Scalar-valued Functions (Linear and Affine)

CE40282-1: Linear Algebra
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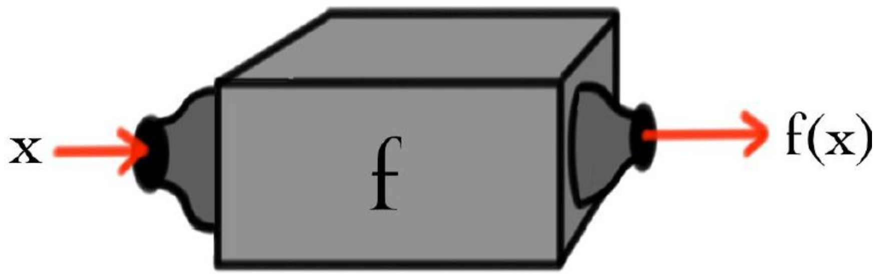
What are Functions?

- Think of a function as a machine f into which one may feed a real number. For each input x this machine outputs a $f(x)$.



(A) What number x satisfies $10x = 3$?

(B) What 3-vector u satisfies $\begin{pmatrix}1\\1\\0\end{pmatrix} \times v = \begin{pmatrix}0\\1\\1\end{pmatrix}$?

What vector X satisfies $f(X) = B$?

(C) What polynomial p satisfies $\int_{-1}^{1} p(y)dy = 0$ and $\int_{-1}^{1} yp(y)dy = 1$?

(D) What power series $f(x)$ satisfies $x\frac{d}{dx}f(x) - 2f(x) = 0$?

(E) What number x satisfies $4x^2 = 1$?

$$\boxed{\begin{array}{lll} u_1 & v_1 & \\ u_2 & v_2 & \\ u_3 & v_3 & u_2v_3 - u_3v_2 \\ u_1 & v_1 & u_3v_1 - u_1v_3 \\ u_2 & v_2 & u_1v_2 - u_2v_1 \end{array}}$$

Note

Linear and affine functions in this session are scalar-valued. We focus on the linear function machine of the previous slide, which outputs are scalar values. Remains will discuss later.

What are Linear Functions?

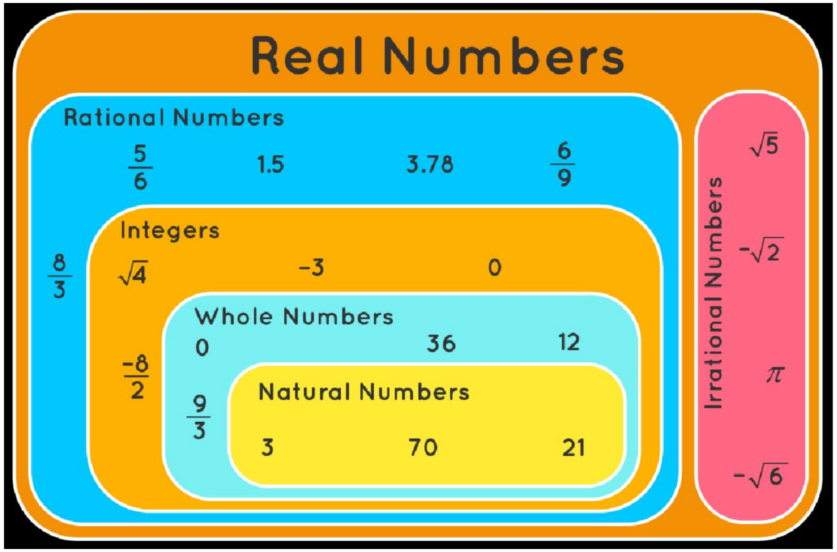
- $f: R^n \rightarrow R$ means that f is a function that maps real n-vectors to real numbers
- $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- $f(x) = (x_1, x_2, \dots, x_n)$: argument
- A function $f: R^n \rightarrow R$ is linear if it satisfies the following two properties:
  - Homogeneity: For any n-vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$
  - Additivity: For any n-vector x and y, $f(x + y) = f(x) + f(y)$
- Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

- A function that satisfies the superposition property is called linear.

