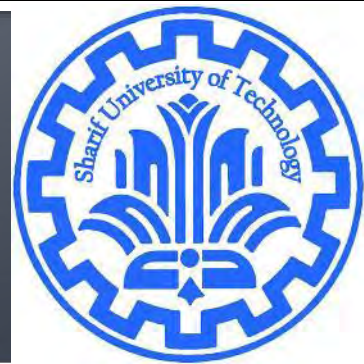


Introduction

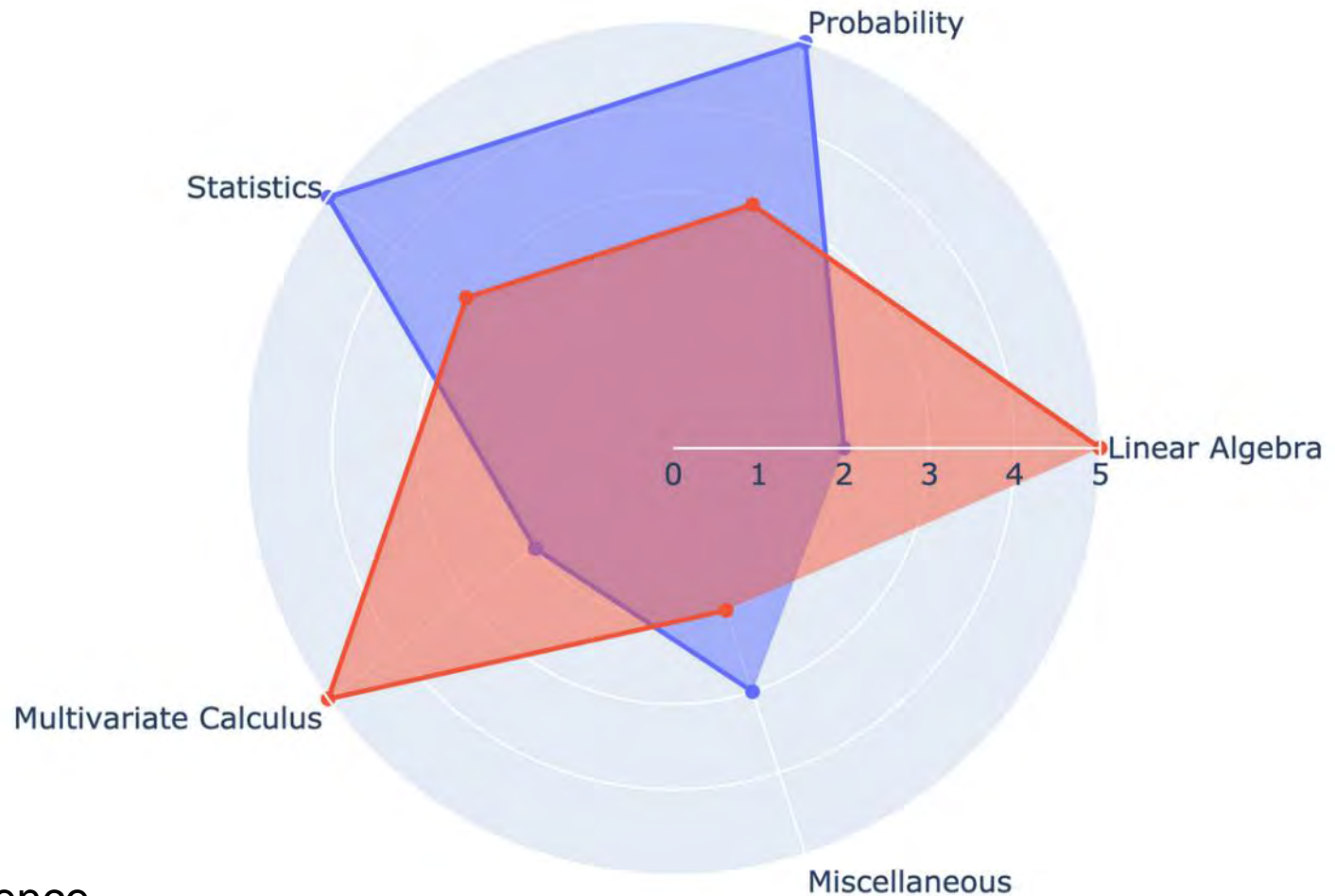
CE40282-1: Linear Algebra
Hamid R. Rabiee and Maryam Ramezani
Sharif University of Technology



Why are we here?

- Linear algebra is a main underlying tool for:
 - Computer Science & Engineering Applications
 - Machine Learning
 - Robotics
 - Operations Research
 - Theoretical Computer Science
- You will learn the basic concepts of Linear Algebra and how they apply to various aspects of computer science.

Why are we here?



Blue: Data Science
Orange: Machine Learning

Reach Us!

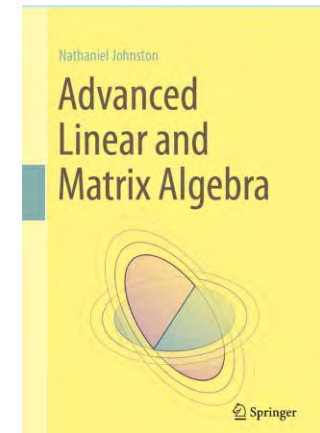
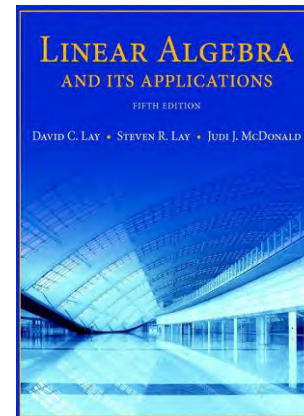
- Office Room: 803-804
- Email:
 - Hamid R. Rabiee: (rabiee@sharif.edu)
 - Maryam Ramezani: (maryam.ramezani@sharif.edu)
- Course notes, homework and solutions, handouts, and other useful resources are available on the Quera page:
 - <https://quera.org/course/13072/>
 - Room: (Sunday & Tuesday: 10:30-12:00)
 - Amozesh 205
 - <https://vc.sharif.edu/ch/rabiee>
- Lead TA:
 - Aryan Ahadinia: (aryan.ahadinia@sharif.edu)
- Feedback
 - <https://forms.gle/ajiQ9v4XxxUEXepQ6>

Resources

■ Textbooks:

- David C. Lay. Linear Algebra and Its Applications. University of Maryland, 2016.
- Stephen Boyd and Lieven Vandenberghe. Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares. Cambridge University Press, 2018.
- Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.
- Gilbert Strang. Linear Algebra and Learning From Data. Wellesley-Cambridge Press, 2019.
- Nathaniel Johnston, Advanced Linear and Matrix Algebra, Springer Nature, 2021.

+Other textbooks and course materials.



Structure of the Course

■ Lectures

- **Goal:** To introduce concepts in linear algebra, and motivate their use and importance.
- **Note:** We try to cover useful materials in class, but we recommend you reading more!

