Eigenvalue – Eigenvector

Linear Algebra

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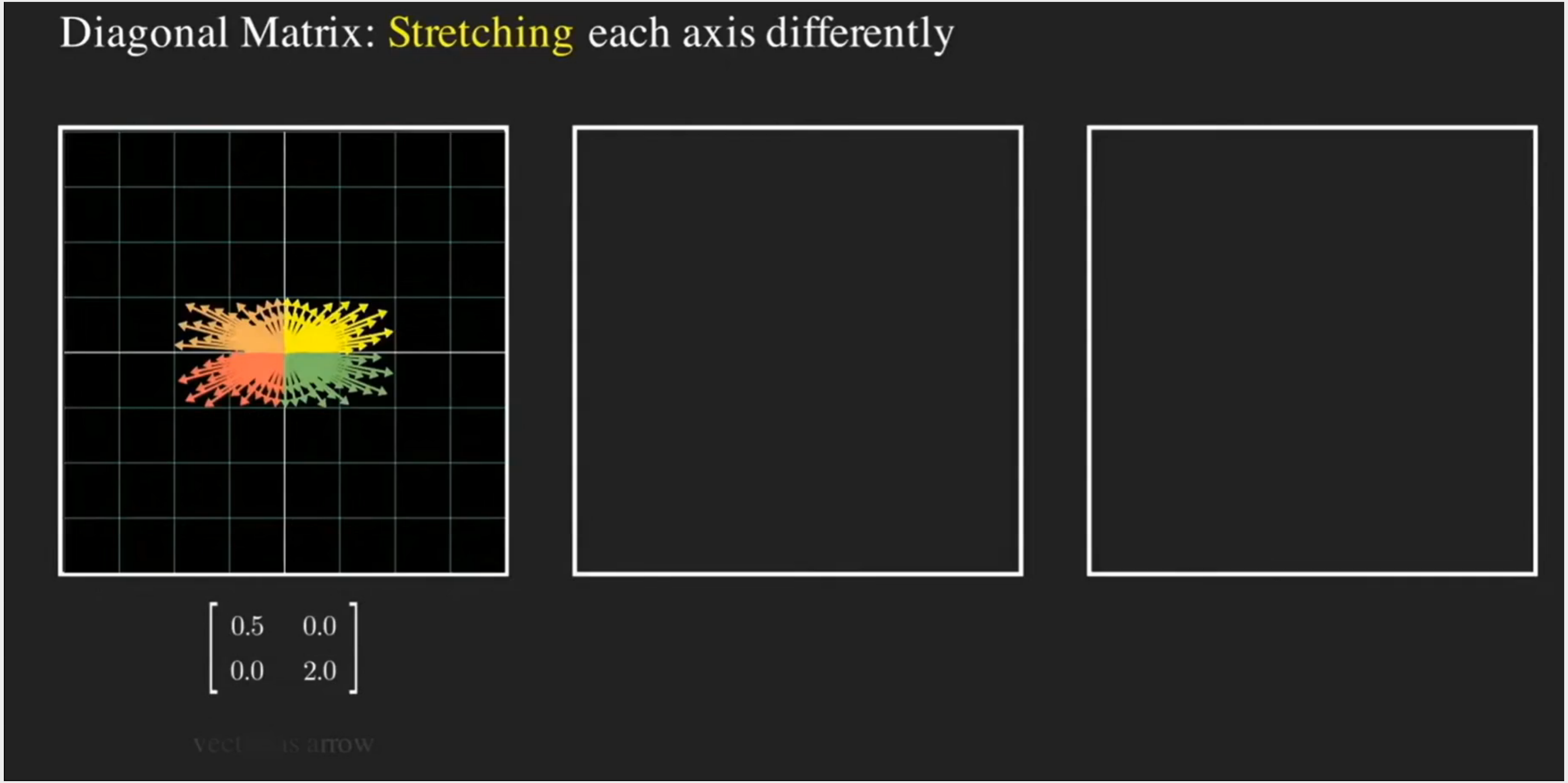
Review


Diagonal Matrix: Stretching each axis differently
0.5 0.0
0.0 2.0

Introduction

Motivation

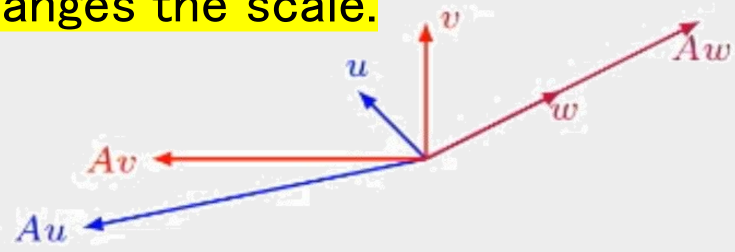
- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

  $$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

  $$v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

  $$w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

- Vector “w” keeps the straight, but changes the scale.



