Linear Transformation

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

- Introduction
- Linear Transformation (Linear Map)
- Rotation-Projection-Reflection
- Onto Linear Transformation
- One-to-One Linear Transformation
- Multiple Transformation
- Inner Product and Linear Transformation

Introduction

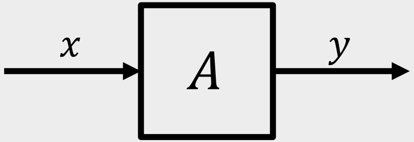
Linear Transformation

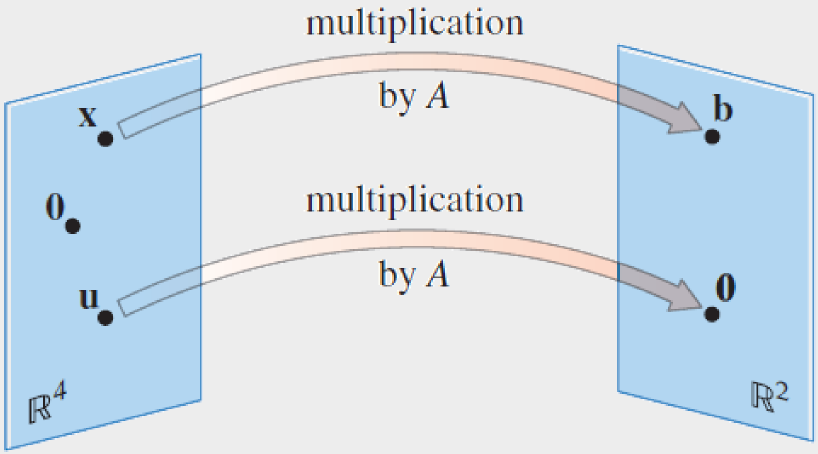
- Matrix is a linear transformation: map one vector to another vector

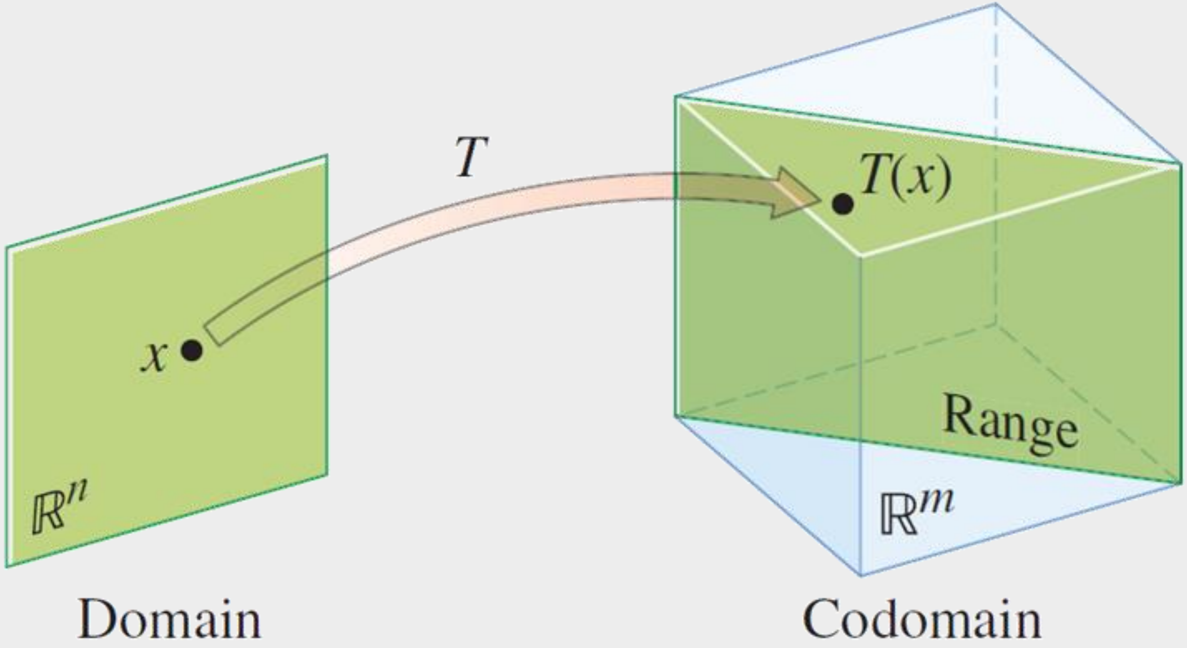
$$A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}: \qquad y_{m\times 1} = A_{m\times n}x_{n\times 1}$$

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

- Input–output







Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, **u** = $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, **b** = $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, **c** = $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Linear Transformation (Linear Map)

Linear mapping

Definition

Let V and W be vector spaces over the field $\mathbb{F}$. A linear transformation (or a linear map) from V into W is a function $\mathbf{T}: V \rightarrow W$ that satisfies following properties for all x, y in V and all scalars a in $\mathbb{F}$:

$$T(x + y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

Notes

- $T(0) = 0$
- Transformation preserves linear combinations

$$T(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 \big(\mathrm{T}(\mathrm{x}_1)\big) + \cdots + \alpha_n \big(\mathrm{T}(\mathrm{x}_\mathrm{n})\big)$$

Example

Which are linear mapping?

