Affine Function

Linear Algebra

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Affine Function

Definition

A function $f: R^n \rightarrow R$ is affine if and only if it can expressed as $f(x) = a^T x + b$ (linear function plus a constant (**offset**))

- Superposition property for affine function which is called restricted superposition

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\,,\;\; \alpha + \beta = 1$$

Affine Function

Theorem

Any scalar–valued function that satisfies the restricted superposition property is affine.

Conclusion

Every affine function can be written as $f(x) = a^T x + b$ with:

$$a^T = [f(e_1) - f(0), f(e_2) - f(0), \dots, f(e_n) - f(0)]$$
$$b = f(0)$$

Conclusion

We can write linear and affine functions in two methods:

- **Method 1:**
  - Linear:

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n),\ \forall \alpha_1, \ldots, \alpha_n$$

  - Affine:

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n),\ \alpha_1 + \cdots + \alpha_n = 1$$

- **Method 2:**
  - Linear:

$$f(x) = a^T x$$

  - Affine:

$$f(x) = a^T x + b$$

Definition

In many applications, scalar-valued functions of n variables, or relations between n variables and a scalar one, can be approximated as linear or affine functions, which is called "**Model**".

Scalar-valued function of a scalar

- **Derivative** of function $f: R \rightarrow R$ at the point $(z, f(z))$:

$$\lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{t}$$

  - It gives the slope of the graph of f at the point $(z, f(z))$.
  - $f'(z)$ is a scalar-valued function of a scalar variable

Review: Scalar-valued function of a vector

- The partial derivative of function $f: R^n \to R$ at the point z, with respect to its ith argument

$$\frac{\partial f}{\partial x_i}(z) = \lim_{t \to 0} \frac{f(z_1, \dots, z_{i-1}, z_i + t, z_{i+1}, \dots, z_n) - f(z)}{t} = \lim_{t \to 0} \frac{f(z + te_i) - f(z)}{t}$$

- The partial derivative is the derivative with respect to the i –th argument, with all other arguments fixed.

- **Gradient**: The partial derivatives of $f(x)$ with respect to its n arguments can be collected into an n vector called the gradient of $f(x)$ (at point z):

$$\nabla f(z) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(z) \\ \vdots \\ \frac{\partial f}{\partial x_n}(z) \end{bmatrix}$$

Theorem

Gradient of a combination of functions:

$$f(z) = ag(z) + bh(z)$$

$$\nabla f(z) = a\nabla g(z) + b\nabla h(z)$$

How to find an approximate affine model

- $f: R^n \rightarrow R$ is differentiable: its partial derivatives exist

Definition

The (first-order) Taylor approximation of f near (or at) the point z:

$$\hat{f}(x) = f(z) + \frac{\partial f}{\partial x_1}(z)(x_1 - z_1) + \cdots + \frac{\partial f}{\partial x_n}(z)(x_n - z_n)$$

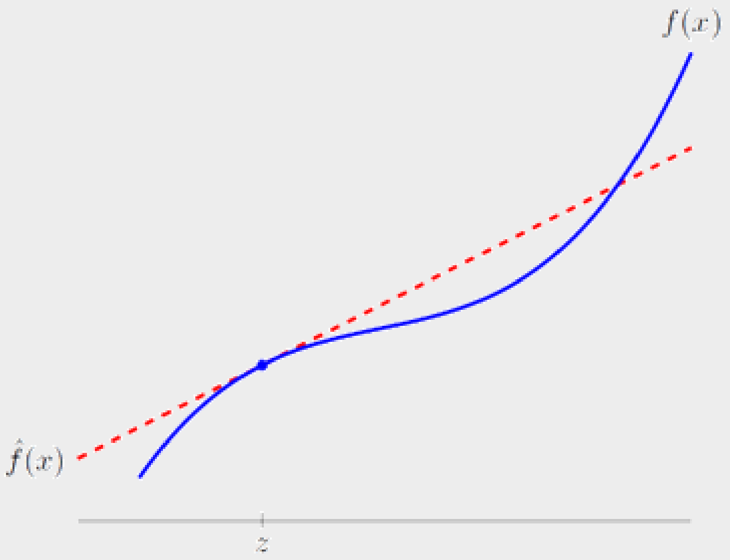
Example

- $\hat{f}(x)$ is a linear function or a affine function?

$$\hat{f}(x) = \underbrace{f(z)}_{\text{Constant– value of function at z}} + \underbrace{\nabla f(z)^T (x - z)}_{\text{Deviation or Perturbation of x from z}}$$

$$\hat{f}(x) = \underbrace{\nabla f(z)^T x}_{\text{Linear function}} + \underbrace{(f(z) - \nabla f(z)^T z)}_{\text{Constant}}$$

- The Taylor approximation is sometimes called the linear approximation or linearized approximation of f (at z)



A function f of one variable, and the first order Taylor approximation $\hat{f}(x) = f(z) + f'(z)(x - z)$ at z