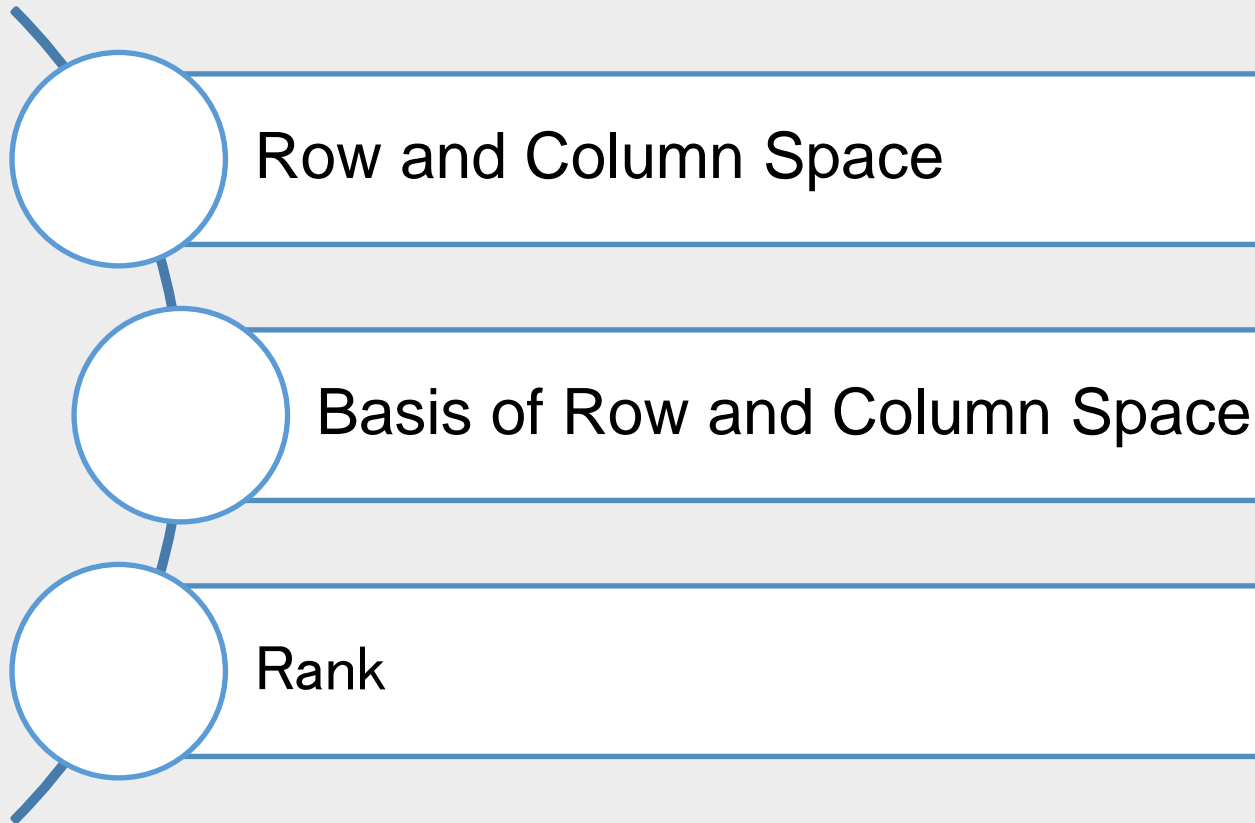# Matrix Rank

Linear Algebra

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Row and Column Space

Basis of Row and Column Space

Rank

Row and Column Space

❏ If we write A by rows, then we can express Ax as,

$$A \in \mathbb{R}^{m \times n} \quad y = Ax = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}. \qquad a_i^T x = \sum_{j=1}^{n} a_{ij} x$$

❏ If we write A by columns, then we have:

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1]x_1 + [a_2]x_2 + \cdots + [a_n]x_n .$$

○ y is a **linear combination** of the columns A.

columns of A are linearly independent if Ax = 0 implies x = 0

It is also possible to multiply on the left by a row vector.