■ Assignments

- **Purpose:** To give you a chance to exercise your mind, and to solidify the concepts introduced to you in class.
- **Structure:** Six theoretical problems, and one linear algebra practical problems.
- **Importance:** Not important unless you want to learn the material and get a good grade.

■ Project (Application)

■ Exams: Midterm + Final

Lecture Notes

- Lecture slide will be uploaded.
- Many times we will write on board, in real-time, during lecture to prove a theory or answer a question or add some additional explanations. It will be your responsibility to take notes.
- Slides links will be provided on Quera.

Homework

- Six series
 - 6 theory questions
 - 1 practical question
 - Basic Python for programming
 - Basic NumPy for array manipulation
 - Basic programming with Pytorch
 - Learning linear algebra and application of machine learning.
- Assignments will be released on Tuesday every two weeks. Students will have 12 days for the theoretical and 14 days for the practical questions to submit answers in Quera
- Homework Upload:
 - <https://quera.org/course/13072/>

Homework

Assignment	Release	Submission	Code Submission	Solution Release
1	1401/11/25	1401/12/06	1401/12/08	1401/12/14
2	1401/12/09	1401/12/20	1401/12/22	1401/12/28
3	1402/01/15	1402/01/26	1402/01/28	1402/02/03
4	1402/01/29	1402/02/09	1402/02/11	1402/02/17
5	1402/02/12	1402/02/30	1402/03/01	1402/03/07
6	1402/03/02	1402/03/13	–	1402/03/19
Final Project	1402/03/02	–	1402/04/09	–

Important: Note that this schedule is tentative and may be affected by unforeseen circumstances.

TA

- TA class: (Tuesdays 12:00 – 13:15)
 - CE??
 - <https://vc.sharif.edu/ch/rabiee-ta>

Date	Title
1401/11/18	Python
1401/11/25	TA Problem Solving
1401/12/02	Python / NumPy
1401/12/09	TA Problem Solving
1401/12/16	NumPy
1401/12/23	TA Problem Solving
1402/01/15	PyTorch (Part 1)
1402/01/22	TA Problem Solving
1402/01/29	<i>No Class</i>
1402/02/05	TA Problem Solving
1402/02/12	<i>No Class</i>
1402/02/19	Exam Exclusive
1402/02/26	PyTorch (Part 2)
1402/03/02	TA Problem Solving
1402/03/09	<i>No Class</i>
1402/03/16	Exam Exclusive

Important: Note that this schedule is tentative and may be affected by unforeseen circumstances.

Grading Scheme

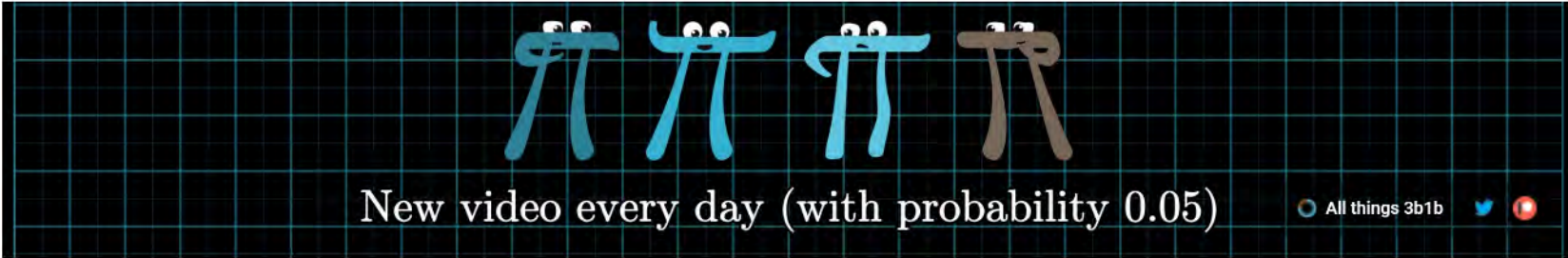
Item	Grade
Homework	50% (10 points)
Final Project	5% (1 point)
Midterm Exam	25% (5 points)
Final Exam	30% (6 points) (Comprehensive)
Class Homework	?
Total	22+?




- Mid-Term Exam: 25% (1402/02/21 9:00 AM)
- Final Exam: 30% (Comprehensive) (1402/02/23 9:00 AM)


Homework Rules

- Homework and Project: 50%
 - Homework Assignments: 10 points, first 5 series with 1.4 points for theoretical questions and 0.32 point for practical questions, last series with 1.40 for theoretical questions
 - Final Project: 1 point
 - You have a total 15 days of allowed late submission (for both theoretical and code part). 0.5% of the assignment grade will be subtracted for each hour of delay
- For each homework, you can use your late submission up to 5 days. No submissions are accepted after TA class.
- Discussing the problems with each other is encouraged. Copying each others assignments or submitting solutions/answers on the web is strictly prohibited. First time, you will receive -100% of grade for the task at hand. If you are caught for a second time, you will fail the course.



Some Useful Links



New video every day (with probability 0.05)  All things 3b1b  



3Blue1Brown ✓
3.94M subscribers

[HOME](#) [VIDEOS](#) [PLAYLISTS](#) [COMMUNITY](#) [CHANNELS](#) [ABOUT](#)  

Description

3Blue1Brown, by Grant Sanderson, is some combination of math and entertainment, depending on your disposition. The goal is for explanations to be driven by animations and for difficult problems to be made simple with changes in perspective.

For more information, other projects, FAQs, and inquiries see the website:
<https://www.3blue1brown.com>

