# Change of basis

## Linear Algebra

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# Overview

- Basis Review
- Change of Basis
- Linear Transformation and Change of Basis

# Basis Review

## Example

- Find the coordinate vector of $2 + 7x + x^2 \in \mathcal{P}^2$ with respect to the basis B = $\{x + x^2, 1 + x^2, 1 + x\}$.

- If C = $\{1,\ x,\ x^2\}$ is the standard basis of $\mathcal{P}^2$ then we have $[2 + 7x + x^2]_C = (2, 7, 1)$.

# Solution

We want to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$2 + 7x + x^2 = c_1(x + x^2) + c_2(1 + x^2) + c_3(1 + x).$$

By matching coefficients of powers of x on the left-hand and right-hand sides above, we arrive at following system of linear equations:

$$\begin{aligned} c_2 + c_3 &= 2 \\ c_1 + c_3 &= 7 \\ c_1 + c_2 &= 1 \end{aligned}$$

This linear system has $c_1 = 3, c_2 = -2, c_3 = 4$ as its unique solution, so our desired coordinate vector is

$$[2 + 7x + x^2] = (c_1, c_2, c_3) = (3, -2, 4)$$

# Change of Basis

# Introduction to change of basis

- $B = \{v_1, \ldots, v_n\}$ are basis of $\mathbb{R}^n$.

- $\mathrm{P} = [v_1 \;\; v_2 \;\; \ldots \;\; v_n]$

- $P[a]_B = a$

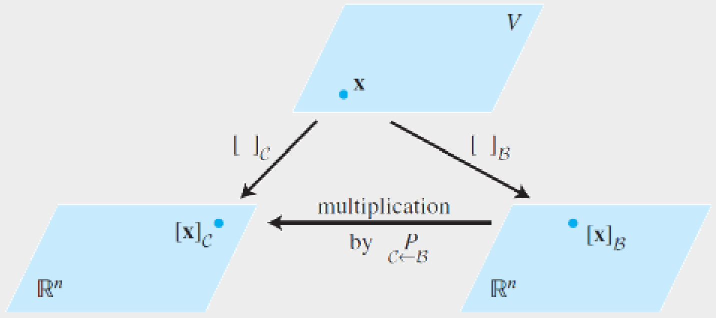
**Theorem**

Let B = $\{b_1, b_2, \dots, b_n\}$ and C = $\{c_1, c_2, \dots, c_n\}$ be basses of a vector space V. Then there is a unique n × n matrix $P_{C\leftarrow B}$ such that

$$[x]_C = P_{C\leftarrow B}[x]_B$$

The columns of $P_{C\leftarrow B}$ are the C−coordinate vectors of the vectors in basis B.
That is ,

$$P_{C\leftarrow B} = [[b_1]_C \quad [b_2]_C \quad \dots \quad [b_n]_C]$$




$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

## Example

Find the change-of-basis matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ for the bases

$$B = \{x + x^2, 1 + x^2, 1 + x\} \ and \ C = \{1, x, x^2\}$$

of $\mathcal{P}^2$. Then find the coordinate vector of $2 + 7x + x^2$ with respect to B.

# Change of basis

**Example**

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for $\mathbb{R}^2$ given by B = $\{b_1, b_2\}$, C = $\{c_1, c_2\}$.

a. Find the change-of-coordinates matrix from C to B.

b. Find the change-of-coordinates matrix from B to C.

# Change of basis

## Example

Find the change-of-basis matrix $P_{C \leftarrow B}$, where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \mathrm{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

# Linear Transformation and Change of Basis

# Linear Transformation

**Example**

We have B = $\{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $\mathcal{P}_3(x)$ and $\mathcal{P}_2(x)$, respectively. Find the matrix of transformation T: $P_3(x) \rightarrow P_2(x)$.

# Solution

Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathcal{P}^3(\mathrm{x})$, we have

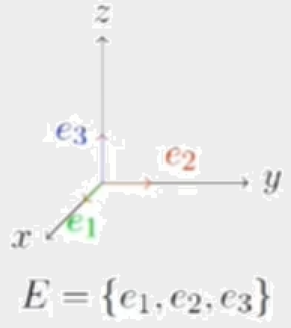
$$\left[\frac{d}{dt}\right]_{\{B,B'\}} = \left[\frac{d}{dt}(x^3) \quad \frac{d}{dt}(x^2) \quad \frac{d}{dt}(x) \quad \frac{d}{dt}(1)\right]$$
$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

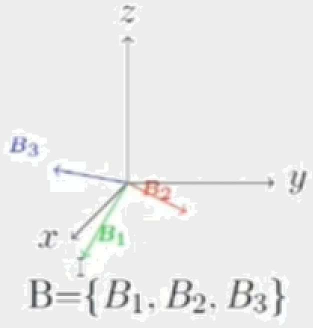
# Matrix representation of linear function

**Important**

Let T: $\mathbb{R}_n \rightarrow \mathbb{R}_m$ be a linear function and u = $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}_n$.