○ If we write A by columns, then we can express $x^T A$ as,

$$y^T = x^T A = x^T \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = [x^T a_1 \quad x^T a_2 \quad \cdots \quad x^T a_n]$$

○ expressing A in terms of rows we have:

$$y^T = x^T A = [x_1 \quad x_2 \quad \cdots \quad x_m] \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

$$= x_1[- \quad a_1^T \quad -] + x_2[- \quad a_2^T \quad -] + \cdots + x_m[- \quad a_m^T \quad -]$$

○ y^T is a linear combination of the rows of A.

1. <mark>As a set of vector–vector products</mark>

$$C = AB = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

2. <mark>As a sum of outer products</mark>

$$C = AB = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^{n} a_i b_i^T$$

Definition

Let A be a $m \times n$ matrix. Then the **column space** of A is $C(A)$:

$$C(A) := \{ Ax : x \in \mathbb{R}^n \}$$

and the **row space** of A is:

$$R(A) := \{ y^T A : y \in \mathbb{R}^m \}$$

❑ **Think about the following facts and prove them:**

The **_row space_** of a matrix is the collection of all linear combinations of its rows.
Equivalently, the row space is the **_span_** of rows.
The elements of a row space are **_row vectors_**.
If a matrix has m columns, its row space is a **_subspace_** of (the row version of) $\mathbb{R}^m$.

Elementary row operations **do not** alter the row space.
Thus a matrix and its echelon form have the same row space. The
pivot rows of an echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & * & * & * \\ & 1 & * & * \\ & & 1 & * \end{bmatrix}$$

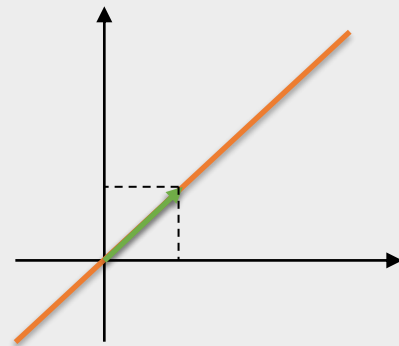$$c_1 [1 \quad 1] + c_2 [0 \quad 0]$$

Row space of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

The **_pivot rows_** of an echelon form span the row space of the original matrix.
The **_dimension_** of the row space is given by the number of pivot rows.
This dimension does not exceed the total row count.

❑ **Think about the following facts and prove them:**

The *column space* of a matrix is the collection of all linear combinations of its columns.
It is the span of columns, the image of the linear transformation carried out by the matrix.
If a matrix has m rows, its column space is a subspace of $\mathbb{R}^m$.

Elementary row operations affect the column space.
So, generally, a matrix and its echelon form have different column spaces.
However, since the row operations preserve the linear relations between columns, the columns of an echelon form and the original columns obey the same relations.

Example in next page!

The pivot columns of a reduced row-echelon form are *linearly independent*.

$$\begin{bmatrix} 1 & * & & * \\ & 1 & & * \\ & & 1 & * \\ & & & \end{bmatrix}$$

Column space of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

The pivot columns of a reduced row-echelon form span its column space.
So the pivot columns of a matrix are linearly independent
and span its column space.
The dimension of the column space is given by the number of pivot columns.
This dimension does not exceed the total column count.

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 1 & 3 & 5 & 8 & 11 \\ 4 & 10 & 16 & 23 & 30 \end{pmatrix} \qquad B_{\text{rref}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= & x_3 - x_5, \\ x_2 &= & -2x_3 + 2x_5, \\ x_4 &= & -2x_5. \end{aligned} \qquad \mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

The column space of B is 3-dimensional, and that a basis is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 8 \\ 23 \end{bmatrix} \right\}$

Note that we do not use the columns of B$_{\text{rref}}$! We use the columns of B.

Basis of Row and Column Space

Theorem

If two matrices $A \ and \ B$ are row-equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows (pivot rows) of B form a basis for the row space of A as well as for that of B.

