Vector Space

CE282: Linear Algebra
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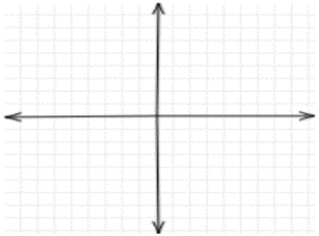
Tuple and Vector Space

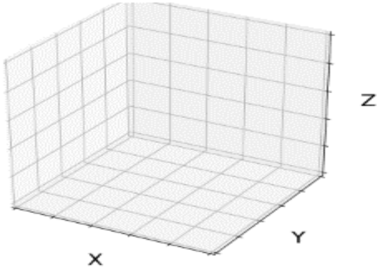
Definition

- A tuple is an ordered list of numbers.
- For example: $\begin{bmatrix} 1 \\ 2 \\ 32 \\ 10 \end{bmatrix}$ is a 4-tuple (a tuple with 4 elements).

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \dots \right\}$$



$$\mathbb{R}^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \dots \right\}$$



Review: Complex Numbers

Numbers:

- Real: Nearly **any number** you can think of is a Real Number!

1	12.38	−0.8625	3/4	√2	1998

- Imaginary: When **squared** give a **negative** result.

The "unit" imaginary number (like 1 for Real Numbers) is i, which is the square root of −1.

Examples of Imaginary Numbers:

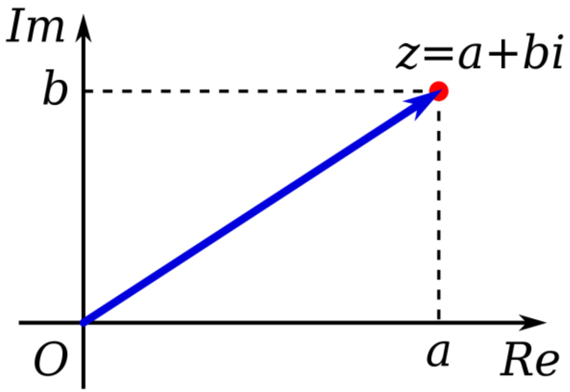
3i	1.04i	−2.8i	3i/4	(√2)i	1998i

And we keep that little "i" there to remind us we need to multiply by $\sqrt{-1}$

- $\mathbb{C}$ is a plane, where number $(a+bi)$ has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$
- Imaginary number: bi , $b \in R$

- Arithmetic with complex numbers $(a+bi)$:

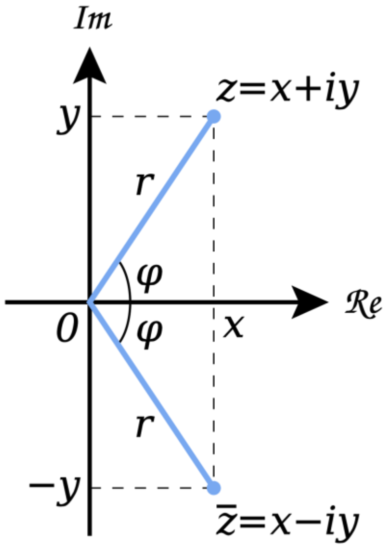
  - $(a+bi)+(c+di)$
  - $(a+bi)(c+di)$
  - $\dfrac{a+bi}{c+di}$



$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{(bc\ -ad)i}{c^2+d^2}$$

Question

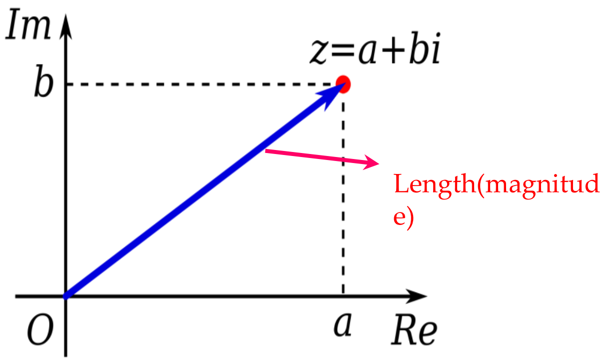
Conjugate of $x + yi$ is noted by $\overline{x + yi}$ .What it'll look like?