- ❑ **zero** map $0 : V \to W$
- ❑ **identity** map $I : V \to V$
- ❑ Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $T_{\mathcal{P}(z)} = \acute{\mathcal{P}}(z)$
- ❑ $\boxed{\text{Let } T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be the map given by } T(x, y) = (x - 2y, 3x + y)}$
- ❑ $T(x) = e^x$ $\qquad$ $T(x_1, \dots, x_n) = (a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{m1}x_1 + \dots + a_{mn}x_n)$
- ❑ $T : \mathbb{F} \to \mathbb{F}$ given by $T(x) = x - 1$

Theorem

Let $(v_1, \ldots, v_n)$ be a ordered basis of finite−dimensional vector space V over the field $\mathbb{F}$ and $(w_1, \ldots, w_n)$ an arbitrary list of any vectors in W.

Then there exists a unique linear map

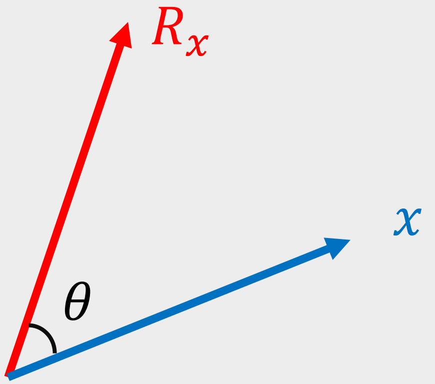
$$T : V \rightarrow W \qquad \text{such that } T(v_i) = w_i.$$

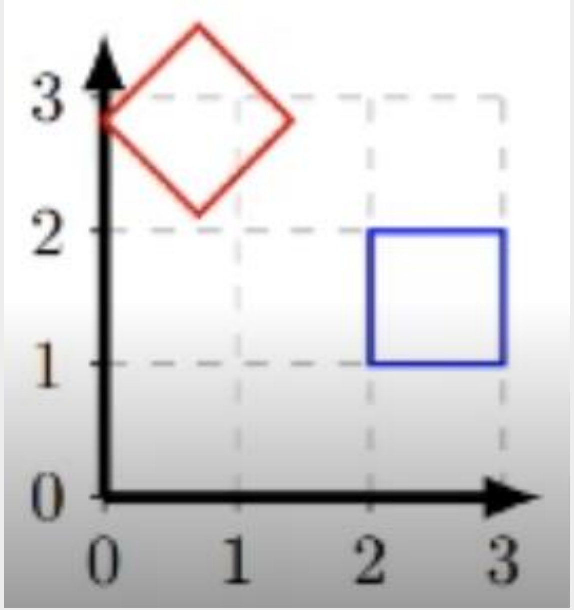
Proof

Rotation–Projection–Reflection

Rotation with $\boldsymbol{\theta}$ degree

- $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

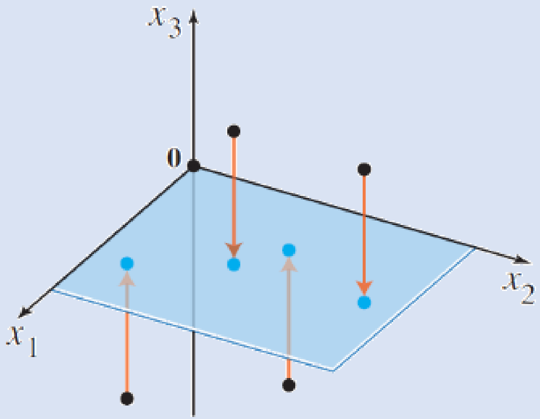




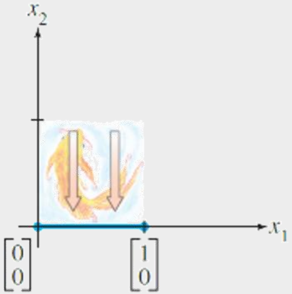
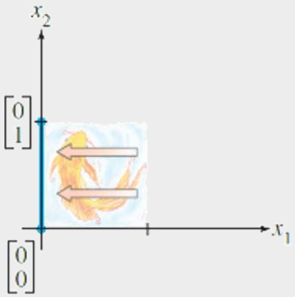
Example

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ onto the x_1x_2–plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$



Projection

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1–axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2–axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Projection

Theorem

Suppose that V is a vector space and $P : V \to V$ is a linear transformation.

a) If $P^2 = P$ then P is called a **projection**.

b) If V is an inner product space and $P^2 = P = P^*$ then P is called an **orthogonal projection**.