Definition

❑ If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

1.   The leading entry in each non-zero row is 1.
2.   Each leading 1 is the only non-zero entry in its columns.
3.   The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
4.   Any row containing only 0's is at the bottom.

$$e_1 \quad e_2 \quad e_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Reduced Echelon form

Theorem

The pivot columns of a matrix A form a basis for $Col(A)$

o   Lemma1: The pivot columns of A are linearly independent

o   Lemma 2. The pivot columns of A span the column space of A

o   From first lectures we know that "The span of the pivot columns is the same as the span of all the columns"

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

span the column space.

We use the reduced echelon matrix of $\mathbf{A}$ in the proof. We designate it by $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ have r pivot columns, then the pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$. For example, consider the reduced echelon matrix

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has pivot columns $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ in its first, third, and fourth columns. These are vectors in $\mathbb{R}^4$.

The solutions of the homogeneous equations $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$ are the same. Therefore, the relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same. (Any non-zero solution of these homogeneous equations gives a relation among these columns.)

Lemma 1. *(a) The pivot columns of $\mathbf{B}$ are linearly independent.*
(b) The pivot columns of $\mathbf{A}$ are linearly independent.

Proof. (a) The pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$ that are linearly independent. (In the example of the matrix given above, $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ are linearly independent in $\mathbb{R}^4$.)

(b) The relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same (as noted above), so the pivot columns of $\mathbf{A}$ are also linearly independent. □

Lemma 2. *The pivot columns of $\mathbf{A}$ span the column space of $\mathbf{A}$*

Proof. Let $\mathbf{b}^k$ be a non-pivot column of the reduced echelon matrix $\mathbf{B}$. Assume that there are j pivot columns to the left of $\mathbf{b}^k$ in $\mathbf{B}$. These pivot columns must be $\mathbf{e}^1, \ldots, \mathbf{e}^j$. The non-pivot column $\mathbf{b}^k$ can only have nonzero entries in the first j components (or it would be a pivot column) and so is a linear combination of $\mathbf{e}^1, \ldots, \mathbf{e}^j$.

Since $\mathbf{A}$ has the same relations among its columns as $\mathbf{B}$, its non-pivot column $\mathbf{a}^k$ is a linear combination of the j pivot columns to the left of it. By Theorem 5(a), the span of the pivot columns is the same as the span of all the columns. This shows that the pivot columns of $\mathbf{A}$ span the column space of $\mathbf{A}$. □

By Lemma 1(b), the pivot columns are linearly independent. By Lemma 2, the pivot columns span the column space of $\mathbf{A}$. Together, these two facts show that the pivot columns form a basis for the column space of $\mathbf{A}$. □

Example

Find:
- ❑ Row Basis
- ❑ Column Basis
- ❑ $\dim(Row(A))$
- ❑ $\dim(Col(A))$
- ❑ $\dim(Null(A))$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = pivot columns

Rank

Definition

❑ We call dim(R(A)) the row rank of A and dim(C(A)) the column rank of A.

❑ We refer to a basis of C(A) consisting of columns of A as a column basis.

A row basis is defined similarly.

Rank of Matrix

Definition

❑ The number of linearly independent rows or columns in the matrix

❑ Dimension of the row (column) space

❑ Number of nonzero rows of the matrix in row echelon form (Ref)

Note

The dimension of the Column Space of A and $rref(A)$ is the same.

Example 1

If Columns of matrix A are linearly independent:

$$nullity(A) = ?$$
$$colrank(A) = ?$$

Example 2

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$nullity(A) = 2, colRank(A) = 2$$

Note

The dimension of $Null(A)$ is the number of free variables in the equation $Ax = 0$, and the dimension of $Col(A)$ is the number of pivot columns in A.

Example

Back to Example!

Find the dimension of the Null Space and the Column Space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(row reduce the Augmented Matrix $[A \; 0]$ to echelon form)

$$T: V \to W \quad \text{Dim}(V) = Nullity(T) + Dim(range(T))$$

Important Note!!! (WHY?)

