



# Subspaces

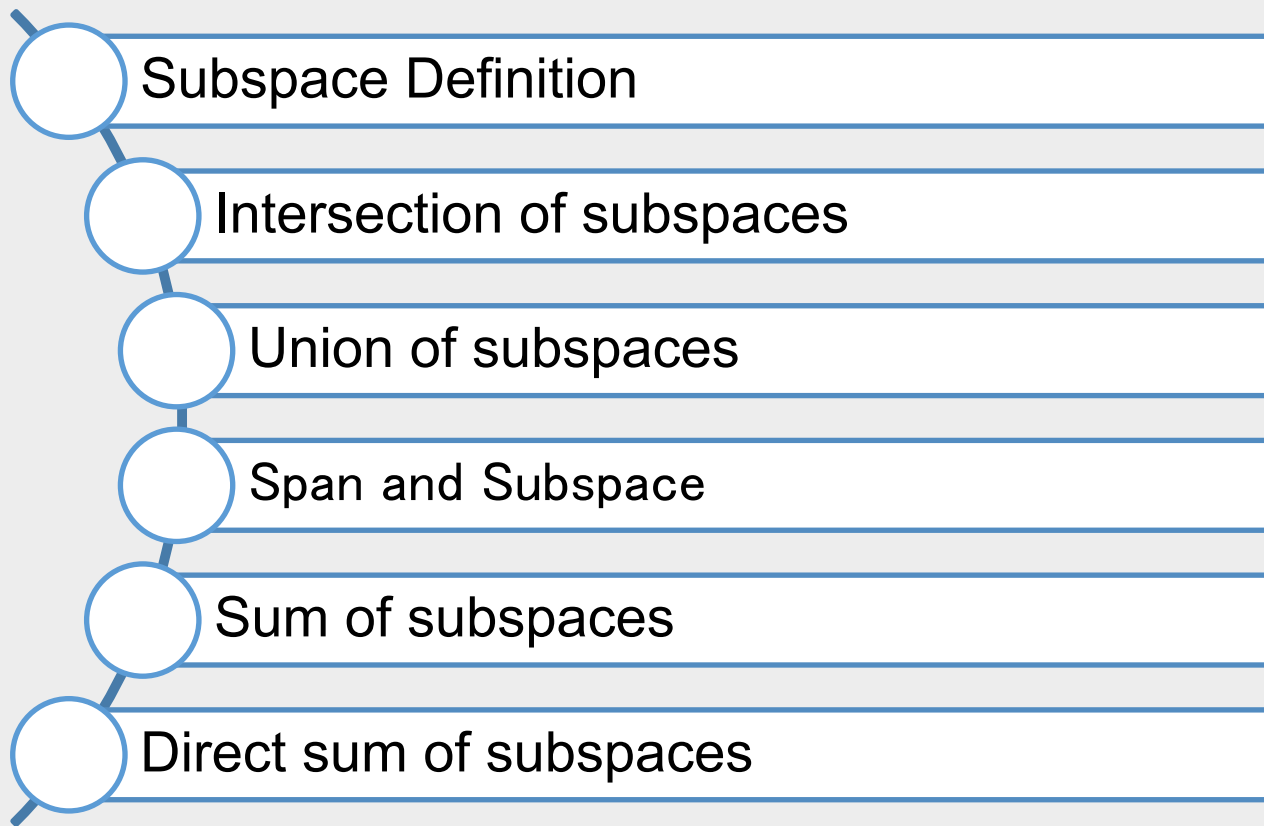
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## Linear Algebra

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# Subspace Definition

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## Definition

A **non-empty subset** of vector space for which closure holds for addition and scalar multiplication is called a subspace.

**Subspace:** If  $V$  is a vector space and **subset**  $U \subseteq V$ , then  **$U$  is itself a vector space** with the **same** addition and scalar multiplication as  $V$ .

- ❑ Zero vector is a subspace of every vector space.
- ❑ Vector space is a subspace of itself.





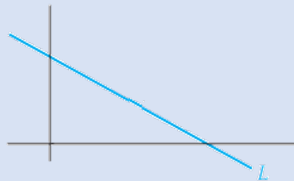
A subspace of  $\mathbb{R}^n$  is any set  $H$  in  $\mathbb{R}^n$  that has these properties:

- The zero vector is in  $H$ .
- For each  $u$  and  $v$  in  $H$ , the sum  $u + v$  is in  $H$ .
- For each  $u$  in  $H$  and each scalar  $c$ , the vector  $cu$  is in  $H$ .

## Example

-  $H = \text{Span} \{x_1, x_2\}$ , then  $H$  is a subspace of  $\mathbb{R}^2$ , where  $x_1, x_2$  are vectors in  $\mathbb{R}^2$ .