(Complex conjugate)

- Length (magnitude): $||a+bi||^2 = \overline{(a+bi)}(a+bi)$

- Inner Product:
  - Real: $< x, y > = x_1y_1 + x_2y_2 + \ ... + x_ny_n$
  - Complex: $< x, y > = \overline{x_1}y_1 + \overline{x_2}y_2 + \ ... + \overline{x_n}y_n$



Extra resource:

If you want to learn more about complex numbers, this video is recommended!

Definition

- Any function from A x A → A is a binary operation.

- **Closure Law:**
  - A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.

$$if\ a \in A, b \in B \ \rightarrow a \ * b \in A$$

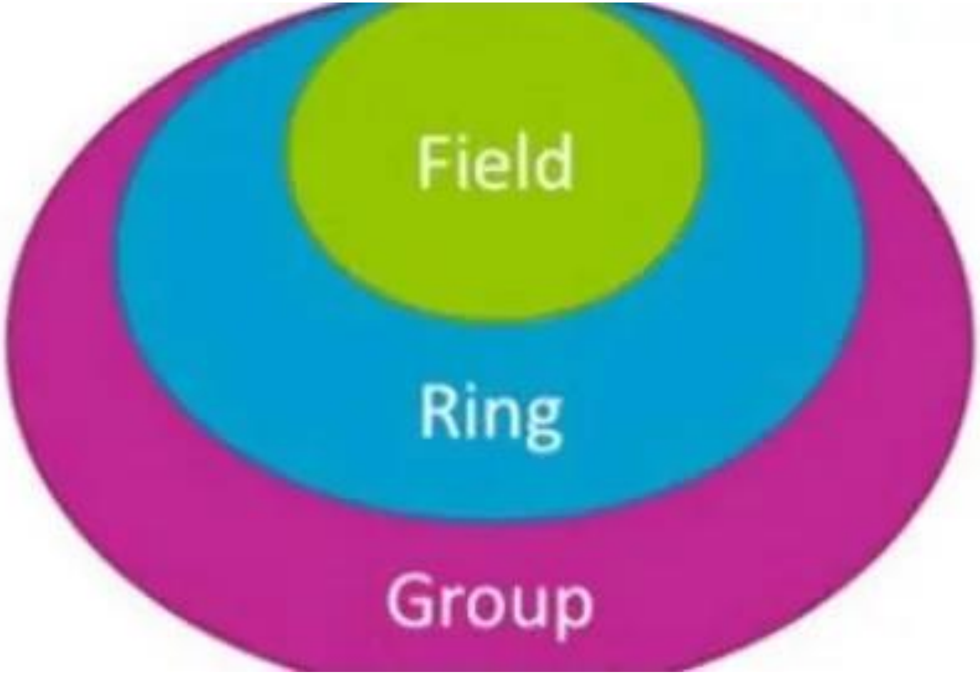
Class Activity

Scan the QR Code and answer the questions (or type the link in your browser)

- ❑ Is “+” a binary operator on natural numbers?
- ❑ Is “x” a binary operator on natural numbers?
- ❑ Is “-” a binary operator on natural numbers?
- ❑ Is “/” a binary operator on natural numbers?

https://forms.gle/KXstkou992to72Ew6

Group – Ring - Field



Definition

- A group G is a pair $(S,\circ)$, where S is a set and $\circ$ is a binary operation on S such that:

1) $\circ$ is **associative**.

2) **(Identity)** There exists an element $e \in S$ such that:

$$e \circ a = a \circ e = a \qquad \forall a \in S$$

3) **(Inverses)** For every $a \in S$ there is $b \in S$ such that:

$$a \circ b = b \circ a = e$$

If $\circ$ is commutative, then G is called a commutative group!

Ring

Definition

- A ring R is a set together with <u>two</u> binary operations + and *, satisfying the following properties:

1. (R,+) is a commutative group
2. * is associative
3. The distributive laws hold in R: (Multiplication is distributive over addition)

$$(a+b)*c=(a*c)+(b*c)$$

$$a*(b+c)=(a*b)+(a*c)$$

Definition

- A field F is a set together with two binary operations + and *, satisfying the following properties:

1. (F,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. (F-{0},*) is a commutative group
3. The distributive law holds in F:

$$(a+b)*c=(a*c)+(b*c)$$
$$a*(b+c)=(a*b)+(a*c)$$

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

Example

(R; +, .) and (Q; +, .) serve as examples of fields.
(Z; +, .) is an example of a ring which is not a field!

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

Fields $\forall\, a, b, c \in F$

Properties	Binary Operations	
	Addition (+)	**Multiplication (.)**
Closure (بسته بودن)	$\exists a + b \in F$	$\exists a.b \in F$
Associative (شرکت‌پذیری)	$a + (b + c) = (a + b) + c$	$a.(b.c) = (a.b).c$
Commutative (جابه‌جایی‌پذیری)	$a + b = b + a$	$a.b = b.a$
Existence of identity $e \in F$	$a + e = a = e + a$	$a.e = a = e.a$
Existence of inverse: For each a in F there <u>must exist</u> b in F	$a + b = e = b + a$	$a.b = e = b.a$ <u>*For any nonzero a*</u>

Multiplication is distributive over addition

$$a.(b + c) = a.b + a.c$$
$$(a + b).c = a.c + b.c$$

Fields

Question

Which are fields? (two binary operations + , *)

$\mathbb{R}$

$\mathbb{C}$

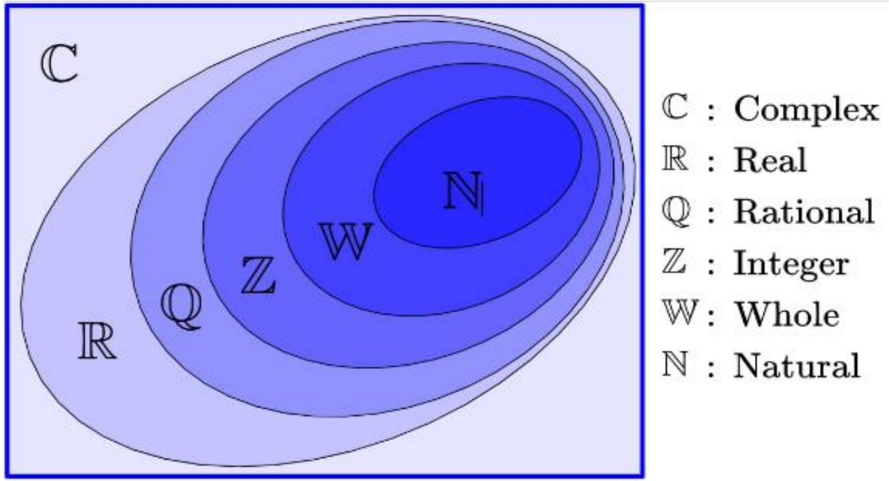
$\mathbb{Q}$

$\mathbb{Z}$

W

$\mathbb{N}$

$\mathbb{R}^{2\times 2}$



- Building blocks of linear algebra.

- A non-empty set V with field F (most of time R or C) forms a vector space with two operations:

  1. $+$: Binary operation on V which is $V \times V \rightarrow V$
  2. $.$: $F \times V \rightarrow V$

Note

In our course, by **default**, field is **R** (real numbers).

Definition

A vector space over a field F is the set V equipped with two operations: $(V, F, +, .)$

i. Vector addition: denoted by "+" adds two elements $x, y \in V$ to produce another element $x + y \in V$

ii. Scalar multiplication: denoted by "." multiplies a vector $x \in V$ with a scalar $\alpha \in F$ to produce another vector $\alpha . x \in V$ .We usually omit the "." and simply write this vector as αx

Vector Space

[A1] Vector addition is commutative: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

[A2] Vector addition is associative: $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ *for every* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.