Column Space (A) = Range(T)

Dim(range(T)) = ColumnRank(A)

Theorem

- $Nullity(A) + ColRank(A) = n$
- $Dim(Null(A)) + Dim(Range(A)) = n$

$$\{number\ of\ pivot\ columns\} + \{number\ of\ non-pivot\ columns\} = \{number\ of\ columns\}$$

Theorem

❑ Let A denote an m × n matrix of rank r. Then:

the n—r basic solutions to the system Ax =0 provided by the gaussian algorithm
are a basis of null (A), so dim[null (A)] = n—r.

Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$nullity(A) = 2, colRank(A) = 2 \quad \text{Nullspace(A)} = \{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \}$$

Theorem (RMRT)

(Rank of a matrix is equal to the rank of its transpose)
Suppose A is an $m \times n$ matrix.
Then $\mathrm{col}rank(A) = colrank(A^T)$

Proof?

Lemma

Suppose A is an $m \times n$ matrix.
- ❑ $Ax = 0 \leftrightarrow A^T Ax = 0$ *Why?*
- ❑ $\mathrm{ColRank}(A^T A) = \mathrm{ColRank}(A)$ *Why?*
- ❑ $\dim(Range(AB)) \leq \dim(Range(A))$ *Why?*

Using the three above lemmas proof that:
- ❑ $\mathrm{ColRank}(A) = \mathrm{ColRank}(A^T)$

Proof?

Theorem

- ❏ $ColRank(A) = RowRank(A)$
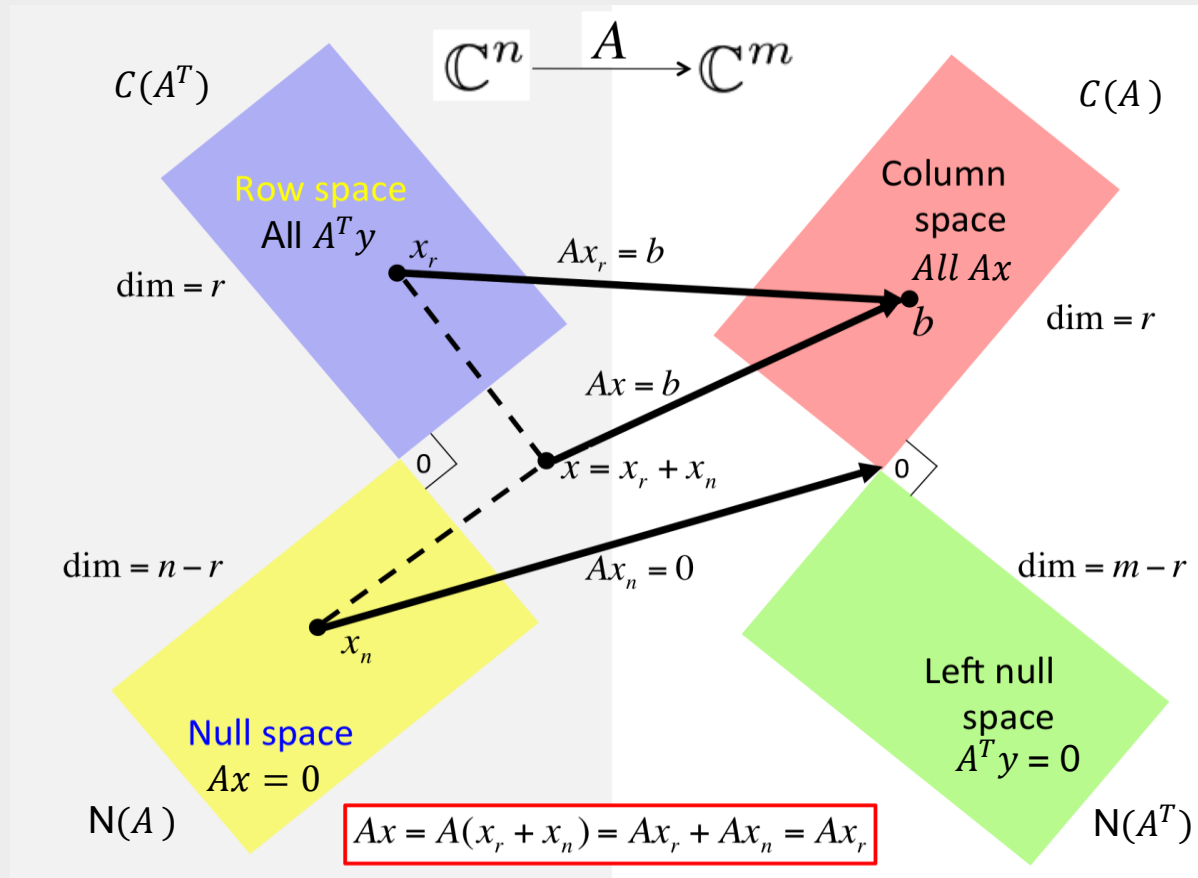- ❏ In general, It's called rank of matrix ($rank(A)$)

