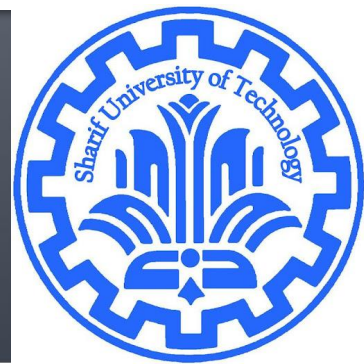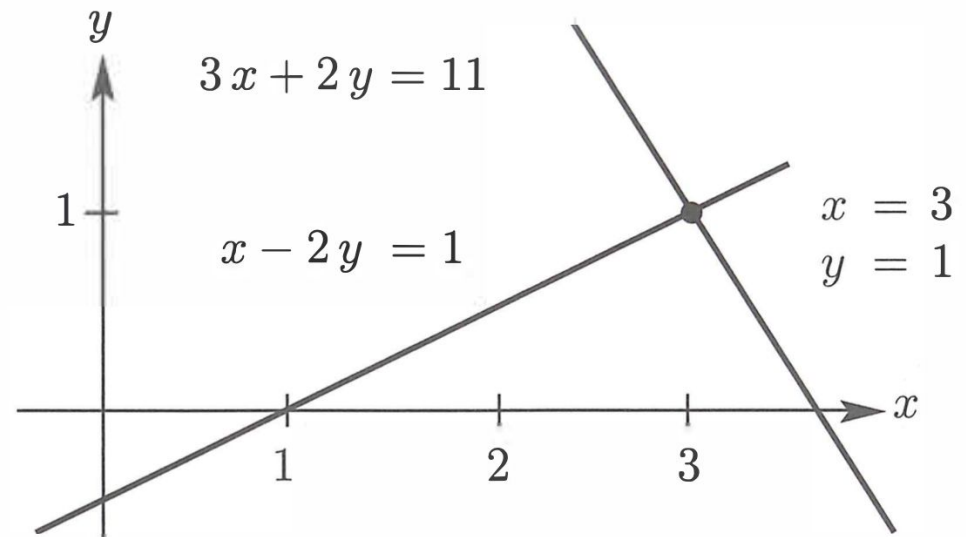# Linear Equations

CE40282-1: Linear Algebra
Hamid R. Rabiee and Maryam Ramezani
Sharif University of Technology

Vectors & Linear Equation

- Consider this simple system of equations,

$$x \quad - \quad 2y \quad = 1$$
$$3x \quad + \quad 2y \quad = 11$$

Vectors & Linear Equation

◼ Can be expressed as matrix-vector multiplication

$$
\begin{aligned}
x &\; - \; 2y &\; = 1 \\
3x &\; + \; 2y &\; = 11
\end{aligned}
\qquad
\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}}
=
\underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}
$$

- Matrix Equation, $A\boldsymbol{x} = \boldsymbol{b}$
- A is often called **coefficient matrix**
- Augmented matrix is: $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

Vectors & Linear Equation

- Also, Can be expressed as linear combination of cols:

$$x \quad - \quad 2y \quad = 1$$
$$3x \quad + \quad 2y \quad = 11$$

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} =$$

- Same for n equation, n variables

Idea of Elimination

■ Subtract a multiple of equation (1) from to (2) to eliminate a variable

$$x - 2y = 1$$
$$3x + 2y = 11$$

(multiply equation 1 by 3)

(subtract to eliminate 3x)

$$x - 2y = 1$$
$$8y = 8$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \end{bmatrix}}_{c}$$
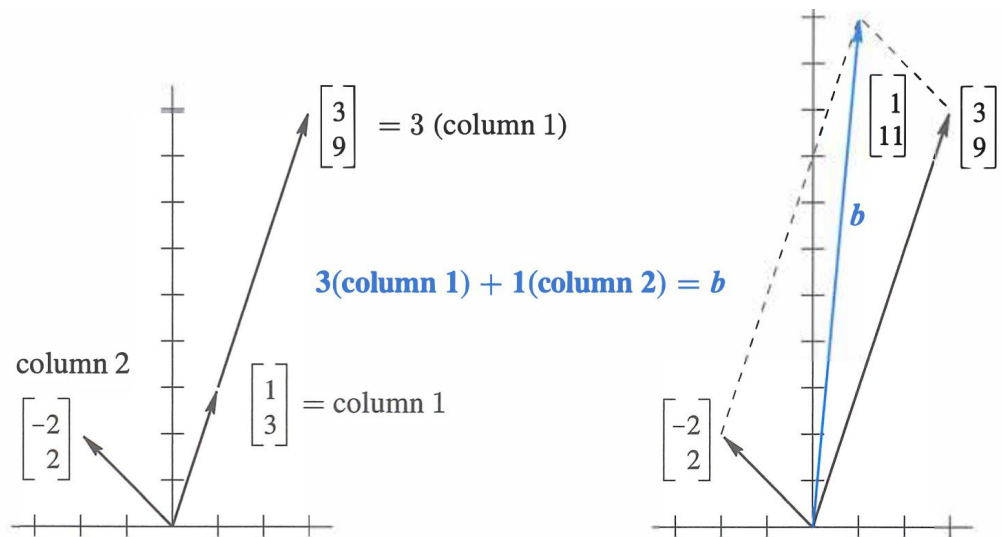
A has become a upper triangle matrix U

Idea of Elimination (Row Reduction Algorithm)

- The **pivots** are on the diagonal of the triangle after elimination (boldface 2 below is the first pivot)

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8$$
$$-2x - 3y + 7z = 10$$

$$\mathbf{2}x + 4y - 2z = 2$$
$$1y + 1z = 4$$
$$4z = 8.$$

- Step 1: by subtracting (1) to (2) eliminate **x**'s in (2)
- Step 2: subtract (1) from (3) and totally eliminate **x**
- Step 3: subtract new(2) from new(3)

- The variables corresponding to pivot columns in the matrix are called basic variables. The other variables are called a free variable.