[A3] Additive identity: *There is an element* $\mathbf{0} \in \mathcal{V}$ *such that* $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

[A4] Additive inverse: *For every* $\boldsymbol{x} \in \mathcal{V}$*, there is an element* $(\boldsymbol{x}) \in \mathcal{V}$ *such that* $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.

Addition of vector space

[M1] Scalar multiplication is associative: $a(b\boldsymbol{x}) = (ab)\boldsymbol{x}$ *for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M2] First Distributive property: $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ *and for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M3] Second Distributive property: $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ *and every* $a \in \Re^1$.

[M4] Unit for scalar multiplication: $1\boldsymbol{x} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

Action of the field of scalars on the vector space

Vector Space

Example

Let V be the set of all real numbers with the operations $u \oplus v = u - v$ ($\oplus$ is an ordinary subtraction) and $c \boxdot u = cu$ ($\boxdot$ is an ordinary multiplication). Is V a vector space? If it's not, which properties fail to hold?

Example

- The n-tuple space,
- The space of m x n matrices
- The space of functions from a set to a field g(s)

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

$$f(t) = 1 + \sin(2t) \quad \text{and} \quad g(t) = 2 + 0.5t$$

- The space of polynomial functions over a field f(x):

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \ \ldots + a_n t^n$$

Vector Space of functions

- Function addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x) \quad and \ (af)(x) = af(x)$$

Non-empty set X and any field F $\longrightarrow$ $F^X = \{f: X \to F\}$

Example

- Set of all polynomials with real coefficients
- Set of all real-valued continuous function on [0,1]
- Set of all real-valued function that are differentiable on [0,1]

P_n ($\mathbb{R}$): Polynomials with max degree (n)

- Scalar multiplication
- Vector addition
- And other 8 properties!

Question

Which are vector spaces?

- ❑ Set $\mathbb{R}^n$ over $\mathbb{R}$
- ❑ Set $\mathbb{C}$ over $\mathbb{R}$
- ❑ Set $\mathbb{R}$ over $\mathbb{C}$
- ❑ Set $\mathbb{Z}$ over $\mathbb{R}$
- ❑ Set of all polynomials with coefficient from field $\mathbb{R}$
- ❑ Set of all polynomials of degree at most n with coefficient from field $\mathbb{R}$
- ❑ Matrix: $M_{m,n}(\mathbb{R})$
- ❑ Function: $f(x): x \longrightarrow \mathbb{R}$

Conclusion

The operations on field F are:

- $+ : F \times F \rightarrow F$
- $\times : F \times F \rightarrow F$

The operations on a vector space V over a field F are:

- $+ : V \times V \rightarrow V$
- $. : F \times V \rightarrow V$

Definition

If $v_1, v_2, v_3, \dots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2, \dots, v_p$ is denoted by Span $\{v_1, v_2, \dots, v_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned (or generated)** by $v_1, v_2, \dots, v_p$.

That is, Span$\{v_1, v_2, \dots, v_p\}$ is the collection of all vectors that can be written in the form:

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

with $c_1, c_2, \dots, c_p$ being scalars.

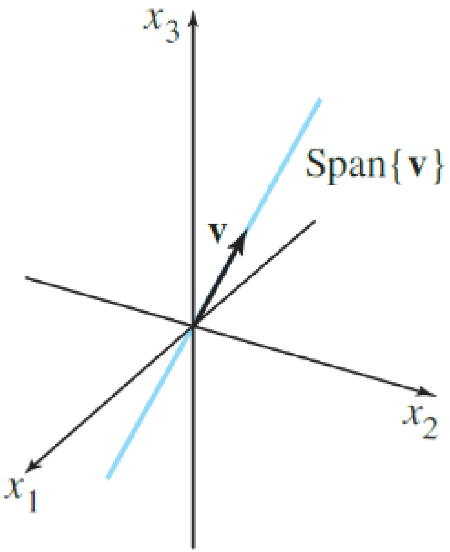
Example

- ❑ Is vector b in Span $\{v_1, v_2, \dots, v_p\}$
- ❑ Is vector v_3 in Span $\{v_1, v_2, \dots, v_p\}$
- ❑ Is vector 0 in Span $\{v_1, v_2, \dots, v_p\}$
- ❑ Span of polynomials: $\{(1+x), (1-x), x^2\}$?
- ❑ Is b in Span $\{a_1, a_2\}$?

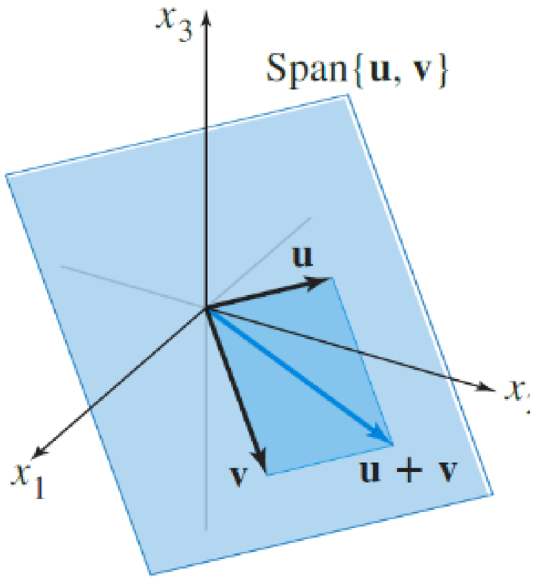
$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Span Geometry

v and u are non-zero vectors in $\mathbb{R}^3$ where v is not a multiple of u

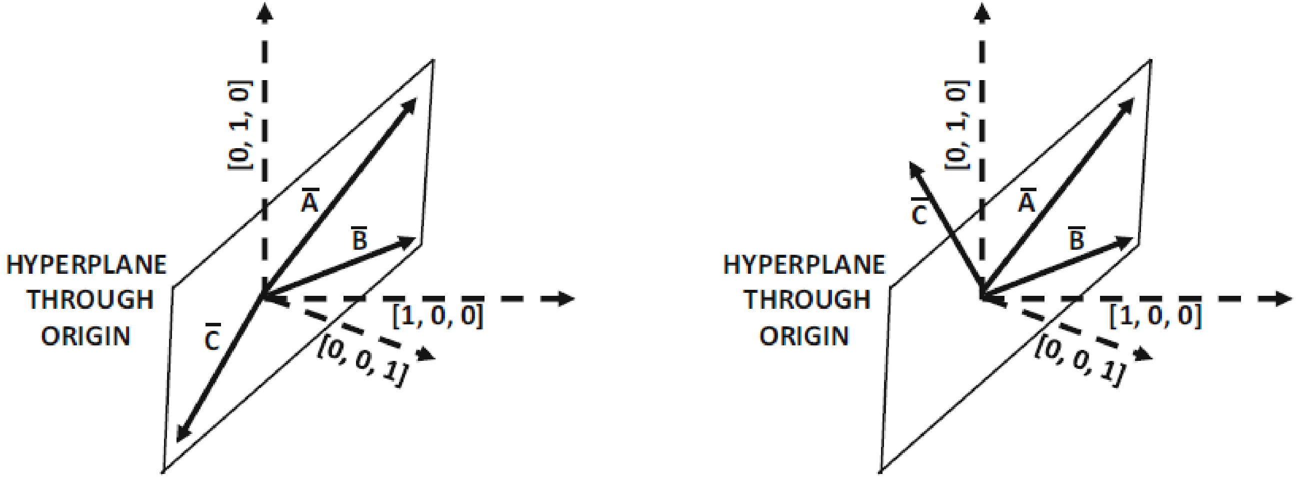


Span {**v**} as a
line through the origin.



Span {**u**, **v**} as a
plane through the origin.