Proof?

> **Theorem**
>
> ❑ $rank(AB) \leq \min\{rank(A), rank(B)\}$

Proof?

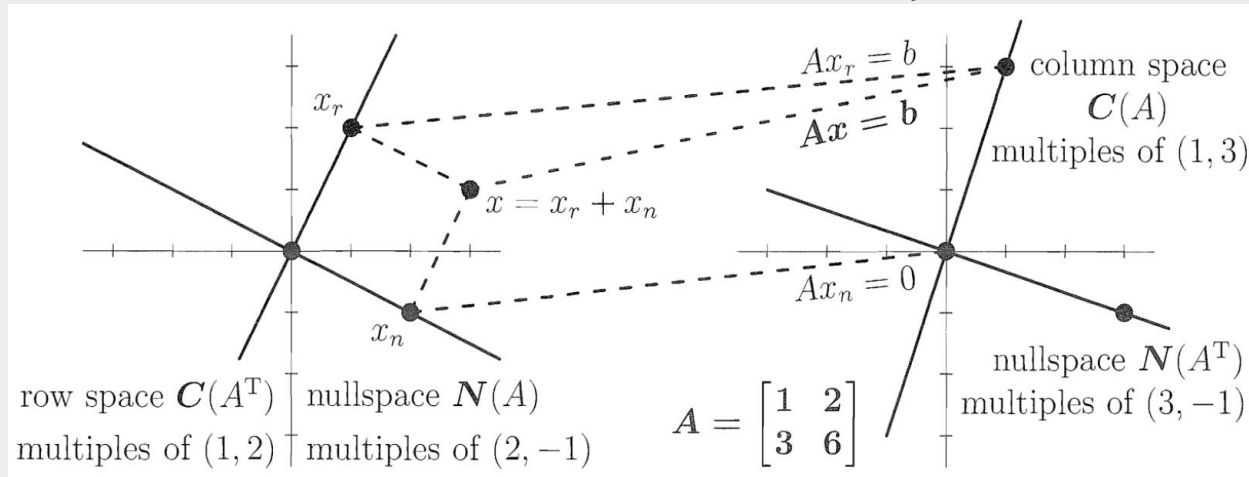**Example**: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has $m = n = 2$, and $rank\ r = 1$

1. The column space has all multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

2. The row space contains all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3. The null space contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

4. The left null space contains all multiples of y = $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

The rows of A with coefficients −3 and 1 add to zero, $A^T y = 0$

Here the row space $C(A^T)$ is the multiples of (1,2), the null space $N(A)$ is the multiples of (2, −1), column space $C(A)$ is the multiples of (1,3) and left null space $N(A^T)$ is multiples of (3,−1).
Also we can see that Row space is orthogonal to Null space and Column space is orthogonal to Left null space.

❑ The summary of the four subspaces associated with m by n matrix A of rank r.

m, n, r	dim R(A)	dim N(A)	dim C(A)	dim $N(A^T)$	Solvability of $Ax = b$
m = n = r	r	0	r	0	Solvable, $x = A^{-1}b$ is unique solution
m > n = r	r	0	r	m − r	Solvable if $b \in C(A)$
n > m = r	r	n − r	r	0	Solvable, infinite solutions $x = x_p + N(A)$
r < min(m,n)	r	n - r	r	m - r	Solvable only if $b \in C(A)$ Infinite solutions