Non-classical correlations in the language of Bayesian game theory

This title would literally make sense by the end!

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Course: 9th semester thesis project

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My initial motivation

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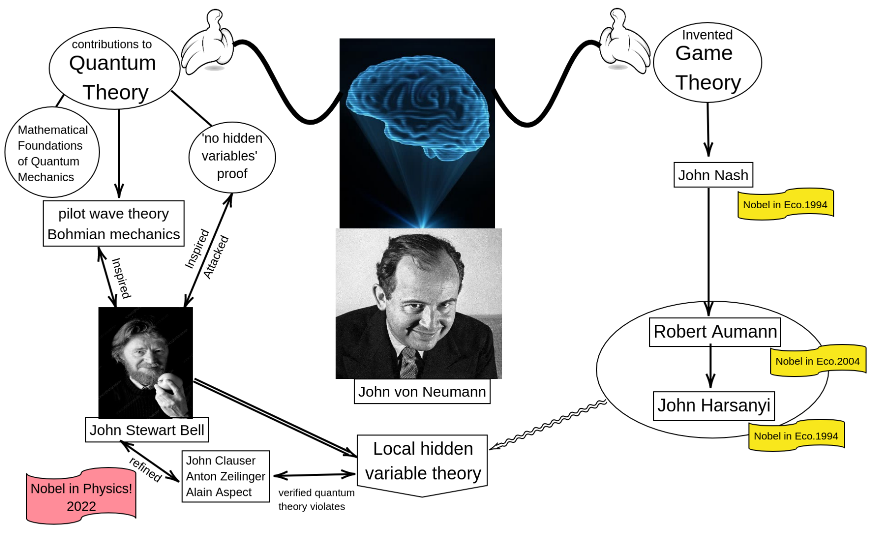


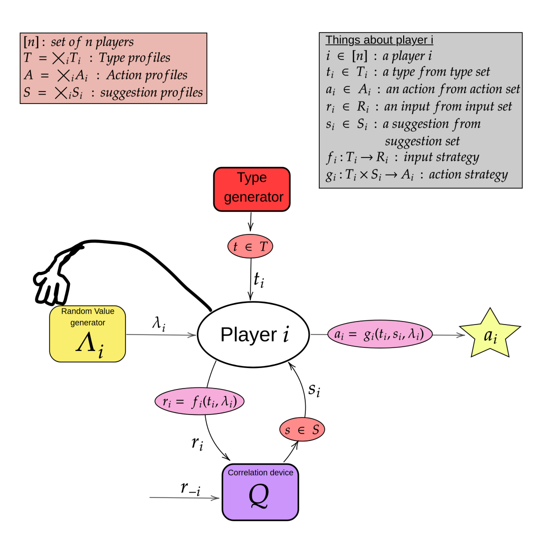
References for now and future

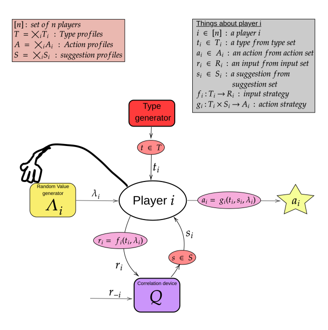
- The main reference until now (more work in progress):
  - Vincenzo Auletta, Diodato Ferraioli, Ashutosh Rai, Giannicola Scarpa, and Andreas Winter. Belief-invariant and quantum equilibria in games of incomplete information. Theoretical Computer Science, 895:151–177, dec 2021.
- An eye to keep on exploring profound implications in foundational physics(for the future):
  - Sayantan Choudhury, Sudhakar Panda, and Rajeev Singh. Bell violation in the sky. The European Physical Journal C, 77(2):1–181, 2017.

The general framework

n-player Bayesian game

- A set of players: $[n] := \{1, 2, \ldots, n\}$
- Action set for each player $i \in [n]$: A_i
  - Action profile $A = \times_i A_i$
- Type set for each player: T_i
  - Type profile $T = \times_i T_i$
  - A joint distribution over type profiles $P(t)$, where $t \in T$
- Utility function for each player i, $v_i : A \times T \to \mathbb{R}$
- A communication device Q:
  - Set of inputs it takes from player i: R_i
  - Set of outputs it gives to player i: S_i
  - The resulting correlation $Q(s \mid r)$, where $s \in S$ and $r \in R$


