Independence (Linear and Affine)

CE282: Linear Algebra

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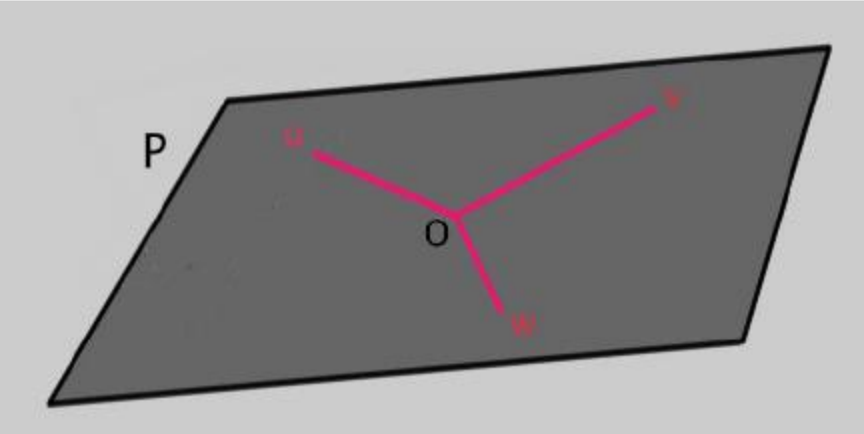
Linear Independence

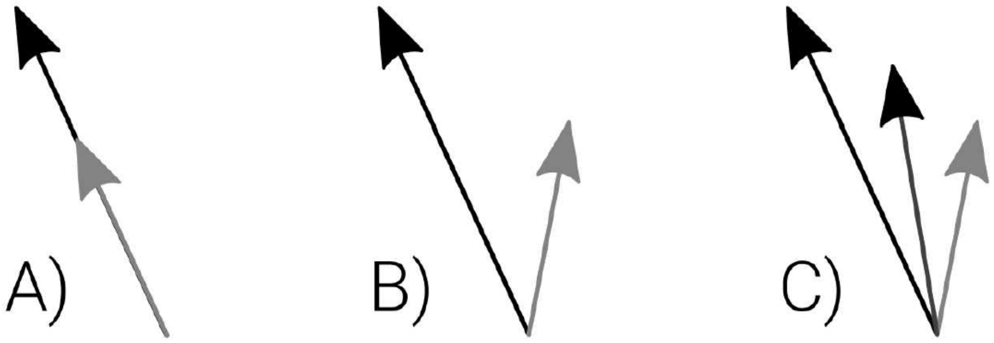


- Plane P includes origin and three non-zero vectors $\{v,u,w\}$ in P
- If no two of $\{v, u, w\}$ are parallel, then P = span{u,v,w}
- Any two vectors determines a plane and express the other as a linear combination of those two:

$$w = d_1 u + d_2 v \ \ (d_1 \& d_2 \ can't \ both \ be \ zero)$$

- $c_1 u + c_2 v + c_3 w = 0 \longrightarrow$ u, w, v are not all independent.
- Independence is a property of a set of vectors.

- Geometry:
  - A set of vectors is linear independent if the subspace dimensionality (its span) equals the number of vectors.
  - Example: 1,2,3 vectors spans?



Geometric sets of vectors in $\mathbb{R}^2$

- Algebra
  - Dependent
    - For at least one $\lambda \neq 0$ $\qquad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \qquad \lambda \in \mathbb{R}$
    - A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.
  - Independence
    - Only when all $\lambda_i = 0$ $\qquad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \qquad \lambda \in \mathbb{R}$
    - No vector in the set is a linear combination of the others (has only the trivial solution)

Example

- Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, and\ v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- A set containing only one vector—say, v—is linearly independent if and only if v is not …

- a) $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b) $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Characterization of Linearly Dependent sets

Theorem

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

- **Does *not* say that *every* vector**

Characterization of Linearly Dependent sets

Proof

If some v_j in S equals a linear combination of the other vectors, then v_j can be subtracted from both sides of the equation, Producing a linear dependence relation with a nonzero weight (-1) on v_j. [For instance, if $v_1 = c_2v_2 + c_3v_3$, then $0 = (-1)v_1 + c_2v_2 + c_3v_3 + 0v_4 + \cdots + 0v_n$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If v_1 is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $v_1 \neq 0$, and there exist weights $c_1, \ldots, c_n$ not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_pv_n = 0$$

Proof

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1 v_1 = 0$, which is impossible because $v_1 \neq 0$. So $j > 1$ and

$$c_1 v_1 + \cdots + c_j v_j + 0 v_{j+1} + 0 v_n = 0$$

$$c_j v_j = -c_1 v_1 - \cdots - c_{j-1} v_{j-1}$$

$$v_j = \left(-\frac{c_1}{c_j}\right) v_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right) v_{j-1}$$

Theorem

Any set of vectors that contains the zeros vector is guaranteed to be linearly dependent.

Properties

- The vectors coming from the vector form of the solution of a matrix equation Ax = 0 are linearly independent

Example

- Vectors related to x_2 and x_3 are linear independent.
- Columns of A related to to x_2 and x_3 are linear dependent.
- Span$\{A_1, A_2, A_3\} = Span\{A_1\}$

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Properties

Important

- If a collection of vectors is linearly dependent, then any superset of it is linearly dependent.

- Any nonempty subset of a linearly independent collection of vectors is linearly independent.

Properties

Theorem

- Any set of $M > N$ vectors in $\mathbb{R}^n$ is necessarily dependent.
- Any set of $M \leq N$ vectors in $\mathbb{R}^n$ could be linearly independent.

Example

Example

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Linear Dependent Properties

- Suppose vectors $v_1, \dots, v_n$ are linearly dependent:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1 \neq 0$. Then:

$$span\{v_1, \dots, v_n\} = span\{v_2, \dots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

Linear combinations of linearly independent vectors

Theorem

Suppose x is linear combination of linearly independent vectors $v_1, \ldots, v_k$:

$$x = \beta_1 v_1 + \cdots + \beta_k v_k$$

The coefficients $\beta_1, \ldots, \beta_k$ are unique.

Proof

Conclusion

Important

- Step 1: Count the number of vectors (call that number M) in the set and compare to N in $\mathbb{R}^n$. As mentioned earlier, if $M > N$, then the set is necessarily dependent. If $M \leq N$ then you have to move on to step 2.

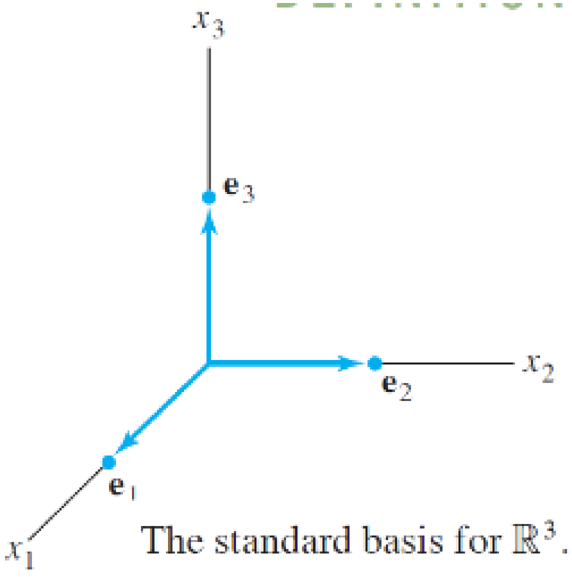
  Step 2: Check for a vector of all zeros. Any set that contains the zeros vector is a dependent set.

- The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.

Basis

- A set of n linearly independent n-vectors is called a basis

- A basis is the combination of span and independence: A set of vectors $\{v_1, \ldots, v_n\}$ forms a basis for some subspace of $\mathbb{R}^n$ if it
  - (1) spans that subspace
  - (2) is an independent set of vectors.



The standard basis for $\mathbb{R}^3$.

Basis

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{b_1, \dots, b_P\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is linearly independent set, and
2. The subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = Span\,\{b_1, \dots, b_P\}$$

Example

Which are unique?

