambda_i(\lambda_i)$$

To understand the procedure, here are some examples I made.

---

<sup>1</sup>The notation means  $(g_i)_{i \in [n]} = (g_1, g_2, \dots, g_n)$

**Examples:**

$$A_i = \{walk, run\}, T_i = \{lazy, testy\}$$

1.  $\Lambda_i \rightarrow$  "a coin".

- Decision:

- $(t_i = lazy)$ : then if  $head \rightarrow walk$ ,  $tail \rightarrow walk$
- $(t_i = testy)$ : then if  $head \rightarrow walk$ ,  $tail \rightarrow run$

- strategy:

$$\begin{aligned} & - g_i(walk|lazy, head) = 1, g_i(walk|lazy, tail) = 1, \\ & \quad g_i(run|lazy, head) = 0, g_i(run|lazy, tail) = 0 \\ & - g_i(walk|testy, head) = 1, g_i(walk|testy, tail) = 0, \\ & \quad g_i(run|testy, head) = 0, g_i(run|testy, tail) = 1 \\ & \implies g_i(walk|lazy) = 1, g_i(run|lazy) = 0, g_i(walk|testy) = 1/2, \\ & \quad g_i(run|testy) = 1/2 \end{aligned}$$

2.  $\Lambda_i \rightarrow$  "a dice"..

- Decision:

- $(t_i = lazy)$ : then if  $\lambda_i = even \rightarrow walk$ ,  $\lambda_i = odd \rightarrow walk$
- $(t_i = testy)$ : then if  $\lambda_i = even \rightarrow walk$ ,  $\lambda_i = odd \rightarrow run$

- strategy:

$$\begin{aligned} & - g_i(walk|lazy, \lambda_i = even) = 1, g_i(walk|lazy, \lambda_i = odd) = 1, \\ & \quad g_i(run|lazy, \lambda_i = even) = 0, g_i(run|lazy, \lambda_i = odd) = 0 \\ & - g_i(walk|testy, \lambda_i = even) = 1, g_i(walk|testy, \lambda_i = odd) = 0, \\ & \quad g_i(run|testy, \lambda_i = even) = 0, g_i(run|testy, \lambda_i = odd) = 1 \end{aligned}$$



$$\begin{aligned} \implies g_i(\text{walk}|\text{lazy}) &= 1, g_i(\text{run}|\text{lazy}) = 0, g_i(\text{walk}|\text{testy}) = 1/2, \\ g_i(\text{run}|\text{testy}) &= 1/2 \end{aligned}$$

The above example also shows how random variables with a different number of outcomes can emulate the same mixed strategies (in this case, a coin and a dice).

### 3.2.3 Nash equilibrium

The strategy  $g$  is a nash equilibrium if for all  $i, t_i$  and  $a_i$ ,

$$\begin{aligned} \sum_{t_{-i}, \lambda} P(t_{-i} | t_i) \Lambda(\lambda) v_i(t, g_{-i}(t_{-i}, \lambda_{-i}) g_i(t_i, \lambda_i)) \\ \geq \sum_{t_{-i}, \lambda} P(t_{-i} | t_i) \Lambda(\lambda) v_i(t, g_{-i}(t_{-i}, \lambda_{-i}) a_i) \end{aligned}$$

where  $\lambda = (\lambda_1, \dots, \lambda_n)$ , and, by independence of  $\lambda_i$ ,  $\Lambda(\lambda) = \prod_{i=1}^n \Lambda_i(\lambda_i)$ . By Nash's theorem [8], every game of incomplete information has an equilibrium.

## 3.3 Strategies and equilibria in Bayesian games: in the presence of correlation device

A more general solution concept than the well-known Nash equilibrium in games of complete information is correlated equilibrium. This powerful concept was first introduced by Robert Aumann in his work [9] in 1974. The idea is that players now look at private signals from a random value generator shared by all players to decide on their actions. This shared resource can be interpreted as a trusted mediator who draws an action profile  $a$  from a joint probability distribution over  $A$  and sends the corresponding action  $a_i$  privately to each player  $i \in [n]$  as advice. If the joint probability distribution over  $A$  is such that no player wouldn't want to deviate from the advised action given rest of the players follow the advice, then the distribution is a correlated equilibrium.

Ever since Harsanyi[7] formulated games of incomplete information or Bayesian games, there have been quite some attempts to bring in the same concept of Aumann to the new framework. I leave the review of those pieces of literature for a future article. For now in this report, I discuss the equilibrium concepts described for Bayesian games in the presence of a correlation device from Ref.[1].

### 3.3.1 The general correlation device $Q$

Since a player's utility in Bayesian games depends on the inputs or types received by every player, including the player's own type, the player should get advice based on the type profile. A general correlation device is an object that privately takes inputs from every player, and based on the input profile, it implements a joint probability distribution over output profiles to generate advice for the players. The general correlation device can be dubbed a communication device if it directly takes private input communicated by the players. Firstly, for better intuition, let's see how it is different from the random value generator that is privately owned by the player to implement mixed strategies by the following points:

- The Correlation device  $Q$  is not something that the players privately own, unlike  $\Lambda_i$ ; it's sort of a public property.
- However, players can only interact with it privately; meaning no player gets to know other player's business with it.
- Unlike the  $\Lambda_i$ , the Correlation device  $Q$  doesn't just spit out a random variable to a player. It takes private input from all the players before doing it.

Now let us see how the general correlation device works with the following points:

- All player  $i \in [n]$  now has to decide the input  $r_i$ , based on their type  $t_i$ . Thus the player now has a new strategy  $f_i : T_i \rightarrow R_i$ .