$[n]$: set of n players
$T = \times_i T_i$: Type profiles
$A = \times_i A_i$: Action profiles
$S = \times_i S_i$: suggestion profiles
Things about player i
$i \in [n]$: a player i
$t_i \in T_i$: a type from type set
$a_i \in A_i$: an action from action set
$r_i \in R_i$: an input from input set
$s_i \in S_i$: a suggestion from suggestion set
$f_i : T_i \to R_i$: input strategy
$g_i : T_i \times S_i \to A_i$: action strategy
Type generator
$t \in T$
t_i
Random Value generator
Λ_i
λ_i
Player i
$a_i = g_i(t_i, s_i, \lambda_i)$
a_i
$r_i = f_i(t_i, \lambda_i)$
s_i
$s \in S$
r_i
r_{-i}
Correlation device
Q



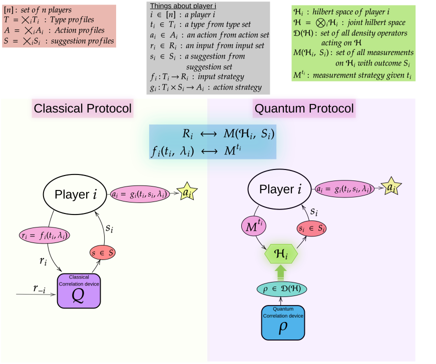
Communication equilibrium: If $\forall i \in [n], \forall t_i \in T_i$

$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f(t,\lambda)\big)v_i\big(t, g(t,s,\lambda)\big) \geq$$

$$\sum_{t_{-i},s,\lambda} P(t_{-i} \mid t_i)\Lambda(\lambda)Q\big(s \mid f_i'(t_i,\lambda_i)f_{-i}(t_{-i},\lambda_{-i})\big)v_i\big(t, g_i'(t_i,s_i,\lambda_i)g_{-i}(t_{-i},s_{-i}\lambda_{-i})\big)$$

Quantum formalism

- Adviser assigns a finite-dimensional Hilbert space $\mathscr{H}_i$ for each player $i \in [n]$
  - They can be any finite-level quantum system (a **qudit** register)
- Implements a state ρ on the joint Hilbert space $\mathscr{H} := \bigotimes_i \mathscr{H}_i := \mathscr{H}_1 \otimes \mathscr{H}_2 \cdots \otimes \mathscr{H}_n$.
  - i.e., $\rho \in \mathscr{D}(\mathscr{H})$
- Sends the assigned qudit register ($\mathscr{H}_i$) to each player $i \in [n]$ privately.
- Players then perform generalized measurement (POVM) $\{M_{a_i}^{t_i}\}_{a_i}$, to obtain the measurement outcome a_i, which would be their advice.
- The correlation formed will be then $Q(a \mid t) = \mathrm{Tr}\left(\rho(M_{a_1}^{t_1} \otimes M_{a_2}^{t_2} \otimes \cdots \otimes M_{a_n}^{t_n})\right)$



The canonical correlation formed:

$$Q(a \mid t) = \mathrm{Tr}\left(\rho(M_{a_1}^{t_1} \otimes M_{a_2}^{t_2} \otimes \cdots \otimes M_{a_n}^{t_n})\right)$$

$\sum_{s_J} \bigotimes_{j\in J} M_{s_j}^{r_j} = \bigotimes_{j\in J} \sum_{s_j} M_{s_j}^{r_j} = \bigotimes_{j\in J} \mathbb{I}$. So,

$$\begin{aligned}
\sum_{s_J \in S_J} q\left(s_I, s_J \mid r_I, r_J\right) &= \sum_{s_J\in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r_j} \right) \\
&= \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} \mathbb{I} \right) \\
&= \sum_{s_J\in S_J} \operatorname{Tr} \rho \left(\bigotimes_{i\in I} M_{s_i}^{r_i} \otimes \bigotimes_{j\in J} M_{s_j}^{r_j'} \right) \\
&= \sum_{s_J\in S_J} q\left(s_I, s_J \mid r_I, r_J'\right),
\end{aligned}$$

Demonstrating Bell violation and Tsirelson's bound with CHSH game in Bayesian game-theoretic language

The CHSH game

Objective:

$$\mathrm{LOC}(S \mid R) \subset \mathrm{Q}(S \mid R) \subset \mathrm{BINV}(S \mid R)$$

The CHSH game

- $N = \{1, 2\}$, $T_i = \{0, 1\}$, $A_i = \{0, 1\}$, $P(t) = \frac{1}{4}\ \forall t \in T$
- the utility function is described as:

$$v_{i=1,2}(t_1 t_2, a_1 a_2) = \begin{cases} 0 & \text{if } t_1 \cdot t_2 \neq a_1 \oplus a_2 \\ 1 & \text{if } t_1 \cdot t_2 = a_1 \oplus a_2 \end{cases}$$

