Inner Product and Orthogonality

CE40282-1: Linear Algebra
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Bilinear Form

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over the same field $\mathbb{F}$. Then a function $f : \mathcal{V} \times \mathcal{W} \to \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   - i) $f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = f(\mathbf{v}_1, \mathbf{w}) + f(\mathbf{v}_2, \mathbf{w})$ and
   - ii) $f(c\mathbf{v}_1, \mathbf{w}) = cf(\mathbf{v}_1, \mathbf{w})$ for all $c \in \mathbb{F}$, $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, and $\mathbf{w} \in \mathcal{W}$.

b) It is linear in its second argument:
   - i) $f(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = f(\mathbf{v}, \mathbf{w}_1) + f(\mathbf{v}, \mathbf{w}_2)$ and
   - ii) $f(\mathbf{v}, c\mathbf{w}_1) = cf(\mathbf{v}, \mathbf{w}_1)$ for all $c \in \mathbb{F}$, $\mathbf{v} \in \mathcal{V}$, and $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}$.

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Then the **dual** of $\mathcal{V}$, denoted by $\mathcal{V}^*$, is the vector space consisting of all linear forms on $\mathcal{V}$.

- Example

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Show that the function $g : \mathcal{V}^* \times \mathcal{V} \to \mathbb{F}$ defined by

$$g(f, \mathbf{v}) = f(\mathbf{v}) \quad \text{for all} \quad f \in \mathcal{V}^*,\ \mathbf{v} \in \mathcal{V}$$

is a bilinear form.

Review: Inner products over real field

An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geqslant 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle w, u + v \rangle = \langle w, u \rangle + \langle w, v \rangle$ for all $u, v, w \in V$.
4. $\langle w, cu \rangle = c \langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle w, v \rangle = \langle v, w \rangle$.

General Inner products

- Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that $\mathcal{V}$ is a vector space over $\mathbb{F}$. Then an **inner product** on $\mathcal{V}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathcal{V}$:

  a) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

  b) $\langle \mathbf{v}, \mathbf{w} + c\mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + c\langle \mathbf{v}, \mathbf{x} \rangle$ (linearity)

  c) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

- $F = \mathbb{R}$ bilinear forms
- $F = \mathbb{C}$ sesquilinear forms—they are linear in their second argument, but only conjugate linear in their first argument

$$\langle \mathbf{v} + c\mathbf{x}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} + c\mathbf{x} \rangle} = \overline{\langle \mathbf{w}, \mathbf{v} \rangle} + \overline{c\langle \mathbf{w}, \mathbf{x} \rangle} = \langle \mathbf{v}, \mathbf{w} \rangle + \overline{c}\langle \mathbf{x}, \mathbf{w} \rangle.$$

Complex Dot Product

- Example

Show that the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^* \mathbf{w} = \sum_{i=1}^{n} \overline{v_i} w_i \quad \text{for all} \quad \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$$

is an inner product on $\mathbb{C}^n$.

Inner Product on Continuous Functions

- Example

Let $a < b$ be real numbers and let $\mathcal{C}[a,b]$ be the vector space of continuous functions on the real interval $[a,b]$. Show that the function $\langle \cdot, \cdot \rangle : \mathcal{C}[a,b] \times \mathcal{C}[a,b] \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx \quad \text{for all} \quad f, g \in \mathcal{C}[a,b]$$

is an inner product on $\mathcal{C}[a,b]$.

Inner Product on Polynomials

- Example
  - Find <p,q>, ||p||, ||p-q||?

    $p = 3 - x + 2x^2$ and $q = 4x + x^2$

Inner product and norm

- Theorem:

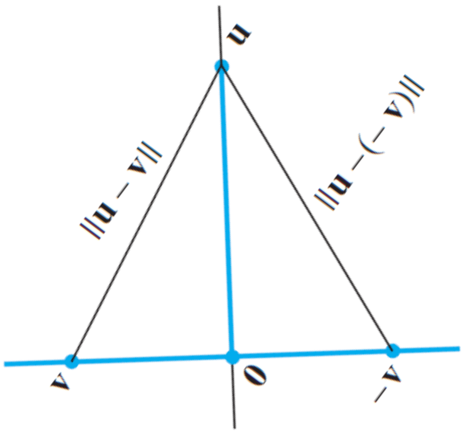
Take any inner product $\langle .,. \rangle$ *and define* $f(x) = \sqrt{\langle x, x \rangle}$. *Then* f *is a norm.*

  - Proof?

