Symmetric Matrices and Quadratic Forms

CE40282-1: Linear Algebra
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Symmetric Matrix

A **symmetric** matrix is a matrix A such that $A^T = A$. Such a matrix is necessarily square. Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

$$\text{Symmetric:} \quad \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$$\text{Nonsymmetric:} \quad \begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Quadratic Form

A quadratic form is any homogeneous polynomial of degree two in any number of variables. In this situation, **homogeneous** means that all the terms are of degree two. For example, the expression $7x_1x_2 + 3x_2x_4$ is homogeneous, but the expression $x_1 - 3x_1x_2$ is not. The square of the distance between two points in an inner-product space is a quadratic form. Quadratic forms were introduced by Hermite, and 70 years later they turned out to be essential in the theory of quantum mechanics! The formal definition follows.

Quadratic Form

- Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, the scalar value $x^T Ax$ is called a ***quadratic form***.

$$x^T Ax = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} A_{ij} x_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

- A **quadratic form** on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector $\mathbf{x}$ in $\mathbb{R}^n$ can be computed by an expression of the form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is an $n \times n$ symmetric matrix. The matrix A is called the **matrix of the quadratic form**.

> Suppose $\mathcal{V}$ is a vector space over $\mathbb{R}$. Then a function $\mathbf{Q}: \mathcal{V} \to \mathbb{R}$ is called a **quadratic form** if there exists a bilinear form $f : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that
>
> $$\mathbf{Q}(\mathbf{v}) = f(\mathbf{v}, \mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

- Simplest example of a nonzero quadratic form is ………

Quadratic Form

- Example
  - Without cross-product term $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$
  - With cross-product term $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Quadratic Form

- Example

  For $\mathbf{x}$ in $\mathbb{R}^3$, let $Q(\mathbf{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$. Write this quadratic form as $\mathbf{x}^T A\mathbf{x}$.

- Quadratic forms are easier to use when they have no cross-product terms—that is, when the matrix of the quadratic form is a diagonal matrix

Change of Variable in a QF

If $\boldsymbol{x}$ represents a variable vector in $\mathbb{R}^n$, then a **change of variable** is an equation of the form

$$\boldsymbol{x} = P\boldsymbol{y} \quad \text{or equivalently,} \quad \boldsymbol{y} = P^{-1}\boldsymbol{x}$$

where P is an **invertible matrix** and $\boldsymbol{y}$ is a new variable vector in $\mathbb{R}^n$.

- $\boldsymbol{y}$ can be regarded as the coordinate vector of $\boldsymbol{x}$ relative to the basis of $\mathbb{R}^n$ determined by the columns of P.

If the change of variable is made in a quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$, then

$$\boldsymbol{x}^T A\boldsymbol{x} = (P\boldsymbol{y})^T A(P\boldsymbol{y}) = \boldsymbol{y}^T P^T AP\boldsymbol{y} = \boldsymbol{y}^T(P^T AP)\boldsymbol{y}$$

- The new matrix of the quadratic form is $P^T AP$.
- A is symmetric, so there is an **orthogonal matrix** P such that $P^T AP$ is a diagonal matrix D.
- Then the quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ becomes $\boldsymbol{y}^T D\boldsymbol{y}$, there is **no cross-product term**.

Quadratic Form

If $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *real matrices connected by the relation*