Homogeneity and Additivity

- Additivity: For any n-vector x and y,
$$f(x+y) = f(x) + f(y)$$
- Homogeneity: For any n-vector x and any scalar $\alpha \in R$:
$$f(\alpha x) = \alpha f(x)$$



Inner product is Linear Function?

- Show that inner product has superposition property so it is a linear function.

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

What are Linear Functions?

- A function defined as the inner product of its argument with some fixed vector is linear.
- If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.
  - Proof
- The representation of a linear function f as $f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.
  - Proof
- Is average a linear function?

Affine Function

- A function $f: R^n \rightarrow R$ is affine if and only if it can expressed as $f(x) = a^T x + b$ (linear function plus a constant (offset))
- Superposition property for affine function:
  $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\alpha + \beta$=1
  - Proof

Affine Function

- Any scalar-valued function that satisfies the restricted superposition property is affine.
  - Proof

- Conclusion: Important note:

every affine function can be written as $f(x) = a^T x + b$ with:

$$
\begin{aligned}
a &= (f(e_1) - f(0),\ f(e_2) - f(0),\ \ldots,\ f(e_n) - f(0)) \\
b &= f(0)
\end{aligned}
$$

Conclusion

- Method 1:
  - Linear:

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n), \forall \alpha_1, \dots, \alpha_n$$

  - Affine:

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n), \alpha_1 + \cdots + \alpha_n = 1$$

- Method 2:
  - Linear $f(x) = a^T x$
  - Affine $f(x) = a^T x + b$

Conclusion

- In many applications, scalar-valued functions of n variables, or relations between n variables and a scalar one, can be approximated as linear or affine functions, which is called "**Model**".

Scalar-valued function of a scalar

- Derivative of function $f: R \rightarrow R$ at the point z: $\left(f'(z)\right)$:

$$\lim_{t \to 0} \frac{f(z+t) - f(z)}{t}$$

  - It gives the slope of the graph of f at the point (z; f(z)).
  - $f'(z)$ is a scalar-valued function of a scalar variable

Review: Scalar-valued function of a vector

- The partial derivative of function $f: R^n \to R$ at the point z, with respect to its ith argument

$$\begin{aligned}\frac{\partial f}{\partial x_i}(z) &= \lim_{t \to 0} \frac{f(z_1, \ldots, z_{i-1}, z_i + t, z_{i+1}, \ldots, z_n) - f(z)}{t} \\ &= \lim_{t \to 0} \frac{f(z + te_i) - f(z)}{t},\end{aligned}$$

  - The partial derivative is the derivative with respect to the ith argument, with all other arguments fixed.

Review: Gradient

- Gradient: The partial derivatives of f with respect to its n arguments can be collected into an n vector called the gradient of f (at z):

$$\nabla f(z) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(z) \\ \vdots \\ \frac{\partial f}{\partial x_n}(z) \end{bmatrix}.$$

- Gradient of a combination of functions

$$f(x) = ag(x) + bh(x)$$

$$\nabla f(z) = a\nabla g(z) + b\nabla h(z)$$

How to find an approximate affine model

- $f: R^n \to R$ is differentiable: its partial derivatives exist
- The (first-order) Taylor approximation of f near (or at) the point z:

$$\hat{f}(x) = f(z) + \frac{\partial f}{\partial x_1}(z)(x_1 - z_1) + \cdots + \frac{\partial f}{\partial x_n}(z)(x_n - z_n),$$

