Vector Space

Linear Algebra

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Overview

- Review Complex Numbers
- Operations
- Group-Ring-Field
- Vector Space
- Linear Combination
- Span – Linear Hull

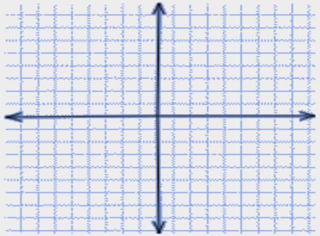
Complex Number Review

Tuple and Vector Space

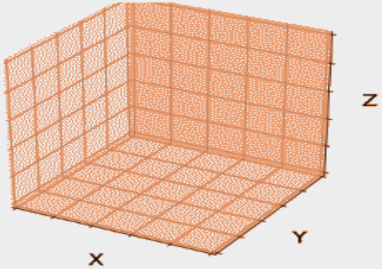
Definition

- A tuple is an ordered list of numbers.
- For example: $\begin{bmatrix} 1 \\ 2 \\ 32 \\ 10 \end{bmatrix}$ is a 4−tuple (a tuple with 4 elements).

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \dots \right\}$$



$$\mathbb{R}^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \dots \right\}$$



Review: Complex Numbers

Numbers:

- Real: Nearly any number you can think of is a Real Number!

1	12.38	−0.8625	3/4	√2	1998

- Imaginary: When squared give a negative result.

The "unit" imaginary number (like 1 for Real Numbers) is "i", which is the square root of −1.

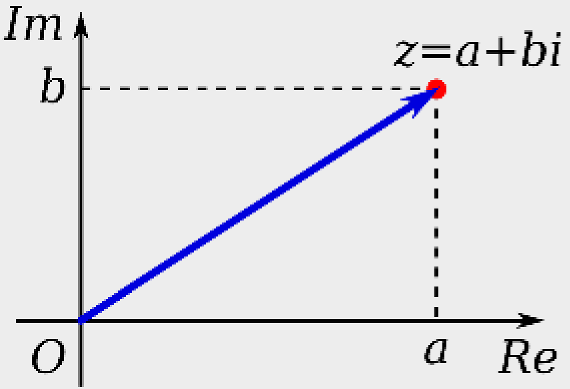
Examples of Imaginary Numbers:

3i	1.04i	−2.8i	3i/4	(√2)i	1998i

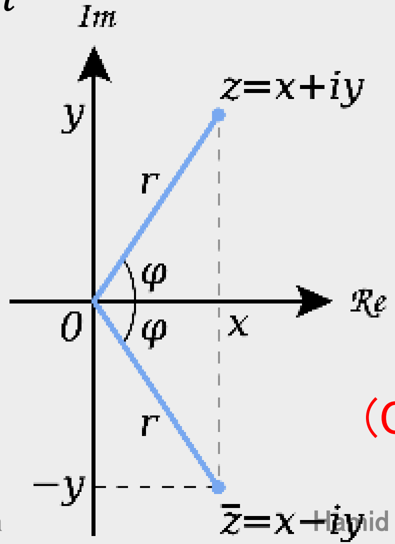
And we keep that little "i" there to remind us we need to multiply by √−1

Review: Complex Numbers

- $\mathbb{C}$ is a plane, where number $(a + bi)$ has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$
- Imaginary number: $bi\ , b \in R$



- Conjugate of $x + yi$ is noted by $\overline{x + yi}$:
  - $x - yi$



(Complex conjugate)

Review: Complex Numbers

- Arithmetic with complex numbers $(a + bi)$:
  - $(a + bi) + (c + di)$
  - $(a + bi)(c + di)$
  - $\frac{a+bi}{c+di}$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

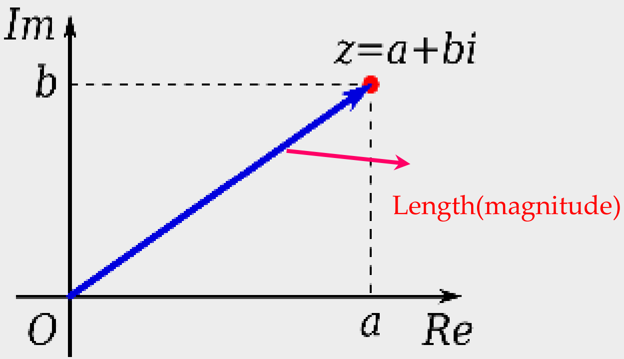
Review: Complex Numbers

- Length (magnitude): $||a+bi||^2 = \overline{(a+bi)}(a+bi) = \mathrm{a}^2 + \mathrm{b}^2$

- Inner Product:

  - Real: $< x, y > = x_1y_1 + x_2y_2 + \ \ldots + x_ny_n$

  - Complex: $< x, y > = \overline{x_1}y_1 + \overline{x_2}y_2 + \ \ldots + \overline{x_n}y_n$



Extra resource:

If you want to learn more about complex numbers, this video is recommended!