- In fact it can be a special case that  $r_i = t_i$  when  $f_i$  is an identity function. The variable  $r_i$  is introduced in order to allow the player to fool the device by sending a false type.
- For mixed strategy the player randomize the function by  $f_i(t_i, \lambda_i)$  by using the random value generator  $\Lambda_i$ .
- Receives private inputs  $r_i \in R_i$  from each player  $i \in [n]$  and holds the input profile  $r \in R$ .
- Based on  $r$  it generates a joint conditional probability distribution  $Q(s | r)$  over suggestion profiles  $S$ , with

$$\sum_{s \in S} Q(s|r) = 1 \quad \forall r \in R$$

- Outputs  $s$  based on  $Q(s|r)$  and privately sends suggestion  $s_i$  to each player  $i$ .
- The only thing a player knows about  $Q$  is the joint probability distribution  $Q(s | r)$  for all possible  $r \in R$ , but doesn't know the current  $r_{-i}$ . And obviously player knows what he sends in and what he receives out from it.
- Based on  $s_i$  and  $t_i$ , the player  $i$  decides on the action  $a_i$  by  $g_i : S_i \times T_i \rightarrow A_i$ .
  - For mixed strategy the player randomize the function by  $g_i(t_i, s_i, \lambda_i)$  using the same random value  $\lambda_i$  from the random value generator  $\Lambda_i$

### **Belief-invariant (aka non-signaling) correlations**

If  $All(S | R)$  denotes the set of all possible joint conditional probability distribution over  $S$  for every given input profiles  $R$ , there is a subset of correlations  $BINV(S | R) \subset All(S | R)$  for which every joint conditional probability distribution  $Q \in BINV(S | R)$

cannot give any information about other players inputs  $r_{-i}$  to each player  $i$  using his/her own input  $r_i$  and  $s_i$  generated by  $Q$ .

Formally, for a set  $I \subset N$ , let  $R_I = \times_{i \in I} R_i$  and  $S_I = \times_{i \in I} S_i$ , a correlation  $Q(s \mid r)$  is belief invariant if for all subsets  $I \subset N$  and  $J = N \setminus I$ ,

$$\sum_{s_J \in S_J} Q(s_I, s_J \mid r_I, r_J) = \sum_{s_J \in S_J} Q(s_I, s_J \mid r_I, r'_J) \quad \forall s_I \in S_I, r_I \in R_I, r_J, r'_J \in R_J$$

Expected utility given  $t_i$

$$\langle v_i(t_i, f, g) \rangle = \sum_{t_{-i}, s, \lambda} P(t_{-i} \mid t_i) \Lambda(\lambda) Q(s \mid f(t, \lambda)) v_i(t, g(t, s, \lambda))$$

### signaling and non-signaling communication equilibrium

$(f, g, Q)$  is communication equilibrium if  $\forall i \in [n]$ ,  $\forall t_i \in T_i$ , and for all random functions  $f'_i \in R_i^{T_i}$  and  $g'_i \in A_i^{T_i \times S_i}$ :

$$\begin{aligned} & \sum_{t_{-i}, s, \lambda} P(t_{-i} \mid t_i) \Lambda(\lambda) Q(s \mid f(t, \lambda)) v_i(t, g(t, s, \lambda)) \geq \\ & \sum_{t_{-i}, s, \lambda} P(t_{-i} \mid t_i) \Lambda(\lambda) Q(s \mid f'_i(t_i, \lambda_i) f_{-i}(t_{-i}, \lambda_{-i})) v_i(t, g'_i(t_i, s_i, \lambda_i) g_{-i}(t_{-i}, s_{-i}, \lambda_{-i})) \end{aligned}$$

The communication equilibrium is called signaling if the conditional probability distribution  $Q$  is signaling. And the equilibrium is called belief-invariant or non-signaling communication equilibrium if  $Q$  is non-signaling.

# Chapter 4

## Quantum correlated strategies and equilibrium for Bayesian games

### 4.1 POVMs

In line with the fact that POVMs are a generalization of projection-valued measures (PVM), quantum measurements described by POVMs are also a generalization of quantum measurements described by PVMs (called projective measurements). A POVM can be compared to a PVM in a rough parallel to what a mixed state is to a pure one. A positive operator-valued measure (POVM) is a set  $\{M_s\}_s$  of operators that satisfy non-negativity and completeness:

$$\forall s : M_s \geq 0, \quad \sum_s M_s = I.$$

The probability for obtaining outcome  $s$  with  $M_s$  acting on a density operator  $\rho$  (pure or mixed) is  $\text{Tr}\{\rho M_s\}$ . We also don't have to worry about post-measurement state for POVMs.

Ref.[1] gives the following form to illustrate classical probabilities in the quantum framework.

$$M_s = \sum_x \mu_s(x) |x\rangle\langle x|$$

where  $\{|x\rangle\}$  is some fixed orthonormal basis, and  $\mu_s \geq 0 \forall s \in S$  with  $\sum_s \mu_s(x) = 1 \forall x$ .

