



Vector Space

CE282: Linear Algebra

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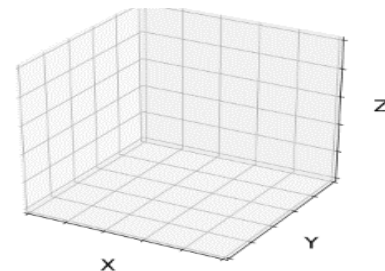
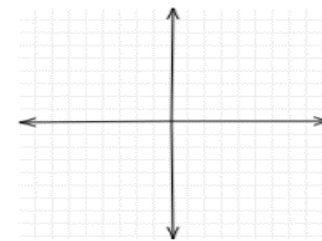


Definition

- A tuple is an ordered list of numbers.
- For example: $\begin{bmatrix} 1 \\ 2 \\ 32 \\ 10 \end{bmatrix}$ is a 4-tuple (a tuple with 4 elements).

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \dots \right\}$$

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \dots \right\}$$



Review: Complex Numbers



Numbers:

- Real: Nearly **any number** you can think of is a Real Number!

1	12.38	-0.8625	3/4	$\sqrt{2}$	1998
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- Imaginary: When **squared** give a **negative** result.

The "unit" imaginary number (like 1 for Real Numbers) is **i**, which is the square root of -1 .

Examples of Imaginary Numbers:

$3i$	$1.04i$	$-2.8i$	$3i/4$	$(\sqrt{2})i$	$1998i$
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And we keep that little "i" there to remind us we need to multiply by $\sqrt{-1}$

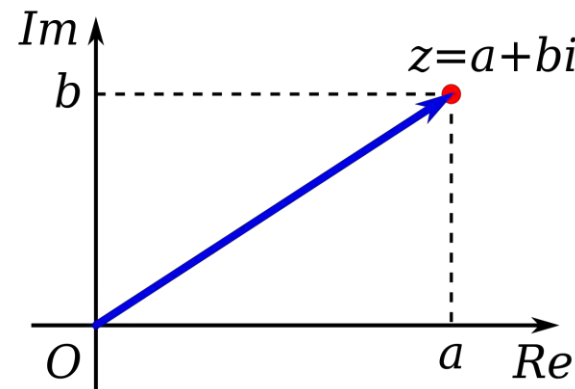
Review: Complex Numbers



- \mathbb{C} is a plane, where number $(a + bi)$ has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$
- Imaginary number: bi , $b \in \mathbb{R}$
- Arithmetic with complex numbers $(a + bi)$:

- $(a + bi) + (c + di)$
- $(a + bi)(c + di)$
- $\frac{a+bi}{c+di}$

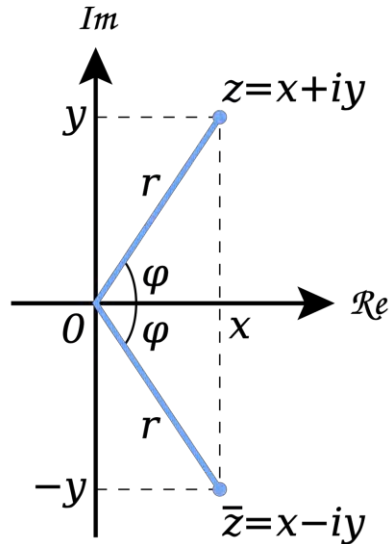
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2}$$





Question

Conjugate of $x + yi$ is noted by $\overline{x + yi}$. What it'll look like?



(Complex conjugate)

Review: Complex Numbers

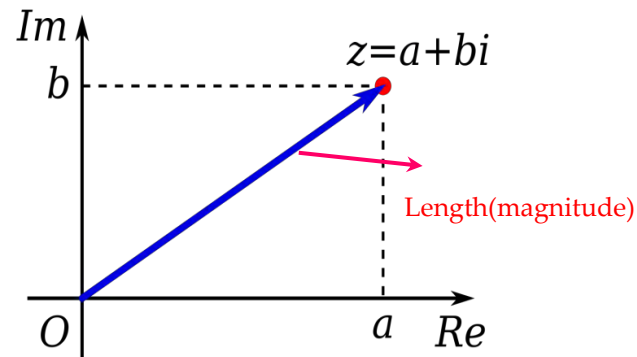


□ Length (magnitude): $||a + bi||^2 = \overline{(a + bi)}(a + bi) = a^2 + b^2$

□ Inner Product:

□ Real: $\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$

□ Complex: $\langle x, y \rangle = \overline{x_1}y_1 + \overline{x_2}y_2 + \dots + \overline{x_n}y_n$



Extra resource:

If you want to learn more about complex numbers, [this](#) video is recommended!



Definition

□ Any function from $A \times A \rightarrow A$ is a binary operation.

□ Closure Law:

□ A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.

$$\text{if } a \in A, b \in B \rightarrow a * b \in A$$



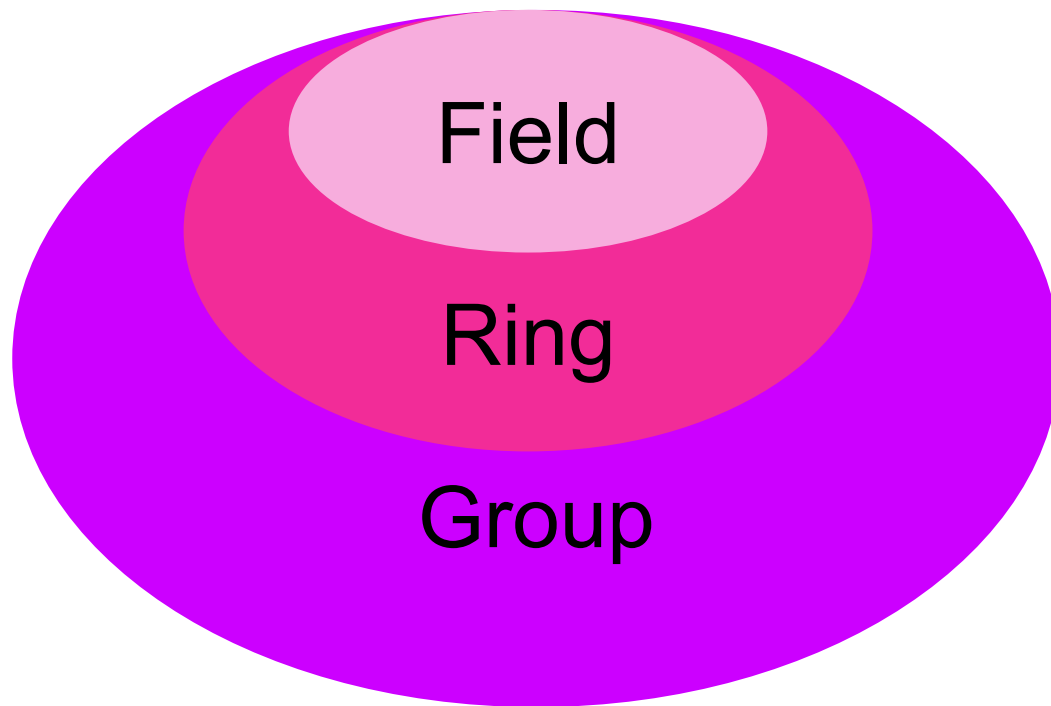
Class Activity

Scan the QR Code and answer the questions (or type the link in your browser)

- ☐ Is “+” a binary operator on natural numbers?
- ☐ Is “ \times ” a binary operator on natural numbers?
- ☐ Is “-” a binary operator on natural numbers?
- ☐ Is “/” a binary operator on natural numbers?



<https://forms.gle/KXstkou992to72Ew6>





Definition

□ A group G is a pair (S, \circ) , where S is a set and \circ is a binary operation on S such that:

1) \circ is **associative**.

2) **(Identity)** There exists an element $e \in S$ such that:

$$e \circ a = a \circ e = a \quad \forall a \in S$$

3) **(Inverses)** For every $a \in S$ there is $b \in S$ such that:

$$a \circ b = b \circ a = e$$

If \circ is commutative, then G is called a **commutative group**!