Vector Operation

Vector Operations

- Vector–Vector Addition
- Vector–Vector Subtraction
- Scalar–Vector Product
- Vector–Vector Products:
  - $\boldsymbol{x}.\boldsymbol{y}$ is called the inner product or dot product or scalar product of the vectors: $x^T y$ $(y^T x)$
    - $< a, b >$ $\quad < a|b >$ $\quad (a, b)$ $\quad a.b$

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i.$$

  - Transpose of dot product:
    - $(a.b)^T = (a^T b)^T = (b^T a) = (b.a) = b^T a$
  - Length of vector

Dot Product Properties

- **Commutativity**
  - The order of the two vector arguments in the inner product does not matter.

$$a^T b = b^T a$$

- **Distributivity with vector addition**
  - The inner product can be distributed across vector addition.

$$(a+b)^T c = a^T c + b^T c$$
$$a^T (b+c) = a^T b + a^T c$$

Dot Product Properties

- Bilinear (linear in both a and b)

$$a^T(\lambda b + \beta c) = \lambda a^T b + \beta a^T c$$

- Positive Definite:

$$(a.a) = a^T a \geq 0$$

  - 0 only if a itself is a zero vector$a = \mathbf{0}$

Dot Product Properties

- Associative
  - Note: the associative law is that parentheses can be moved around, e.g., $(x+y)+z = x+(y+z)$ and $x(yz) = (xy)z$

1) Associative property of the vector dot product with a scalar (scalar–vector multiplication embedded inside the dot product)

scalar

$$\gamma(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = (\gamma\mathbf{u}^{\mathrm{T}})\mathbf{v} = \mathbf{u}^{\mathrm{T}}(\gamma\mathbf{v}) = (\mathbf{u}^{\mathrm{T}}\mathbf{v})\gamma$$

$$= (\gamma\boldsymbol{u})^{T}\boldsymbol{v} = \gamma\boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{v}$$

Dot Product Properties

- Associative

  2) Does vector dot product obey the associative property?

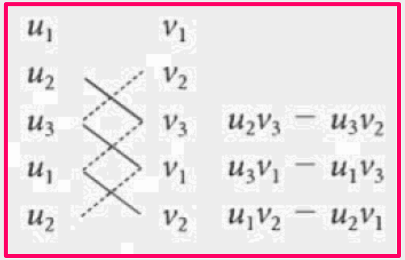
$$\underbrace{\mathbf{u}^{\mathrm{T}}(\mathbf{v}^{\mathrm{T}}\mathbf{w})}_{\substack{\text{vector–scalar product} \\ \text{row vector}}} = \underbrace{(\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{w}}_{\substack{\text{scalar–vector product} \\ \text{column vector}}}$$

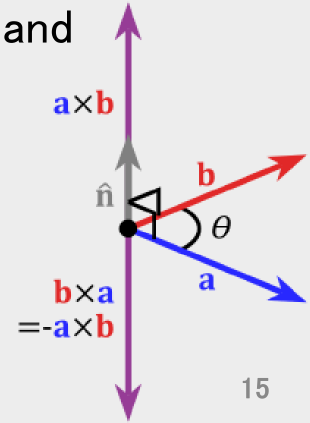
Cross product

- The cross product is defined only for two 3-element vectors, and the result is another 3-element vector. It is commonly indicated using a multiplication symbol (×).

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta_{ab}) \qquad \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

- It used often in geometry, for example to create a vector c that is orthogonal to the plane spanned by vectors a and b. It is also used in vector and multivariate calculus to compute surface integrals.





