Combinations (Linear, Affine, Convex)

CE40282-1: Linear Algebra
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Linear Combinations

- The linear combinations of m vectors $a_1, \dots a_m$, each with size n is:

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

where $\beta_1, \dots, \beta_m$ are scalars and called the coefficients of the linear combination

- We can write any n-vector b as a linear combination of the standard unit vectors, as:

$$b = b_1 e_1 + \cdots + b_n e_n$$

  - Example: What are the coefficients and combination for this vector? $\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

Linear Combinations

- For vectors $x_1, x_2, \dots, x_k$:any point y is a linear combination of them iff:

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \in \mathbb{R}$$

- If we restrict α_i's to be positive then we get a conic combination.

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \geq 0 \in \mathbb{R}$$

- Instead of being positive, if we put the restriction that α_i's sum up to 1, it is called an affine combination

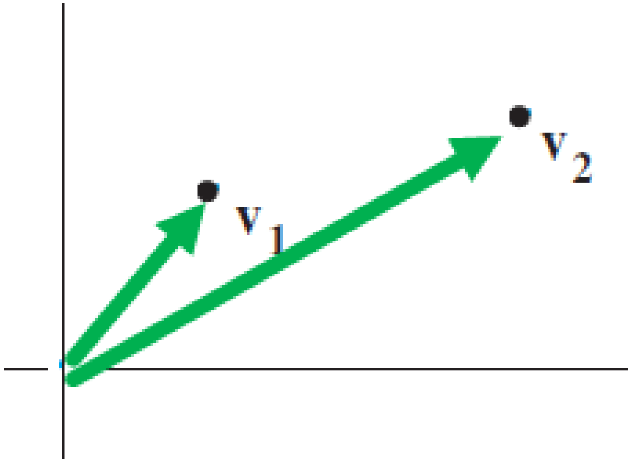
$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \in \mathbb{R},\ \sum \alpha_i = 1$$

- When a combination is affine as well as conic, it is called a convex combination

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \quad \forall i,\ \alpha_i \geq 0 \in \mathbb{R},\ \sum_i \alpha_i = 1$$

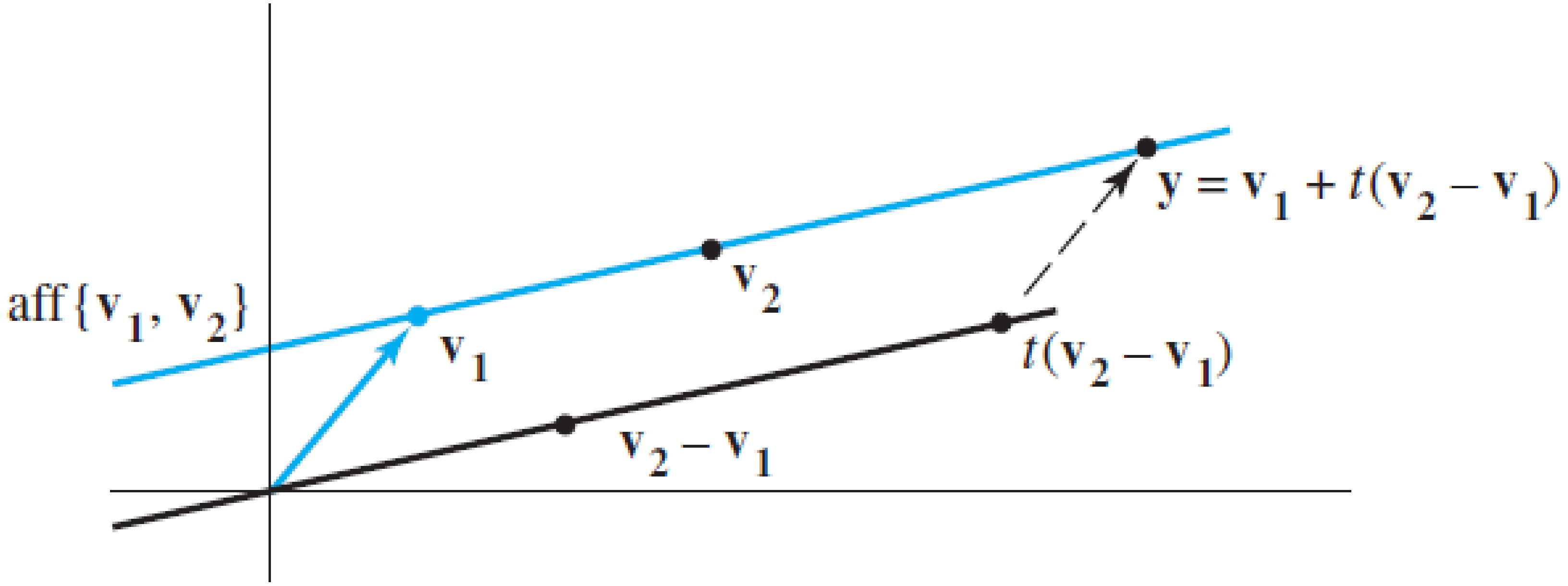
Affine

- The set of all affine combinations of points in a set S is called the affine hull (or affine span) of S, denoted by aff(S).
- What is the affine hull of single point v_1?
- What is the affine hull of two distinct $\{v_1, v_2\}$?



Affine

- What is the affine hull of two distinct $\{v_1, v_2\}$?
  - Answer: the blue line



Affine

- Theorem:

A point y in R^n is an affine combination of $v_1, \ldots, v_p$ in R^n if and only if $y - v_i$ is a linear combination of the translated points $v_2 - v_i, \ldots, v_p - v_i$

Proof?

