Linear Equations

CE40282-1: Linear Algebra
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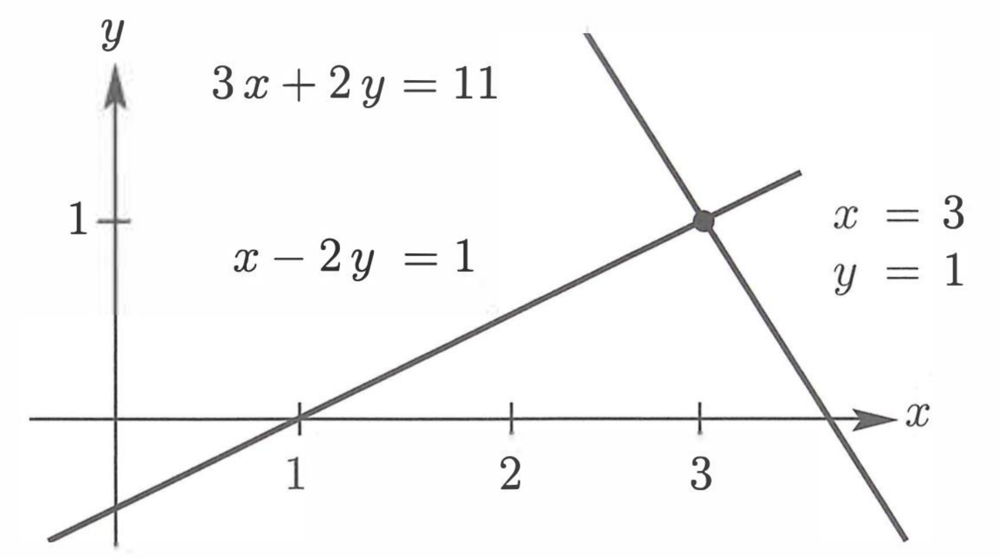
Vectors & Linear Equation

- Consider this simple system of equations,

$$\begin{aligned} x \quad - \quad 2y \quad &= 1 \\ 3x \quad + \quad 2y \quad &= 11 \end{aligned}$$



Vectors & Linear Equation

- Can be expressed as matrix-vector multiplication

$$\begin{aligned} x \quad &- \quad 2y &= 1 \\ 3x \quad &+ \quad 2y &= 11 \end{aligned} \qquad \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}$$

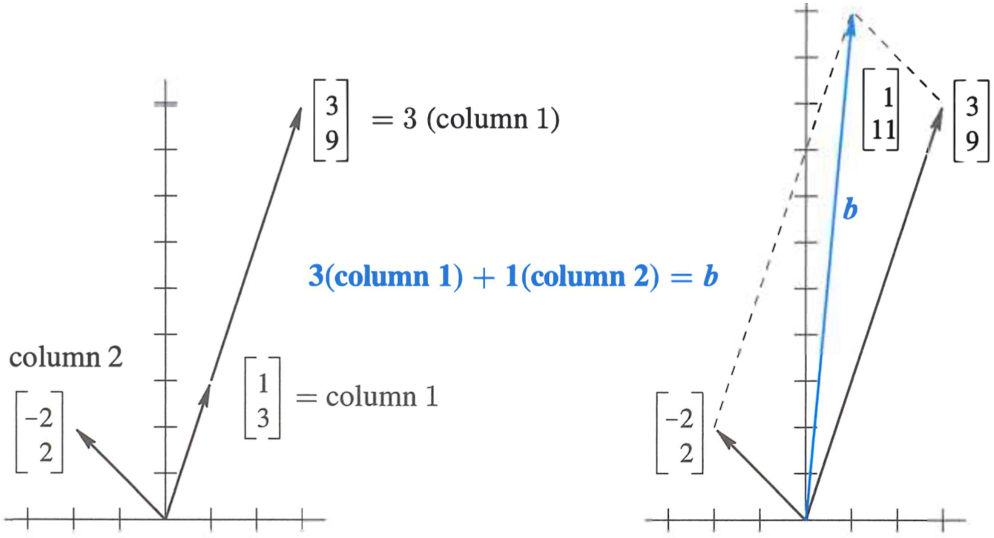
- Matrix Equation, $A\boldsymbol{x} = \boldsymbol{b}$
- A is often called **coefficient matrix**
- **Augmented matrix** is: $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

Vectors & Linear Equation

- Also, Can be expressed as linear combination of cols:

$$\begin{aligned} x \quad - \quad 2y \quad &= 1 \\ 3x \quad + \quad 2y \quad &= 11 \end{aligned}$$

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \boldsymbol{b}$$



- Same for n equation, n variables

Idea of Elimination

- Subtract a multiple of equation (1) from to (2) to eliminate a variable

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \xrightarrow[\textit{(subtract to eliminate } 3x\textit{)}]{\textit{(multiply equation 1 by 3)}} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \end{bmatrix}}_{\boldsymbol{c}}$$

A has become a upper triangle matrix U

Idea of Elimination (Row Reduction Algorithm)

- The **pivots** are on the diagonal of the triangle after elimination (boldface 2 below is the first pivot)

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8. \end{aligned}$$

  - Step 1: by subtracting (1) to (2) eliminate x's in (2)
  - Step 2: subtract (1) from (3) and totally eliminate x
  - Step 3: subtract new(2) from new(3)
- The variables corresponding to pivot columns in the matrix are called basic variables. The other variables are called a free variable.

Idea of Elimination (Row Reduction Algorithm)

- The whole idea is how to make U out of A by eliminate variables column by column

Column 1. ***Use the first equation to create zeros below the first pivot.***

Column 2. ***Use the new equation 2 to create zeros below the second pivot.***

Columns 3 to n. ***Keep going to find all n pivots and the upper triangular U.***

$$\begin{bmatrix} \boldsymbol{x} & x & x & x \\ 0 & \boldsymbol{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} \boldsymbol{x} & x & x & x \\ & \boldsymbol{x} & x & x \\ & & \boldsymbol{x} & x \\ & & & \boldsymbol{x} \end{bmatrix}$$

Elementary Row Operations

- **ELEMENTARY ROW OPERATIONS**
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

- It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange

> If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example

- Augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad\quad - 5x_3 &= 1 \\ x_2 + \ \ x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

- x_1, x_2: basic x_3: free
- This system is consistent, because the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

Definition

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

  If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

  4. The leading entry in each nonzero row is 1.
  5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Reduced echelon form

Existence and Uniqueness Questions

- Two fundamental question about a linear system:
  - Is the system consistent; that is, does at least one solution exist?
  - 2. If a solution exists, is it the only one; that is, is the solution unique?

Existence and Uniqueness Questions

- Theorem: A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \ \cdots \ 0 \ b]$ with b nonzero
- If a linear system is consistent, then the solution set contains either:
  - a unique solution, when there are no free variables
  - infinitely many solutions, when there is at least one free variable.

Find all solutions of a linear system

- 1. Write the augmented matrix of the system.
- 2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
- 3. Continue row reduction to obtain the reduced echelon form.
- 4. Write the system of equations corresponding to the matrix obtained in step 3.
- 5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Existence of Solutions

- The equation Ax=b has a solution if and only if b is a linear combination of the columns of A.

EXAMPLE Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if it can be written in the form $Ax=0$, where A is a matrix and 0 is the zero vector.
- Trivial solution: $Ax=0$ always has at least one solution, namely, $x=0$ (the zero vector).
- Nontrivial solution: The non-zero solution for $Ax=0$.
- The homogeneous equation $Ax=0$ has a nontrivial solution if and only if the equation has at least one free variable.

Nonhomogeneous Systems

Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad\quad &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R})$$

Conclusion

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Reference

- Linear Algebra and Its Applications, David C. Lay
- Linear Algebra Done Right, Axler, Chapter 3.D
- Introduction to Linear Algebra, Strange, Chapter 2.1,2.2
- http://vmls-book.stanford.edu/vmls-slides.pdf