- spread it out:

$$V_a^t = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

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- The strategy set of this ”Bayesian game” is then:

$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\}$$

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

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$$A_i^{T_i} = \{g_i^1 : x \mapsto 0, g_i^2 : x \mapsto x, g_i^3 : x \mapsto x \oplus 1, g_i^4 : x \mapsto 1\}$$

- The Nash equlibria of this game are:

$$(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$$

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

- The Nash equilibria of this game are(all are product distribution(deterministic), not a Local correlation!):

$$(g_1^1 g_2^1), (g_1^1, g_2^2), (g_1^2, g_2^1), (g_1^2, g_2^3), (g_1^3, g_2^2), (g_1^3, g_2^4), (g_1^4, g_2^3), (g_1^4, g_2^4)$$

- All gives the payoff profile $(3/4, 3/4)$:
  - Although the number is correct, conceptually, this doesn't define the actual classical bound! (when it comes to conflicting interest games)

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

- The **convex hull** of Nash equilibria forms a specific class of correlated equilibria.
- Convex combination by maximal distribution:

$$\begin{aligned} Q(a \mid t) = {} & \frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^1(t_2)} + \frac{1}{8}\delta_{a_1,g_1^1(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^1(t_2)} \\ & + \frac{1}{8}\delta_{a_1,g_1^2(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^2(t_2)} + \frac{1}{8}\delta_{a_1,g_1^3(t_1)}\delta_{a_1,g_2^4(t_2)} \\ & + \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^3(t_2)} + \frac{1}{8}\delta_{a_1,g_1^4(t_1)}\delta_{a_1,g_2^4(t_2)} \end{aligned} \tag{1}$$

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

- spreading out the conditional probability distribution:

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \tag{2}$$

- This is the Local correlation!
- Conceptually, when speaking about Aumann's correlated advice of functions (strategies) in bayesian games, this is it!
-

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & 3/8 & 3/8 & 3/8 & 1/8 \\ 01 & 1/8 & 1/8 & 1/8 & 3/8 \\ 10 & 1/8 & 1/8 & 1/8 & 3/8 \\ 11 & 3/8 & 3/8 & 3/8 & 1/8 \end{array} \tag{3}$$

- Of course, the maximum payoff profile we get with this correlation is $(3/4, 3/4)$

$N = \{1, 2\}$, $T_i = \{0, 1\}$, $A_i = \{0, 1\}$, $P(t) = \frac{1}{4}$ $\forall t \in T$

Consider the quantum strategy (ρ, M^{t_1}, M^{t_2})

- $\rho = |\phi^+\rangle\langle\phi^+|$ where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$
- POVMs:

$$
\begin{aligned}
M^0_{a_1} &= |\phi_{a_1}(\theta^0_1)\rangle\langle\phi_{a_1}(\theta^0_1)|, \quad & M^1_{a_1} &= |\phi_{a_1}(\theta^1_1)\rangle\langle\phi_{a_1}(\theta^1_1)| \\
M^0_{a_2} &= |\phi_{a_2}(\theta^0_2)\rangle\langle\phi_{a_2}(\theta^0_2)|, \quad & M^1_{a_2} &= |\phi_{a_2}(\theta^1_2)\rangle\langle\phi_{a_2}(\theta^1_2)|
\end{aligned}
$$

where $\{|\phi_0(\theta^{t_i}_i)\rangle = \cos\theta^{t_i}_i|0\rangle + \sin\theta^{t_i}_i|1\rangle, \quad |\phi_1(\theta^{t_i}_i)\rangle = -\sin\theta^{t_i}_i|0\rangle + \cos\theta^{t_i}_i|1\rangle\}$.

- payoff tensor:

$$V_a^t = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & (1,1) & (1,1) & (1,1) & (0,0) \\ 01 & (0,0) & (0,0) & (0,0) & (1,1) \\ 10 & (0,0) & (0,0) & (0,0) & (1,1) \\ 11 & (1,1) & (1,1) & (1,1) & (0,0) \end{array}$$

-

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \\ 01 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 10 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 11 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \end{array} \tag{4}$$

- $$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \\ 01 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 10 & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\sin^2(\theta_1^1 - \theta_2^1) \\ 11 & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^0 - \theta_2^1) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^0) & \frac{1}{2}\cos^2(\theta_1^1 - \theta_2^1) \end{array} \tag{5}$$
- $$|\theta_1^0 - \theta_2^0| = |\theta_1^0 - \theta_2^1| = |\theta_1^1 - \theta_2^0| = \frac{\pi}{2} - |\theta_1^1 - \theta_2^1| \tag{6}$$
- solution that gives maximum $(\theta_1^0, \theta_1^1) = (0, \frac{\pi}{4})$ and $(\theta_2^0, \theta_2^1) = (\frac{\pi}{8}, -\frac{\pi}{8})$

- $$Q(a \mid t) = \frac{1}{2}\begin{bmatrix} \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} \\ \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \cos^2\frac{\pi}{8} & \sin^2\frac{\pi}{8} \end{bmatrix} \tag{7}$$

- $Q(a \mid t) =$ (8)

a \ t	00	01	10	11
00	0.43	0.43	0.43	0.07
01	0.07	0.07	0.07	0.43
10	0.07	0.07	0.07	0.43
11	0.43	0.43	0.43	0.07

- And the maximum payoff profile is: $(0.85, 0.85)$

The CHSH game can be won with an average payoff profile (1,1) with the following non-signaling strategy:

$$Q(a \mid t) = \begin{array}{c|cccc} {}_{a}\diagdown^{t} & 00 & 01 & 10 & 11 \\ \hline 00 & 0.5 & 0.5 & 0.5 & 0 \\ 01 & 0 & 0 & 0 & 0.5 \\ 10 & 0 & 0 & 0 & 0.5 \\ 11 & 0.5 & 0.5 & 0.5 & 0 \end{array} \tag{9}$$

- This correlation can only be achieved if the types are communicated to the advisor.

Conclusion

Conclusion

- We had a brief look at the general framework for games with incomplete information and quantum formalism.
- We saw a comprehensive demonstration of Bell violation and Tsirelson's bound using the popular CHSH game, but speaking in the language of Bayesian game theory and Aumman's concept.
- Although the CHSH game is simple and already well analyzed, my attempt was to propose a unique conceptual and comprehensive methodology for approaching any sophisticated games with incomplete information of **conflicting interest**.

A new idea!

- Games of complete information apparently have no actual quantum advantage.
- However Games of incomplete information can be somehow converted to games of complete information (induced normal form and agent normal form).
- What are the games with complete information that is actually in its induced or agent normal form of a game with incomplete information that must have a quantum advantage of non-local correlation?

A quest on focus:

- Quantum advantage of separable states in conflicting interest games?

A quest on exploring(perhaps for the future):

- What will be the implication of the framework we are studying here for foundational physics experiments?

Thank You!

Games!

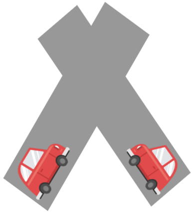
Games!

What is a game?

- An abstract object that has a finite number of "**players.**"
- Each player has a set of "**actions**" to take.
- A player will have a "**consequence.**" But what does that consequence depend on?
  1. The **action** that the **player** decides to take.
  2. The **actions** the "**rest of the players**" take.

Example:

$$
\begin{aligned}
(\text{stay}, \text{stay}) &\implies (\text{mad}, \text{mad}) \\
(\text{stay}, \text{cross}) &\implies (\text{mad}, \text{happy}) \\
(\text{cross}, \text{stay}) &\implies (\text{happy}, \text{mad}) \\
(\text{cross}, \text{cross}) &\implies (\text{crash}, \text{crash})
\end{aligned}
$$



Games!

What is Aumann's concept then?

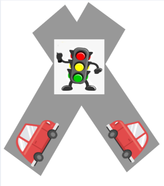
What is this thing about Aumann's concept then?

An adviser comes in...

- Of course, to advise the players
  - on what actions to take.
- But advice each players **privately**
  - For the betterment of all of them.
- Of course, then every player's actions will be **correlated** since they are listening to the same advisor.
  - So the advisor implements the advice in a way that any player disobeying the advice faces a bad consequence (**Aummann's correlated equilibrium!**).

Example:

Half of the time, advises:
(stay, cross) $\implies$ (mad, happy)
Half of the time, advises:
(cross, stay) $\implies$ (happy, mad)



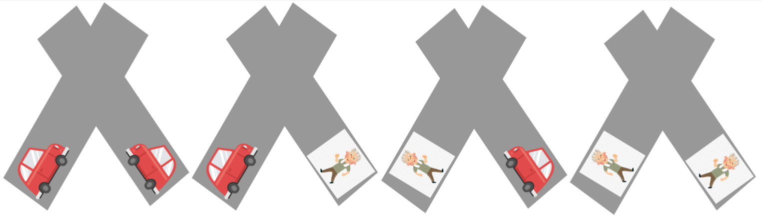
Games!

