Surjection and Injection
Change of basis

CE282: Linear Algebra

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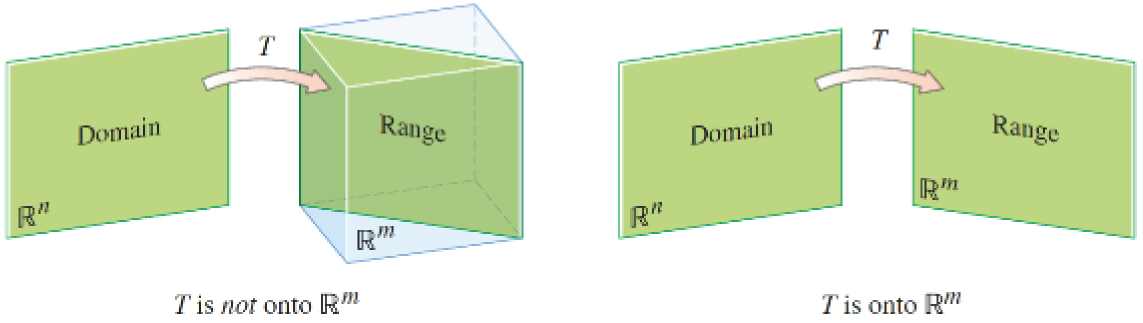
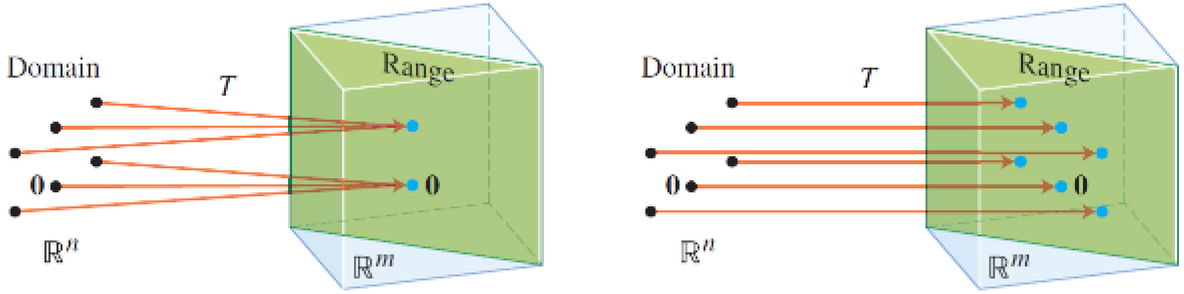
Mapping

- A mapping T : $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in R^n



- A mapping T : $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$



Onto (surjective) Transformation

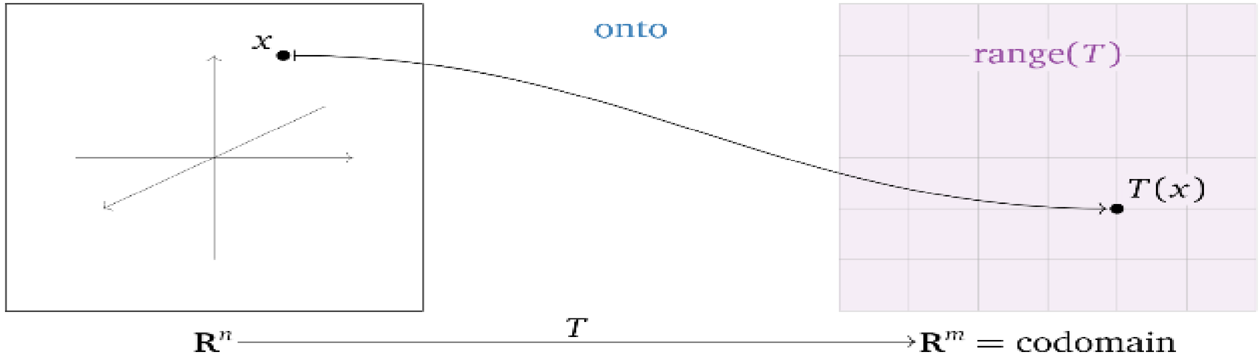
Definition

A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto if, for every vector b in $\mathbb{R}^m$, the equation T(x) = b has at least one solution x in $\mathbb{R}^n$.