Stats

Joined Mar 4, 2015

217,063,334 views

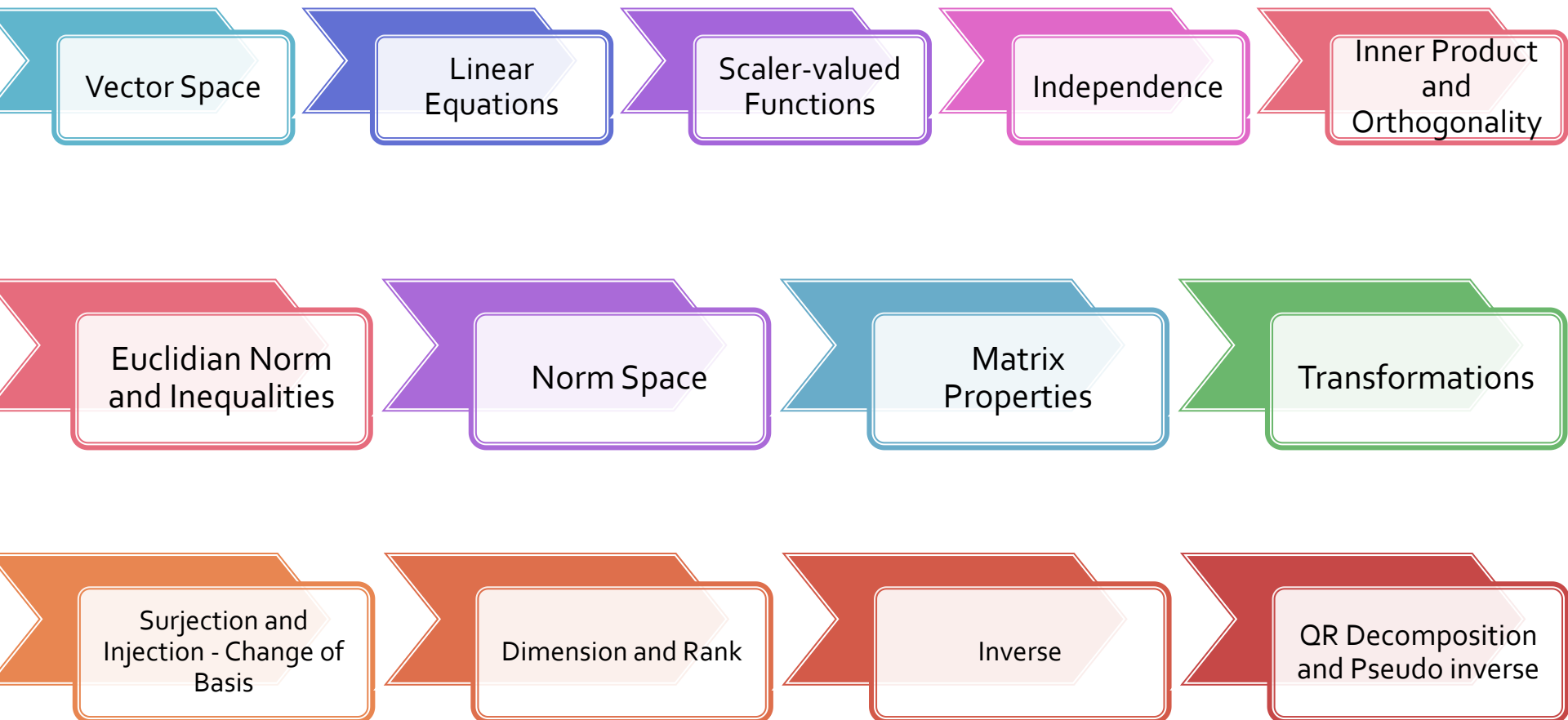
Details

Location: [United States](#)