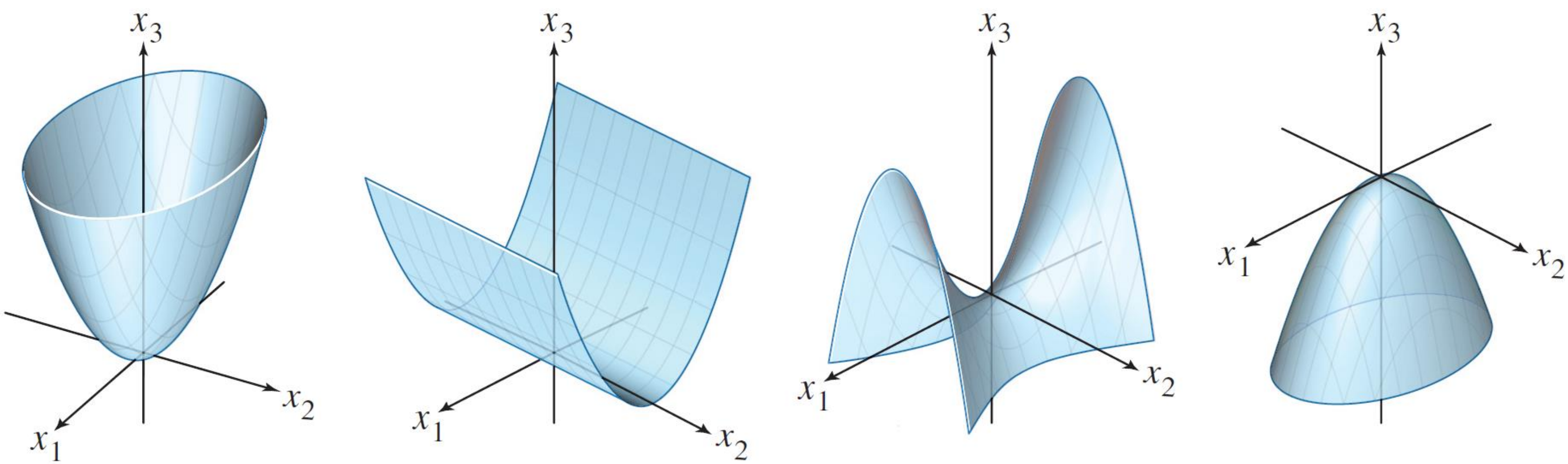
$$\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

then the corresponding quadratic forms of $\mathbf{A}$ *and* $\mathbf{B}$ *are identical, and* $\mathbf{B}$ *is symmetric.*

Classifying Quadratic Forms

When A is an $n \times n$ matrix, the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is a real-valued function with domain $\mathbb{R}^n$.

$$\text{point } (x_1, x_2, z) \text{ where } z = Q(\mathbf{x})$$



(a) $z = 3x_1^2 + 7x_2^2$ (b) $z = 3x_1^2$ (c) $z = 3x_1^2 - 7x_2^2$ (d) $z = -3x_1^2 - 7x_2^2$

Classifying Quadratic Forms

Definition

A quadratic form Q is:

a) **positive definite** if $Q(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$;
b) **negative definite** if $Q(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \neq \mathbf{0}$;
c) **indefinite** if $Q(\boldsymbol{x})$ assumes both positive and negative values;
d) **positive semidefinite** if $Q(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$;
e) **negative semidefinite** if $Q(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$.