- Is  $L$  subspace of  $\mathbb{R}^2$  ?



- The vector space  $\mathbb{R}^2$  is a subspace of  $\mathbb{R}^3$ ?

- Is  $H$  a subspace of  $\mathbb{R}^3$  ?  $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$

# Vector Space vs Subspace



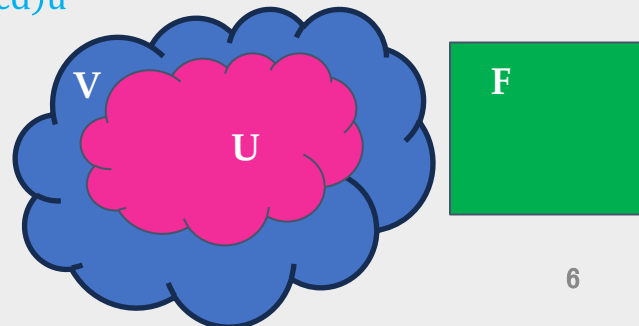
Let  $V$  be a vector subspace and let  $U \subseteq V$ :

## Vector Space

1.  $u + v \in V$
2.  $u + v = v + u$
3.  $(u + v) + w = u + (v + w)$
4. There is a vector  $0 \in V$  such that  $u + 0 = u$
5. Foreach  $u \in V$ , there is a vector  $-u \in V$  such that  $u + (-u) = 0$
6.  $cu \in V$
7.  $c(u + v) = cu + cv$
8.  $(c + d)u = cu + du$
9.  $c(du) = (cd)u$
10.  $1u = u$

## Subspace

1.  $u + v \in U$
2.  $u + v = v + u$
3.  $(u + v) + w = u + (v + w)$
4. **There is a vector  $0 \in U$  such that  $u + 0 = u$**
5. **Foreach  $u \in U$ , there is a vector  $-u \in U$  such that  $u + (-u) = 0$**
6.  **$cu \in U$**
7.  $c(u + v) = cu + cv$
8.  $(c + d)u = cu + du$
9.  $c(du) = (cd)u$
10.  $1u = u$





## Theorem

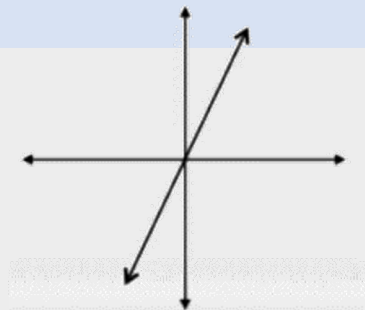
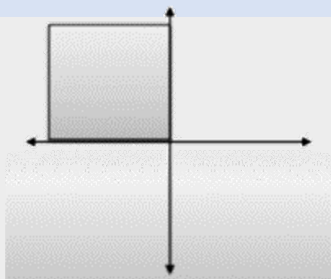
A non-empty subset  $U$  of  $V$  is a subspace of  $V$  if and only if for each pair of vectors  $b, c$  in  $U$  and each scalar  $\alpha$  in  $F$  the vector  $\alpha b + c$  is again in  $U$ .

Proof:



## Example

- In  $F^n$ , the set of  $n$ -tuples  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  with  $x_1 = 0$
- In  $F^n$ , the set of  $n$ -tuples  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  with  $x_1 = 1 + x_2$  ( $n \geq 2$ )
- Every vector space with more than one member has at least \_\_\_\_ subspaces.
- Name subspace for  $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$
- Following figures are subspace of  $\mathbb{R}^2$ ?







## Example

Let  $H$  be the set of all vectors of the form  $\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix}$  where  $a, b$  are arbitrary scalars.

That is, let  $H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} : a, b \text{ in } R \right\}$ . Show that  $H$  is a subspace of  $\mathbb{R}^4$ .



## Example

- Set of all continuous real-valued functions on  $\mathbb{R}$  is a subspace of the vector space of all functions on  $\mathbb{R}$ .
- Set of all differentiable real-valued functions on  $\mathbb{R}$  is a subspace of the vector space of all functions on  $\mathbb{R}$ .
- Set of all functions  $D(f(x)) = f'(x)$  is a subspace of the vector space of all functions on  $\mathbb{R}$ .

# Intersection of subspaces

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## Theorem

If  $W_1$  and  $W_2$  are subspaces of  $V$ , then  $W_1 \cap W_2$  is a subspace.

**Proof:**

**$W_1 \cap W_2$  is the largest subspace contained in  $W_1$  and  $W_2$  both.**



## Theorem

Intersection of any collection of subspaces of a vector space  $V$ , is a subspace of  $V$ .