Games with incomplete information

Games with incomplete information

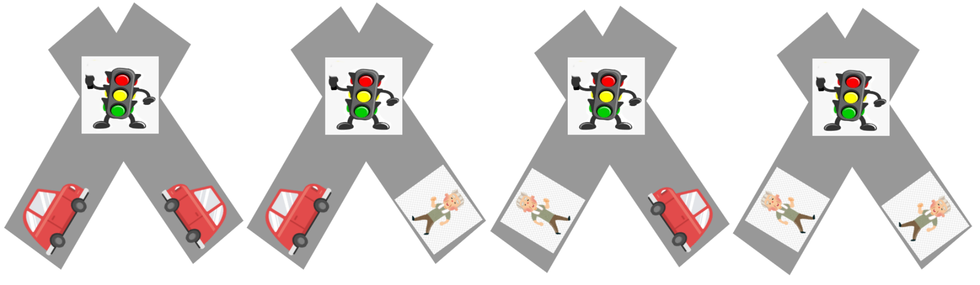
What is this ”**incomplete information**”?

- Now, each player may have a ”**type**”.
- The **consequence** to a player now depends on:
  - the **type** of the player as well.
  - the **types** of **other players** as well.
- But a player does not know what type of players they are facing. But however, they may have a belief about other players’ types (the incomplete information).

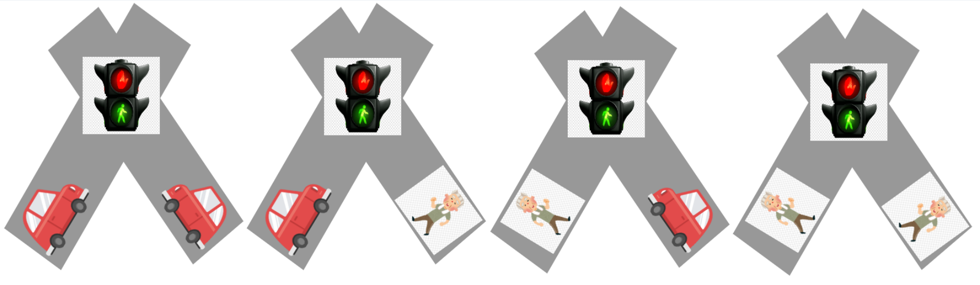


How would we bring in Aumann's concept here?

Would this be right?



Or this?



Correlations

Correlations

So what is a "**correlation**" here?

- Say Player 1 is "blue," and Player 2 is "red."
- Players can be of **types**: {driver, walker}
- Players **actions** to choose: {stay, cross}

$Q(a \mid t) =$

a \ t	d d	d w	w d	w w
s s	0	0	0	0
s c	1/2	1/2	1/2	0
c s	1/2	1/2	1/2	0
c c	0	0	0	1

- Something that a player can learn about the other player's types
  - by observing the advice they receive multiple times on repeated events

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 0 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1 \end{array}$$

$$Q(a_1 \mid t_1\, \text{d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1\, \text{w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

Signalling!

$$Q(a \mid t) = \begin{array}{c|cccc} a \backslash t & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 0 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1 \end{array}$$

$$Q(a_1 \mid t_1 \text{ d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1 \text{ w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

$$Q(a_2 \mid t_2 \text{ d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_2 \mid t_2 \text{ w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 0 \\ \text{c} & 1/2 & 1 \end{array}$$

Non-signalling!

$$Q(a \mid t) = \begin{array}{c|cccc} \text{a} \backslash \text{t} & \text{d d} & \text{d w} & \text{w d} & \text{w w} \\ \hline \text{s s} & 0 & 0 & 0 & 1/2 \\ \text{s c} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c s} & 1/2 & 1/2 & 1/2 & 0 \\ \text{c c} & 0 & 0 & 0 & 1/2 \end{array}$$

$$Q(a_1 \mid t_1 \text{ d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_1 \mid t_1 \text{ w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}$$

$$Q(a_2 \mid t_2 \text{ d}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}, \quad Q(a_2 \mid t_2 \text{ w}) = \begin{array}{c|cc} a_1 \backslash t_1 & \text{d} & \text{w} \\ \hline \text{s} & 1/2 & 1/2 \\ \text{c} & 1/2 & 1/2 \end{array}$$

General :

For a subset $I \subset [n]$, let $R_I = \times_{i \in I} R_i$ and $S_I = \times_{i \in I} S_i$, a correlation $Q(s \mid r)$ is belief invariant if for all subsets $I \subset [n]$ and $J = [n] \backslash I$,

$$\sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r_J\right) = \sum_{s_J \in S_J} Q\left(s_I, s_J \mid r_I, r_J'\right) \quad \forall s_I \in S_I, r_I \in R_I, r_J, r_J' \in R_J$$