❑ Vector–Vector Products:

- Given two vectors $x \in R^m, y \in R^n$:
  - $x \otimes y = xy^T \in R^{m\times n}$ is called the outer product of the vectors: $(xy^T)_{ij} = x_i y_j$

$$xy^T \in \mathbb{R}^{m\times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

Example

❑ Represent $A \in R^{m\times n}$ with outer product of two vectors:

$$A = \begin{bmatrix} | & | & & | \\ x & x & \cdots & x \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m & \cdots & x_m \end{bmatrix}$$

Outer Product Properties

- Properties:
  - $(u \otimes v)^T = (v \otimes \mathrm{u})$
  - $(v + w) \otimes u = v \otimes u + w \otimes u$
  - $u \otimes (v + w) = u \otimes v + u \otimes w$
  - $c(v \otimes u) = (cv) \otimes u = v \otimes (cu)$
  - $(u. v) = trace(u \otimes v) \ \ (u, v\, \epsilon R^n)$
  - $(u \otimes v)w = (v. w)u$

Vector Operations

- Vector-Vector Products:
  - Hadamard
  - Element-wise product

$$c = a \odot b = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ \cdot \\ \cdot \\ a_n b_n \end{bmatrix}$$

- Hadamard product is used in image compression techniques such as JPEG. It is also known as Schur product

- Hadamard Product is used in LSTM (Long Short-Term Memory) cells of Recurrent Neural Networks (RNNs).

Hadamard Product Properties

- Properties:

    - $a \odot b = b \odot a$
    - $a \odot (b \odot c) = (a \odot b) \odot c$
    - $a \odot (b + c) = a \odot b + a \odot c$
    - $(\theta a) \odot b = a \odot (\theta b) = \theta (a \odot b)$
    - $a \odot \mathbf{0} = \mathbf{0} \odot a = \mathbf{0}$

Binary Operation

Definition

- Any function from A x A → A is a binary operation.

- **Closure Law:**
  - A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.

$$if\ a \in A, b \in A \ \rightarrow a * b \in A$$

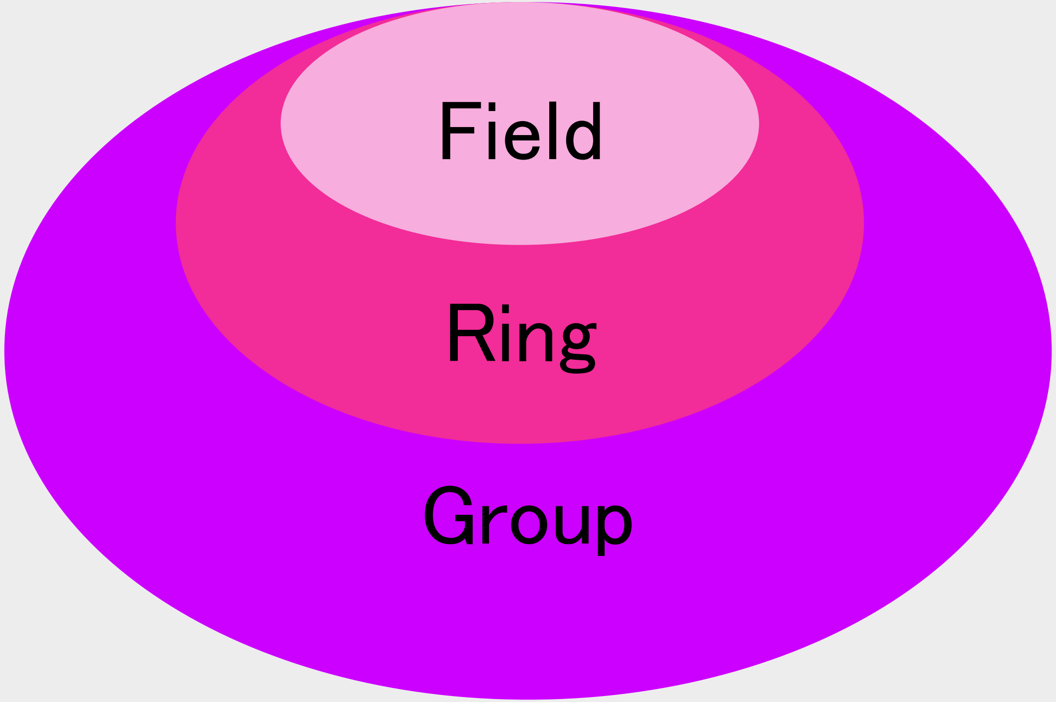
Binary Operations

Example

- Is “+” a binary operator on natural numbers?
- Is “x” a binary operator on natural numbers?
- Is “−” a binary operator on natural numbers?
- Is “/” a binary operator on natural numbers?

Group–Ring–Field

Group – Ring – Field



Definition

- A group G is a pair $(S,\circ)$, where S is a set and $\circ$ is a binary operation on S such that:

- $\circ$ is **associative**
- **(Identity)** There exists an element e $\in S$ such that:

$$e \circ a = a \circ e = a \quad \forall a \in S$$

- **(Inverses)** For every $a \in S$ there is b $\in S$ such that:

$$a \circ b = b \circ a = e$$

If $\circ$ is commutative, then G is called a commutative group!

