I. CLASSICAL HAWK-DOVE GAME

Lets consider a two player strategic form Hawk-Dove game with complete information[3]. Hawks are aggressive and always fight to take possession of a resource. These fights are brutal and the loser is one who first sustains the injury. The winner takes sole possession of the resource. However, Doves never fight for the resource, displaying patience and if attacked immediately withdraw to avoid injury. Thus, Doves will always lose a conflict against Hawk but without sustaining any injury. In case, two Doves face each other there will be a period of displaying patience with some cost (time or energy for display) to both but without any injury. It is assumed that both the Doves are equally good in displaying and waiting for random time. In a Dove-Dove contest, both have equal probability of winning. The winner would be the one with more patience. The classical payoff matrix is represented as follows:

		Bob	
		H	D
	H	$\left(\frac{v}{2}-\frac{i}{2}, \frac{v}{2}-\frac{i}{2}\right)$	(v,0)
Alice	D	(0,v)	$(\frac{v}{2}-d, \frac{v}{2}-d)$

(1)

where v and i are the value of resource and cost of injury, respectively. The cost of displaying patience and waiting is d. Let $v = 50$, $i = 100$ and $d = 10$. Then for the said set of values the payoff table when both Alice and Bob pursue pure strategies is given as-

		Bob	
		H	D
	H	(-25,-25)	(50,0)
Alice	D	(0,50)	(15,15)

(2)

The Nash equilibrium for the classical Hawk-Dove game is (H, D) and (D, H). And (D, D) is the Pareto optimal.

Since, the Hawk-Dove game give rise to two asymmetric equilibrium, there must exist a third mixed strategy nash equilibrium[3]. Such equilibrium can be found by equating the fitness of Hawk, W(H) and the fitness of Dove, W(D) given as,

$$\begin{aligned} W(H) &= p \times \$(H,H) + (1-p) \times \$(H,D) \\ W(D) &= p \times \$(D,H) + (1-p) \times \$(D,D) \end{aligned} \tag{3}$$

where p is the probability of Alice playing Hawk. The equilibrium is when W(H) = W(D). ie,

$$-25p + 50(1-p) = 0 + 15(1-p) \tag{4}$$

$$\implies p = 7/12 \tag{5}$$

And Since the game is symmetric, the probability of Bob playing Hawk, q is also 7/12 at equilibrium. Thus, the mixed strategy Nash equilibrium is when (p, q) = (7/12, 7/12).

The payoff matrix with an added mixed strategy equilibrium "MR_{CD}" with $p = 7/12$ and $q = 7/12$ is,

		Bob		
		H	D	MR_{CD}
	H	(-25,-25)	(50,0)	(6.25, -14.58)
Alice	D	(0,50)	(15,15)	(6.25, 35.42)
	MR_{CD}	(-14.58, 6.25)	(35.42, 6.25)	(6.25, 6.25)

(6)

Hence, the equilibrium is (H, D) and (D, H) and (MR_{CD}, MR_{CD})

II. QUANTUM HAWK-DOVE GAME

To introduce the quantum version of Hawk-Dove game we follow Marinatto and Weber's scheme[12]. This scheme has been extended to include various forms of Hawk-Dove game with initially entangled states in Refs.[13–15]. The initial state ρ_{in} is taken to be a maximally entangled state, $\rho_{in} = |\psi\rangle\langle\psi|$ with

$$|\psi\rangle = \hat{J}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) \tag{7}$$

The two entangled qubits are then forwarded to Alice and Bob respectively who perform unitary operation on the initial state to get-

$$\rho = (U_A \otimes U_B)\rho_{in}(U_A \otimes U_B)^\dagger \tag{8}$$

and then disentangled,

$$\rho_{final} = \hat{J}^\dagger (U_A \otimes U_B)\rho_{in}(U_A \otimes U_B)^\dagger \hat{J} \tag{9}$$

U describes a general strategy represented by a 2×2 unitary matrix parametrized by two parameters $\theta \in [0, \pi]$ and $\phi \in [0, \pi/2]$, and is given as

$$U(\theta, \phi) = \begin{bmatrix} e^{i\phi}\cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\phi}\cos(\theta/2) \end{bmatrix} \tag{10}$$

The strategy operator for the players can be formulated using this unitary operator, with Hawk (H) and Dove(D) being,

$$H = U(\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{11}$$

$$D = U(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{12}$$

Thus, the basis vectors of the Hilbert space will be, $|DD\rangle = (1,\ 0,\ 0,\ 0)$, $|DH\rangle = (0,\ 1,\ 0,\ 0)$, $|HD\rangle = (0,\ 0,\ 1,\ 0)$, $|HH\rangle = (0,\ 0,\ 0,\ 1)$.

