

Matrix Inverse

Linear Algebra

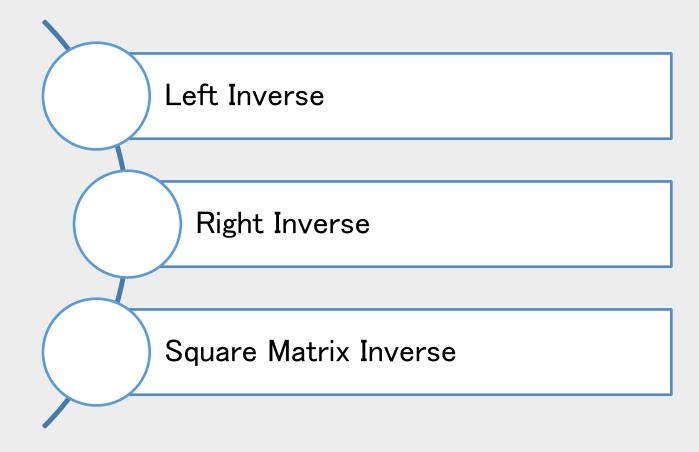
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Overview





Left Inverse

Left Inverse



Definition

- \square A number x that satisfies xa = 1 is called the inverse of a
- \square Inverse (i.e., $\frac{1}{a}$) exists if and only if $a \neq 0$, and is unique
- \square A matrix X that satisfies XA = I is called a left inverse of A
- \Box If a left inverse exists we say that A is left-invertible
- \square $A: m \times n \Rightarrow I: n \times n \Rightarrow X: n \times m$

Example

The matrix
$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

Has two different left inverses:

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix},$$

$$C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

Solving linear equations with a left inverse



Method

- \square Suppose Ax = b, and A has a left inverse C
- \Box Then Cb = C(Ax) = (CA)x = Ix = x
- ☐ So multiplying the right-hand side by a left inverse yields the solution

Left inverse of vector



Note

- ☐ A non-zero column vector always has a left inverse.
- ☐ Left inverse is not unique.

Example

$$\Box = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \text{Three ways: } (1)a^{-1} = \frac{1}{a_i}e_i^T \quad (2)a^Ta = 1 \Rightarrow \frac{a^T}{||a||^2} \quad (3)a^{-1}a = 1$$

 \square Matrix with orthonormal columns $A^{-1} = A^T$

Example

□ Row vector does not have left inverse

$$A = [1 \ 0 \ 3]$$

Think about rank(BA), rank(I) with this theory: $rank(BA) \le min(rank(A), rank(B))$

Left inverse and column independence



Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof

Left inverse and column independence



Theorem

- \square If A has a left inverse C then the columns of A are linearly independent
- ☐ We'll see later that the converse is also true, so:

A matrix is left-invertible if and only if its columns are linearly independent

☐ Matrix generalization of

A number is invertible if and only if it is nonzero

From Previous Theorem

Left-invertible matrices are all tall or square

- Wide matrix is not always left invertible
- ☐ Tall or square matrices can be left invertible

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Right Inverse

Right inverses



Definition

- \square A matrix X that satisfies AX = I is a right inverse of A
- ☐ If a right inverse exists we say that A is right-invertible
- $\square A$ is right-invertible if and only if A^T is left-invertible:

$$AX = I \Longrightarrow (AX)^T = I \Longrightarrow X^T A^T = I$$

□ so we conclude:

A is right invertible if and only if its rows are linearly independent

☐ Right-invertible matrices are wide or square

Solving linear equations with a right inverse



Method

- \square Suppose A has a right inverse B
- \Box Consider the (square or underdetermined) equations of Ax = b
- $\square x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

 \Box So Ax = b has a solution for any b

Example

- \square Same A, B, C in last example.
- \square C^T and B^T are both right inverses of A^T
- \Box Under-determined equations $A^Tx=(1,2)$ has (different) solutions.

$$B^{T}(1,2) = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}), \quad C^{T}(1,2) = (0, \frac{1}{2}, -1)$$

there are many other solutions as well

Conclusion: Left and Right Inverse

Linear equations and matrix inverse



Definition

Left-Invertible matrix: if *X* is a left inverse of *A*, then

$$Ax = b \Longrightarrow x = XAx = Xb$$

There is at most one solution using X (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

Why "at most"??

$$XA = I$$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases} \qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1\\ 0 & -1\\ 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Linear equations and matrix inverse



Note

- If the system of equations Ax = b is consistent, and if a matrix B exists such that BA = I, then the system of equations has a unique solution, namely x = Bb.
- \square Right-inversible matrix: if X is a right inverse of A, then there is <u>at least one</u> solution (x=Xb):

$$x = Xb \implies Ax = AXb = b$$

- To pursue these ides further, suppose that again we want to solve a system of linear equations, Ax = b. Assume now that we have another matrix, B, such that AB = I. Then we can write A(Bb) = (AB)b = Ib = b; hence Bb solves the equations Ax = b. This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether Bb is the only solution. There may be others.
- ☐ Invertible matrix: if A is invertible, then

$$Ax = b \iff x = A^{-1}b$$

There is a unique solution

Conclusion



 \Box System of linear equations Ax = b:

- \circ A right inverse of A, say AB = I. Then Bb is a solution, as is verified by nothing A(Bb) = (AB)b = Ib = b.
- Why don't need to check the consistency for using right inverse?
- $_{\circ}$ A left inverse of A, say CA = I, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