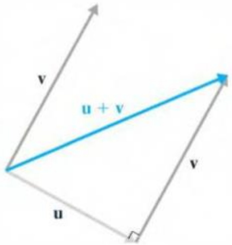
  - Note: Every inner product gives rise to a norm, but not every norm comes from an inner product (Think about norm 2 and norm max ☺)

Orthogonal vectors

- Geometry:

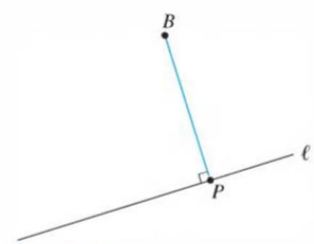


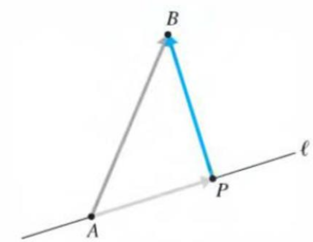
- Algebra:
  - Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.
  - Suppose $\mathcal{V}$ is an inner product space. Two vectors $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ are called **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
  - **The Pythagorean Theorem**
    Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$



Projection

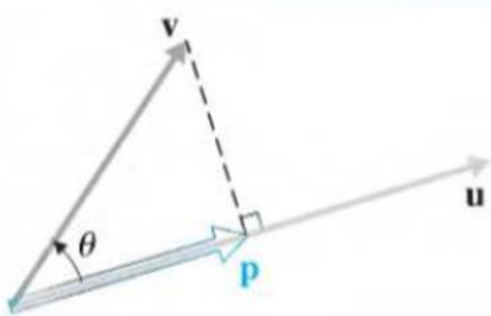
- Finding the distance from a point B to line l = Finding the length of line segment BP
- AP: projection of AB onto the line l





Definition If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, then the ***projection of* v *onto* u** is the vector $\text{proj}_{\mathbf{u}}(\mathbf{v})$ defined by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$



The projection of **v** onto **u**

Orthogonal Sets

- A set of vectors $\{a_1, \ldots, a_k\}$ in R^n is orthogonal set if each pair of distinct vectors is orthogonal (mutually orthogonal vectors).
- Theorem:
  - If $S = \{a_1, \ldots, a_k\}$ is an orthogonal set of nonzero vectors in R^n, then S is linearly independent and is a basis for the subspace spanned by S.
    - Proof?

If $k = n$, then prove that S is a basis for $\mathbb{R}^n$

Orthonormal vectors

Orthonormal Bases

A basis B of an inner product space $\mathcal{V}$ is called an **orthonormal basis** of $\mathcal{V}$ if

a) $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{v} \neq \mathbf{w} \in B$, and (mutual orthogonality)

b) $\|\mathbf{v}\| = 1$ for all $\mathbf{v} \in B$. (normalization)

- set of n-vectors $a_1, \ldots, a_k$ are *(mutually) orthogonal* if $a_i \perp a_j$ for $i \neq j$
- they are *normalized* if $\|a_i\| = 1$ for $i = 1, \ldots, k$
- they are *orthonormal* if both hold
- can be expressed using inner products as

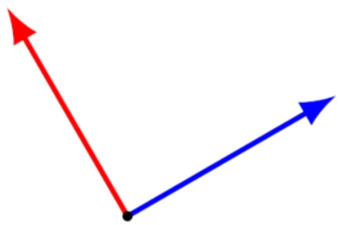
$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- orthonormal sets of vectors are linearly independent
- by independence-dimension inequality, must have $k \leq n$
- when $k = n$, $a_1, \ldots, a_n$ are an *orthonormal basis*

Independence-dimension inequality. If the n-vectors $\overrightarrow{a_1}, \cdots, \overrightarrow{a_k}$ are linearly independent, then $k \leq n$.

Examples of orthonormal bases

- Standard unit n-vectors $e_1, .., e_n$
- The 3-vectors $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$
- The 2-vectors shown below



- The standard basis in $P^p\ [a, b]$ (be the set of real-valued polynomials of degree at most p.)

Linear combinations of orthonormal vectors

- A simple way to check if an n-vector y is a linear combination of the orthonormal vectors $a_1, \dots, a_k$, if and only if:

$$y = (a_1^T y)a_1 + \cdots + (a_k^T y)a_k$$

- For orthogonal vectors $a_1, \dots, a_k$:

$$y = c_1 a_1 + \cdots + c_k a_k$$

$$c_j = \frac{y.a_j}{a_j.a_j}$$

Example

- Write x as a linear combination of a_1, a_2, a_3?