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$$

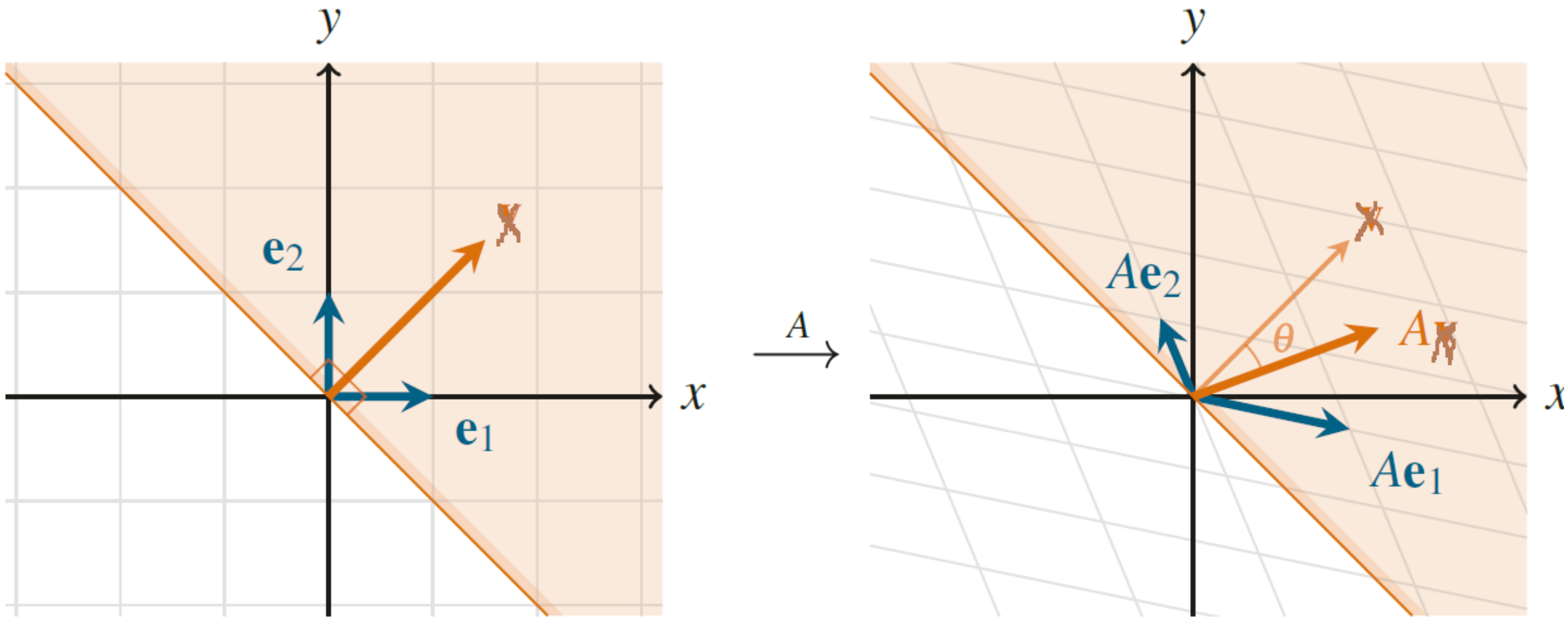
- A symmetric matrix $A \in \mathbb{S}^n$ is ***positive definite*** (PD) if for all non-zero vectors $x \in \mathbb{R}^n$, $x^T A x > 0$. This is usually denoted $A \succ 0$, and often times the set of all positive definite matrices is denoted $\mathbb{S}^n_{++}$.
- A symmetric matrix $A \in \mathbb{S}^n$ is ***positive semidefinite*** (PSD) if for all vectors $x^T A x \geq 0$. This is written $A \succeq 0$, and the set of all positive semidefinite matrices is often denoted $\mathbb{S}^n_+$.
- Likewise, a symmetric matrix $A \in \mathbb{S}^n$ is ***negative definite*** (ND), denoted $A \prec 0$ if for all non-zero $x \in \mathbb{R}^n$, $x^T A x < 0$.
- Similarly, a symmetric matrix $A \in \mathbb{S}^n$ is ***negative semidefinite*** (NSD), denoted $A \preceq 0$ if for all $x \in \mathbb{R}^n$, $x^T A x \leq 0$.
- Finally, a symmetric matrix $A \in \mathbb{S}^n$ is ***indefinite***, if it is neither positive semidefinite nor negative semidefinite — i.e., if there exists $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T A x_1 > 0$ and $x_2^T A x_2 < 0$.

- For diagonal matrix $A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \Longrightarrow x^T A x = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2$

Geometric interpretation

- $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$

- $\theta = \arccos\left(\frac{(Ax).x}{||x||||Ax||}\right)$



Characterization of Positive Semidefinite Matrices

- Suppose $A \in \mathcal{M}_n(\mathbb{F})$ is self-adjoint. The following are equivalent:
  a) A is positive semidefinite,
  b) All of the eigenvalues of A are non-negative,
  c) There is a matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = B^*B$, and
  d) There is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ with non-negative diagonal entries and a unitary matrix $U \in \mathcal{M}_n(\mathbb{F})$ such that $A = UDU^*$.

Characterization of Positive Definite Matrices

- Suppose $A \in \mathcal{M}_n(\mathbb{F})$ is self-adjoint. The following are equivalent:

  a) A is positive *definite*,

  b) All of the eigenvalues of A are *strictly positive*,

  c) There is an *invertible* matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = B^*B$, and

  d) There is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ with *strictly positive* diagonal entries and a unitary matrix $U \in \mathcal{M}_n(\mathbb{F})$ such that $A = UDU^*$.

Quadratic form

Theorem

Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\boldsymbol{x}^T A \boldsymbol{x}$ is:

a) **positive definite** if and only if the eigenvalues of A are **all positive**;
b) **negative definite** if and only if the eigenvalues of A are **all negative**;
c) **indefinite** if and only if A has **both positive and negative** eigenvalues.
d) How about semidefinite?

Positive Definite Matrices

- For a symmetric matrix the signs of the pivots are the signs of the eigenvalues.

$$\text{number of positive pivots} = \text{number of positive eigenvalues.}$$

A symmetric matrix **A** is to be **positive definite** if

1. all the eigenvalues are positive
2. all the pivots are positive
3. all the determinants are positive
4. $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \ \forall \mathbf{x}$ except $\mathbf{x} = 0$.

