Classification of saturated fusion systems on cyclic and Klein four groups.

Kadir İhsan Sayıcı

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Abstract

The notion of saturated fusion systems is an important concept in the representation theory of finite groups and homotopy theory. Loosely speaking, a (saturated) fusion system on a p-group P, where p is a prime, is a category whose objects are subgroups of P and whose morphisms are injective group homomorphisms satisfying certain axioms. In this report, we first introduce saturated fusion systems and give an important class of examples arising from Sylow p-subgroup of finite groups. We then study a well-known theorem describing all saturated fusion systems on an abelian p-group. As applications of this theorem, we study the classification of saturated fusion systems on Klein four group and some cyclic p-groups.

1 Introduction

Fusion system is an important subject in the areas of group theory, representation theory, and topology[Cra11]. Classifying all possible fusion system on a given p-group is an important task in group theory. In this project, we classify saturated fusion systems on abelian p-groups. We then focus on Klein four-group and cyclic groups. Specifically, in Section 2, we study on some definitions related to introduction part of category theory, since we study on a category called fusion system. Moreover, we learn four definitions related to group theory, not only for refreshing our memory but also for using in the research. In Section 3, we study on a definition, that is a fusion system on a finite p-group P is a category with objects as subgroups of P and morphisms are injective group homomorphisms with certain axioms. Then, we study definitions and a proposition in order to define saturated fusion system with two axioms. After this, we learn our main theorem and its proof that states if we have a *Sylow* p-subgroup, then there exists a saturated fusion system on it. Then, we try to find saturated fusion systems on Klein four-group and try to classify the isomorphism classes, by the help of Theorem 3.7. Besides, we study on a theorem and its proof that states every morphism is restriction of an automorphism of abelian P. Finally, we try to find more saturated fusion systems on Klein four group, by Proposition 3.9, Theorem 3.10, and Proposition 3.11. In Section 4, we try to find saturated fusion systems on some cyclic groups.

2 Preliminaries

In this section, we recall basic notions about category theory and group theory. Definition 2.1, Definition 2.2, and Definition 2.3 can be found in Section 1 of [Bol19]. The other definitions can be found in [DF04].

Definition 2.1 A category C is a structure consisting of

• a class of *objects*, denoted by $Ob(C)$

• for any two objects $X, Y \in Ob(C)$, a set $Hom_C(X, Y)$ called *morphisms* in C from X to Y.

• for any three objects $X, Y, Z \in Ob(C)$, a map

$$Hom_C(X, Y) \times Hom_C(Y, Z) \to Hom_C(X, Z)$$

$$(g, f) \to f \circ g$$

called *composition* satisfying the following:

(i) Associativity of Composition : $(h \circ g) \circ f = h \circ (g \circ f)$.

(ii) For every $X \in Ob(C)$, there exists a morphism called *identity morphism* and denoted by $Id_X \in Hom_C(X, X)$ with the property that for all $W, Y \in Ob(C)$ and all $f \in Hom_C(W, X)$ and $g \in Hom_C(X, Y)$ one has

$$id_X \circ f = f \text{ and } g \circ id_X = g.$$

(iii) If $(X, Y, X^{'}, Y^{'}) \in Ob(C)$ and $(X, Y) \neq (X^{'}, Y^{'})$ then one has

$$Hom_C(X, Y) \cap Hom_C(X^{'}, Y^{'}) = \emptyset.$$

Remark If $f \in Hom_C(X, Y)$, it can be denoted as

$$f : X \to Y$$

By (iii) in Definition 2.1, every morphism $f : X \to Y$ in category C has a *unique* domain and codomain.

Example 2.2 (i) The category *Sets* of sets is the category whose

• Objects : Sets,

• Morphisms : Functions $f : X \to Y$,

• Composition : Usual composition of functions, that is $\circ$.

(ii) The category *Group* of groups is the category whose

• Objects : Groups,

• Morphisms : Group Homomorphisms $f : X \to Y$,

• Composition : Usual composition $\circ$.

Definition 2.3 Let $f : X \to Y$ in a category C.