Then the payoff operators for Alice and Bob are defined as

$$P_A = \left(\frac{v}{2} - \frac{i}{2}\right)|HH\rangle\langle HH| + v|HD\rangle\langle HD| + \left(\frac{v}{2} - d\right)|DD\rangle\langle DD| \tag{13}$$

$$P_B = \left(\frac{v}{2} - \frac{i}{2}\right)|HH\rangle\langle HH| + v|DH\rangle\langle DH| + \left(\frac{v}{2} - d\right)|DD\rangle\langle DD| \tag{14}$$

The expected payoff for either Alice or Bob are calculated as $\$ = Tr(P\rho_f)$

A. When Alice and Bob are restricted to play only the Quantum Strategies

Now, lets consider a scenario where both the players are restricted to play the quantum strategy "Q" as defined in[6]:

$$Q = U(0, \pi/2) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{15}$$

and the Miracle move defined as-

$$M = U(\pi/2, \pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix} \tag{16}$$

The expected payoffs is calculated using above scheme, giving the payoff table,

		Bob	
		Q	M
	Q	(15, 15)	(32.5, 7.5)
Alice	M	(7.5, 32.5)	(10, 10)

(17)

In this scenario we can see that (Q, Q) becomes the Nash equilibrium and Pareto optimal too. But since, this is the only one dominant strategy here, there cant exist a mixed strategy nash equillibiurm.

B. Random strategy in quantum HAWK-DOVE Game

We now establish that using a random strategy on a maximally entangled state in a purely quantum game scenario that yields a solution and payoff's which cannot be replicated in the classical Hawk-Dove game. Let $|\psi_{in}\rangle$ be a maximally entangled state represented by-

$$|\psi_{in}\rangle = \hat{J}|DD\rangle = \frac{1}{\sqrt{2}}(|DD\rangle + i|HH\rangle) \tag{18}$$

If Alice plays quantum move Q, with probability p and the miracle move M with probability $(1-p)$ and Bob uses these operators with probability q and $(1-q)$, respectively, then the final density matrix of the bipartite system takes the form:

$$\rho_f = pq[\hat{J}^\dagger (Q_A \otimes Q_B)\rho_{in}(Q_A^\dagger \otimes Q_B^\dagger)\hat{J}] + p(1-q)[\hat{J}^\dagger (Q_A \otimes M_B)\rho_{in}(Q_A^\dagger \otimes M_B^\dagger)\hat{J}]$$

$$+(1-p)q[\hat{J}^\dagger (M_A \otimes Q_B)\rho_{in}(M_A^\dagger \otimes Q_B^\dagger)\hat{J}] + (1-p)(1-q)[\hat{J}^\dagger (M_A \otimes M_B)\rho_{in}(M_A^\dagger \otimes M_B^\dagger)\hat{J}] \tag{19}$$

The expected payoff functions for both players is calculated as $\$_A = Tr(P_A\rho_f)$ and $\$_B = Tr(P_B\rho_f)$

We will consider the random strategy as "R_{QM}" where Q and M are played by Alice with probability p = 1/2 and Bob with probability q = 1/2. The payoff table in the larger strategic space which includes pure and the random strategy is as follows-

		Bob		
		Q	M	R_{QM}
Alice	Q	(15, 15)	(32.5, 7.5)	(23.75, 11.25)
	M	(7.5, 32.5)	(10, 10)	(8.75, 21.25)
	R_{QM}	(11.25, 23.75)	(21.25, 8.75)	(16.25, 16.25)

(20)

We can see (Q,Q) is still the Nash equillibrium after adding R_{QM} to the payoff table. But the game is totally different when played with only Quantum strategies with Nash equilibrium being (15, 15) and new Pareto optimal (R_{QM}, R_{QM}). One must also notice that, the Pareto optimal (16.25, 16.25) is slightly greater than Pareto optimal of the classical game. And this cannot be replicated in a Purely classical game. Thereby, negating the Van Enk-Pikes assertion.

[1] Wikipedia entry on Euler's problem, https://en.wikipedia.org/wiki/Eulerian_path.
[2] Neumann, John von and Morgenstern, Oskar, Theory of Games and Economic Behavior. Princeton, NJ. Princeton University Press, 1953.
[3] Game Theory in Action: An Introduction to Classical and Evolutionary Models, Stephen Schecter and Herbert Gintis, Princeton Univ. Press (2016).
[4] Evolutionary game theory and complex networks of scientific information, M Hanauske - Models of Science Dynamics, 2012 - Springer
[5] D. Challet, M. Marsili, Y.-C. Zhang, Minority Games: Interacting Agents in Financial Markets, Oxford University Press, Oxford (2005)
[6] Jens Eisert & Martin Wilkens, Quantum games, Journal of Modern Optics 47:14-15, 2543-2556 (2000); Jens Eisert, Martin Wilkens, and Maciej Lewenstein, Quantum Games and Quantum Strategies, Phys. Rev. Lett. 83, 3077.
[7] D. Meyer, Quantum Strategies, Phy. Rev. Lett. 82, 1052 (1999).
[8] S.J. van Enk and R. Pike, Classical Rules in Quantum Games, Phys. Rev. A 66, 024306 (2002).
[9] Namit Anand and Colin Benjamin, Do quantum strategies always win?, Quantum Information Processing, 14, 4027-4038 (2015). When one of us(CB) submitted the aforesaid paper to Phys. Rev. Lett., the editor asked us to address the fundamental criticism of quantum game theory research as elucidated in Ref. 8.
[10] Azhar Iqbal, PhD thesis, Studies in the Theory of Quantum Games, arXiv:quant-ph/0503176 (2006).
[11] Ken Prestwich, Course on teaching game theory to students of animal behavior, http://college.holycross.edu/faculty/kprestwi/behavior/ESS/ESS_index_frmset.html
[12] L. Marinatto and T. Weber, A Quantum Approach To Static Games Of Complete Information, Phys. Lett. A 272, 291 (2000).
[13] A. Nawaz and A. H. Toor, "Generalized quantization scheme for non-zero sum games" J. Phys. A 37, 11457–11463 (2004).
[14] E. A. B. Lindsell, K. Wiesner, Entanglement gives rise to Pareto optimality in the generalised quantum Hawk-Dove Game, arXiv:1410.3631 [quant-ph]
[15] H. Situ, A quantum approach to play asymmetric coordination games, Quantum Inf. Process. 13, 591 (2014).

