# Linear Equations

**CE282: Linear Algebra**

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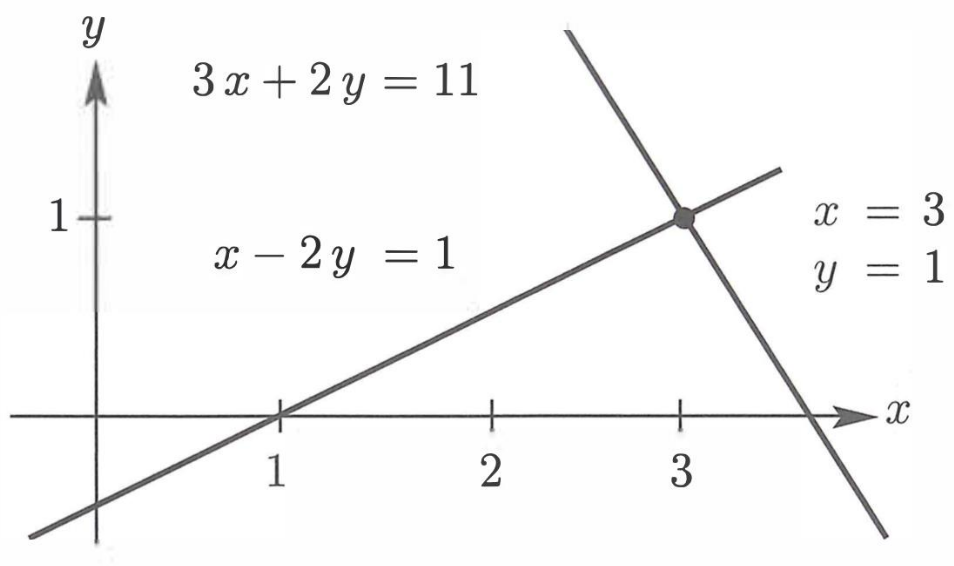
❑ Consider this simple system of equations,

$$x - y = 1$$
$$3x + 2y = 11$$



- Can be expressed as a matrix-vector multiplication

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}$$

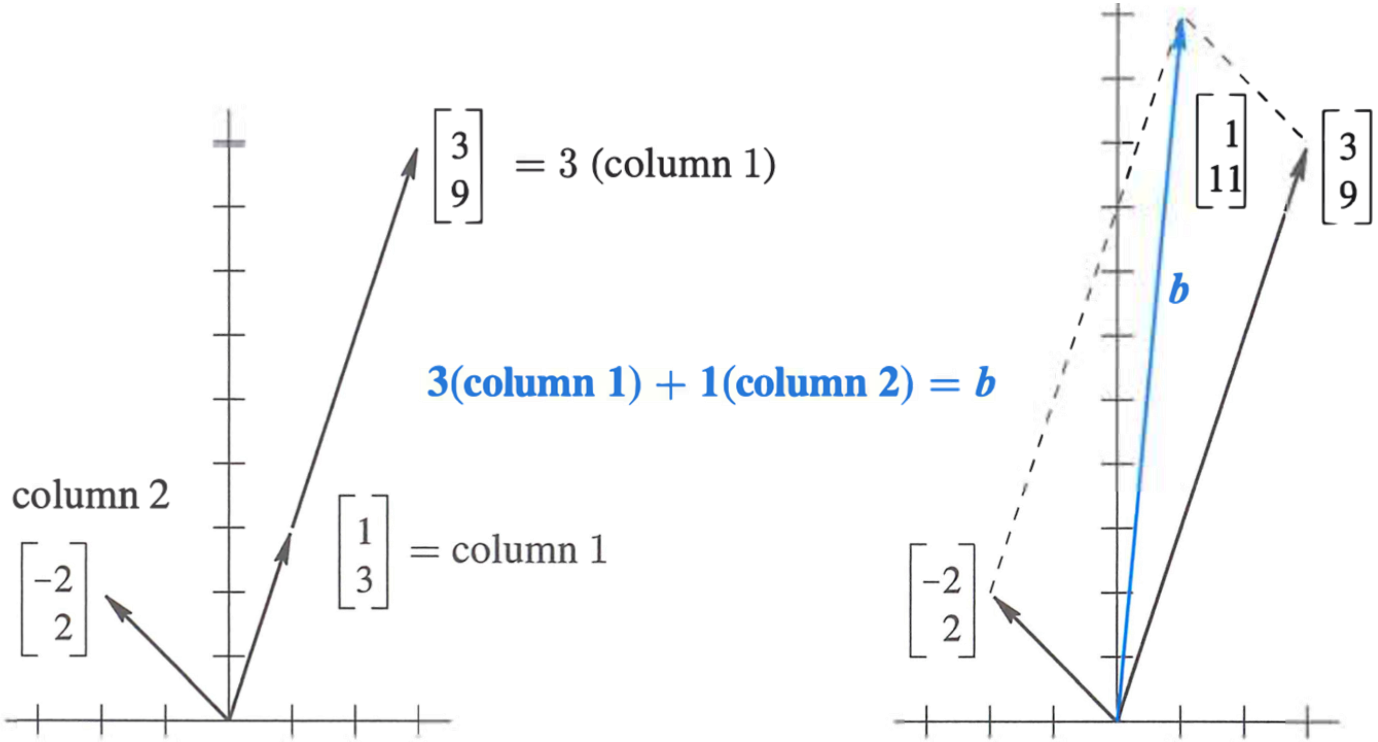
- Matrix Equation: $Ax = b$
- $A$ is often called **coefficient matrix**
- **Augmented matrix** is: $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

- Also, Can be expressed as linear combination of cols:

$$\begin{aligned} x \;-2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}$$

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = b$$




- Same for $n$ equation, $n$ variable

# Idea Of Elimination

- Subtract a multiple of equation (1) from (2) to eliminate a variable

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \xrightarrow[\text{Subtract to eliminate } 3x]{\text{multiply equation 1 by 3}} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 8 \end{bmatrix}}_{c}$$

A has become a upper triangle matrix $U$

## ❑ Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

### Definition

Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

### Note

- ❑ It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange.
- ❑ If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

# Idea Of Elimination (Row Reduction Algorithm)

- The **pivots** are on the diagonal of the triangle after elimination (boldface 2 below is the first pivot)

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8 \end{aligned}$$

  - Step 1: subtract (1) from (2) to eliminate x’s in (2)
  - Step 2: subtract (1) from (3) to totally eliminate x
  - Step 3: subtract new (2) from new (3)

**Definition**

The variables corresponding to pivot columns in the matrix are called basic variables.
The other variables are called a free variable.

$$\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix} \quad \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix}$$

# Echelon form

## Definition

A leading entry of a row refers to **the leftmost** nonzero entry in a nonzero row

## Definition

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

$$\underbrace{\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}}_{\text{Echelon form}}$$

## Definition

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
  1. The leading entry in each non-zero row is 1.
  2. Each leading 1 is the only non-zero entry in its columns.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon form}}$$

# Existence and Uniqueness Questions

**Two fundamental questions about a linear system:**

1. Is the system consistent? That is, does at least one solution exist?
2. If a solution exists, is it the only one? That is, is the solution unique?

## Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x - 5z &= 1 \\ y + z &= 4 \\ 0 &= 0 \end{aligned} \qquad \begin{cases} x = 1 + 5z \\ y = 4 - z \\ z \text{ is free variable} \end{cases}$$

- $x, y$: basic variable $\qquad$ $z$: free variable
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

# Existence and Uniqueness Questions

**Theorem**

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with nonzero $b$.

- If a linear system is consistent, then the solution set contains either:
  - A unique solution, when there are no free variables
  - Infinitely many solutions, when there is at least one free variable

# Find all solutions of a linear system

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

# Existence Of Solutions

## Fact

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of A.

## Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible $b_1, b_2, b_3$?

## Solution

Row reduce the augmented matrix for $Ax = b$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)$. The equation $Ax = b$ is not consistent for every b because some choices of $b$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

# Homogeneous Linear Systems

## Definition

- A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where $A$ is a matrix and $0$ is the zero vector.
- Trivial solution: $Ax = 0$ always has at least one solution, namely, $x = 0$ (the zero vector)
- Nontrivial solution: The non-zero solution for $Ax = 0$.

## Fact

The homogenous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

| Lines in $\mathbb{R}^2$ | | | |
|---|---|---|---|
| **Normal Form** | **General Form** | **Vector Form** | **Parametric Form** |
| $\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$ | $ax + by = c$ | $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$ | $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$ |

| Lines and Planes in $\mathbb{R}^3$ | | | | |
|---|---|---|---|---|
| | **Normal Form** | **General Form** | **Vector Form** | **Parametric Form** |
| Lines | $\begin{cases} \boldsymbol{n_1} \cdot \boldsymbol{x} = \boldsymbol{n_1} \cdot \boldsymbol{p_1} \\ \boldsymbol{n_2} \cdot \boldsymbol{x} = \boldsymbol{n_2} \cdot \boldsymbol{p_2} \end{cases}$ | $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$ | $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$ | $\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$ |
| Planes | $\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$ | $ax + by + cz = d$ | $\boldsymbol{x} = \boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{v}$ | $\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$ |

# Nonhomogeneous Systems & General Solution

## Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

## Example

Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$, where: $A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

# Nonhomogeneous Systems & General Solution

## Question

Can we change the order of columns in an augmented matrix???

$$\left\{\begin{array}{rcrcrcl} ax & + & by & + & cz & = & d \\ a'x & + & b'y & + & c'z & = & d' \\ a''x & + & b''y & + & c''z & = & d'' \end{array}\right.$$

Is equivalent to

$$\left\{\begin{array}{rcrcrcl} ax & + & cz & + & by & = & d \\ a'x & + & c'z & + & b'y & = & d' \\ a''x & + & c''z & + & b''y & = & d'' \end{array}\right.$$

# Conclusion

## Theorem

Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
b. Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
c. The columns of $A$ span $\mathbb{R}^m$.
*d.* $A$ has a pivot position in every row.

## Note

If $A$ does not have a pivot in every row, that does not mean that $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution for some given vector $\boldsymbol{b}$. It just means that there are some vectors $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution.

## Extra Resource

If you want to learn more about elementary row operations and echelon form, this video is recommended!

# Reference

- Linear Algebra and Its Applications, David C. Lay
- Linear Algebra Done Right, Axler, Chapter 3.D
- Introduction to Linear Algebra, Strange, Chapter 2.1, 2.2
- http://vmls-book.Stanford.edu/vmls-slides.pdf