- ❑ Express a vector in terms of any particular basis
- ❑ Bases for $\mathbb{R}^2$
- ❑ Bases with unit length for $\mathbb{R}^2$

Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called linearly dependent if there are nonzero constants c_1 and c_2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called linearly independent.

Example

Linearly dependent or independent?

- Functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - cos^2 t$
- Functions $\{\sin^2 x, \cos^2 x, \cos(2x)\} \subset \mathcal{F}$

Vector Space of Polynomials

Example (Linear independence)

Are $(1-x), (1+x), x^2$ linearly independent?

Example (Basis)

- ❑ Standard bases for $P_n(\mathbb{R})$?
- ❑ Are $(1-x), (1+x), x^2$ basis for $P_2(\mathbb{R})$?

Coordinate Systems

- The main reason for selecting a basis for a subspace H; instead of merely a spanning set, is that each vector in H can be written in only one way as a linear combination of the basis vectors.

Note

Suppose the set $\mathcal{B} = \{\boldsymbol{b_1}, \dots, \boldsymbol{b_P}\}$ is a basis for a subspace H. For each x in H, the **coordinates of $\boldsymbol{x}$ relative to the basis $\boldsymbol{\mathcal{B}}$** are the weights $c_1, \dots, c_P$ such that $\mathrm{x} = c_1 b_1 + \cdots + c_P b_p$, and the vector in $\mathbb{R}^p$

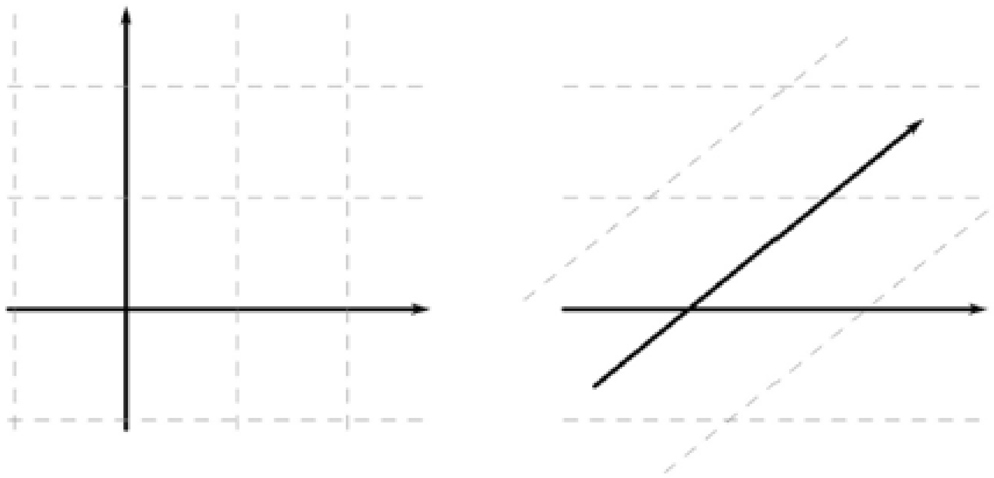
$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_P \end{bmatrix}$$

is called the **coordinate vector of $\boldsymbol{x}$ (relative to $\boldsymbol{\mathcal{B}}$)** or the $\mathcal{B}$-coordinate vector of x.

Example

Coordinate vector of $p(x) = 4 - x + 3x^2$ respect to basis $\{1, x, x^2\}$

Coordinate axes



- The familiar Cartesian plane (left) has orthogonal coordinate axes. However, axes in linear algebra are not constrained to be orthogonal (right), and non-orthogonal axes can be advantageous.

Linearly Independent Sets versus Spanning Sets

Theorem

Let V be a vector space with a basis B of size n. Then

a) Any set of more than n vectors in V must be linearly dependent, and

b) Any set of fewer than n vectors cannot span V.

Span	Linearly Independent
Want many vectors in small space	Want few vectors in big space
Adding vectors to list only helps	Deleting vectors from list only helps
Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX= b has solution $\Leftrightarrow b \in span\{v_1, \ldots, v_k\}$	Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX = 0 has only trivial solution(X=0) $\Leftrightarrow v_1, \ldots, v_k$ are linearly independent.

