Matrix Transformation

CE282: Linear Algebra

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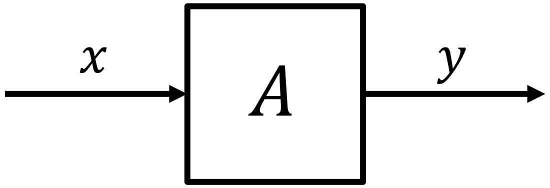
Linear Transformation

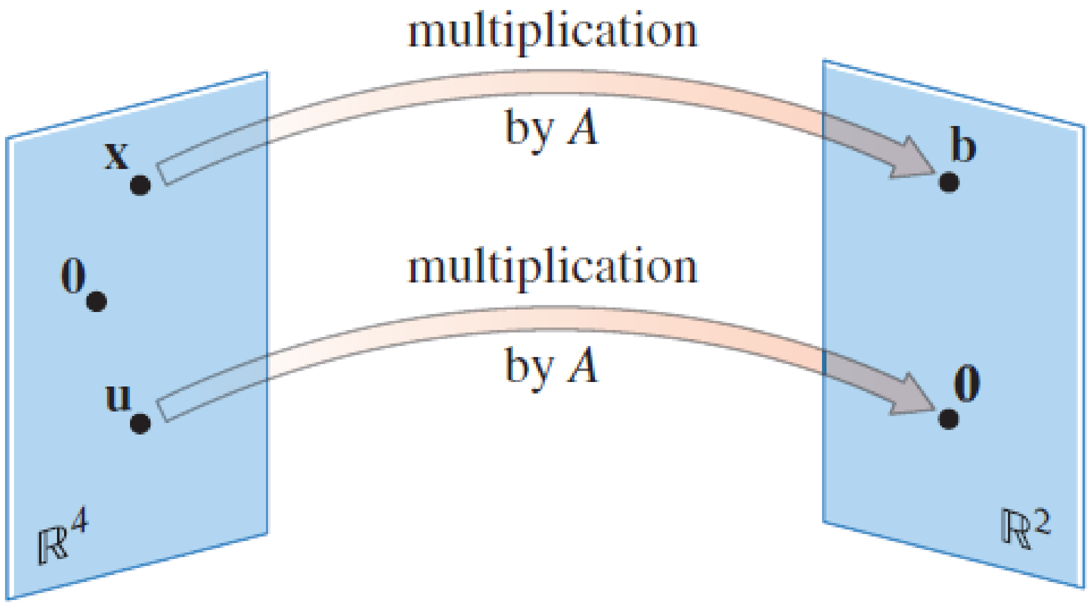
- Matrix is a linear transformation: map one vector to another vector

$$A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}: \qquad y_{m\times 1} = A_{m\times n}x_{n\times 1}$$

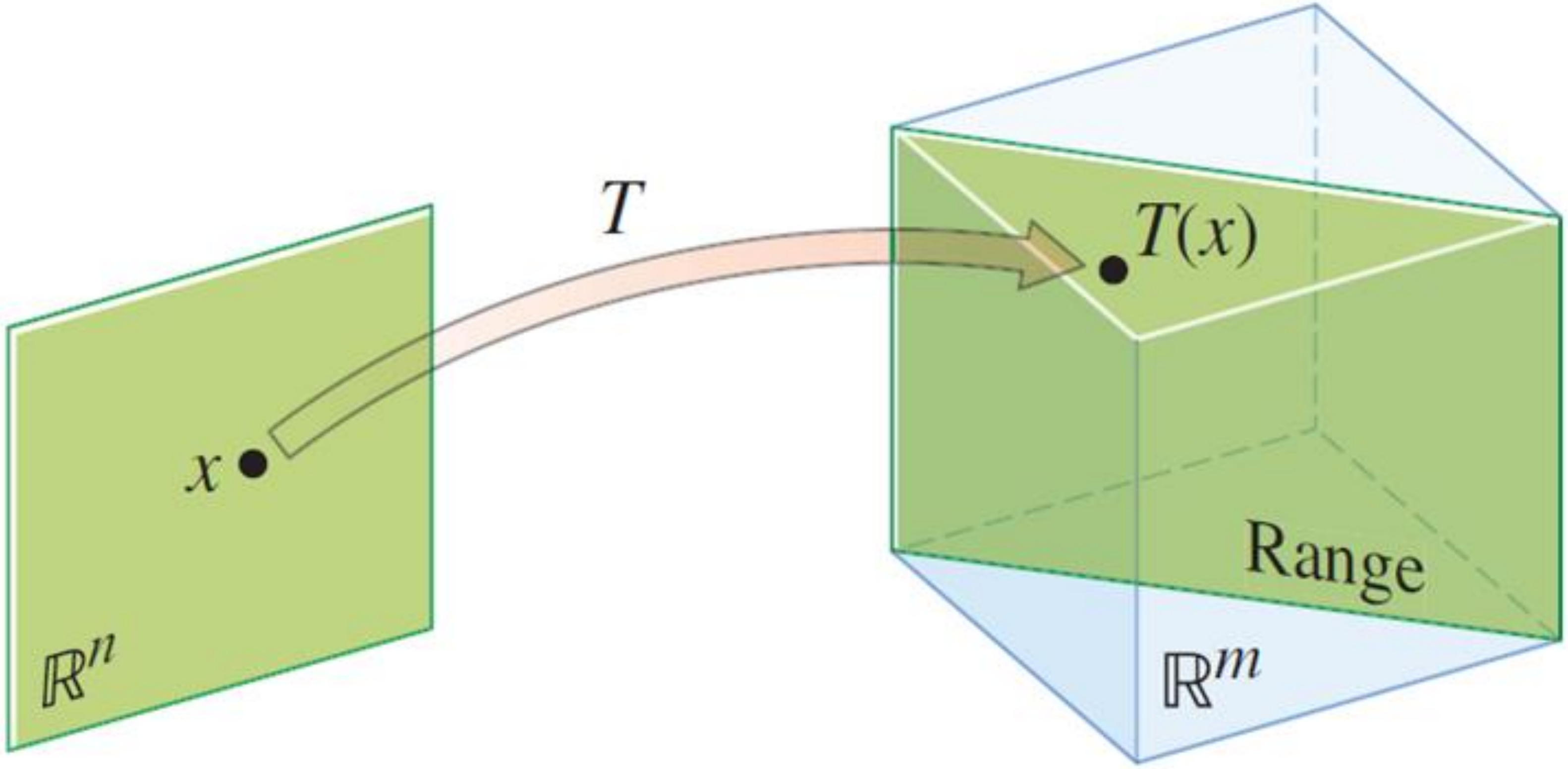
$$A : \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$$

- Input-output





Linear Transformation



Domain

Codomain

Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$

Linear Transformation

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Linear mapping

Theorem

A linear transformation (or a linear map) is a function $\mathbf{T} : \mathbb{R}^{\mathrm{n}} \rightarrow \mathbb{R}^{\mathrm{m}}$ that satisfies following properties:

1. $\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$
2. $\mathbf{T}(\mathrm{a}\mathbf{x}) = \mathrm{a}\mathbf{T}(\mathbf{x})$

for any vectors $\mathbf{x}, \mathbf{y} \epsilon \mathbb{R}^{\mathrm{n}}$ and any scalar $a \epsilon \mathbb{R}$.

Linear mapping

Example

Which are linear mapping?

- ❑ **zero** map $0 : V \to W$
- ❑ **identity** map $I : V \to V$
- ❑ Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $T_{\mathcal{P}(z)} = \acute{\mathcal{P}}(z)$
- ❑ $\boxed{\text{Let } T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be the map given by } T(x, y) = (x - 2y, 3x + y)}$
- ❑ $T(x) = e^x$
- ❑ $T : \mathbb{F} \to \mathbb{F}$ given by $T(x) = x - 1$

Linear mapping

Theorem

Let $(v_1, \ldots, v_n)$ be a basis of V and $(w_1, \ldots, w_n)$ an arbitrary list of vectors in W. Then there exists a unique linear map