Ring

Definition

- A ring R is a set together with two binary operations + and ∗, satisfying the following properties:

1. (R,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. ∗ is associative
3. The distributive laws hold in R: (Multiplication is distributive over addition)

$$(a+b)*c=(a*c)+(b*c)$$

$$a*(b+c)=(a*b)+(a*c)$$

Fields

Definition

- A field F is a set together with two binary operations + and ∗, satisfying the following properties:

1. (F,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. (F−{0},∗) is a commutative group
3. The distributive law holds in F:

$$(a+b)*c=(a*c)+(b*c)$$
$$a\ *(b+c)=(a*b)+(a*c)$$

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

Example

(R; +, .) and (Q; +, .) serve as examples of fields.
(Z; +, .) is an example of a ring which is not a field!

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

Fields $\forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in F$

Properties	Binary Operations	
	Addition (+)	**Multiplication (.)**
Closure (بسته بودن)	$\exists a + b \in F$	$\exists a.b \in F$
Associative (شرکت‌پذیری)	$a + (b + c) = (a + b) + c$	$a.(b.c) = (a.b).c$
Commutative (جابه‌جایی‌پذیری)	$a + b = b + a$	$a.b = b.a$
Existence of identity $\mathrm{e} \in F$	$a + e = a = e + a$	$a.e = a = e.a$
Existence of inverse: For each a in F there <u>must exist</u> b_1 in F	$a + b = e = b + a$	$a.b = e = b.a$ <u>*For any nonzero a*</u>
Multiplication is distributive over addition $a.(b + c) = a.b + a.c$ $(a + b).c = a.c + b.c$		

Example

Set $B = \{0,1\}$ under following operations is a field?

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Fields

Example

Which are fields? (two binary operations + , *)

$\mathbb{R}$

$\mathbb{C}$

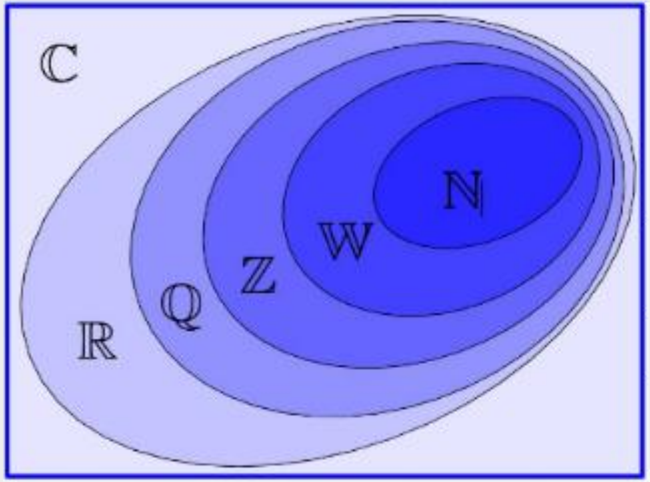
$\mathbb{Q}$

$\mathbb{Z}$

W

$\mathbb{N}$

$\mathbb{R}^{2\times 2}$



ℂ : Complex
ℝ : Real
ℚ : Rational
ℤ : Integer
W : Whole
ℕ : Natural

Vector Space

Vector Space

- Building blocks of linear algebra.

- A non-empty set V with field F (most of time R or C) forms a vector space with two operations:

  1. + : Binary operation on V which is $V \times V \rightarrow V$
  2. . : $F \times V \rightarrow V$

Note

In our course, by **default**, field is **R** (real numbers).

Vector Space

Definition

A vector space over a field F is the set V equipped with two operations: $(V, F, +, .)$

i. Vector addition: denoted by "+" adds two elements $x, y \in V$ to produce another element $x + y \in V$

ii. Scalar multiplication: denoted by "." multiplies a vector $x \in V$ with a scalar $\alpha \in F$ to produce another vector $\alpha . x \in V$. We usually omit the "." and simply write this vector as αx

Vector Space Properties

- Addition of vector space $(x + y)$
  - **Commutative** $x + y = y + x \ \forall x, y \in V$
  - **Associative** $(x + y) + z = x + (y + z) \ \forall x, y, z \in V$
  - **Additive identity** $\exists \mathbf{0} \in V$ such that $x + \mathbf{0} = x, \forall x \in V$
  - **Additive inverse** $\exists (-x) \in V$ such that $x + (-x) = 0, \forall x \in V$

Vector Space Properties

- **Action of the scalars field on the vector space** (αx)