Dimensions

- The dimensionality of a vector is the number of coordinate axes in which that vector exists.
- If a vector space is spanned by a finite number of vectors, it is said to be finite-dimensional. Otherwise it is infinite-dimensional.
- The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted dim(V).

Definition

A vector space V is called...

a) **finite-dimensional** if it has a finite basis, and its **dimension**, denoted by $\dim(V)$, is the number of vectors in one of its bases.

b) **infinite-dimensional** if it has no finite basis, and we say that $\dim(V) = \infty$.

Example

Let's compute the dimension of some vector spaces that we've been working with.

Vector space	**Basis**	**Dimension**
F^n		
P^p		
$M_{m,n}$		
P (all polynomials)		
F $(functions)$		
C (continues functions)		

Dimensionality and Properties of Bases

Note

Let V be a finite dimensional vector space over field F. Below are some properties of bases:

1. Any linearly independent list can be extended to a basis (a maximal linearly independent list is spanning).
2. Any spanning list contains a basis (a minimal spanning list is linearly independent).
3. Any linearly independent list of length $\dim V$ is a basis.
4. Any spanning list of length $\dim V$ is a basis.

- **We will learn about change of basis in matrix transformation lecture!**

Independent ≤ spanning

Note

In a finite-dimensional space,

$$\begin{matrix} \textit{the length of every linearly} \\ \textit{independent list of vectors} \end{matrix} \quad \leq \quad \begin{matrix} \textit{the length of every} \\ \textit{spanning list of vectors} \end{matrix}$$

Proof

Affine Independence

Theorem

An indexed set of points $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is **affinely dependent** if there exists real numbers $c_1, \dots, c_p$, not all zero, such that

$$c_1 + \cdots + c_p = 0 \quad \textit{and} \qquad c_1 v_1 + \cdots + c_p v_p = 0$$

Otherwise, the set is **affinely independent.**

Example

- ❑ $\{v_1\}$

Affine Independence

Note

Given an indexed set $S = \{v_1, \dots, v_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. S is affinely dependent.

b. One of the points in S is an affine combination of other points in S.

c. The set $\{v_2 - v_1, \dots, v_p - v_1\}$ in $\mathbb{R}^n$ is linearly dependent.

Example

Let $v_1 = \begin{bmatrix}1\\3\\7\end{bmatrix}, v_2 = \begin{bmatrix}2\\7\\6.5\end{bmatrix}, v_3 = \begin{bmatrix}0\\4\\7\end{bmatrix}$, and $v_4 = \begin{bmatrix}0\\14\\6\end{bmatrix}$, and let $S = \{v_1, \dots, v_4\}$. Is S affinely dependent?

https://forms.gle/PRPepskcrsodHQKN6

https://docs.google.com/forms/d/e/1FAIpQLSfC_nasehBd-wePexo4b_32rhlLgz7uP5bdterYx7L96_EpOw/viewform?usp=sf_link

Theorem

Let set $S = \{v_1, \dots, v_k\}$ be an affinely independent set in $\mathbb{R}^n$. Then each $\mathbf{p}$ in aff S has a unique representation as an affine combination of $v_1, \dots, v_k$. That is, for each $\mathbf{p}$ there exists a unique set of scalers $c_1, \dots, c_k$ such that

$$\mathbf{p} = c_1 v_1 + \cdots + c_k v_k \quad \textit{and} \quad c_1 + \cdots + c_k = 1$$

Note

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} v_k \\ 1 \end{bmatrix}$$

Involving the homogeneous forms of the points. Row reduction of the augmented matrix $[\widetilde{v_1} \ \dots \ \widetilde{v_k} \quad \tilde{\mathbf{p}}]$ produces the Barycentric coordinates of $\mathbf{p}$.

