Echelon Forms and Row Reduction

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

- Row-Reduced Matrix
- Echelon form
- Row-Reduced Echelon Form
- Solutions of a Linear System

Row-Reduced Matrix

Row-Reduced Matrix

Definition

A leading entry of a row refers to **the leftmost** nonzero entry in a nonzero row.

Definition

- A $m \times n$ matrix R is called row-reduced if:
  1. Leading entries=1: The first non-zero entry in each non-zero row of R is equal to 1.
  2. Each column of R which contains the leading non-zero entry of some row has all its other entries 0.

Row-Reduced Matrix

Example

- Are following matrices Row-Reduced Matrix?
  - $a.$ $n \times n$ identity matrix
  - $b.$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

  - $a.$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

  - $b.$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

Row-Reduced matrix for Every Matrix

Theorem

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

Echelon Form

Echelon form

Definition

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

$$\underbrace{\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}}_{\text{Echelon form}}$$

Row-Reduced Echelon Form

Definition

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
  1. The leading entry in each non-zero row is 1.
  2. Each leading 1 is the only non-zero entry in its columns.
  3. The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
  4. Any row containing only 0's is at the bottom.

$$\underbrace{\begin{array}{cccc} e_1 & e_2 & e_3 & \\ \multicolumn{4}{c}{\begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \begin{matrix}29\\16\\3\end{matrix} \end{bmatrix}} \end{array}}_{\text{Reduced Echelon form}}$$

Row-Reduced echelon matrix for Every Matrix

Theorem

Every $m \times n$ matrix over the field F is row-equivalent to a row-reduced echelon matrix.

Example

- Are following matrices RREF?

$$a.\quad 0_{m\times n}$$

$$b.\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c.\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$d.\quad \begin{bmatrix} 0 & 1 & -3 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

- Consider the system of three equations in four unknowns represented by the augmented matrix, find RREF:

$$\begin{bmatrix} -1 & 2 & 6 & 7 & 15 \\ 3 & -6 & 0 & -3 & -9 \\ 1 & 0 & 6 & -1 & 5 \end{bmatrix}$$

Existence and Uniqueness Questions

Two fundamental questions about a linear system:

1. Is the system consistent? That is, does at least one solution exist?
2. If a solution exists, is it the only one? That is, is the solution unique?

Solutions of a Linear System

Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x - 5z &= 1 \\ y + z &= 4 \\ 0 &= 0 \end{aligned} \qquad \begin{cases} x = 1 + 5z \\ y = 4 - z \\ z \text{ is free variable} \end{cases}$$

- x, y: basic variable $\quad z$: free variable
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

Existence and Uniqueness Questions

Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with nonzero b.

- If a linear system is consistent, then the solution set contains either:
  - A unique solution, when there are no free variables
  - Infinitely many solutions, when there is at least one free variable

Find all solutions of a linear system

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3?

Solution

Row reduce the augmented matrix for $Ax = b$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)$. The equation $Ax = b$ is not consistent for every b because some choices of b can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Existence of solutions

Example

True or False?

Equation $Ax = b$ is consistent, if its augmented matrix $[A\ b]$ has one pivot column in each rows? (Having one leading entry in each rows)

Definition

- A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where A is a matrix and 0 is the zero vector.
- Trivial solution: $Ax = 0$ always has at least one solution, namely, $x = 0$ (the zero vector)
- Nontrivial solution: The non-zero solution for $Ax = 0$.

Fact

The homogenous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Existence Of Solutions

Theorem

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $Ax = 0$ has a non-trivial solution.

Homogenous system

Theorem

If A and B are row-equivalent $m \times n$ matrices, the homogenous systems of linear equations $Ax = 0$ and B$x = 0$ have exactly the same solutions.

Proof:

Theorem

If A is an $n \times n$ square matrix, then A is row-equivalent to the $n \times n$ identity matrix if and only if the system of equations $Ax = 0$ has only the trivial solution.

Existence Of Solutions

Fact

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

The line ℓ with equation $2x + y = 0$

$\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then the equation becomes $\mathbf{n} \cdot \mathbf{x} = 0$.

ℓ as $\mathbf{x} = t\mathbf{d}$.

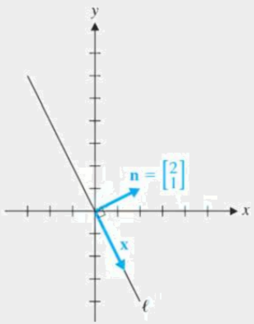
Definition The *normal form of the equation of a line* ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

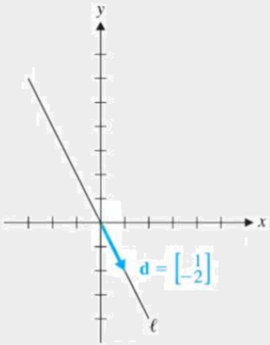
where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ.

