Matrix Algebra: Dimension and Rank

CE282: Linear Algebra

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Review: Dimension

Definition

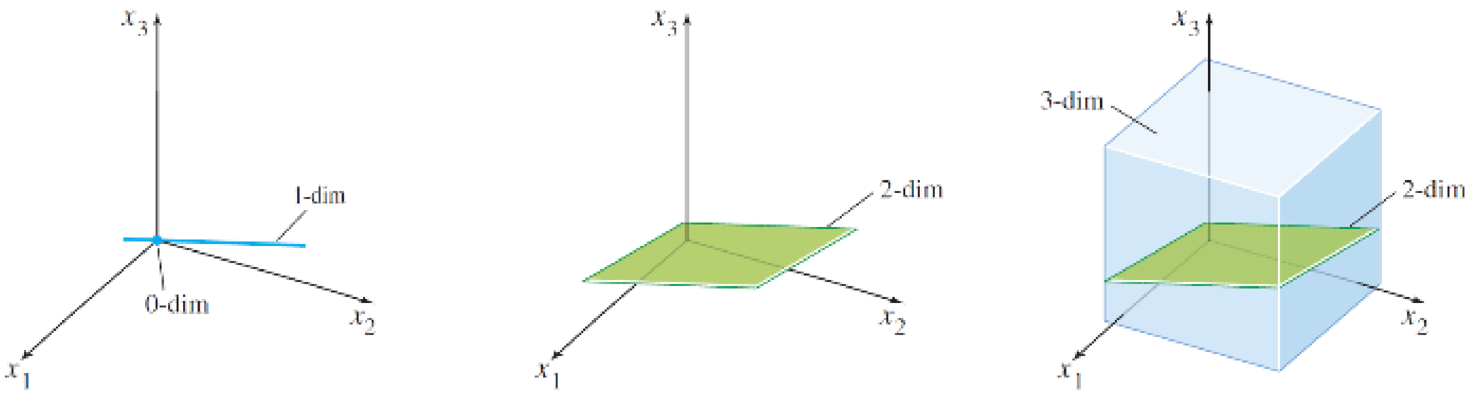
If V has a finite basis, then $\dim(V)$ is the number of elements (vectors) of any basis of V

- Note: $Dim(\{0\}) = 0$

Note

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim(V)$**,** is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ *is defined to be zero.* If V is not spanned by a finite set, then V is said to be **infinite-dimensional.**

Review: Dimension



Extra Resource!

To see some good discussions about "infinite-dimensional" vector spaces and some examples of it, see [here!]

Finite-Dimensional Space

Theorem

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and:

$$\dim(H) \leq \dim(V)$$

When the dimension of a vector space or subspace is known, the search for a basis is simplified by the next theorem.

Theorem (The Basis Theorem)

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that span V is automatically a basis for V.

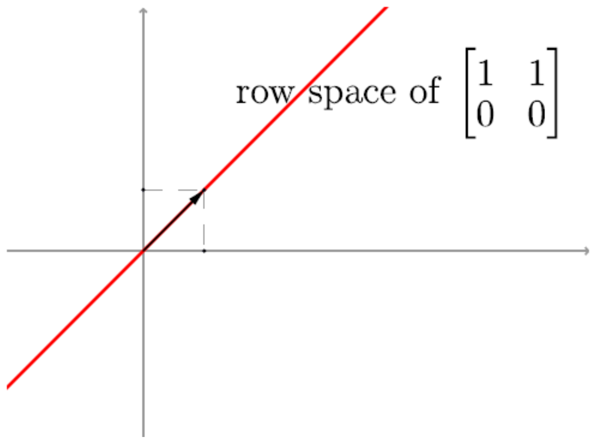
Row Space

The *row space* of a matrix is the collection of all linear combinations of its rows.

Equivalently, the row space is the span of rows.

The elements of a row space are *row* vectors.

If a matrix has m columns, its row space is a subspace of (the row version of) $\mathbb{R}^m$



$$c_1 \begin{bmatrix} 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Elementary row operations **do not** alter the row space.

Thus a matrix and its echelon form have the same row space.

The pivot rows of an echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

The pivot rows of an echelon form span the row space of the original matrix.

The dimension of the row space is given by the number of pivot rows.

This dimension does not exceed the total row count.

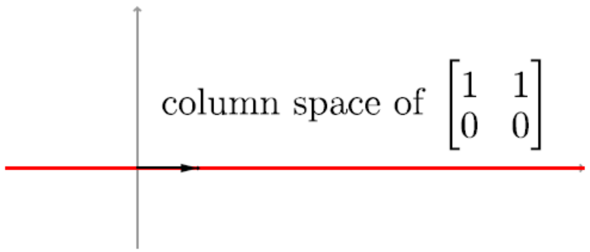
Row Space

Theorem

Let A and B denote m×n matrices:

$$If\ A \to B\ by\ elementary\ row\ operations, then\ row\ A\ =\ row\ B.$$

Column Space



$$c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix}$$

The *column space* of a matrix is the collection of all linear combinations of its columns.

It is the span of columns, the range of the linear transformation carried out by the matrix.

If a matrix has n rows, its column space is a subspace of $\mathbb{R}^n$.

Elementary row operations **affect** the column space.

So, generally, a matrix and its echelon form have different column spaces.

However, since the row operations preserve the linear relations between columns,

the columns of an echelon form and the original columns obey the *same* relations.

Example

The pivot columns of a reduced row-echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & \\ & & & & \end{bmatrix}$$

The pivot columns of a reduced row-echelon form **span** its column space.

So the pivot columns of a matrix are linearly independent and span its column space.

The dimension of the column space is given by the number of pivot columns.

This dimension does not exceed the total column count.

Example

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 1 & 3 & 5 & 8 & 11 \\ 4 & 10 & 16 & 23 & 30 \end{pmatrix} \qquad B_{\text{rref}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= x_3 - x_5, \\ x_2 &= -2x_3 + 2x_5, \\ x_4 &= - 2x_5. \end{aligned} \qquad \mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

The column space of B is 3-dimensional, and that a basis is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 8 \\ 23 \end{bmatrix} \right\}$

Note that we do not use the columns of B_{rref}! We use the columns of B.

Theorem

If two matrices A *and* B are row-equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as for that of B.

Review (Reduced Echelon Form (RREF))

Definition

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
  1. The leading entry in each non-zero row is 1.
  2. Each leading 1 is the only non-zero entry in its columns.
  3. The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
  4. Any row containing only 0's is at the bottom.

$$\underbrace{\begin{array}{cccc} e_1 & e_2 & e_3 & \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon form}}$$

Row and Column Space

Theorem

The pivot columns of a matrix A form a basis for $Col(A)$

- Lemma1: The pivot columns of A are linearly independent
- Lemma 2. The pivot columns of A span the column space of A

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Find:
- ❑ Row Basis
- ❑ Column Basis
- ❑ $\dim(Row(A))$
- ❑ $\dim(Col(A))$
- ❑ $\dim(Null(A))$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = pivot columns

Rank of Matrix

Definition

- The number of linearly independent rows or columns in the matrix
- Dimension of the row (column) space
- Number of nonzero rows of the matrix in row echelon form (Ref)

Note

$Row\ rank\ =\ Column\ rank$ for a matrix in reduced row echelon form.

Note

The dimension of the Column Space of A and $rref(A)$ is the same.

Rank of a Matrix

Theorem (RMRT)

(Rank of a matrix is equal to the rank of its transpose)
Suppose A is an $m \times n$ matrix. Then $rank(A) = rank\,(A^T)$

Definition

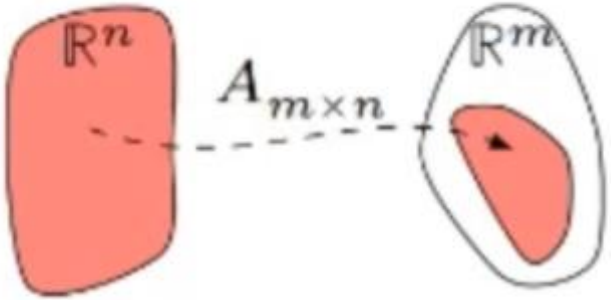
- For $A_{m\times n} = [a_1\ a_2\ \ ...\ a_n]$:

$$Range(A) = Span(a_1, a_2, \dots, a_n) = \{y \mid y = \alpha_1 a_1 + \alpha_2 a_2 + \ \dots + \alpha_n a_n,\ \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$$
$$= \{y \mid y = Ax, x \in \mathbb{R}^n\}$$

- Range is a Vector Space
- Range of A is a subspace of $\mathbb{R}^m$
- Is dim(A) = m ?

$$\dim(Range(A)) = ColRank(A)$$

number of linearly independent columns



Class Activity

Scan The QR Code or type the below link in your browser:

https://forms.gle/kQCEVaq8V4xeHqFL8

Class Activity

Find dim(Range(A)):

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

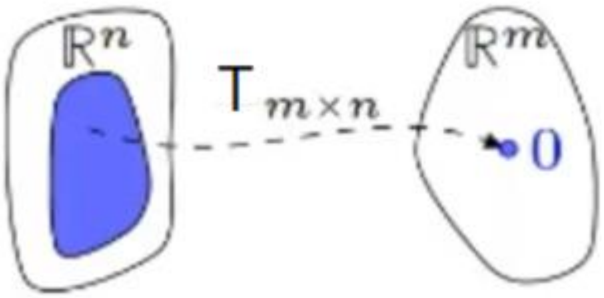
Timer: (2.5 minutes)

Definition

Let $T: V \to W$ be a linear map. Then the null space or kernel of T is the set of all vectors in V that map to zero:

$$N(T) = Null(T) = \{v \in V \mid Tv = 0\}$$

- Null Space is a Vector Space
- Null Space T is a sub-space of V $(\mathbb{R}^n)$
- Is $Dim(Null(T)) = 0$?
- $Nullity(T) := Dim(Null(T))$



Null Space

Question

What is the null space for differentiation mapping?

Note

$$Nullity(A) = number\ of\ free\ variables$$

Null Space

Example 1

If Columns of matrix A are linearly independent:

$$
\begin{aligned}
nullity(A) &= ? \\
colrank(A) &= ?
\end{aligned}
$$

Nullity

Example 2

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$nullity(A) = 2, colRank(A) = 2$$

Conclusion

Theorem

The vectors attached to the free variables in the parametric vector form of the solution set of $Ax = 0$ form a basis of $Null(A)$

Important

Let A be an $n \times n$ matrix. Then the following statements are equivalent to the statement that A is an invertible matrix:

- The columns of A form a basis of $\mathbb{R}^n$.
- $Col(A) = \mathbb{R}^n$
- $Dim(Col(A)) = n$
- $Rank(A) = n$
- $Null\ (A) = \{0\}$
- $Dim\ (Null(A)) = 0$

Conclusion

Note

The dimension of $Null(A)$ is the number of free variables in the equation $Ax = 0$, and the dimension of $Col(A)$ is the number of pivot columns in A.

Example

Back to Example!

Find the dimension of the Null Space and the Column Space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(row reduce the Augmented Matrix $[A\ 0]$ to echelon form)

Rank Theorem

Theorem

- $ColRank(A) = RowRank(A)$
- In general, It's called rank of matrix ($rank(A)$)

Proof?

Rank-Nullity Theorem

Theorem

- Let A denote an m×n matrix of rank r. Then:

  the n−r basic solutions to the system Ax =0 provided by the gaussian algorithm are a basis of null (A), so dim[null (A)] = n−r.

Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$nullity(A) = 2, colRank(A) = 2$ Nullspace(A)={$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$}

Rank-Nullity Theorem

Theorem

- $Nullity(A) + ColRank(A) = n$
- $Dim(Null(A)) + Dim(Range(A)) = n$

$$\{\,number\ of\ pivot\ columns\} + \{number\ of\ non - pivot\ columns\} = \{number\ of\ columns\}$$

Rank Properties

Important

- $ColRank(A_{m \times n}) \leq \min(m,n)$
- $RowRank(A_{m \times n}) \leq \min(m,n)$
- $Dim(Range(A)) = Rank(A)$
- $Nullity(A) + Rank(A) = n$
- $Rank(A) \leq \min(m,n)$
- $Rank(A) \leq \min(m,n)$
- Rank(AB) ≤Rank(A) and Rank(AB) ≤Rank(B)

Rank Properties

Important

- For $A, B \in \mathbb{R}^{m \times n}$:
  1. $rank(A) \leq \min(m, n)$
  2. $rank(A) = rank(A^T)$
  3. $rank(AB) \leq \min(rank(A), rank(B))$
  4. $rank(A + B) \leq rank(A) + rank(B)$
- A has full rank if $rank(A) = \min(m, n)$
- If $rank(A) < m$, rows are not linearly independent (same for columns if $rank(A) < n$)