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

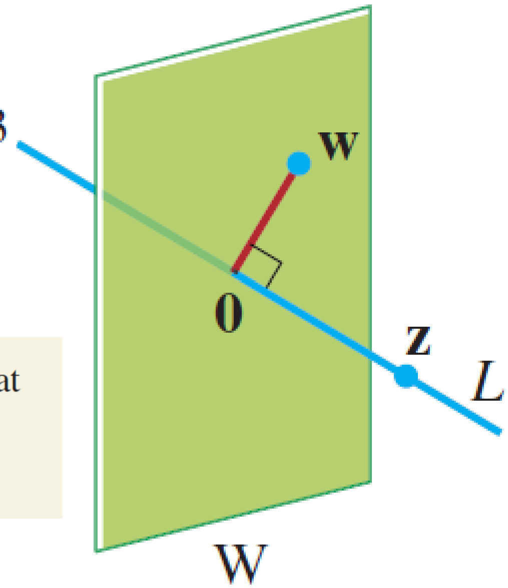
Orthogonal Complements

- If a vector z is orthogonal to every vector in a subspace W of R^n, then z is said to be orthogonal to W.
- The set of all vectors z that are orthogonal to W is called the orthogonal complement of W and is denoted by $w^{\perp}$

W be a plane through the origin in $\mathbb{R}^3$

$L = W^{\perp}$ and $W = L^{\perp}$

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $\boxed{W^{\perp} \text{ is a subspace of } \mathbb{R}^n.}$



Orthogonal Complements

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $\boxed{W^{\perp} \text{ is a subspace of } \mathbb{R}^n.}$

- Proof?

We emphasize that W_1 and W_2 can be orthogonal without being complements.

$W_1 = \text{span}((1,0,0))$ and $W_2 = \text{span}((0,1,0))$.

Orthogonal Projection of y onto W:

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \qquad (1)$$

$\text{proj}_W\, \mathbf{y}$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

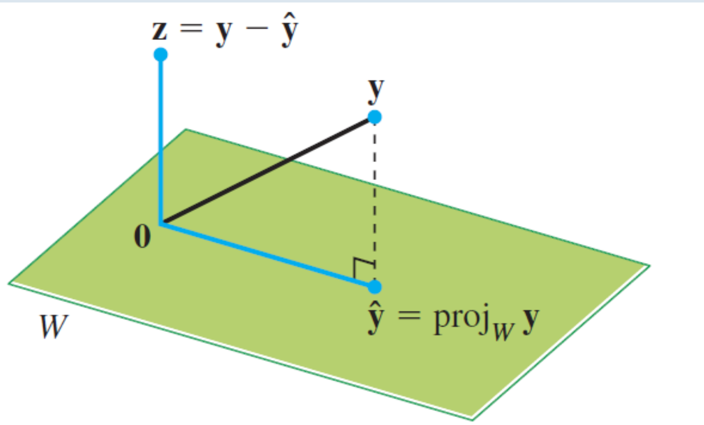
$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad (2)$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

$\hat{y}$: **orthogonal projection of y**

Proof? Z is in W^T

The uniqueness of the decomposition (1) shows that the orthogonal projection $\hat{\mathbf{y}}$ depends only on W and not on the particular basis used in (2).



The orthogonal projection of $\mathbf{y}$ onto W.

Best Approximation

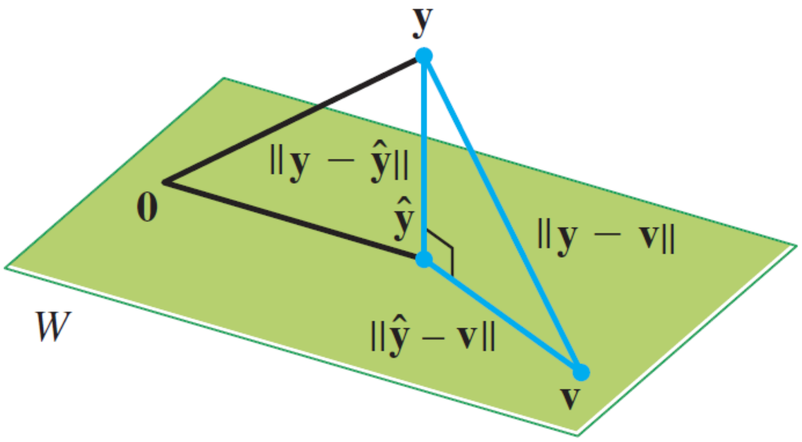
The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.