We furthermore say that P **projects onto** $\mathrm{range}(P)$.

- Projection of vector v on:
  - Two orthogonal vectors
  - Two non-orthogonal vectors

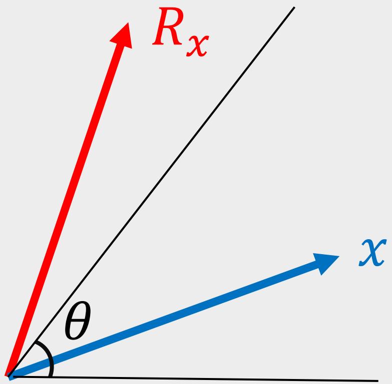
Projection on $\boldsymbol{\theta}$ Line

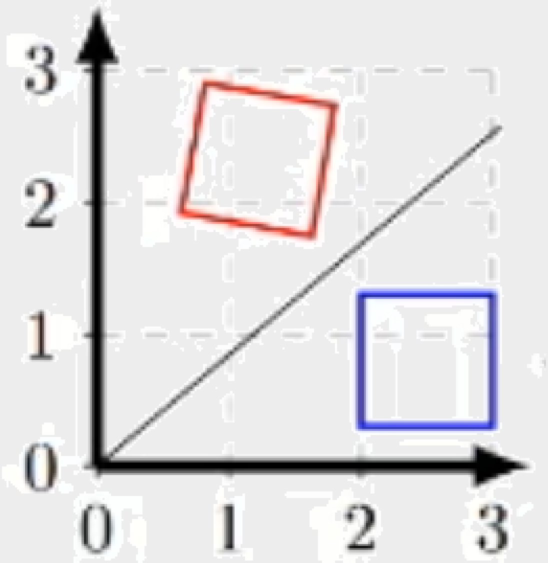
$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$P^2 = P$$

Reflection in the θ Line

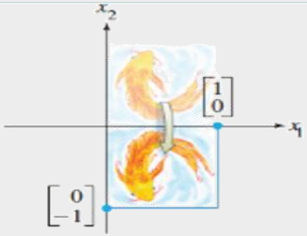
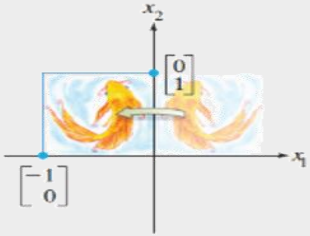
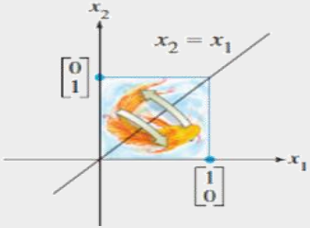
- $R = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$





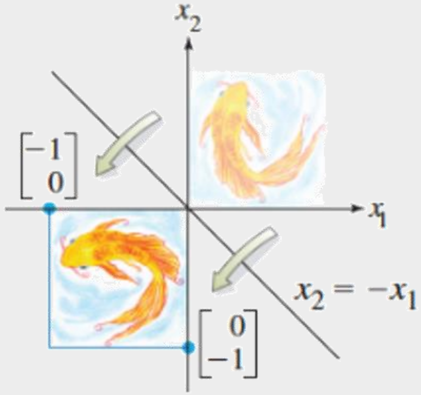
$$R^2 = I$$

Reflection

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1–axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2–axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

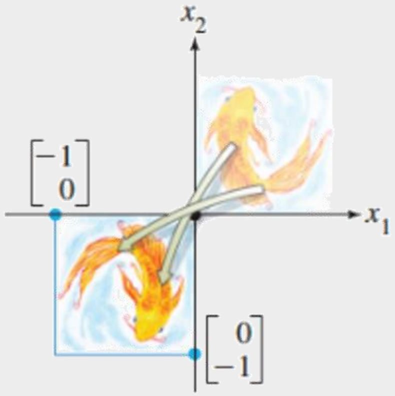
Reflection

Reflection through the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Reflection through the origin