  - **Associative** $\alpha(\beta x) = (\alpha\beta)x \qquad \forall \alpha, \beta \in F; \forall x \in V$

  - **Distributive over** ……
    scalar addition: $(\alpha + \beta)x = \alpha x + \beta x \qquad \forall \alpha, \beta \in F; \forall x \in V$
    vector addition: $\alpha(x + y) = \alpha x + \alpha y \qquad \forall \alpha \in F; \forall x, y \in V$

  - **Scalar identity** $1x = x \qquad \forall x \in V$

Vector Space

Example

Let V be the set of all real numbers with the operations $u \oplus v = u - v$, $\oplus$ is an ordinary subtraction) and $c \boxdot u = cu$ ($\boxdot$ is an ordinary multiplication).
Is V a vector space? If it's not, which properties fail to hold?

Vector Space

Example: Fields are R in this example:

- The n-tuple space,
- The space of m x n matrices
- The space of functions:

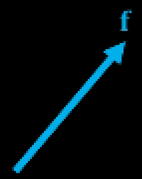
  $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$

  $f(t) = 1 + \sin(2t)$ and $g(t) = 2 + 0.5t$

- The space of polynomial functions over a field f(x):

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

Vector Space



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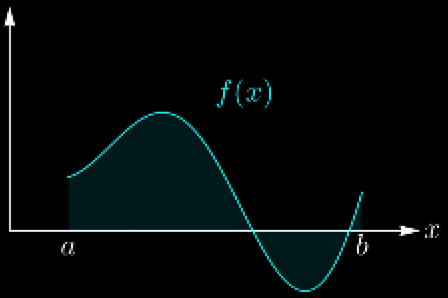

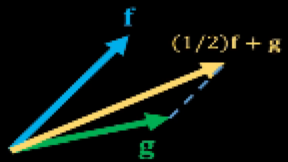

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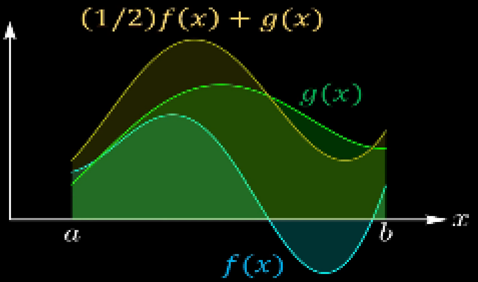
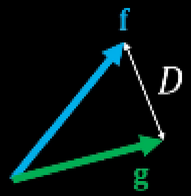


$$\mathbf{v} = \alpha \mathbf{f} + \beta \mathbf{g}$$

$$v(x) = \alpha f(x) + \beta g(x)$$

“Vector”



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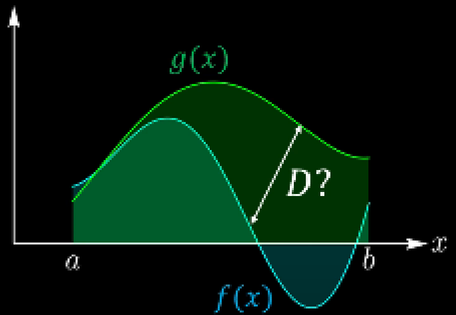

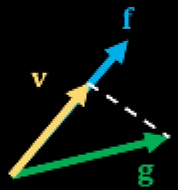

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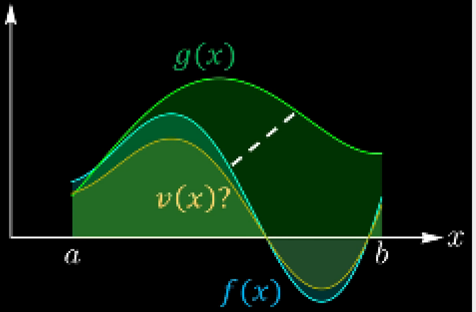


$$\mathbf{v} = \langle \mathbf{g}, \hat{\mathbf{f}} \rangle \, \hat{\mathbf{f}}$$

$$\hat{\mathbf{f}} = \frac{\mathbf{f}}{|\mathbf{f}|}$$

$$v(x) = \langle g(x), \hat{f}(x) \rangle \, \hat{f}(x)$$

(we will see how)

$$D = \|\mathbf{f} - \mathbf{g}\| = \sqrt{\langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle}$$

$$D = \|f(x) - g(x)\|$$

Vector Space of functions

- Function addition and scalar multiplication

$$(f + g)(x) = f(x) + g(x) \quad and \; (af)(x) = af(x)$$

Non-empty set X and any field F $\longrightarrow$ $F^{x} = \{f: X \to F\}$

Example

- Set of all polynomials with real coefficients
- Set of all real-valued continuous function on [0,1]
- Set of all real-valued function that are differentiable on [0,1]

$P_n\ (\mathbb{R})$: Polynomials with max degree (n)

- Vector addition
- Scalar multiplication
- And other 8 properties!

