Matrix Transformation

CE40282-1: Linear Algebra
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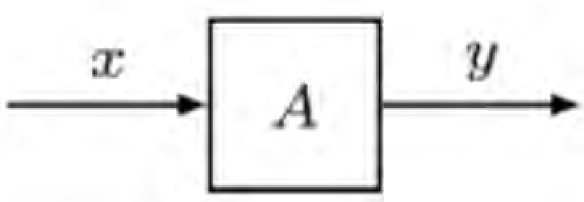
Linear Transformation

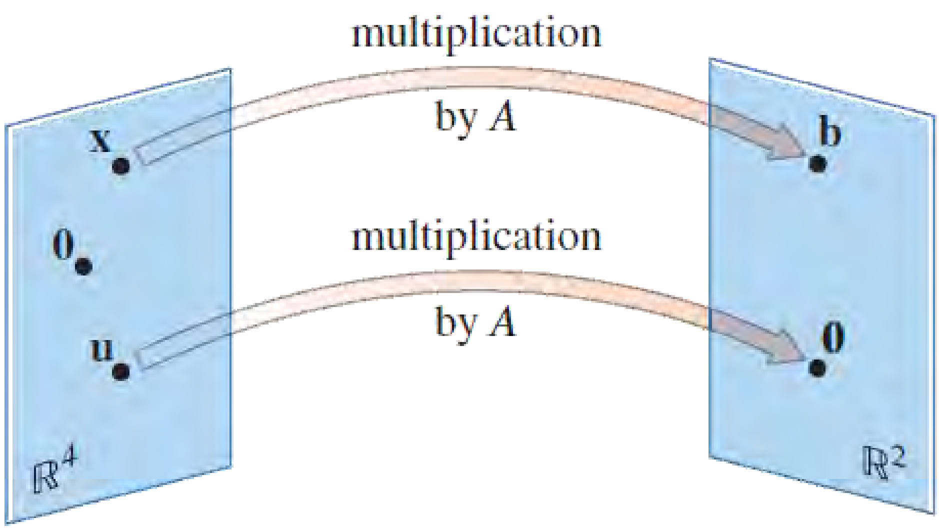
- Matrix is a linear transformation: map one vector to another vector

$$A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^m : \qquad y_{m\times 1} = A_{m\times n}x_{n\times 1}$$

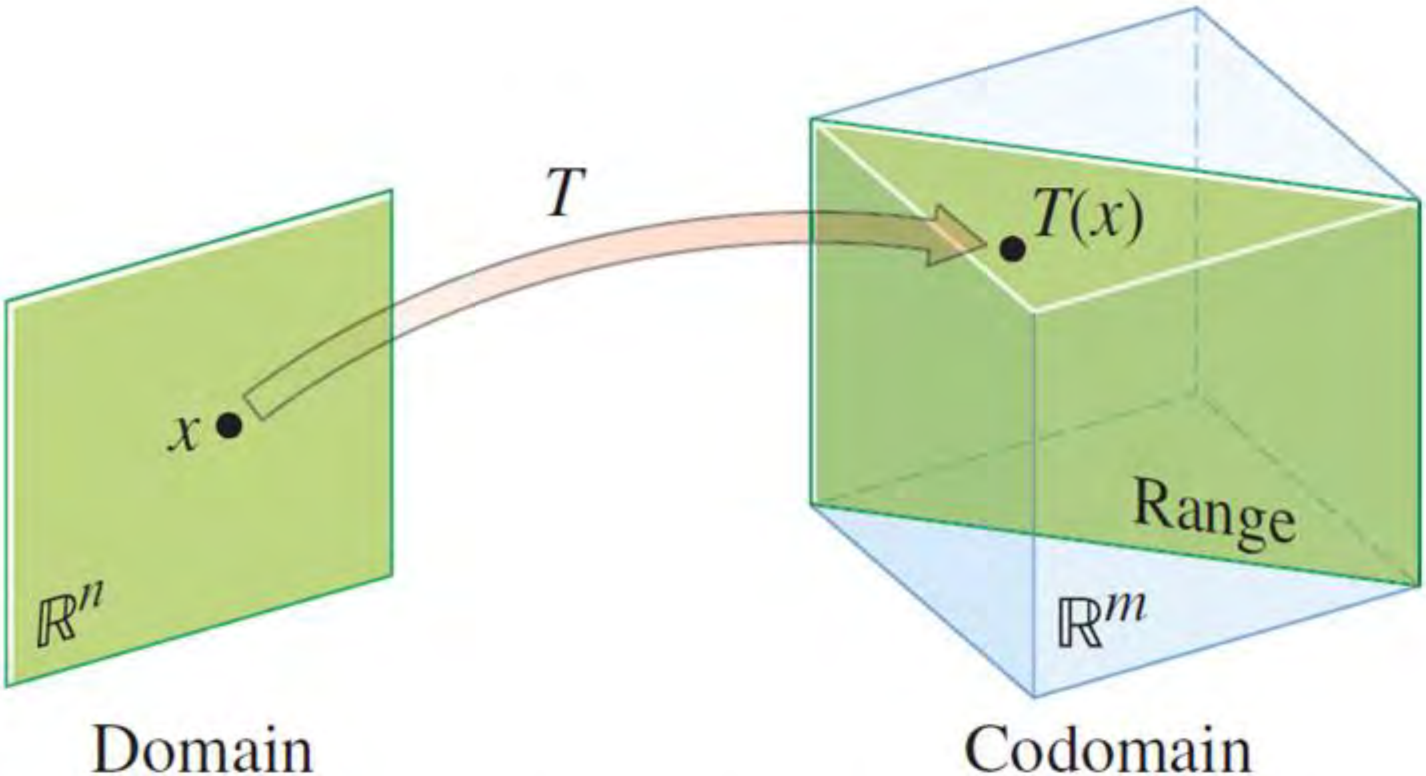
$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

- Input-output





Linear Transformation



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$

Linear Transformation

EXAMPLE 1 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Linear mapping

A linear transformation (or a linear map) is a function $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^m$ that satisfies the following properties:

1. $\mathbf{T}(\mathbf{x} + \mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$
2. $\mathbf{T}(a\mathbf{x}) = a\mathbf{T}(\mathbf{x})$

for any vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and any scalar $a \in \mathbf{R}$.

