Matrix Inner Product and Norm

CE282: Linear Algebra

Computer Engineering Department

Sharif University of Technology

Hamid R. Rabiee

Maryam Ramezani

Inner Products

Definition

- The standard inner product is:

$$\langle x, y\rangle = x^T y = \sum x_i y_i\,, \qquad x, y \in \mathbb{R}^n$$

- The standard inner product between matrices is:$(X, Y \in \mathbb{R}^{m\times n})$

$$\langle X, Y\rangle = Tr(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

Example

Example

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Definition (Norm)

- A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a norm if
  1. $f(x) \geq 0, f(x) = 0 \Leftrightarrow x = 0$ (positivity)
  2. $f(\alpha x) = |\alpha| f(x), \forall \alpha \in \mathbb{R}$ (homogeneity)
  3. $f(x + y) \leq f(x) + f(y)$ (triangle inequality)

Entry-wise matrix norms

Definition

$$\|A\|_{p,p} = \|vec(A)\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}$$

Special Cases

- Frobenius (Eucilian, Hilbert Schmidt) norm:(p = 2)

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} = \sqrt{trace(A^*A)}$$

- Max norm ($p = \infty$)

$$\|A\|_{max} = \max_{ij}|a_{ij}|$$

- Sum-absolute-value norm

$$\|A\|_{sav} = \sum_{i,j}|A_{i,j}|$$

Frobenius (Eucilian, Hilbert Schmidt) norm

Special Cases

- Invariant under rotations (unitary operations)

$$\|A\|_F = \|AU\|_F = \|UA\|_F$$

$$\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + 2\langle A, B\rangle$$

$$\|A^*A\|_F = \|AA^*\|_F \leq \|A\|_F^2$$

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{\frac{1}{2}} = \sqrt{trace(A^*A)} = \sqrt{\sigma_1^2 + \cdots . + \sigma_r^2}$$

$$\text{show that } \sigma_{\max}(A) \leq \|A\|_{\mathrm{F}} \leq \sqrt{r}\sigma_{\max}(A)$$

Frobenius (Euclidean norm)

Theorem

Let $b_1, b_2, \cdots, b_n$ denote the columns of B. Then

$$\|AB\|_{HS}^2 = \sum_{i=1}^{n} \|Ab_i\|^2 \leq \sum_{i=1}^{n} \|A\|^2 \|b_i\|^2 = \|A\|^2 \|B\|_{HS}^2$$

Using Cauchy-Schawrtz Inequality

Definition

$$\|A\|_p = \max_{\vec{x}\neq\vec{0}} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p} = \max_{\|\vec{x}\|_p=1} \|A\vec{x}\|_p$$

Theorem

1. $\|Ax\| \leq \|A\|\|x\|$ for all vectors $\|x\|$
2. For all matrices A, B: $\|AB\| \leq \|A\|\|B\|$

Definition

- The norm of a matrix is a real number which is a measure of the magnitude of the matrix.
- Norm 1:

$$\|A\|_1 = \max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|\right)$$

- Norm max:

$$\|A\|_\infty = \max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)$$

- Spectral norm (l2) is the largest singular value (the square root of the largest eigenvalue of the matrix gram A)

$$\|A\|_2 = \sqrt{\max\{eigenvalue(A^TA)\}} = \max\{sing(A)\}$$

Example

$$B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

Norms Compare

Note

The 2d-norm (spectral norm) of a matrix is the greatest distortion of the unit circle/sphere/hyper-sphere. It corresponds to the largest singular value (or eigenvalue if the matrix is symmetric/Hermitian).
The frobenius norm is the "diagonal" between all the singular values.
i.e.

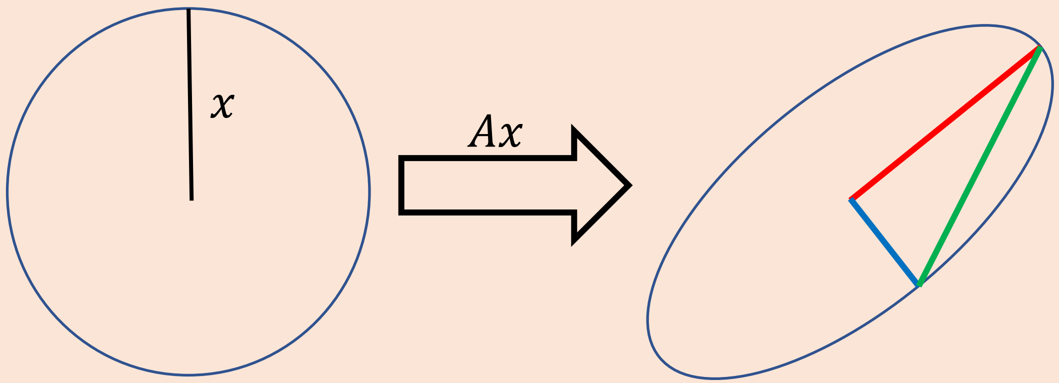
$$\|A\|_2 = s_1, \ \|A\|_F = \sqrt{s_1^2 + s_2^2 + \cdots + s_r^2}$$

(r being the rank of A)

Norms Compare

Note

Here's a $2D$ version of it: x is any vector on the unit circle. Ax is the deformation of all those vectors. The length of the red line is the 2-norm (biggest singular value). And the length of the green line is the Frobenius norm (diagonal)



2-norm: s_1

s_2

Frobenius norm:

$\sqrt{s_1^2 + s_2^2}$