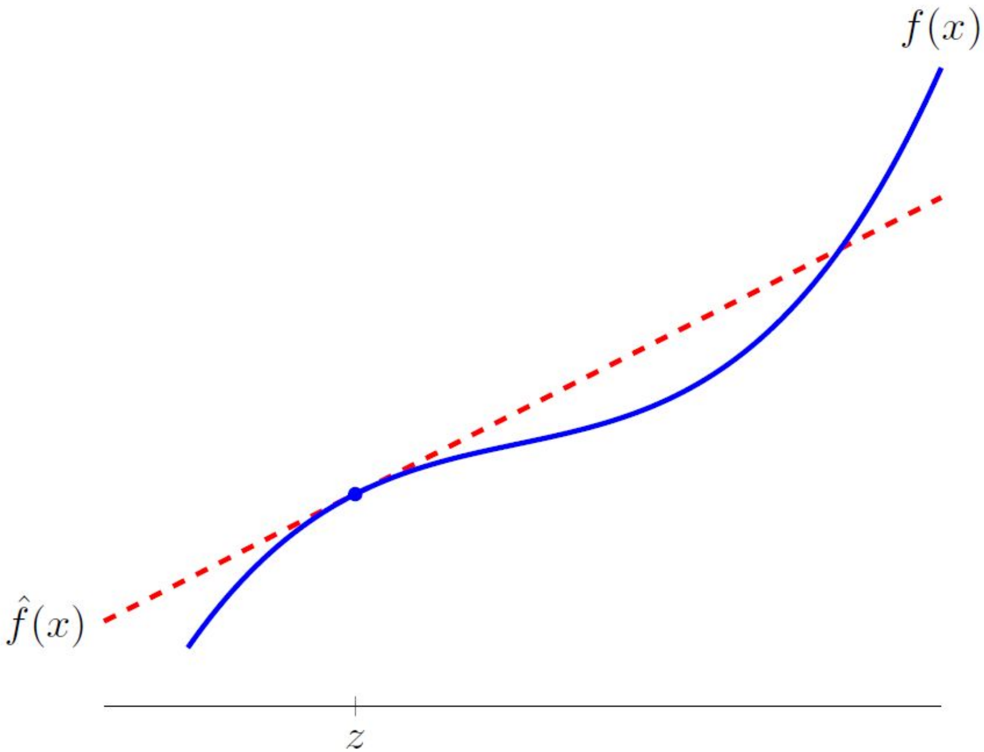
  - z is n-vector
- $\hat{f}(x)$ is a linear function or a affine function?

$$\hat{f}(x) = \underbrace{f(z)}_{\text{Constant- value of function at z}} + \nabla f(z)^T \underbrace{(x - z)}_{\text{Deviation or Perturbation of x from z}}$$

$$\hat{f}(x) = \underbrace{\nabla f(z)^T x}_{\text{Linear function}} + \underbrace{(f(z) - \nabla f(z)^T z)}_{\text{Constant}}$$

Taylor approximation

- The Taylor approximation is sometimes called the linear approximation or linearized approximation of f (at z)



A function f of one variable, and the first-order Taylor approximation $\hat{f}(x) = f(z) + f'(z)(x - z)$ at z

Taylor approximation

- Example:

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x) = x_1 + \exp(x_2 - x_1)$, the Taylor approximation $\hat{f}$ near the point $z = (1, 2)$

$e \sim 2.71$

$e^{-1} \sim 0.367$

x	$f(x)$	$\hat{f}(x)$	$\lvert\hat{f}(x) - f(x)\rvert$
$(1.00, 2.00)$	3.7183	3.7183	0.0000
$(0.96, 1.98)$	3.7332	3.7326	0.0005
$(1.10, 2.11)$	3.8456	3.8455	0.0001
$(0.85, 2.05)$	4.1701	4.1119	0.0582
$(1.25, 2.41)$	4.4399	4.4032	0.0367

Regression model

- Regression model is (the affine function of x)

$$\hat{y} = x^T w + w_0$$

Regression model

- Example

  - y is selling price of house in $\$1000$ (in some location, over some period)
  - regressor is

$$x = (\text{house area, \# bedrooms})$$

    (house area in 1000 sq.ft.)