Completeness check:

$$\sum_s M_s = \sum_s \sum_x \mu_s(x) |x\rangle\langle x| = \sum_x \sum_s \mu_s(x) |x\rangle\langle x| = \sum_x |x\rangle\langle x| = I$$

**Examples:** Here are some examples I came up with during the internship for better understanding.

1.  $\left\{ M_1 = |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{4}|2\rangle\langle 2|, \quad M_2 = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|, \quad M_3 = \frac{1}{4}|2\rangle\langle 2| \right\}$
2.  $\left\{ M_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \quad M_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \right\}$

Here are some other examples of POVMs that are not of the above form:

Consider two arbitrary non-orthonormal states  $|a\rangle$  and  $|b\rangle$  so that  $\langle a|b\rangle \neq 0$ . Since  $\langle \psi|a\rangle\langle a|\psi\rangle \geq 0$  for any  $|\psi\rangle \in \mathcal{H}$ , the following can be a POVM:

$$M_1 = \alpha|a\rangle\langle a|, \quad M_2 = \beta|b\rangle\langle b|, \quad M_3 = I - M_1 - M_2$$

Considering normalized states,  $\alpha, \beta > 0$  just has to satisfy,

$$\alpha + \beta|\langle a|b\rangle|^2, \quad \alpha|\langle a|b\rangle|^2 + \beta, \quad \beta|\langle a^\perp|b\rangle|^2, \quad \alpha|\langle a|b^\perp\rangle|^2 < 1$$

Then the following is a POVM,

$$M_1 = \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}|a\rangle\langle a|, \quad M_2 = \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}|b\rangle\langle b|, \quad M_3 = I - \frac{1}{1+|\langle a^\perp|b^\perp\rangle|}(|a\rangle\langle a| + |b\rangle\langle b|).$$

## 4.2 Quantum formalism

The procedure of the Bayesian game setup with the quantum correlated device can be understood with the following points:

1. The quantum correlation device assigns Hilbert spaces  $\mathcal{H}_i$  (a qudit register) to each player  $i$  and implements a density operator  $\rho$  on the joint Hilbert space  $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ .

- The state  $\rho$  can be pure or mixed. In fact, the ensemble of the composite pure states can be either formed from a joint or disjoint distribution.
2. Sends the respective qudit registers to its assigned players so they can do the local measurement  $M^{t_i}$  based on their type to get their suggestions  $s_i$ .
  3. So in this setup, the information about the player's type stays with the player itself as its not communicated to the device (or the mediator). And the device implements the state  $\rho$  independent of the players type profile.
- This is the quantum analog of classical correlated strategy where the correlation device implements joint probability distribution independent of the player's type profile. The reason why the equilibrium that arises of it is called quantum correlated equilibrium.

So the formulation now is,

$$Q(s \mid r) = \text{Tr}(\rho M_s^r)$$

Where  $M_s^r = \bigotimes_i M_{s_i}^{r_i}$ . Again, for better intuition I created the following depiction Fig.[4.1].

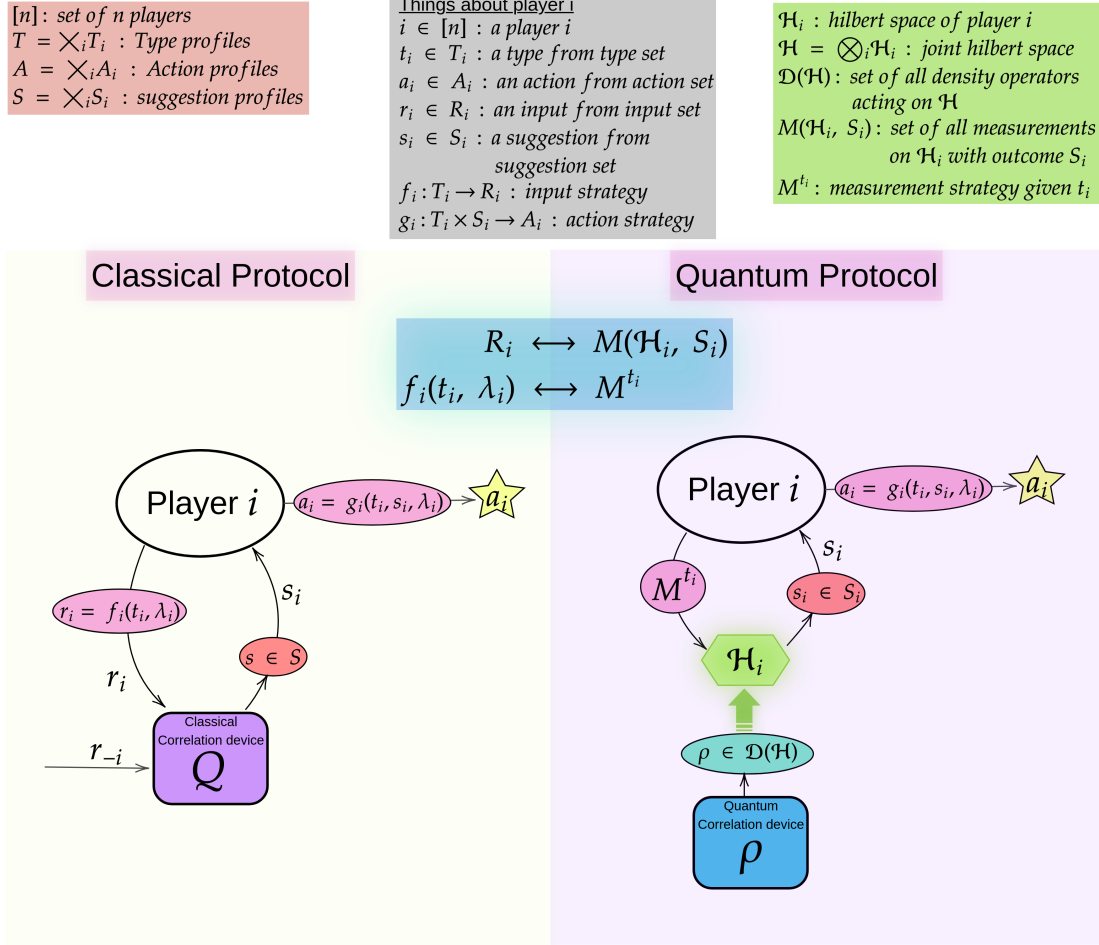


Figure 4.1: Easy depiction of the game setup with quantum correlation device and its comparison with the classical setup. Assume the type generator part in Fig.[3.1] is already there. Its been removed here just to give emphasis to the part of the image where the distinction is made.