Linear mapping

- Example: which are linear mapping?
  - **zero** map $0 : V \to W$
  - **identity** map $I : V \to V$
  - Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $Tp(z) = p'(z)$.
  - $\boxed{\text{Let } T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be the map given by } T(x, y) = (x - 2y, 3x + y)}$
  - $T(x) = e^x$
  - $T : \mathbb{F} \to \mathbb{F}$ given by $T(x) = x - 1$

Linear mapping

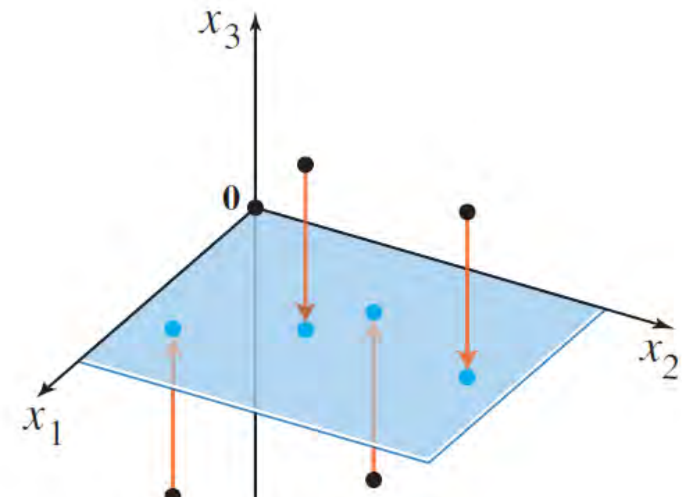
- Theorem

Let $(v_1, \ldots, v_n)$ *be a basis of* V *and* $(w_1, \ldots, w_n)$ *an arbitrary list of vectors in* W. *Then there exists a unique linear map*

$$T : V \to W \quad \text{such that } T(v_i) = w_i.$$

Projection

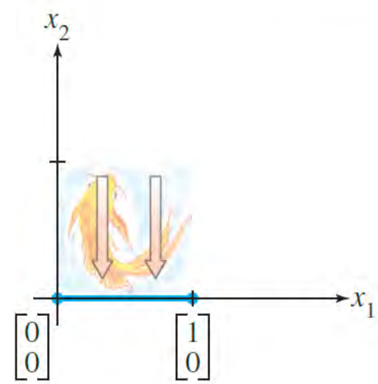
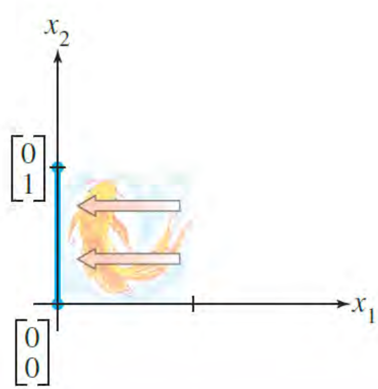
- Example:



If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ *projects* points in $\mathbb{R}^3$ onto the x_1x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

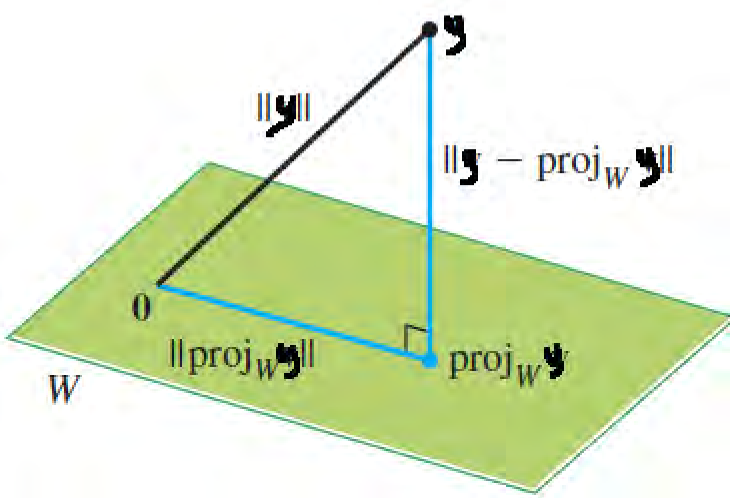
Projection

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projection

The ***projection*** of a vector $y \in \mathbb{R}^m$ onto the span of $\{x_1, \ldots, x_n\}$ is the vector $v \in \mathrm{span}(\{x_1, \ldots x_n\})$, such that v is as close as possible to y, as measured by the Euclidean norm $\|v - y\|_2$.

$$\mathrm{Proj}(y; \{x_1, \ldots x_n\}) = \mathrm{argmin}_{v \in \mathrm{span}(\{x_1,\ldots,x_n\})} \|y - v\|_2.$$



Projection

Suppose that $\mathcal{V}$ is a vector space and $P : \mathcal{V} \to \mathcal{V}$ is a linear transformation.

a) If $P^2 = P$ then P is called a **projection**.

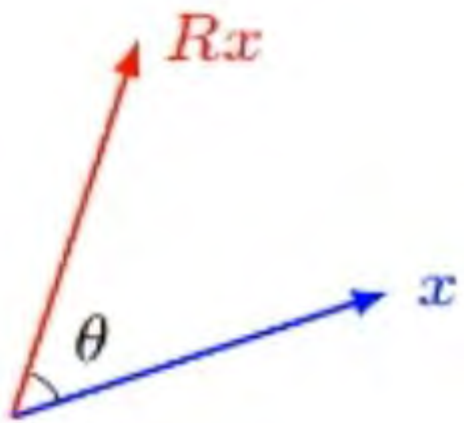
b) If $\mathcal{V}$ is an inner product space and $P^2 = P = P^*$ then P is called an **orthogonal projection**.

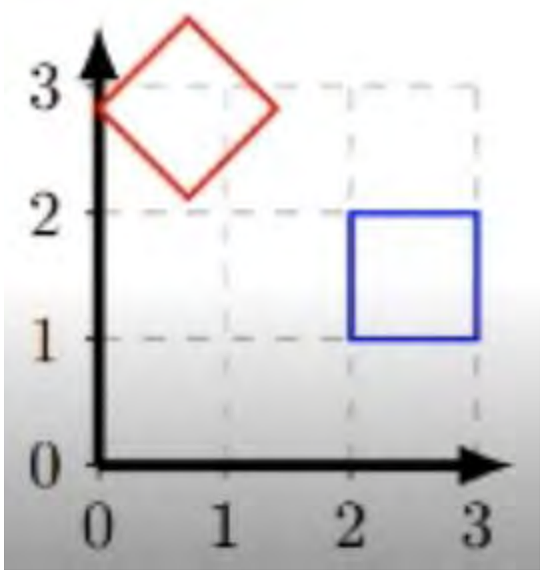
We furthermore say that P **projects onto** range(P).