Example 2.4.7. Find a vector equation and parametric equations of the plane in $\mathbb{R}^4$ that passes through $(-17, 6, 29, 0)$, $(-13, 3, 25, -2)$, and $(-15, 6, 25, -1)$.

Affine

- A set is called affine iff for any two points in the set, the line through them is contained in the set. In other words, for any two points in the set, their affine combinations are in the set itself.

- The affine span of the vectors $v_1, \dots, v_p$ is the smallest affine set containing these specific vectors.

Affine

- Theorem: A set (S) is affine iff any affine combination of points in the set is in the set itself.
  S is affine if and only if S=aff(S)

- Which are affine sets?
  - Square in R^2
  - R^2
  - $\{x: Ax = b\}$
  - $\{x\}$

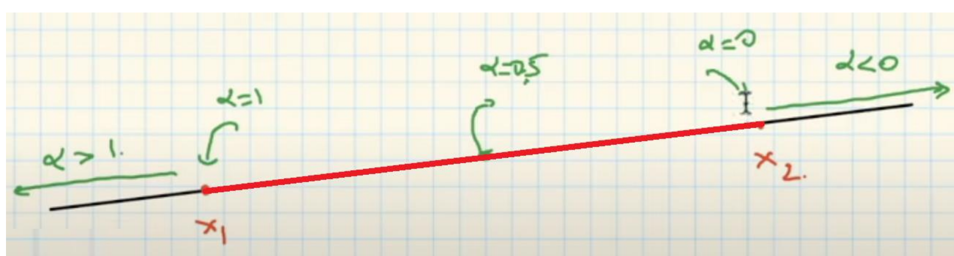
Convex combination

A **convex combination** of points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is a linear combination of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

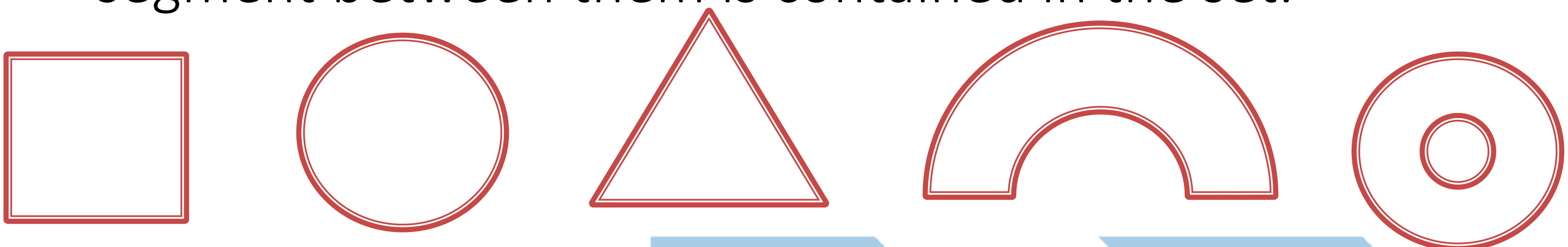
such that $c_1 + c_2 + \cdots + c_k = 1$ and $c_i \geq 0$ for all i. The set of all convex combinations of points in a set S is called the **convex hull** of S, denoted by conv S.

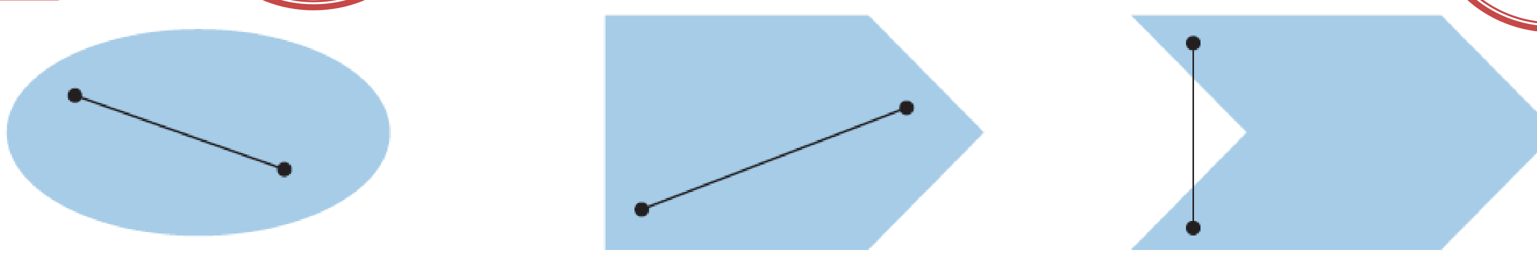
- The convex hull of a single point v_1?
- What are the points in the $conv(v_1, v_2\}$?
  - $y = (1-\alpha)v_1 + \alpha v_2 \quad 0 \leq \alpha \leq 1$
  - The line segment between v_1, v_2 denoted by $\overline{v_1 v_2}$



Convex Set

- A set S is convex any two points in the set, the line segment between them is contained in the set.





Convex | Convex | Not convex

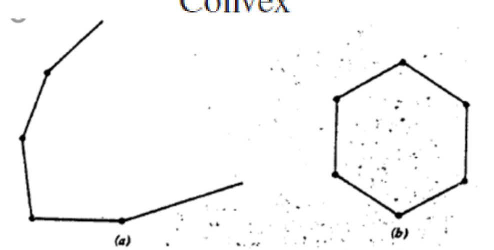


Figure 3.11 (a) A convex polyhedron that is not a polytope.
(b) A convex polytope.