Definition

□ A **ring** R is a set together with two binary operations $+$ and $*$, satisfying the following properties:

1. $(R, +)$ is a commutative group
2. $*$ is associative
3. The distributive laws hold in R : (Multiplication is distributive over addition)

$$(a + b) * c = (a * c) + (b * c)$$

$$a * (b + c) = (a * b) + (a * c)$$



Definition

□ A **field** F is a set together with two binary operations $+$ and $*$, satisfying the following properties:

1. $(F, +)$ is a commutative group $\left\{ \begin{array}{l} \bullet \text{ Associative} \\ \bullet \text{ Identity} \\ \bullet \text{ Inverses} \\ \bullet \text{ Commutative} \end{array} \right.$
2. $(F - \{0\}, *)$ is a commutative group
3. The distributive law holds in F :

$$(a + b) * c = (a * c) + (b * c)$$

$$a * (b + c) = (a * b) + (a * c)$$



- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: $(F, +, \cdot)$

Example

$(\mathbb{R}; +, \cdot)$ and $(\mathbb{Q}; +, \cdot)$ serve as examples of fields.

$(\mathbb{Z}; +, \cdot)$ is an example of a ring which is not a field!

- Field is a set (F) with two binary operations $(+ , \cdot)$ satisfying following properties:



Properties	Binary Operations	
	Addition (+)	Multiplication (.)
Closure (بسته بودن)	$\exists a + b \in F$	$\exists a. b \in F$
Associative (شرکت پذیری)	$a + (b + c) = (a + b) + c$	$a. (b. c) = (a. b). c$
Commutative (جابه جایی پذیری)	$a + b = b + a$	$a. b = b. a$
Existence of identity $e \in F$	$a + e = a = e + a$	$a. e = a = e. a$
Existence of inverse: For each a in F there <u>must exist</u> b in F	$a + b = e = b + a$	$a. b = e = b. a$ <u>For any nonzero a</u>
Multiplication is distributive over addition $a. (b + c) = a. b + a. c$ $(a + b). c = a. c + b. c$		



Question

Which are fields? (two binary operations $+$, $*$)

\mathbb{R}

\mathbb{C}

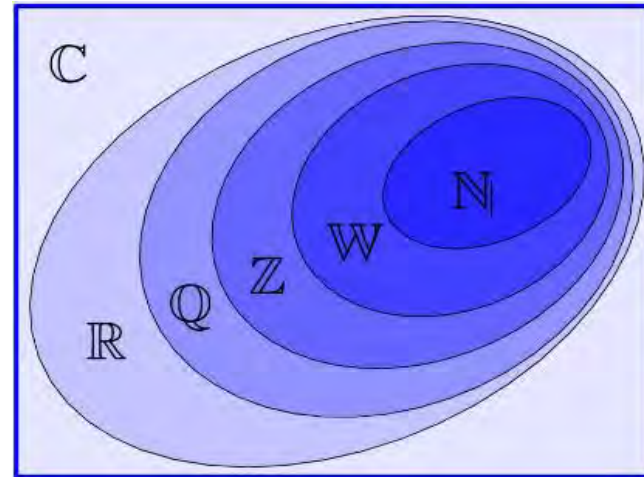
\mathbb{Q}

\mathbb{Z}

\mathbb{W}

\mathbb{N}

$\mathbb{R}^{2 \times 2}$



\mathbb{C} : Complex

\mathbb{R} : Real

\mathbb{Q} : Rational

\mathbb{Z} : Integer

\mathbb{W} : Whole

\mathbb{N} : Natural



- ❑ Building blocks of linear algebra.
- ❑ A **non-empty set V** with **field F** (most of time \mathbb{R} or \mathbb{C}) forms a vector space with two operations:
 1. **$+$: Binary operation on V which is $V \times V \rightarrow V$**
 2. **\cdot : $F \times V \rightarrow V$**

Note

In our course, by **default**, field is \mathbb{R} (real numbers).