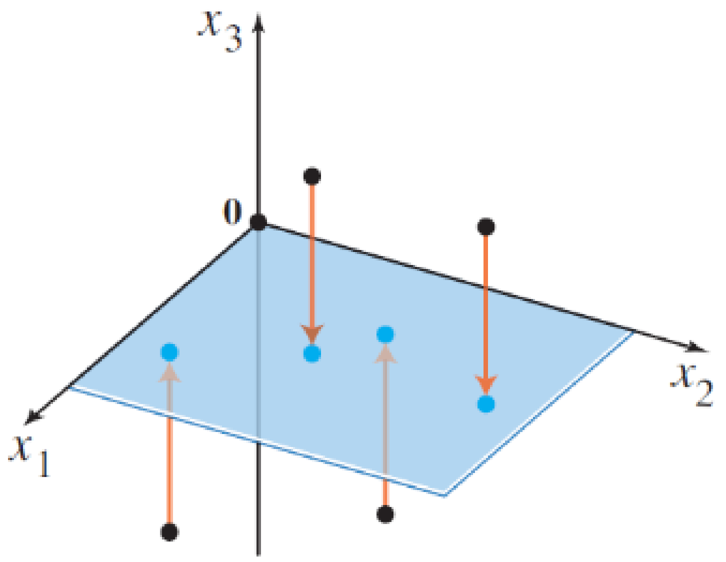
$$T : V \to W \qquad \text{such that } T(v_i) = w_i.$$

Projection

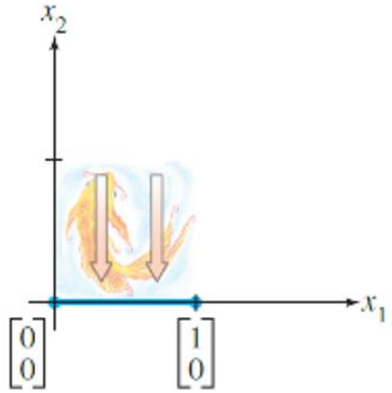
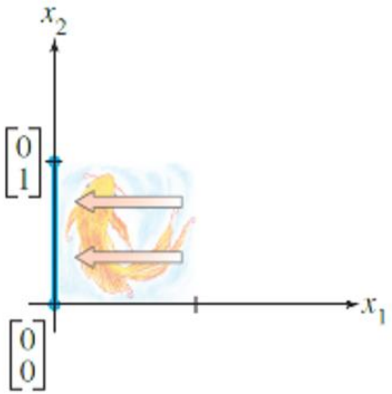
Example

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ onto the x_1x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$



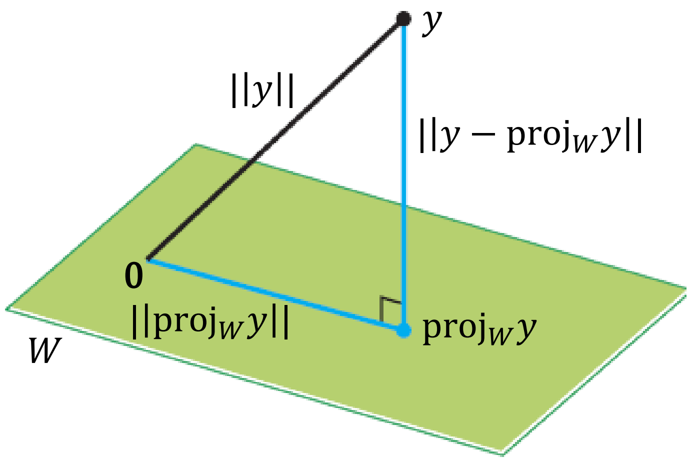
Projection

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Definition

The ***projection*** of a vector $y \in \mathbb{R}^m$ onto the span of $\{x_1, \dots, x_n\}$ is the vector $v \in span(\{x_1, \dots, x_n\})$, such that v is as close as possible to y, as measured by the Euclidean norm $||v - y||_2$.

$$\mathrm{Proj}(y; \{x_1, \dots, x_n\}) = \mathrm{argmin}_{v \in \mathrm{span}(\{x_1,\dots,x_n\})} ||y - v||_2.$$



Projection

Theorem

Suppose that V is a vector space and $P : V \to V$ is a linear transformation.

a) If $P^2 = P$ then P is called a **projection**.

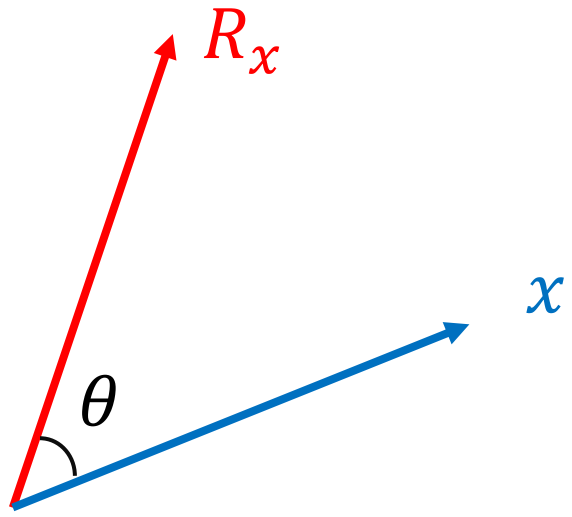
b) If V is an inner product space and $P^2 = P = P^*$ then P is called an **orthogonal projection**.