Vector Space

Example

Which are vector spaces?

- ❑ Set $\mathbb{R}^n$ over $\mathbb{R}$
- ❑ Set $\mathbb{C}$ over $\mathbb{R}$
- ❑ Set $\mathbb{R}$ over $\mathbb{C}$
- ❑ Set $\mathbb{Z}$ over $\mathbb{R}$
- ❑ Set of all polynomials with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- ❑ Set of all polynomials of degree at most n with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- ❑ Matrix: $M_{m,n}(\mathbb{R})$ over $\mathbb{R}$
- ❑ Function: $f(x): x \longrightarrow \mathbb{R}$ over $\mathbb{R}$

Conclusion

The operations on field F are:

- $+ : F \times F \rightarrow F$
- $\times : F \times F \rightarrow F$

The operations on a vector space V over a field F are:

- $+ : V \times V \rightarrow V$
- $. : F \times V \rightarrow V$

Linear Combination

Linear Combinations

- The linear combinations of m vectors $a_1, \dots a_m$, each with size n is:

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

where $\beta_1, \dots, \beta_m$ are scalars and called the coefficients of the linear combination

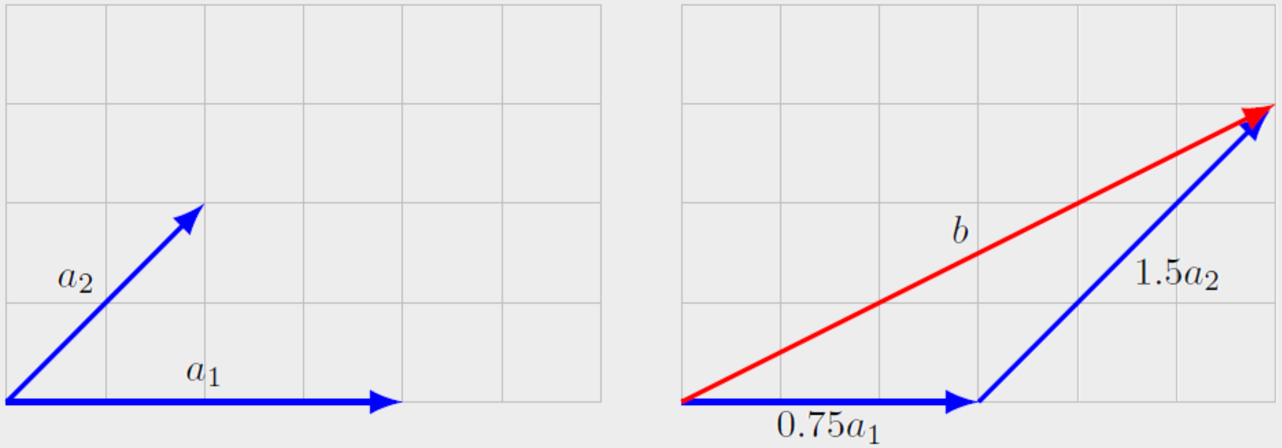
- **Coordinates**: We can write any n−vector b as a linear combination of the standard unit vectors, as:

$$b = b_1 e_1 + \cdots + b_n e_n$$

  - Example: What are the coefficients and combination for this vector?

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

Linear Combinations



Left. Two 2-vectors a_1 and a_2. *Right.* The linear combination $b = 0.75a_1 + 1.5a_2$

Special Linear Combinations

- Sum of vectors
- Average of vectors

Span – Linear Hull

Span or linear hull

Definition

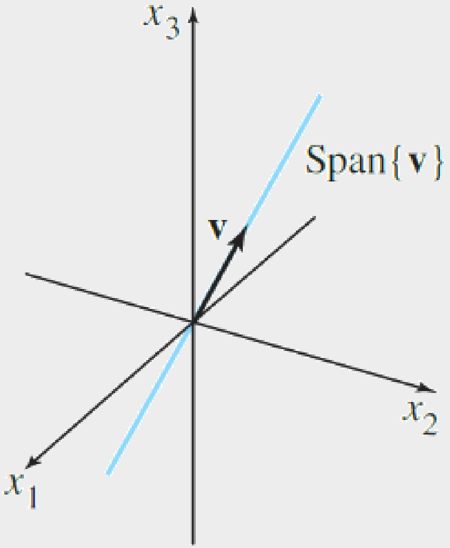
If $v_1, v_2, v_3, \dots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2, \dots, v_p$ is denoted by Span $\{v_1, v_2, \dots, v_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned (or generated)** by $v_1, v_2, \dots, v_p$.