Note

Here are some equivalent ways of saying that T is onto:

- The range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.

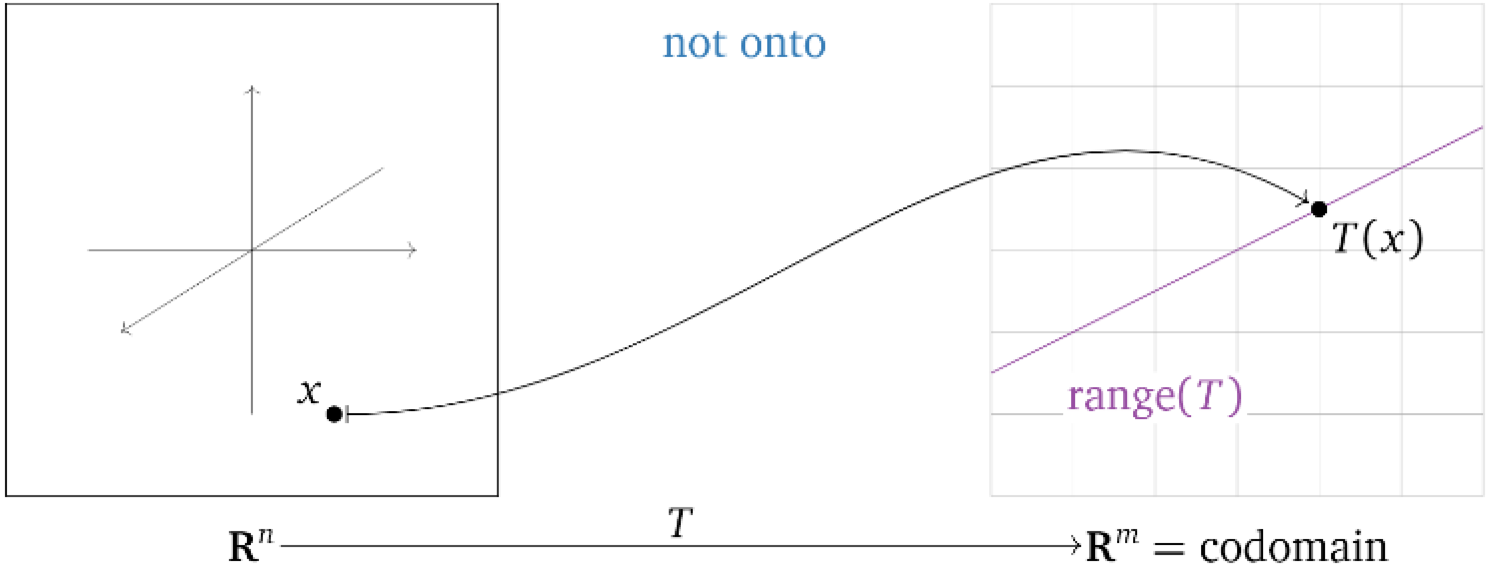


Onto Transformations

Note

Here are some equivalent ways of saying that T is not onto:

- The range of T is smaller to the codomain of T.
- There exists a vector b in $\mathbb{R}^m$ such that the equation T(x) = b does not have a solution
- There is a vector in the codomain that is not the output of any input vector.



Onto Transformation

Theorem

Let A be an $m \times n$ matrix and let T(x) = Ax be the associated matrix transformation. The following statement are equivalent:

- T in onto.
- T(x) = b has at least one solution for every b in $\mathbb{R}^m$.
- Ax = b is consistent for every b in $\mathbb{R}^m$.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot in every row.
- The range of T has dimension m.

Onto Transformations

Important

Tall matrices do not have onto transformations.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is an onto matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every row, it must have at least as many columns as rows: $m \leq n$.

This says that for instance, $\mathbb{R}^2$ is **too small** to admit an onto linear transformation to $\mathbb{R}^3$**.**

Note that there exist wide matrices that are not onto, for example,

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Does not have a pivot in every row.

Example

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

One-to-One (injective) Linear Transformation

Theorem

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then t is one-to-one if and only if the equation T(x) = 0 has only the trivial solution.