Definition

An **eigenvector** of a square $n \times n$ matrix A is nonzero vector v such that $Av = \lambda v$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution v of $Av = \lambda v$; such an v is called an *eigenvector corresponding to* λ.

- An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.

Example

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$
- Show that 7 is an eigenvalue of matrix B, and find the corresponding eigenvectors.

$$B = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

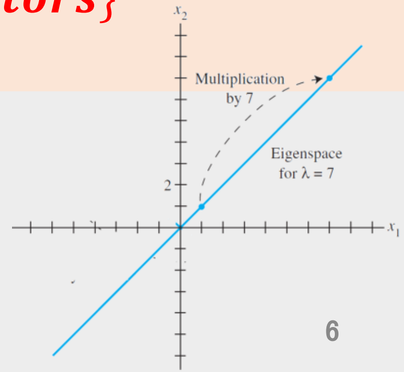
Note

λ is an eigenvalue of an $n \times n$ matrix:

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Eigenspace: A vector space formed by eigenvectors corresponding to the same eigenvalue and the origin point. $span\{corresponding\ eigenvectors\}$



Note

- $Av = \lambda v \Rightarrow Av - \lambda vI = 0 \Rightarrow (\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{v} = \boldsymbol{0}\ \ \boldsymbol{v} \neq \boldsymbol{0}$
  - $\boldsymbol{v} \in \boldsymbol{N}(\boldsymbol{A} - \boldsymbol{\lambda I})$
  - $\boldsymbol{A} - \boldsymbol{\lambda I}$ **must be singular.**
  - Proof that for finding the eigenvalue we should solve the determinate zero equation. Look at nullspace, rank and nullity theorem, singular matrix, and det zero!
- Characteristic polynomial $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I})$
- Characteristic equation $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}) = 0$
- If λ is an eigenvalue of A, then the subspace E_{λ} = {span{v} | Av = λ v} is called the eigenspace of A associated with λ. (This subspace contains all the span of eigenvectors with eigenvalue λ, and also the zero vector.)
- **Eigenvector is basis for eigenspace.**
- Set of all eigenvalues of matrix is $\sigma(A)$ *named spectrum of a matrix*

Definitions

Note

- Instead of $\det(A - \lambda I)$, we will compute $\mathbf{det}(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A})$. Why?
  - $\det(A - \lambda I) = (-1)^{\mathrm{n}} \det(\lambda I - \mathrm{A})$
  - Matrix $n \times n$ with real values has ⋯⋯ eigenvalues.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)\,X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example

Find eigenvalues and eigenvectors, eigenspace (E), and *spectrum* of matrix $A = \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$:

$$\det(A - \lambda \mathrm{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{matrix} \lambda_1 = 1 \\ (A - \lambda_1 I)q_1 = 0 \end{matrix}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \lambda_2 = 2 \\ (A - \lambda_2 I)q_2 = 0 \end{matrix}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues={1,2}

Eigenvectors=$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$E_1(A) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $E_2(A) = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\sigma(A)$={1,2}

$$\boldsymbol{AQ = Q\Lambda} \Rightarrow \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

Eigenvalues

Theorem

To have (1) scalar for largest degree instead of $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$, consider $|\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A}|$

$$f(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$ Proof?

- The n roots of this polynomial are eigenvalues!
  - $\boldsymbol{f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)}$
- What is c_{n-1}?
  - $c_{n-1} = -trace(A)$
- What is c_0?
  - $c_0 = -\det(A)$

Sum and Product of eigenvalues

Theorem

If A is an n × n matrix, then the sum of the n eigenvalues of A is the trace of A.
(coefficient c_{n-1} in expanded characteristic equation)

Other view: $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$

Proof?

Theorem

If A is an n × n matrix, then the product of the n eigenvalues is the determinant of A.
(coefficient c_0 in expanded characteristic equation)

Proof?

Theorem

$$0 \in \sigma(A) \iff |A| = 0$$

Proof?

Conclusion: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- ❑ The number 0 is not an eigenvalue of A .
- ❑ The determinant of A is not zero.

An Important Theorem!

Theorem

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal. The eigenvectors are e_is.

Proof?

Real Eigenvalues of different matrices

- Projection matrix
  - 0 , 1
  - If rank(P)=r with n columns, what are the repetition of the eigenvalues?
    - 0: n−r 1:r
- Reflection matrix
  - 1 , −1
- Permutation matrix
  - 1 , −1

Example

Find the eigenvalues with their repetition and eigenvectors:

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.
- $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$
- $C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$
- $D = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Eigenvalues of matrix products

Theorem

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA.

Proof?

Why Diagonalization?

Conclusion from pervious theorems

- Theorem "The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal." can leads to if we have matrix A and B that $D = B^{-1}AB$ be a diagonal matrix:

$$\det(\lambda I - A) = \det\left(\lambda I - B^{-1}AB\right)$$

Proof?