That is, Span$\{v_1, v_2, \dots, v_p\}$ is the collection of all vectors that can be written in the form:

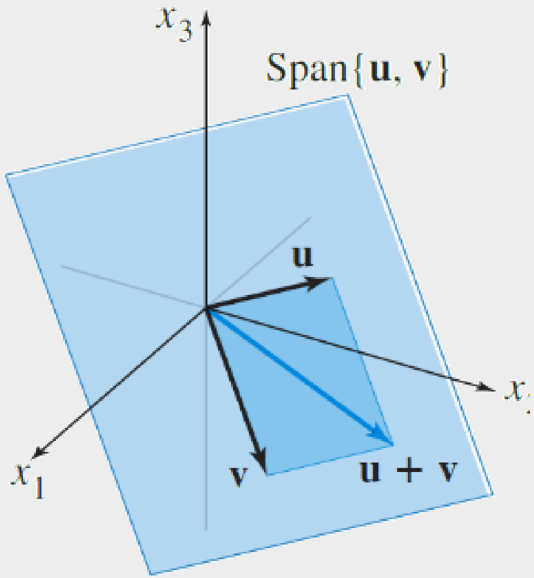
$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

with $c_1, c_2, \dots, c_p$ being scalars.

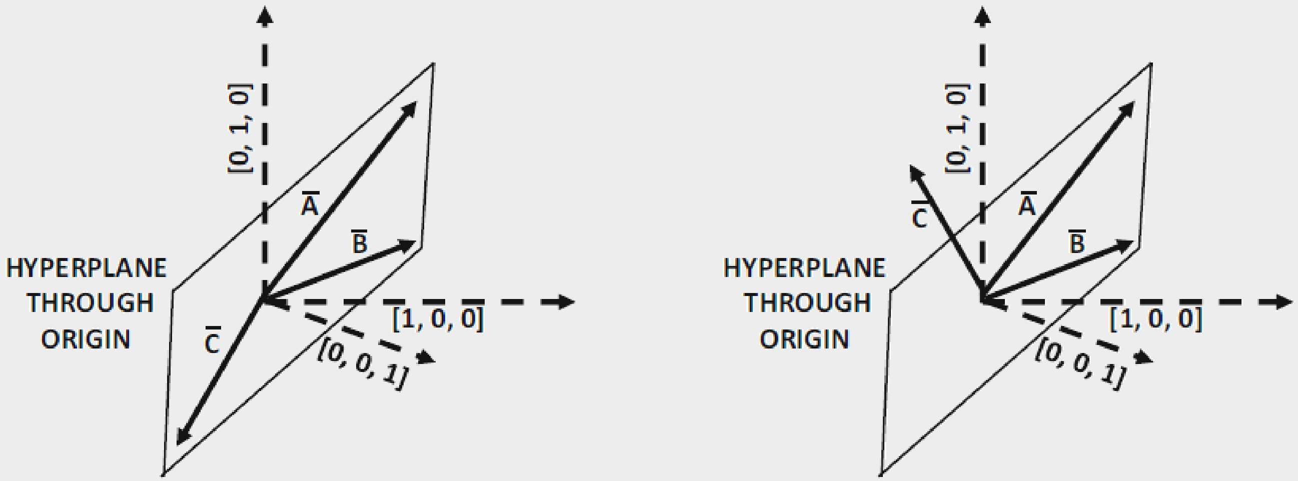
v and u are non-zero vectors in $\mathbb{R}^3$ where v is not a multiple of u



Span {**v**} as a
line through the origin.



Span {**u**, **v**} as a
plane through the origin.