One-to-One Linear Transformation

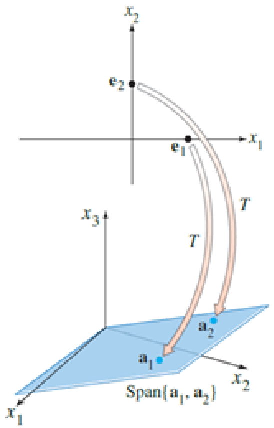
Important

Let $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
b. T is one-to-one if and only if the columns of A are linearly independence.

Example

Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?



One-to-One Transformations

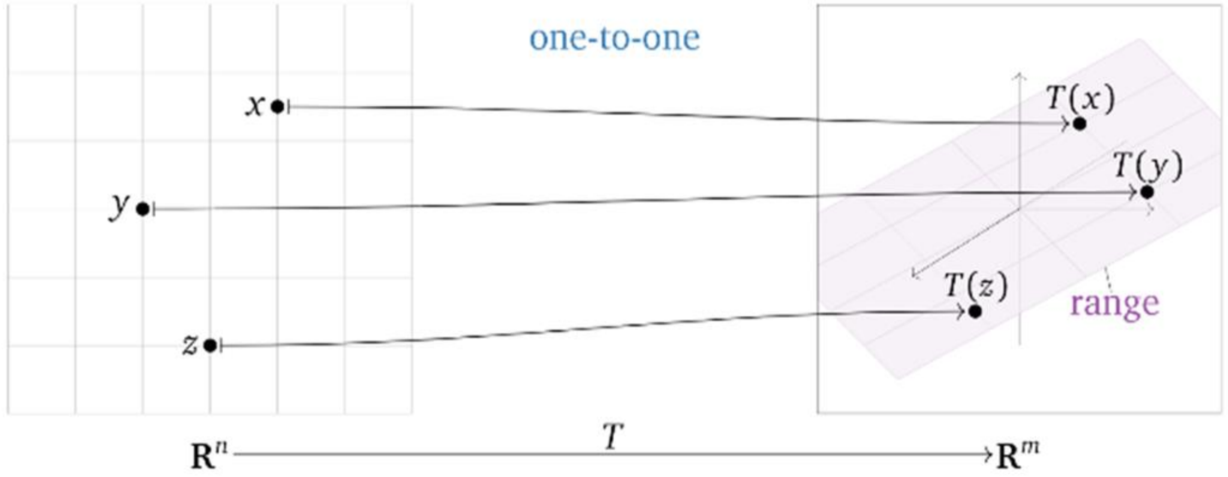
Definition

One-to-one transformations: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if, for every vector b in $\mathbb{R}^m$, the equation T(x) = b has at most one solution x in $\mathbb{R}^n$.

Remark

Here are some equivalent ways of saying that T is one-to-one:

- For every vector b in $\mathbb{R}^m$, the equation T(x) = b has zero or one solution x in $\mathbb{R}^n$.
- Different inputs of T have different outputs.
- If T(u) = T(v) then u = v.

