Eigenvalue – Eigenvector

Linear Algebra

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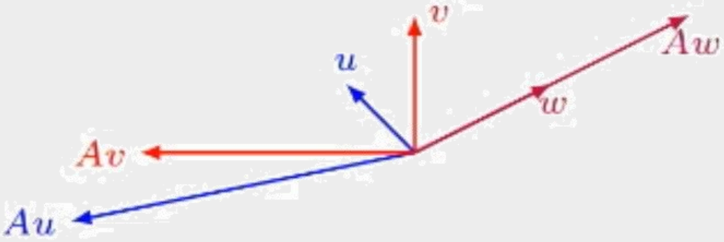
Motivation

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

  $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

  $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

  $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



Definition

An **eigenvector** of an $n \times n$ matrix A is nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

- An eigenvector must be nonzero, by definition, but an eigenvalue may be

Example

$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$

Show that 7 is an eigenvalue of matrix A, and find the corresponding eigenvectors.

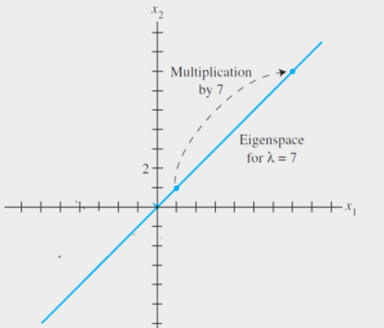
$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Note

λ is an eigenvalue of an $n \times n$ matrix if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.



Characteristic Equation

Note

- $Av \ = \ \lambda v \Rightarrow Av - \lambda v I = 0 \Rightarrow (A - \lambda \mathrm{I})v = 0 \ \ v \neq 0$
- Characteristic equation $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$ = **0**
- Characteristic polynomial $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$ $\boldsymbol{\Delta_A(\lambda), \Delta(\lambda)}$
  - Matrix $n \times n$ has ⋯. eigenvalue

Example

The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplications.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$$

Matrix Spectrum

Definition

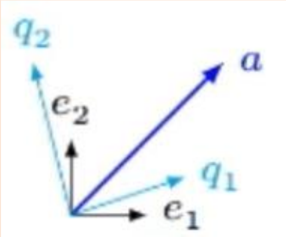
Set of all eigenvalues of matrix is $\sigma(A)$.

Theorem

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal.

- Proof?
- $0 \in \sigma(A) \Leftrightarrow |A| = 0$
- A is invertible if and only if $\cdots$
- 0 is an eigenvalue of A if and only if A is not invertible.

Note



- n-vector a based on basis $\{e_1, \dots, e_n\}$

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

- n-vector a based on new basis $\{q, \dots, q_n\}$

$$a = \overline{a_1} q_1 + \overline{a_2} q_2 + \cdots + \overline{a_n} q_n = \underbrace{[q_1 \quad \dots \quad q_n]}_{Q} \begin{bmatrix} \overline{a_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{a_n} \end{bmatrix}$$

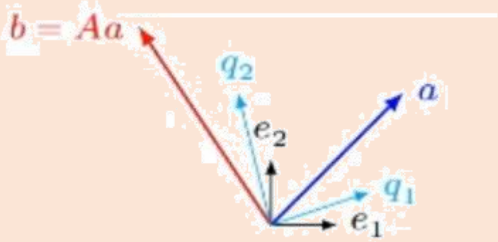
- Matrix Q is invertible.
- Any invertible matrix is a basic matrix.