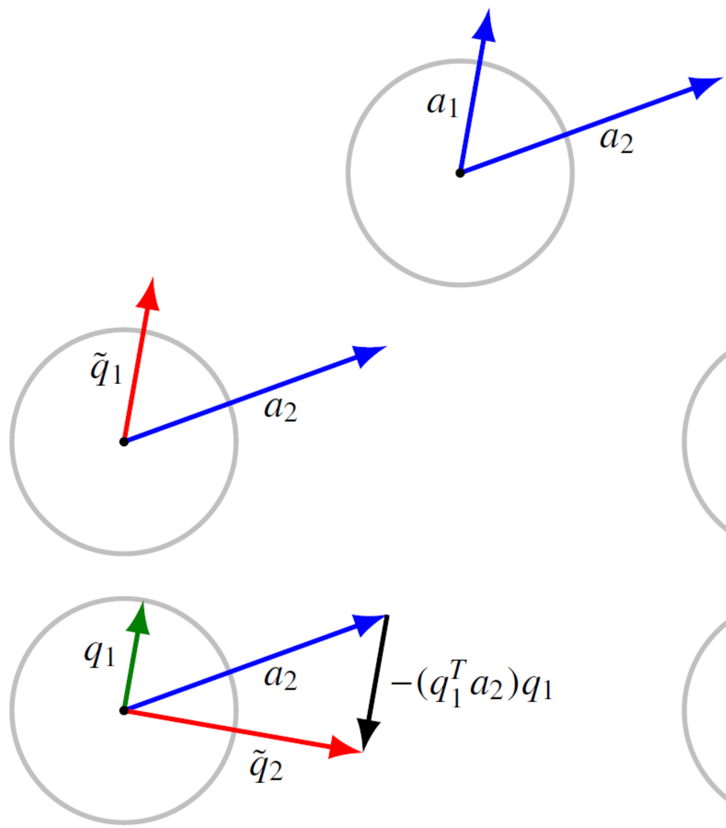
- Proof?



The orthogonal projection of $\mathbf{y}$ onto W is the closest point in W to $\mathbf{y}$.

Gram–Schmidt (orthogonalization) algorithm

- Find orthonormal basis for span$\{a_1, a_2, \ldots, a_k\}$
- Geometry:



Gram–Schmidt (orthogonalization) algorithm

- Find orthonormal basis for span$\{a_1, a_2, \dots, a_k\}$
- Algebra:

1. $$q_1 = \frac{a_1}{\|a_1\|}$$
2. $$\tilde{q}_2 = a_2 - (q_1^T a_2)q_1 \rightarrow q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$
3. $$\tilde{q}_3 = a_3 - (q_1^T a_3)q_1 - (q_2^T a_3)q_2 \rightarrow q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|}$$

$$\vdots$$

k. $$\tilde{q}_k = a_k - (q_1^T a_k)q_1 - \dots - (q_{k-1}^T a_k)q_{k-1} \rightarrow q_k = \frac{\tilde{q}_k}{\|\tilde{q}_k\|}$$

Gram–Schmidt (orthogonalization) algorithm

- Why $\{q_1, q_2, \dots, q_k\}$ is a orthonormal basis for span$\{a_1, a_2, \dots, a_k\}$?
  - $\{q_1, q_2, \dots, q_k\}$ are normalized.
  - $\{q_1, q_2, \dots, q_k\}$ is a orthogonal set
  - a_i is a linear combination of $\{q_1, q_2, \dots, q_i\}$

$$\Downarrow$$

span$\{q_1, q_2, \dots, q_k\}$=span$\{a_1, a_2, \dots, a_k\}$

- q_i is a linear combination of $\{a_1, a_2, \dots, a_i\}$

Gram–Schmidt (orthogonalization) algorithm

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given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$

1. *Orthogonalization:* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence:* if $\tilde{q}_i = 0$, quit
3. *Normalization:* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

- if G–S does not stop early (in step 2), $a_1, \ldots, a_k$ are linearly independent
- if G–S stops early in iteration $i = j$, then a_j is a linear combination of $a_1, \ldots, a_{j-1}$ (so $a_1, \ldots, a_k$ are linearly dependent)

$$a_j = (q_1^T a_j)q_1 + \cdots + (q_{j-1}^T a_j)q_{j-1}$$

Complexity of Gram–Schmidt algorithm

given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$

1. *Orthogonalization:* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence:* if $\tilde{q}_i = 0$, quit
3. *Normalization:* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

Gram–Schmidt

- Suppose $\mathrm{B} = \{a_1, a_2, \dots, a_n\}$ is a basis of an inner product space A. Then $\mathrm{C} = \{q_1, q_2, \dots, q_n\}$ is an orthonormal basis of $span\{a_1, a_2, \dots, a_n\}$.

$$q_1 = \frac{a_1}{||a_1||} \qquad \mathrm{q_k} = \frac{a_k - \sum_{i=1}^{k-1} < q_i, a_k > q_i}{\left\| a_k - \sum_{i=1}^{k-1} < q_i, a_k > q_i \right\|} \quad \text{for } 2 \le k \le n$$

  - Proof? We prove this result by induction on k.

TAKE HOME QUESTION

- Example

Find an orthonormal basis for $\mathcal{P}^2[-1,1]$ *with respect to the inner product*

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\ dx.$$

Conclusion

- Existence of Orthonormal Bases

Every finite-dimensional inner product space has an orthonormal basis.

Since finite-dimensional inner product spaces (by definition) have a basis consisting of finitely many vectors, and the Gram–Schmidt process tells us how to convert that basis into an orthonormal basis, we now know that every finite-dimensional inner product space has an orthonormal basis:

Reference

- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra Devid Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares