Inner Product Space

Linear Algebra

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Overview

- Linear Form
- Bilinear Form
- Bilinear Form on Complex Vector Space
- Inner Product
- Inner Product Space

Linear Form

What are Linear Functions?

- ❑ $f: R^n \rightarrow R$ means that f is a function that maps real $n-$vectors to real numbers
- ❑ $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- ❑ $f(x) = (x_1, x_2, \ldots, x_n)$: argument

Definition

A function $f: R^n \rightarrow R$ is linear if it satisfies the following two properties:

- ❑ Additivity: For any $n-$vector x and y, $f(x + y) = f(x) + f(y)$
- ❑ Homogeneity: For any $n-$vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$

Superposition property:

Definition

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Note

- ❑ A function that satisfies the superposition property is called linear

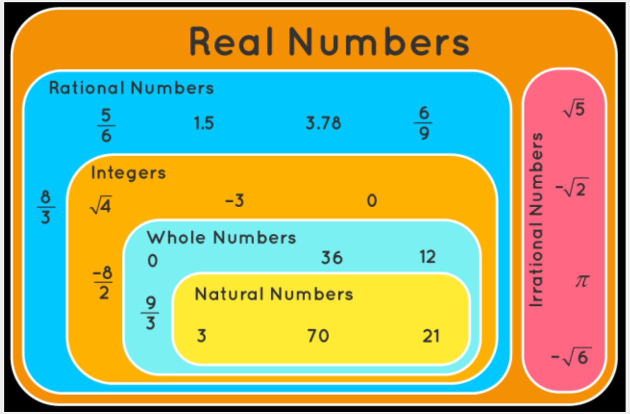
Definition

- Additivity:

For any n−vector x and y, $f(x+y) = f(x) + f(y)$

- Homogeneity:

For any n−vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$



Counterexample:

$f\left(a + \sqrt{5}\,b\right) \rightarrow a + b + \sqrt{5}\,b$

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

Inner product is Linear Function?

Theorem

A function defined as the inner product of its argument with some fixed vector is linear.

Proof?

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

What are Linear Functions?

Theorem

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof?

What are Linear Functions?

Theorem

The representation of a linear function f as $f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.

Proof?

Linear Form Examples

Example

- Is average a linear function?
- Is maximum a linear function?

Bilinear Form

Bilinear Form over a real vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \to \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and
   ii. $f(c\mathbf{v_1}, \mathbf{w}) = cf(\mathbf{v_1}, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is linear in its second argument:
   i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and
   ii. $f(\mathbf{v}, c\mathbf{w_1}) = cf(\mathbf{v}, \mathbf{w_1})$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Bilinear Form

Note

Let V be a vector space over a field $\mathbb{F}$. Then the **dual** of V, denoted by V^*, is the vector space consisting of all linear forms on V.

Example

Let V be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \rightarrow \mathbb{F}$ defined by

$$g(f, \mathbf{v}) = f(\mathbf{v}) \text{ for all } f \in V^*, \mathbf{v} \in V$$

is a bilinear form.

Positive Definite Bilinear Form

Definition

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V is called positive definite if for all $v \in V, v \neq 0$:

$$f(v, v) > 0$$

Example

Which one is a positive definite bilinear form?

- ❑ $f(x, y) = x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2$
- ❑ $f(x, y) = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 3x_2 y_2$

Symmetric Bilinear Form

Definition

A **bilinear form** function $f: V \times V \to \mathbb{F}$ over a real vector space V is called symmetric if for all $v, w \in V$:

$$f(v, w) = f(w, v)$$

Bilinear Form arises from a matrix

Theorem

Every **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V arises from a matrix for all $v, w \in V$:

$$f(v, w) = v^T A w$$

Proof?

Associated Matrices

Definition

If V is a finite-dimensional vector space, $B = \{b_1, \ldots, b_n\}$ is a basis of V, and $f: V \times V \to \mathbb{F}$ be a **bilinear form** function the associated matrix A of f with respect to B is the matrix $[f]_B \in \mathbb{F}^{n \times n}$ whose (i, j)-entry is the value $f(b_i, b_j)$.

$$f(v, w) = v^T A w = v^T [f]_B w$$

$$[f]_{\mathcal{B}} = \begin{pmatrix} f(b_1, b_1) & \ldots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \ldots & f(b_n, b_n) \end{pmatrix}$$

Note

The associated matrix changes if we use a different basis.

Example

For the bilinear form $f\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = 2ac + 4ad - bc$ on $\mathbb{F}^2$, find $[f]_B$ for basis $B = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}\}$ and $[f]_P$ for basis P= $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

Bilinear Form
Over Complex Vector Space

Bilinear Form over a real vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $f: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and
   ii. $f(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda f(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is conjugate linear in its second argument:
   i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and
   ii. $f(\mathbf{v}, \lambda \mathbf{w_1}) = \bar{\lambda} f(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Bilinear forms (real vs complex vector space)

Bilinear forms on $\mathbb{R}^n$	**Bilinear forms on $\mathbb{C}^n$**
Linear in the first variable	Linear in the first variable
Linear in the second variable	Conjugate linear in the second variable

Inner product

Inner product over real vector space

Definition

An inner product is a positive-definite symmetric bilinear form.

- An inner product on V is a function $\langle , \rangle : V \times V \ \rightarrow \ \mathbb{R}$ such that $v, w \in V, c \in \mathbb{R}$:

1. $\langle v, v \rangle = 0$ if and only if $v = 0$.
2. $\langle w, v \rangle = \langle v, w \rangle$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v \rangle \geq 0$ for all $v \in V$.

Why for bilinear form I wrote just two properties instead of four properties?

- Using properties (2) and (4) and again (2)

$$\langle w, cu\rangle = \langle cu, w\rangle = c\langle u, w\rangle = c\langle w, u\rangle$$

- Using properties (2), (3) and again (2)

$$\langle w, u+v\rangle = \langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = \langle w, u\rangle + \langle w, v\rangle$$

1. $\langle v, v\rangle = 0$ if and only if $v = 0$.
2. $\langle w, v\rangle = \langle v, w\rangle$.
3. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.
4. $\langle cw, u\rangle = c\langle w, u\rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v\rangle \geq 0$ for all $v \in V$.

Inner Products

Note

- For $v \in V, \langle 0, v \rangle = 0$, $\langle v, 0 \rangle$ =0.

General Inner product

Definition

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Then an **inner product** on V is a function $\langle \cdot,\cdot \rangle : V \times V \rightarrow \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$:

a) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

$b)$ $\langle \mathbf{v}+c\mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + c\langle \mathbf{x}, \mathbf{w} \rangle$ (linearity)

$c)$ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

Note

- The standard inner product between vectors is: $(x, y \in \mathbb{R}^n)$

$$\langle x, y\rangle = x^T y = \sum x_i y_i$$

- Euclidean inner product: The function $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ defined by

$$\langle \mathrm{v}, \mathrm{w}\rangle = v\mathrm{w}^* = \sum_{i=1}^{n} v_i \overline{w_i}$$

for all $\mathrm{v}, \mathrm{w} \in \mathbb{C}^n$ is an inner product on $\mathbb{C}^n$.

Inner Product for matrices

Note

- The standard inner product between two matrices is: $(X, Y \in \mathbb{R}^{m\times n})$

$$\langle X, Y\rangle = trace(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

Example

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Inner Product for functions

Note

- Let $a < b$ be real numbers and let $C[a,b]$ be the vector space of continuous functions on the real interval $[a,b]$. The function $\langle \cdot,\cdot \rangle : C[a,b] \times C[a,b] \to \mathbb{R}$ defined by

$$\langle f,g \rangle = \int_a^b f(x)g(x)\,dx \qquad \text{for all} \qquad f,g \in C[a,b]$$

is and inner product on $C[a,b]$.

Inner Product for polynomials

Note

- For $p(x)$ and $q(x)$ with at most degree n:

$$\langle p(x), q(x)\rangle = \mathrm{p}(0)\mathrm{q}(0) + \mathrm{p}(1)\mathrm{q}(1) + \cdots + \mathrm{p}(\mathrm{n})\mathrm{q}(\mathrm{n})$$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \mathrm{p}(0)\mathrm{q}(0) + \int_{-1}^{1} p'q'$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}dx$

Inner product space

Definition

An inner product space is a finite-dimensional real or complex vector space V along with an inner product on V.

Euclidean Space Unitary Space

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