Convex Set

- Theorem: A set (S) is convex iff every convex combination of points of S lines in S.
  S is convex if and only if S=conv(S)

  - Note: $C_{k+1} = 1$

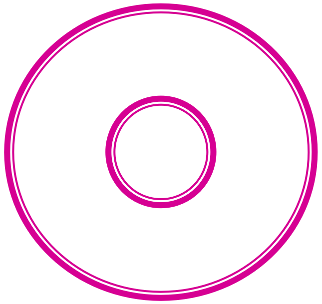
Affine and Convex set

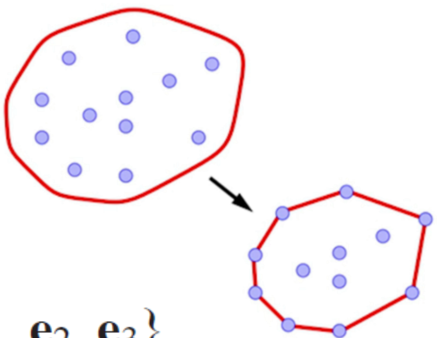
- Theorem: Let $\{S_\alpha : \alpha \in A\}$ be any collection of convex sets. Then $\bigcap_{\alpha \in A} S_\alpha$ is convex.

- Theorem: Let $\{S_\alpha : \alpha \in A\}$ be any collection of affine sets. Then $\bigcap_{\alpha \in A} S_\alpha$ is affine.

Convex Hull

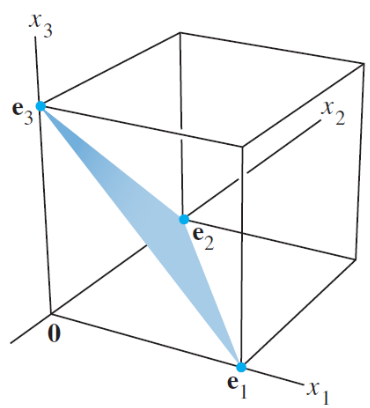
- Convex Hull Conv(S) is set of all convex combinations of points in S
- Convex Hull is the smallest convex polygon containing a given set of points





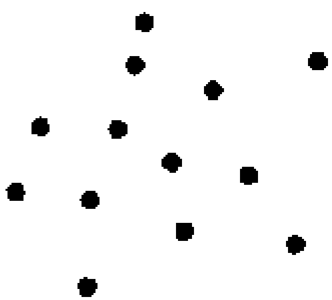
- S be the set consisting of the standard basis for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

  the conv(S) is:

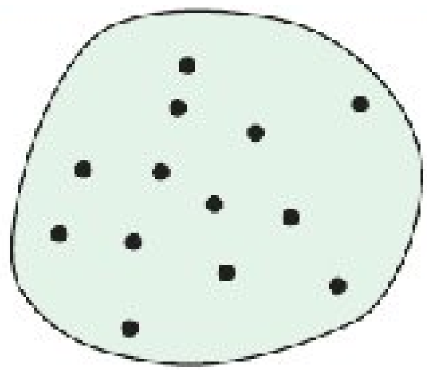


Convex Hull

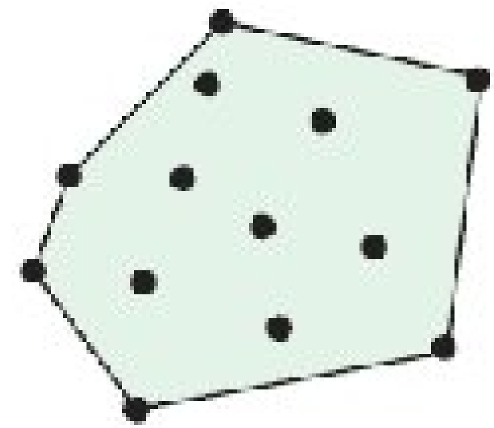
- Theorem: For any set P, the convex hull of P (conv(P)) is the intersection of all the convex sets that contain P.



set P



convex set



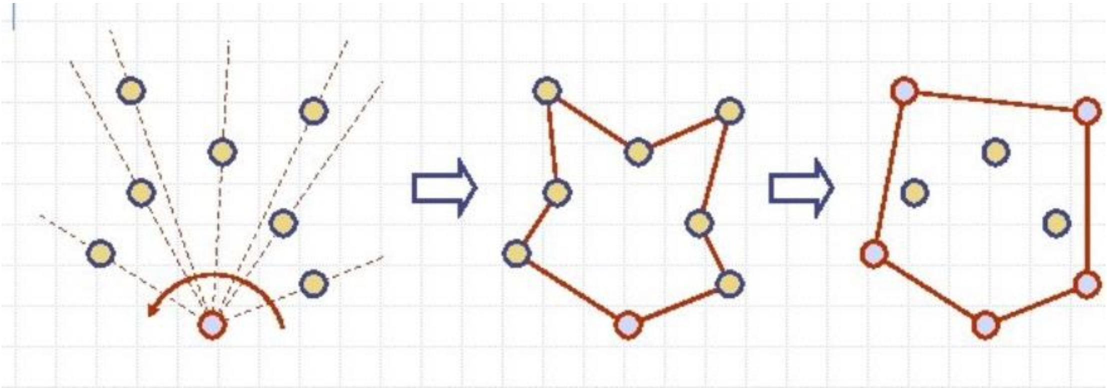
convex hull (conv P)

For any $P \subseteq \mathbb{R}^d$ *we have*

$$\operatorname{conv}(P) = \left\{ \sum_{i=1}^{n} \lambda_i p_i \;\middle|\; n \in \mathbb{N} \wedge \sum_{i=1}^{n} \lambda_i = 1 \wedge \forall i \in \{1, \ldots, n\} : \lambda_i \geqslant 0 \wedge p_i \in P \right\}$$

Convex Hull

- One way:
  - Phase 1: Find the lowest point (anchor point)
  - Phase 2: Form a nonintersecting polygon by sorting the points counterclockwise around the anchor point
  - Phase 3: While the polygon has a nonconvex vertex, remove it



- Algorithms for finding a convex hull:
  - Graham Scan
  - Jarvis March
  - Divide & Conquer

Convex Hull

- Theorem:

(Caratheodory) If S is a nonempty subset of $\mathbb{R}^n$, then every point in conv S can be expressed as a convex combination of $n+1$ or fewer points of S.