The *general form of the equation of* ℓ is $ax + by = c$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for ℓ.

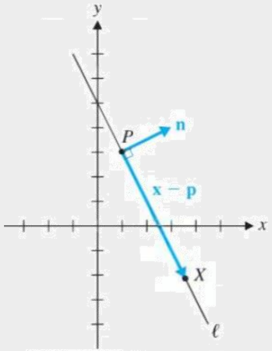
The line $2x + y = 0$



A normal vector $\mathbf{n}$



A direction vector $\mathbf{d}$

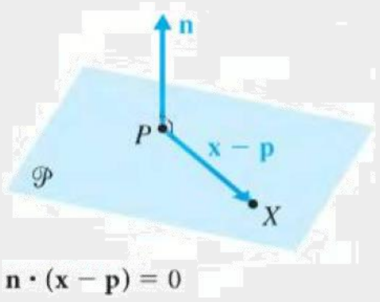


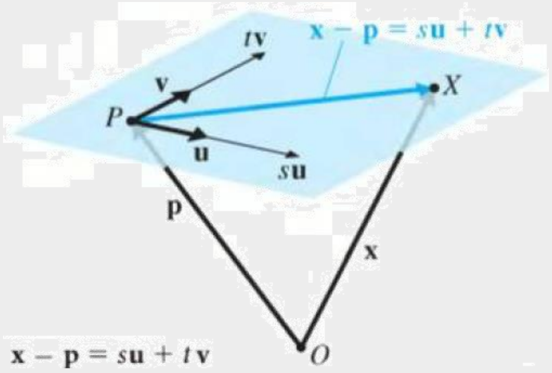
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

Lines in $\mathbb{R}^2$			
Normal Form	**General Form**	**Vector Form**	**Parametric Form**
$\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$	$ax + by = c$	$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$ax + by + cz = d \;(\text{where } d = \mathbf{n} \cdot \mathbf{p})$$





Lines and Planes in $\mathbb{R}^3$				
	Normal Form	**General Form**	**Vector Form**	**Parametric Form**
Lines	$\begin{cases} \boldsymbol{n_1} \cdot \boldsymbol{x} = \boldsymbol{n_1} \cdot \boldsymbol{p_1} \\ \boldsymbol{n_2} \cdot \boldsymbol{x} = \boldsymbol{n_2} \cdot \boldsymbol{p_2} \end{cases}$	$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$	$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$
Planes	$\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$	$ax + by + cz = d$	$\boldsymbol{x} = \boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{v}$	$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

Nonhomogeneous Systems & General Solution

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example

Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$, where: $A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $[A\ b] = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Question

Can we change the order of columns in an augmented matrix???

$$\begin{cases} ax & + & by & + & cz & = & d \\ a'x & + & b'y & + & c'z & = & d' \\ a''x & + & b''y & + & c''z & = & d'' \end{cases}$$

Is equivalent to

$$\begin{cases} ax & + & cz & + & by & = & d \\ a'x & + & c'z & + & b'y & = & d' \\ a''x & + & c''z & + & b''y & = & d'' \end{cases}$$

Conclusion

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
b. Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Note

If A does not have a pivot in every row, that does not mean that $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution for some given vector $\boldsymbol{b}$. It just means that there are some vectors $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution.

Example

Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where: $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

Here A is the matrix of coefficients. Row Operations on $[A \quad |b]$ produce:

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$ and x_3 is free. As a vector, the general solution of $A\boldsymbol{x} = \boldsymbol{b}$ has the form:

General Solution written in vector form

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{p} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{v}$$

The equation $\boldsymbol{x} = \boldsymbol{p} + x_3\boldsymbol{v}$, or, writing t as a general parameter,

$$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v} \text{ (t in } \mathbb{R})$$

Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3? If not, describe under which circumstances, this system of equation can be consistent ?!

Example

Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$, where: $A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 - 8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

Thus $x_1 = 8x_3 + 7x_4$, $x_2 = -4x_3 - 3x_4$ and x_3, x_4 are free.
As a vector, the general solution of $A\boldsymbol{x} = \mathbf{0}$ has the form:

General Solution written in vector form

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

References

- Chapter 1: Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.
- Chapter 1: David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016
- Chapter 2: David Poole, Linear Algebra: A Modern Introduction. Cengage Learning, 2014.
- Chaper1: Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016