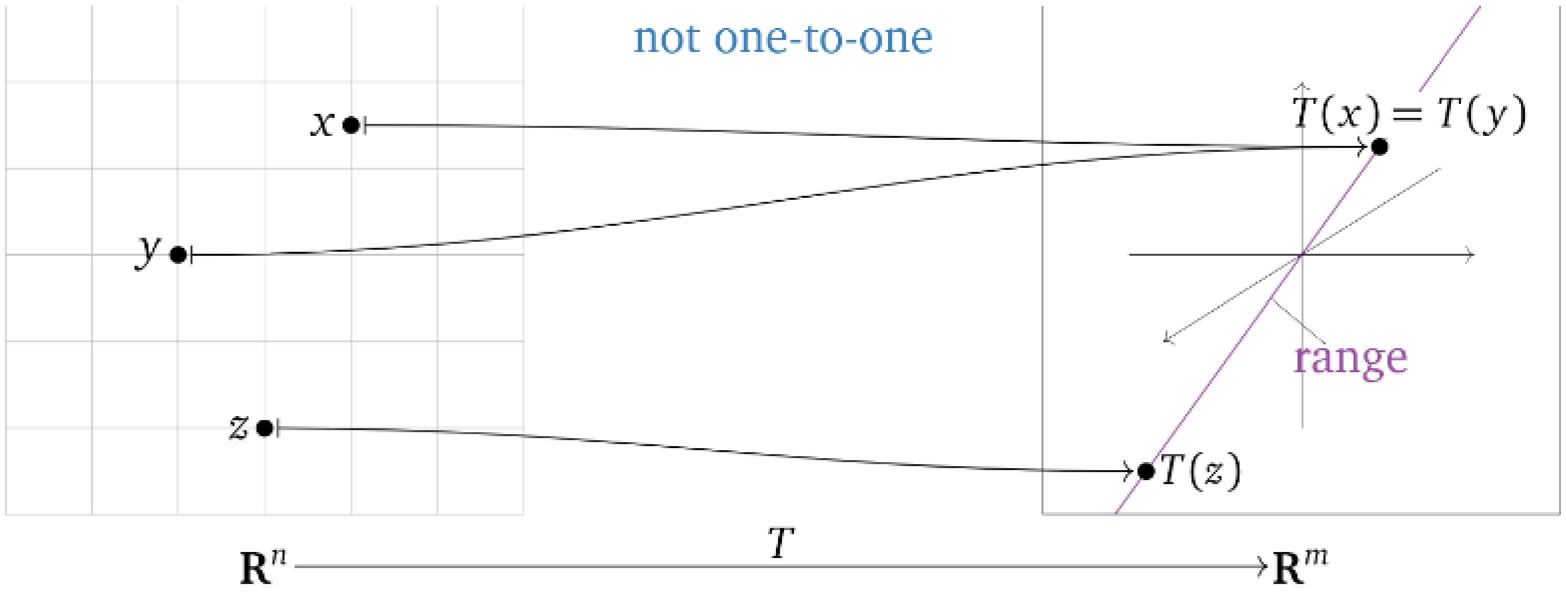


One-to-one Transformations

Remark

Here are some equivalent ways of saying that T is not one-to-one:

- There exist some vector b in $\mathbb{R}^m$ such that the equation T(x) = b has more than one solution x in $\mathbb{R}^n$.
- There are two different inputs of T with the same output.
- There exist vectors u, v such that $u \neq v$ but T(u) = T(v).



One-to-one Transformations

Theorem

Let A be an $m \times n$ matrix and let T(x) = Ax be the associated matrix transformation. The following statements are equivalent:

1. T is one-to-one.
2. For every b in $\mathbb{R}^m$, the equation T(x) = b has at most one solution.
3. For every b in $\mathbb{R}^m$, the equation T(x) = b has a unique solution or is inconsistent.
4. Ax = 0 has only the trivial solution.
5. The columns of A are linearly independent.
6. A has a pivot in every column.
7. The range of T has dimension n.

One-to-one Transformations

Important

Wide matrices do not have one-to-one transformations.

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an one-to-one matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every column, it must have at least as many rows as columns : $n \leq m$.

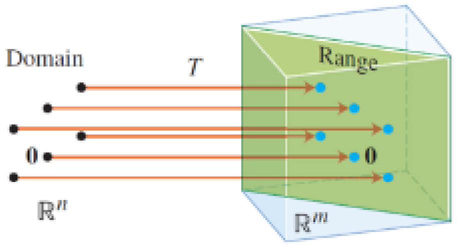
This says that for instance, $\mathbb{R}^3$ is **too big** to admit a one-to-one linear transformation into $\mathbb{R}^2$**.**

Note that there exist tall matrices that are not one-to-one, for example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Does not have a pivot in every column.

A is an m × n matrix, and T: $\mathbb{R}^n \to \mathbb{R}^m$ is the matrix transformation T(x) = Ax.



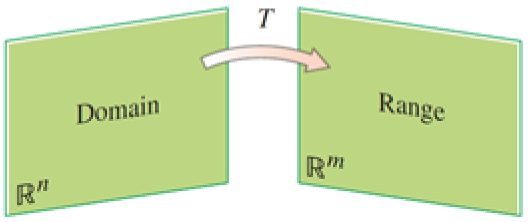
T is one-to-one

$T(x) = b$ has *at most one solution* for every b.

The columns of A are linearly independent.

A has a pivot in every column.

The range of T has dimension n.



T is onto

$T(x) = b$ has *at least one solution* for every b.

The columns of A span $\mathbf{R}^m$.

A has a pivot in every row.

The range of T has dimension m.

One-to-one and onto

Important

One-to-one is the same as onto for square matrices. We observed that a square has a pivot in every row if and only if it has a pivot in every column. Therefore, a matrix transformation T from $\mathbb{R}^n$ to itself is one-to-one if and only if it is onto : in this case, the two notations are equivalent.

Conversely, by this note, if a matrix transformation T: $\mathbb{R}^m \rightarrow \mathbb{R}^n$ is both one-to-one and onto, then m = n.

Note that in general, a transformation T is both one-to-one and onto if and only if T(x) = b has exactly one solution for all b in $\mathbb{R}^m$.

Bijective

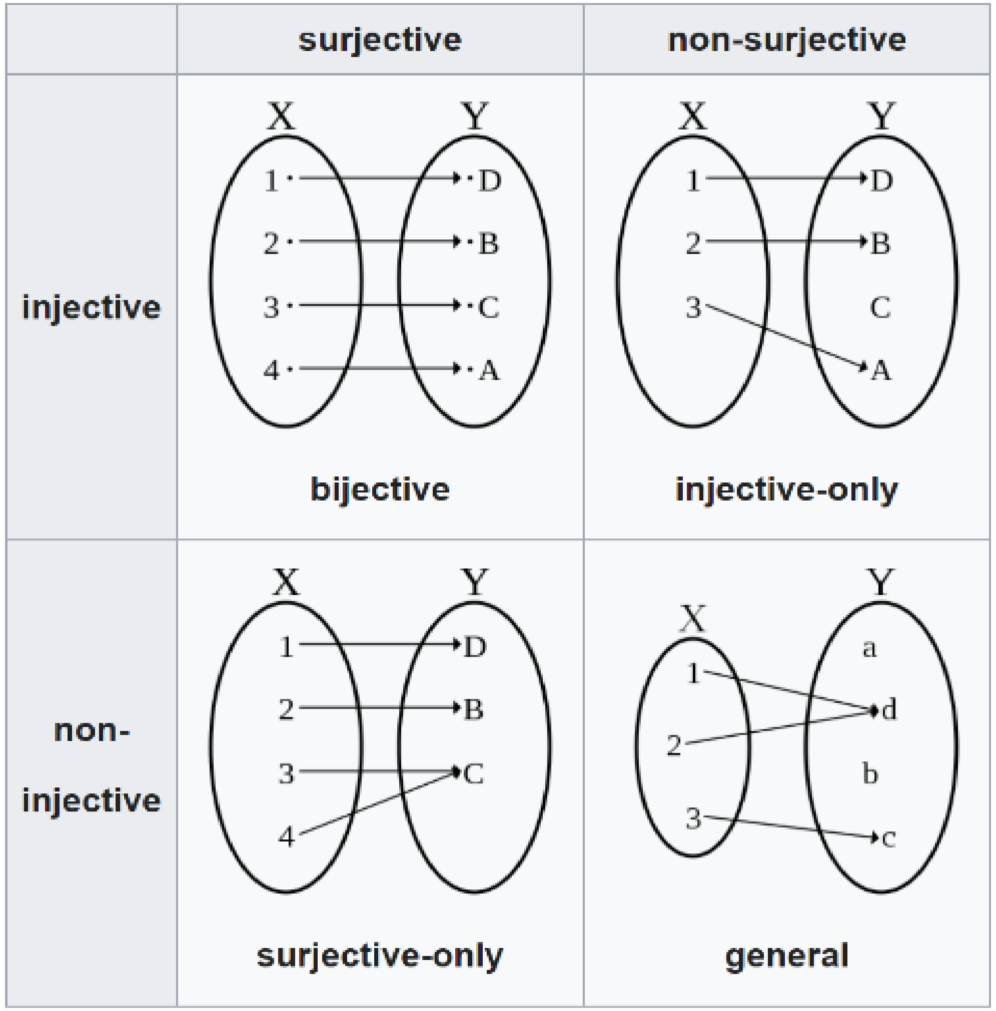
Note

- **One-to-one and onto.**
- **If and only if every possible image is mapped to by exactly one argument.**

Conclusion

onto

One-to-one



Machine learning application

- The central problem in machine learning and deep learning is to meaningfully transform data; in other words, to learn useful representations of the input data at hand – representations that get us to the expected output.

Inner Product

Note

$$< Ax, y > = < x, A^T y >$$

What about symmetric matrix?

Example

Show that unitary matrix preserves inner product. $< Ux, Uy > = < x, y >$

Example

- Find the coordinate vector of $2 + 7x + x^2 \in \mathcal{P}^2$ with respect to the basis B = $\{x + x^2, 1 + x^2, 1 + x\}$.
- If C = $\{1,\ x,\ x^2\}$ is the standard basis of $\mathcal{P}^2$ then we have $[2 + 7x + x^2]_C = (2, 7, 1)$.

Introduction to change of basis

- $B = \{v_1, \dots, v_n\}$ are basis of $\mathbb{R}^n$.

- $\mathrm{P} = [v_1 \;\; v_2 \quad \dots \quad v_n]$

- $P[a]_B = a$

Change of Basis

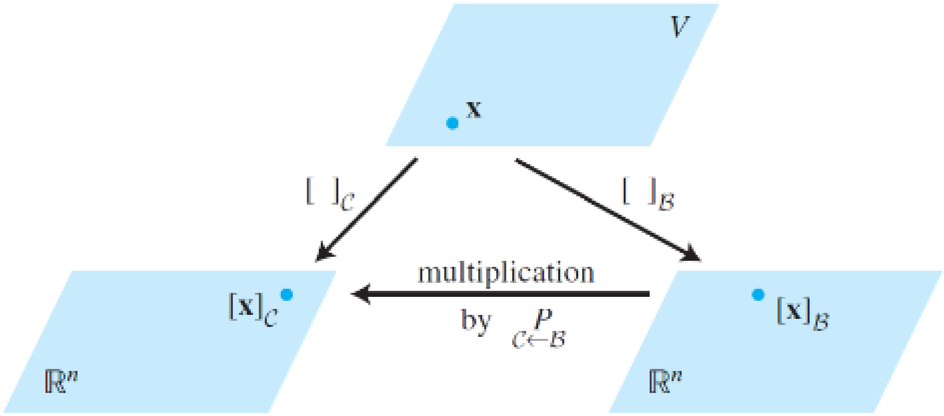
Theorem

Let B = $\{b_1, b_2, \ldots, b_n\}$ and C = $\{c_1, c_2, \ldots, c_n\}$ be basses of a vector space V. Then there is a unique n × n matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B}[x]_B$$

The columns of $P_{C \leftarrow B}$ are the C-coordinate vectors of the vectors in basis B.
That is ,

$$P_{C \leftarrow B} = [[b_1]_C \quad [b_2]_C \quad \ldots \quad [b_n]_C]$$



$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$$

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

Example

Find the change-of-basis matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ for the bases

$$B = \{x + x^2, 1 + x^2, 1 + x\} \;\; and \;\; C = \{1, x, x^2\}$$

of $\mathcal{P}^2$. Then find the coordinate vector of $2 + 7x + x^2$ with respect to B.

Change of basis

Example

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for $\mathbb{R}^2$ given by B = $\{b_1, b_2\}$, C = $\{c_1, c_2\}$.

a. Find the change-of-coordinates matrix from C to B.
b. Find the change-of-coordinates matrix from B to C.

Change of basis

Example

Find the change-of-basis matrix $P_{C \leftarrow B}$, where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \mathrm{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Important

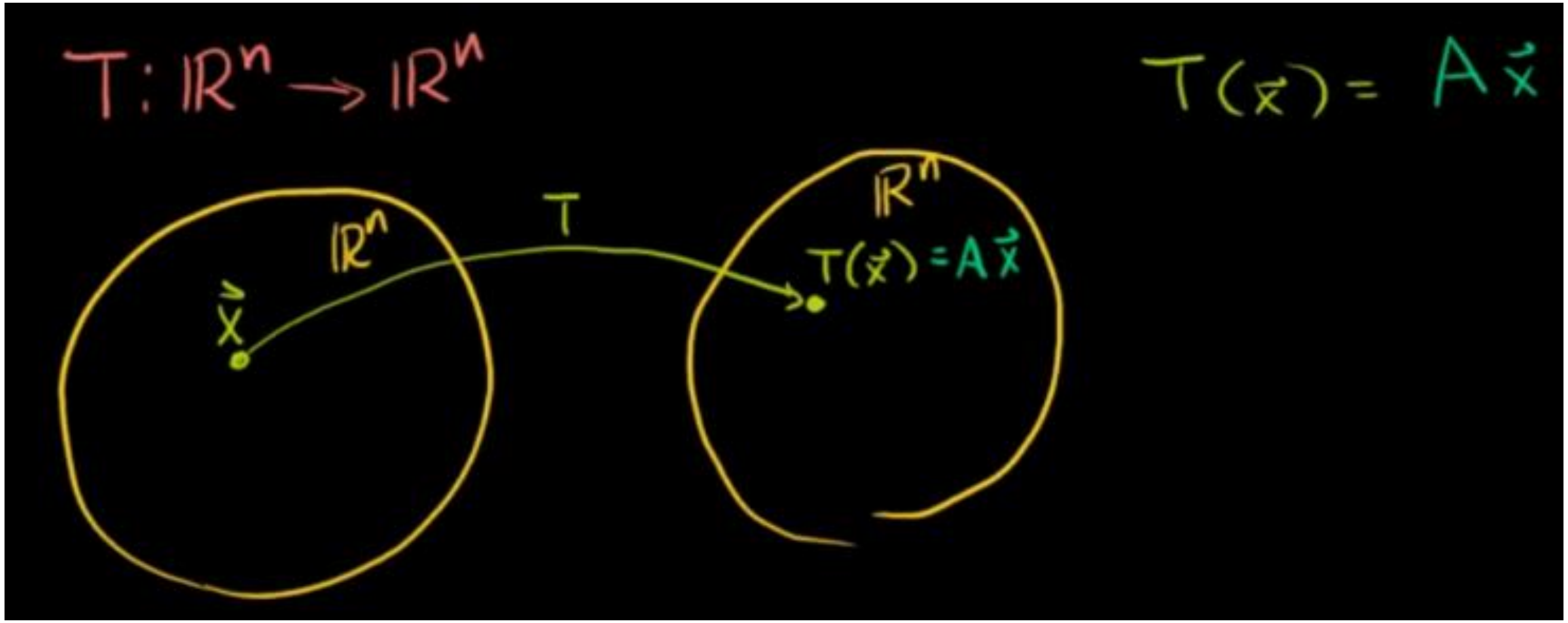
Let T: $\mathbb{R}_n \rightarrow \mathbb{R}_m$ be a linear function and u = $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}_n$.

The matrix $[A(e_1) \quad \dots \quad A(e_n)]$ is called the matrix representation of linear function (transformation)T which is denoted by $[A]_E$.

$E = \{e_1, e_2, e_3\}$ B=$\{B_1, B_2, B_3\}$

What is the relation between $[A]_B$ and $[A]_E$?

Transformation with change of basis



- $B = \{v_1, v_2, \ldots, v_n\}$ *are basis of* $\mathbb{R}^n$.
- $P = [v_1 \quad v_2 \quad \ldots \quad v_n]$
- $[T(x)]_B = P^{-1}AP[x]_B$

Change of basis

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[A]_B}"] \arrow[d, "P"'] & \mathbb{R}^n \\ \mathbb{R}^n \arrow[r, "{[A]_E}"'] & \mathbb{R}^n \arrow[u, "P^{-1}"'] \end{tikzcd}$$

$$[A]_B = P^{-1}[A]_E P$$

Example (Take-home)

Example

We have B = $\left\{x^3, x^2, x, 1\right\}$ and $B' = \{x^2, x, 1\}$ are bases for $\mathcal{P}_3(x)$ and $\mathcal{P}_2(x)$, respectively. Find the matrix of transformation T: $P_3(x) \rightarrow P_2(x)$.

Isomorphisms

Definition

Suppose V and W are vector spaces over the same field. We say that V and W are **isomorphic**, denoted by $V \cong W$, if there exists an invertible linear transformation T: $V \to W$(called an **isomorphism** from V to W).

- If T: $V \to W$ is an isomorphism then so is $T^{-1}: W \to V$.
- If T: $V \to W$ and S: $W \to X$ are isomorphism then so is $S \circ T: V \to X$.
  in particular, if $V \cong W$ and W $\cong X$ then $V \cong X$.

Example

Show that the vector space V = span(e^x, xe^x, x^2e^x) and $\mathbb{R}^3$ are isomorphic.