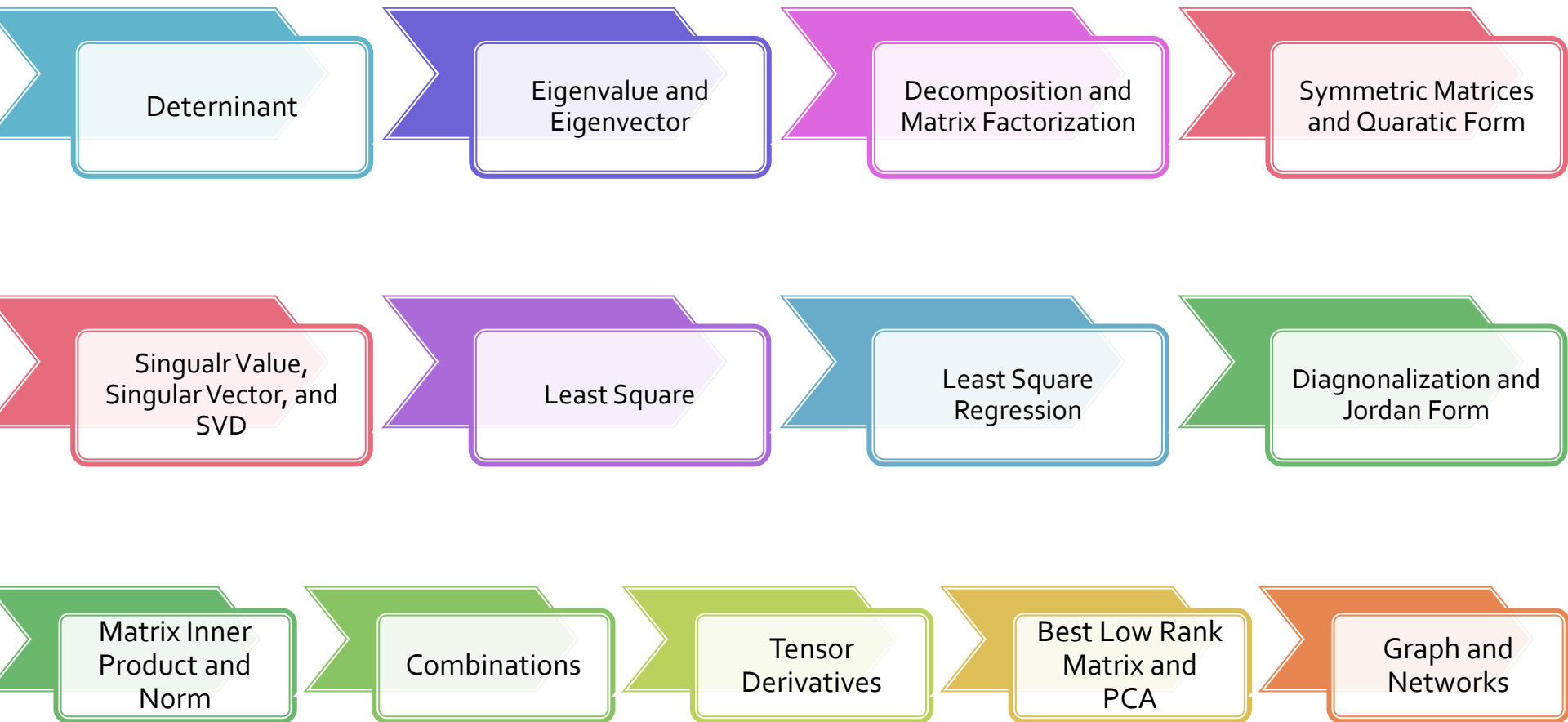
What we'll cover

Date	Title	Instructor	Description	Assignments
1401/11/16	S01	Ramezani	Introduction	HW1
1401/11/18	S02	Ramezani	Vector Space (Introduction and Operations)	
1401/11/23	S03	Ramezani	Vector Space (Field and subspace)	
1401/11/25	S04	Ramezani	Linear Equations	
1401/11/30	S05	Ramezani	Scalar-valued Functions (Linear and Affine)	HW2
1401/12/02	S06	Ramezani	Independence (Linear and Affine)	
1401/12/07	S07	Ramezani	Inner Product and Orthogonality	
1401/12/09	S08	Ramezani	Euclidian Norm and Inequalities	
1401/12/14	S09	Ramezani	Norm Space	
1401/12/16	S10	Ramezani	Matrix Properties	HW3
1401/12/21	S11	Ramezani	Transformations	
1401/12/23	S12	Ramezani	Surjection and Injection - Change of Basis	
1401/12/25	S13	Ramezani	Dimension and Rank (Online)	
1402/01/15	S14	Ramezani	Inverse	HW4
1402/01/20	S15	Ramezani	QR Decomposition and Pseudo inverse	
1402/01/22	S16	Ramezani	Determinant	
1402/01/27	S17	Ramezani	Eigenvalue and Eigenvector	
1402/01/29	S18	Ramezani	Decomposition and Matrix Factorization	HW5
1402/02/03	National Holiday		Eid Al-Fitr	
1402/02/05	S19	Ramezani	Symmetric Matrices and Quaratic Form	
1402/02/10	S20	Ramezani	Singular Value, Singular Vector, and SVD	
1402/02/12	S21	Ramezani	Least Square	
1402/02/17	S22	Rabiee	Least Square Regression	HW6
1402/02/19	S23	Rabiee	Diagonalization and Jordan Form	
1402/02/21	Midterm Exam		9:00 AM	
1402/02/24	S24	Rabiee	Matrix Inner Product and Norm	
1402/02/26	National Holiday		Martyrdom of Imam Jafar Sadiq	HW6
1402/02/31	Survey Start		Educational Calendar	
1402/02/31	S25	Rabiee	Combinations	
1402/03/02	S26	Ramezani	Tensor Derivatives	
1402/03/07	S27	Ramezani	Best Low Rank Matrix and PCA	
1402/03/08	Withdraw Deadline		Educational Calendar	
1402/03/09	S28	Ramezani	Graph and Networks	
1402/03/16	Makeup Session	N/A	Makeup Session	
1402/03/23	Final Exam		9:00 AM, Comprehensive	
1402/04/15	Grade Submission		Educational Calendar	

Course Roadmap



Course Roadmap



Some (Interesting) and Important Applications

- [Coding theory](#)
- [Cryptography](#)
- [Tomography](#)
- [Vector graphics](#)
- [Games Theory](#)
- [Quantum computing](#)
- [Economics](#)
- [Chemistry](#)
- [Genetics](#)
- [Image processing](#)
- [Markov chains](#)
- [Linear Regression](#)

What is linear algebra?

- Linear algebra is a branch of mathematics that studies vector spaces, also called linear spaces, along with linear functions that input one vector and output another.
- It has extensive applications in engineering, physics, natural sciences, computer science, and the social sciences (particularly in economics).
- Nonlinear mathematical models can often be approximated by linear ones.
- Linear algebra is a branch of mathematics, but the truth of it is that linear algebra is the mathematics of data.
- Matrices and vectors are the language of data

What is linear algebra?

- The study of linear algebra and matrices has a long history. It first emerged from determinants, which were used to solve systems of linear equations.
- Linear algebra is the study of lines and planes, vector spaces and mappings that are required for linear transforms.

$$\begin{aligned}y_1 &= 0.1 \times x_1 + 0.4 \times x_2 \\y_2 &= 0.3 \times x_1 + 0.9 \times x_2 \\y_3 &= 0.2 \times x_1 + 0.3 \times x_2 \\&\vdots\end{aligned}$$



$$y = A \cdot b$$

Numerical Linear Algebra

- The application of linear algebra in computers is often called **numerical linear algebra**
- Computers are good at performing linear algebra calculations, and much of the dependence on Graphical Processing Units (GPUs) by modern machine learning methods such as deep learning is because of their ability to compute linear algebra operations fast.

Numerical Linear Algebra and Statistics

- The application of linear algebra in computers is often called **numerical linear algebra**: Graphical Processing Units (GPUs)
- Some clear fingerprints of linear algebra on **statistics** and statistical methods include:
 - Use of vector and matrix notation, especially with multivariate statistics.
 - Solutions to least squares and weighted least squares, such as for linear regression.
 - Estimates of mean and variance of data matrices.
 - The covariance matrix that plays a key role in multinomial Gaussian distributions.
 - Principal component analysis for data reduction that draws many of these elements together.

Applications of Linear Algebra

- Matrices in Engineering
- Graphs and Networks, such as analyzing networks
- Markov Matrices, Population, and Economics, such as population growth
- Linear Programming, the simplex optimization method
- Fourier Series: Linear Algebra for functions, used widely in signal processing
- Linear Algebra for statistics and probability, such as least squares for regression
- Computer Graphics, such as the various translation, rescaling and rotation of images

Linear Algebra in Machine Learning

- Images and Photographs
 - Each image that you work with is itself a table structure with a width and height and one pixel value in each cell for black and white images or 3 pixel values in each cell for a color image.
 - A photo is yet another example of a matrix from linear algebra.
 - Operations on the image, such as cropping, scaling, shearing, and so on are all described using the notation and operations of linear algebra.

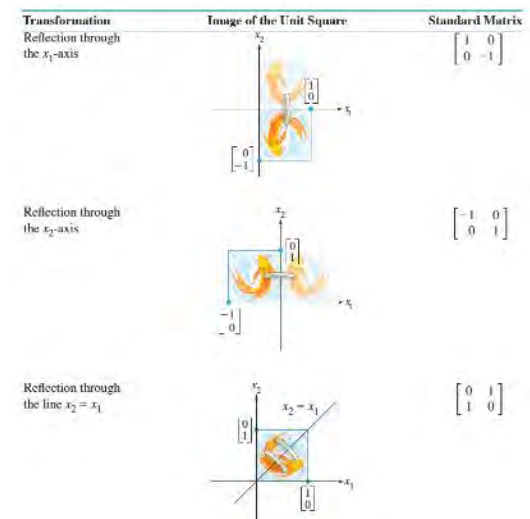
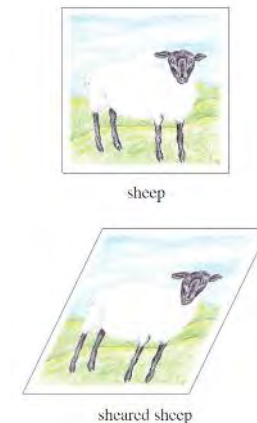


Image Processing

- Image Rotation



(a)



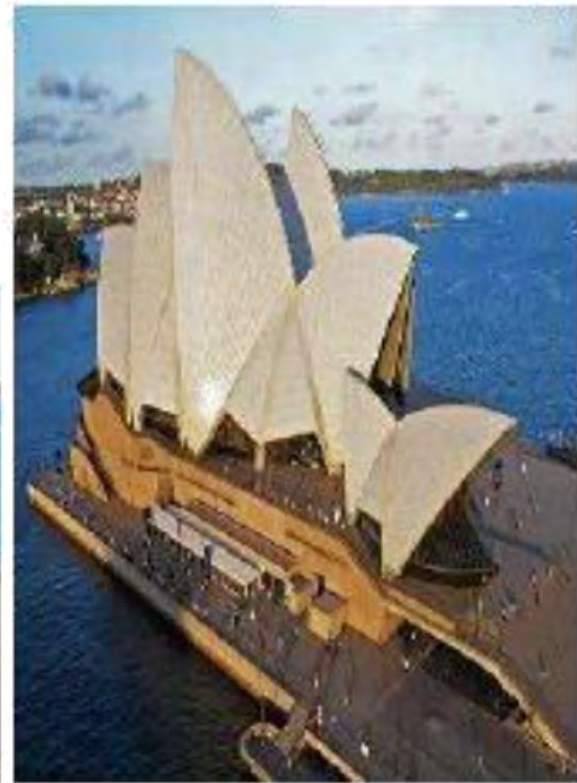
(b)

Image Processing

- Image Scaling



(a)



(b)

Linear Algebra in Machine Learning

■ Dataset and Data Files



Attribute Information:

1. sepal length in cm
2. sepal width in cm
3. petal length in cm
4. petal width in cm
5. class:
 - Iris Setosa
 - Iris Versicolour
 - Iris Virginica

Linear Algebra in Machine Learning

■ Dataset and Data Files

- In machine learning, you fit a model on a dataset.
- This is the table-like set of numbers where each row represents an observation and each column represents a feature of the observation.

Attribute Information:

1	5.1,3.5,1.4,0.2,Iris-setosa
2	4.9,3.0,1.4,0.2,Iris-setosa
3	4.7,3.2,1.3,0.2,Iris-setosa
4	4.6,3.1,1.5,0.2,Iris-setosa
5	5.0,3.6,1.4,0.2,Iris-setosa

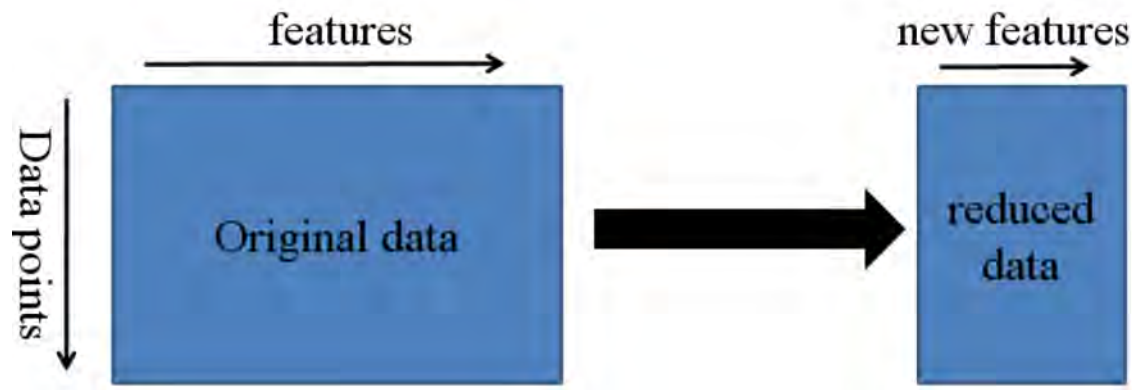
1. sepal length in cm
2. sepal width in cm
3. petal length in cm
4. petal width in cm
5. class:
-- Iris Setosa
-- Iris Versicolour
-- Iris Virginica

- This data is in fact a matrix: a key data structure in linear algebra.

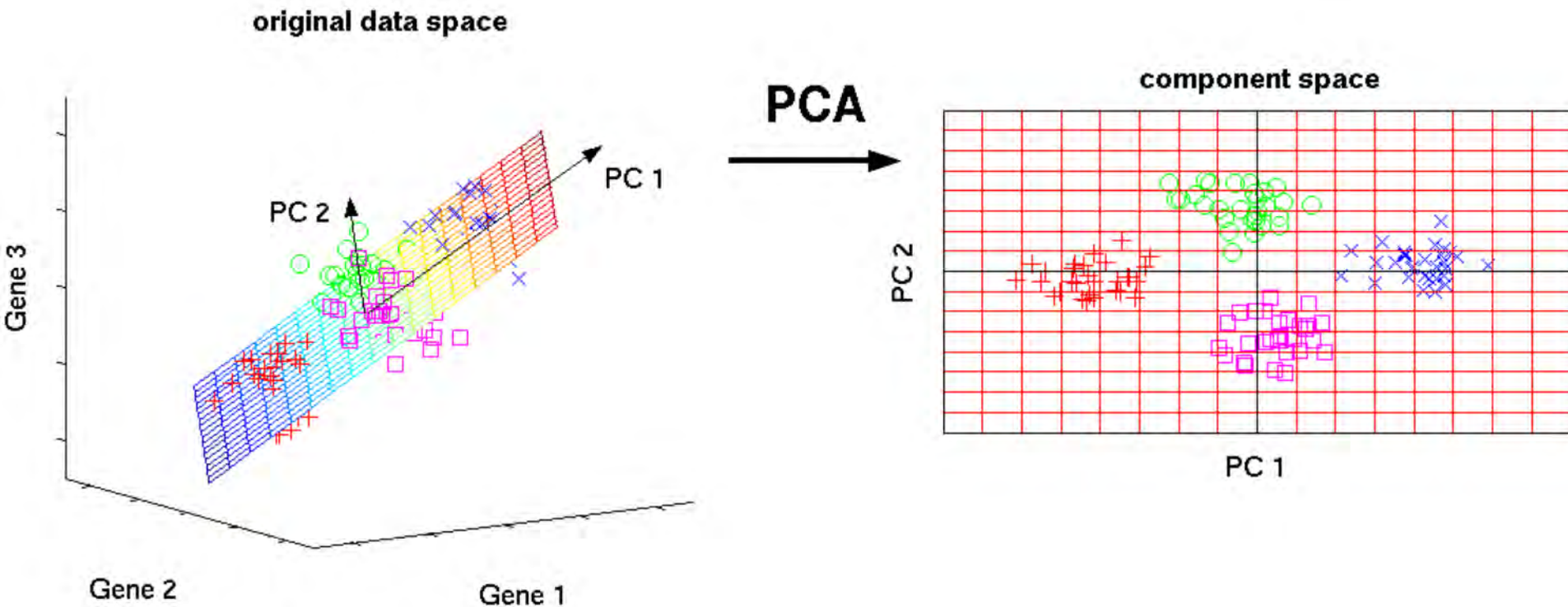
Linear Algebra in Machine Learning

■ Principal Component Analysis

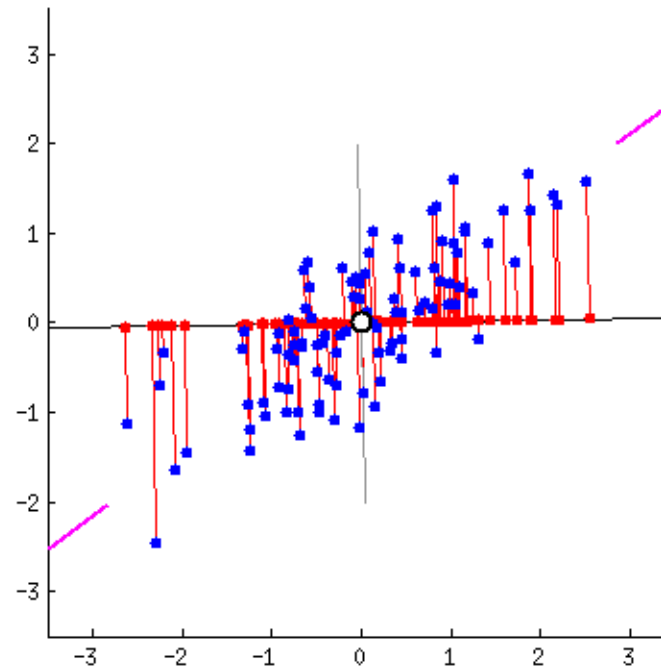
- Often, a dataset has many columns, perhaps tens, hundreds, thousands, or more.
- Methods for automatically reducing the number of columns of a dataset are called dimensionality reduction, and perhaps the most popular method is called the principal component analysis, or PCA for short
- The core of the PCA method is a matrix factorization method from linear algebra.



Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Linear Algebra in Machine Learning

■ Singular-Value Decomposition

- It is a matrix factorization method from the field of linear algebra.
- It has wide use in linear algebra and can be used directly in applications such as feature selection, visualization, noise reduction, and more.

Linear Algebra in Machine Learning

■ One Hot Encoding

- Sometimes you work with categorical data in machine learning.
- It is common to encode categorical variables to make them easier to work with and learn by some techniques. A popular encoding for categorical variables is the one hot encoding.
- A one hot encoding is:

For example, the color variable with the 3 rows:

1	red
2	green
3	blue
4	...

Might be encoded as:

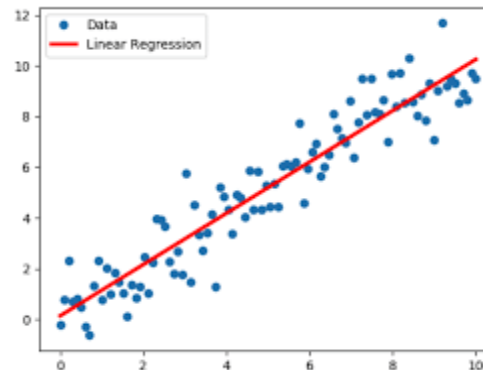
1	red, green, blue
2	1, 0, 0
3	0, 1, 0
4	0, 0, 1
5	...

Linear Algebra in Machine Learning

■ Linear Regression

- Describing the relationships between variables.
- Predicting numerical values in simpler regression problems.
- The most common way of solving linear regression is via a least squares optimization that is solved using matrix factorization methods from linear regression, such as an LU decomposition or a singular-value decomposition, or SVD.

$$y = A \cdot b$$



Linear Algebra in Machine Learning

■ Latent Semantic Analysis

- In the sub-field of machine learning for working with text data called natural language processing, it is common to represent documents as large matrices of word occurrences
- Matrix factorization methods, such as the singular-value decomposition can be applied to this sparse matrix. Documents processed in this way are much easier to compare, query, and use as the basis for a supervised machine learning model.

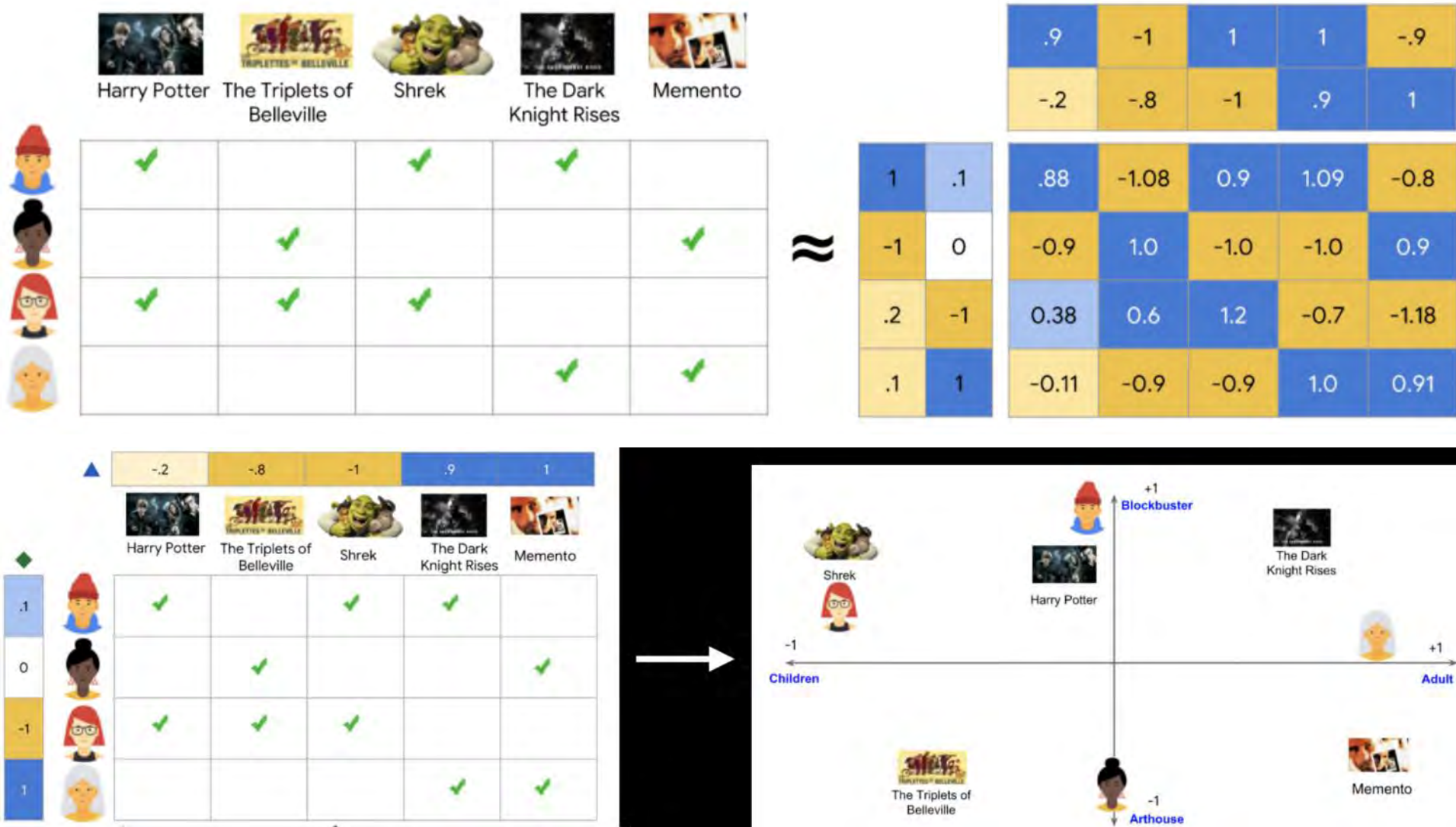
	Quick	Brown	Fox	Jumps	Over	Lazy	Dog
The quick brown fox jumps over the lazy dog	1	1	1	1	1	1	1
If the fox is quick he can jump over the dog.	1	0	1	0	1	0	1
Foxes are quick. Dogs are lazy.	0	1	1	0	0	1	1
Can a fox jump over a dog?	0	0	1	1	1	0	1

Linear Algebra in Machine Learning

■ Recommender Systems

- Predictive modeling problems that involve the recommendation of products are called recommender systems, a sub-field of machine learning.
- The development of recommender systems is primarily concerned with linear algebra methods. A simple example is in the calculation of the similarity between sparse customer behavior vectors using distance measures such as Euclidean distance or dot products.
- Matrix factorization methods like the singular-value decomposition are used widely in recommender systems to distill item and user data to their essence for querying and searching and comparison.

