Proof of theorems - Linear Equations

Theorem

For every matrix A, there is a sequence of row operations taking to a matrix A in row reduced echelon form

Proof. We prove this by induction on the number of columns of A. When A has one column, either A is the zero vector (in which case it is already in RREF) or it has a nonzero entry a. Swap a to the top row, multiply the top row by $1/a$, and use the $1, 1$ entry as a pivot to eliminate the other entries of A. The result is the vector with a 1 at the top and zeroes elsewhere, which is in RREF.

For the inductive step, suppose that A is $m \times n$ and that the result is true for all matrices with $n - 1$ columns. We then know that there is a series of row operations we can do to A that result in a matrix X whose first $n - 1$ columns form a RREF matrix. Suppose the matrix formed by these $n - 1$ columns has k rows of zeroes at the bottom. If the final column has zeroes in its bottom k entries, the matrix is in RREF. If not, swap a nonzero entry to the top of these k rows, use it as a pivot to eliminate all other nonzero entries in the final column, and multiply by a scalar so that its entry is 1. The result is in RREF. ∎

Theorem

Let A be a matrix. If R and S are RREF matrices that can be obtained by doing row operations to , then $R = S$.

This theorem says that there is only one RREF matrix which can be obtained by doing row operations to A, so we are justified in calling the unique RREF matrix reachable from A *the* row reduced echelon form of A.

Proof. Again, the proof is by induction on the number n of columns of A. There are only two RREF column vectors: the zero vector and a vector with a 1 at the top and all other entries zero. Clearly no sequence of row operations takes one of these to the other, so the base case of the induction holds.

For the inductive step, suppose that R and S are RREF matrices reachable from A. Let A', R', and S' be the matrices formed by the first $n - 1$ columns of A, R, and S respectively. The matrices R' and S' are RREF matrices formed by doing row operations to A', so by induction they are equal. Suppose for a contradiction that $R \neq S$, so that there is some j such that the jth entry r_{jn} in the last column of R which differs from the corresponding entry s_{jn} of S.

The equations $A\mathbf{x} = \mathbf{0}$, $R\mathbf{x} = \mathbf{0}$, and $S\mathbf{x} = \mathbf{0}$ all have exactly the same solutions.

Let $\mathbf{u}$ be any solution of $A\mathbf{x} = \mathbf{0}$. We have $R\mathbf{u} = S\mathbf{u} = \mathbf{0}$, so $(R - S)\mathbf{u} = \mathbf{0}$. Since the first $n - 1$ columns of $R - S$ are all zeroes, we get $(r_{jn} - s_{jn})u_n = 0$, so $u_n = 0$. In other words, every solution of $A\mathbf{x} = \mathbf{0}$ has last entry zero.

The RREF matrix R' has some nonzero rows and then some zero rows. Say there are k zero rows. We can then write R' like this

$$\begin{pmatrix} X \\ \mathbf{0}_{k \times (n-1)} \end{pmatrix}$$

where X has no zero rows. Then R and S have the form

$$R = \begin{pmatrix} X & \mathbf{r} \\ \mathbf{0}_{k \times (n-1)} & \mathbf{t} \end{pmatrix}, S = \begin{pmatrix} X & \mathbf{s} \\ \mathbf{0}_{k \times (n-1)} & \mathbf{u} \end{pmatrix}$$

I claim that $\mathbf{t} \neq \mathbf{0}$. Suppose for a contradiction that $\mathbf{t} = \mathbf{0}$. The first $m - k$ rows of R all have leading entries. For $1 \leqslant i \leqslant m - k$, let the leading entry in row i of R occur in column number c_i. Let $\mathbf{r}$ have entries $r_1, \ldots, r_{m-k}$, where m is the number of rows of A. Notice that column c_i of R has a 1 in position i and zeroes everywhere else, so if we add up r_1 times column c_1, r_2 times column c_2, and so on, up to r_{m-k} times column c_{m-k} we get the vector

$$\begin{pmatrix} r_1 \\ \vdots \\ r_{m-k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which is the last column of R. It follows that the vector with -1 in position n, with r_i in position c_i for $1 \leqslant i \leqslant m - k$, and with zeroes elsewhere is a solution to $R\mathbf{x} = \mathbf{0}$. This contradicts every solution to $A\mathbf{x} = \mathbf{0}$ having last entry zero.

Since R is in RREF, $\mathbf{t}$ must have a 1 at the top and all other entries zero, and $\mathbf{r} = \mathbf{0}$. The same argument applies to S, so $\mathbf{u} = \mathbf{t}$ and $\mathbf{s} = \mathbf{0}$. This shows $R = S$. ∎

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.
a. For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
b. Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Statements (a), (b), and (c) are equivalent because of the definition of $A\boldsymbol{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$. The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[\,A \quad \boldsymbol{b}\,]$ has a pivot position in every row, then the equation $A\boldsymbol{x} = \boldsymbol{b}$ may or may not be consistent.