Matrix Algebra: Dimension and Rank

CE40282-1: Linear Algebra
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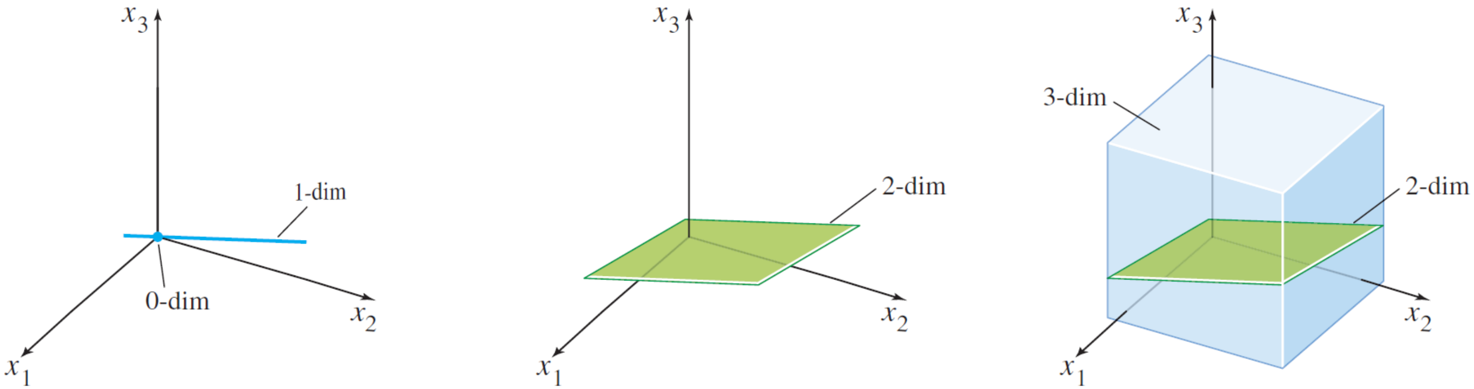
Review: Dimension

- If V has a finite basis, then $\mathbf{dim}(V)$ is the number of elements (vectors) of any basis of V
- Dim({0})=0

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.



Finite-Dimensional Space

THEOREM

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

THEOREM

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Row and Column Space

THEOREM If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

The pivot columns of a matrix A form a basis for Col A.

Row and Column Space

- Example
  - Row Basis
  - Column Basis
  - dim(row(A))
  - dim(column(A))
  - dim(null(A))

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

number of non-zero rows=pivot columns

Rank

- Rank of matrix: (row or column)
  - The number of linearly independent rows or columns in the matrix
  - Dimension of the row (column) space
  - Number of nonzero rows of the matrix in row echelon form (Ref)

1) Row rank = column rank for a matrix in reduced row echelon form.

2) The dimension of the column space of A and $\mathrm{rref}(A)$ is the same.

Rank

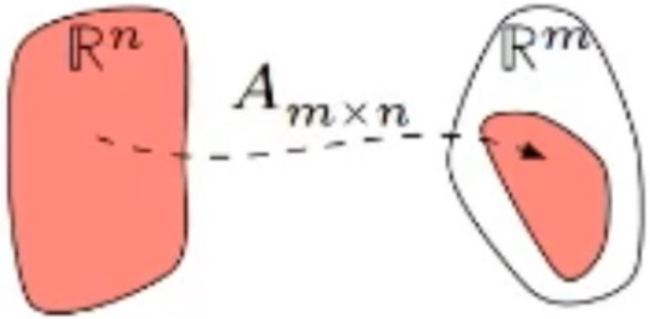
Theorem RMRT: Rank of a Matrix is the Rank of the Transpose. *Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$*

Range Space

- For $A_{m \times n} = [a^1 \dots a^n] = [a_1 \dots a_n]$

$$\begin{aligned}\text{range}(A) &= \text{span}(a_1, \dots, a_n) \\ &= \left\{ y \mid y = \alpha_1 a_1 + \cdots + \alpha_n a_n,\ \alpha_1, \dots, \alpha_n \in \mathbb{R} \right\} \\ &= \left\{ y \mid y = Ax,\ x \in \mathbb{R}^n \right\}\end{aligned}$$

- Range is a vector space
- Range of A is a subspace of R^m
- Is Dim(A)=m?