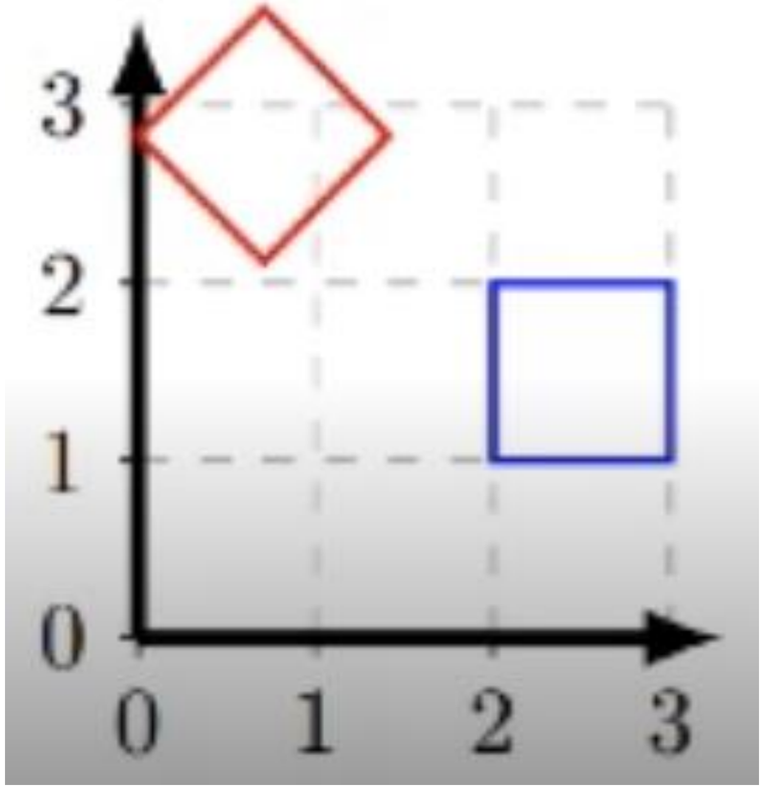
We furthermore say that P **projects onto** $\text{range}(P)$.

- Projection of vector v on:
  - Two orthogonal vectors
  - Two non-orthogonal vectors

Rotation

- $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$





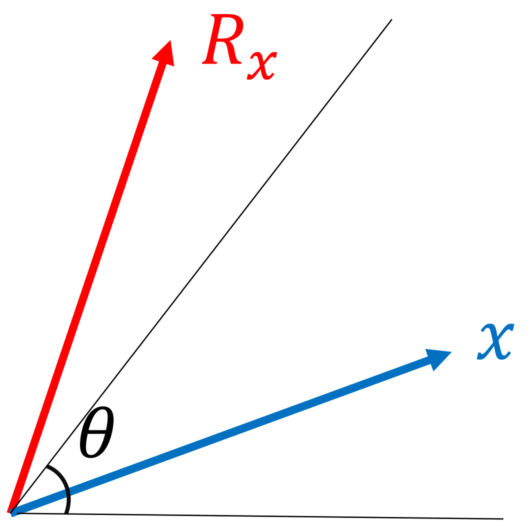
Projection on $\boldsymbol{\theta}$ line