Idea of Elimination (Row Reduction Algorithm)

- The whole idea is how to make U out of A by eliminate variables column by column

Column 1. *Use the first equation to create zeros below the first pivot.*

Column 2. *Use the new equation 2 to create zeros below the second pivot.*

Columns 3 to n. *Keep going to find all n pivots and the upper triangular U.*

$$\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \Longrightarrow \begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

Elementary Row Operations

- **ELEMENTARY ROW OPERATIONS**

  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

- It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange

  If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad\; -5x_3 &= 1 \\ x_2 + \; x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

- x_1, x_2: basic x_3: free
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

Definition

A leading entry of a row refers to **the leftmost nonzero entry in a nonzero row**

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Reduced echelon form

Existence and Uniqueness Questions

- Two fundamental question about a linear system:

  - Is the system consistent; that is, does at least one solution exist?

  - If a solution exists, is it the only one; that is, is the solution unique?

Existence and Uniqueness Questions

- **Theorem**: A linear system is **consistent** if and only if the rightmost column of the augmented
matrix is not a pivot column—that is, if and only if an echelon $\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix}$ with b nonzero
matrix has no row of the form

- If a linear system is consistent, then the solution set contains either:

  - a unique solution, when there are no free variables

  - infinitely many solutions, when there is at least

Find all solutions of a linear system

- 1. Write the augmented matrix of the system.
- 2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
- 3. Continue row reduction to obtain the reduced echelon form.
- 4. Write the system of equations corresponding to the matrix obtained in step 3.
- 5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Existence of Solutions

- The equation Ax=b has a solution if and only if b is a linear combination of the columns of A.

EXAMPLE Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

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SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if it can be written in the form Ax=0, where A is a matrix and 0 is the zero vector.
- Trivial solution: Ax=0 always has at least one solution, namely, x =0 (the zero vector).
- Nontrivial solution: The non-zero solution for Ax=0.
- The homogeneous equation Ax=0 has a nontrivial solution if and only if the equation has at least one free variable.

$$\begin{pmatrix} 1 & 3 & 4 & \bigm| & 0 \\ 2 & -1 & 2 & \bigm| & 0 \\ 1 & 0 & 1 & \bigm| & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & \bigm| & 0 \\ 0 & 1 & 0 & \bigm| & 0 \\ 0 & 0 & 1 & \bigm| & 0 \end{pmatrix}$$

Review

Equations of Lines in $\mathbb{R}^2$

parametric vector form

Normal Form	General Form	Vector Form	Parametric Form
$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by = c$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

Table 1.3   Lines and Planes in $\mathbb{R}^3$

parametric vector form

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases} \mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}_1 \\ \mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{p}_2 \end{cases}$	$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$
Planes	$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by + cz = d$	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

Nonhomogeneous Systems & General Solution

Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{rcl} x_1 \qquad -\frac{4}{3}x_3 &=& -1 \\ x_2 \qquad &=& 2 \\ 0 &=& 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3, x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

General Solution in written in vector form (parametric vector form)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underset{\mathbf{p}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + x_3 \underset{\mathbf{v}}{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}$$

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})$$

General Solution in written in vector form (parametric vector form)

$$Ax = 0$$

$$A = \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{cases} x_1 & -8x_3 - 7x_4 = 0 \\ & x_2 + 4x_3 + 3x_4 = 0. \end{cases}$$

$$\begin{cases} x_1 = & 8x_3 + 7x_4 \\ x_2 = & -4x_3 - 3x_4 \\ x_3 = & x_3 \\ x_4 = & x_4. \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Note

Can we change the order of columns???

$$\begin{cases} ax & + & b\,y & + & c\,z = & d \\ a'x & + & b'\,y & + & c'\,z = & d' \\ a''x & + & b''y & + & c''z = & d'' \end{cases}$$

is equivalent to

$$\begin{cases} a\,x & + & c\,z & + & b\,y = & d \\ a'\,x & + & c'\,z & + & b'\,y = & d' \\ a''\,x & + & c''z & + & b''y = & d''. \end{cases}$$

Conclusion

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Note: If A does not have a pivot in every row, that does not mean that Ax = b does not have a solution for some given vector b. It just means that there are some vectors b for which Ax = b does not have a solution.

[A b] is augmented matrix

Reference

- Linear Algebra and Its Applications, David C. Lay

- Linear Algebra Done Right, Axler, Chapter 3.D

- Introduction to Linear Algebra, Strange, Chapter 2.1,2.2

- http://vmls-book.stanford.edu/vmls-slides.pdf