### 4.3 Belief invariance of quantum correlated strategy

It can be easily showed that the quantum correlated strategy is non-signalling as follows:  $\sum_{s_J} \bigotimes_{j \in J} M_{s_j}^{r_j} = \bigotimes_{j \in J} \sum_{s_j} M_{s_j}^{r_j} = \bigotimes_{j \in J} \mathbb{I}$ . So,

$$\begin{aligned} \sum_{s_J \in S_J} q(s_I, s_J \mid r_I, r_J) &= \sum_{s_J \in S_J} \text{Tr} \rho \left( \bigotimes_{i \in I} M_{s_i}^{r_i} \otimes \bigotimes_{j \in J} M_{s_j}^{r_j} \right) \\ &= \text{Tr} \rho \left( \bigotimes_{i \in I} M_{s_i}^{r_i} \otimes \bigotimes_{j \in J} \mathbb{I} \right) \\ &= \sum_{s_J \in S_J} \text{Tr} \rho \left( \bigotimes_{i \in I} M_{s_i}^{r_i} \otimes \bigotimes_{j \in J} M_{s_j}^{r'_j} \right) \\ &= \sum_{s_J \in S_J} q(s_I, s_J \mid r_I, r'_J), \end{aligned}$$

This fact naturally arises from non-signalling and non-cloning theorems of quantum theory.

### 4.4 Emulating classical correlations.

1. Local correlation:  $Q(\mathbf{s} \mid \mathbf{r}) = \sum_{\gamma} V(\gamma) L_1(s_1 \mid r_1 \gamma_1) \cdots L_n(s_n \mid r_n \gamma_n)$

$$\rho = \sum_{\gamma} V(\gamma) |\gamma_1\rangle\langle\gamma_1| \otimes \cdots \otimes |\gamma_n\rangle\langle\gamma_n|, \quad M_{s_i}^{r_i} = \sum L_i(s_i \mid r_i \gamma_i) |\gamma_i\rangle\langle\gamma_i|$$

So,

$$\text{LOC}(S \mid R) \subset Q(S \mid R) \subset \text{BINV}(S \mid R)$$

This relation will be demonstrated in the following section 5.1.

2. canonical correlation:

$$\rho = \sum_{\mathbf{s}} Q(\mathbf{s}) |s_1\rangle\langle s_1| \otimes \cdots \otimes |s_n\rangle\langle s_n|, \quad M_{a_i}^{t_i} = \sum_{s_i} \delta_{g_i(t_i, s_i), a_i} |s_i\rangle\langle s_i|$$

So classical correlations can be emulated with mixed density operator of pure orthonormal states and suitable POVMs.

## 4.5 Quantum correlated equilibrium

Before we move on, let's introduce the following notation that I came up with for unilateral deviation of measurement strategy:

$$[X_{s_i}^{r_i}] M_{-s_i}^{-r_i} = \bigotimes_{j < i} M_{s_j}^{r_j} \otimes X_{s_i}^{r_i} \otimes \bigotimes_{j > i} M_{s_j}^{r_j}$$

### 4.5.1 what $f_i$ is now on this setup?

Since the quantum correlation device does not take any inputs there is no point of  $R_i$  here. So players can choose any measurements they want from the infinite set  $M(\mathcal{H}_i, S_i)$  based on their type. So the prior input strategy  $f_i$  transforms to POVM strategy in the quantum setup as follows:

$$\{f_i : T_i \rightarrow R_i\} \longleftrightarrow \{M_i : T_i \rightarrow M(\mathcal{H}_i, S_i)\}$$

So now  $M^{r_i}$  is re-denoted as  $M_i(t_i) = M^{t_i}$ .

### 4.5.2 Canonical correlation and solution

If we have  $M_{a_i}^{t_i} = \sum_{s_i} \delta_{g_i(t_i, s_i), a_i} |s_i\rangle\langle s_i|$  or directly the observables as actions  $A_i$ , then we can write the representation of canonical correlation:

$$Q(a | t) = \text{Tr}(\rho M_a^t)$$

We now have  $(M^t, \rho)$  as equilibrium for all  $i, t_i$  and  $X_{a_i}^{t_i} \in M(\mathcal{H}_i, A_i)$  if,

$$\begin{aligned} \langle v_{i, t_i}(M^t, \rho) \rangle = \\ \sum_{t_{-i}, \mathbf{a}} P(t_{-i} | t_i) \text{Tr}(\rho M_a^t) v_i(t, a) \geq \sum_{t_{-i}, \mathbf{a}} P(t_{-i} | t_i) \text{Tr}(\rho [X_{a_i}^{t_i}] M_{-a_i}^{-t_i}) v_i(t, a) \end{aligned}$$

# Chapter 5

## Non-local games and their conversion to conflicting interest games

### 5.1 Demonstrating Bell violation and Tsirelson's bound with CHSH game in Bayesian game-theoretic language

The CHSH game in Bayesian game-theoretic language as in 3.1 can be described as follows:

$N = \{1, 2\}$ ,  $T_i = \{0, 1\}$ ,  $A_i = \{0, 1\}$ ,  $P(t) = \frac{1}{4} \forall t \in T$ , and the utility function is described as:

$$v_{i=1,2}(t_1 t_2, a_1 a_2) = \begin{cases} 0 & \text{if } t_1 \cdot t_2 \neq a_1 \oplus a_2 \\ 1 & \text{if } t_1 \cdot t_2 = a_1 \oplus a_2 \end{cases} \quad (5.1)$$

Remember, in Bayesian game theory, the information about a player's type and the action they are going to execute stays within the player itself. And there is no communication between the players. The CHSH game is a simple game to analyze. However, with this example, I present a much more comprehensive method to approach the Bayesian game, which would come in handy while analyzing more sophisticated conflicting interest games. I start by expanding the payoff function Eq.(5.1), to a payoff tensor:

$$V_a^t = \begin{array}{c|cccc} & \begin{array}{c} t \\ \hline a \end{array} & 00 & 01 & 10 & 11 \\ \hline 00 & (1, 1) & (1, 1) & (1, 1) & (0, 0) \\ 01 & (0, 0) & (0, 0) & (0, 0) & (1, 1) \\ 10 & (0, 0) & (0, 0) & (0, 0) & (1, 1) \\ 11 & (1, 1) & (1, 1) & (1, 1) & (0, 0) \end{array} \quad (5.2)$$

To clarify the notation,  $v_{a_1 a_2}^{t_1 t_2} = (v_1(t_1 t_2, a_1 a_2), v_2(t_1 t_2, a_1 a_2))$ . Where  $v_1$  is the payoff for player 1, and  $v_2$  is the payoff function for player 2.

Now, for the next step, I would write out every possible strategies for the player. Remember, strategy in a Bayesian is a "function" mapping from types to action. So the strategy set to both players  $i \in \{1, 2\}$  is:

$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\} \quad (5.3)$$

To show you an example, a strategy profile  $g = (g_1^\gamma, g_2^\gamma)$  would give the payoff profile  $V_{g_1^\gamma(t_1)g_2^\gamma(t_2)}^{t_1 t_2} = (v_1(t_1 t_2, g_1^\gamma(t_1)g_2^\gamma(t_2)), v_2(t_1 t_2, g_1^\gamma(t_1)g_2^\gamma(t_2)))$ . For example, the strategy profile  $g = (g_1^1, g_2^3)$ , the type profile  $t = (0, 1)$  would give the payoff profile,  $V_{g_1^1(t_1)g_2^3(t_2)}^{t_1 t_2} = V_{g_1^1(0)g_2^3(1)}^{01} = V_{00}^{01} = (1, 1)$ .

So now we would further expand the payoff tensor of Eq.(5.2) as follows:

$$V_a^t = \begin{array}{c|cccc} & \begin{array}{c} t \\ \hline g \end{array} & 00 & 01 & 10 & 11 \\ \hline g_1^1 g_2^1 & (1, 1) & (1, 1) & (1, 1) & (0, 0) \\ g_1^1 g_2^2 & (1, 1) & (0, 0) & (1, 1) & (1, 1) \\ g_1^1 g_2^3 & (0, 0) & (1, 1) & (0, 0) & (0, 0) \\ g_1^1 g_2^4 & (0, 0) & (0, 0) & (0, 0) & (1, 1) \\ g_1^2 g_2^1 & (1, 1) & (1, 1) & (0, 0) & (1, 1) \\ g_1^2 g_2^2 & (1, 1) & (0, 0) & (0, 0) & (0, 0) \\ g_1^2 g_2^3 & (0, 0) & (1, 1) & (1, 1) & (1, 1) \\ g_1^2 g_2^4 & (0, 0) & (0, 0) & (1, 1) & (0, 0) \\ g_1^3 g_2^1 & (0, 0) & (0, 0) & (1, 1) & (0, 0) \\ g_1^3 g_2^2 & (0, 0) & (1, 1) & (1, 1) & (1, 1) \\ g_1^3 g_2^3 & (1, 1) & (0, 0) & (0, 0) & (0, 0) \\ g_1^3 g_2^4 & (1, 1) & (1, 1) & (0, 0) & (1, 1) \\ g_1^4 g_2^1 & (0, 0) & (0, 0) & (0, 0) & (1, 1) \\ g_1^4 g_2^2 & (0, 0) & (1, 1) & (0, 0) & (0, 0) \\ g_1^4 g_2^3 & (1, 1) & (0, 0) & (1, 1) & (1, 1) \\ g_1^4 g_2^4 & (1, 1) & (1, 1) & (1, 1) & (0, 0) \end{array} \quad (5.4)$$

Now we can take the average over the distribution of type profiles to obtain the payoff of each strategy profile.

$$V_a^t = \begin{array}{c|c} g & \sum_t V_g^t P(t) \\ \hline g_1^1 g_2^1 & (3/4, 3/4) \\ g_1^1 g_2^2 & (3/4, 3/4) \\ g_1^1 g_2^3 & (1/4, 1/4) \\ g_1^1 g_2^4 & (1/4, 1/4) \\ g_1^2 g_2^1 & (3/4, 3/4) \\ g_1^2 g_2^2 & (1/4, 1/4) \\ g_1^2 g_2^3 & (3/4, 3/4) \\ g_1^2 g_2^4 & (1/4, 1/4) \\ g_1^3 g_2^1 & (1/4, 1/4) \\ g_1^3 g_2^2 & (3/4, 3/4) \\ g_1^3 g_2^3 & (1/4, 1/4) \\ g_1^3 g_2^4 & (3/4, 3/4) \\ g_1^4 g_2^1 & (1/4, 1/4) \\ g_1^4 g_2^2 & (1/4, 1/4) \\ g_1^4 g_2^3 & (3/4, 3/4) \\ g_1^4 g_2^4 & (3/4, 3/4) \end{array} \quad (5.5)$$

### 5.1.1 Nash equilibrium

It is obvious that the Nash equilibrium for common interest games (where players always get equal payoffs) is the strategy profile that gives the highest payoff for each player. However, we further our analysis as this section emphasizes the methodology for approaching a Bayesian game, which should incredibly help in analyzing far more sophisticated games of conflicting interest.