(a) f is called *monomorphism* if for all arbitrarily chosen objects $A \in C$ and for all $a_1, a_2 \in \mathrm{Hom}_C(A, X)$ one has

$$f \circ a_1 = f \circ a_2 \implies a_1 = a_2.$$

(b) f is called *epimorphism* if for all arbitrarily chosen objects $B \in C$ and for all $b_1, b_2 \in \mathrm{Hom}_C(Y, B)$ one has

$$b_1 \circ f = b_2 \circ f \implies b_1 = b_2.$$

(c) f is called *isomorphism* if there exists $g \in \mathrm{Hom}_C(Y, X)$ with

$$g \circ f = id_X \text{ and } f \circ g = id_Y.$$

Furthermore, for two objects $X, Y \in Ob(C)$, we say that they are isomorphic if there exists an isomorphism $f : X \to Y$ in C.

Definition 2.4 A group is called p-group if all elements of this group has an order p or power of p.

Definition 2.5 If G is a group of order $p^\alpha \cdot m$, where $p \nmid m$, then a subgroup of order p^α is called *Sylow p-subgroups*, and the set of *Sylow p-subgroups* denoted as $Syl_p(G)$.

Definition 2.6 The Klein 4-group, V_4, is the group of order 4

$$V_4 = \{e, a, b, c\}.$$

with $e = identity$, $a \neq b \neq c$, $ab = c$, $ac = b$, $bc = a$, $a^2 = b^2 = c^2 = e$. Note that V_4 is abelian.

Definition 2.7 A group H is *cyclic* if it is generated by a single element. In other words, there is some element $x \in H$ such that

$$H = \{x^n |\ n \in \mathbb{Z}\}.$$

3 Fusion System

In this section, we introduce the notion of fusion systems. All results and definitions in this section can be found in Section 8.1 of [Lin18].

Let p be a prime, G be a finite group, $P, Q, R \leq G$. We set

• $Hom_P(Q, R) = \{\phi : Q \to R,\ group\ homomorphisms | \exists x \in P\ such\ that\ \phi(u) = xux^{-1}, \forall u \in Q\}$,
• $Hom_P(Q, Q) =: Aut_P(Q, Q) \cong N_P(Q) \big/ C_P(Q)$,
where the last isomorphism can be found in [DF04].

Definition 3.1 A *fusion system* on a finite p-group P is a category $\mathcal{F}$ having objects as subgroups of P, and for two subgroups $Q, R \leq P$, $Hom_{\mathcal{F}}(Q, R)$ a set of injective group homomorphisms with

(i) $Hom_P(Q, R) \subseteq Hom_{\mathcal{F}}(Q, R) \subseteq Inj(Q, R)$,

(ii) If $Q \leq R$, then the inclusion map $Q \hookrightarrow R$ is in $\mathcal{F}$,

(iii) If $\phi : Q \to R$ is in $\mathcal{F}$, then induced isomorphism $\phi : Q \cong \phi(Q)$ is in $\mathcal{F}$ and its inverse also is in $\mathcal{F}$,

(iv) Composition is the usual composition of group homomorphism, $\circ$.

Example 3.2 Let G be a finite group and $P \leq G$ be a p-subgroup. Denote $\mathcal{F}_P(G)$ be the category with objects of P and morphism sets

$$Hom_{\mathcal{F}_P(G)}(Q, R) = Hom_G(Q, R).$$

Hence, $\mathcal{F}_P(G)$ is a fusion system on P.

Definition 3.3 Let $\mathcal{F}$ be a fusion system on P. Then a subgroup Q of P is called

• *fully* $\mathcal{F}$-centralized if $|C_P(Q)| \geq |C_P(Q^{'})|$ for any subgroup $Q^{'} \leq$ P such that there exists an isomorphism $\phi : Q \cong Q^{'}$ in $\mathcal{F}$.

• *fully* $\mathcal{F}$-normalized if $|N_P(Q)| \geq |N_P(Q^{'})|$ for any subgroup $Q^{'} \leq$ P such that there exists an isomorphism $\phi : Q \cong Q^{'}$ in $\mathcal{F}$.