Span Geometry



(a) $\text{Span}(\{\overline{A}, \overline{B}\}) = \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors on hyperplane

(b) $\text{Span}(\{\overline{A}, \overline{B}\}) \neq \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors in $\mathcal{R}^3$

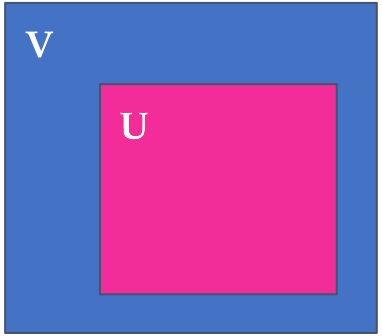
Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set

Subspace

Definition

A **non-empty subset** of vector space for which closure holds for addition and scalar multiplication is called a subspace.

Subspace: If V is a vector space and subset $U \subseteq V$, then U is itself a vector space with the **same addition and scalar multiplication as V**.



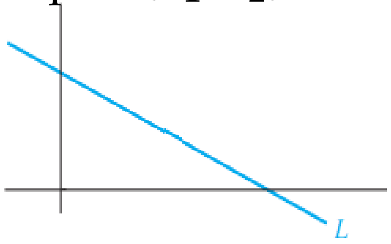
Subspace

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has these properties:

a. The zero vector is in H.
b. Foreach u and v in H, the sum u + v is in H.
c. Foreach u in H and each scalar c, the vector cu is in H.

Example 1. H = Span $\{x_1, x_2\}$, then H is a subspace

Example 2.



Example 3. The vector space $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$?

Example 4. Is H a subset of $\mathbb{R}^3$? $H = \{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real}\}$

Vector Space vs Subspace

$Let\ V\ be\ a\ vector\ subspace\ and\ let\ U \subseteq V$:

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. $There\ is\ a\ vector\ 0 \in V\ such\ that\ u + 0 = u$
5. $For each\ u \in V, there\ is\ a\ vector\ -u \in V\ such\ that\ u + (-u) = 0$
6. $cu \in V$
7. $c\ (u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c\ (du) = (cd)u$
10. $1u = u$

1. $u + v \in U$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
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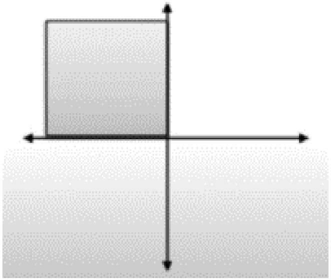
5. $For each\ u \in U, there\ is\ a\ vector\ -u \in U\ such\ that\ u + (-u) = 0$
6. $cu \in U$
7. $c\ (u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

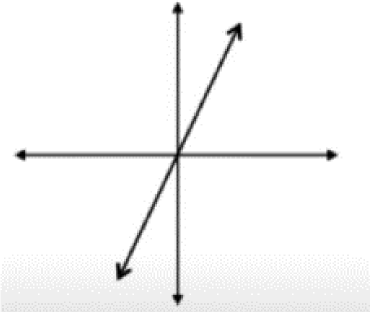
Subspace Summary

- A subspace is a subset of vector space that holds closure under addition and scalar multiplication.
- Zero vector is a subspace of every vector space.
- Vector space is a subspace of itself.

Example

- Set of all continuous real-valued functions on R.
- **Set of all differentiable real-valued functions on R.**
- Every vector space with more than one member has at least _____ subspaces.
- Name subspace for $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$
- Following figures:





Example

$$for\ vector\ space\ \mathbb{R}^4 (4\ dimensional), subspaces\ are:$$

a. $\mathbb{R}^4\ itself$
b. $zero\ vector\ ([0, 0, 0, 0])$
c. $Line\ passing\ through\ zero\ vector\ (1 - dimensional)$
d. $Plane\ passing\ through\ zero\ vector\ (2 - dimensional)$
e. $3D\ figure\ containing\ zero\ vector\ (3 - dimensional)$

Intersection of subspaces

Theorem

If U and W are subspaces of V, then $U \cap V$ is a subspace.