For demonstration, we can now convert Eq.(5.5) to the good old payoff matrix:

$$V_a^t = \begin{array}{c|cccc} & \text{pl 2} & g_2^1 & g_2^2 & g_2^3 & g_2^4 \\ \hline \text{pl 1} & & & & & \\ g_1^1 & (3/4, 3/4) & (3/4, 3/4) & (1/4, 1/4) & (1/4, 1/4) \\ g_1^2 & (3/4, 3/4) & (1/4, 1/4) & (3/4, 3/4) & (1/4, 1/4) \\ g_1^3 & (1/4, 1/4) & (3/4, 3/4) & (1/4, 1/4) & (3/4, 3/4) \\ g_1^4 & (1/4, 1/4) & (1/4, 1/4) & (3/4, 3/4) & (3/4, 3/4) \end{array} \quad (5.6)$$

It is now easy to see which all strategy profiles are Bayesian Nash equilibrium. They are  $(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$ .

With the above payoff matrix, we now have kind of converted a game of incomplete information (bayesian game) to complete information. As a matter of fact, one can actually convert a bayesian game to its induced normal form [10] and also agent normal form [11], which we will discuss in the final thesis. Games of complete information don't necessarily have any actual quantum advantage [12]. However, I find it's a great idea to look into certain games of complete information that are in their induced normal form or agent form of a particular bayesian game that would have an actual quantum advantage.

### 5.1.2 Local correlation: Aumann's correlated equilibrium

Most of us may have heard the maximum classical winning probability is well known to be  $\frac{3}{4}$ . This is, of course, apparent from the previous subsection. Even though same number, it is not true that the classical boundary of local realism is set by what we have discussed in the previous section. Nash equilibria are a result of independent decision-making and are never locally correlated. The concept of Aumann's correlated equilibria is a far more general and profound concept than that of Nash.

Now coming back to our analysis, if a game has multiple Nash equilibria, then a convex hull of all of it forms a specific set of correlated equilibria (I will discuss these theorems in a separate chapter for Aumann's concept and as well for Harsanyi's framework in the final thesis). Since there are eight Nash equilibria, let us consider the convex combination by the maximal distribution,

$$g = \frac{1}{8}(g_1^1 \otimes g_2^1) + \frac{1}{8}(g_1^1 \otimes g_2^2) + \frac{1}{8}(g_1^2 \otimes g_2^1) + \frac{1}{8}(g_1^2 \otimes g_2^3) + \frac{1}{8}(g_1^3 \otimes g_2^2) + \frac{1}{8}(g_1^3 \otimes g_2^4) + \frac{1}{8}(g_1^4 \otimes g_2^3) + \frac{1}{8}(g_1^4 \otimes g_2^4) \quad (5.7)$$

The interpretation is that an adviser draws these strategy profiles from a uniform distribution and recommends the strategy ("the whole function") corresponding to each player privately. Now, writing out the expanded payoff tensor is a painful process

for this case. Thus, a better and more efficient approach is to form the correlation  $Q(a | t)$ . To do so, we represent each function as deterministic channels  $g_i^\gamma(a_i | t_i) = \delta_{a_i, g_i^\gamma(t_i)}$ . So the correlation by Eq.(5.7) is,

$$\begin{aligned}
 Q(a | t) = & \frac{1}{8} \delta_{a_1, g_1^1(t_1)} \delta_{a_1, g_2^1(t_2)} + \frac{1}{8} \delta_{a_1, g_1^1(t_1)} \delta_{a_1, g_2^2(t_2)} + \frac{1}{8} \delta_{a_1, g_1^2(t_1)} \delta_{a_1, g_2^1(t_2)} \\
 & + \frac{1}{8} \delta_{a_1, g_1^2(t_1)} \delta_{a_1, g_2^2(t_2)} + \frac{1}{8} \delta_{a_1, g_1^3(t_1)} \delta_{a_1, g_2^3(t_2)} + \frac{1}{8} \delta_{a_1, g_1^3(t_1)} \delta_{a_1, g_2^4(t_2)} \\
 & + \frac{1}{8} \delta_{a_1, g_1^4(t_1)} \delta_{a_1, g_2^3(t_2)} + \frac{1}{8} \delta_{a_1, g_1^4(t_1)} \delta_{a_1, g_2^4(t_2)} \quad (5.8)
 \end{aligned}$$

Expanding the above conditional probability distribution strategy  $Q(a | t)$  can be expanded by correlation matrix for each input and output:

$$Q(a | t) = \begin{array}{c|cccc} & \text{t} & 00 & 01 & 10 & 11 \\ \hline \text{a} & 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ & 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ & 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ & 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \quad (5.9)$$

Taking the marginals:

$$\begin{aligned}
 Q(a_1 | t_1(0)) &= \begin{array}{c|cc} a_1 \backslash t_1 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}, \quad Q(a_1 | t_1(1)) = \begin{array}{c|cc} a_1 \backslash t_1 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array} \\
 Q(a_2 | t_2(1)) &= \begin{array}{c|cc} a_2 \backslash t_2 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}, \quad Q(a_2 | t_2(1)) = \begin{array}{c|cc} a_2 \backslash t_2 & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{array}
 \end{aligned}$$

Thus,  $Q(a | t)$  in Eq.(5.9) is clearly not a product distribution as  $Q(a | t) \neq Q(a_1 | t_1)Q(a_2 | t_2)$ , and hence this is not a product correlation formed by the independent choice of mixed strategy. It's a local correlation formed by correlated advice of full function strategy. And of the expected payoff profile is  $\sum_t Q(a | t)v(a, t)P(t) = (3/4, 3/4)$ . Hence, when speaking about the classical bound of a bayesian game, we can say it by the maximum payoff profile obtainable by Ammann's correlated

equilibrium.