SUPPLEMENTARY MATERIAL: RANDOM STRATEGIES IN QUANTUM PRISONER'S DILEMMA

A similar calculations as done for quantum Hawk-Dove can be done for Prisoner's dilemma, where Alice and Bob are allowed to play "C" (I) and "D" (X) (representing the strategies corresponding to *Confess(C)* and *Defect(D)*)

III. CLASSICAL GAME

The classical payoff table for pure strategies in PD is represented as-

		Bob	
		C	D
	C	$(3,3)$	$(0,5)$
Alice	D	$(5,0)$	$(1,1)$

(21)

The Nash equilibrium is (D, D) here.

Since, (D, D) is the only one dominant strategy in PD, there can't exists a Mixed strategy Nash equilibrium. But we may expand the strategic space with a Random strategy with R_{CD} where, p = q = 1/2

		Bob		
		C	D	R_{CD}
	C	(3,3)	(0,5)	(1.5, 4)
Alice	D	(5,0)	(1,1)	(3, 0.5)
	RCD	(4, 1.5)	(0.5, 3)	(2.25, 2.25)

(22)

It's clear that (D, D) is still the Nash equilibrium after a adding mixed strategy to the payoff table.

IV. QUANTUM GAME

When Alice and Bob are restricted to play only the Quantum Strategies Q and M in prisoners dilemma the following payoff matrix can be calculated using the Marinatto and Weber's scheme[12] we incorporated in section II,

		Bob	
		Q	M
	Q	(3,3)	(4, 1.5)
Alice	M	(1.5, 4)	(2.25, 2.25)

(23)

In this scenario we can see that (Q, Q) becomes the Nash equilibrium and Pareto optimal too. And there can't exists a mixed strategy Nash equilibrium since there is only one dominant strategy.

A. Random strategy in quantum Prisoner's Dilemma

Now let $|\psi_{in}\rangle$ be a maximally entangled state represented by:

$$|\psi_{in}\rangle = \hat{J}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) \tag{24}$$

If Alice uses Q, the quantum strategy, with probability p and M with probability $(1-p)$ and Bob uses these operators with probability q and $(1-q)$, respectively. Then the final density matrix of the bipartite system takes the form:

$$\rho_f = pq[\hat{J}^\dagger(Q_A \otimes Q_B)\rho_{in}(Q_A^\dagger \otimes Q_B^\dagger)\hat{J}] + p(1-q)[\hat{J}^\dagger(Q_A \otimes M_B)\rho_{in}(Q_A^\dagger \otimes M_B^\dagger)\hat{J}]$$

$$+(1-p)q[\hat{J}^\dagger(M_A \otimes Q_B)\rho_{in}(M_A^\dagger \otimes Q_B^\dagger)\hat{J}] + (1-p)(1-q)[\hat{J}^\dagger(M_A \otimes M_B)\rho_{in}(M_A^\dagger \otimes M_B^\dagger)\hat{J}] \tag{25}$$

Here $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. The payoff operators for Alice and Bob are defined as

$$P_A = 3|00\rangle\langle 00| + 5|10\rangle\langle 10| + |11\rangle\langle 11| \tag{26}$$

$$P_B = 3|00\rangle\langle 00| + 5|01\rangle\langle 01| + |11\rangle\langle 11| \tag{27}$$

The payoff functions for Alice and Bob are the mean values of the above operators, i.e.,

$$\$_A(p,q) = Tr(P_A\rho_f) \quad and \quad \$_B(p,q) = Tr(P_B\rho_f) \tag{28}$$

Quantum payoff table with random strategy is:

		Bob		
		Q	M	R_{QM}
Alice	Q	(3,3)	(4, 1.5)	(3.5, 2.25)
	M	(1.5, 4)	(2.25, 2.25)	(1.88, 3.12)
	RQM	(2.25, 3.5)	(3.12, 1.88)	(2.69, 2.69)

(29)

In this case, clearly (Q, Q) is NE and Pareto optimal with an added Mixed Quantum strategy in the payoff table. But its clear the that payoff obtained here cannot be replicated in a pure classical game. Thereby, negating the Van Enk-Pikes assertion.