Vector Space

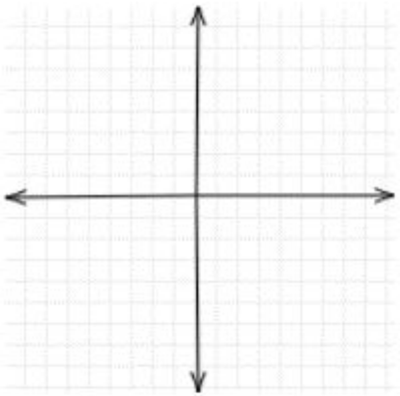
CE40282-1: Linear Algebra
Hamid R. Rabiee and Maryam Ramezani
Sharif University of Technology

Tuple and Vector Space

- A tuple is an ordered list of numbers.
- For example: (1,22,3,8) is a 4-tuple (a tuple with 4 elements).

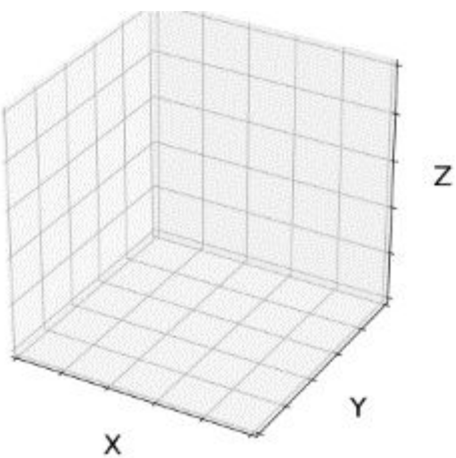
$$R^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ 2/3 \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \ldots\ldots \right\}$$

Here, e is exponential constant. Its value is approximately 2.718



$$R^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \ldots\ldots \right\}$$

Set of R^3



Review: Complex number

- Numbers:
  - Real: Nearly any number you can think of is a Real Number!

1	12.38	−0.8625	3/4	√2	1998

  - Imaginary: When **squared** give a **negative** result.
- The "unit" imaginary number (like 1 for Real Numbers) is i, which is the square root of −1.

Examples of Imaginary Numbers:

3i	1.04i	−2.8i	3i/4	(√2)i	1998i

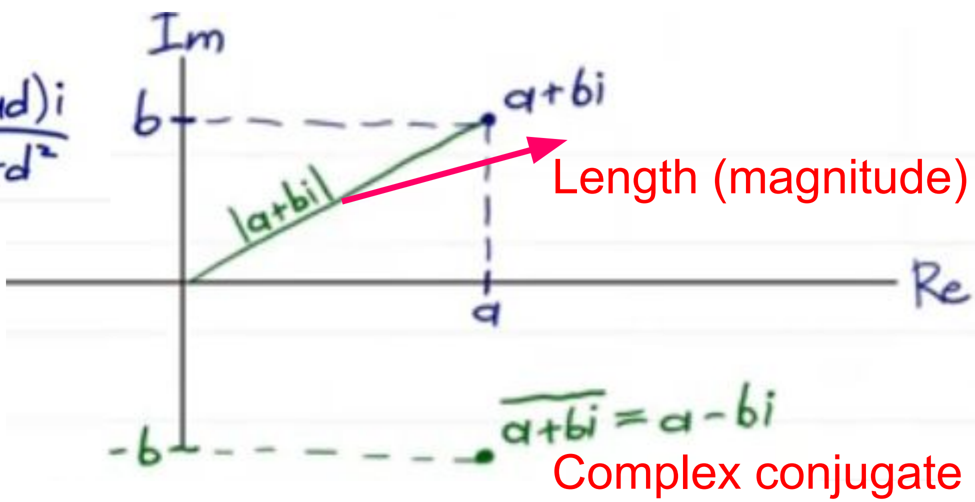
And we keep that little "i" there to remind us we need to multiply by √−1

Review: Complex number

- $i^2 = -1$
- Imaginary number: $bi\ , b \in R$
- Arithmetic with complex numbers $(a + bi)$:
  - $(a + bi) + (c + di)$
  - $(a + bi)(c + di)$
- $\mathbb{C}$ is a plane, where number $(a + bi)$ has coordinates (a, b)