### 5.1.3 Non-local correlation

The quantum strategy  $(\rho, M^{t_1}, M^{t_2})$  that gives the maximum winning probability is as follows:

$$\rho = |\phi^+\rangle\langle\phi^+|$$

where  $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$  is the maximally entangled state. And the measurement strategy is,

We now consider the POVM strategy as a projection operator as a special case. Say the players perform measurements on an arbitrary linear independent basis,  $\{|\phi_0(\theta_i^{t_i})\rangle = \cos \theta_i^{t_i}|0\rangle + \sin \theta_i^{t_i}|1\rangle, |\phi_1(\theta_i^{t_i})\rangle = -\sin \theta_i^{t_i}|0\rangle + \cos \theta_i^{t_i}|1\rangle\}$ , the projection operators are,

$$M_{a_1}^0 = |\phi_{a_1}(\theta_1^0)\rangle\langle\phi_{a_1}(\theta_1^0)|, \quad M_{a_1}^1 = |\phi_{a_1}(\theta_1^1)\rangle\langle\phi_{a_1}(\theta_1^1)|$$

$$M_{a_2}^0 = |\phi_{a_2}(\theta_2^0)\rangle\langle\phi_{a_2}(\theta_2^0)|, \quad M_{a_2}^1 = |\phi_{a_2}(\theta_2^1)\rangle\langle\phi_{a_2}(\theta_2^1)|$$

The conditional distribution for quantum correlation is given by  $Q(a \mid t) = \text{Tr } \rho M_{a_1}^{t_1} \otimes M_{a_2}^{t_2}$ .

Expanding out the distribution,

t a	00	01	10	11
00	$\frac{1}{2} \cos^2(\theta_1^0 - \theta_2^0)$	$\frac{1}{2} \cos^2(\theta_1^0 - \theta_2^1)$	$\frac{1}{2} \cos^2(\theta_1^1 - \theta_2^0)$	$\frac{1}{2} \cos^2(\theta_1^1 - \theta_2^1)$
01	$\frac{1}{2} \sin^2(\theta_1^0 - \theta_2^0)$	$\frac{1}{2} \sin^2(\theta_1^0 - \theta_2^1)$	$\frac{1}{2} \sin^2(\theta_1^1 - \theta_2^0)$	$\frac{1}{2} \sin^2(\theta_1^1 - \theta_2^1)$
10	$\frac{1}{2} \sin^2(\theta_1^0 - \theta_2^0)$	$\frac{1}{2} \sin^2(\theta_1^0 - \theta_2^1)$	$\frac{1}{2} \sin^2(\theta_1^1 - \theta_2^0)$	$\frac{1}{2} \sin^2(\theta_1^1 - \theta_2^1)$
11	$\frac{1}{2} \cos^2(\theta_1^0 - \theta_2^0)$	$\frac{1}{2} \cos^2(\theta_1^0 - \theta_2^1)$	$\frac{1}{2} \cos^2(\theta_1^1 - \theta_2^0)$	$\frac{1}{2} \cos^2(\theta_1^1 - \theta_2^1)$

(5.10)

Now looking at the payoff tensor of the game Eq.(5.4) what can we make out of the above quantum correlation distribution? My technique would be to somehow convert the sin in the last column ( $t = 11$ ) to cos and the cos to sin without disturbing the rest of the columns. So we would have the equation,



$$|\theta_1^0 - \theta_2^0| = |\theta_1^0 - \theta_2^1| = |\theta_1^1 - \theta_2^0| = \frac{\pi}{2} - |\theta_1^1 - \theta_2^1| \quad (5.11)$$

So a solution of the above relation that maximizes  $\cos^2(\theta_1^0 - \theta_2^0)$  and the rest is  $(\theta_1^0, \theta_1^1) = (0, \frac{\pi}{4})$  and  $(\theta_2^0, \theta_2^1) = (\frac{\pi}{8}, -\frac{\pi}{8})$ . So substituting these angles to the correlation matrix, we get,

$$Q(a | t) = \frac{1}{2} \begin{bmatrix} \cos^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} \\ \sin^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} \\ \sin^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} \\ \cos^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} & \cos^2 \frac{\pi}{8} & \sin^2 \frac{\pi}{8} \end{bmatrix} = \begin{bmatrix} 0.43 & 0.43 & 0.43 & 0.07 \\ 0.07 & 0.07 & 0.07 & 0.43 \\ 0.07 & 0.07 & 0.07 & 0.43 \\ 0.43 & 0.43 & 0.43 & 0.07 \end{bmatrix} \quad (5.12)$$

$$Q(a | t) = \begin{array}{c|cccc} & \begin{array}{c} t \\ \hline a \end{array} & 00 & 01 & 10 & 11 \\ \hline 00 & 0.43 & 0.43 & 0.43 & 0.07 \\ 01 & 0.07 & 0.07 & 0.07 & 0.43 \\ 10 & 0.07 & 0.07 & 0.07 & 0.43 \\ 11 & 0.43 & 0.43 & 0.43 & 0.07 \end{array} \quad (5.13)$$

We can now evidently witness the bell violation in the conditional probability distribution of Eq.(5.13), as it significantly crosses the limit of local correlation Eq.(5.8). And the maximum payoff profile with this quantum advice is (0.85, 0.85)! And this establishes the Tsirelson's bound.