- Proof later
- Example

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

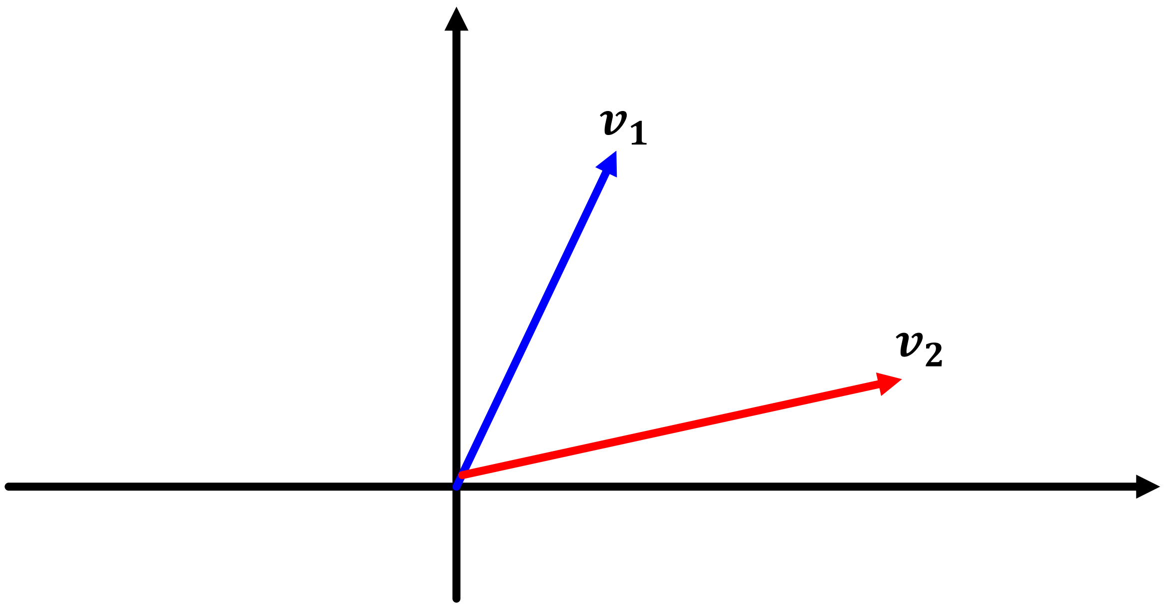
$$\tfrac{1}{4}\mathbf{v}_1 + \tfrac{1}{6}\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \tfrac{1}{12}\mathbf{v}_4 = \mathbf{p} = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix}$$

  - Then:

$$\tfrac{17}{48}\mathbf{v}_1 + \tfrac{4}{48}\mathbf{v}_2 + \tfrac{27}{48}\mathbf{v}_3 = \mathbf{p}$$

Example

- Find the convex combination and convex hull of following two vectors:



Properties of Convex Hull

- Convex hull is always a convex set.
- Convex hull is the smallest set that contains the underlying set.

Conic hull

Definition *A conic hull of a set C is the minimum set of all conic combination:*

$$\textbf{cone}\, C = \{\sum_i \theta_i x_i \mid x_i \in C,\ \theta_i \ge 0,\ i = 1, \dots, n\}.$$

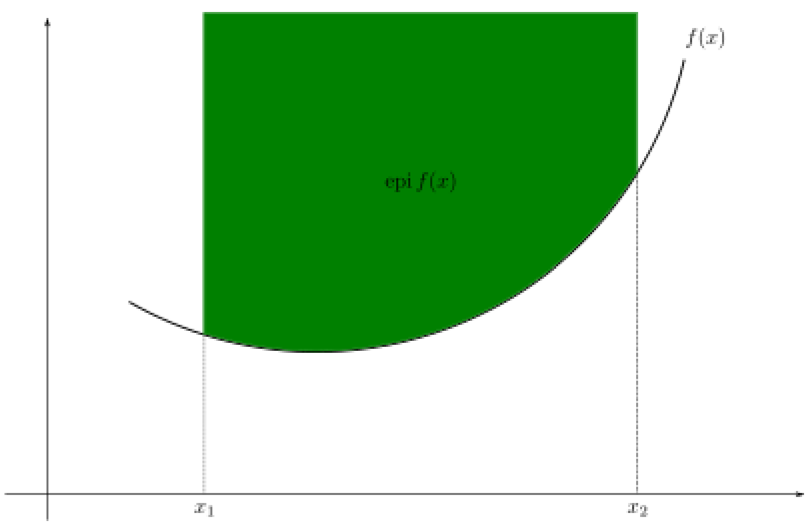
Proof. If $y \in \textbf{cone}\ C, \alpha \ge 0$, then $\alpha y = \alpha \sum_i \theta_i x_i = \sum_i (\alpha\theta_i) x_i \in \textbf{cone}\ C$. And if $y_1, y_2 \in \textbf{cone}\ C$ then $\alpha y_1 + (1-\alpha) y_2 = \alpha \sum_i \theta_i x_i + (1-\alpha) \sum_i \mu_i x_i = \sum_i (\alpha\theta_i + (1-\alpha)\mu_i) x_i \in \textbf{cone}\ C$