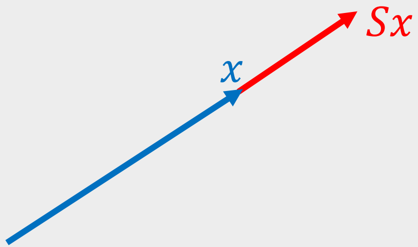


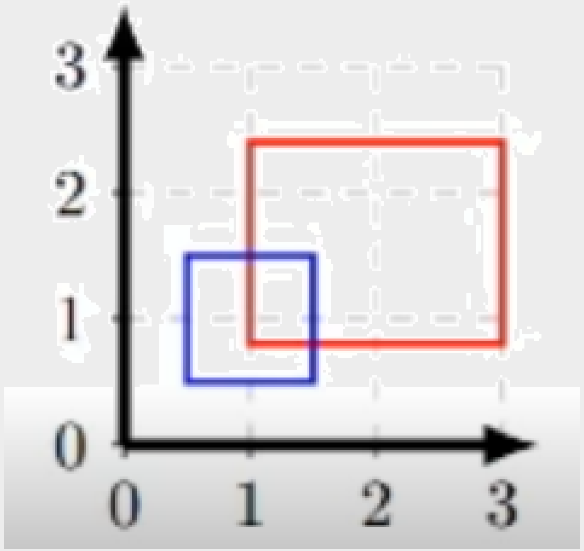
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Applications

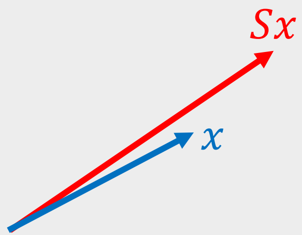
Uniform Scaling

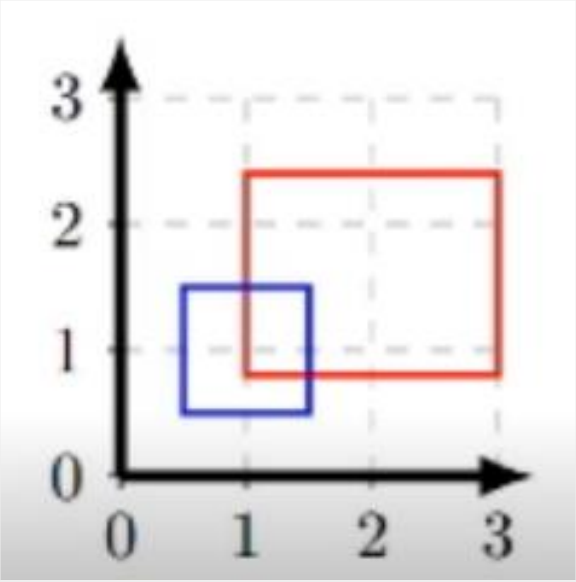
- $S = sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$





- $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$





Shearing

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

A typical shear matrix is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

sheep

sheared sheep

A shear parallel to the x axis results in $\acute{x} = x + \lambda y$ and $\acute{y} = y$.

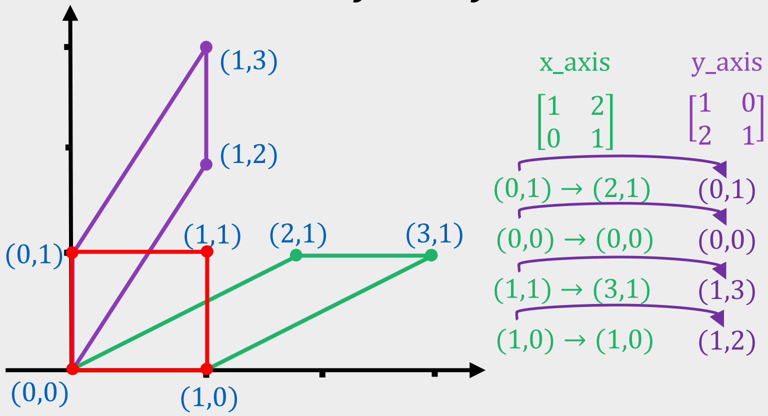
In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Similarly, a shear parallel to the y axis has $\acute{x} = x$ and $\acute{y} = y + \lambda x$.

In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



Difference Matrix

Note

$$D_{(n-1)\times n} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

$$D: \mathbb{R}^n \to \mathbb{R}^{n-1} \quad \Rightarrow \quad D\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Example

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1-0 \\ 3-(-1) \\ 2-3 \\ 5-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$$

Selectors

- an $m \times n$ *selector matrix*: each row is a unit vector (transposed)

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix}$$

multiplying by A selects entries of x:

$$Ax = (x_{k_1}, x_{k_2}, \ldots, x_{k_m})$$

- $A : \mathbb{R}^n \to \mathbb{R}^m \quad \Rightarrow \quad A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_m} \end{bmatrix}$

Selectors

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- Selecting first and last elements of vector:
- Reversing the elements of vector:

Slicing

- Keeping m elements from r to s (m=s−r+1)

$$\begin{bmatrix} 0_{m\times(r-1)} & I_{m\times m} & 0_{m\times(n-s)} \end{bmatrix}$$

Example

- Slicing two first and one last elements:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Down Sampling