If any of the eigenvalues or pivots or determinants is zero, that matrix is called a **Positive semidefinite** matrix.

Positive Definite Matrices

- Eigenvalue and Eigenvector
- A positive definite matrix S has positive eigenvalues, positive pivots, positive determinants, and positive energy v^TSv for every vector v. $S = A^TA$ is always positive definite if A has independent columns.
- Positive Definite Matrix
  - Five Tests
    - $x^TSx > 0$ for all x (other than zero-vector)
    - If S is positive definite $S = A^TA$ (A must have independent columns)
    - All eigenvalues are greater than 0
    - Sylvester's Criterion: All upper left determinants must be > 0
    - Every pivot must be > 0

Positive Definite Matrix

- If S is positive definite $S = A^T A$ (A must have independent columns): $A^T A$ is positive definite iff the columns of A are linearly independent.
  - Proof?

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Eigenvalues & Positive Definite Matrices

- POSITIVE DEFINITE $\Rightarrow$ POSITIVE EIGENVALUES
  - Proof?
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  - Proof?

Positive Definite Matrices

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Left determinants & Positive Definite Matrix

- All upper left determinants must be > 0

$$A = \begin{bmatrix} \boxed{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- POSITIVE DEFINITE $\Rightarrow$ POSITIVE DETERMINANT

  - Proof?

Sylvester's Criterion

- Suppose $A \in \mathcal{M}_n$ is self-adjoint. Then A is positive definite if and only if, for all $1 \leq k \leq n$, the determinant of the top-left $k \times k$ block of A is strictly positive.

Sylvester's Criterion for Positive SemiDefinite Matrices

- A principal minor of a square matrix is the determinant of a submatrix of A that is obtained by deleting some (or none) of its rows as well as the corresponding columns.
- A matrix is positive semidefinite if and only if all of its principal minors are non-negative

$$B = \begin{bmatrix} a & b & c \\ \overline{b} & d & e \\ \overline{c} & \overline{e} & f \end{bmatrix}$$

are a, d, f, $\det(B)$ itself, as well as

$$\det\left(\begin{bmatrix} a & b \\ \overline{b} & d \end{bmatrix}\right) = ad - |b|^2, \quad \det\left(\begin{bmatrix} a & c \\ \overline{c} & f \end{bmatrix}\right) = af - |c|^2, \quad \text{and}$$

$$\det\left(\begin{bmatrix} d & e \\ \overline{e} & f \end{bmatrix}\right) = df - |e|^2.$$

Positive Definite Matrices

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- A positive definite matrix S has positive eigenvalues, positive pivots, positive determinants, and positive energy v^TSv for every vector v. $S = A^TA$ is always positive definite if A has independent columns.
- Positive Definite Matrix
  - Five Tests
    - $x^TSx > 0$ for all x (other than zero-vector)
    - If S is positive definite $S = A^TA$ (A must have independent columns)
    - All eigenvalues are greater than 0
    - Sylvester's Criterion: All upper left determinants must be > 0
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Pivots & Positive Definite Matrix

- Every pivot must be > 0
  - Pivots are, in general, way easier to calculate than eigenvalues.
  - Just perform elimination and examine the diagonal terms.
  - Example: Is the following matrix positive definite matrix?
    $$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$
  - Note: number of positive (negative) pivots = number of positive (negative) eigenvalue

Pivots & Positive Definite Matrix

- $\text{POSITIVE PIVOTS} \Rightarrow \text{POSITIVE DEFINITE}$

  - Proof?

Properties

- If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ will also be positive definite.

Properties

- If $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices, $\mathbf{A} + \mathbf{B}$ will also be a positive definite matrix.

Properties

- Positive definite and negative definite matrices are always full rank, and hence, invertible.

- For $A \in \mathbb{R}^{m \times n}$ gram matrix is always positive semidefinite. Further, if $m \geq n$ (and we assume for convenience that A is full rank), then gram matrix is positive definite.

Properties

- Suppose $A, B \in \mathcal{M}_n$ are positive (semi)definite, $P \in \mathcal{M}_{n,m}$ is any matrix, and $c > 0$ is a real scalar. Then
  a) $A + B$ is positive (semi)definite,
  b) cA is positive (semi)definite,
  c) A^T is positive (semi)definite, and
  d) P^*AP is positive semidefinite. Furthermore, if A is positive definite then P^*AP is positive definite if and only if $\operatorname{rank}(P) = m$.