Proposition 3.4 [Lin18, p. 166]. Let P be a *Sylow* p-subgroup of G. Let $Q \leq$ P be a subgroup. Set $\mathcal{F} = \mathcal{F}_P(G)$. Then Q is *fully* $\mathcal{F}$-centralized if and only if $C_P(Q)$ is a *Sylow* p-subgroup of $C_G(Q)$. Also, Q is *fully* $\mathcal{F}$-normalized if and only if $N_P(Q)$ is a *Sylow* p-subgroup of $N_G(Q)$.

Proof Let S be a *Sylow* p-subgroup of $C_G(Q)$ containing $C_G(P)$. By *Sylow* Theorem, if P is a *Sylow* p-subgroup of G and Q is any p-subgroup of G, then there exists $x \in G$ such that $Q \leq xPx^{-1}$. In other words, Q is contained in some conjugate of P. In particular, any two *Sylow* p-subgroup of G are conjugate in G. Hence, QS is a p-subgroup of G and hence there is $x \in G$ such that ${}^{x}(\mathrm{QS}) \leq P$. Then conjugation by x induces an isomorphism

$$\phi : Q \cong {}^{x}\mathrm{Q}.$$

belonging to the category $\mathcal{F}$, since both Q and ${}^{x}\mathrm{Q}$ are contained in P. Moreover, ${}^{x}\mathrm{S} \leq C_P({}^{x}\mathrm{Q})$. Thus, $|C_P(Q)| \leq |S| \leq |C_P({}^{x}\mathrm{Q})|$. Hence, Q is *fully* $\mathcal{F}$-centralized if and only if $|C_P(Q)| = |S|$. Therefore, $C_P(Q) = S$ itself.

Definition 3.5 Let $\mathcal{F}$ be a fusion system on P and $\phi : Q \to P$ in $\mathcal{F}$. We set

$$N_\phi = \{y \in N_P(Q) | \exists z \in N_P(\phi(Q)) \text{ such that } \phi(yuy^{-1}) = z\phi(u)z^{-1}, \forall u \in Q\}.$$

Definition 3.6 A fusion system $\mathcal{F}$ on P is called *saturated* if the following holds:

(i) *Sylow Axiom* : The group $Aut_P(P) = Hom_P(P,P)$ is a *Sylow* p-subgroup of $Aut_{\mathcal{F}}(P)$.
(ii) *Extension Axiom* : For every morphism $\phi : Q \to P$ in $\mathcal{F}$ such that $\phi(Q)$ is *fully* $\mathcal{F}$-normalized, there exists $\psi : N_\phi \to P$ in $\mathcal{F}$ such that $\psi|_Q = \phi$.

Theorem 3.7[Lin18, p. 167]. Let P be a *Sylow* p-subgroup of G. Then $\mathcal{F}_P(G)$ is a *saturated* fusion system on P.
Proof Set $\mathcal{F} = \mathcal{F}_P(G)$. Clearly, $\mathcal{F}$ is a category of P, and we have $\mathcal{F}_P(G) \subseteq \mathcal{F}$. Thus, we just only to check the axioms of saturated fusion system.
$Aut_{\mathcal{F}}(P) \cong N_G(P) \big/ C_G(P)$, and the image of P in this quotient group is a *Sylow* p-subgroup, which implies that

$$Aut_P(P) \cong P \big/ Z(P).$$

is a *Sylow* p-subgroup of $Aut_{\mathcal{F}}(P)$. This shows that *Sylow Axiom* holds.
Let $Q \leq P$ and let $\phi : Q \to P$ in $\mathcal{F}$. Suppose that $\phi(Q)$ is *fully* $\mathcal{F}$-normalized. Also let $x \in G$ such that $\phi(u) = {}^{x}(\mathrm{u}) = xux^{-1}$ for all $u \in Q$. We get

$$N_\phi = \{y \in N_P(Q) | \exists z \in N_P(\phi(Q)) \text{ such that } {}^{xy}u = {}^{zx}u \ \forall u \in Q\}.$$

Thanks to this definition, we get

$${}^{xy}u = {}^{zx}u \Leftrightarrow {}^{x^{-1}z^{-1}xy}u = u \Leftrightarrow x^{-1}z^{-1}xy \text{ centralizes } Q.$$