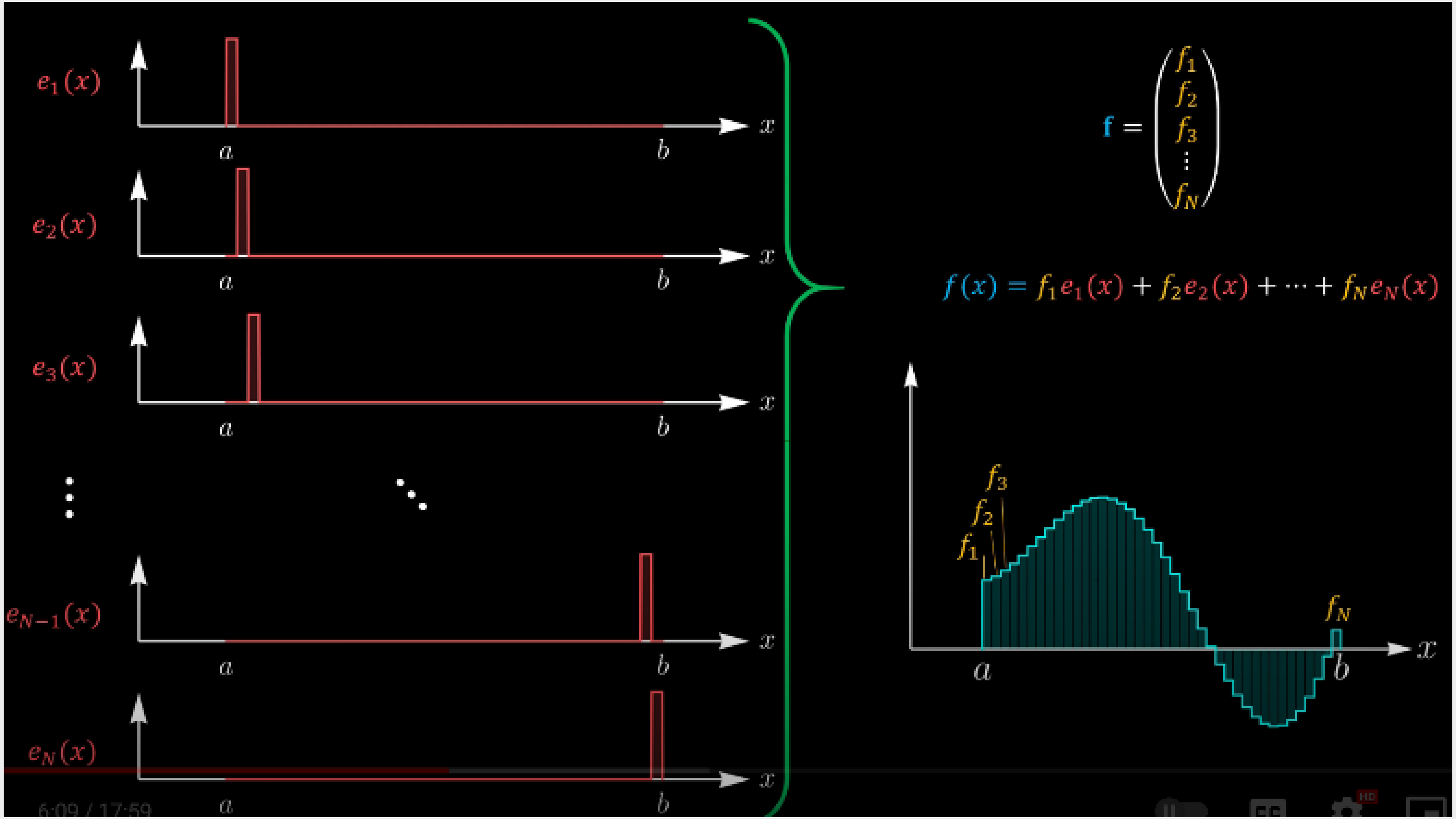


(a) $\text{Span}(\{\overline{A}, \overline{B}\}) = \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) = \text{All vectors on hyperplane}$

(b) $\text{Span}(\{\overline{A}, \overline{B}\}) \neq \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) = \text{All vectors in } \mathcal{R}^3$

Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set

Span or linear hull



Example

- Is vector b in Span $\{v_1, v_2, \ldots, v_p\}$
- Is vector v_3 in Span $\{v_1, v_2, \ldots, v_p\}$
- Is vector 0 in Span $\{v_1, v_2, \ldots, v_p\}$
- Span of polynomials: $\{(1+x), (1-x), x^2\}$?
- Is b in Span $\{a_1, a_2\}$?

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Conclusion

- Vector–Vector Operations
- Binary operations
- Field
- Vector space
- Linear combination and introduction to affine combination
- Span of vectors (linear hull)

References

- LINEAR ALGEBRA: Theory, Intuition, Code
- LINEAR ALGEBRA, KENNETH HOFFMAN.
- LINEAR ALGEBRA, Jim Hefferon
- David C. Lay, Linear Algebra and Its Applications
- Online Courses!
- Chapter 4 of Elementary Linear Algebra with Applications
- Chapter 3 of Applied Linear Algebra and Matrix Analysis