  - regression model weight vector and offset are

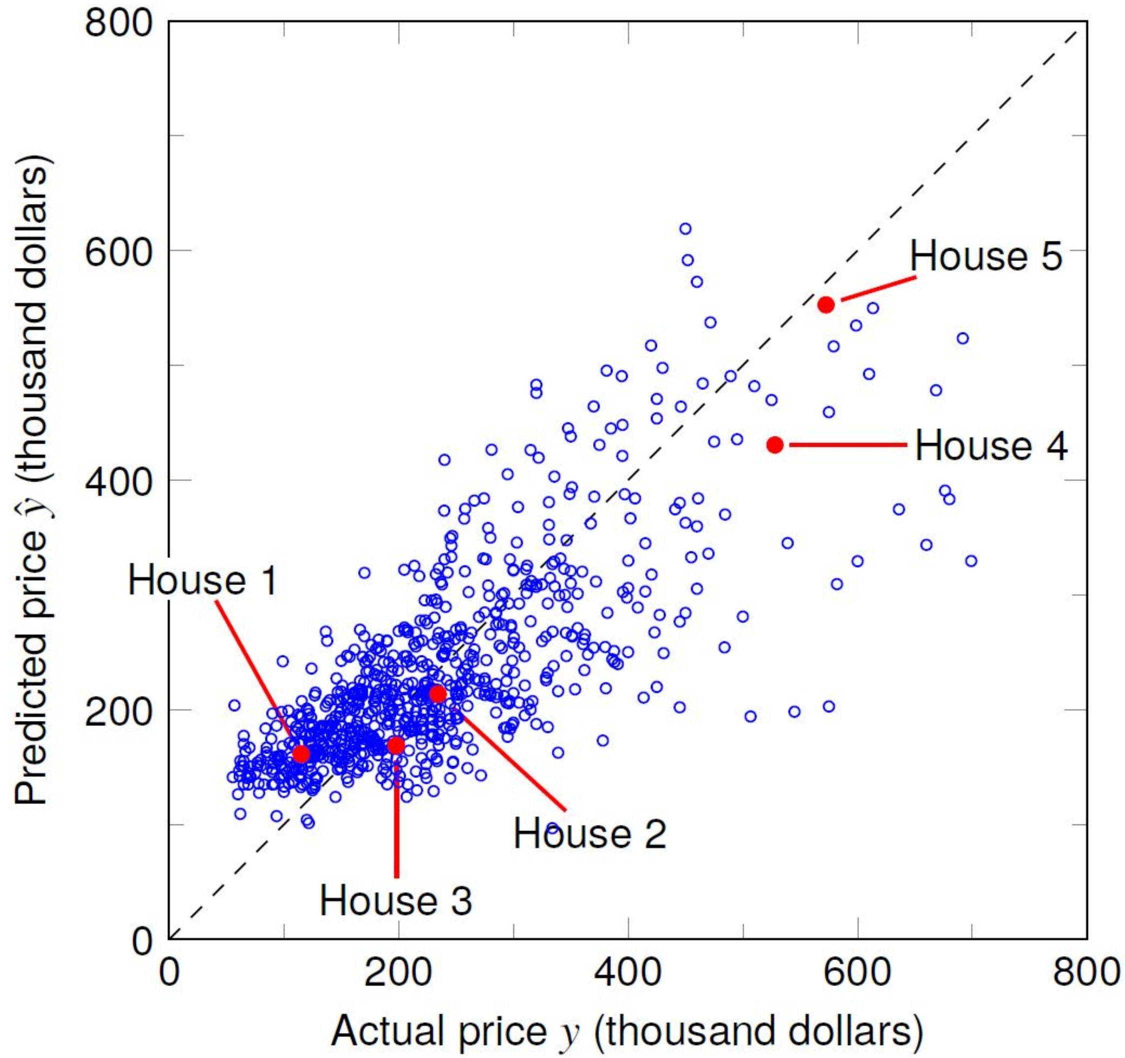
$$\beta = (148.73, -18.85), \qquad v = 54.40$$

  - we'll see later how to guess β and v from sales data

Regression model

House	x_1 (area)	x_2 (beds)	y (price)	$\hat{y}$ (prediction)
1	0.846	1	115.00	161.37
2	1.324	2	234.50	213.61
3	1.150	3	198.00	168.88
4	3.037	4	528.00	430.67
5	3.984	5	572.50	552.66

Regression model



Example

- Features:
- Y:

Reference

- Chapter 2: Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares
- Part of chapter 1 and chapter 6: Linear Algebra by David Cherney, etc.