- Projection of vector v on:
  - Two orthogonal vectors
  - Two non-orthogonal vectors

Rotation

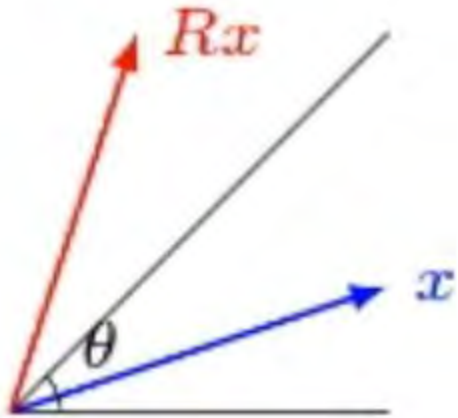
- $$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

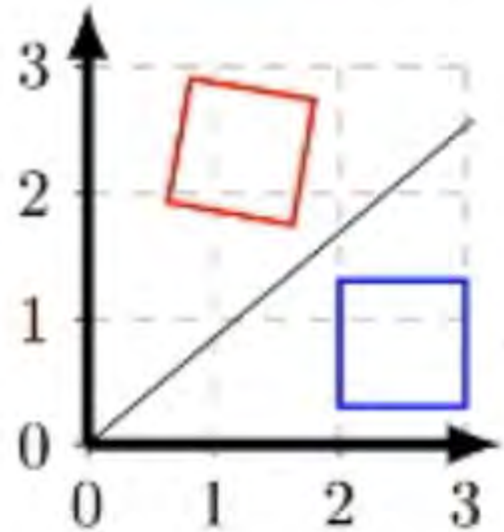




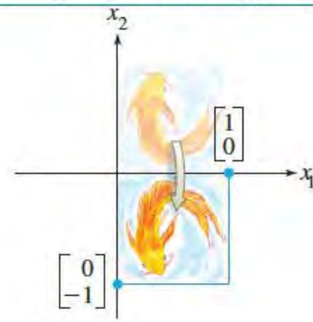
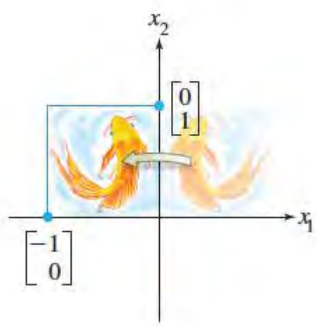
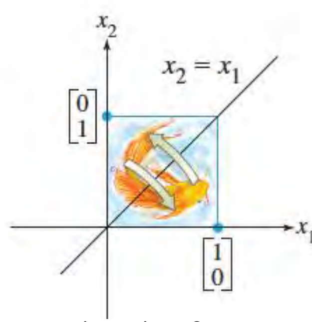
Reflection

- $R = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$



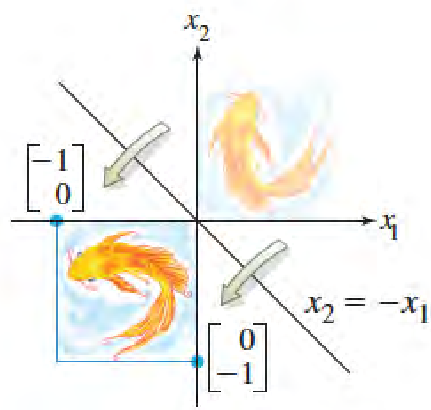


Reflection

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

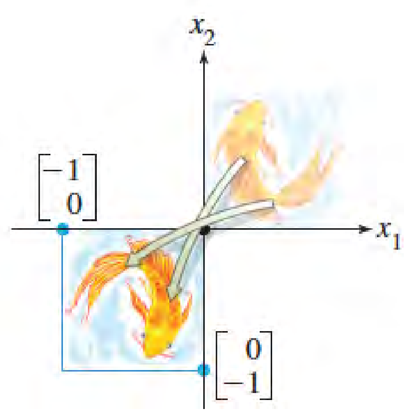
Reflection

Reflection through the line $x_2 = -x_1$



$$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

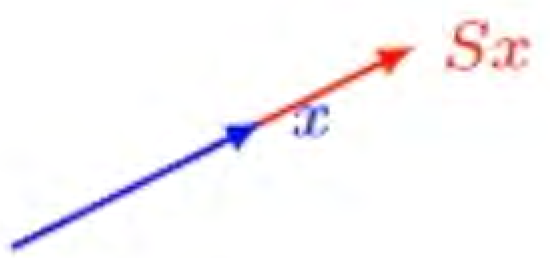
Reflection through the origin

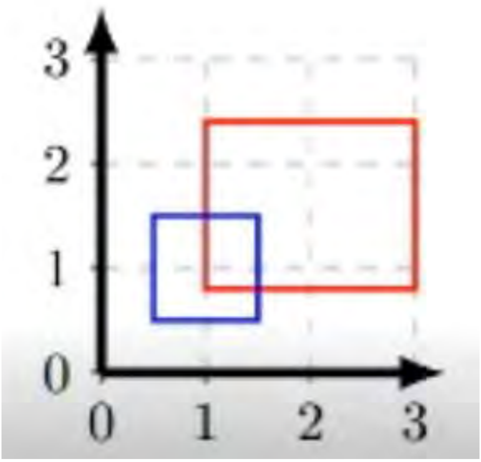


$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

Uniform Scaling

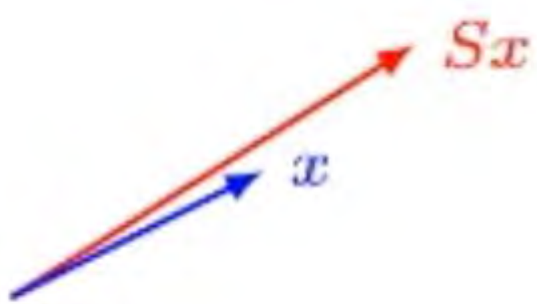
- $S = sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

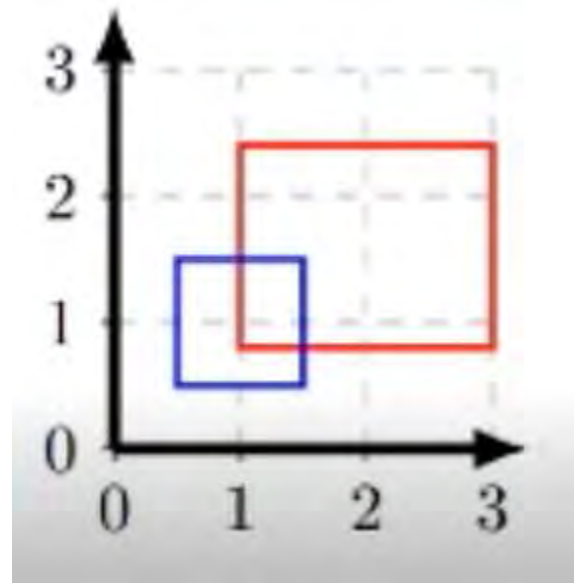




Non-uniform Scaling

- $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$





Shearing

- Example

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$

sheep

A typical shear matrix is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

sheared sheep

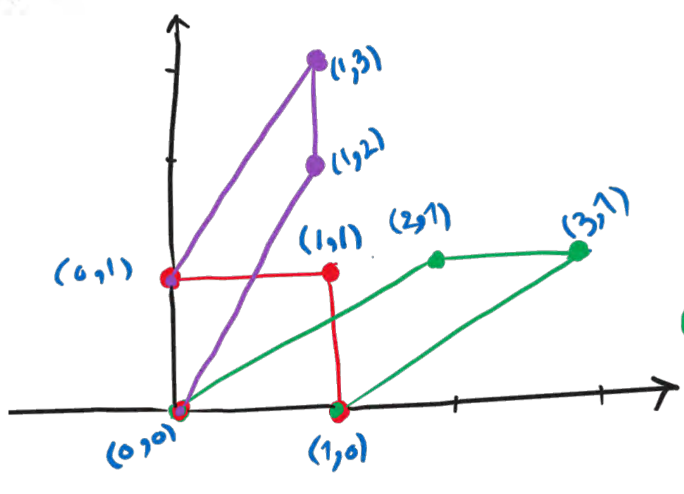
Shearing

A shear parallel to the *x* axis results in $x' = x + \lambda y$ and $y' = y$. In matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Similarly, a shear parallel to the *y* axis has $x' = x$ and $y' = y + \lambda x$. In matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