Convex function

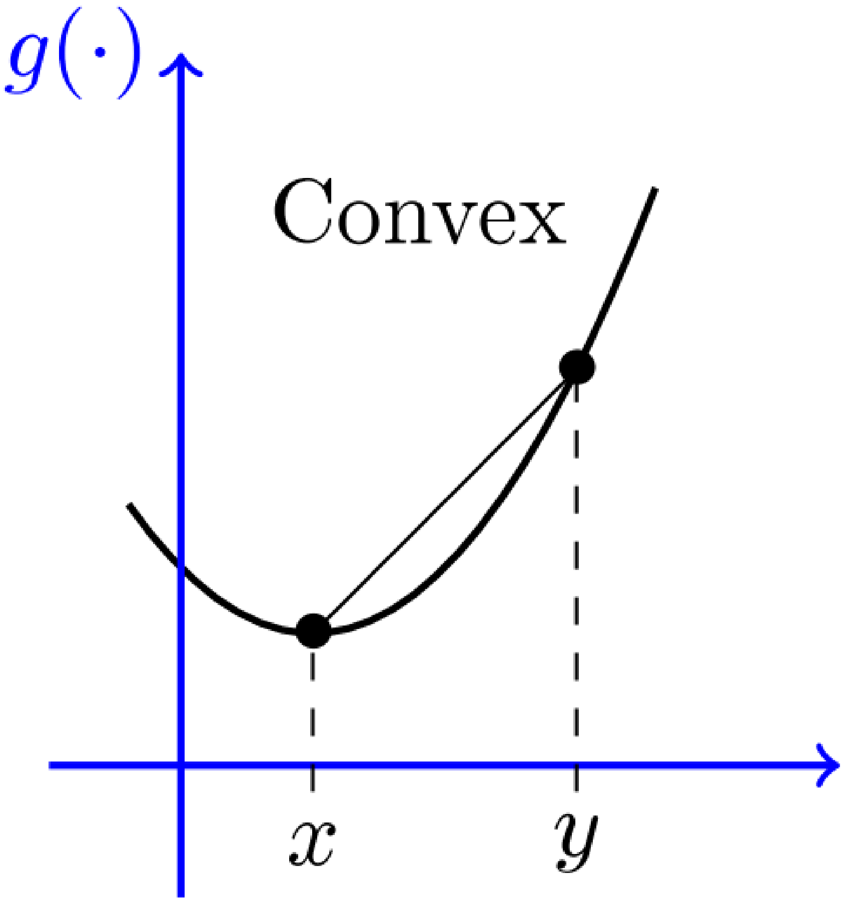
- A function is convex iff its epigraph is a convex set.
- Epigraph or supergraph

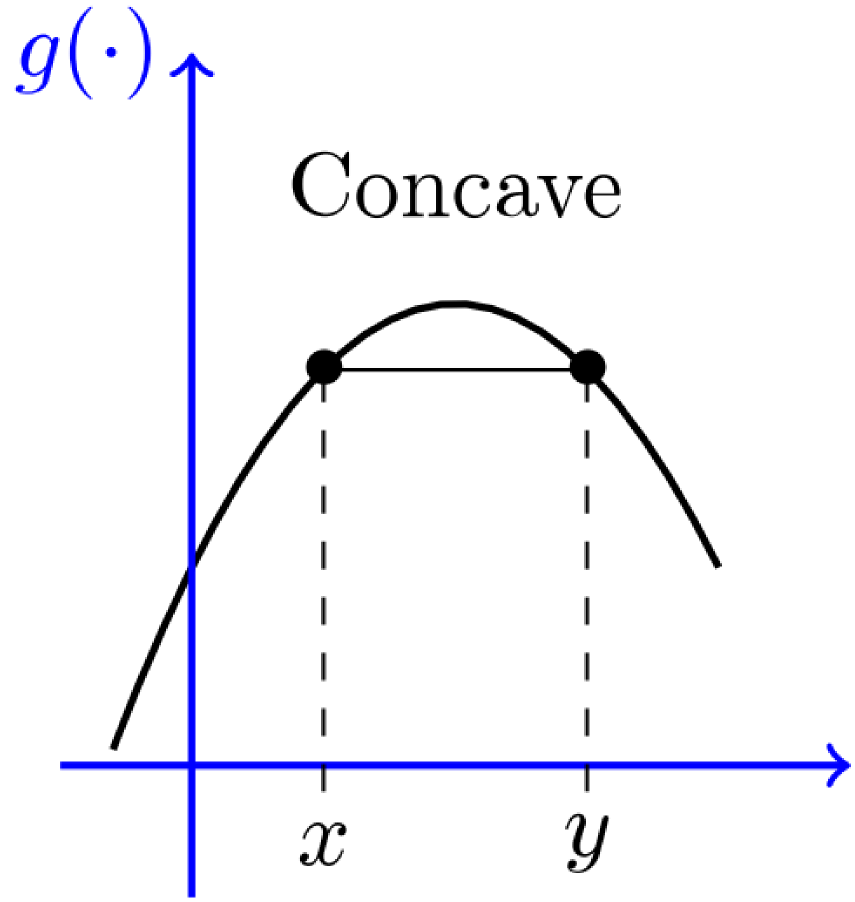
$$\mathbf{epi} f = \{(x, \mu) \, : \, x \in \mathbb{R}^n, \, \mu \in \mathbb{R}, \, \mu \geq f(x)\} \subseteq \mathbb{R}^{n+1}$$

$$f\left((1-\theta)x^{(0)} + \theta x^{(1)}\right) \leq (1-\theta) f\left(x^{(0)}\right) + \theta f\left(x^{(1)}\right), \quad \forall \theta \in [0, 1]$$



Convex and Concave Function





second derivative is nonnegative on its entire domain

Convexity

- The following operations preserve convexity:
  - The intersection of (a possibly in finite) set of convex sets
  - The sum two convex sets
  - The product of two convex sets
  - The image of a convex set under an affine function (a linear function plus an set). Similarly, the inverse image of a convex set under an affine function
  - The projection of a convex set onto some of its coordinates.