$$\dim(range(A)) = colrank(A)$$

number of linear independent columns

- Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

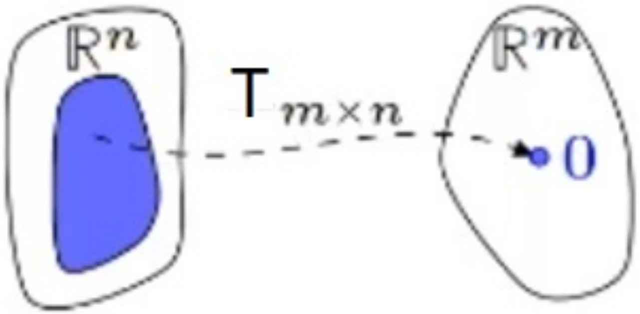
Null space (kernel)

- Definition

Let $T : V \to W$ be a linear map. Then the **null space** or **kernel** of T is the set of all vectors in V that map to zero:

$$N(T) = \operatorname{null} T = \{v \in V \mid Tv = 0\}.$$

- Null space is a vector space
- Null space T is a subspace of (V) R^n
- Is Dim(null (T))=n?
- Nullity(T): Dim(null (T))
- Question:
  - What is null space for differentiation mapping?



Null space (kernel)

- Nullity(A)=the number of free variables
- Example
  - If columns of matrix (A) are linearly independent

$$nullity(A) = ?$$
$$col\big(rank(A)\big) = ?$$

  - $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{nullity}(A) = 2 \quad \text{col rank}(A) = 2$$

Conclusion

Theorem 2.7.17. *The vectors attached to the free variables in the parametric vector form of the solution set of $Ax = 0$ form a basis of* $\mathrm{Nul}(A)$.

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\mathrm{Col}\, A = \mathbb{R}^n$

o. $\dim \mathrm{Col}\, A = n$

p. $\mathrm{rank}\, A = n$

q. $\mathrm{Nul}\, A = \{\mathbf{0}\}$

r. $\dim \mathrm{Nul}\, A = 0$

Conclusion

The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

- Examples:
  - Go to slide 6
  - Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Row reduce the augmented matrix $[\,A \quad \mathbf{0}\,]$ to echelon form

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank-Nullity Theorem

- $nullity(A) + col\ rank(A) = n$
- $\dim\big(null(A)\big) + \dim\big(range(A)\big) = n$
  - Proof?

$$\left\{\begin{matrix}\text{number of}\\ \text{pivot columns}\end{matrix}\right\} + \left\{\begin{matrix}\text{number of}\\ \text{nonpivot columns}\end{matrix}\right\} = \left\{\begin{matrix}\text{number of}\\ \text{columns}\end{matrix}\right\}$$

Rank Theorem

- Theorem:
  - $col\ rank(A) = \text{row}\ rank(A)$
  - In general it is called rank of matrix! rank(A)
  - Proof?

Rank Properties

- $col\ rank(A_{m\times n}) \leq \min(m, n)$
- row $rank(A_{m\times n}) \leq \min(m, n)$
- $\dim(\text{range}(A)) = \text{rank}(A)$

$$\text{nullity}(A) + \text{rank}(A) = n$$

$$\text{rank}(A) \leq \min(m, n)$$

Rank Properties

- For $A, B \in \mathbb{R}^{m\times n}$
  1. $\operatorname{rank}(A) \leq \min(m, n)$
  2. $\operatorname{rank}(A) = \operatorname{rank}(A^T)$
  3. $\operatorname{rank}(AB) \leq \min\big(\operatorname{rank}(A), \operatorname{rank}(B)\big)$
  4. $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$
- A has *full rank* if $\operatorname{rank}(A) = \min(m, n)$
- If $m > \operatorname{rank}(A)$ rows not linearly independent
  – Same for columns if $n > \operatorname{rank}(A)$

Rank Properties

- The ***range*** or the columnspace of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $\mathcal{R}(A)$, is the the span of the columns of A. In other words,

$$\mathcal{R}(A) = \{v \in \mathbb{R}^m : v = Ax, x \in \mathbb{R}^n\}.$$

- Assuming A is full rank and $n < m$, the projection of a vector $y \in \mathbb{R}^m$ onto the range of A is given by,

$$\mathrm{Proj}(y; A) = \mathrm{argmin}_{v \in \mathcal{R}(A)} \|v - y\|_2 = A(A^T A)^{-1} A^T y \quad .$$

- When A contains only a single column, $a \in \mathbb{R}^m$, this gives the special case for a projection of a vector on to a line:

$$\mathrm{Proj}(y; a) = \frac{aa^T}{a^T a} y \quad .$$