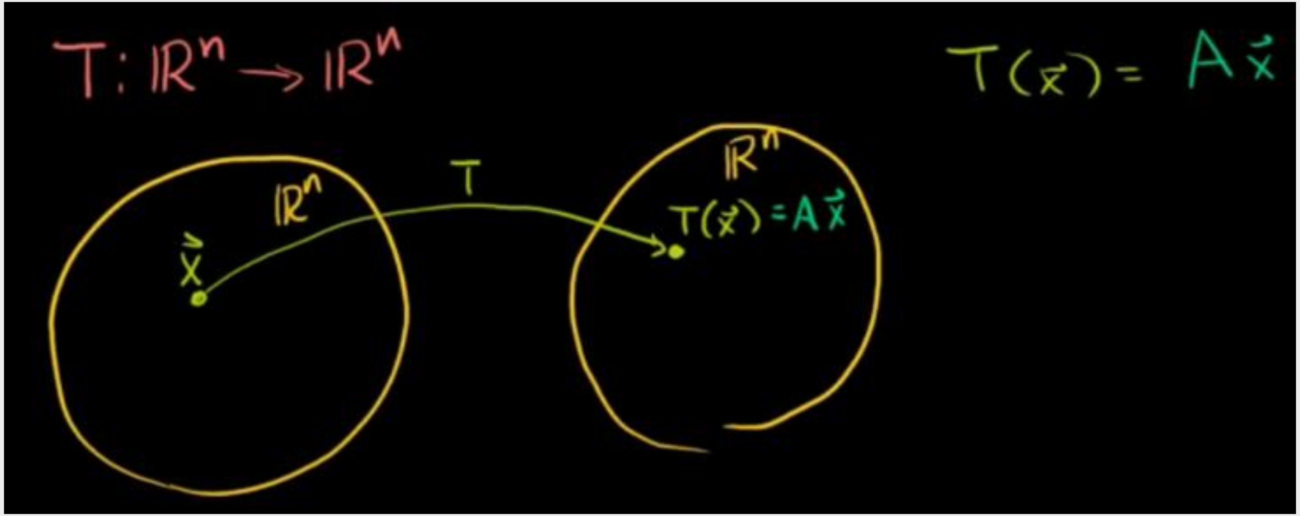
The matrix $[A(e_1) \quad \ldots \quad A(e_n)]$ is called the matrix representation of linear function (transformation)T which is denoted by $[A]_E$.




What is the relation between $[A]_B$ and $[A]_E$?

# Transformation with change of basis




- B = $\{v_1, v_2, \ldots, v_n\}$ *are basis of* $\mathbb{R}^n$.
- P = $[v_1 \quad v_2 \quad \ldots \quad v_n]$
- $[T(x)]_B = P^{-1}AP[x]_B$

# Change of basis

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "{[A]_B}"] \arrow[d, "P"'] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "{[A]_E}"'] & \mathbb{R}^n \arrow[u, "P^{-1}"']
\end{tikzcd}
$$

$$[A]_B = P^{-1}[A]_E P$$

# Isomorphisms

## Definition

Suppose V and W are vector spaces over the same field. We say that V and W are **isomorphic**, denoted by $V \cong W$, if there exists an invertible linear transformation T: $V \to W$(called an **isomorphism** from V to W).

- If T: $V \to W$ is an isomorphism then so is $T^{-1}: W \to V$.
- If T: $V \to W$ and S: $W \to X$ are isomorphism then so is $S \circ T: V \to X$.
  in particular, if $V \cong W$ and W $\cong X$ then $V \cong X$.

## Example

Show that the vector space V = span($e^x, xe^x, x^2e^x$) and $\mathbb{R}^3$ are isomorphic.

# Solution

The standard way to show that two space are isomorphic is to construct an isomorphism between them. To this end, consider the linear transformation T: $\mathbb{R}^3 \to V$ defined by

$$T(a,b,c) = ae^x + bxe^x + cx^2e^x.$$

It is straightforward to show that this function is linear transformation, so we just need to convince ourselves that it is invertible. We can construct the standard matrix $[T]_{B\leftarrow E}$, where $E = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$:

$$[T]_{B\leftarrow E} = \big[[T(1,0,0)]_B, [T(0,1,0)]_B, [T(0,0,1)]_B\big]$$

$$= \big[[e^x]_B, [xe^x]_B, [x^2e^x]_B\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $[T]_{B\leftarrow E}$ is clearly invertible (the identity matrix is its own inverse), T is invertible too and is thus an isomorphism.

# References


- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra David Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares