



Bases and Dimension

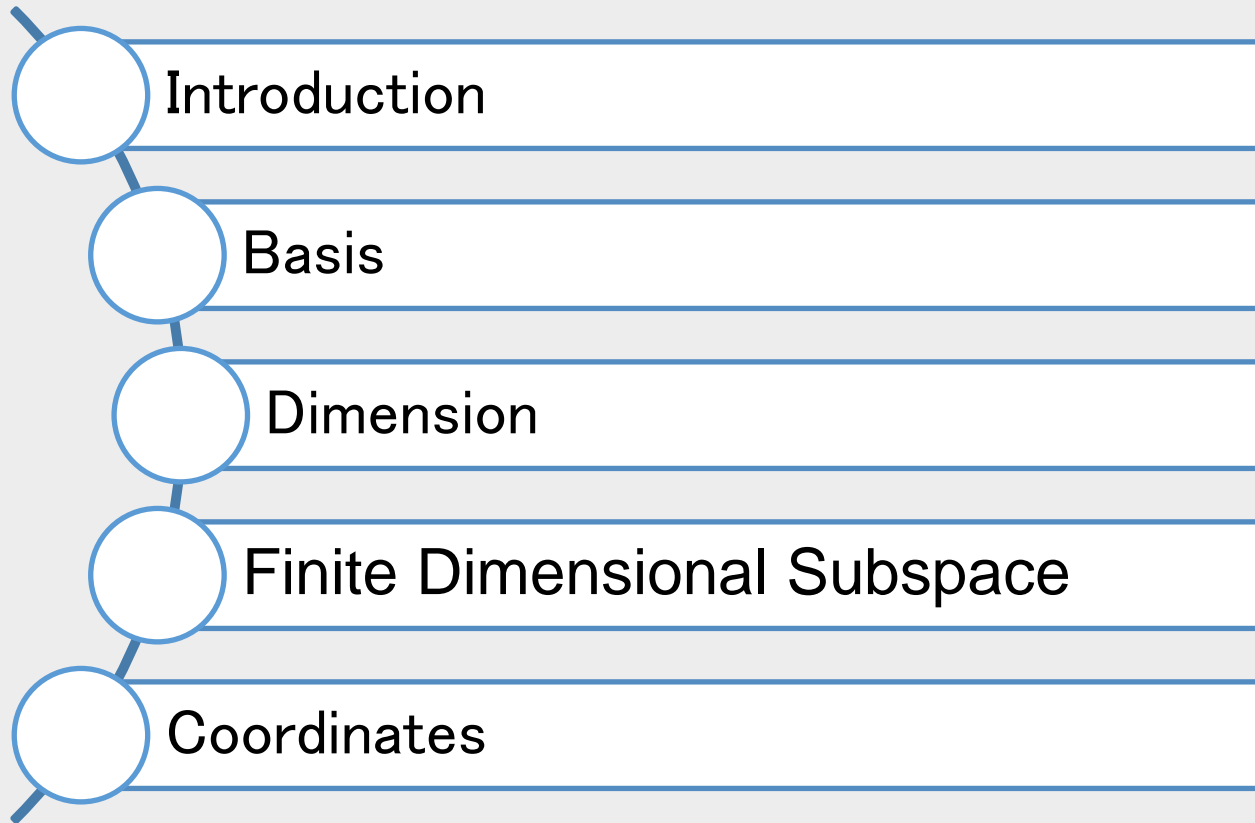
Linear Algebra

Department of Computer Engineering

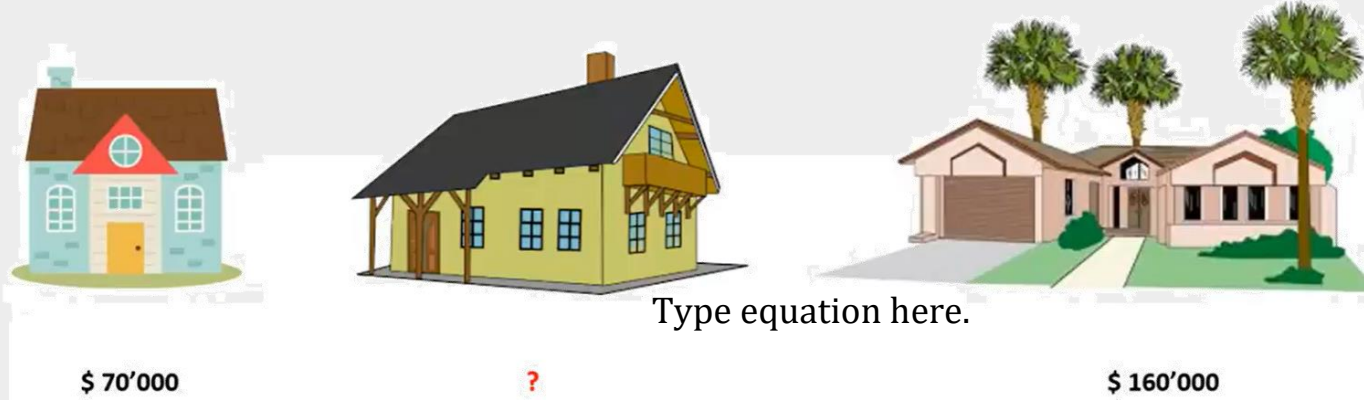
Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu



Introduction

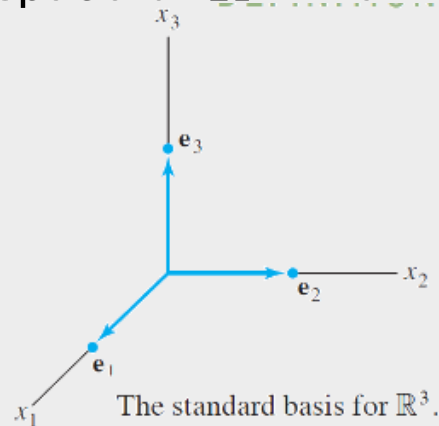


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Basis



- A set of n linearly independent n -vectors is called a basis.
- A basis is the combination of span and independence: A set of vectors $\{v_1, \dots, v_n\}$ forms a basis for some subspace of \mathbb{R}^n if it
 - (1) spans that subspace
 - (2) is an independent set of vectors.





Definition

Let H be a subspace of a vector space V . An indexed set of vectors $\mathcal{B} = \{b_1, \dots, b_n\}$ in V is a **basis** for H if

1. \mathcal{B} is linearly independent set, and
2. The subspace spanned by \mathcal{B} coincides with H ; that is,

$$H = \text{Span} \{b_1, \dots, b_n\}$$

Example

Which are unique?

- ☐ Express a vector in terms of any particular basis
- ☐ Bases for \mathbb{R}^2
- ☐ Bases with unit length for \mathbb{R}^2



Be careful: A vector space can have many bases that look very different from each other!

Example (Basis)

- ❑ Standard bases for $P_n(\mathbb{R})$?
- ❑ Are $(1 - x), (1 + x), x^2$ basis for $P_2(\mathbb{R})$?

Dimension



- ❑ The dimensionality of a vector is the number of coordinate axes in which that vector exists.
- ❑ If a vector space is spanned by a finite number of vectors, it is said to be **finite-dimensional**. Otherwise it is **infinite-dimensional**.
- ❑ The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted $\dim(V)$.



Theorem **

Let V be a vector space which is spanned by a finite set of vectors x_1, x_2, \dots, x_m . Then any independent set of vectors in V is finite and contains no more than m elements.

Proof

Conclusion

Every basis of V is finite and contains no more than m elements.



Theorem

If V is a finite-dimensional vector space, then any two bases of V has the same (finite) number of elements.

Proof



The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted as $\dim(V)$.

Theorem **

Let V be a vector space which is spanned by a finite set of vectors x_1, x_2, \dots, x_m . Then any independent set of vectors in V is finite and contains no more than m elements.



Theorem

Let V be a vector space with a basis B of size n . Then

- a) Any set of more than n vectors in V must be linearly dependent, and
- b) Any set of fewer than n vectors cannot span V .



Definition

A vector space V is called...

- a) **finite-dimensional** if it has a finite basis, and its **dimension**, denoted by $\dim(V)$, is the number of vectors in one of its bases.
- b) **infinite-dimensional** if it has no finite basis, and we say that $\dim(V) = \infty$.

Note

Dimension of subspace $\{0\}$?



Example

Let's compute the dimension of some vector spaces that we've been working with.

Vector space	Basis	Dimension
F^n		
P^p		
$M_{m,n}$		
P (all polynomials)		
F (functions)		
C (continues functions)		

← Note!

Finite Dimensional Subspace



Theorem

If W is a subspace of a finite-dimensional vector space V , every linearly independent subset of W is finite and is part of a (finite) basis for W .

Proof

Theorem (Lemma)

Let S be a linearly independent subset of a vector space V . Suppose u is a vector in V which is not in the subspace spanned by S . Then the set obtained by adjoining u to S is linearly independent.

Proof



Corollary

If W is a proper subspace of a finite-dimensional vector space V , then W is finite-dimensional and $\dim(W) < \dim(V)$

Proof

Corollary

In a finite-dimensional vector space V , every non-empty linearly independent set of vectors is part of basis.



Theorem

If W_1 and W_2 are finite-dimensional subspaces of a vector space V , the $W_1 + W_2$ is a finite-dimensional and

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Proof



Note

Let V be a finite dimensional vector space over field F . Below are some properties of bases:

1. Any linearly independent list can be extended to a basis (a maximal linearly independent list is spanning).
2. Any spanning list contains a basis (a minimal spanning list is linearly independent).
3. Any linearly independent list of length $\dim V$ is a basis.
4. Any spanning list of length $\dim V$ is a basis.

❑ **We will learn about change of basis after linear transformation lecture!**



Note

In a finite-dimensional space,

$$\begin{array}{l} \text{the length of every} \\ \text{linearly independent list} \\ \text{of vectors} \end{array} \leq \begin{array}{l} \text{the length of every} \\ \text{spanning list of} \\ \text{vectors} \end{array}$$

Proof

Coordinates



Definition

If V is a finite-dimensional vector space, an **ordered basis** for V is a finite **sequence** of vectors which is linearly independent and spans V .

Be careful: The order in which the basis vectors appear in B affects the order of the entries in the coordinate vector. This is kind of janky (technically, sets don't care about order), but everyone just sort of accepts it.



- The main reason for selecting a basis for a subspace H ; instead of merely a spanning set, is that **each vector in H can be written in only one way as a linear combination of the basis vectors.**

Note

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for a subspace H . For each x in H , the **coordinates of x relative to the basis \mathcal{B}** are the weights c_1, \dots, c_p such that $x = c_1 b_1 + \dots + c_p b_p$, and the vector in \mathbb{R}^p

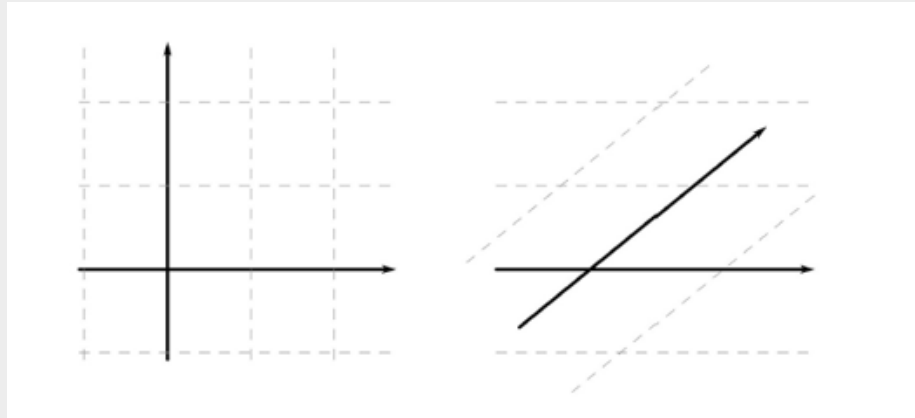
$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of x (relative to \mathcal{B})** or the \mathcal{B} -coordinate vector of x .



Example

Coordinate vector of $p(x) = 4 - x + 3x^2$ respect to basis $\{1, x, x^2\}$



- ❑ The familiar Cartesian plane (left) has orthogonal coordinate axes. However, axes in linear algebra are not constrained to be orthogonal (right), and non-orthogonal axes can be advantageous.



Theorem

Let set $S = \{v_1, \dots, v_k\}$ be an affinely independent set in \mathbb{R}^n . Then each \mathbf{p} in $\text{aff } S$ has a unique representation as an affine combination of v_1, \dots, v_k . That is, for each \mathbf{p} there exists a unique set of scalars c_1, \dots, c_k such that

$$\mathbf{p} = c_1 v_1 + \dots + c_k v_k \quad \text{and} \quad c_1 + \dots + c_k = 1$$

Note

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} v_1 \\ 1 \end{bmatrix} + \dots + c_k \begin{bmatrix} v_k \\ 1 \end{bmatrix}$$

Involving the homogeneous forms of the points. Row reduction of the augmented matrix $[\widetilde{v}_1 \ \dots \ \widetilde{v}_k \ \widetilde{\mathbf{p}}]$ produces the Barycentric coordinates of \mathbf{p} .



Definition

Let set $S = \{v_1, \dots, v_k\}$ be an affinely independent set. Then for each point \mathbf{p} in $\text{aff } S$, the coefficients c_1, \dots, c_k in the unique representation

$$\mathbf{p} = c_1 v_1 + \dots + c_k v_k \quad \text{and} \quad c_1 + \dots + c_k = 1$$

of \mathbf{p} are called the **Barycentric** (or, sometimes **affine**) **coordinates** of \mathbf{p}



Example

Let $a = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $p = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the Barycentric Coordinates of p determined by the affinely independent set $\{a, b, c\}$.

Note

$S = \{v_1, \dots, v_k\}$ are affinely independent, if $\begin{bmatrix} v_1 \\ 1 \end{bmatrix} \dots \begin{bmatrix} v_k \\ 1 \end{bmatrix}$ are linear independent.



- ❑ Page 97 LINEAR ALGEBRA: Theory, Intuition, Code
- ❑ Page 213: David Cherney,
- ❑ Page 54: Linear Algebra and Optimization for Machine Learning