We also get

$$z^{-1}xyx^{-1} \text{ centralizes } {}^{x}\mathrm{Q} = \phi(Q).$$

Hence, we get $xyx^{-1} = zc$ for some $c \in C_G(\phi(Q))$. This shows that we have

$${}^{x}\mathrm{N}_\phi \subseteq N_P(\phi(Q))C_G(\phi(Q)).$$

Since $\phi(Q)$ is *fully* $\mathcal{F}$-normalized, $N_P(\phi(Q))$ is a *Sylow* p-subgroup of $N_G(\phi(Q))$, so of $N_P(\phi(Q))C_G(\phi(Q))$. Thus, $\exists d \in C_G(\phi(Q))$ such that

$$^{dx}\mathrm{N}_\phi \subseteq N_P(\phi(Q)).$$

Define $\psi : N_\phi \to P$ by $\psi(u) = {}^{dx}y$, for all $y \in N_\phi$. We claim that ψ extends ϕ. Indeed, if $u \in Q$, then $\psi(u) = {}^{dx}u = {}^{d}(\phi(u)) = \phi(u)$, since d centralizes $\phi(Q)$. This completes the proof.

Before application of the theorem on Klein four-group, we need to give a remark which is important.

Remark $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$. Moreover, $V_4 \cong H \leq A_4$ where A_4 is *alternating group of* 4 and H is a subgroup of A_4.

Example 3.8 Describe $\mathcal{F}_{V_4}(V_4)$ and $\mathcal{F}_{V_4}(A_4)$.

Proof (i) First of all, we describe $\mathcal{F}_{V_4}(V_4)$. The subgroup lattice of V_4 is

$$\begin{tikzcd}[arrows=dash]
 & V_4 \arrow[dl] \arrow[d] \arrow[dr] & \\
< (12)(34) > \arrow[dr] & < (13)(24) > \arrow[d] & < (14)(23) > \arrow[dl] \\
 & e &
\end{tikzcd}$$

By Example 3.2 and Theorem 3.7, let $P = V_4$ and $G = V_4$. By Definition 3.1, subgroups of $P = V_4$ are objects of this fusion system and morphisms are

$$Hom_{\mathcal{F}_{V_4}(V_4)}(Q,R) = Hom_{V_4}(Q,R).$$

where $Q, R \leq P = V_4$. Then we can say that $\mathcal{F}_{V_4}(V_4)$ is a saturated fusion system on V_4, by Theorem 3.7. Furthermore, since V_4 is abelian and all maps are conjugations with elements of V_4, all maps are identity maps. Therefore, there is only identity map from any object to itself. Moreover, the only other maps are inclusions. Hence, there are five isomorphism classes of objects in $\mathcal{F}_{V_4}(A_4)$.

(ii) By again Example 3.2 and Theorem 3.7, let $P = V_4$ and $G = A_4$. Since $V_4 \in Syl_2(A_4)$, $\mathcal{F}_{V_4}(A_4)$ is saturated fusion system on V_4.

Moreover, $Aut_{\mathcal{F}}(V_4) \cong {N_{A_4}(V_4)}\Big/{C_{A_4}(V_4)} = {A_4}\big/{V_4}$. Since the order of A_4 is

12 and the order of V_4 is 4, the quotient group has order 3. Hence,

$$A_4 \big/ V_4 \cong C_3.$$

where C_3 is a cyclic group of order 3.

Furthermore, $(<(12)(34)>)$,$(<(13)(24)>)$,$(<(14)(23)>)$ are A_4-conjugate and they are isomorphic to C_2. Hence, the subgroup lattice of V_4 is

$$\begin{tikzcd}
 & V_4 \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
<(12)(34)> \arrow[r, phantom, "="] \arrow[dr, no head] & <(13)(24)> \arrow[r, phantom, "="] \arrow[d, no head] & <(14)(23)> \arrow[dl, no head] \\
 & e &
\end{tikzcd}$$

Therefore, in $\mathcal{F}_{V_4}(A_4)$, there are three isomorphism classes of objects.