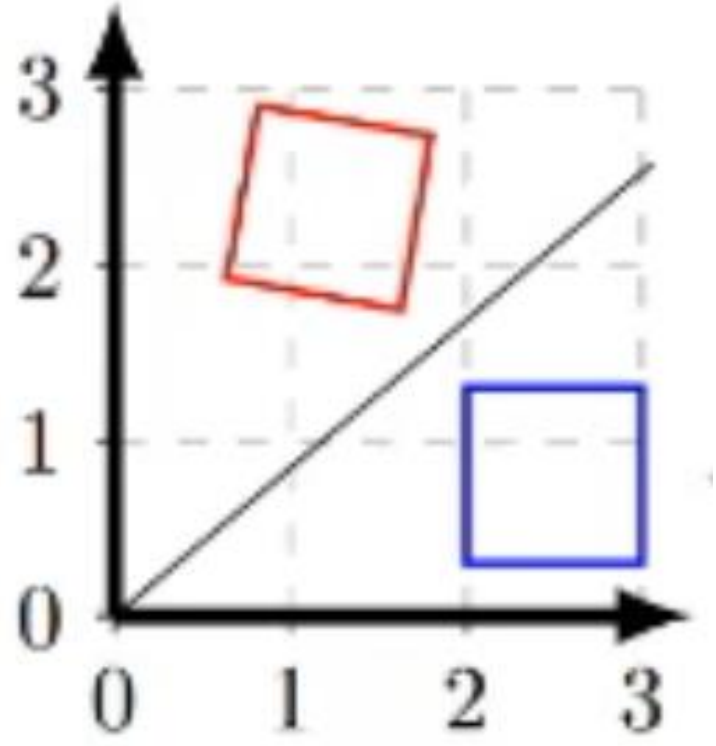
$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$P^2 = P$$

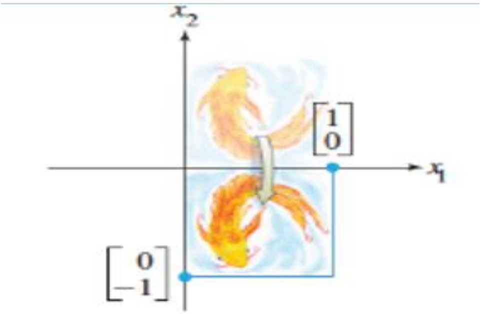
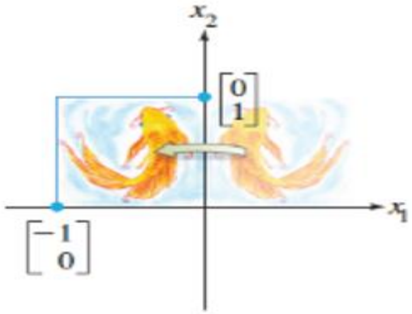
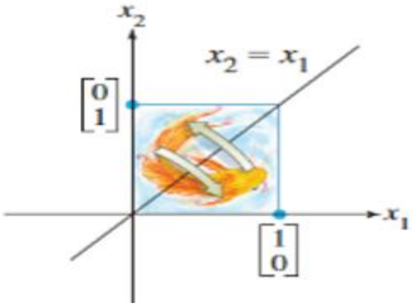
Reflection