x_axis $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ y_axis $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(0,1) → (2,1) (0,1)
(0,0) → (0,0) (0,0)
(1,1) → (3,1) (1,3)
(1,0) → (1,0) (1,2)

Difference Matrix

- $$D_{(n-1)\times n} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

$$D : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1} \quad \Longrightarrow \quad D \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

- Example

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1-0 \\ 3-(-1) \\ 2-3 \\ 5-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$$

Selectors

- an $m \times n$ *selector matrix*: each row is a unit vector (transposed)

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix}$$

multiplying by A selects entries of x:

$$Ax = (x_{k_1}, x_{k_2}, \ldots, x_{k_m})$$

- $$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_m} \end{bmatrix}$$

Selectors

- Example
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

  - Selecting first and last elements of vector:
  - Reversing the elements of vector:

Slicing

- Keeping m elements from r to s (m=s-r+1)

$$\begin{bmatrix} 0_{m\times(r-1)} & I_{m\times m} & 0_{m\times(n-s)} \end{bmatrix}$$

- Example: Slicing two first and one last elements:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Down Sampling

- Down sampling with k: selecting one sample in every k samples
- Example: k=2?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix}$$

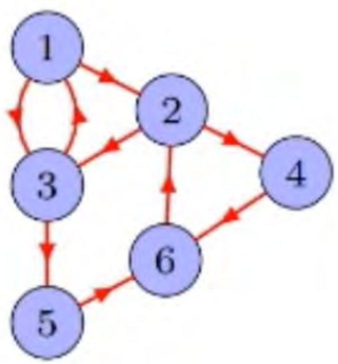
Applications

- Rotation matrix

(i) sin 2A = 2 sin A cos A
(ii) cos 2A = cos² A − sin² A

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Longrightarrow R^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

- Adjacency matrix



$$A = \begin{array}{c} \begin{array}{cccccc} n1 & n2 & n3 & n4 & n5 & n6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiple Transformation

- $x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Longrightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Longrightarrow z = B(Ax) = BAx$
- Example
  - Difference Matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \xrightarrow[4\times 5]{D} y = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ x_5 - x_4 \end{bmatrix} \xrightarrow[3\times 4]{D} z = \begin{bmatrix} x_3 - x_2 - (x_2 - x_1) \\ x_4 - x_3 - (x_3 - x_2) \\ x_5 - x_4 - (x_4 - x_3) \end{bmatrix} = \begin{bmatrix} x_3 - 2x_2 + x_1 \\ x_4 - 2x_3 + x_2 \\ x_5 - 2x_4 + x_3 \end{bmatrix}$$

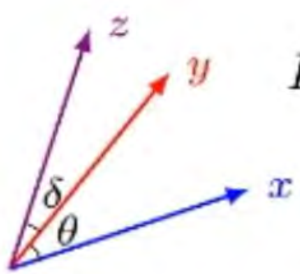
$$x \longrightarrow z \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}_{3\times 5}$$

$$x \longrightarrow y \longrightarrow z$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{4\times 5} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Multiple Transformation

- $x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Longrightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Longrightarrow z = B(Ax) = BAx$
- Example
  - Rotation



$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

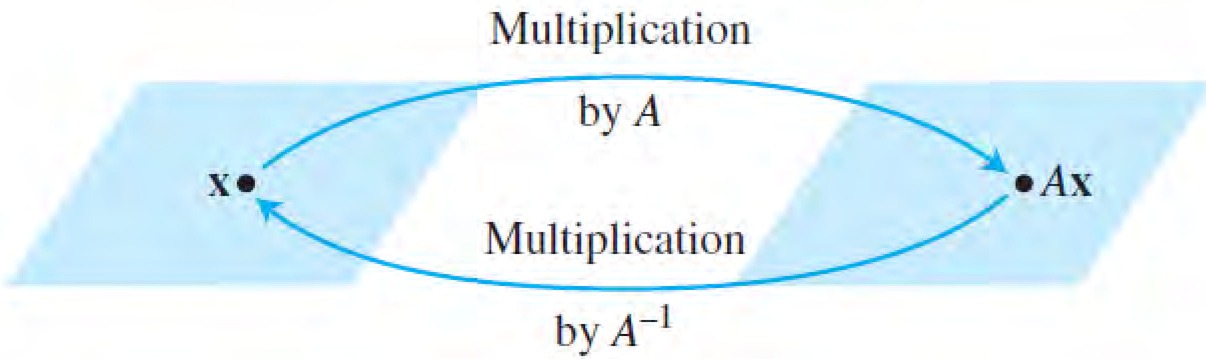
$$x \longrightarrow z \qquad z = R_{\delta+\theta}x \qquad \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$

$$x \longrightarrow y \longrightarrow z \begin{cases} y = R_\theta x \\ z = R_\delta y \end{cases} \Longrightarrow z = R_\delta R_\theta x \qquad \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\delta\cos\theta - \sin\delta\sin\theta & -\cos\delta\sin\theta - \sin\delta\cos\theta \\ \sin\delta\cos\theta + \cos\delta\sin\theta & -\sin\delta\sin\theta + \cos\delta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$

Invertible Linear Transformations



- Definition:

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$
$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Invertible Linear Transformations

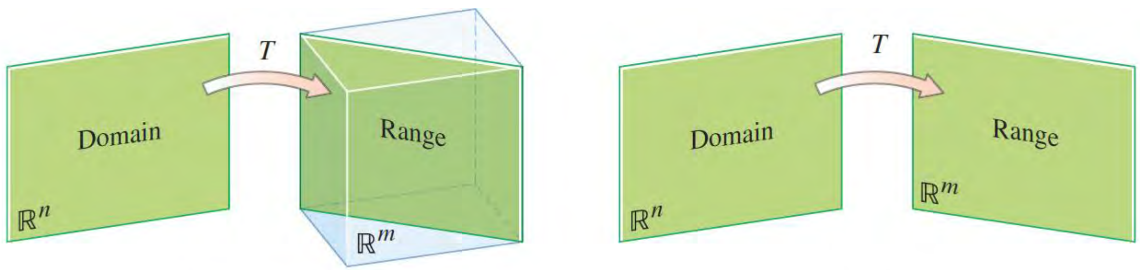
- Theorem:

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$
$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

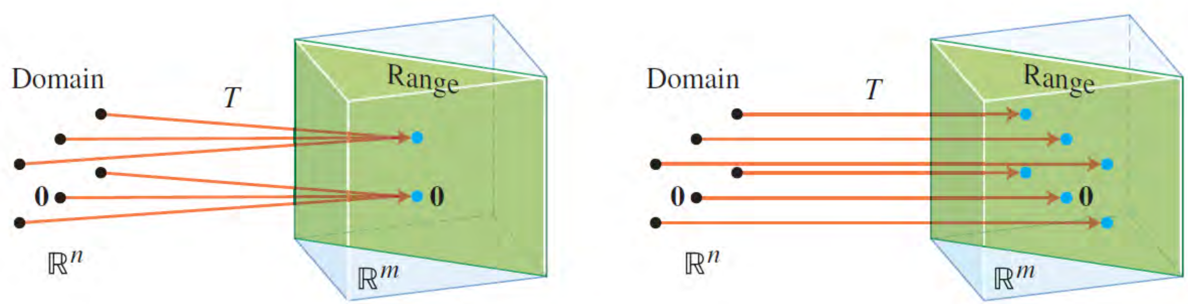
Mapping

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

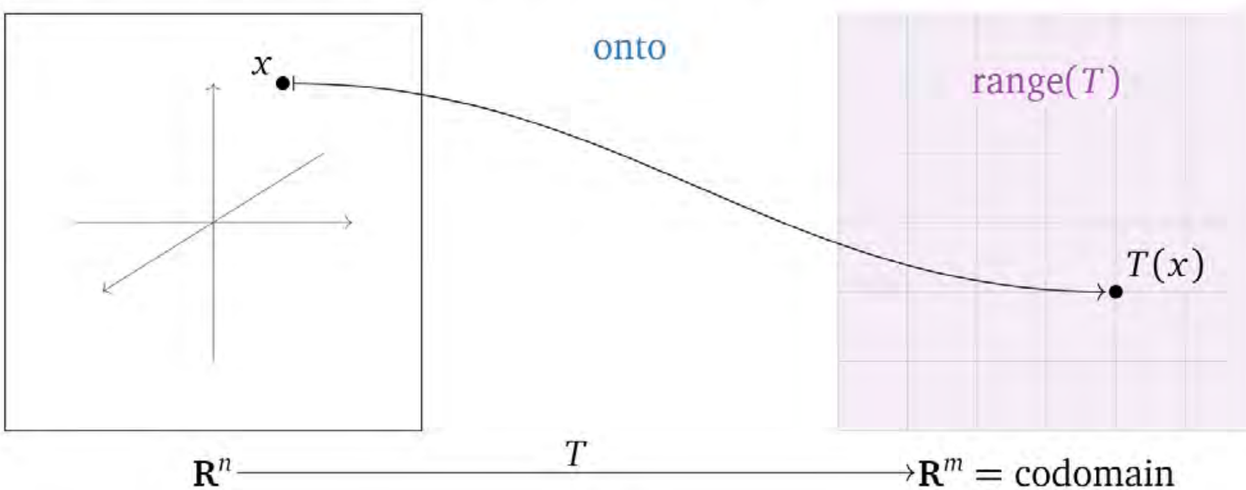


Onto (surjective) Transformations

Definition (Onto transformations). A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is ***onto*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at least one* solution x in $\mathbf{R}^n$.

Here are some equivalent ways of saying that T is onto:

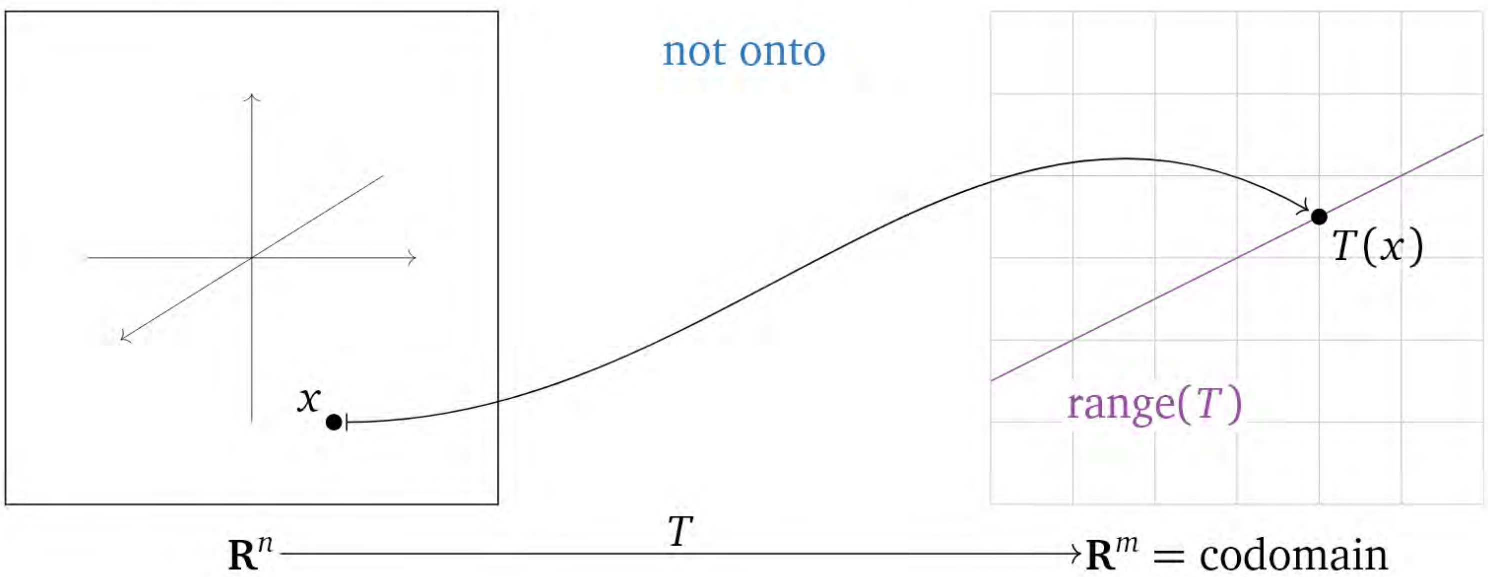
- The range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.



Onto Transformations

Here are some equivalent ways of saying that T is *not* onto:

- The range of T is smaller than the codomain of T.
- There exists a vector b in $\mathbf{R}^m$ such that the equation $T(x) = b$ does not have a solution.
- There is a vector in the codomain that is not the output of any input vector.