In this example, we show the saturated fusion systems on Klein four-group. Is there any possible saturated fusion systems on Klein four-group other than we found? In order to answer this question, we need prove propositions.

Proposition 3.9 [Lin18, p. 170]. Suppose that P is abelian and $\mathcal{F}$ is a saturated fusion system on P. Let $Q \leq P$ and $\phi : Q \to P \ in \ \mathcal{F}$. Then,

$$N_\phi = P.$$

Moreover, every subgroup of P is $fully$ $\mathcal{F}$-centralized.

Proof We know that if P is abelian, then $N_P(Q) = P$. Since P is abelian, we can say that $yuy^{-1} = u$ or $yu = uy$,$\forall u \in Q$. Hence, $\exists z \in N_P(Q)$ such that $\phi(yuy^{-1}) = \phi(u) = z\phi(u)z^{-1}$. Thus, $\phi(u)z = z\phi(u)$,$\forall u \in Q$. Therefore, $N_\phi = P$.

When P is non-abelian, the order of multiplication is important, however, if P is abelian, we do not care the order of multiplication. Since P is abelian, every element in P commutes with every other element in Q, hence $C_P(Q) = P$. This is valid for all other subgroups of P. Therefore, by Definition 3.3, we get $|C_P(Q)| = |C_P(Q')|$ for all Q' in P. This shows that Q is $fully$ F-centralized.

Theorem 3.10 [Lin18, p. 170]. Suppose that P is abelian and $\mathcal{F}$ is a saturated fusion system on P. Let $Q \leq P$ and $\phi : Q \to P \ in \ \mathcal{F}$. Then, every morphism $\phi : Q \to P$ such that $\phi(Q)$ is $fully$ $\mathcal{F}$-centralized extends to a morphism

$\psi : N_\phi \to P$ in $\mathcal{F}$. Moreover, $\mathcal{F} = \mathcal{F}_P(P \rtimes E)$, where E :=$Aut_\mathcal{F}(P)$.

Proof Define $\phi : Q \to P$. Then, by Proposition 3.9, $\phi(Q)$ is *fully* $\mathcal{F}$-centralized, since P is abelian. Furthermore, again by Proposition 3.9, every morphism $\psi : N_\phi = P \to P$ in $\mathcal{F}$ such that $\psi|_Q = \phi$. Hence, every morphism in $\mathcal{F}$ is restriction of an automorphism of P. Therefore, $\mathcal{F} = \mathcal{F}_P(P \rtimes E)$.

Proposition 3.11 [Lin18, p. 170]. Suppose that P is abelian and $\mathcal{F}$ is a saturated fusion system on P. Then E :=$Aut_\mathcal{F}(P)$ is a p'-group. In other words, if P is a *Sylow* p-subgroup, then the order of $Aut_\mathcal{F}(P)$ is prime to p.

Proof By *Sylow axiom* of saturated fusion systems, $Aut_P(P) = Hom_P(P, P)$ is a *Sylow* p-subgroup of $Aut_\mathcal{F}(P)$. Since P is abelian, $Aut_P(P) = Hom_P(P, P)$ is equal to identity morphism. Since identity morphism, $\{id\}$, is a *Sylow* p-subgroup of $Aut_\mathcal{F}(P)$, it follows that $Aut_\mathcal{F}(P)$ is a p'-group.

Before showing whether there exists saturated fusion systems on V_4 other than we found, we need to prove a lemma.

Lemma 3.12 $Aut(V_4) \cong Sym(3)$, where $Sym(3)$ is the *symmetric group of order* 6.