Definition

A vector space over a field F is the set V equipped with two operations:
 $(V, F, +, \cdot)$

- i. **Vector addition:** denoted by “+” adds two elements $x, y \in V$ to produce another element $x + y \in V$
- ii. **Scalar multiplication:** denoted by “.” multiplies a vector $x \in V$ with a scalar $\alpha \in F$ to produce another vector $\alpha \cdot x \in V$. We usually omit the “.” and simply write this vector as αx



□ Addition of vector space ($x + y$)

□ **Commutative** $x + y = y + x \quad \forall x, y \in V$

□ **Associative** $(x + y) + z = x + (y + z) \quad \forall x, y, z \in V$

□ **Additive identity** $\exists \mathbf{0} \in V$ such that $x + \mathbf{0} = x, \forall x \in V$

□ **Additive inverse** $\exists (-x) \in V$ such that $x + (-x) = 0, \forall x \in V$



□ Action of the scalars field on the vector space (αx)

□ **Associative** $\alpha(\beta x) = (\alpha\beta)x$ $\forall \alpha, \beta \in F; \forall x \in V$

□ **Distributive over**

scalar addition: $(\alpha + \beta)x = \alpha x + \beta x$ $\forall \alpha, \beta \in F; \forall x \in V$

vector addition: $\alpha(x + y) = \alpha x + \alpha y$ $\forall \alpha \in F; \forall x, y \in V$

□ **Scalar identity** $1x = x$ $\forall x \in V$



Example

Let V be the set of all real numbers with the operations $u \oplus v = u - v$ (\oplus is an ordinary subtraction) and $c \boxdot u = cu$ (\boxdot is an ordinary multiplication). Is V a vector space? If it's not, which properties fail to hold?



Example

- The n-tuple space,
- The space of $m \times n$ matrices
- The space of functions from a set to a field $g(s)$

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

$$f(t) = 1 + \sin(2t) \quad \text{and} \quad g(t) = 2 + 0.5t$$

- The space of polynomial functions over a field $f(x)$:

$$p_n(t) = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$$



- Function addition and scalar multiplication

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (af)(x) = af(x)$$

Non-empty set X and any field F



$$F^X = \{f: X \rightarrow F\}$$

Example

- Set of all polynomials with real coefficients
- Set of all real-valued continuous function on $[0,1]$
- Set of all real-valued function that are differentiable on $[0,1]$



$P_n(\mathbb{R})$: Polynomials with max degree (n)

- ❑ Vector addition
- ❑ Scalar multiplication
- ❑ And other 8 properties!



Question

Which are vector spaces?

- ☐ Set \mathbb{R}^n over \mathbb{R}
- ☐ Set \mathbb{C} over \mathbb{R}
- ☐ Set \mathbb{R} over \mathbb{C}
- ☐ Set \mathbb{Z} over \mathbb{R}
- ☐ Set of all polynomials with coefficient from field \mathbb{R}
- ☐ Set of all polynomials of degree at most n with coefficient from field \mathbb{R}
- ☐ Matrix: $M_{m,n}(\mathbb{R})$
- ☐ Function: $f(x): x \rightarrow \mathbb{R}$



The operations on field F are:

- $+: F \times F \rightarrow F$
- $\times: F \times F \rightarrow F$

The operations on a vector space V over a field F are:

- $+: V \times V \rightarrow V$
- $\cdot: F \times V \rightarrow V$



Definition

If $v_1, v_2, v_3, \dots, v_p$ are in \mathbb{R}^n , then the set of all linear combinations of v_1, v_2, \dots, v_p is denoted by $\text{Span} \{v_1, v_2, \dots, v_p\}$ and is called the **subset of \mathbb{R}^n spanned (or generated) by v_1, v_2, \dots, v_p .**

That is, $\text{Span}\{v_1, v_2, \dots, v_p\}$ is the collection of all vectors that can be written in the form:

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

with c_1, c_2, \dots, c_p being scalars.



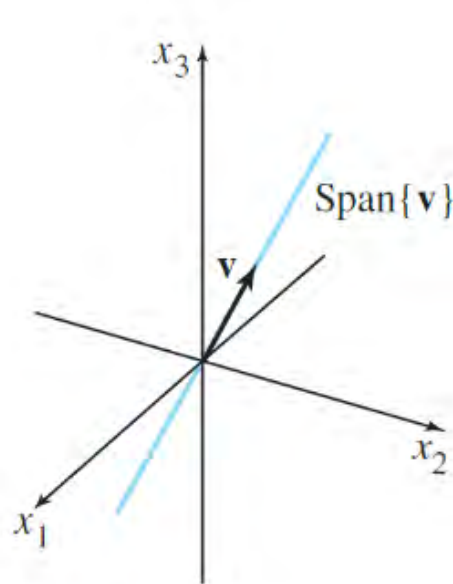
Example

- ❑ Is vector b in $\text{Span} \{v_1, v_2, \dots, v_p\}$
- ❑ Is vector v_3 in $\text{Span} \{v_1, v_2, \dots, v_p\}$
- ❑ Is vector 0 in $\text{Span} \{v_1, v_2, \dots, v_p\}$
- ❑ Span of polynomials: $\{(1+x), (1-x), x^2\}$?
- ❑ Is b in $\text{Span} \{a_1, a_2\}$?

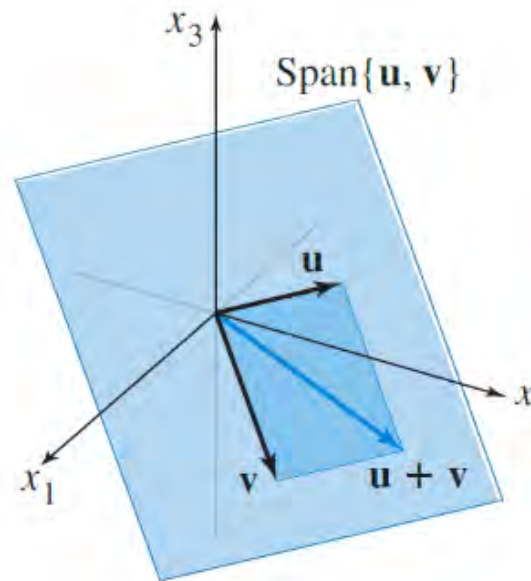
$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$



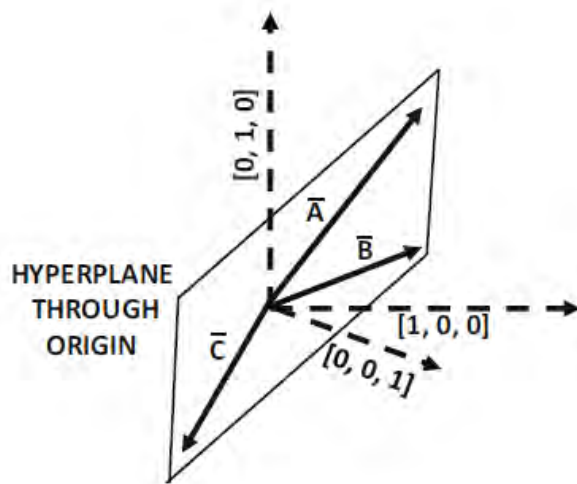
\mathbf{v} and \mathbf{u} are non-zero vectors in \mathbb{R}^3 where \mathbf{v} is not a multiple of \mathbf{u}



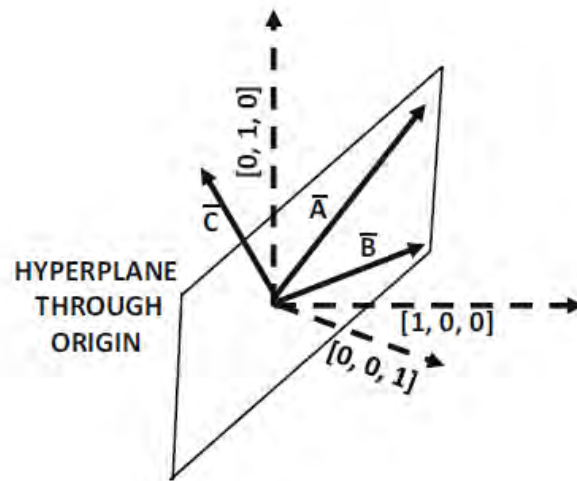
$\text{Span}\{\mathbf{v}\}$ as a
line through the origin.



$\text{Span}\{\mathbf{u}, \mathbf{v}\}$ as a
plane through the origin.



(a) $\text{Span}(\{\vec{A}, \vec{B}\}) = \text{Span}(\{\vec{A}, \vec{B}, \vec{C}\})$
 $\text{Span}(\{\vec{A}, \vec{B}, \vec{C}\}) = \text{All vectors on hyperplane}$



(b) $\text{Span}(\{\vec{A}, \vec{B}\}) \neq \text{Span}(\{\vec{A}, \vec{B}, \vec{C}\})$
 $\text{Span}(\{\vec{A}, \vec{B}, \vec{C}\}) = \text{All vectors in } \mathcal{R}^3$

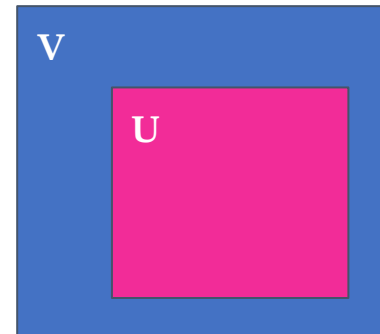
Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set



Definition

A **non-empty subset** of vector space for which closure holds for addition and scalar multiplication is called a subspace.

Subspace: If V is a vector space and **subset** $U \subseteq V$, then **U is itself a vector space** with the **same** addition and scalar multiplication as V .



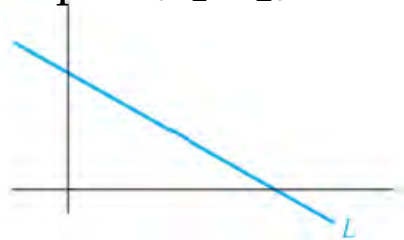


A subspace of \mathbb{R}^n is any set H in \mathbb{R}^n that has these properties:

- a. The zero vector is in H .
- b. For each u and v in H , the sum $u + v$ is in H .
- c. For each u in H and each scalar c , the vector cu is in H .

Example 1. $H = \text{Span} \{x_1, x_2\}$, then H is a subspace

Example 2.



Example 3. The vector space \mathbb{R}^2 is a subspace of \mathbb{R}^3 ?

Example 4. Is H a subset of \mathbb{R}^3 ? $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$



Let V be a vector subspace and let $U \subseteq V$:

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. *There is a vector $0 \in V$ such that $u + 0 = u$*
5. *Foreach $u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$*
6. $cu \in V$
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

1. $u + v \in U$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. *There is a vector $0 \in U$ such that $u + 0 = u$*
5. *Foreach $u \in U$, there is a vector $-u \in U$ such that $u + (-u) = 0$*
6. $cu \in U$
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

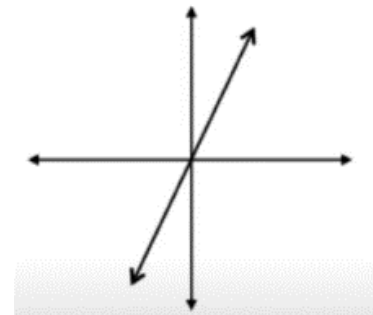
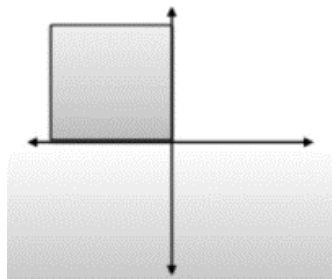


- ❑ A subspace is a subset of vector space that holds closure under addition and scalar multiplication.
- ❑ Zero vector is a subspace of every vector space.
- ❑ Vector space is a subspace of itself.



Example

- Set of all continuous real-valued functions on \mathbb{R} .
- **Set of all differentiable real-valued functions on \mathbb{R} .**
- Every vector space with more than one member has at least _____ subspaces.
- Name subspace for $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$
- Following figures:





Example

for vector space \mathbb{R}^4 (4 dimensional), subspaces are:

- a. \mathbb{R}^4 itself*
- b. zero vector $([0, 0, 0, 0])$*
- c. Line passing through zero vector (1 – dimensional)*
- d. Plane passing through zero vector (2 – dimensional)*
- e. 3D figure containing zero vector (3 – dimensional)*



Theorem

If U and W are subspaces of V , then $U \cap W$ is a subspace.

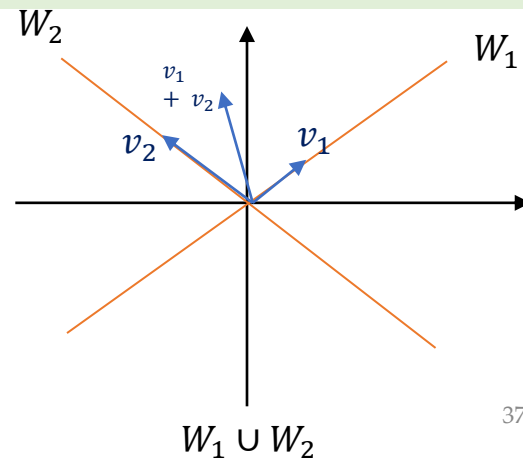
Proof:



Theorem

Fact: The union of two sub-spaces is not a subspace unless **one is contained in the other**.

W_1 and W_2 are subspaces of V , then $W_1 \cup W_2$ is subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$





Theorem

If v_1, v_2, \dots, v_p are in a vector space V , then $\text{Span} \{v_1, v_2, \dots, v_p\}$ is a subspace of V .

Proof:

Example

Let H be the set of all vectors of the form $\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix}$ where a, b are arbitrary scalars.

That is, let $H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} : a, b \text{ in } R \right\}$. Show that H is a subspace of \mathbb{R}^4 .



Theorem

Let v_1, v_2, \dots, v_n be vectors in vector space V and let w_1, w_2, \dots, w_k be vectors in $\text{Span} \{v_1, v_2, \dots, v_n\}$. Then:

$$\text{Span} \{w_1, w_2, \dots, w_k\} \subseteq \{v_1, v_2, \dots, v_n\}$$



- There are two reasons to use the sum of two vector spaces.
 - to build new vector spaces from old ones.
 - to decompose the known vector space into sum of two (smaller) spaces.
- Since we consider linear transformations between vector spaces, these sums lead to representations of these linear maps and corresponding matrices into forms that reflect these sums. In many very important situations, we start with a vector space V and can identify subspaces “internally” from which the whole space V can be built up using the construction of sums.



Definition

Let A and B be non-empty subsets of a vector space V . The **sum of A and B** , denoted $A+B$, is the set of all possible sums of elements from both subsets: $A + B = \{a + b : a \in A, b \in B\}$

Theorem

If W_1, \dots, W_m are subspaces of V , then $W_1 + \dots + W_m$ is a subspace of V .

A vector space V is called the **direct sum** of V_1 and V_2 if V_1 and V_2 are subspaces of V such that $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = V$. This means that every vector v from V is **uniquely represented via sum of two vectors** $v = v_1 + v_2, v_1 \in V_1, v_2 \in V_2$. We denote that V is the direct sum of V_1 and V_2 by writing $V = V_1 \oplus V_2$



Definition

$U + W$ is called a **direct sum**, if any element in $U + W$ can be written uniquely as $u + w$ where $u \in U$ and $w \in W$ (Notation: $U \oplus W$)

Example

Check where $U \oplus W$ exists?

$$\text{a) } W = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}, U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

$$\text{b) } W = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix}, U = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$



Theorem

If U and W are subspaces of V , then $U \oplus W$ is a subspace, if and only if $U \cap W = \{0\}$

Proof:



Example

Let E denote the set of all polynomials of even powers.

$E = \{a_n t^{2n} + a_{n-1} t^{2n-2} + \dots + a_0\}$, and O be the set of all polynomials of odd powers :

$O = \{a_n t^{2n+1} + a_{n-1} t^{2n-1} + \dots + a_0\}$. The set of all polynomials P is a direct sum of E and O :

$$P = E \oplus O$$

It is easy to see that any polynomial (or function) can be uniquely decomposed into direct sum of its even and odd counterparts:

$$p(t) = \frac{p(t) + p(-t)}{2} + \frac{p(t) - p(-t)}{2}$$

Example

Prove set of all bound functions such as

$$W = \{f(x) \mid \exists M \in R \text{ such that } |f(x)| \leq M, \forall x \in R\}$$

is a subspace of $V = \{\text{all functions from } R \text{ to } R\}$



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- David Cherney,
- Online Courses!
- Chapter 4 of Elementary Linear Algebra with Applications
- Chapter 3 of Applied Linear Algebra and Matrix Analysis