Matrix Algebra: Dimension and Rank

CE282: Linear Algebra

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Review: Dimension

Definition

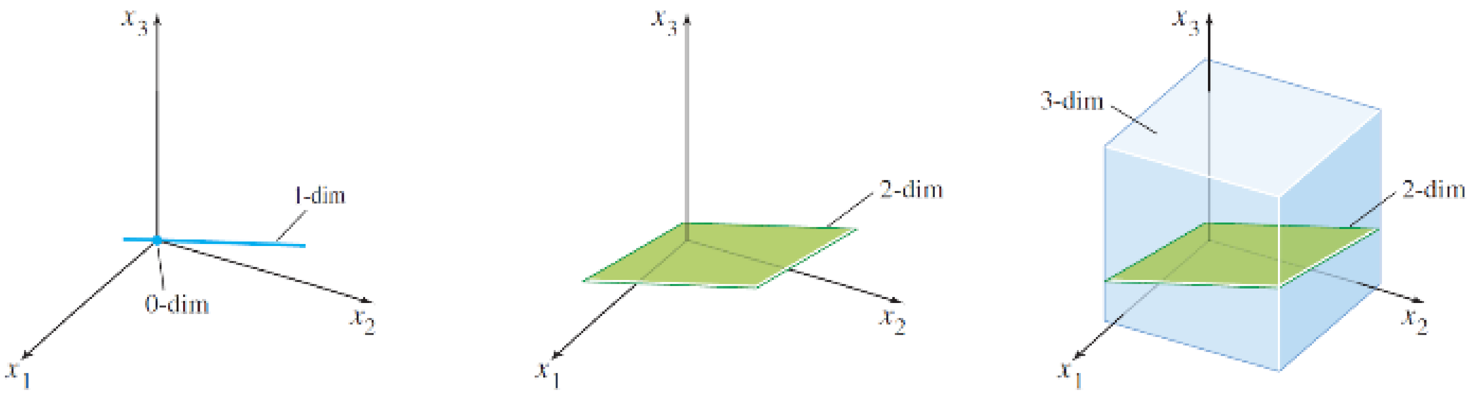
If V has a finite basis, then $\dim(V)$ is the number of elements (vectors) of any basis of V

❑ Note: $Dim(\{0\}) = 0$

Note

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim(V)$**,** is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ *is defined to be zero.* If V is not spanned by a finite set, then V is said to be **infinite-dimensional.**

Review: Dimension



Extra Resource!

To see some good discussions about “infinite-dimensional” vector spaces and some examples of it, see here!

Finite-Dimensional Space

Theorem

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and:

$$\dim(H) \leq \dim(V)$$

Theorem (The Basis Theorem)

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that span V is automatically a basis for V.

Row and Column Space

Theorem

If two matrices $A \text{ and } B$ are row-equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as for that of B.

Theorem

The pivot columns of a matrix A form a basis for $Col(A)$

Row and Column Space

Example

Find:

- ❑ Row Basis
- ❑ Column Basis
- ❑ $\dim(Row(A))$
- ❑ $\dim(Col(A))$
- ❑ $\dim(Null(A))$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = pivot columns

Rank of Matrix

Definition

- The number of linearly independent rows or columns in the matrix
- Dimension of the row (column) space
- Number of nonzero rows of the matrix in row echelon form (Ref)

Theorem 1

$Row\ rank\ =\ Column\ rank$ for a matrix in reduced row echelon form.

Theorem 2

The dimension of the Column Space of A and $rref(A)$ is the same.

Rank of a Matrix

Theorem (RMRT)

(Rank of a matrix is equal to the rank of its transpose)
Suppose A is an $m \times n$ matrix. Then $rank(A) = rank\,(A^T)$

Definition

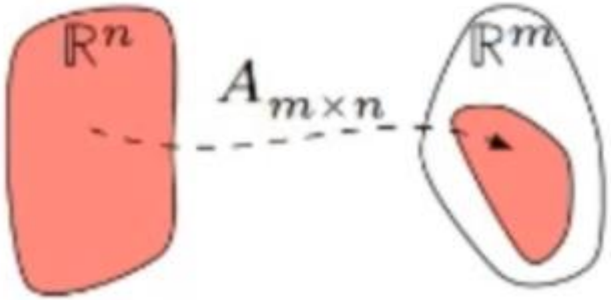
- For $A_{m\times n} = \left[a^1\ a^2\ \ \dots a^n\right] = [a_1\ a_2\ \ \dots\ a_n]$:

$$Range(A) = Span(a_1, a_2, \dots, a_n)\ = \{y \mid y = \alpha_1 a_1 + \alpha_2 a_2 + \ \dots + \alpha_n a_n,\ \ \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$$
$$= \{y \mid y = Ax, x \in \mathbb{R}^n\}$$