- Down sampling with k: selecting one sample in every k samples

Example

$\text{K} = 2?$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix}$$

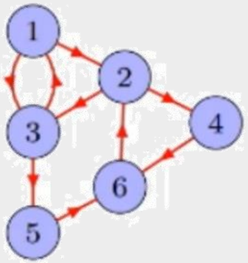
(i) $\sin 2A = 2 \sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

- Rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow R^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

- Adjacency matrix



$$\begin{array}{c} \begin{array}{cccccc} n1 & n2 & n3 & n4 & n5 & n6 \end{array} \\ A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

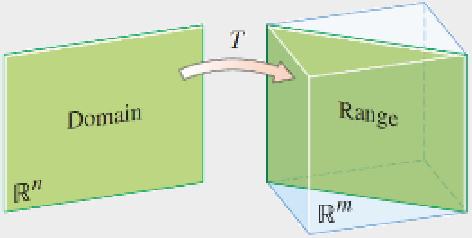
$$A^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

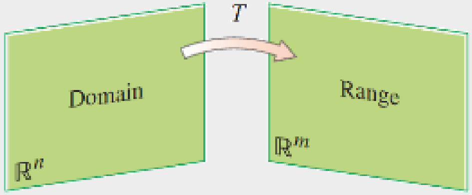
Onto Linear Transformation

Onto Mapping

- A mapping T : $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto (surjective) $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in R^n



T is *not* onto $\mathbb{R}^m$



T is onto $\mathbb{R}^m$

Onto (surjective) Transformation

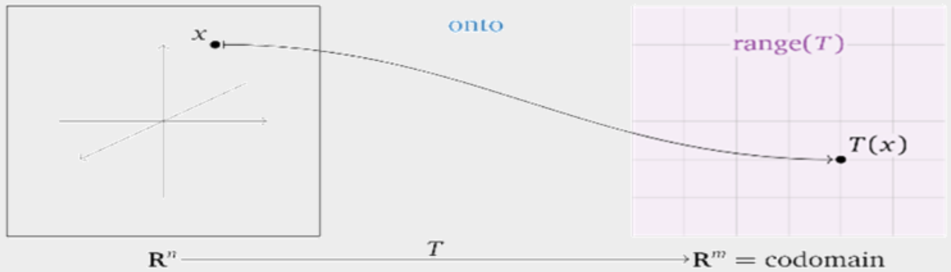
Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if, for every vector b in $\mathbb{R}^m$, the equation T(x) = b has at least one solution x in $\mathbb{R}^n$.

Note

Here are some equivalent ways of saying that T is onto:

- The range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.

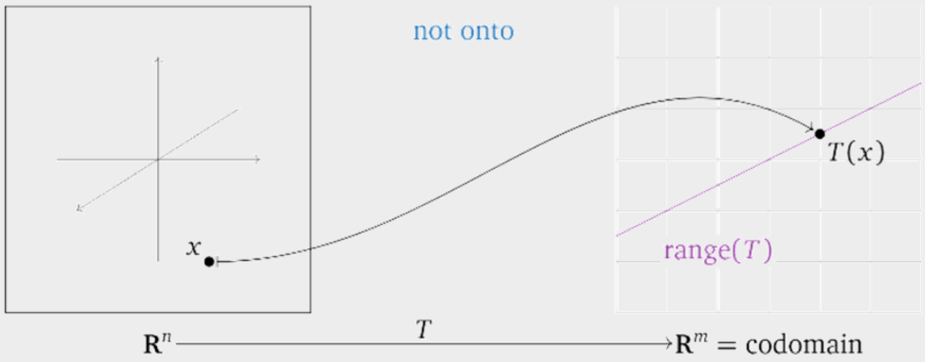


Onto Transformations

Note

Here are some equivalent ways of saying that T is not onto:

- The range of T is smaller to the codomain of T.
- There exists a vector b in $\mathbb{R}^m$ such that the equation T(x) = b does not have a solution
- There is a vector in the codomain that is not the output of any input vector.



Onto Transformation

Theorem

Let A be an $m \times n$ matrix and let T(x) = Ax be the associated matrix transformation. The following statement are equivalent:

- T in onto.
- T(x) = b has at least one solution for every b in $\mathbb{R}^m$.
- Ax = b is consistent for every b in $\mathbb{R}^m$.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot in every row.
- The range of T has dimension m.

Onto Transformations

Important

Tall matrices do not have onto transformations.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is an onto matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every row, it must have at least as many columns as rows: $m \leq n$.

This says that for instance, $\mathbb{R}^2$ is **too small** to admit an onto linear transformation to $\mathbb{R}^3$.

Note that there exist wide matrices that are not onto, for example,

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Does not have a pivot in every row.

