# Independence (Linear and Affine)

CE40282-1: Linear Algebra
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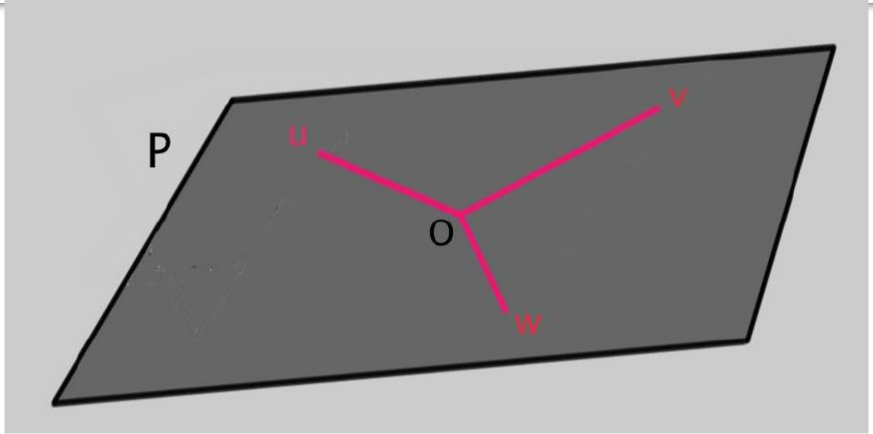
# Linear Independence



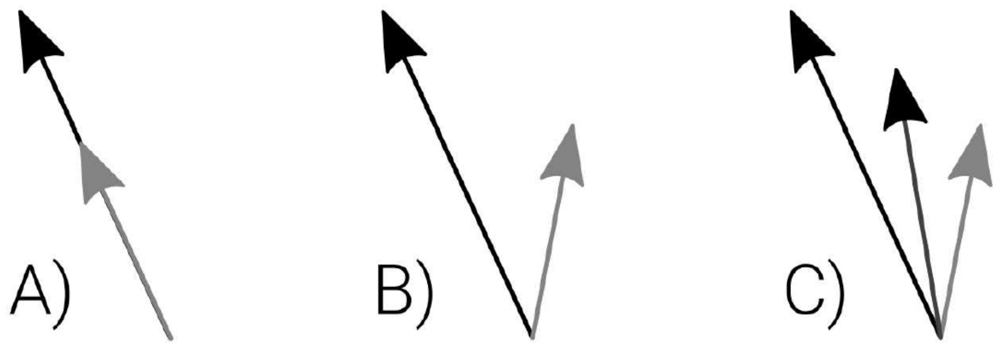

- Plane $P$ includes origin and three non-zero vectors $\{v, u, w\}$in $P$
- If no two of $\{v, u, w\}$ are parallel, then P=*span{u,v,w}*
- Any two vectors determines a plane and express the other as a linear combination of those two:

$$w = d_1 u + d_2 v \ \ (d_1 \& d_2\ can't\ both\ be\ zero)$$

- $c_1 u + c_2 v + c_3 w = 0 \longrightarrow u, w, v$ are not all independent.
- Independence is a property of a set of vectors.

# Definition

- Geometry:
  - A set of vectors is linear independent if the subspace dimensionality (its span) equals the number of vectors.
  - Example: 1,2,3 vectors spans?




Geometric sets of vectors in $\mathbb{R}^2$

# Definition

- Algebra
  - Dependent
    - For at least one $\lambda \neq 0$ $\quad \mathbf{0} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n, \quad \lambda \in \mathbb{R}$
    - A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.
  - Independence
    - Only when all $\lambda_i = 0$ $\quad \mathbf{0} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n, \quad \lambda \in \mathbb{R}$
    - No vector in the set is a linear combination of the others (has only the trivial solution)

# Example

- Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

- A set containing only one vector—say, v—is linearly independent if and only if v is not

  ...............

- a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$  b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

# Characterization of Linearly Dependent sets

## Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

- **does *not* say that *every* vector**

# Properties

- Theorem:

Any set of vectors that contains the zeros vector is guaranteed to be linearly dependent

# Properties

- The vectors coming from the parametric vector form of the solution of a matrix equation Ax = 0 are linearly independent.
- Example:
  - Vectors related to $x_2$ and $x_3$ are linear independent.
  - Columns of A related to to $x_2$ and $x_3$ are linear dependent.
  - Span$\{A_1, A_2, A_3\} = Span\{A_1\}$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}? \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

# Properties

- If a collection of vectors is linearly dependent, then any superset of it is linearly dependent.

- Any nonempty subset of a linearly independent collection of vectors is linearly independent.

# Properties

- Theorem:

Any set of $M > N$ vectors in $\mathbb{R}^{\mathrm{N}}$ is necessarily linearly dependent.

Any set of $M \leq N$ vectors in $\mathbb{R}^{\mathrm{N}}$ *could be* linearly independent.

# Example

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

# Linear Dependent Properties

- Suppose vectors $v_1, \ldots, v_n$ are linearly dependent:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1 \neq 0$. Then:

$$\text{span}\{v_1, \ldots, v_n\} = \text{span}\{v_2, \ldots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

# Linear combinations of linearly independent vectors

- suppose $x$ is linear combination of linearly independent vectors $a_1, \ldots, a_k$:

$$x = \beta_1 a_1 + \cdots + \beta_k a_k$$

- the coefficients $\beta_1, \ldots, \beta_k$ are *unique*

- proof

# Conclusion

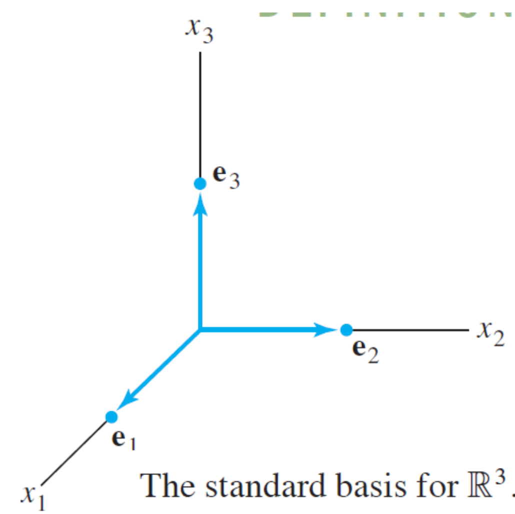
Step 1: Count the number of vectors (call that number $M$) in the set and compare to $N$ in $\mathbb{R}^{\mathrm{N}}$. As mentioned earlier, if $M > N$, then the set is necessarily dependent. If $M \leq N$ then you have to move on to step 2.

Step 2: Check for a vector of all zeros. Any set that contains the zeros vector is a dependent set.

- The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.

# Basis

- A set of n linearly independent n-vectors is called a basis
- A basis is the combination of span and independence: A set of vectors $\{v_1, \ldots, v_n\}$ forms a basis for some subspace of $R^n$ if it
  - (1) spans that subspace
  - (2) is an independent set of vectors.




The standard basis for $\mathbb{R}^3$.

# Basis

Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in $V$ is a **basis** for $H$ if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with $H$; that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

- Which are unique?
  - express a vector in terms of any particular basis
  - bases for $R^2$
  - bases with unit length for $R^2$

# Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called linearly dependent if there are nonzero constants c1 and c2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called linearly independent.

- Example: (linearly dependent or independent?)

$$\text{functions } f(t) = 2\sin^2 t \text{ and } g(t) = 1 - \cos^2(t)$$

$$functions\ \{\sin^2(x), \cos^2(x), \cos(2x)\} \subset \mathcal{F}$$

# Vector Space of Polynomials

- Linear independence
  - Example: Are $(1-x), (1+x), x^2$ linearly independent?

- Basis
  - Standard bases for $P_n(\mathbb{R})$?
  - Example: Are $(1-x), (1+x), x^2$ basis for $P_2(\mathbb{R})$?

# Coordinate Systems

- The main reason for selecting a basis for a subspace H; instead of merely a spanning set, is that each vector in H can be written in only one way as a linear combination of the basis vectors.

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for a subspace $H$. For each $\mathbf{x}$ in $H$, the **coordinates of x relative to the basis** $\mathcal{B}$ are the weights $c_1, \dots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of x (relative to** $\mathcal{B}$**)** or the $\mathcal{B}$**-coordinate vector of x**.[1]

- Example
  - Coordinate vector of $p(x) = 4 - x + 3x^2$ respect to basis $\{1, x, x^2\}$

# Coordinate axes

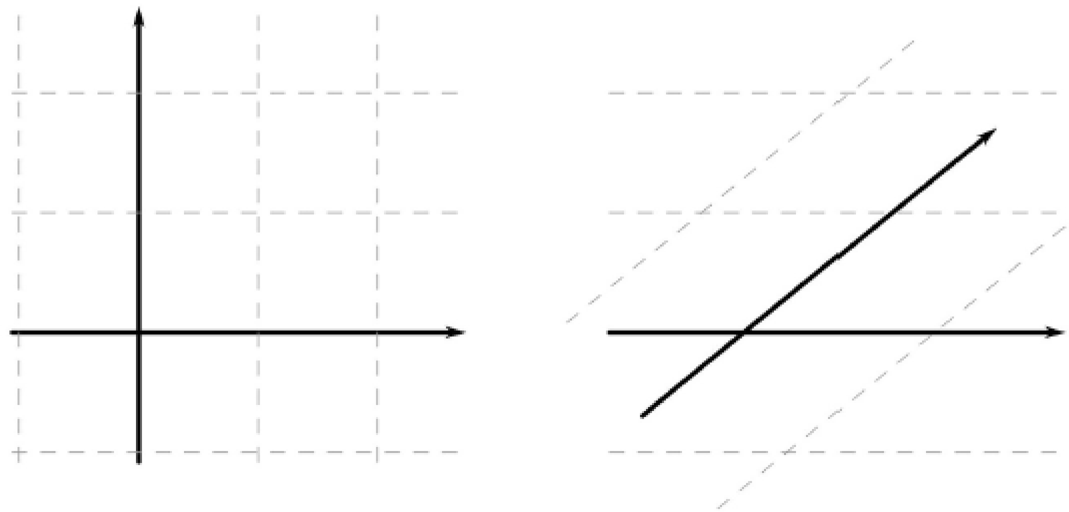


Figure 4.3: The familiar Cartesian plane (left) has orthogonal coordinate axes. However, axes in linear algebra are not constrained to be orthogonal (right), and non-orthogonal axes can be advantageous.

# Linearly Independent Sets versus Spanning Sets

**Theorem 2.2 — Linearly Independent Sets versus Spanning Sets**

Let $\mathcal{V}$ be a vector space with a basis $B$ of size $n$. Then

**a)** Any set of more than $n$ vectors in $\mathcal{V}$ must be linearly dependent, and

**b)** Any set of fewer than $n$ vectors cannot span $\mathcal{V}$.

Span vs Lin Indep

| Span | Lin Indep |
|---|---|
| Want many vectors in small space | Want few vectors in big space. |
| Adding vectors to list only helps | Deleting vectors from list only helps |
| $A = \begin{bmatrix} \vert & & \vert \\ \underline{v}_1 & \cdots & \underline{v}_k \\ \vert & & \vert \end{bmatrix}$<br>$A\underline{x}=\underline{b}$ has soln<br>$\iff \underline{b} \in \mathrm{Span}\{\underline{v}_1, \ldots, \underline{v}_k\}$ | $A\underline{x}=\underline{0}$ has only triv soln $\underline{x}=\underline{0}$<br>$\iff \underline{v}_1, \ldots, \underline{v}_k$ lin indep |

# Dimensions

- The dimensionality of a vector is the number of coordinate axes in which that vector exists.
- If a vector space is spanned by a finite number of vectors, it is said to be finite-dimensional. Otherwise it is infinite-dimensional.
- The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted dim(V).

# Dimensions

**Definition 2.3 — Dimension of a Vector Space**

A vector space $\mathcal{V}$ is called...

a) **finite-dimensional** if it has a finite basis, and its **dimension**, denoted by $\dim(\mathcal{V})$, is the number of vectors in one of its bases.

b) **infinite-dimensional** if it has no finite basis, and we say that $\dim(\mathcal{V}) = \infty$.

- Example: *Let's compute the dimension of some vector spaces that we've been working w*

| Vector space | Basis | Dimension |
|---|---|---|
| $F^n$ | | |
| $P^p$ | | |
| $M_{m,n}$ | | |
| $P$ | | |
| $F\ (functions)$ | | |
| $C$ (continues functions) | | |

# Dimensionality and Properties of Bases

Let $V$ be a finite dimensional vector space over a field $F$. Below are some properties of bases:

1. Any linearly independent list can be extended to a basis (a maximal linearly independent list is spanning).
2. Any spanning list contains a basis (a minimal spanning list is linearly independent).
3. Any linearly independent list of length $\dim V$ is a basis.
4. Any spanning list of length $\dim V$ is a basis.

- We will learn about change of basis in matrix transformation lecture!

# Independent ≤ spanning

In a finite-dimensional space,

| | | |
|---|---|---|
| the length of every linearly independent list of vectors | $\leq$ | the length of every spanning list of vectors |

- proof

# Affine Independence

An indexed set of points $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is **affinely dependent** if there exist real numbers $c_1, \ldots, c_p$, not all zero, such that

$$c_1 + \cdots + c_p = 0 \quad \text{and} \quad c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

Otherwise, the set is **affinely independent**.

- Example:
  - $\{v_1\}$

# Affine Independence

Given an indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. $S$ is affinely dependent.

b. One of the points in $S$ is an affine combination of the other points in $S$.

c. The set $\{\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1\}$ in $\mathbb{R}^n$ is linearly dependent.

- Example:
  - Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 14 \\ 6 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_4\}$. Is $S$ affinely dependent?

# Barycentric Coordinates

**THEOREM 6**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set in $\mathbb{R}^n$. Then each $\mathbf{p}$ in aff $S$ has a unique representation as an affine combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. That is, for each $\mathbf{p}$ there exists a unique set of scalars $c_1, \ldots, c_k$ such that

$$\mathbf{p} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad c_1 + \cdots + c_k = 1 \tag{7}$$

**DEFINITION**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an affinely independent set. Then for each point $\mathbf{p}$ in aff $S$, the coefficients $c_1, \ldots, c_k$ in the unique representation (7) of $\mathbf{p}$ are called the **barycentric** (or, sometimes, **affine**) **coordinates** of $\mathbf{p}$.

Observe that (7) is equivalent to the single equation

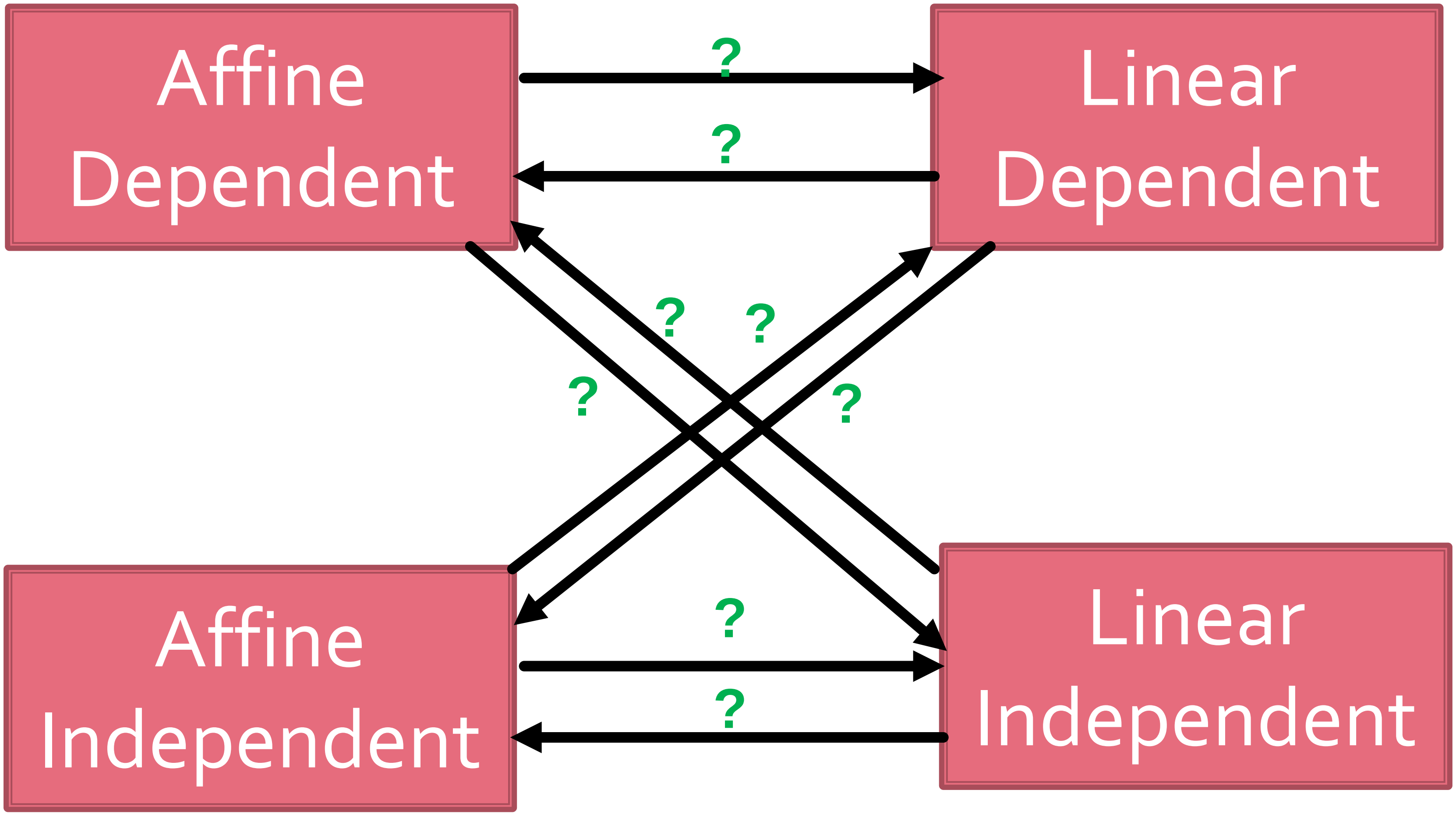
$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} \mathbf{v}_k \\ 1 \end{bmatrix} \tag{8}$$

involving the homogeneous forms of the points. Row reduction of the augmented matrix $\begin{bmatrix} \tilde{\mathbf{v}}_1 & \cdots & \tilde{\mathbf{v}}_k & \tilde{\mathbf{p}} \end{bmatrix}$ for (8) produces the barycentric coordinates of $\mathbf{p}$.

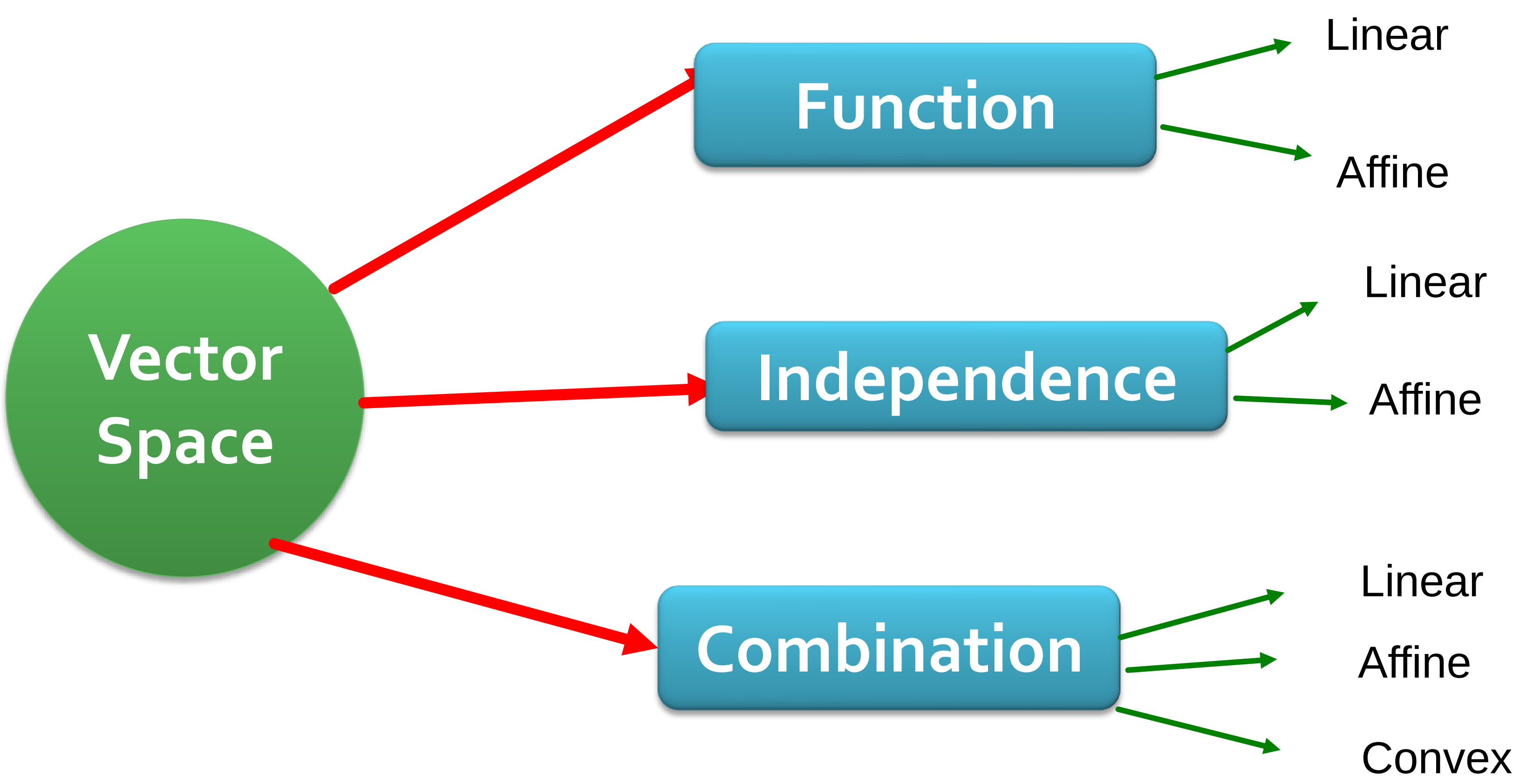
# Barycentric Coordinates

**EXAMPLE 4** Let $\mathbf{a} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the barycentric coordinates of $\mathbf{p}$ determined by the affinely independent set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

# Conclusion : Linear and Affine



# Conclusion and Review



# Reference

- Page 97 LINEAR ALGEBRA: Theory, Intuition, Code
- Page 213: David Cherney,
- Page 54: Linear Algebra and Optimization for Machine Learning