Square Matrix Inverse



Definition

For $A \in M_{n \times n}$, if there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$, then:

- ☐ A is invertible (or nonsingular)
- ☐ B is the inverse of A
- \Box The inverse of A is denoted by $B = A^{-1}$

A square matrix that does not have an inverse is called non-invertible (or singular)

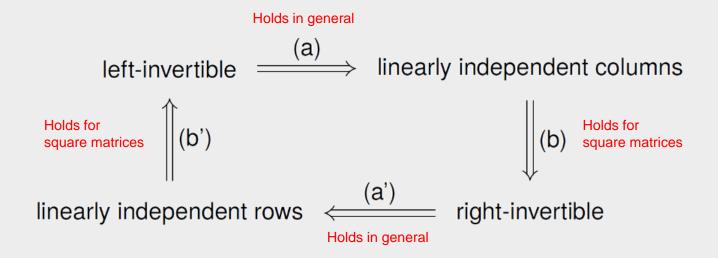
For a square matrix left and right inverse are the same. Rows and columns are linear independent.

Theorem

The inverse of a matrix is unique

Invertible Matrices





Gauss-Jordan Elimination for finding the Inverse of a matrix



Method

- \square Let A be a $n \times n$ matrix:
 - \square Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A:I_n]$.
 - \square Compute the reduced echelon form of $[A:I_n]$.
- \square If the reduced echelon form is of the type $[I_n:B]$, then B is the inverse of A.
- \square If the reduced echelon form is not the type $[I_n:B]$, in that the first $n\times n$ submatrix is not I_n then A has no inverse.

 $[A \mid I]$ Gauss—Jordan elimination $[I \mid A^{-1}]$

Important

An $n \times n$ matrix is invertible if and only if its reduced echelon form is I_n .

A is row equivalent to I_n

Inverse (Example)



Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \implies \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases}$$
(1)
$$x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1$$
(2)

This two system of linear equations have the same coefficient matrix, which is exactly the matrix \boldsymbol{A}

Inverse (Example)



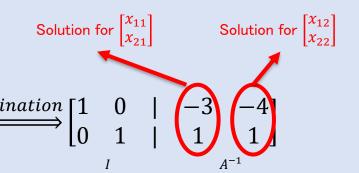
Rest of The Example

$$(1) \Rightarrow \begin{bmatrix} 1 & 4 & | & 1 \\ -1 & -3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \begin{bmatrix} 1 & 4 & | & 0 \\ -1 & -3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -4 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus
$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ -1 & -3 & | & 0 & 1 \end{bmatrix} \xrightarrow{Guass-Jordan\ elimination} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Using Gauss-Jordan Elimination on the



Definition

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):

- \square A^{-1} is invertible and $(A^{-1})^{-1} = A$
- \square A^k is invertible and $(A^k)^{-1} = A^{-k} = (A^{-1})^k$
- \Box cA is invertible if $c \neq 0$ and $(cA)^{-1} = \frac{1}{c}A^{-1}$ \Box A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Theorem

If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1 A_2 A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1} A_2^{-1} A_1^{-1}$$



Theorem

Let Ax = b be a system of n linear equations in n variable. If A^{-1} exists, the solution is unique and is given by $x = A^{-1}b$



Theorem

The solution set K of any system Ax=b of m linear in n unknows is, s is a particular solution:

$$K = s + Null(T_A)$$

Theorem

Let Ax = b be a system of n linear equations in n variable.

The system has exactly one solution $A^{-1}b$ if and only if A is invertible.

Invertible Matrix



Definition

Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If ad - bc = 0, then A is not invertible

Note

Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. det $A = ad - bc$.

 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Elementary Matrices



Definition

Each Elementary Matrix is E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example

Find the inverse of
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Solving square systems of linear equations



Method

- \square Suppose *A* is invertible
- \Box For any b, Ax = b has the unique solution

$$x = A^{-1}b$$

- ☐ Matrix generalization of simple scalar equation ax = b having solution $x = \left(\frac{1}{a}\right)b$ (for $a \neq 0$)
- \square Simple-looking formula $x = A^{-1}b$ is basis for many applications

Invertible (Nonsingular) matrices



Conclusion

The following are equivalent for a square matrix A:

- ☐ A is invertible
- \square Columns of A are linearly independent
- \square Rows of A are linearly independent
- $\square A$ has a left inverse
- $\square A$ has a right inverse

$$row \, rank(A) = col \, rank(A) = n$$

If any of these hold, all others do

Invertible matrices



Examples

- $\Box I^{-1} = I$
- \square If Q is orthogonal, i.e., square with $Q^TQ = I$, then $Q^{-1} = Q^T$
- \square 2 × 2 matrix A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

- You need to know this formula
- There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)
- ☐ Consider matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$
 - > A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

- \triangleright Verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)
- > We'll soon see how to compute the inverse

Properties



Properties

- $\Box (AB)^{-1} = B^{-1}A^{-1}$
- ☐ If A is nonsingular, then A^T is nonsingular $(A^T)^{-1} = (A^{-1})^T$ (sometimes denoted A^{-T})
- \square Negative matrix powers: $(A^{-1})^k$ is denoted by A^{-k}
- \square With $A^0 = I$, Identity $A^k A^l = A^{k+l}$ holds for any integers k, l

Triangular matrices



Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??

Question?



Why Matrix of Change of Basis is invertible?