**Proof:**

# Union of subspaces

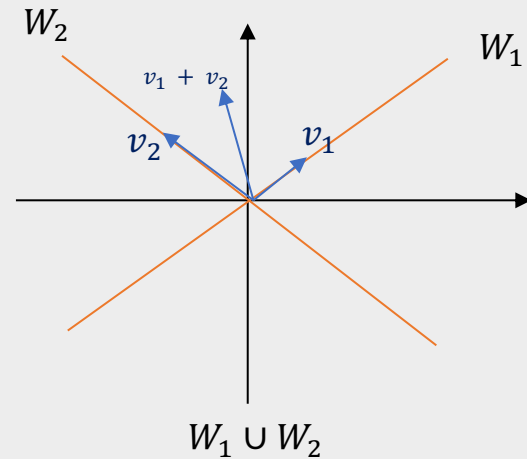
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## Theorem

The union of two sub-spaces may not be a subspace.

**Proof:**





## Theorem

Fact: The union of two sub-spaces is not a subspace unless **one is contained in the other**.

$W_1$  and  $W_2$  are subspaces of  $V$ , then  $W_1 \cup W_2$  is subspace of  $V$  **if and only if**  
 $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$

**Proof:**



# Span and Subspace

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## Theorem

If  $v_1, v_2, \dots, v_p$  are in a vector space  $V$ , then  $\text{Span} \{v_1, v_2, \dots, v_p\}$  is a subspace of  $V$ .

Proof:

# Sum of subspaces

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- There are two reasons to use the sum of two vector spaces.
  - to build new vector spaces from old ones.
  - to decompose the known vector space into sum of two (smaller) spaces.
  
- Since we consider linear transformations between vector spaces, these sums lead to representations of these linear maps and corresponding matrices into forms that reflect these sums. In many very important situations, we start with a vector space  $V$  and can identify subspaces “internally” from which the whole space  $V$  can be built up using the construction of sums.



## Definition

Let  $A$  and  $B$  be non-empty subsets of a vector space  $V$ . The **sum of  $A$  and  $B$** , denoted  $A+B$ , is the set of all possible sums of elements from both subsets:  $A + B = \{a + b : a \in A, b \in B\}$

## Example

–  $A = \{(2,3)\}$   $B = \{t(3,1) | t \text{ is scalar}\}$ ,  $A+B$ ?

–  $A = \{t_1(1,2,0) | t_1 \text{ is scalar}\}$   $B = \{t_2(0,1,2) | t_2 \text{ is scalar}\}$ ,  $A+B$ ?



## Theorem

If  $W_1, \dots, W_m$  are subspaces of  $V$ , then  $W_1 + \dots + W_m$  is a subspace of  $V$ .

# Direct sum of subspaces

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## Definition

$U + W$  is called a **direct sum**, if any element in  $U + W$  can be written uniquely as  $u + w$  where  $u \in U$  and  $w \in W$  (Notation:  $U \oplus W$ )

## Example

Check where  $U \oplus W$  exists?

$$\text{a) } U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, W = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$\text{b) } U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, W = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix}$$





## Theorem

If  $U$  and  $W$  are subspaces of  $V$ , then  $U \oplus W$  is a subspace, if and only if  $U \cap W = \{0\}$

Proof:



## Example

Let  $E$  denote the set of all polynomials of even powers,

$$E = \{a_n t^{2n} + a_{n-1} t^{2n-2} + \dots + a_0\},$$

and  $O$  be the set of all polynomials of odd powers,

$$O = \{a_n t^{2n+1} + a_{n-1} t^{2n-1} + \dots + a_0\}.$$

The set of all polynomials  $P$  is a direct sum of  $E$  and  $O$ :

$$P = E \oplus O$$

It is easy to see that any polynomial (or function) can be uniquely decomposed into direct sum of its even and odd counterparts:

$$p(t) = \frac{p(t) + p(-t)}{2} + \frac{p(t) - p(-t)}{2}$$



## Example

Prove set of all bound functions such as

$$W = \{f(x) \mid \exists M \in \mathbb{R} \text{ such that } |f(x)| \leq M, \forall x \in \mathbb{R}\}$$

is a subspace of  $V = \{\text{all functions from } \mathbb{R} \text{ to } \mathbb{R}\}$

## Note

Triangle Inequality for Real Numbers

$$|a + b| \leq |a| + |b|$$



- ❑ LINEAR ALGEBRA: Theory, Intuition, Code
- ❑ David Cherney,
- ❑ Online Courses!
- ❑ Chapter 4 of Elementary Linear Algebra with Applications
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