Thus the non-local/quantum-correlated strategies for particular entangled state  $\rho$  of this game can be described as follows:

$$Q(a | t) = \begin{bmatrix} \frac{3}{8} < x_{00}^{00} \leq 0.43 & \frac{3}{8} < x_{00}^{01} \leq 0.43 & \frac{3}{8} < x_{00}^{10} \leq 0.43 & \frac{1}{8} > x_{00}^{11} \geq 0.07 \\ \frac{1}{8} > x_{01}^{00} \geq 0.07 & \frac{1}{8} > x_{01}^{01} \geq 0.07 & \frac{1}{8} > x_{01}^{10} \geq 0.07 & \frac{3}{8} < x_{01}^{11} \leq 0.43 \\ \frac{1}{8} > x_{10}^{00} \geq 0.07 & \frac{1}{8} > x_{10}^{01} \geq 0.07 & \frac{1}{8} > x_{10}^{10} \geq 0.07 & \frac{3}{8} < x_{10}^{11} \leq 0.43 \\ \frac{3}{8} < x_{11}^{00} \leq 0.43 & \frac{3}{8} < x_{11}^{01} \leq 0.43 & \frac{3}{8} < x_{11}^{10} \leq 0.43 & \frac{1}{8} > x_{11}^{11} \geq 0.07 \end{bmatrix} \quad (5.14)$$

As we can see, to classically emulate the above non-local correlation, the players must communicate their type to the correlation device or the trusted mediator. And finally, there you have the most "moral" solution to settle the dispute between your friends (or siblings)! By implementing a quantum resource, you can not only avoid

extracting private information from them, but you can also avoid directly giving them "manipulative" advice to settle the irritating fight.

#### 5.1.4 Super Quantum correlation

The CHSH game can be won with probability 1 with the following non-signaling strategy:

$$Q(a | t) = \begin{array}{c|cccc} & \begin{array}{c} t \\ \hline a \end{array} & 00 & 01 & 10 & 11 \\ \hline 00 & 0.5 & 0.5 & 0.5 & 0 \\ 01 & 0 & 0 & 0 & 0.5 \\ 10 & 0 & 0 & 0 & 0.5 \\ 11 & 0.5 & 0.5 & 0.5 & 0 \end{array} \quad (5.15)$$

We can see the above correlation significantly crosses the limits of non-local quantum correlation and gives the expected profile (1,1). Such belief invariant correlations that cannot be emulated by quantum mechanics but require the players to communicate with the device is called super quantum correlation. The upper limit 0.43 and lower limit 0.07 for corresponding  $x_a^t$ s of Eq.(5.14) is the Tsilerson bound of this game.

#### 5.1.5 Modifications to conflicting interest games

To be continued for the final thesis...

# Chapter 6

## Conclusion

Chapters 3 and 4 gave a brief review of the broad frame framework formulated and studied in [1] in an intuitive way, which also addresses why quantum non-local correlations are non-signaling or belief invariant. Although the influence of the collapse of wavefunction by measurement on one of the entangled pairs done by one party affects the state of another party far away instantaneously (the spooky action at a distance), one can never communicate information by non-local correlation since it is nonsignalling. In chapter 5, I demonstrated bell's violation and Tsilerson bound comprehensively using the popular CHSH game speaking in the language of Bayesian game theory. Although the CHSH game is simple and already well analyzed, my attempt in chapter 5 was to propose a unique conceptual and comprehensive methodology for approaching any sophisticated games of incomplete information.

### 6.1 Works on progress

In chapter 5 we have only demonstrated the well-familiar CHSH game. Due to lack of time, I wasn't able to give a similarly detailed analysis for non-local games, such as the GHZ pseudo-telepathic games and their modifications to conflicting interest games. More analysis on some of the original conflicting interest games I am devising that should have a quantum advantage is missed out in this current report. All of them would definitely come in the final thesis. One latest Idea is to explore specific games of complete information that would be in its agent normal or is in its induced normal form of some game of incomplete information that would have non-local quantum

equilibrium. I find it super fascinating to look at it. For example, Football can be viewed as a Bayesian game that is in its agent normal form.

## 6.2 A quest on focus:

Whether entanglement is required for quantum advantage in Bayesian games is a significant outstanding topic at the basic level. This is unquestionably accurate for cooperative games (Bell inequalities). But is it feasible to develop a conflict-of-interest game in which a separable quantum-advice equilibrium exists and outperforms all classically correlated equilibria? Or is it the case that all classically correlated equilibria can mimic quantum?

- To approach this problem, an extensive review of Harsanyi's framework on Bayesian games and Aumann's concept of correlated equilibrium is a must. I would be writing separate detailed chapters for both of them for the final thesis.
- The methodology of analyzing a bayesian I demonstrated in Chapter 5 should give us some interesting clues.

## 6.3 Future quest

Ref.[2] titled "Bell violation in the sky" gives an extensive review of bells theory in its original language and proposes experiments to test bell violations in context to string theory and cosmology. The framework we study here gives a simple and comprehensive language to approach bell-violating scenarios. Non-local games can, in fact, be used for designing experimental setups [13]. I would love to explore more, and I find it fascinating to know how the framework we are studying here can be used in pioneering experimental tests in foundational physics.

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