- $R = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$



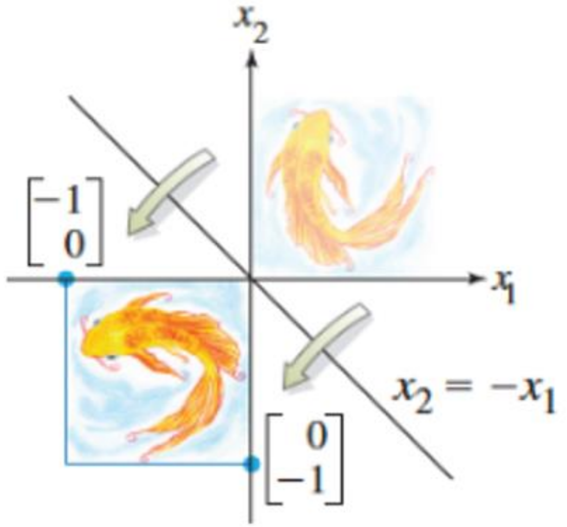


$$R^2 = I$$

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

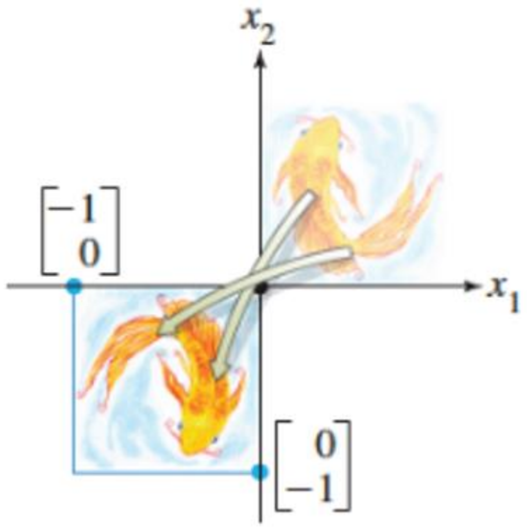
Reflection

Reflection through the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

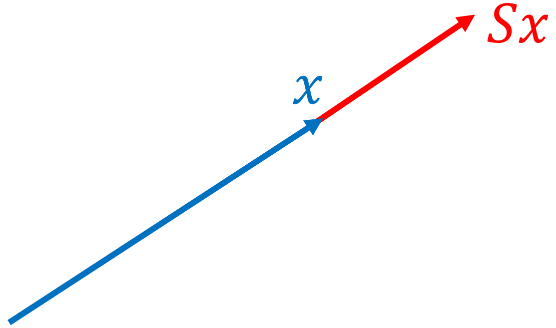
Reflection through the origin

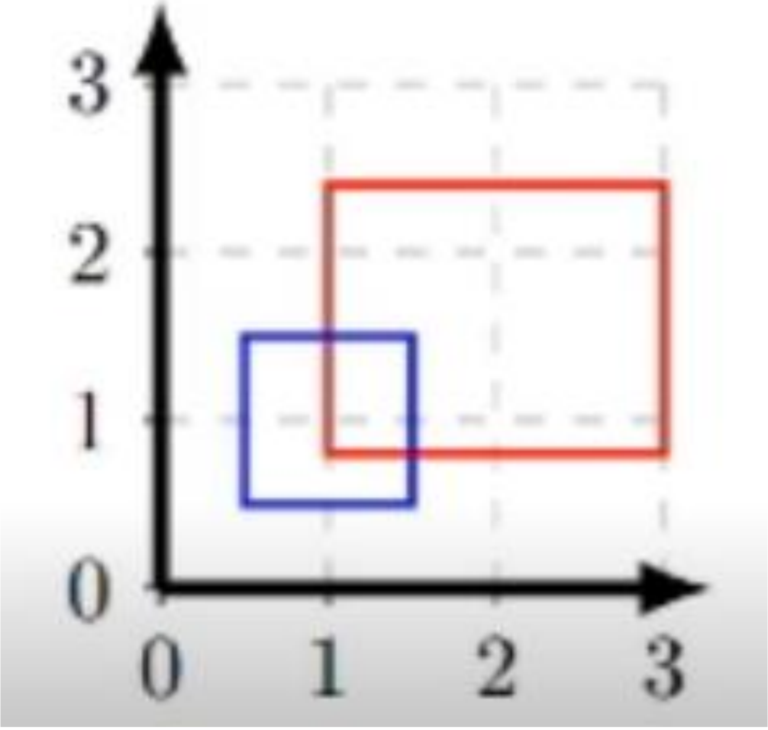


$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Uniform Scaling

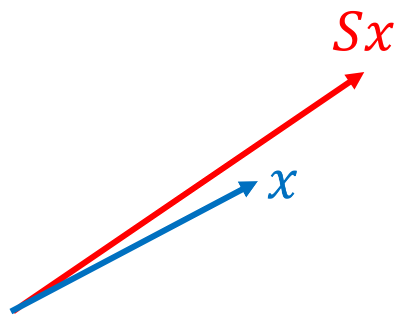
- $S = sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

