Matrix Inverse

CE282: Linear Algebra

Computer Engineering Department

Sharif University of Technology

Hamid R. Rabiee

Maryam Ramezani

Definition

❑ A number x that satisfies $xa = 1$ is called the inverse of a
❑ Inverse (i.e., $\frac{1}{a}$) exists if and only if $a \neq 0$, and is unique
❑ A matrix X that satisfies $XA = I$ is called a left inverse of A
❑ If a left inverse exists we say that A is left-invertible
❑ $A: m \times n \Rightarrow I: n \times n \Rightarrow X: n \times m$

Example

The matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$

Has two different left inverses:

$$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \qquad C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

Definition

❑ A non-zero column vector always has a left inverse.
❑ Left inverse is not unique.

Example

❑ $A = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

❑ Matrix with orthonormal columns

Definition

❑ Row vector does not have left inverse

$$A = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$$

Definition

❑ If A has a left inverse C then the columns of A are linearly independent
❑ We'll see later that the converse is also true, so:

A matrix is left-invertible if and only if its columns are linearly independent

❑ Matrix generalization of

A number is invertible if and only if it is nonzero

❑ Left-invertible matrices are all tall or square
  ❑ Wide matrix is not always left invertible
  ❑ Tall or square matrices can be left invertible

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solving linear equations with a left inverse

Method

❑ Suppose $Ax = b$, and A has a left inverse C
❑ Then $Cb = C(Ax) = (CA)x = Ix = x$
❑ So multiplying the right-hand side by a left inverse yields the solution

Definition

□ A matrix X that satisfies $AX = I$ is a right inverse of A
□ If a right inverse exists we say that A is right-invertible
□ A is right-invertible if and only if A^T is left-invertible:
$$AX = I \Longrightarrow (AX)^T = I \Longrightarrow X^T A^T = I$$

□ so we conclude:

A is right invertible if and only if its rows are linearly independent
□ Right-invertible matrices are wide or square

Solving linear equations with a right inverse

Method

- Suppose A has a right inverse B
- Consider the (square or underdetermined) equations of $Ax = b$
- $x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- So $Ax = b$ has a solution for any b

Example

- Same A, B, C in last example.
- C^T and B^T are both right inverses of A^T
- Under-determined equations $A^T x = (1, 2)$ has (different) solutions.

$$B^T(1,2) = (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}), \qquad C^T(1,2) = \left(0, \tfrac{1}{2}, -1\right)$$

  there are many other solutions as well

Definition

Left-Invertible matrix: if X is a left inverse of A, then
$$Ax = b \implies x = XAx = Xb$$
There is at most one solution (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

Why "at most"??

$XA = I$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases} \quad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Definition

❑ If the system of equations $Ax = b$ is consistent, and if a matrix B exists such that $BA = I$, then the system of equations has a unique solution, namely $x = Bb$.

❑ **Right-inversible matrix:** if X is a right inverse of A, then
$$x = Xb \implies Ax = AXb = b$$

❑ To pursue these ides further, suppose that again we want to solve a system of linear equations, $Ax = b$. Assume now that we have another matrix, B, such that $AB = I$. Then we can write $A(Bb) = (AB)b = Ib = b$; hence Bb solves the equations $Ax = b$. This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether Bb is the only solution. There may be others.

❑ **Invertible matrix:** if A is invertible, then
$$Ax = b \iff x = A^{-1}b$$

There is a unique solution

Linear equations and matrix inverse

Definition

System of linear equations $Ax = b$:

- ❑ A right inverse of A, say $AB = I$. Then Bb is a solution, as is verified by nothing $A(Bb) = (AB)b = Ib = b$.
- ❑ A left inverse of A, say $CA = I$, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

Class Activity

The only matrix having both left and right inverse is ………

Inverse

Theorem

For a square matrix left and right inverse are the same. Rows and columns are linear independent

Proof

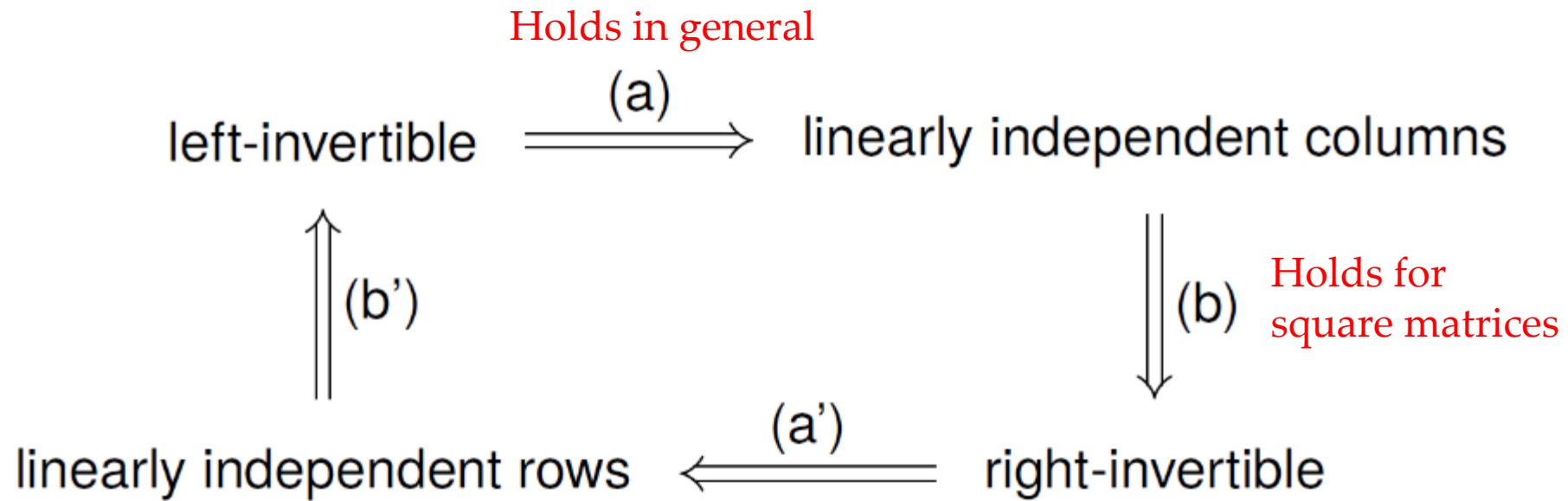If A has a left and a right inverse, they are unique and equal (and we say that A is invertible)
So A must be square
to see this: if $AX = I, YA = I$
We denote them by A^{-1}:

$$A^{-1}A = AA^{-1} = I$$

Inverse of inverse: $\left(A^{-1}\right)^{-1} = A$

Holds in general

left-invertible $\xrightarrow{\text{(a)}}$ linearly independent columns

(b') Holds for square matrices (b)

linearly independent rows $\xleftarrow{\text{(a')}}$ right-invertible

Definition

For $A \in M_{n \times n}$, if there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$, then:
- ❏ A is invertible (or nonsingular)
- ❏ B is the inverse of A

A square matrix that does not have an inverse is called non-invertible (or singular)

Note

- ❏ The definition of the inverse of a matrix is similar to that of the inverse of a scalar, i.e. $c \cdot \left(\frac{1}{c}\right) = 1$
- ❏ Since there is no inverse of a matrix (or said multiplicative inverse for real number 0 you can "imagine" that noninvertible matrices act a similar role to the real number 0 is some sense.

Inverse

Definition

The inverse of A is denoted by A^{-1}

Theorem

The inverse of a matrix is unique

Important

Finding the inverse of a matrix by the Gauss-Jordan elimination:
$$[A \quad | \quad I] \quad \text{Gauss–Jordan elimination} \quad [I \quad | \quad A^{-1}]$$

Method

- ❏ Let A be a $n \times n$ matrix:
  - ❏ Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
  - ❏ Compute the reduced echelon form of $[A : I_n]$.
- ❏ If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.
- ❏ If the reduced echelon form is not the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n then A has no inverse.

Important

An $n \times n$ matrix is invertible if and only if its reduced echelon form is I_n.

A is row equivalent to I_n

Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \implies \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad (1)$$
$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases} \quad (2)$$

This two system of linear equations have the same coefficient matrix, which is exactly the matrix A

Rest of The Example

$$(1) \Rightarrow \begin{bmatrix} 1 & 4 & | & 1 \\ -1 & -3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \begin{bmatrix} 1 & 4 & | & 0 \\ -1 & -3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -4 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus $X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Solution for $\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$ Solution for $\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ -1 & -3 & | & 0 & 1 \end{bmatrix}_{A \quad\quad I} \xrightarrow{Guass-Jordan\ elimination} \begin{bmatrix} 1 & 0 & | & -3 & -4 \\ 0 & 1 & | & 1 & 1 \end{bmatrix}_{I \quad\quad A^{-1}}$$

Definition

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):
- ❑ A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$
- ❑ A^k is invertible and $\left(A^k\right)^{-1} = A^{-k} = \left(A^{-1}\right)^k$
- ❑ cA is invertible if $c \neq 0$ and $(cA)^{-1} = \frac{1}{c}A^{-1}$
- ❑ A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Theorem

If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1 A_2 A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1} A_2^{-1} A_1^{-1}$$

Theorem

Let $AX = B$ be a system of n linear equations in n variable.
If A^{-1} exists, the solution is unique and is given by $X = A^{-1}B$

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Note

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$.

2×2 matrix A is invertible if and only if $\det A \neq 0$.

Definition

Each Elementary Matrix is E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Method

❑ Suppose A is invertible
❑ For any b, $Ax = b$ has the unique solution
$$x = A^{-1}b$$

❑ Matrix generalization of simple scalar equation $ax = b$ having solution $x = \left(\frac{1}{a}\right)b$ (for $a \neq 0$)
❑ Simple-looking formula $x = A^{-1}b$ is basis for many applications

Definition

The following are equivalent for a square matrix A:
- ❏ A is invertible
- ❏ Columns of A are linearly independent
- ❏ Rows of A are linearly independent
- ❏ A has a left inverse
- ❏ A has a right inverse

$$row\ rank(A) = col\ rank(A) = n$$

If any of these hold, all others do

Invertible matrices

Examples

- $I^{-1} = I$
- If Q is orthogonal, i.e. , square with $Q^T Q = I$, then $Q^{-1} = Q^T$
- 2×2 matrix A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

  - You need to know this formula
  - There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)

- Consider matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

  - A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

  - Verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)
  - We'll soon see how to compute the inverse

Properties

- $(AB)^{-1} = B^{-1}A^{-1}$
- If A is nonsingular, then A^T is nonsingular
$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T \text{ (sometimes denoted } A^{-T})$$
- Negative matrix powers: $\left(A^{-1}\right)^k$ is denoted by A^{-k}
- With $A^0 = I$, Identity $A^k A^l = A^{k+l}$ holds for any integers k, l

Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??