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

$$\overline{(a+bi)}(a+bi) = |a+bi|^2$$



Review: Complex number

- Inner Product
  - real $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots\cdots + x_n y_n$
  - complex $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{x}_1 y_1 + \overline{x}_2 y_2 + \cdots\cdots + \overline{x}_n y_n$

Binary Operations

- Any function from $A \times A \longmapsto A$ is a binary operation.
- Closure Law $\text{if } a \in A, b \in A \Rightarrow a * b \in A$
  - A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.
  - Is + operator a binary on natural numbers?
  - Is $\times$ operator a binary on natural numbers?
  - Is - operator a binary on natural numbers?
  - Is / operator a binary on natural numbers?

Groups

A group G is a pair $(S, \blacklozenge)$, where S is a set and $\blacklozenge$ is a binary operation on S such that:

1. $\blacklozenge$ is associative $\quad \forall a,b,c \in S$

   $(a \lozenge b) \lozenge c = a \lozenge (b \lozenge c)$

2. (Identity) There exists an element $e \in S$ such that:

   $e \blacklozenge a = a \blacklozenge e = a, \qquad \text{for all } a \in S$

3. (Inverses) For every $a \in S$ there is $b \in S$ such that: $a \blacklozenge b = b \blacklozenge a = e$

If $\blacklozenge$ is commutative, then ~~G is~~ called a commutative group

(Abelian Group)

Ring

A ring R is a set together with <u>two</u> binary operations + and *, satisfying the following properties:

1. (R,+) is a commutative group
2. * is associative
3. The distributive laws hold in R: (Multiplication is distributive over addition)

$$(a + b) * c = (a * c) + (b * c)$$

$$a * (b + c) = (a * b) + (a * c)$$

Fields

A field F is a set together with two binary operations + and *, satisfying the following properties:

1. (F,+) is a commutative group
2. (F-{0},*) is a commutative group

- **Associative**
- **Identity**
- **Inverses**
- **Commutative**

3. The distributive law holds in F:

$$(a + b) * c = (a * c) + (b * c)$$

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

$(\mathbb{R};+,\cdot)$ and $(\mathbb{Q};+,\cdot)$ serve as examples of **fields**,
$(\mathbb{Z};+,\cdot)$ is an example of a **ring** which is not a field.

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

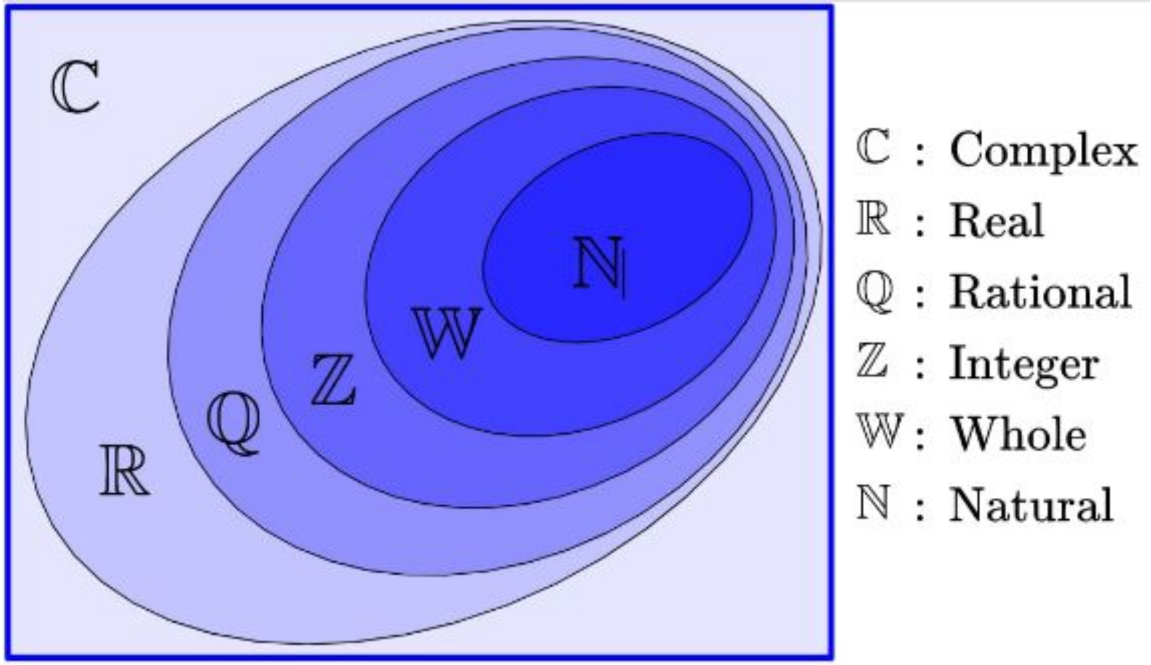
Fields

$\forall\ a, b, c \in \boldsymbol{F}$

Properties	Binary Operations	
	Addition (+)	Multiplication (.)
Closure(بسته بودن)		
Associative (شرکت‌پذیری)		
Commutative (جابه‌جایی‌پذیری)		

Fields

- Which are fields? (two binary operations + and *)
  - $\mathbb{R}$
  - $\mathbb{C}$
  - $\mathbb{Q}$
  - $\mathbb{Z}$
  - W
  - $\mathbb{N}$
  - $\mathbb{R}^{2\times2}$



Vector Space

- Building blocks of linear algebra

- A non empty set V with field F (most of time $\mathbb{R}$ or $\mathbb{C}$) forms a vector space with two operations:
  - $" + "$: Binary operation on V which is $\mathrm{V} \times V \rightarrow V$
  - $"."$: $F \times V \rightarrow V$

- NOTE: In our course by default Field is $\mathbb{R}$

Vector Space

Definition *A* **vector space** *over a field $\mathcal{F}$ is the set $\mathcal{V}$ equipped with two operations: $(\mathcal{V}, \mathcal{F}, +, \cdot)$*

(i) Vector addition: *denoted by "$+$" adds two elements $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ to produce another element $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{V}$;*

(ii) Scalar multiplication: *denoted by "$\cdot$" multiplies a vector $\boldsymbol{x} \in \mathcal{V}$ with a scalar $\alpha \in \mathcal{F}$ to produce another vector $\alpha \cdot \boldsymbol{x} \in \mathcal{V}$. We usually omit the "$\cdot$" and simply write this vector as $\alpha\boldsymbol{x}$.*

[A1] Vector addition is commutative: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

[A2] Vector addition is associative: $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ *for every* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.

[A3] Additive identity: *There is an element* $\mathbf{0} \in \mathcal{V}$ *such that* $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

[A4] Additive inverse: *For every* $\boldsymbol{x} \in \mathcal{V}$, *there is an element* $(\boldsymbol{x}) \in \mathcal{V}$ *such that* $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.

Addition of vector space

[M1] Scalar multiplication is associative: $a(b\boldsymbol{x}) = (ab)\boldsymbol{x}$ *for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M2] First Distributive property: $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ *and for every* $a, b \in \mathcal{F}$ *and for every* $\boldsymbol{x} \in \mathcal{V}$.

[M3] Second Distributive property: $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ *and every* $a \in \Re^1$.

[M4] Unit for scalar multiplication: $1\boldsymbol{x} = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in \mathcal{V}$.

Action of the field of scalars on the vector space

Example

- Let V be the set of all real numbers with the operations $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} - \mathbf{v}$ ($\oplus$ is ordinary subtraction) and $c \odot \mathbf{u} = c\mathbf{u}$ ($\odot$ is ordinary multiplication). Is V a vector space? If it is not, which properties fail to hold?

Vector Space

- Examples:
  - The n-tuple space, R^n
  - The space of $m \times n$ matrices
  - The space of functions from a set to a field $g(s)$

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

$$\mathbf{f}(t) = 1 + \sin 2t, \text{ and } \mathbf{g}(t) = 2 + .5t$$

  - The space of polynomial functions over a field $f(x)$

$$\mathbf{p}_n(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Vector spaces of functions

- Function addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x) \text{ and } (af)(x) = af(x)$$

non-empty set X and any field $\mathcal{F}$, $\rightarrow$ $\mathcal{F}^X = \{f : X \to \mathcal{F}\}$

for all $x \in X$, $f, g \in \mathcal{F}^X$ and $a \in \mathcal{F}$

- Example
  - Set of all polynomials with real coefficients
  - Set of all real-valued continuous function on [0,1]
  - Set of all real-valued function that are differentiable on [0,1]

Vector Space of Polynomials

- $P_n(\mathbb{R})$
  - Scalar multiplication
  - Vector addition
  - And other 8 properties!

Vector Space

- Which are vector space?
  - Set $\mathbb{R}^n$ over $\mathbb{R}$
  - Set $\mathbb{C}$ over $\mathbb{R}$
  - Set $\mathbb{R}$ over $\mathbb{C}$
  - Set $\mathbb{Z}$ over $\mathbb{R}$
  - Set of all polynomials with coefficient from field $\mathbb{R}$
  - Set of all polynomials of degree at most n with coefficient from field $\mathbb{R}$
  - Matrix: $M_{m,n}(\mathbb{R})$
  - Function: $f(x): x \longrightarrow \mathbb{R}$

Conclusion

The operations on a field $\mathbb{F}$ are

- $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
- $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

The operations on a vector space $\mathbb{V}$ over a field $\mathbb{F}$ are

- $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$
- $\cdot : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$

Span or linear hull

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

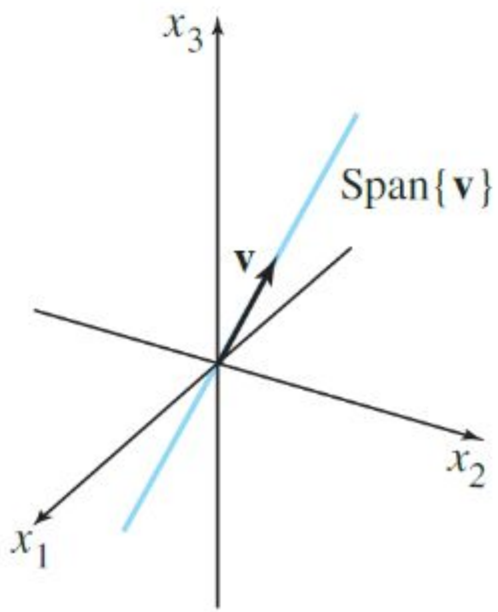
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

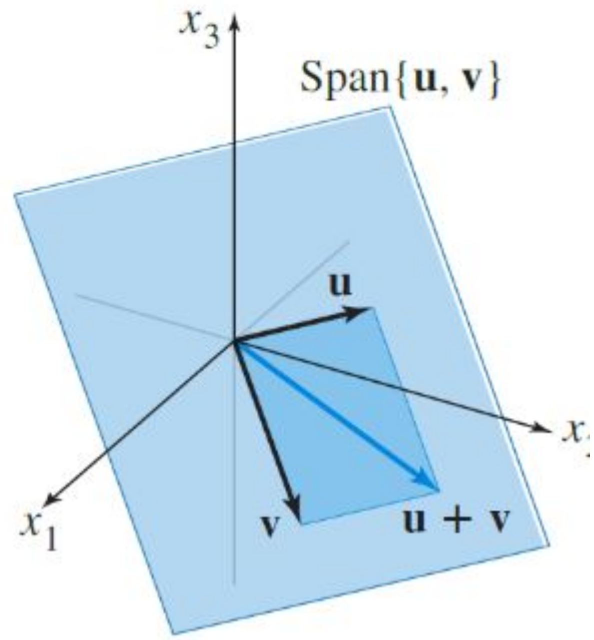
- Examples:
  - Is vector b in Span$\{v_1, \dots, v_p\}$?
  - Is vector v_3 in Span$\{v_1, \dots, v_p\}$?
  - Is vector $\mathbf{0}$ in Span$\{v_1, \dots,$
  - Is b in Span$\{a_1, a_2\}$? $\quad \mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$
  - Span of polynomials: $\{(1+x), (1-x), x^2\}$?

Span

- v and u are non-zero vectors in R^3 where v not a multiple of u

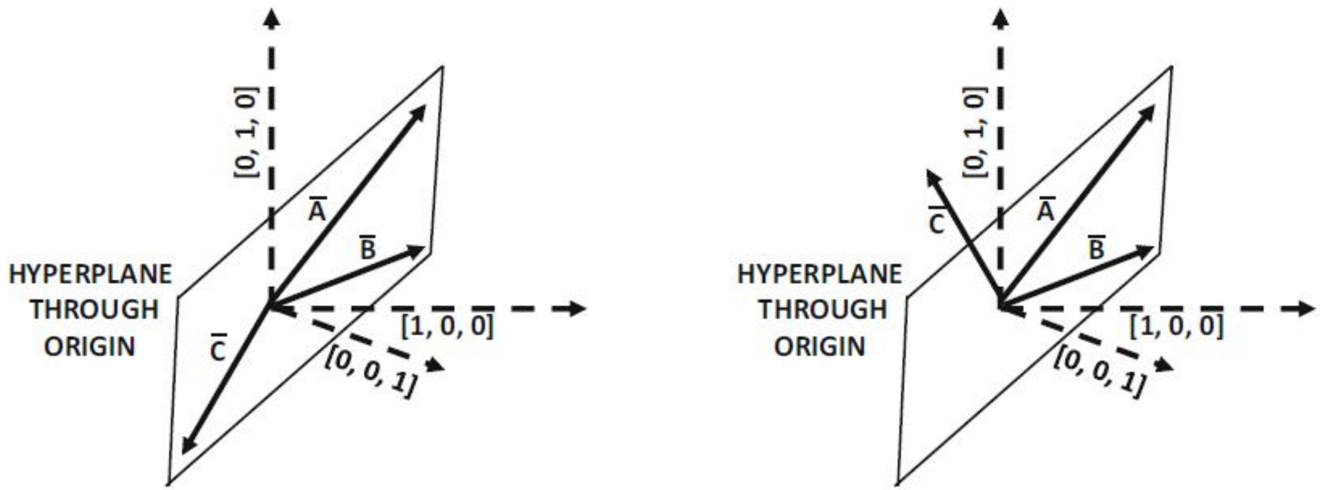


Span {**v**} as a line through the origin.



Span {**u**, **v**} as a plane through the origin.

Span



(a) $\text{Span}(\{\overline{A}, \overline{B}\}) = \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors on hyperplane

(b) $\text{Span}(\{\overline{A}, \overline{B}\}) \neq \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors in $\mathcal{R}^3$

Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set

Subspace

- A nonempty subset of vector space for which closure holds for addition and scalar multiplication is called a subspace.
- Subspace: If V is a vector space and subset $\mathrm{U} \subseteq V$, then U is itself a vector space with the same addition and scalar multiplication as V.

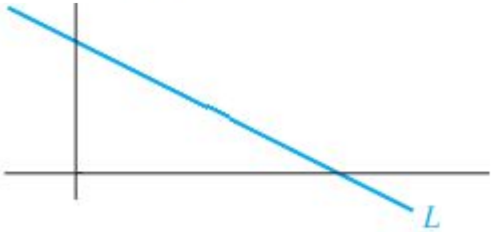
Subspace

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- Examples
  - $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace
  - 

  - The vector space R^2 is a subspace of R^3?
  - Is H a subset of R^3?

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

Vector space vs Subspace Properties

Let V be a vector space and let $U \subseteq V$...

1. $\boldsymbol{u} + \boldsymbol{v} \in V$
2. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
3. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
4. There is a vector $\mathbf{0} \in V$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
5. For each $\boldsymbol{u} \in V$, there is a vector $-\boldsymbol{u} \in V$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
6. $c\boldsymbol{u} \in V$
7. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
10. $1\boldsymbol{u} = \boldsymbol{u}$

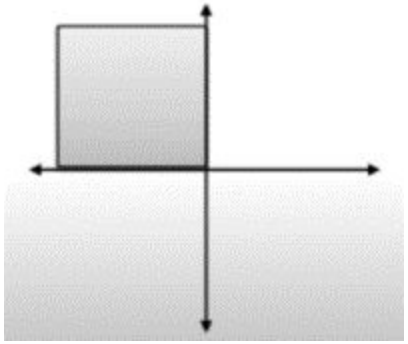
1. $\boldsymbol{u} + \boldsymbol{v} \in U$
2. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
3. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
4. There is a vector $\mathbf{0} \in U$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
5. For each $\boldsymbol{u} \in U$, there is a vector $-\boldsymbol{u} \in U$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
6. $c\boldsymbol{u} \in U$
7. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
10. $1\boldsymbol{u} = \boldsymbol{u}$

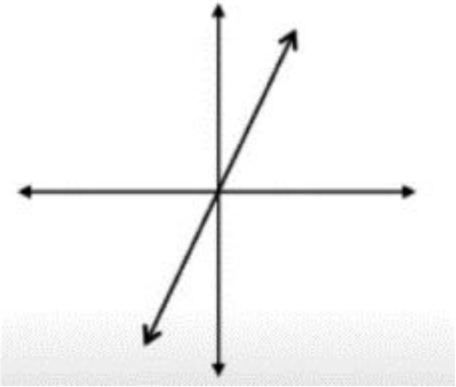
Subspace Summarize

- A subspace is a subset of vector space that holds closure under addition and scalar multiplication.
- Zero vector is a subspace of every vector space.
- Vector space is a subspace of itself.

Subspace

- Example:





  - Set of all continuous real-valued functions on R.
  - Set of all differentiable real-valued function on R.
  - Every vector space with more than one member has at least ……. subspaces.
  - Name subspace for:
    - R^2
    - R^3

Intersection of subspaces

- Theorem: If U and W are subspaces of V, then $U \cap W$ is a subspace.
  - Proof?

Subspace

THEOREM 1 If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

- Proof?

- Example:
  - Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where a and b are arbitrary scalars. That is, let $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$.

Subspace

- Theorem 3.4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in the vector space V and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ be vectors in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then

$$\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\} \subseteq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

Sum of vector spaces

- There are two reasons to use the sum of two vector spaces.
  - to build new vector spaces from old ones.
  - to decompose the known vector space into sum of two (smaller) spaces.
- Since we consider linear transformations between vector spaces, these sums lead to representations of these linear maps and corresponding matrices into forms that reflect these sums. In many very important situations, we start with a vector space V and can identify subspaces “internally” from which the whole space V can be built up using the construction of sums.

Sum of vector spaces

Let A and B be nonempty subsets of a vector space V. The **sum** of A and B, denoted $A + B$, is the set of all possible sums of elements from both subsets: $A + B = \{a + b \,:\, a \in A,\ b \in B\}$.

A vector space V is called the **direct sum** of V_1 and V_2 if V_1 and V_2 are subspaces of V such that $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = V$. This means that every vector **v** from V is uniquely represented via sum of two vectors $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_1 \in V_1,\ \mathbf{v}_2 \in V_2$. We denote that V is the direct sum of V_1 and V_2 by writing $V = V_1 \oplus V_2$.

Direct Sum

- Definition: $U + W$ is called a **direct sum**, if any element in $U + W$ can been written uniquely as $\mathbf{u} + w$ where $\mathbf{u} \in U$ and $w \in W$
(Notation: $\mathrm{U} \oplus W$)

Example: Let E denote the set of all polynomials of even powers: $E = \left\{a_n t^{2n} + a_{n-1} t^{2n-2} + \cdots + a_0\right\}$, and O be the set of all polynomails of odd powers: $O = \left\{a_n t^{2n+1} + a_{n-1} t^{2n-1} + \cdots + a_0 t\right\}$. Then the set of all polynomials P is the direct sum of these sets: $P = O \oplus E$.

It is easy to see that any polynomial (or function) can be ubiquely decomposed into direct sum of even and odd counterparts:

$$p(t) = \frac{p(t) + p(-t)}{2} + \frac{p(t) - p(-t)}{2}.$$

Direct Sum

- Theorem: If U and W are subspaces of V, then $U \oplus W$ is a subspace, if and only if $U \cap W = \{0\}$

Examples

- Prove set of all bound function such as
$$W = \{f(x) | \exists M \in R \; s.t. |f(x)| \leq M, \forall x \in R\}$$
is a subspace of $V = \{all\ functions\ from\ R\ to\ R\}$.

Reference

- LINEAR ALGEBRA: Theory, Intuition, Code
- David Cherney,
- Online Courses!
- Chapter 4 of Elementary Linear Algebra with Applications
- Chapter 3 of Applied Linear Algebra and Matrix Analysis

The elements of $\mathcal{V}$ can be functions. Define function addition and scalar multiplication as

$$(f+g)(x) = f(x) + g(x) \text{ and } (af)(x) = af(x) .$$

Then the following are examples of vector spaces of functions.

The set of all polynomials with real coefficients.

The set of all real-valued continuous functions defined on $[0, 1]$.

The set of real-valued functions that are differentiable on $[0, 1]$.

Each of the above examples are special cases for an even more general construction. For *any* non-empty set X and any field $\mathcal{F}$, define $\mathcal{F}^X = \{f : X \to \mathcal{F}\}$ to be a space of functions with addition and scalar multiplication defined for all $x \in X$, $f, g \in \mathcal{F}^X$ and $a \in \mathcal{F}$. Then F^X is a vector space of functions over the field $\mathcal{F}$. This vector space is denoted by $\Re^X$ when the field is chosen to be the real line.

Example :

For vector space $\mathbf{R}^4$ (4 Dimensional), subspaces are :

a. $\mathbf{R}^4$ itself

b. Zero vector ([0,0,0,0])

c. Line passing through zero vector (1 Dimensional)

d. Plane passing though zero vector (2 Dimensional)

e. 3D figure containing zero vector (3 Dimensional)

Anchor vectors selection. By following the treatment in [20, 21, 22], we first set $b_k = 0$ and divide anchor vectors into two categories:

- DC anchor vector $\mathbf{a}_0 = \frac{1}{\sqrt{N}}(1, \cdots, 1)^T$.
- AC anchor vectors $\mathbf{a}_k$, $k = 1, \cdots K-1$.

The terms "DC" and "AC" are borrowed from circuit theory, and they denote the "direct current" and the "alternating current", respectively. Based on the two categories of anchor vectors, we decompose the input vector space, $\mathcal{S} = R^N$, into the direct sum of two subspaces:

$$\mathcal{S} = \mathcal{S}_{DC} \oplus \mathcal{S}_{AC}, \tag{8}$$

where $\mathcal{S}_{DC}$ is the subspace spanned by the DC anchor and and $\mathcal{S}_A C$ is the subspace spanned by the AC anchors. They are called the DC and AC subspaces accordingly. For any vector $\mathbf{x} \in R^N$, we can project $\mathbf{x}$ to $\mathbf{a}_0$ to get its DC component. That is, we have

$$\mathbf{x}_{DC} = \mathbf{x}^T \mathbf{a}_0 = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n. \tag{9}$$

Subspace $\mathcal{S}_{AC}$ is the orthogonal complement to $\mathcal{S}_{DC}$ in $\mathcal{S}$. We can express the AC component of $\mathbf{x}$ as

$$\mathbf{x}_{AC} = \mathbf{x} - \mathbf{x}_{DC}. \tag{10}$$

Clearly, we have $\mathbf{x}_{DC} \in \mathcal{S}_{DC}$ and $\mathbf{x}_{AC} \in \mathcal{S}_{AC}$. We conduct the PCA on all possible $\mathbf{x}_{AC}$ and, then, choose the first $(K-1)$ principal components as AC anchor vectors $\mathbf{a}_k$, $k = 1, \cdots K-1$.