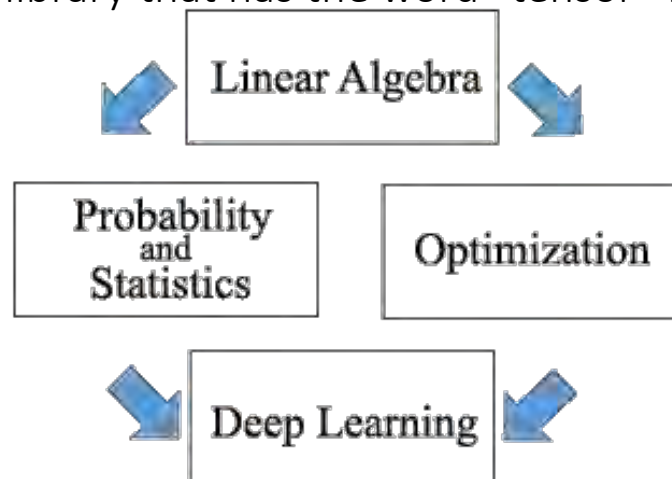
Recommender Systems



Linear Algebra in Machine Learning

■ Deep Learning

- The execution of neural networks involves linear algebra data structures multiplied and added together. Scaled up to multiple dimensions, deep learning methods work with vectors, matrices, and even tensors of inputs and coefficients, where a tensor is a matrix with more than two dimensions.
- Linear algebra is central to the description of deep learning methods via matrix notation to the implementation of deep learning methods such as Google's TensorFlow Python library that has the word "tensor" in its name.





تعداد واحد: ۳	نوع درس: نظری
پیش نیاز: ریاضی عمومی ۲	هم نیاز: —

اهداف درس

هدف از این درس، آشنایی دانشجویان با مفاهیم پایه‌ای نظری جبر خطی و همچنین نحوه‌ی استفاده و پیاده‌سازی آن‌ها در بستر نرم‌افزاری مناسب است. آشنایی با مفاهیم این درس امکان تحلیل نگاشت‌ها و سیستم‌های خطی از طریق ماتریس‌ها و اعمال، اپراتورها و مفاهیم تعریف شده مرتبط با آنها را فراهم می‌کند. همین‌طور مساله بهینه‌سازی به عنوان یکی از کاربردهای پر استفاده جبر خطی مورد بررسی قرار می‌گیرد.

ریز مواد

- فضاهای برداری
- نگاشت خطی و ماتریس
 - فضای برداری نگاشت خطی
 - ساختار جبری نگاشت خطی
 - ماتریس و رنک
 - معکوس نگاشت خطی
 - دوگانگی
 - دستگاه‌های خطی
 - حجم و دترمینان
- چندجمله‌ای‌ها
 - صفرهای چندجمله‌ای
 - فاکتورگیری چندجمله‌ای در میدان‌های مختلط و حقیقی
- مقادیر ویژه و بردارهای ویژه
 - زیرفضاهای ثابت
 - بردارهای ویژه و مقادیر ویژه
 - بردارهای ویژه‌ی مستقل خطی
 - فضاهای ویژه و قطری‌سازی ماتریس‌ها
- فضاهای ضرب داخلی
 - ضرب داخلی و تعریف فاصله
 - پایه‌های متعامد
 - عملگرهای فضاهای ضرب داخلی
- عملگرها و تجزیه

- تجزیه‌ی قطبی
- تجزیه‌ی مقدارهای منفرد
- تجزیه‌ی چولسکی
- تجزیه‌ی ال یو
- تجزیه‌ی کیو آر
- عملگرهای الحاقی
- عملگرهای نرمال
- عملگرهای یکانی و ایزومتري
- عملگرهای مثبت

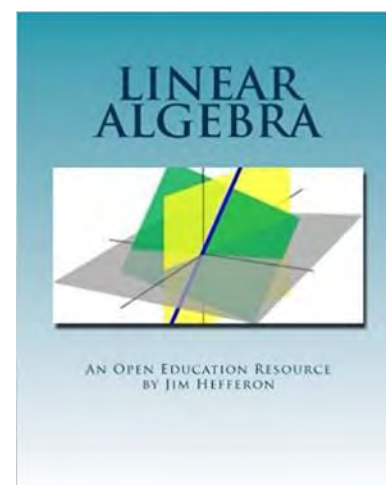
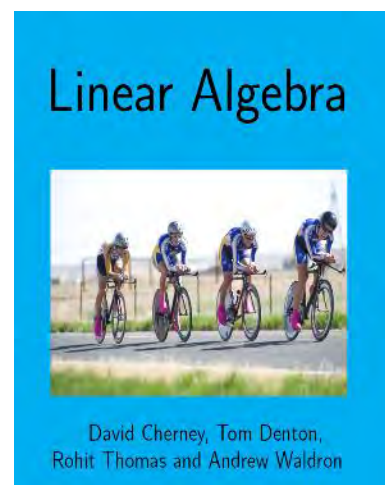
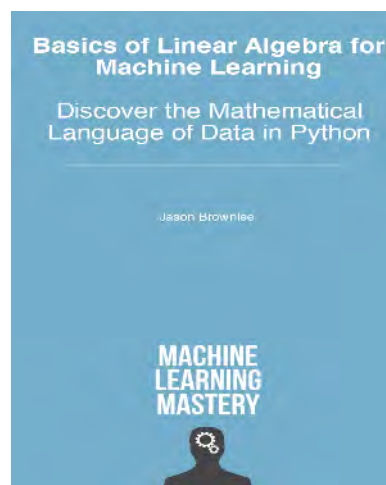
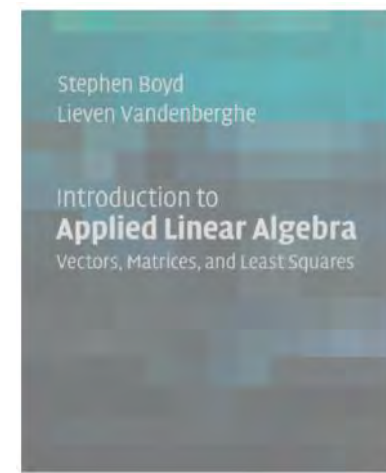
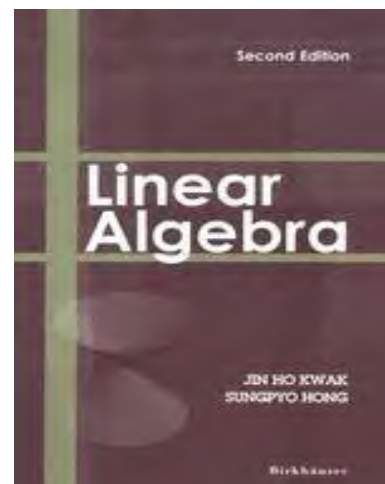
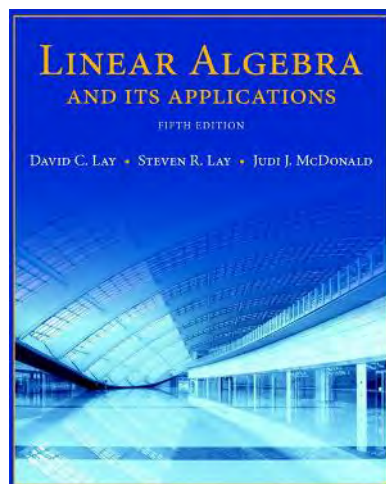
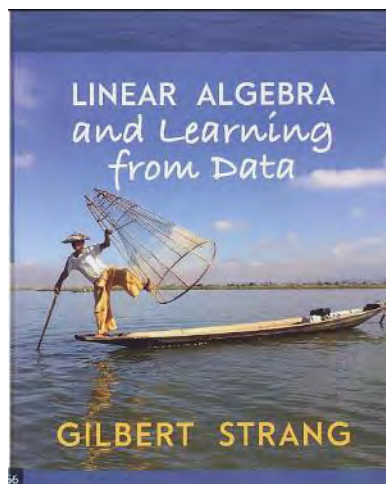
ارزیابی