Definition

Let set $S = \{v_1, \dots, v_k\}$ be an affinely independent set. Then for each point $\mathbf{p}$ in aff S, the coefficients $c_1, \dots, c_k$ in the unique representation

$$\mathbf{p} = c_1 v_1 + \cdots + c_k v_k \quad \textit{and} \quad c_1 + \cdots + c_k = 1$$

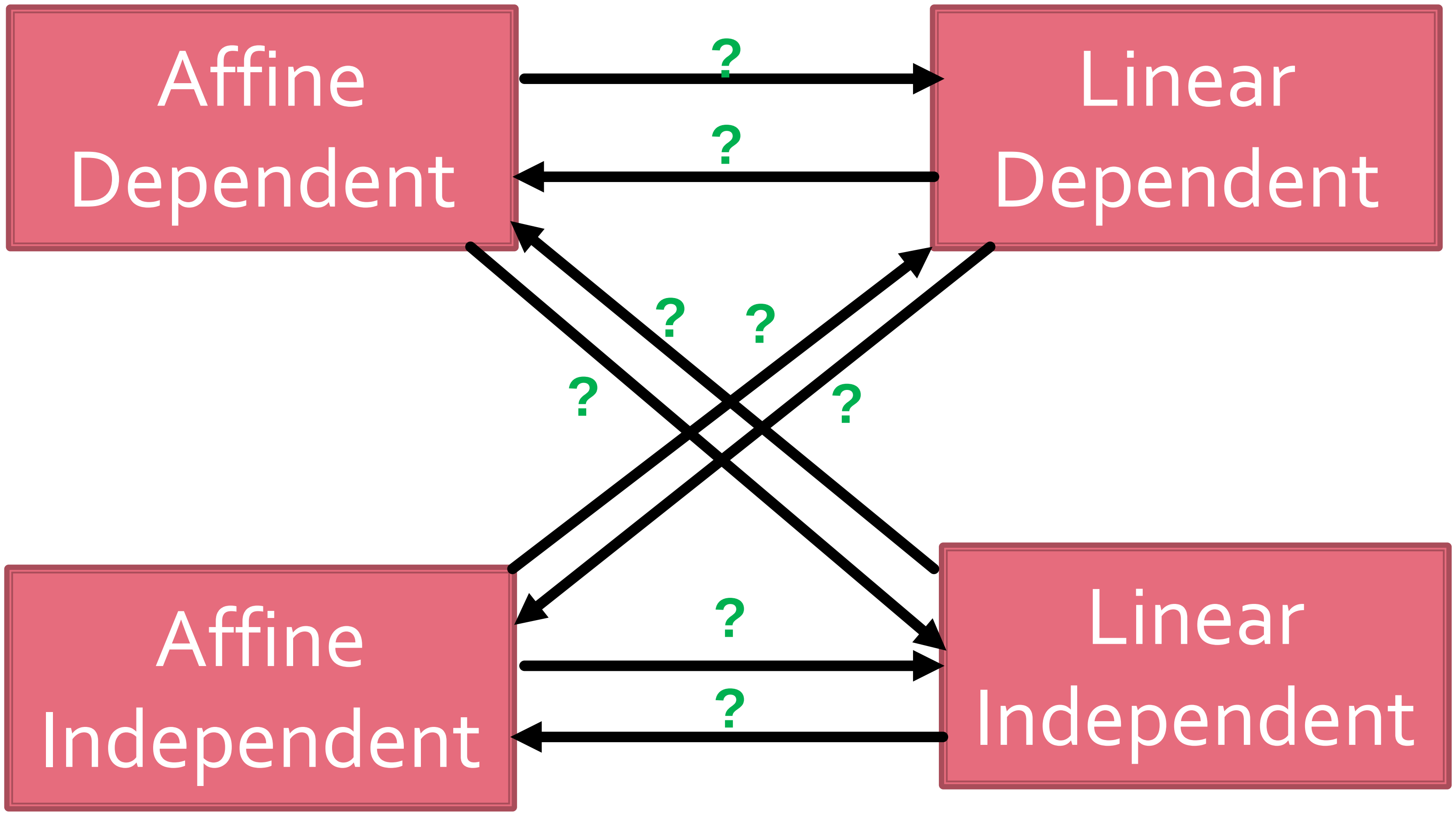
of $\mathbf{p}$ are called the **Barycentric** (or, sometimes **affine**) **coordinates** of $\mathbf{p}$

Barycentric Coordinates

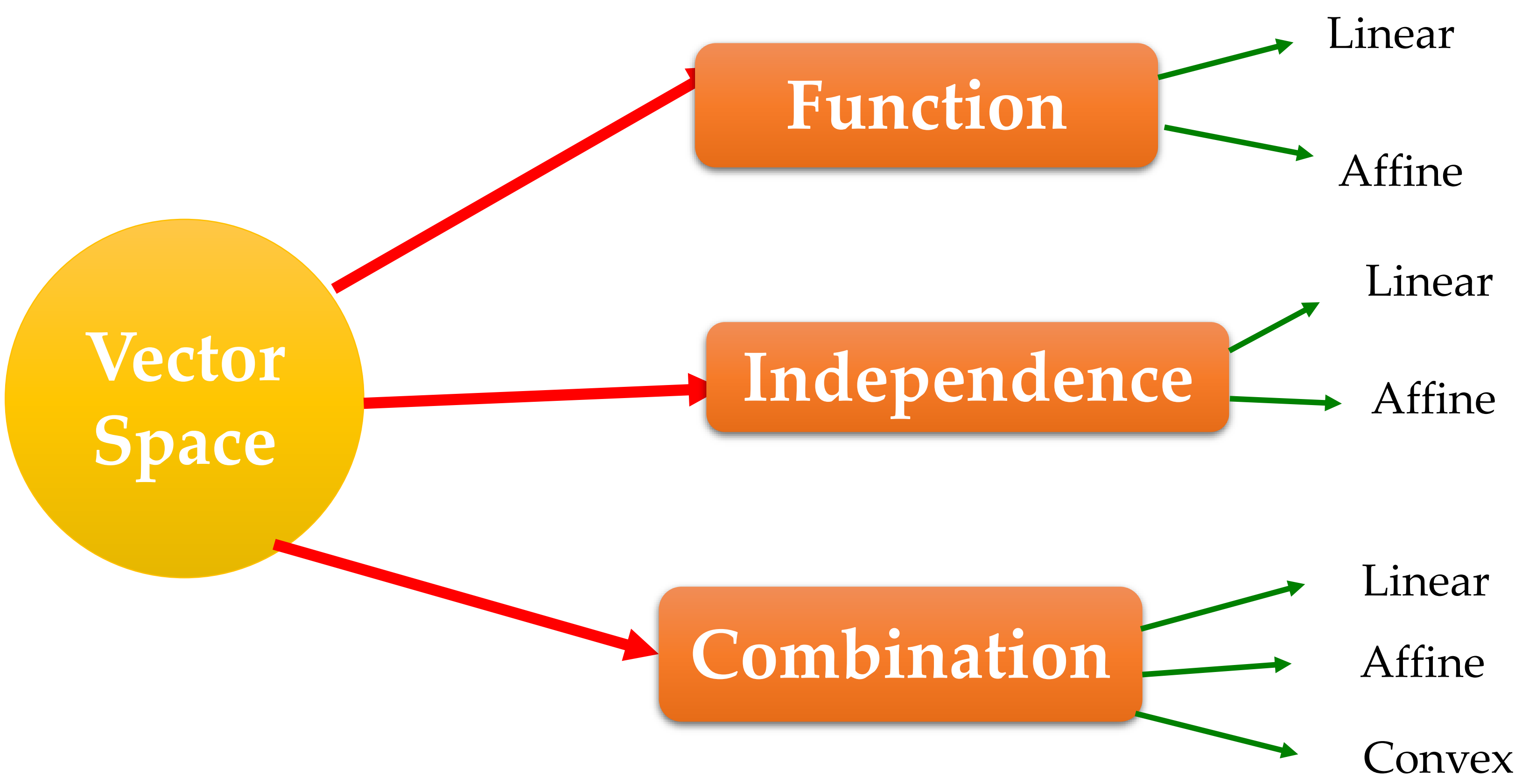
Example

Let $a = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $p = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the Barycentric Coordinates of p determined by the affinely independent set $\{a, b, c\}$.

Conclusion : Linear and Affine



Conclusion and Review



Reference

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