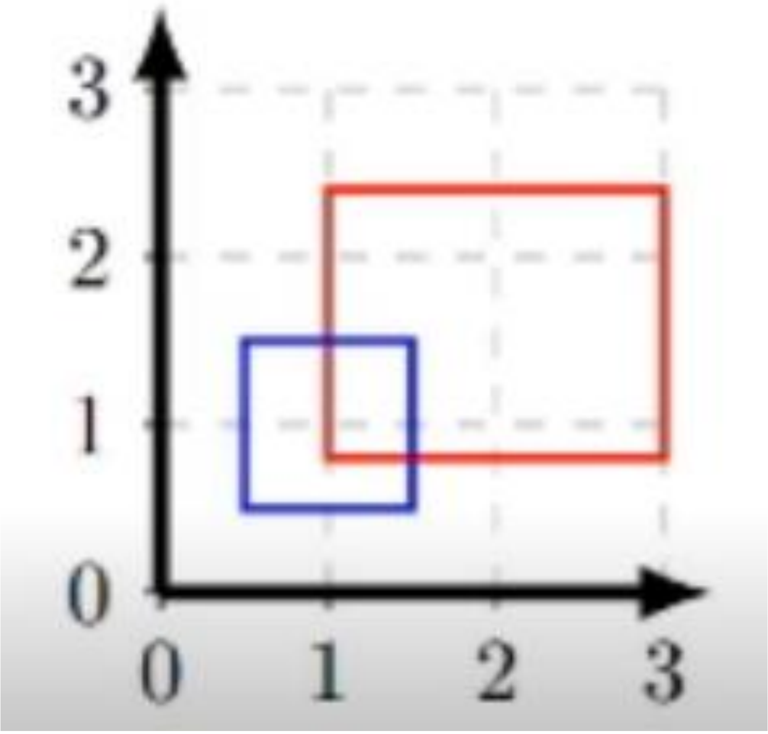




Non-uniform Scaling

- $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$





Example

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

A typical shear matrix is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

sheep

sheared sheep

A shear parallel to the x axis results in $\acute{x} = x + \lambda y$ and $\acute{y} = y$.

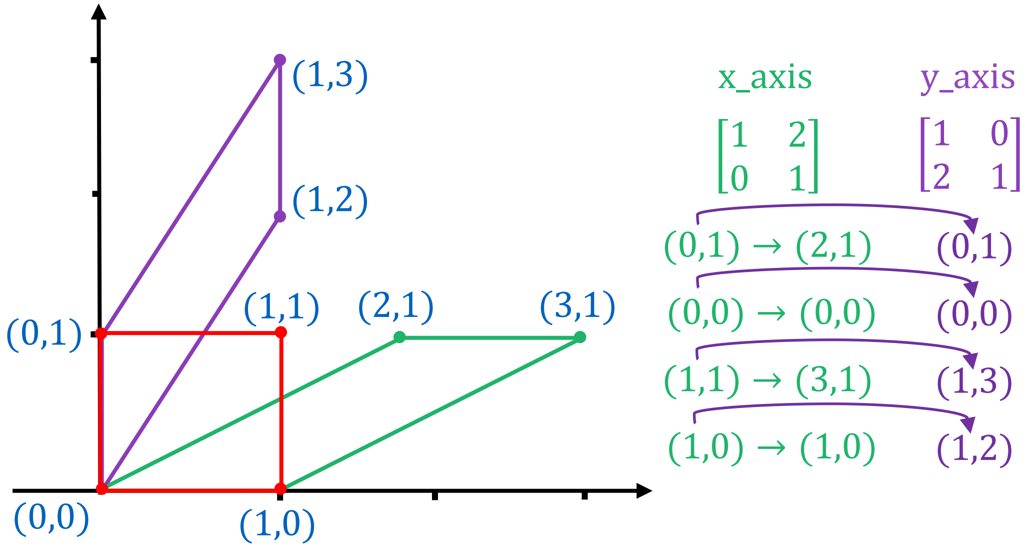
In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Similarly, a shear parallel to the y axis has $\acute{x} = x$ and $\acute{y} = y + \lambda x$.

In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



Difference Matrix

Note

$$D_{(n-1)\times n} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

$$D : \mathbb{R}^n \to \mathbb{R}^{n-1} \quad \Rightarrow \quad D\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Example

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 - 0 \\ 3 - (-1) \\ 2 - 3 \\ 5 - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$$

Selectors

- an $m \times n$ *selector matrix*: each row is a unit vector (transposed)

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix}$$

multiplying by A selects entries of x:

$$Ax = (x_{k_1}, x_{k_2}, \dots, x_{k_m})$$

- $A : \mathbb{R}^n \to \mathbb{R}^m \quad \Rightarrow \quad A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_m} \end{bmatrix}$

Selectors

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- Selecting first and last elements of vector:
- Reversing the elements of vector:

Slicing

- Keeping m elements from r to s (m=s-r+1)

$$\begin{bmatrix} 0_{m\times(r-1)} & I_{m\times m} & 0_{m\times(n-s)} \end{bmatrix}$$

Example

- Slicing two first and one last elements:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Down Sampling