- تمرین‌ها: ۶ نمره
- دو آزمون میان‌ترم: ۸ نمره
- آزمون پایانی: ۶ نمره
- آزمونک‌ها: ۱ نمره

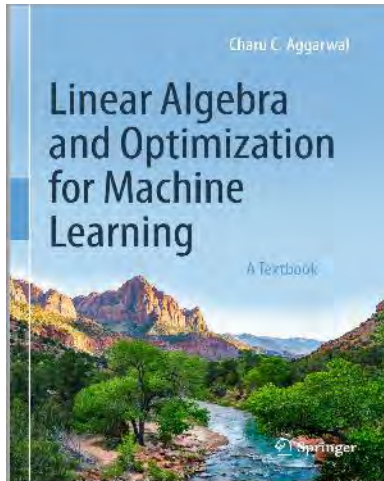
مراجع

- [1] Right Sheldon Axler. *Linear Algebra*. Springer, 2015.
- [2] Gilbert Strang. *Linear Algebra and Its Application*. 4th edition, Cengage Learning, 2006.
- [3] David Clay. *Linear Algebra and Its Application*. 4th edition, Pearson, 2011.

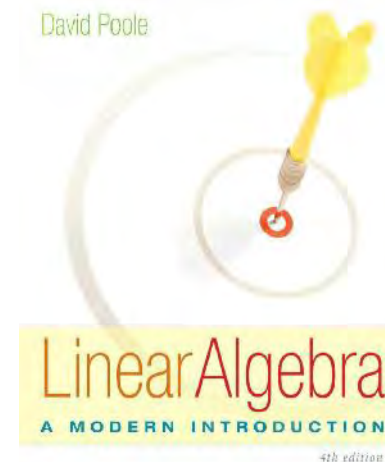
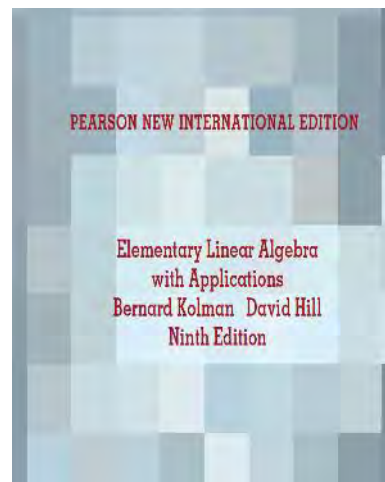
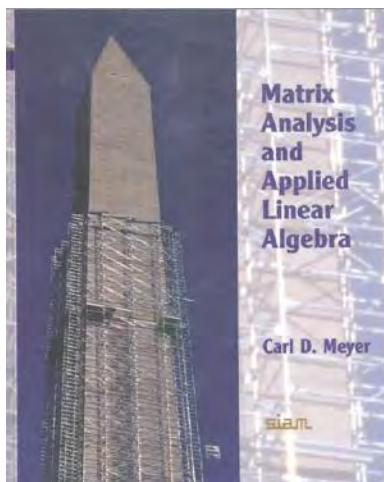
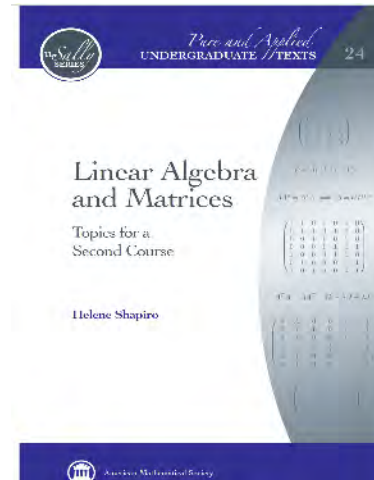
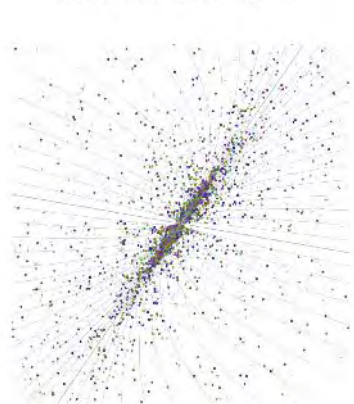
Textbooks



Textbooks



Interactive Linear Algebra



Textbooks

PROBLEMS AND THEOREMS IN LINEAR ALGEBRA

V. PRASOLOV

ABSTRACT. This book contains the basics of linear algebra with an emphasis on non-standard and neat proofs of known theorems. Many of the theorems of linear algebra obtained mainly during the past 30 years are usually ignored in text-books but are quite accessible for students majoring or minoring in mathematics. These theorems are given with complete proofs. There are about 230 problems with solutions.

LINEAR ALGEBRA

Second Edition

KENNETH HOFFMAN

Professor of Mathematics
Massachusetts Institute of Technology

RAY KUNZE

Professor of Mathematics
University of California, Irvine

Next Session

- Vectors

- Background:

Who knows what is a vector?
How about Vector Space?)