Proof

$$\phi_1 = e \longmapsto e\ ,\ a \longmapsto a\ ,\ b \longmapsto b\ ,\ c \longmapsto c\ =\ e.$$

$$\phi_2 = e \longmapsto e\ ,\ a \longmapsto b\ ,\ b \longmapsto a\ ,\ c \longmapsto c\ =\ (ab).$$

$$\phi_3 = e \longmapsto e\ ,\ a \longmapsto c\ ,\ b \longmapsto b\ ,\ c \longmapsto a\ =\ (ac).$$

$$\phi_4 = e \longmapsto e\ ,\ a \longmapsto a\ ,\ b \longmapsto c\ ,\ c \longmapsto b\ =\ (bc).$$

$$\phi_5 = e \longmapsto e\ ,\ a \longmapsto c\ ,\ b \longmapsto a\ ,\ c \longmapsto b\ =\ (acb).$$

$$\phi_6 = e \longmapsto e\ ,\ a \longmapsto b\ ,\ b \longmapsto c\ ,\ c \longmapsto a\ =\ (abc).$$

Hence, we get

$$Aut(V_4) = \{e, (ab), (ac), (bc), (abc), (acb)\}.$$

$$Sym(3) = \{e, (12), (13), (23), (123), (132)\}.$$

Therefore, by mapping each element in $Aut(V_4)$ with each element in $Sym(3)$, we get a bijection. This completes the proof.

Application of Proposition 3.11 By Proposition 3.11, if $\mathcal{F}$ is a saturated fusion system on V_4, then E :=$Aut_\mathcal{F}(V_4)$ can be either C_3 or 1. Since E $\subseteq Aut(P)$, it can form $P \rtimes$ E. In other words, since $\mathcal{F}_P(G) = \mathcal{F}_P(P \rtimes \text{E})$, we can write G as in the form of $P \rtimes$E.

(i) $\mathcal{F}_{V_4}(V_4) = \mathcal{F}_{V_4}(V_4 \rtimes 1)$, where E $= Aut_\mathcal{F}(V_4) = 1$.

(ii)$\mathcal{F}_{V_4}(A_4) = \mathcal{F}_{V_4}(V_4 \rtimes C_3)$, where E $= Aut_\mathcal{F}(V_4) \cong N_{A_4}(V_4) \big/ C_{A_4}(V_4) = A_4 \big/ V_4 \cong \mathrm{C}_3$.

Since $|A_4| = 12 = 2^2.3$, A_4 has *Sylow* 2-subgroup and *Sylow* 3-subgroup. Also we know that V_4 is a $Sylow$ 2-subgroup of A_4, and hence $\mathcal{F}_{V_4}(A_4)$ is a saturated fusion system on V_4. By Proposition 3.11, E:= $Aut_{\mathcal{F}}(P) \subseteq Aut(V_4) \cong Sym(3)$ and $|Aut(V_4)| = |Sym(3)| = 6 = 2.3$. Hence, the order of E $= Aut_{\mathcal{F}}(P)$ cannot be 2 because V_4 is a $Sylow$ 2-subgroup of A_4. So the order of E $= Aut_{\mathcal{F}}(P)$ can be 1 or 3, since those are not divisible by 2. Hence, E $= 1$ or E $= C_3$. This implies that $\mathcal{F}_{V_4}(V_4) = \mathcal{F}_{V_4}(V_4 \rtimes 1)$ for E $= 1$, and $\mathcal{F}_{V_4}(A_4) = \mathcal{F}_{V_4}(V_4 \rtimes C_3)$ for E $= C_3$. Those are the only possible saturated fusion systems. That's why, we cannot find more saturated fusion system on V_4 other than these two saturated fusion systems.

4 Saturated Fusion Systems on Cyclic Groups

In this section, we classify saturated fusion systems on cyclic groups, but before doing that, we need to explain a proposition which is helpful.

Proposition 4.1 [DF04]. $Aut(C_p) \cong C_{p-1}$, where p is a prime.

Proof Let $C_p = < x >$. Define $\phi \in Aut(C_p)$ such that $\phi(x) = x^k$ for some integer k with $gcd(k, p) = 1$ in which $1 \leq k < p$. So the number of k's is equal to $p-1$. Hence, $|Aut(C_p)| = p-1$. In fact, the automorphism group of C_n has order $\phi(n)$ where $\phi(n)$ is $Euler - Phi$ function, in which $\phi(n)$ is the number of $gcd(n, k) = 1$ with $1 \leq k < n$. If we take $n = p$, then $\phi(p)$ is equal to $p-1$.

Also let $Aut(C_p) = \{\phi_1, \phi_2, ..., \phi_{p-1}\}$ and $C_{p-1} = \{1, x, x^2, ..., x^{p-2}\}$ with $|Aut(C_p)| = |C_{p-1}| = p-1$. Define $\psi : Aut(C_p) \to C_{p-1}$, $\phi_k \mapsto x^{k-1}$. Clearly, ψ is injective, surjective and group homomorphism. Therefore, $Aut(C_p) \cong C_{p-1}$.

Example 4.2.1

Let $P = C_2$ and $p = 2$. By Proposition 4.1, $Aut(C_2) \cong \{1\}$. Therefore, there is only one saturated fusion system on C_2, which is $\mathcal{F}_{C_2}(C_2)$, by Theorem 3.10.

Example 4.2.2

Let $P = C_3$ and $p = 3$. By Proposition 4.1, $Aut(C_3) \cong C_2$. Hence, by Proposition 3.11, E $= 1$ or E $= C_2$ .Therefore, there exist two saturated fusion systems on C_3, which are $\mathcal{F}_{C_3}(C_3)$ and $\mathcal{F}_{C_3}(C_3 \rtimes C_2) = \mathcal{F}_{C_3}(S_3)$, by Theorem 3.10.

Example 4.2.3

Let $P = C_4$ and $p = 2$. By Proposition 4.1, since C_4 has order 4, we need to find the number of k's in which $gcd(4, k) = 1$. So $k = 1$ *or* $k = 3$. Hence, $\phi(4) = 2$. Therefore, $Aut(C_4) \cong C_2$, thanks to *Euler - Phi* function. Hence, by Proposition 3.11, E $= 1$. Therefore, there exists one saturated fusion system on C_4, which is $\mathcal{F}_{C_4}(C_4)$.

Example 4.2.4

Let $P = C_5$ and $p = 5$. By Proposition 4.1, $Aut(C_5) \cong C_4$. Hence, by Proposition 3.11, E = 1 or E = C_2 or E = C_4. Therefore, there exist three saturated fusion systems on C_5, which are $\mathcal{F}_{C_5}(C_5)$, $\mathcal{F}_{C_5}(C_5 \rtimes C_2) = \mathcal{F}_{C_5}(D_{10})$, and $\mathcal{F}_{C_5}(C_5 \rtimes C_4)$.

Example 4.2.5 Let $P = C_7$ and $p = 7$. By Proposition 4.1, $Aut(C_7) \cong C_6$. Hence, by Proposition 3.11, E = 1 or E = C_2 or E = C_3 or E = C_6. Therefore, there exist four saturated fusion systems on C_7, which are $\mathcal{F}_{C_7}(C_7)$, $\mathcal{F}_{C_7}(C_7 \rtimes C_2)$, $\mathcal{F}_{C_7}(C_7 \rtimes C_4)$, and $\mathcal{F}_{C_7}(C_7 \rtimes C_6)$.

Example 4.2.6 Let $P = C_8$ and $p = 2$. Let $C_8 = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. So x, x^3, x^5, x^7 has order 8. Thus, $Aut(C_8)$ is equal to C_4 or $C_2 \times C_2$. Since $1^2 \equiv 3^2 \equiv 5^2 \equiv 7^2 \ (mod\ 8)$, the order of any element of $Aut(C_8)$ is 2. Therefore, $Aut(C_8) \cong C_2 \times C_2 \cong V_4$, thanks to the Remark giving before Example 3.8. Hence, by Proposition 3.11, E = 1 . Therefore, there exists one saturated fusion system on C_8, which is $\mathcal{F}_{C_8}(C_8)$.

Example 4.2.7 Let $P = C_9$ and $p = 3$. Let $C_9 = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$. So x, x^2, x^4, x^5, x^7, x^8 has order 9. So the number of automorphisms are 6. Therefore, $Aut(C_9) \cong C_6$. Hence, by Proposition 3.11, E = 1 or E = C_2. Therefore, there exist two saturated fusion system on C_9, which are $\mathcal{F}_{C_9}(C_9)$, and $\mathcal{F}_{C_9}(C_9 \rtimes C_2)$.

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