- Range is a Vector Space
- Range of A is a subspace of $\mathbb{R}^m$
- Is dim(A) = m ?

$$\dim(Range(A))\ =\ ColRank(A)$$

number of linearly independent columns



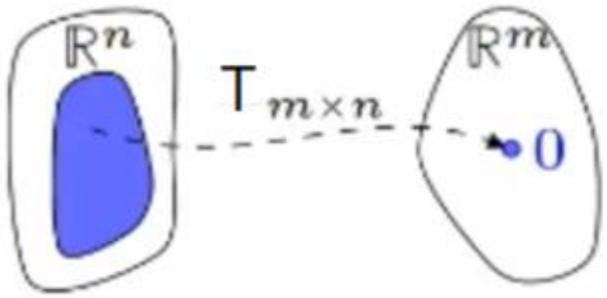
Null Space

Definition

Let $T: V \to W$ be a linear map. Then the null space or kernel of T is the set of all vectors in V that map to zero:

$$N(T) = Null(T) = \{v \in V \mid Tv = 0\}$$

- Null Space is a Vector Space
- Null Space T is a sub-space of V $(\mathbb{R}^n)$
- Is $Dim(Null(T)) = 0$?
- $Nullity(T) := Dim(Null(T))$



Question

What is the null space for differentiation mapping?

Note

$$Nullity(A) \ = \ number\ of\ free\ variables$$

Example 1

If Columns of matrix A are linearly independent:

$$\begin{aligned} nullity(A) &= ? \\ col(rank(A)) &= ? \end{aligned}$$

Example 2

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$nullity(A) = 2, colRank(A) = 2$$

Conclusion

Theorem

The vectors attached to the free variables in the parametric vector form of the solution set of $Ax = 0$ form a basis of $Null(A)$

Important

Let A be an $n \times n$ matrix. Then the following statements are equivalent to the statement that A is an invertible matrix:

- The columns of A form a basis of $\mathbb{R}^n$.
- $Col(A) = \mathbb{R}^n$
- $Dim(Col(A)) = n$
- $Rank(A) = n$
- $Null\ (A) = \{0\}$
- $Dim\ (Null(A)) = 0$

Conclusion

Note

The dimension of $Null(A)$ is the number of free variables in the equation $Ax = 0$, and the dimension of $Col(A)$ is the number of pivot columns in A.

Example

Find the dimension of the Null Space and the Column Space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(row reduce the Augmented Matrix $[A\ 0]$ to echelon form)

Rank-Nullity Theorem

Theorem

- $Nullity(A) + ColRank(A) = n$
- $Dim(Null(A)) + Dim(Range(A)) = n$

$$\{\, number\ of\ pivot\ columns\} + \{number\ of\ non-pivot\ columns\} = \{number\ of\ columns\}$$

Proof?

Rank Theorem

Theorem

- ❑ $ColRank(A) \; = \; RowRank(A)$
- ❑ In general, It’s called rank of matrix ($rank(A)$)

Proof?

Rank Properties

Important

- $ColRank(A_{m \times n} \leq \min(m,n)$
- $RowRank(A_{m \times n} \leq \min(m,n)$
- $Dim(Range(A)) = Rank(A)$
- $Nullity(A) + Rank(A) = n$
- $Rank(A) \leq \min(m,n)$

Rank Properties

Important

- For $A, B \in \mathbb{R}^{m \times n}$:
  1. $rank(A) \leq \min(m, n)$
  2. $rank(A) = rank(A^T)$
  3. $rank(AB) \leq \min(rank(A), rank(B))$
  4. $rank(A + B) \leq rank(A) + rank(B)$
- A has full rank if $rank(A) = \min(m, n)$
- If $rank(A) < m$, rows are not linearly independent (same for columns if $rank(A) < n$)

Important

- The Range or Column Space of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $\mathcal{R}(A)$, is the span of the columns of A. in other words:

$$\mathcal{R}(A) = \{v \in \mathbb{R}^m : v = Ax \,, x \in \mathbb{R}^n\}.$$

- Assuming A is full rank and $n < m$, the projection of a vector $y \in \mathbb{R}^m$ onto the range of A is given by:

$$Proj(y; A) = armin_{v \in \mathcal{R}(A)} ||v - y||_2 = A\left(A^T A\right)^{-1} A^T y$$

- When A contains only a single column, $a \in \mathbb{R}^m$, this gives the special case for a projection of a vector to a line:

$$Proj(y; a) = \frac{aa^T}{a^T a} y$$