Hyperplanes

- Hyperplanes play a special role in the geometry of R^n because they divide the space into two disjoint pieces, just as a plane $(ax + by + cz = d)$ separates R^3 into two parts and a line $(ax + by = d)$ cuts through R^2.
- Hyperplane: $\{x|a^T x = b\}$ affine and convex
- Halfspace: $\{x|a^T x \leq b\}$ convex

Linear Combinations

	Linear	Affine	Convex	Conic
x in R^2				
x_1, x_2 in R^2				
x_1, x_2 in R^3				

Flat

DEFINITION

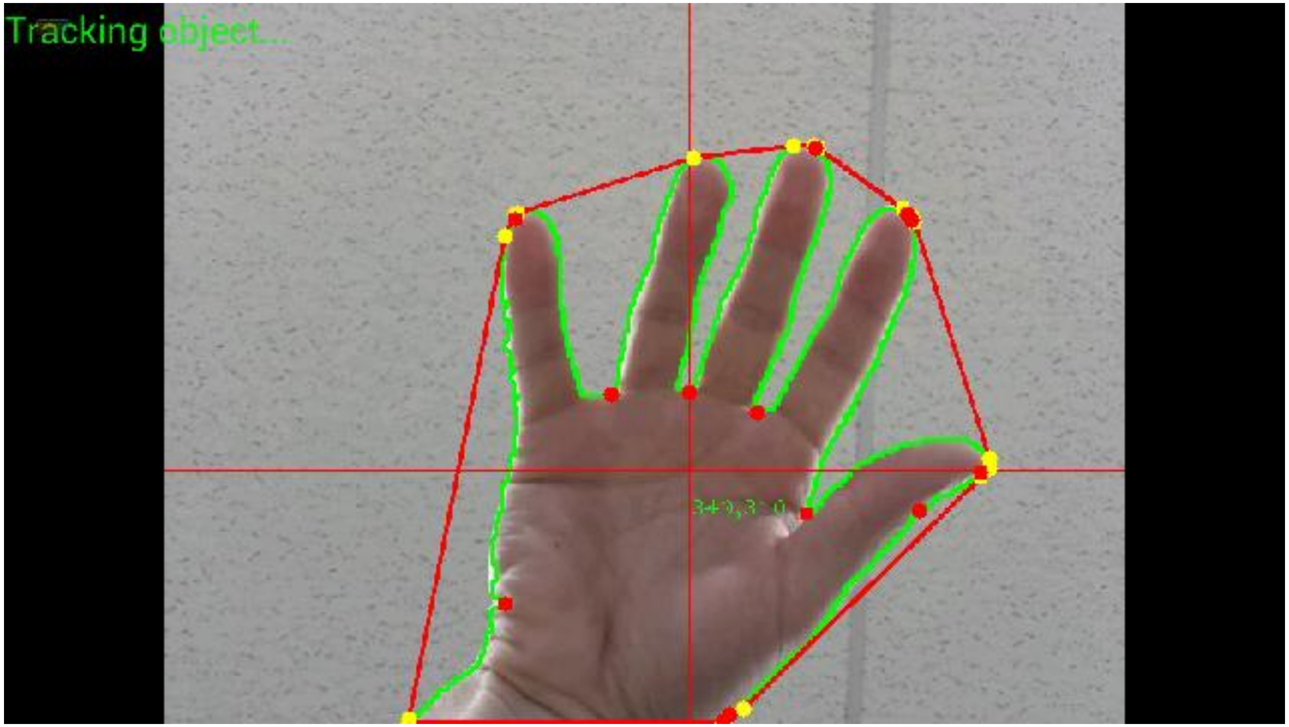
A translate of a set S in $\mathbb{R}^n$ by a vector $\mathbf{p}$ is the set $S + \mathbf{p} = \{\mathbf{s} + \mathbf{p} : \mathbf{s} \in S\}$.[2] A **flat** in $\mathbb{R}^n$ is a translate of a subspace of $\mathbb{R}^n$. Two flats are **parallel** if one is a translate of the other. The **dimension of a flat** is the dimension of the corresponding parallel subspace. The **dimension of a set** S, written as $\dim S$, is the dimension of the smallest flat containing S. A **line** in $\mathbb{R}^n$ is a flat of dimension 1. A **hyperplane** in $\mathbb{R}^n$ is a flat of dimension $n - 1$.

THEOREM

A nonempty set S is affine if and only if it is a flat.

Convex Hull Applications

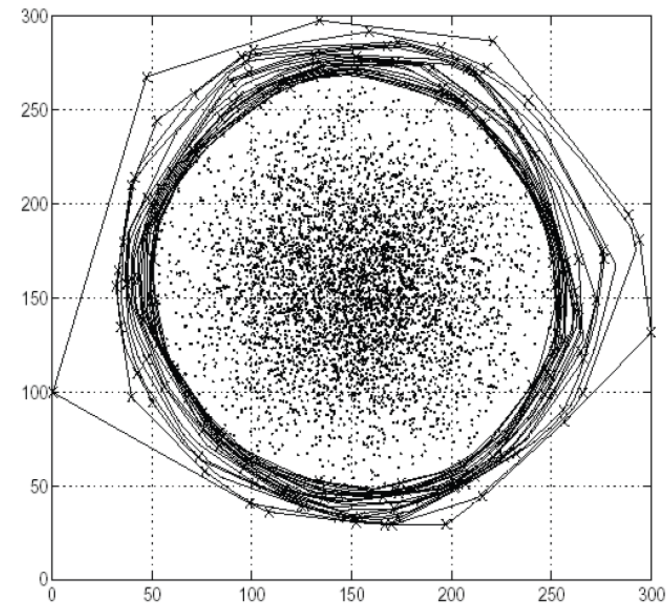
- Computer Vision



Convex Hull Applications

- Detecting outliers:

- General idea
  - Search for outliers at the border of the data space but independent of statistical distributions
  - Organize data objects in convex hull layers
  - Outliers are objects on outer layers



Picture taken from [Johnson et al. 1998]

- Basic assumption
  - Outliers are located at the border of the data space
  - Normal objects are in the center of the data space

Convex Hull Applications

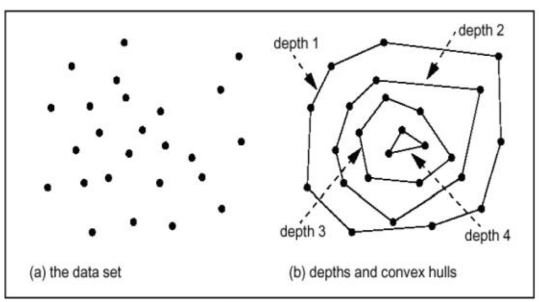
- Detecting outliers:

Depth-based Approaches

DATABASE SYSTEMS GROUP

LMU

- Model [Tukey 1977]
  - Points on the convex hull of the full data space have depth = 1
  - Points on the convex hull of the data set after removing all points with depth = 1 have depth = 2
  - …
  - Points having a depth $\leq k$ are reported as outliers



Picture taken from [Preparata and Shamos 1988]

Convex Hull Applications

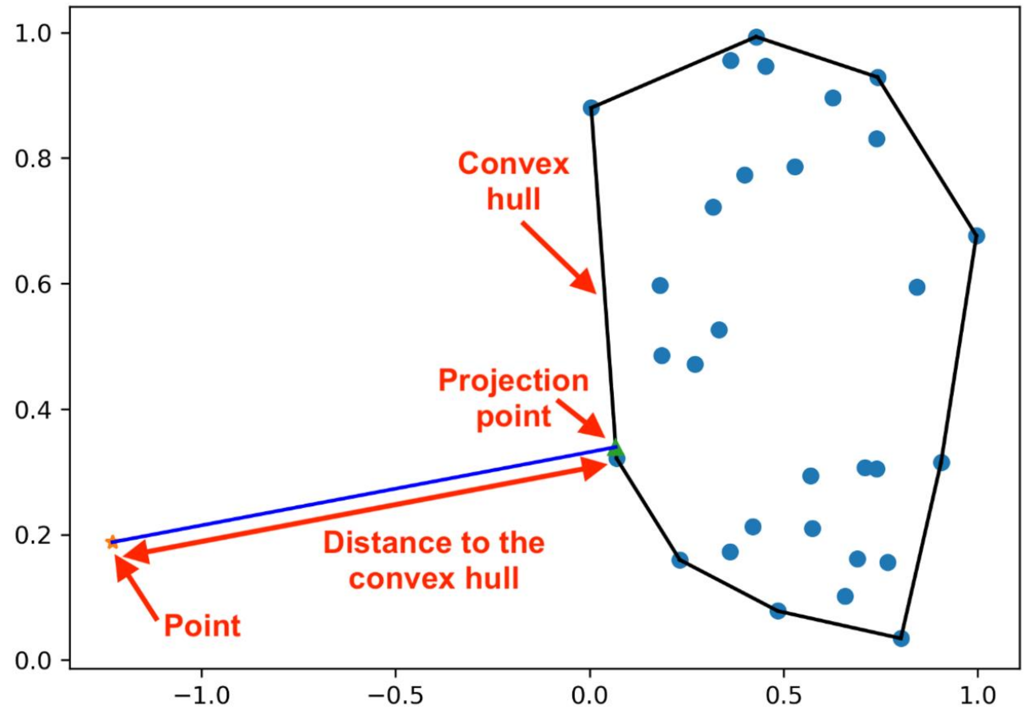
- Detecting outliers:

- Sample algorithms
  - ISODEPTH [Ruts and Rousseeuw 1996]
  - FDC [Johnson et al. 1998]

- Discussion
  - Similar idea like classical statistical approaches (k = 1 distributions) but independent from the chosen kind of distribution
  - Convex hull computation is usually only efficient in 2D / 3D spaces
  - Originally outputs a label but can be extended for scoring easily (take depth as scoring value)
  - Uses a global reference set for outlier detection

Convex Hull Applications

- Outlier Detection for new sample of data:



Convex Hull Applications

- Anomaly Detection
  - The use of a convex hull make it possible to draw the boundary between normal and abnormal data behaviour.

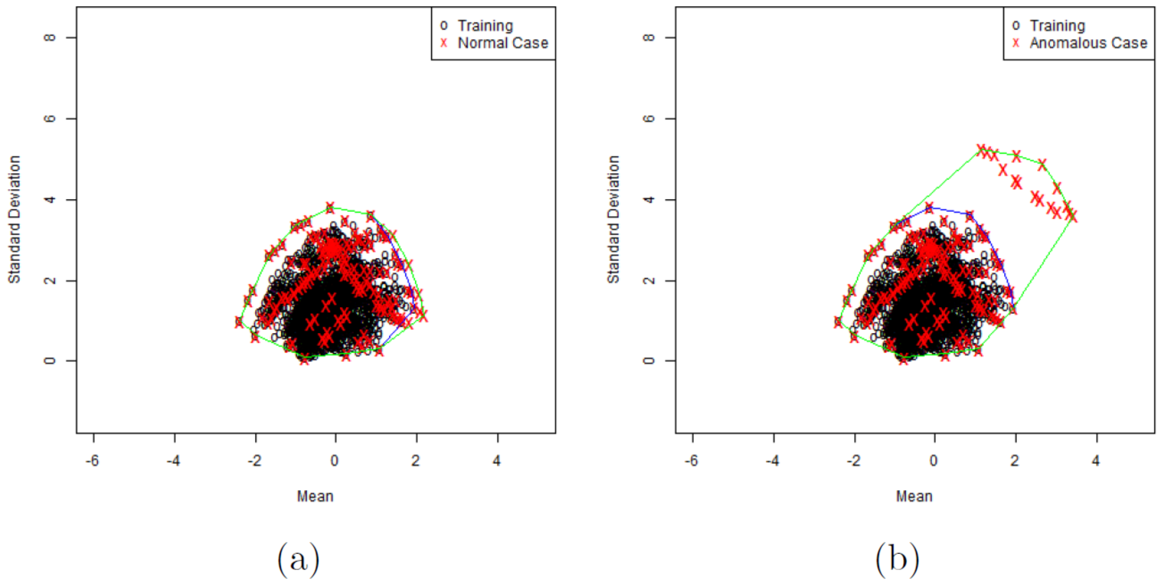
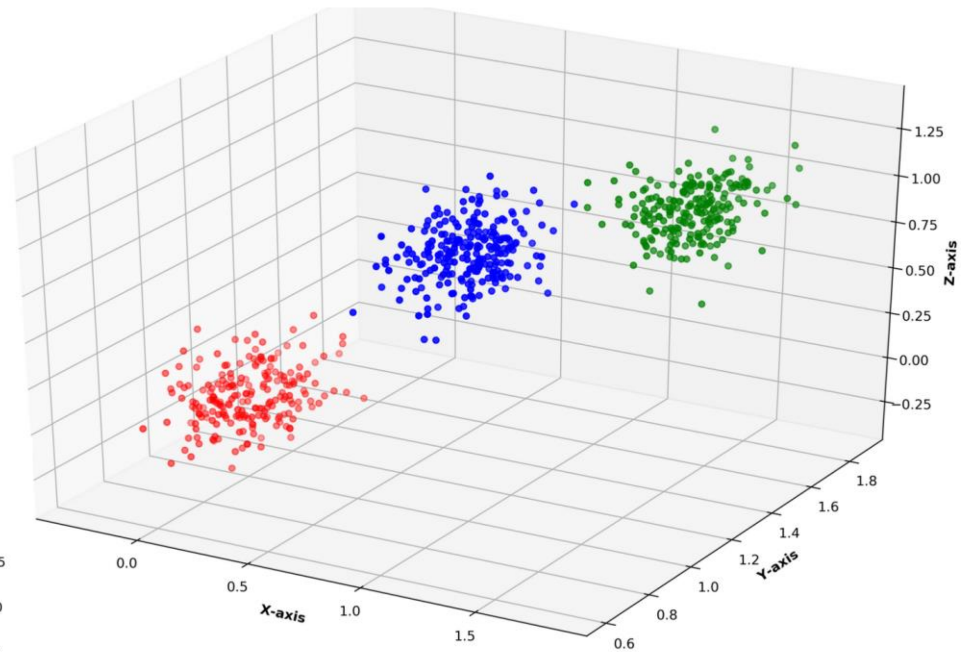
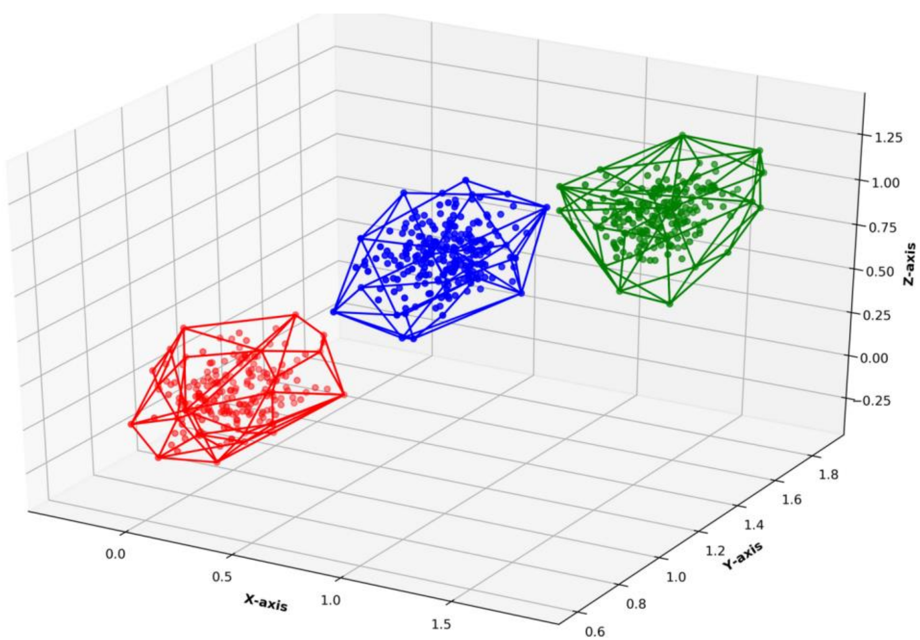


Fig. 2. Examples of convex hulls on parameter spaces using the dataset Normal-vs-2. The green line shows H_U and the blue line H_N. A normal sample detection is shown in (a) and an anomaly detection shown in (b).

Convex Hull Applications

- Clustering data





Reference

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- Chapter 8: Linear Algebra and its applications
- Chapter 2: Linear Algebra Jim Hefferon
- Chapter 4: Linear Algebra Devid Cherney