Onto Transformations

Theorem (Onto matrix transformations). *Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *T is onto.*
2. *$T(x) = b$ has at least one solution for every b in $\mathbf{R}^m$.*
3. *$Ax = b$ is consistent for every b in $\mathbf{R}^m$.*
4. *The columns of A span $\mathbf{R}^m$.*
5. *A has a pivot in every row.*
6. *The range of T has dimension m.*

Onto Transformations

Tall matrices do not have onto transformations. If $T : \mathbf{R}^n \to \mathbf{R}^m$ is an onto matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every row, it must have *at least as many columns* as rows: $m \leq n$.

This says that, for instance, $\mathbf{R}^2$ is "too small" to admit an onto linear transformation to $\mathbf{R}^3$.

Note that there exist wide matrices that are not onto: for example,

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

does not have a pivot in every row.

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

One-to-One (injective) Linear Transformation

THEOREM

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

One-to-One Linear Transformation

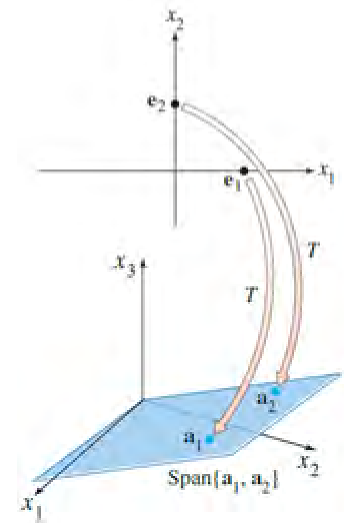
- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

  a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

  b. T is one-to-one if and only if the columns of A are linearly independent.

- Example

Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?



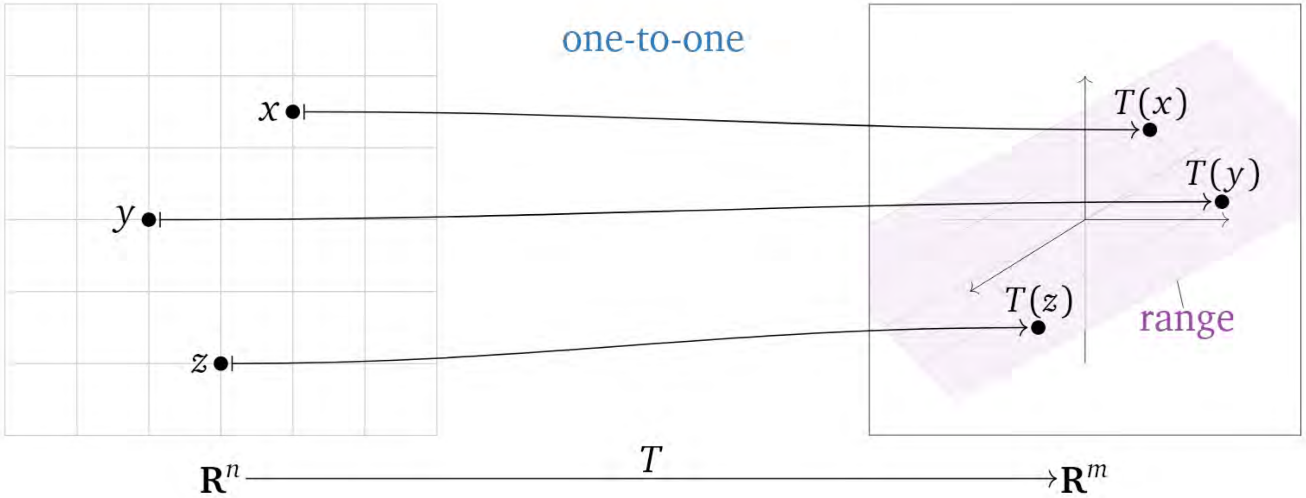
One-to-one Transformations

Definition (One-to-one transformations). A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is ***one-to-one*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x in $\mathbf{R}^n$.

Remark.

Here are some equivalent ways of saying that T is one-to-one:

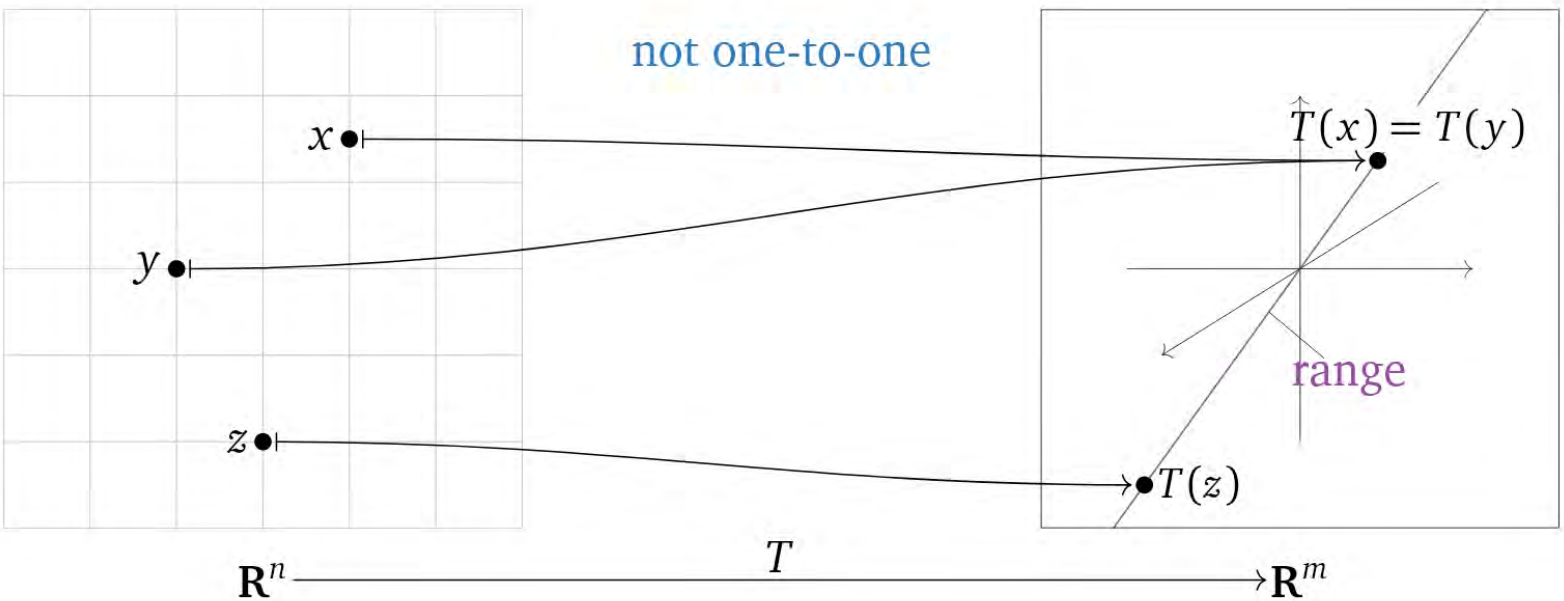
- For every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has *zero or one* solution x in $\mathbf{R}^n$.
- Different inputs of T have different outputs.
- If $T(u) = T(v)$ then $u = v$.



One-to-one Transformations

Here are some equivalent ways of saying that T is *not* one-to-one:

- There exists some vector b in $\mathbf{R}^m$ such that the equation $T(x) = b$ has *more than one* solution x in $\mathbf{R}^n$.
- There are two different inputs of T with the same output.
- There exist vectors u, v such that $u \neq v$ but $T(u) = T(v)$.



One-to-one Transformations

Theorem (One-to-one matrix transformations). *Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *T is one-to-one.*
2. *For every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.*
3. *For every b in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.*
4. *$Ax = 0$ has only the trivial solution.*
5. *The columns of A are linearly independent.*
6. *A has a pivot in every column.*
7. *The range of T has dimension n.*

One-to-one Transformations

Wide matrices do not have one-to-one transformations. If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every column, it must have *at least as many rows* as columns: $n \leq m$.

This says that, for instance, $\mathbf{R}^3$ is "too big" to admit a one-to-one linear transformation into $\mathbf{R}^2$.

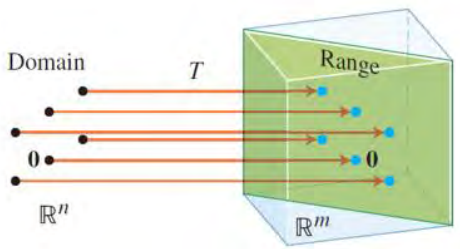
Note that there exist tall matrices that are not one-to-one: for example,

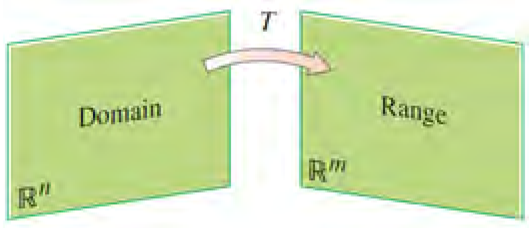
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

does not have a pivot in every column.

Comparison

A is an $m \times n$ matrix, and $T: \mathbf{R}^n \to \mathbf{R}^m$ is the matrix transformation $T(x) = Ax$





T **is one-to-one**	T **is onto**
$T(x) = b$ has *at most one solution* for every b.	$T(x) = b$ has *at least one solution* for every b.
The columns of A are linearly independent.	The columns of A span $\mathbf{R}^m$.
A has a pivot in every column.	A has a pivot in every row.
The range of T has dimension n.	The range of T has dimension m.

One-to-one and onto

One-to-one is the same as onto for square matrices. We observed in the previous example that a square matrix has a pivot in every row if and only if it has a pivot in every column. Therefore, a matrix transformation T from $\mathbf{R}^n$ to itself is one-to-one if and only if it is onto: in this case, the two notions are equivalent.

Conversely, by this note and this note, if a matrix transformation $T : \mathbf{R}^m \to \mathbf{R}^n$ is both one-to-one and onto, then $m = n$.

Note that in general, a transformation T is both one-to-one and onto if and only if $T(x) = b$ has *exactly one* solution for all b in $\mathbf{R}^m$.

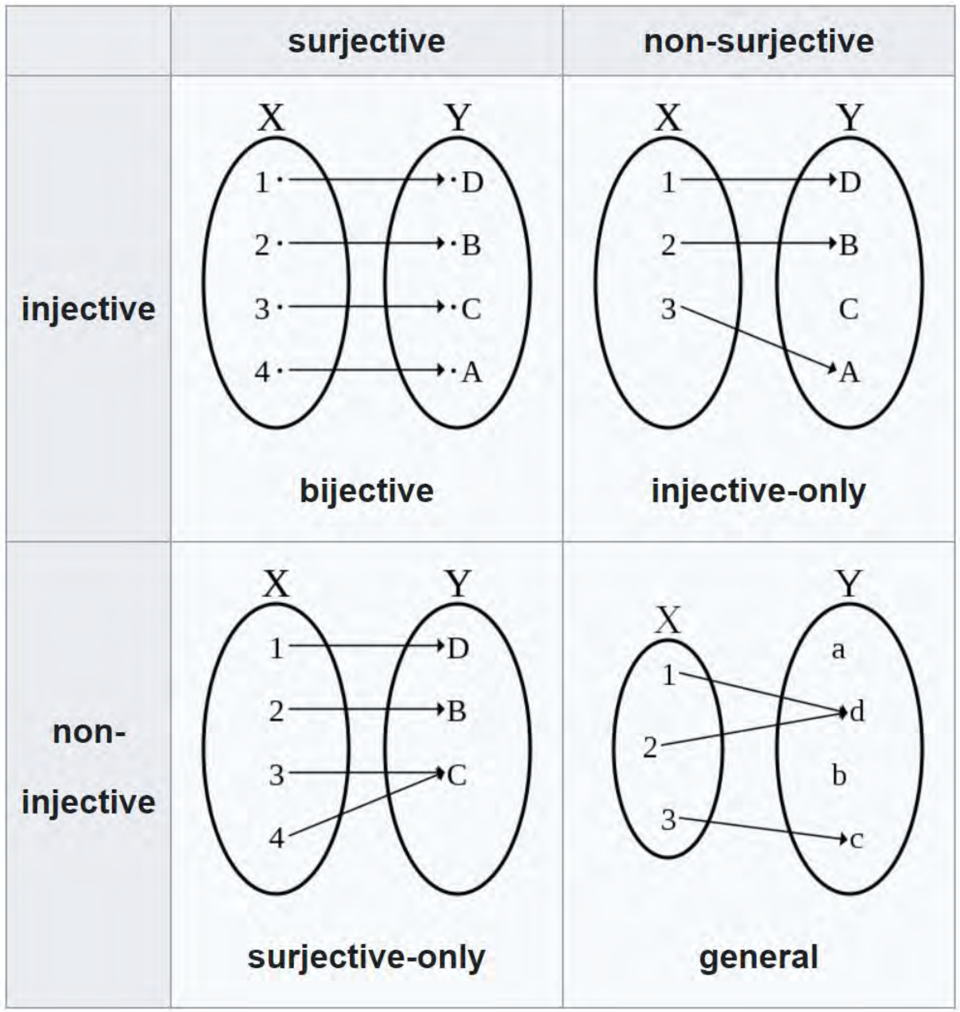
Bijective

- one-to-one and onto
- if and only if every possible image is mapped to by exactly one argument

Conclusion

On to

One-to-One



Machine Learning Application

- The central problem in machine learning and deep learning is to meaningfully transform data: in other words, to learn useful representations of the input data at hand — representations that get us closer to the expected output.

Inner Product

- $< Ax, y > = < x, A^T y >$
  - What about symmetric matrix?
- Show that unitary matrix preserves inner product. $< Ux, Uy > = < x, y >$

Introduction to change of basis

- $B = \{v_1, \ldots, v_n\}$ are basis of R^n
- $C[a]_B = a$
- $C = [v_1 \quad v_2 \quad \cdots \quad v_n]$

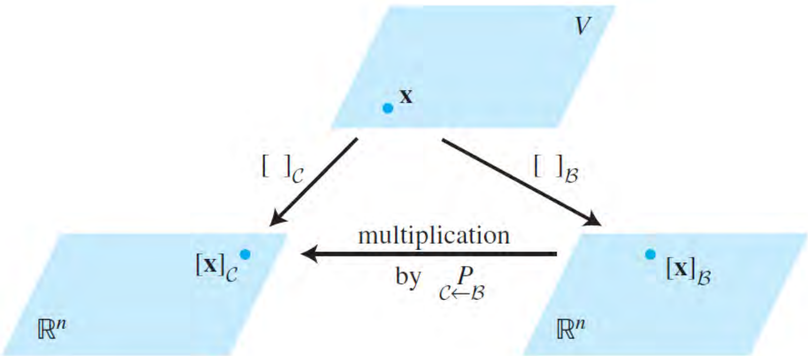
Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\,\mathbf{x}\,]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$



$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Change of Basis

- Example

Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

- Final Review!

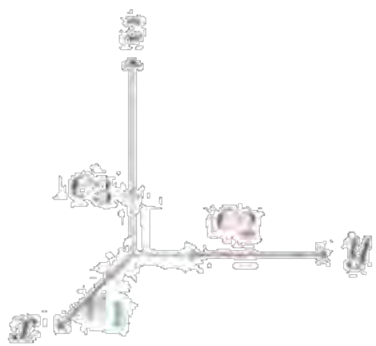
$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

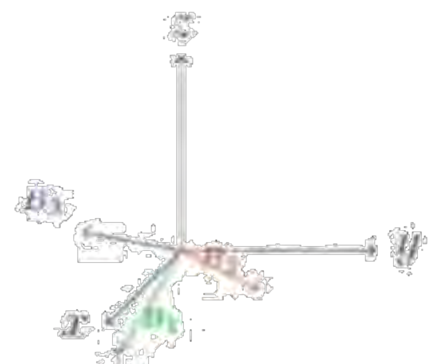
Matrix Representation of Linear Function

- Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear function and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$.

  The matrix $\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$ is called the matrix representation of linear function (transformation) T which is denoted by $[T]_E$.



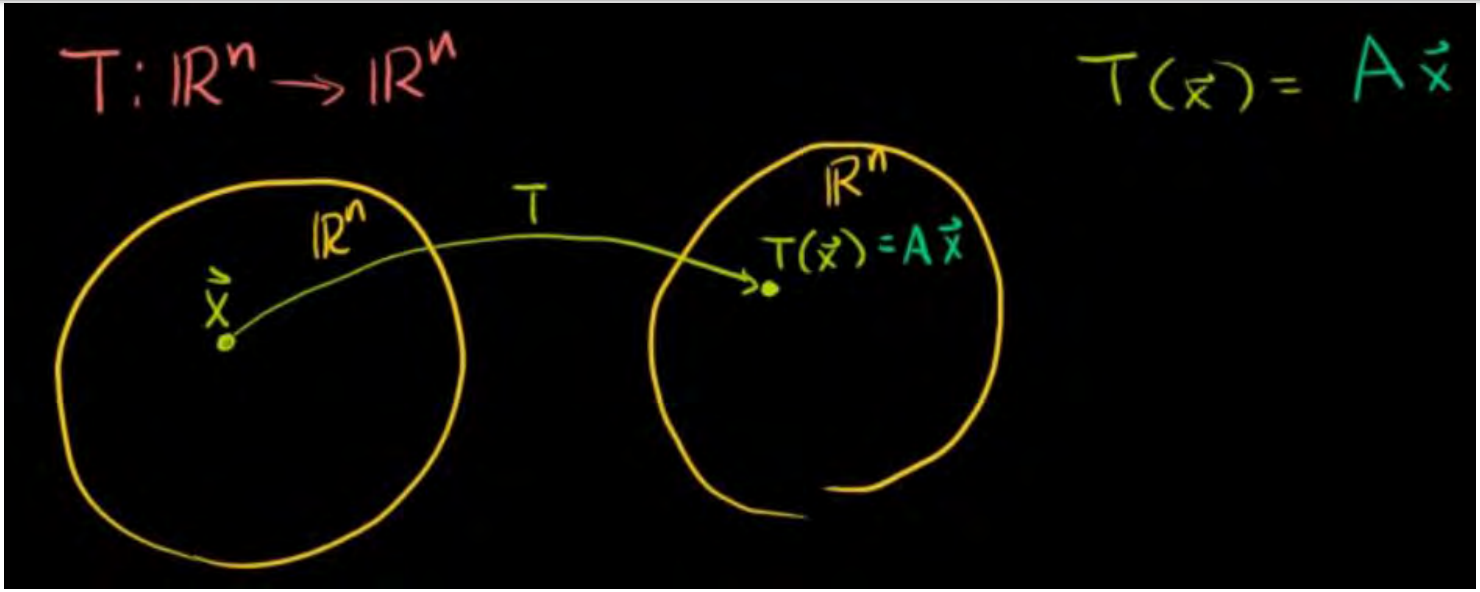
$E = \{e_1, e_2, e_3\}$



B=$\{B_1, B_2, B_3\}$

What is the relation between $[T]_B$ and $[T]_E$?

Transformation with Change of Basis



- $B = \{v_1, \dots, v_n\}$ are basis of R^n
- $C = [v_1 \quad v_2 \quad \cdots \quad v_n]$
- $[T(x)]_B = C^{-1}AC[x]_B$

Change of Basis

- $$\begin{tikzcd} \mathbf{R}^n \arrow[r, "{[T]_B}"] \arrow[d, "P"'] & \mathbf{R}^n \\ \mathbf{R}^n \arrow[r, "{[T]_E}"'] & \mathbf{R}^n \arrow[u, "P^{-1}"'] \end{tikzcd}$$

$$[T]_B = P^{-1}[T]_E P$$

Example

- We have $B = \{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $P_3(x)$ and $P_2(x)$, respectively.

- Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in P_3(x)$, we have

$$\left[\frac{d}{dt}\right]_{\{B,B'\}} = \begin{bmatrix} \frac{d}{dt}(x^3) & \frac{d}{dt}(x^2) & \frac{d}{dt}(x) & \frac{d}{dt}(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$