- Down sampling with k: selecting one sample in every k samples

Example

$K = 2?$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix}$$

Applications

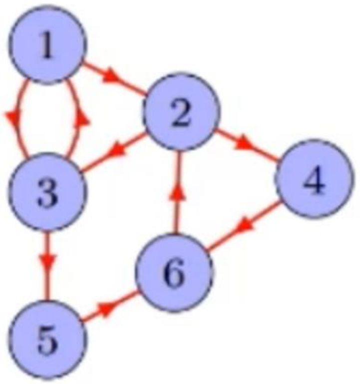
- ❑ Rotation matrix

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow R^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

- ❑ Adjacency matrix



$$A = \begin{array}{c} \begin{matrix} n1 & n2 & n3 & n4 & n5 & n6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiple Transformation

- $x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Rightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Rightarrow \ z = B(Ax) = BAx$

Example

- Difference Matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \xrightarrow[4\times 5]{D} y = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ x_5 - x_4 \end{bmatrix} \xrightarrow[3\times 4]{D} z = \begin{bmatrix} x_3 - x_2 - (x_2 - x_1) \\ x_4 - x_3 - (x_3 - x_2) \\ x_5 - x_4 - (x_4 - x_3) \end{bmatrix} = \begin{bmatrix} x_3 - 2x_2 + x_1 \\ x_4 - 2x_3 + x_2 \\ x_5 - 2x_4 + x_3 \end{bmatrix}$$

$$x \ \rightarrow z \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}_{3\times 5}$$

$$x \ \rightarrow y \ \rightarrow z$$

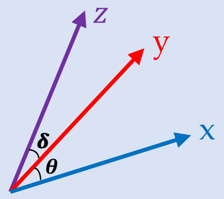
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{4\times 5} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Multiple Transformation

- $x_{n\times 1} \xrightarrow{A_{m\times n}} y_{m\times 1} \xrightarrow{B_{p\times m}} z_{p\times 1} \Rightarrow \begin{cases} y = Ax \\ z = By \end{cases} \Rightarrow z = B(Ax) = BAx$

Example

- Rotation



$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

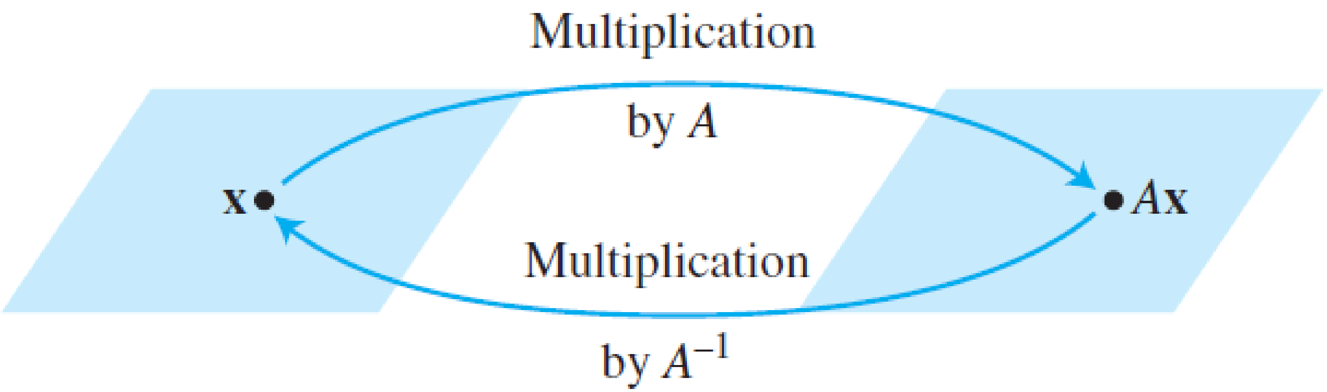
$$x \to z \qquad z = R_{\delta+\theta}x \qquad \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$

$$x \to y \to z \begin{cases} y = R_\theta x \\ z = R_\delta y \end{cases} \Rightarrow z = R_\delta R_\theta x \qquad \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\delta\cos\theta - \sin\delta\sin\theta & -\cos\delta\sin\theta - \sin\delta\cos\theta \\ \sin\delta\cos\theta + \cos\delta\sin\theta & -\sin\delta\sin\theta + \cos\delta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\delta+\theta) & -\sin(\delta+\theta) \\ \sin(\delta+\theta) & \cos(\delta+\theta) \end{bmatrix}$$

Invertible Linear Transformations



Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Invertible Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$