Similarity and Diagonalizable

Definition

Two n-by-n matrices A and B are called similar if there exists an invertible n-by-n matrix Q such that

$$A = Q^{-1}BQ$$

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix D: $\mathrm{D} = Q^{-1}AQ$, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

Similarity

Note

- A square matrix for a linear transform

$$A: n \times n \qquad T: R^n \to R^n \quad \Rightarrow \quad \boldsymbol{Aa = b} \qquad a, b \in R^n$$

$$\left.\begin{array}{l} a = Q\bar{a} \\ \mathrm{b} = Q\bar{b} \end{array}\right\} \Rightarrow \mathrm{AQ}\bar{a} = Q\bar{b} \Rightarrow \underbrace{Q^{-1}AQ}_{\bar{A}}\, \bar{a} = \bar{b} \Rightarrow \bar{A}\bar{a} = \bar{b}$$

- Linear transform in new basis $\bar{A} = Q^{-1}AQ$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Think!

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

- A matrix is a similarity invariant, meaning it remains unchanged under a similarity transformation.

- Why trace is a similarity invariant?
- Why rank is a similarity invariant?

Theorem

- Similar matrices have:
  - same determinant
  - equal characteristic equations
  - same trace
  - same rank
  - inverse of A and B are similar (if exists)

Proof?

Example

Find the similarity matrix of A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\boldsymbol{i} & \boldsymbol{i} \end{bmatrix}$$

Diagonalization

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

- The columns of Q is called an eigenvector basis of $\mathbb{R}^n$.

Corollary

- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- Distinct eigenvalues -> eigenvectors are Linear Independent
- Duplicate eigenvalues -> 🧐🧐

- Not all matrices are diagonalizable.
  - Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The diagonalizing matrix S is not unique.

Diagonalisable and Non-Diagonalisable Matrices

- For matrix $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$AS = SD$$

$$\begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & -4 & 3 \\ 0 & 6 & -9 \end{pmatrix}$$

Diagonal Matrix

S is not invertible!

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

So what's going on here?

Diagonal Matrix

R is invertible!

❑ Details for matrix A:

(i) For the eigenvalue -3, we have

$$\begin{pmatrix} 3 & -6 & -4 \\ 5 & -8 & -6 \\ -6 & 9 & 7 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) For the repeated eigenvalue -2, we have

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which equally straightforwardly gives the eigenvector

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Details for matrix B:

(i) For the eigenvalue -3, we have

$$\begin{pmatrix} 7 & 8 & -2 \\ -3 & -3 & 1 \\ 9 & 12 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which, as before, straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) This time, for the repeated eigenvalue -2, we have

$$\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, here things are different, because all three of the rows of this matrix may be reduced to the equation

$$3X + 4Y - Z = 0.$$

- Details for matrix B:

This represents a **plane** in 3D space, and any vector in this plane is an eigenvector. We may therefore form our diagonalising matrix S out of

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

together with any two non-parallel vectors of the form

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

that satisfy

$$3X + 4Y - Z = 0;$$

that is, that are perpendicular to the vector

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Both of the choices

$$S = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$$

will work fine, as will infinitely many others.

Diagonalisable and Non-Diagonalisable Matrices

General considerations

1. In general, any n by n matrix whose eigenvalues are distinct can be diagonalised.
2. If there is a repeated eigenvalue, whether or not the matrix can be diagonalised depends on the eigenvectors.
   - (i) If there $k<n$ eigenvectors (up to multiplication by a constant), then the matrix cannot be diagonalised.
   - (ii) If the unique eigenvalue corresponds to an eigenvector e, but the repeated eigenvalue corresponds to an entire plane, then the matrix can be diagonalised, using e together with any two vectors that lie in the plane.
3. If all n eigenvalues are repeated, then things are much more straightforward: the matrix can't be diagonalised unless it's already diagonal.

Example

Find A^n?

Conclusion

Another Notation

- With similarity transformation Q, matrix A changed to a diagonal matrix $diag(\lambda_{1,}, \lambda_2)$
- Matrix A has n linear independent eigenvectors

- $Aq_1 = \lambda_1 q_1 = [q_1\ q_2\ \cdots\ \ q_n]\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix} \cdots Aq_n = \lambda_n q_n = [q_1\ q_2\ \cdots\ \ q_n]\begin{bmatrix}0\\0\\\vdots\\\lambda_n\end{bmatrix}$

- $[Aq_1\ \ Aq_2\ \ \cdots\ \ Aq_n] = \underbrace{[q_1\ \ q_2\ \cdots\ \ q_n]}_{Q}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\0 & \lambda_2 & \dots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \dots & \lambda_n\end{bmatrix}}_{\Lambda}$

- $A\,[q_1\ q_2\ \cdots\ \ q_n] = Q\Lambda \Longrightarrow AQ = Q\Lambda$
- $\Lambda = Q^{-1}AQ^T$
- $A = Q\Lambda Q^{-1}$