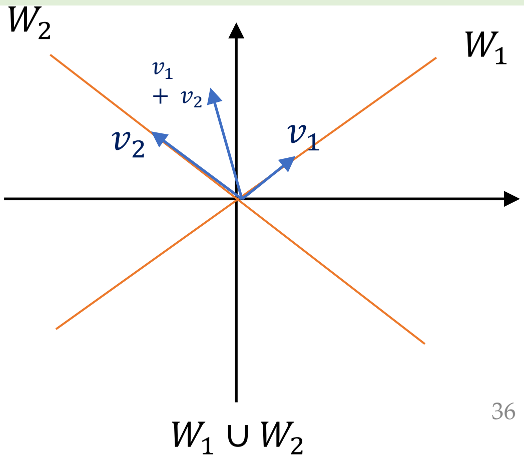
Proof:

Union

Theorem

Fact: The union of two sub-spaces is not a subspace unless one is contained in the other.

W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ is subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$



Subspace

Theorem

If $v_1, v_2, \ldots, v_p$ are in a vector space V, then Span $\{v_1, v_2, \ldots, v_p\}$ is a subspace of V.

Proof:

Example

Let H be the set of all vectors of the form $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix}$ where a,b are arbitrary scalars. That is, let $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \text{ in } R \right\}$. Show that H is a subspace of $\mathbb{R}^4$.

Subspace

Theorem

Let $v_1, v_2, \dots, v_n$ be vectors in vector space V and let $w_1, w_2, \dots, w_k$ be vectors in Span $\{v_1, v_2, \dots, v_n\}$. Then:

$$Span\ \{w_1, w_2, \dots, w_k\} \subseteq \ \{v_1, v_2, \dots, v_n\}$$

Sum of vector spaces

- There are two reasons to use the sum of two vector spaces.
  - to build new vector spaces from old ones.
  - to decompose the known vector space into sum of two (smaller) spaces.
- Since we consider linear transformations between vector spaces, these sums lead to representations of these linear maps and corresponding matrices into forms that reflect these sums. In many very important situations, we start with a vector space V and can identify subspaces “internally” from which the whole space V can be built up using the construction of sums.

Definition

Let A and B be non-empty subsets of a vector space V. The sum of A and B, denoted A+B, is the set of all possible sums of elements from both subsets: $A + B = \{a + b : a \in A, b \in B\}$

Theorem

If $W_1, \dots, W_m$ are subspaces of V, then $W_1 + \cdots + W_m$ is a subspace of V.

A vector space V is called the **direct sum** of $V_1 \, and \; V_2$ if V_1 and V_2 are subspaces of V such that $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = V$. This means that every vector v from V is uniquely represented via sum of two vectors $v = v_1 + v_2, v_1 \in V, v_2 \in V$. We denote that V is the direct sum of $V_1 \, and \; V_2$ by writing $V = V_1 \oplus V_2$

Definition

U + W is called a **direct sum**, if any element in U + W can be written uniquely as u + w where $u \in U$ *and* $w \in W$ (Notation: $U \oplus W$)

Example

Check where $U \oplus W$ exists?

a) $W = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}, U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

b) $W = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix}, U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

Theorem

If U and W are subspaces of V, then $U \oplus W$ is a subspace, if and only if $U \cap W = \{0\}$

Proof:

Example

Let E denote the set of all polynomials of even powers.
$E = \{a_n t^{2n} + a_{n-1} t^{2n-2} + \ \ldots + a_0\}$, and O be the set of all polynomials of odd powers :
$O = \{a_n t^{2n+1} + a_{n-1} t^{2n-1} + \ \ldots + a_0\}$. The set of all polynomials P is a direct sum of E and O :

$$P = E \oplus O$$

It is easy to see that any polynomial (or function) can be uniquely decomposed into direct sum of its even and odd counterparts:

$$p(t) = \frac{p(t) + p(-t)}{2} + \frac{p(t) - p(-t)}{2}$$

Example

Prove set of all bound functions such as

$$W = \{f(x) \mid \exists M \in R \ such\ that\ |f(x)| \le M, \forall x \in R\}$$

is a subspace of $V = \{all\ functions\ from\ R\ to\ R\}$

References

- LINEAR ALGEBRA: Theory, Intuition, Code
- David Cherney,
- Online Courses!
- Chapter 4 of Elementary Linear Algebra with Applications
- Chapter 3 of Applied Linear Algebra and Matrix Analysis