Note

- A square matrix for a linear transform

$$A: n \times n \qquad A: R^n \to R^n \;\Rightarrow\; Aa = b \qquad a.b \in R^n$$

$$\left.\begin{array}{l} a = Q\bar{a} \\ \mathrm{b} = \mathrm{Q}\bar{b} \end{array}\right\} \Rightarrow \mathrm{AQ}\bar{a} = Q\bar{b} \Rightarrow \underbrace{Q^{-1}AQ}_{\bar{A}}\,\bar{a} = \bar{b} \;\Rightarrow\; \bar{A}\bar{a} = \mathrm{b}$$



- Linear transform in new basis $\bar{A} = Q^{-1}AQ$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Similar Matrices

Note

- Two n-by-n matrices A and B are called similar if there exists an **invertible n-by-n matrix $\boldsymbol{Q}$** such that

$$A = Q^{-1}BQ$$

- A and B are similar if $QA = BQ$
- $A = Q^{-1}BQ \ \rightarrow B = QAQ^{-1}$
- Same determinant
- Inverse of A and B are similar (if exists)

Similarity Transformation

- We can use similarity transformation for changing the standard matrix of linear transformation

Example

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \bar{A} = Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Think!

- Why trace is a similarity invariant?

- Why rank is a similarity invariant?

- Similar matrices have equal characteristic equations.
  - vice versa?

Example

- $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvectors Linear Independence

The Invertible Matrix Theorem

If $v_1, \cdots, v_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ of an $n \times n$ matrix A , then the set $\{\lambda_1, \cdots, \lambda_n\}$ is linearly independent.

- One way to prove the statement "If P then Q" is to show that P and the negation of Q leads to a contradiction
- Distinct eigenvalues -> eigenvectors are LI
- Duplicate eigenvalues -> ???
  - Example

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- The number 0 is not an eigenvalue of A .
- The determinant of A is not zero.

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

Example

Find eigenvalues and eigenvectors?

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda \mathrm{I}| = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^3 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{matrix} \lambda_1 = 1 \\ (A - \lambda_1 I)q_1 = 0 \end{matrix}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \lambda_2 = 2 \\ (A - \lambda_2 I)q_2 = 0 \end{matrix}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Definition

- With similarity transformation Q, matrix A changed to a diagonal matrix $diag(\lambda_{1,}\lambda_2)$
- Matrix A has n linear independent eigenvectors

- $Aq_1 = \lambda_1 q_1 = [q_1\ q_2\ \cdots\ \ q_n]\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix} \cdots Aq_n = \lambda_n q_n = [q_1\ q_2\ \cdots\ \ q_n]\begin{bmatrix}0\\0\\\vdots\\\lambda_n\end{bmatrix}$

- $[Aq_1\ \ Aq_2\ \ \cdots\ \ Aq_n] = \underbrace{[q_1\ \ q_2\ \cdots\ \ q_n]}_{Q}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\0 & \lambda_2 & \dots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \dots & \lambda_n\end{bmatrix}}_{\Lambda}$

- $A\,[q_1\ q_2\ \cdots\ \ q_n] = Q\Lambda \Longrightarrow AQ = Q\Lambda$
- $\Lambda = Q^{-1}AQ^T$
- $A = Q\Lambda Q^{-1}$

Diagonalizable

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if A
$= PDP^{-1}$
for some invertible matrix P and some diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

- The columns of P is called an eigenvector basis of $\mathbb{R}^n$.
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Symmetric Matrix

Theorem

If A is symmetric, then any two eigenvectors from different eigenspace are orthogonal.

$$\left.\begin{array}{c} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow {v_1}^T v_2 = 0$$

Symmetric Matrix

Important

- Eigenvalues of a real symmetric matrix are real.
- If A is diagonalizable by an orthogonal matrix, then A is a symmetric matrix.
- A symmetric matrix is always diagonalizable.
- A similar transform that diagonalized the symmetric matrix is orthogonal.

- $Q^TQ = I$ $\quad\quad A = Q\Lambda Q^T,\quad\quad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \lambda_i \in \mathbb{R}$

Orthogonally Diagonalizable

Theorem

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

($\Rightarrow$):

$$A = A^T \Rightarrow A = Q\Lambda Q^T, \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}$$

($\Leftarrow$):

$$A = A^T \Leftarrow A = Q\Lambda Q^T, \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}$$

$$A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$$

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

Gram Matrix

- Eigenvalues are real.
- Eigenvalues are nonnegative.