Solution

The reduction row echelon form of A is :

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

There is not a pivot in every row, so T is not onto. The range of T is the column space of A which is equal to

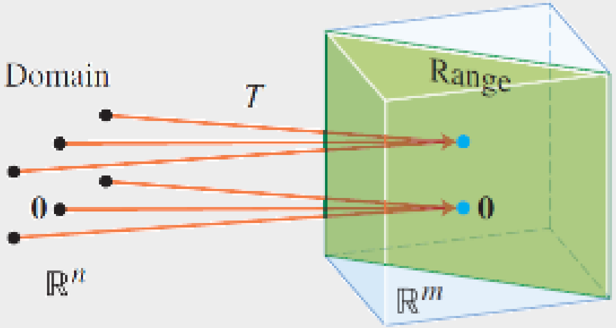
$$span\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right\} = span\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

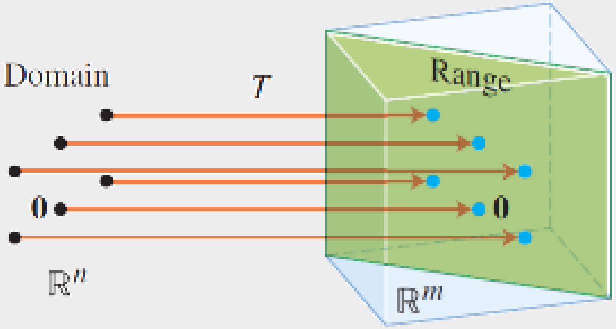
since all three columns of A are collinear. Therefore, any vector not on the line through $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is not in the range of T. for instance, if b = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then T(x) = b has no solution.

One-to-one

One-to-One Mapping

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one (injective)** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$





One-to-One (injective) Linear Transformation

Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then t is one-to-one if and only if the equation T(x) = 0 has only the trivial solution.

Proof

Example

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

One-to-One Linear Transformation

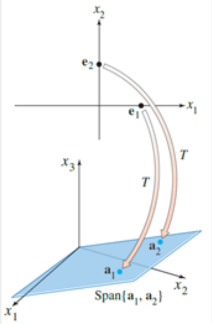
Important

Let $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
b. T is one-to-one if and only if the columns of A are linearly independence.

Example

Let T$(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?



One-to-One Transformations

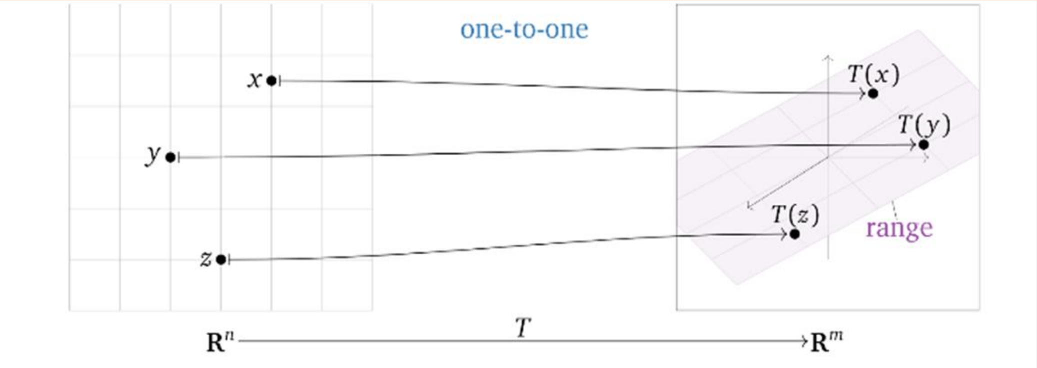
Definition

One-to-one transformations: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if, for every vector b in $\mathbb{R}^m$, the equation T(x) = b has at most one solution x in $\mathbb{R}^n$.

Remark

Here are some equivalent ways of saying that T is one-to-one:

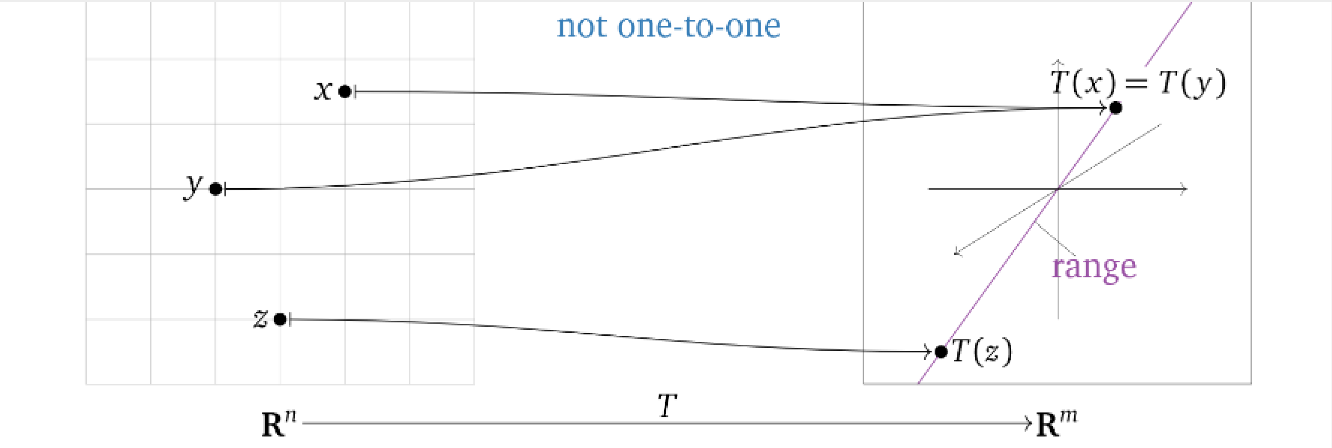
- For every vector b in $\mathbb{R}^m$, the equation T(x) = b has zero or one solution x in $\mathbb{R}^n$.
- Different inputs of T have different outputs.
- If T(u) = T(v) then u = v.



Remark

Here are some equivalent ways of saying that T is not one-to-one:

- There exist some vector b in $\mathbb{R}^m$ such that the equation T(x) = b has more than one solution x in $\mathbb{R}^n$.
- There are two different inputs of T with the same output.
- There exist vectors u, v such that $u \neq v$ but T(u) = T(v).



One-to-one Transformations

Theorem

Let A be an $m \times n$ matrix and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:

1. T is one-to-one.
2. For every b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution.
3. For every b in $\mathbb{R}^m$, the equation $T(x) = b$ has a unique solution or is inconsistent.
4. $Ax = 0$ has only the trivial solution.
5. The columns of A are linearly independent.
6. A has a pivot in every column.
7. The range of T has dimension n.

Important

Wide matrices do not have one-to-one transformations.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is an one-to-one matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every column, it must have at least as many rows as columns:

$n \leq m$.

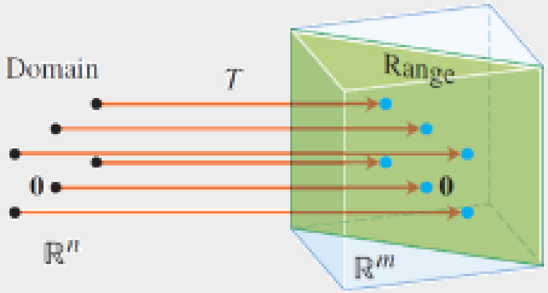
This says that for instance, $\mathbb{R}^3$ is **too big** to admit a one-to-one linear transformation into $\mathbb{R}^2$.

Note that there exist tall matrices that are not one-to-one, for example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Does not have a pivot in every column.

A is an m × n matrix, and T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is the matrix transformation T(x) = Ax.



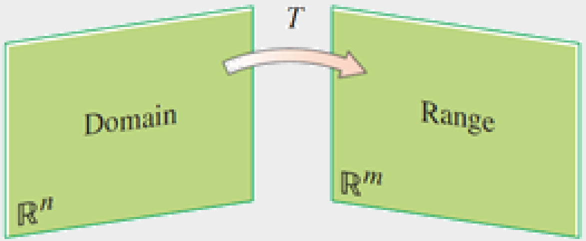
T **is one-to-one**

$T(x) = b$ has *at most one solution* for every b.

The columns of A are linearly independent.

A has a pivot in every column.

The range of T has dimension n.



T **is onto**

$T(x) = b$ has *at least one solution* for every b.

The columns of A span $\mathbf{R}^m$.

A has a pivot in every row.

The range of T has dimension m.

One-to-one and onto

Important

One-to-one is the same as onto for square matrices. We observed that a square has a pivot in every row if and only if it has a pivot in every column. Therefore, a matrix transformation T from $\mathbb{R}^n$ to itself is one-to-one if and only if it is onto : in this case, the two notations are equivalent.

Conversely, by this note, if a matrix transformation T: $\mathbb{R}^m \rightarrow \mathbb{R}^n$ is both one-to-one and onto, then
m = n.

Note that in general, a transformation T is both one-to-one and onto if and only if T(x) = b has exactly one solution for all b in $\mathbb{R}^m$.

Bijective

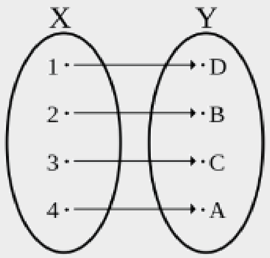
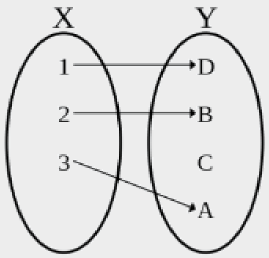
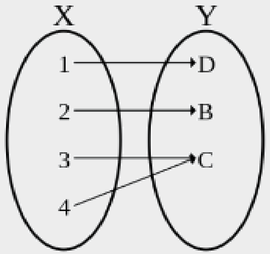
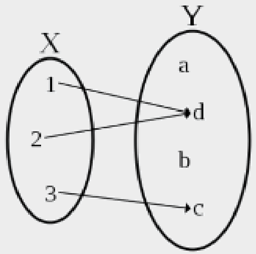
Note

- One-to-one and onto.
- If and only if every possible image is mapped to by exactly one argument.

Conclusion

onto

One-to-one

	surjective	non-surjective
injective	 bijective	 injective-only
non-injective	 surjective-only	 general

Machine learning application

- The central problem in machine learning and deep learning is to meaningfully transform data; in other words, to learn useful representations of the input data at hand – representations that get us to the expected output.

Multiple Transformation

Multiple Transformation

- $$x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Rightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Rightarrow \; z = B(Ax) = BAx$$

Example

- Difference Matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \xrightarrow[4\times 5]{} y = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ x_5 - x_4 \end{bmatrix} \xrightarrow[3\times 4]{} z = \begin{bmatrix} x_3 - x_2 - (x_2 - x_1) \\ x_4 - x_3 - (x_3 - x_2) \\ x_5 - x_4 - (x_4 - x_3) \end{bmatrix} = \begin{bmatrix} x_3 - 2x_2 + x_1 \\ x_4 - 2x_3 + x_2 \\ x_5 - 2x_4 + x_3 \end{bmatrix}$$

$$x \; \to z \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}_{3\times 5}$$

$$x \; \to y \; \to z$$

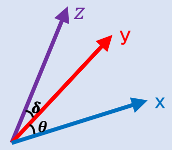
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{4\times 5} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Multiple Transformation

- $x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Rightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Rightarrow z = B(Ax) = BAx$

Example

- Rotation



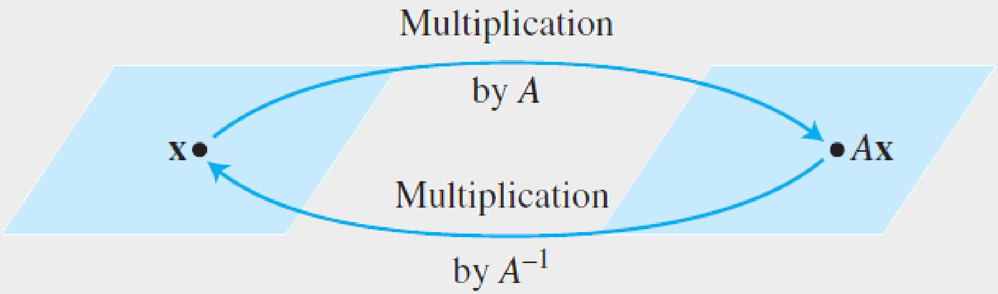
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$x \to z \qquad z = R_{\delta+\theta}x \qquad \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$

$$x \to y \to z \begin{cases} y = R_\theta x \\ z = R_\delta y \end{cases} \Rightarrow z = R_\delta R_\theta x \qquad \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\delta\cos\theta - \sin\delta\sin\theta & -\cos\delta\sin\theta - \sin\delta\cos\theta \\ \sin\delta\cos\theta + \cos\delta\sin\theta & -\sin\delta\sin\theta + \cos\delta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$



Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Invertible Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Inner Product and Linear Transformation

Hermitian Matrix

- Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own **conjugate transpose**

$$A\ Hermitian \quad \Leftrightarrow \quad A = A^H$$

- **conjugate transpose**

$$\boldsymbol{A^H = A^* = (\bar{A})^T}$$

Unitary matrix

- $U^*U = UU^* = UU^{-1} = I$

Note

If U is a square, complex matrix, then the following conditions are equivalent:

1. U is unitary.
2. U^* is unitary.
3. U is invertible with $U^{-1} = U^*$.
4. The columns of U form an orthonormal basis of $\mathbb{C}^n$ with respect to usual inner product. In other words, $U^*U = 1$.
5. The rows of U form an orthonormal basis of $\mathbb{C}^n$ with respect to usual inner product. In other words, $UU^* = 1$.

Normal Matrix

- A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is called **normal** if $A^*A = AA^*$

- A normal and upper triangle matrix is a diagonal matrix.

Inner Product

Note

$$< Ax, y > = < x, A^T y >$$

What about symmetric matrix?

Example

Show that unitary matrix preserves inner product. $< Ux, Uy > = < x, y >$

References

- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra David Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares