Singular Values and Singular Vectors

Linear Algebra

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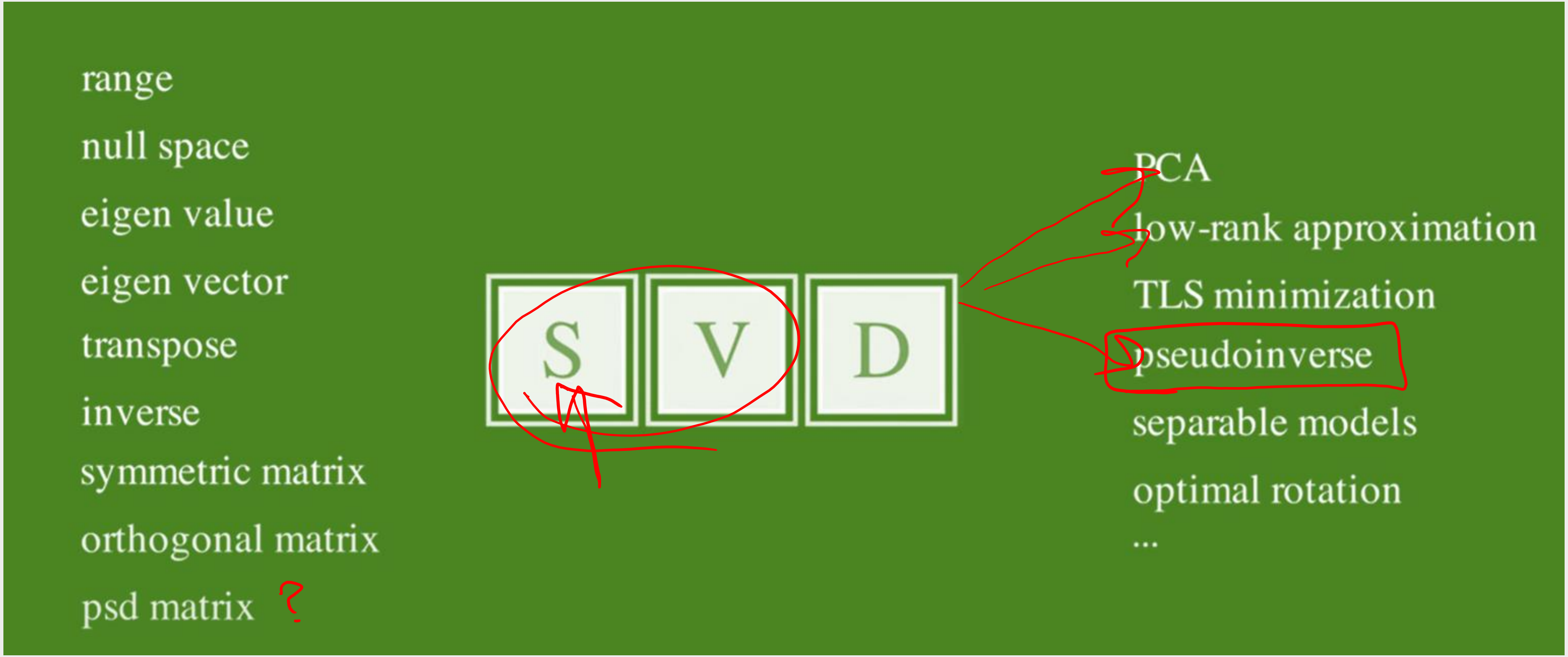
Introduction



REVIEW: Eigenvectors of a Symmetric Matrix

$\lambda u^T v = \gamma u^T v$

$\boxed{A^T = A}$

Theorem

$v \neq 0$, $u \neq 0$

Orthogonality of Eigenvectors of a Symmetric Matrix Corresponding to Distinct Eigenvalues

$\boxed{\begin{aligned} Au &= \lambda u \\ Av &= \gamma v \end{aligned}}$

$\forall u, v \in$

$\lambda u^T v$

$u \perp v$

Proof?

$A = A^T$

$\boxed{\lambda \neq \gamma}$

$\boxed{\lambda \langle u, v \rangle}$

$\boxed{\langle u, v \rangle = 0}$

$= \langle Au, v \rangle$

Au

$\boxed{(\lambda - \gamma)}\, u^T v = 0$

$\neq 0$

$(Au)^T v = u^T A^T v$

$\boxed{Av}$

$= u^T \gamma v = \gamma u^T v$

Theorem

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA.

Proof?

$(AB)v = \lambda v$

$\lambda Bv = B\lambda v = B(AB)v = BA(Bv)$

عدد λ, z ($Bv = z$)

BA بردار ویژه $= z$

BA مقدار ویژه $= \lambda$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Singular Value

- $S_{m\times n}$ Non–Square!!
- $\sigma_i = \sqrt{\lambda_i} \quad \lambda_i \in \sigma(S^T S), i = 1, \dots, n$
- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{m-1} \geq \sigma_m$

Example

$$S = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$S^T S = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \Rightarrow \lambda(S^T S) = \{360, 90, 0\}$$

$$\Rightarrow \begin{cases} \sigma_1 = \sqrt{360} = 6\sqrt{10} \\ \sigma_2 = \sqrt{90} = 3\sqrt{10} \\ \sigma_3 = 0 \end{cases}$$

Singular value and eigenvalue

Theorem

$\{v_1, \dots, v_n\}$ are orthonormal eigenvectors of matrix $S^T S$ then singular values of matrix S are norm of Sv_i vectors:

$$||Sv_i|| = \sigma_i$$

Proof?

Example

$$S = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \rightarrow S^T S = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \rightarrow \sigma_1 = \sqrt{360}, \sigma_2 = \sqrt{90}, \sigma_3 = 0$$

$$v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$Sv_1 = \begin{bmatrix} 18 \\ 6 \end{bmatrix} \Rightarrow \|Sv_1\| = \sqrt{18^2 + 6^2} = \sigma_1$$

$$Sv_2 = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \Rightarrow \|Sv_2\| = \sqrt{3^2 + (-9)^2} = \sigma_2$$

$$Sv_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \|Sv_3\| = 0 = \sigma_3$$

Singular value and Rank

Theorem

$\{v_1, \dots, v_n\}$ are orthonormal eigenvectors of matrix $S^T S$ and S has r non-zero singular value:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \qquad \sigma_{r+1} = \cdots = \sigma_n = 0$$

$\{Sv_1, \dots, Sv_r\}$ is a orthogonal basis for range of S

rank(S)=r

Rank of Matrix = Number of nonzero singular values

How to find $\{u_1, \dots, u_r\}$ is a orthonormal basis for range of S

Singular Value Decomposition (SVD)

- Given any $m \times n$ matrix **A**, algorithm to find matrices **U**, **V**, and $\sum$ such that (**always exists**)
- $A = U\Sigma V^T$ $A = U\Sigma V^*$

  U is $m \times m$ and orthogonal (always real)

  $\sum$ is $m \times n$ and diagonal with non-negative (always real) called singular values.

  V is $n \times n$ and orthogonal (always real)
- Columns of U are eigenvectors of AA^T (called the left singular vectors).
- Columns of V are eigenvectors of $A^T A$ (called the right singular vectors).
- The non-zero singular vectors are the positive square roots of non-zero eigenvalues of AA^T or $A^T A$.

SVD Introduction

- Generalization of the spectral decomposition that applies to all matrices, rather than just normal matrices.
- Applications:
  - Compute the size of a matrix (in a way that typically makes more sense than norm)
  - Provide a new geometric interpretation of linear transformations
  - Solve optimization problems
  - Construct an “almost inverse” for matrices that do not have an inverse.

SVD Comparison

SVD	Diagonalization	Spectral decomposition	Schur triangularization
applies to every single matrix (even rectangular ones).	only applies to matrices with a basis of eigenvectors	only applies to normal matrices	only applies to square matrices
matrix Σ in the middle of the SVD is diagonal (and even has real non-negative entries)	do not guarantee an entrywise non-negative matrix	do not guarantee an entrywise non-negative matrix	only results in an upper triangular middle piece
It requires two unitary matrices U and V	only required one invertible matrix	only required one unitary matrix	only required one unitary matrix

SVD

- The Σ_i are called the singular values of **A**
- If **A** is singular, some of the Σ_i will be 0
- In general *rank*(**A**) = number of nonzero Σ_i
- SVD is mostly unique (up to permutation of singular values, or if some Σ_i are equal)

SVD for Square Matrix

- The SVD is a factorization of a m x n matrix into

$$A = U\Sigma V^T$$

Where U is a m x m orthogonal matrix, V^T is a n x n orthogonal matrix and Σ is a m x n diagonal matrix.

For a square matrix (m=n):

$$A = \begin{pmatrix} \vdots & \cdots & \vdots \\ u_1 & \cdots & u_n \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} \cdots & v_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_n^T & \cdots \end{pmatrix}$$

$$A = \begin{pmatrix} \vdots & \cdots & \vdots \\ u_1 & \cdots & u_n \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} \vdots & \cdots & \vdots \\ v_1 & \cdots & v_n \\ \vdots & \cdots & \vdots \end{pmatrix}^T$$

Reduced SVD

$$[Sv_1 \quad \dots \quad Sv_r \quad 0 \quad \dots \quad 0]_{m\times n} = [\sigma_1 u_1 \quad \dots \quad \sigma_r u_r \quad 0 \quad \dots \quad 0]_{m\times n}$$
$$[Sv_1 \quad \dots \quad Sv_r \quad Sv_{r+1} \quad \dots \quad Sv_n]_{m\times n} = [\sigma_1 u_1 \quad \dots \quad \sigma_r u_r \quad 0 \quad \dots \quad 0]_{m\times n}$$

$$S[v_1 \quad \dots \quad v_n] = [u_1 \quad \dots \quad u_m]\left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & & \vdots & 0 \\ 0 & \cdots & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

$$S_{m\times n}V_{n\times n} = U_{m\times m}\Sigma_{m\times n}$$

$$S = U\Sigma V^T$$

- what happens when A is not a square matrix?
- n > m

$$A = U\Sigma V^T$$

$$= \begin{pmatrix} \vdots & \cdots & \vdots \\ u_1 & \cdots & u_m \\ \vdots & \cdots & \vdots \end{pmatrix}_{m\times m} \begin{pmatrix} \sigma_1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \sigma_m & & & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} \cdots & v_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_m^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_n^T & \cdots \end{pmatrix}_{n\times n}$$

We can instead rewrite the above as:

$$A = U\Sigma_R V_R^T$$

where V_R is n x m matrix and Σ_R is a m x m matrix

In general:

$$A = U_R\Sigma_R V_R^T$$

Now U and V are not orthogonal.
But their columns are orthonormal.

U_R is a m x k matrix
Σ_R is a k x k matrix
V_R is a n x k matrix

k = min(m, n)

Reduced SVD

- m > n

$$A = U\Sigma V^T$$

$$= \begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ u_1 & \cdots & u_n & \cdots & u_m \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix}_{m\times m} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} \cdots & v_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_n^T & \cdots \end{pmatrix}_{n\times n}$$

We can instead rewrite the above as:

$$A = U\Sigma_R V_R^T$$

Now U and V are not orthogonal.
But their columns are orthonormal.

where U_R is m x n matrix and Σ_R is a n x n matrix

Reduced SVD

- Let's take a look at the product of $\Sigma^T\Sigma$ where Σ has the singular values of a A, a m x n matrix.

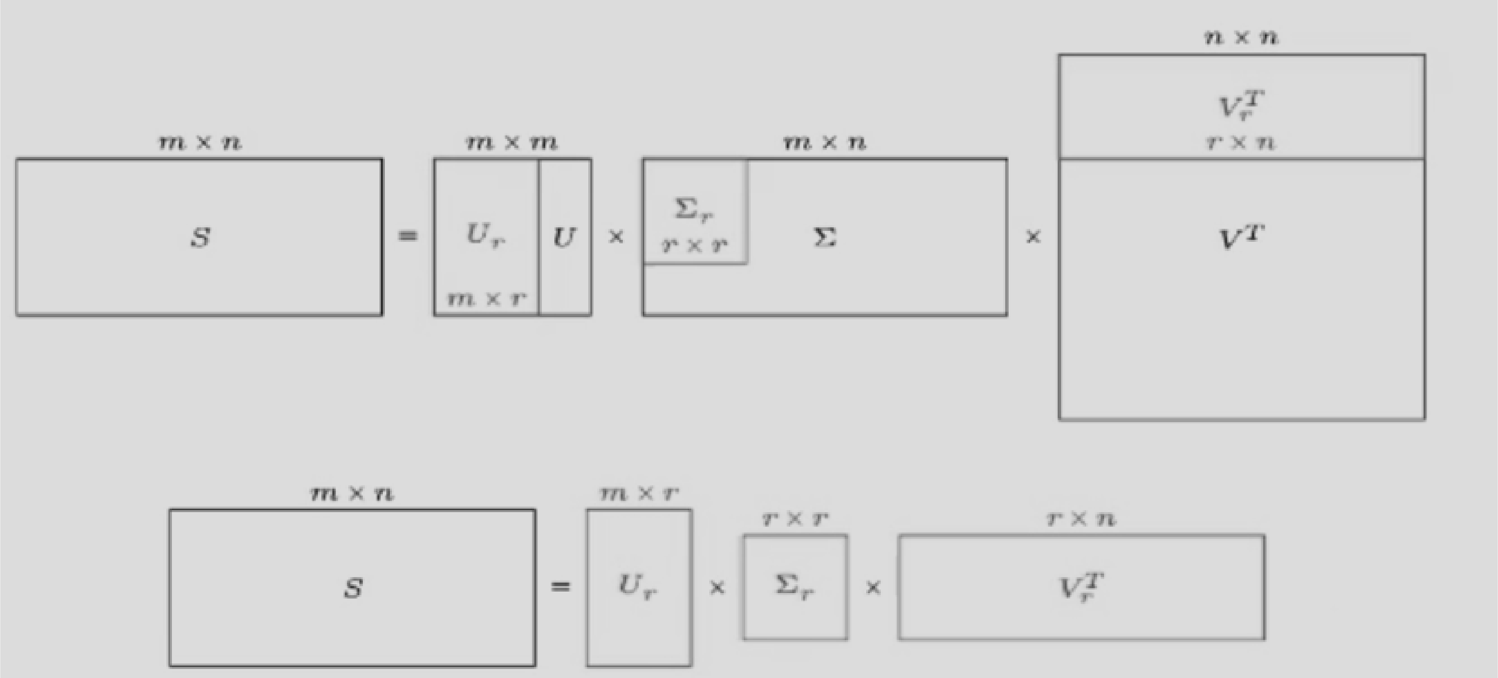
  - **m > n:**

$$\Sigma^T\Sigma = \begin{pmatrix} \sigma_1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \sigma_n & & & 0 \end{pmatrix}_{n\times m} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n} = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}_{n\times n}$$

  - **n > m:**

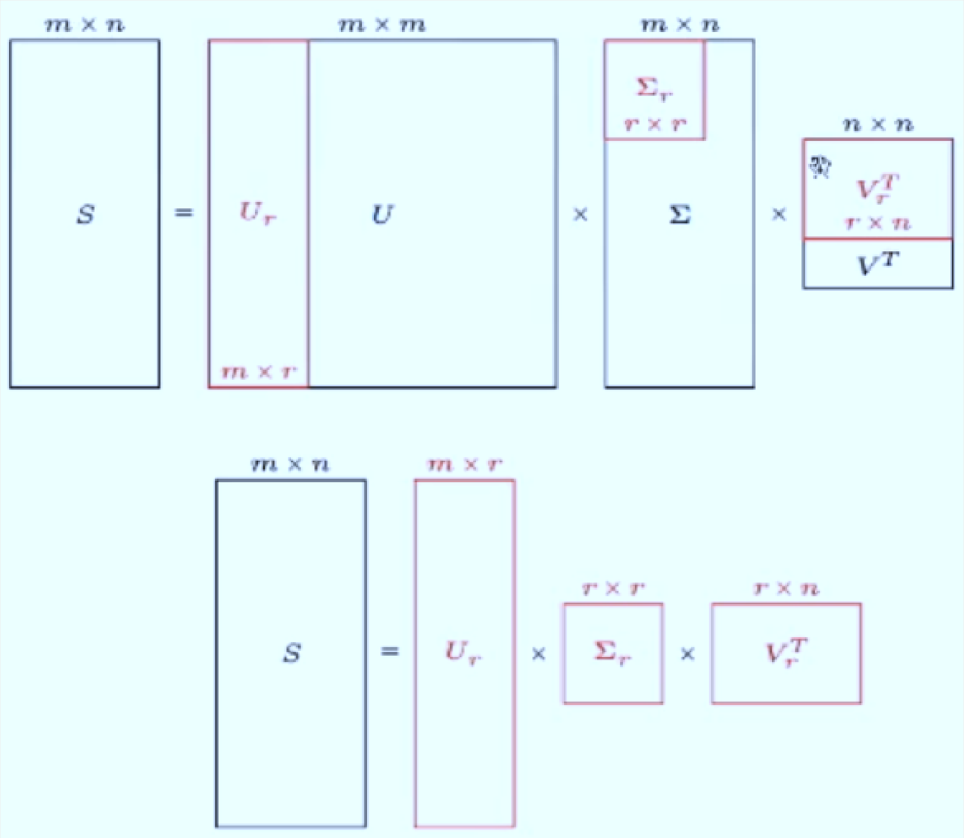
$$\Sigma^T\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times m} \begin{pmatrix} \sigma_1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \sigma_m & & & 0 \end{pmatrix}_{m\times n} = \begin{pmatrix} \sigma_1^2 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & \sigma_m^2 & & & 0 \\ 0 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 0 \end{pmatrix}_{n\times n}$$

- Wide Matrix


$m \times n$
S
$=$
$m \times m$
U_r
U
$m \times r$
$\times$
$m \times n$
Σ_r
$r \times r$
Σ
$\times$
$n \times n$
V_r^T
$r \times n$
V^T
$m \times n$
S
$=$
$m \times r$
U_r
$\times$
$r \times r$
Σ_r
$\times$
$r \times n$
V_r^T

Reduced SVD

- Tall Matrix



How can we compute an SVD of a matrix A?

- Assume A with singular value decomposition $A = U\Sigma V^T$. Let's take a look at the eigenpairs corresponding to $A^T A$:

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T)$$
$$(V^T)^T (\Sigma)^T U^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$$

Hence $A^T A = V\Sigma^2 V^T$

- Recall that columns of V are all linear independent (orthogonal matrix), then from diagonalization ($B = XDX^{-1}$), we get:
    - The columns of V are the eigenvectors of the matrix $A^T A$
    - The diagonal entries of Σ^2 are the eigenvalues of $A^T A$
- Let's call λ the eigenvalues of $A^T A$, then $\sigma_i^2 = \lambda_i$

How can we compute an SVD of a matrix A?

- In a similar way,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T$$

$$(U\Sigma V^T)(V^T)^T(\Sigma)^T U^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$$

Hence $AA^T = U\Sigma^2 U^T$

- Recall that columns of U are all linear independent (orthogonal matrix), then from diagonalization ($B = XDX^{-1}$), we get:
  - The columns of U are the eigenvectors of the matrix AA^T

How can we compute an SVD of a matrix A?

1. Evaluate the n eigenvectors v_i and eigenvalues λ_i of $A^T A$
2. Make a matrix V from the normalized vectors v_i. The columns are called "right singular vectors".

$$V = \begin{pmatrix} \vdots & \cdots & \vdots \\ v_1 & \cdots & v_n \\ \vdots & \cdots & \vdots \end{pmatrix}$$

3. Make a diagonal matrix from the square roots of the eigenvalues.

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \quad \sigma_i = \sqrt{\lambda_i} \quad \text{and } \sigma_1 \geq \sigma_2 \geq \cdots$$

4. Find $U: A = U\Sigma V^T \Rightarrow U\Sigma = AV \Rightarrow U = AV\Sigma^{-1}$. The columns are called "left singular values".

Example

$$S=\begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix} \rightarrow S^TS=\begin{bmatrix}9 & -9\\ -9 & 9\end{bmatrix}, rank(S)=1$$

$$\Delta(\lambda)=\lambda^2-18\lambda=0 \Rightarrow \sigma_1=\sqrt{18}, \sigma_2=0 \Rightarrow \boldsymbol{\Sigma}=\begin{bmatrix}\mathbf{3\sqrt{2}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$$

$$v_1=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}, v_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} \Rightarrow \boldsymbol{V}=\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{-1} & \mathbf{1}\end{bmatrix}$$

$$Sv_1=\begin{bmatrix}2/\sqrt{2}\\ -4/\sqrt{2}\\ 4/\sqrt{2}\end{bmatrix} \Rightarrow u_1=\frac{1}{\sigma_1}Sv_1=\frac{1}{3\sqrt{2}}\begin{bmatrix}2/\sqrt{2}\\ -4/\sqrt{2}\\ 4/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1/3\\ -2/3\\ 2/3\end{bmatrix}$$

$$u_2=\begin{bmatrix}2/3\\ 2/3\\ 1/3\end{bmatrix}, u_3=\begin{bmatrix}-2/3\\ 1/3\\ 2/3\end{bmatrix} \Rightarrow \boldsymbol{U}=\frac{\mathbf{1}}{\mathbf{3}}\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-2}\\ \mathbf{-2} & \mathbf{2} & \mathbf{1}\\ \mathbf{2} & \mathbf{1} & \mathbf{2}\end{bmatrix}$$

$$S=\begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ -2 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}3\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=U\Sigma V^T$$

Unitary Freedom of PSD Decompositions

Suppose $B, C \in \mathcal{M}_{m,n}(\mathbb{F})$. The following are equivalent:

a. There exists a unitary matrix $U \in \mathcal{M}_m(\mathbb{F})$ such that $C = UB$,

b. $B^*B = C^*C$,

c. $(B\mathbf{v}).(B\mathbf{w}) = (C\mathbf{v}).(C\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$, and

d. $\|B\mathbf{v}\| = \|C\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{F}^n$.

Example

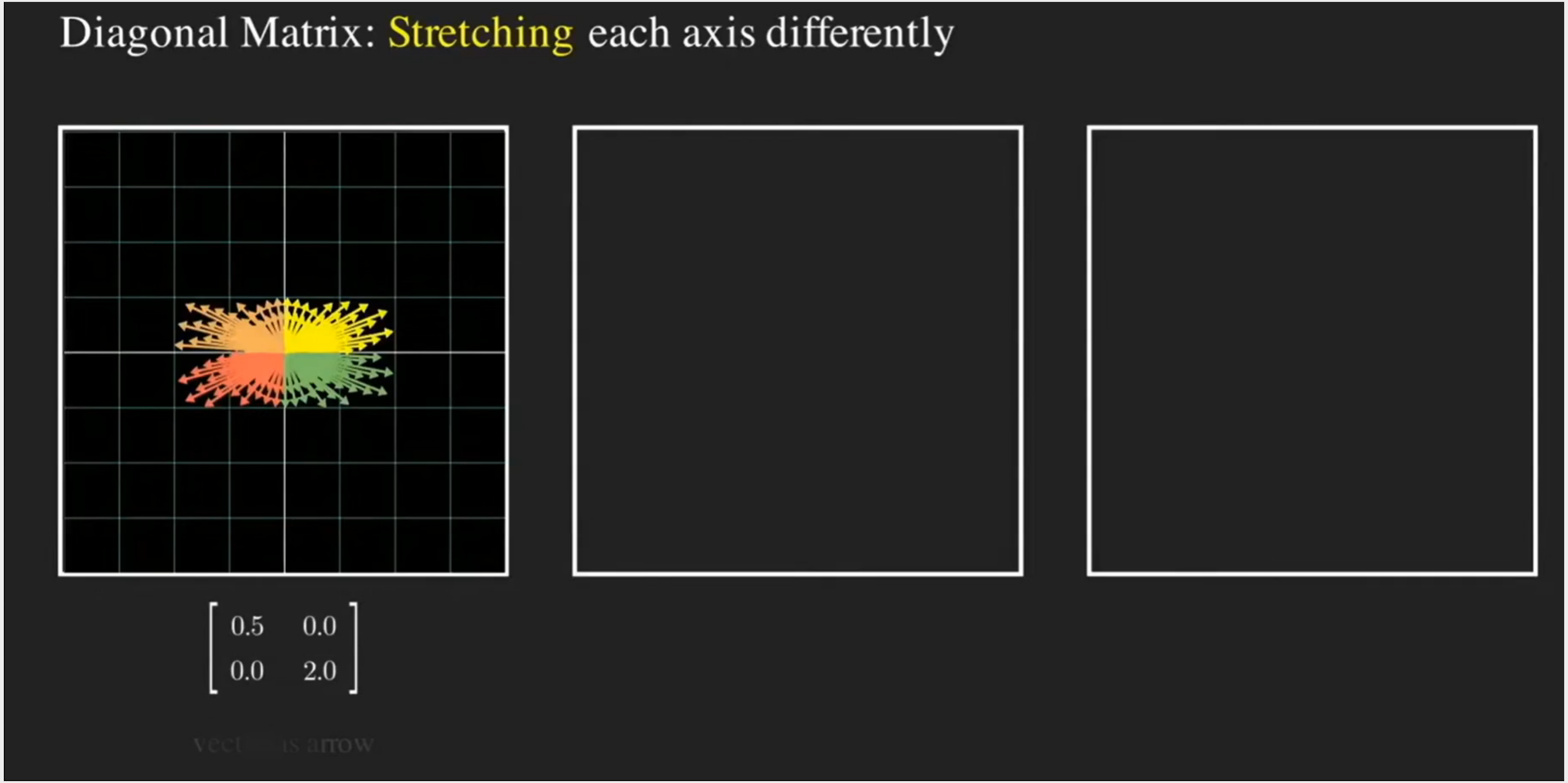
$$\begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

SVD Proof

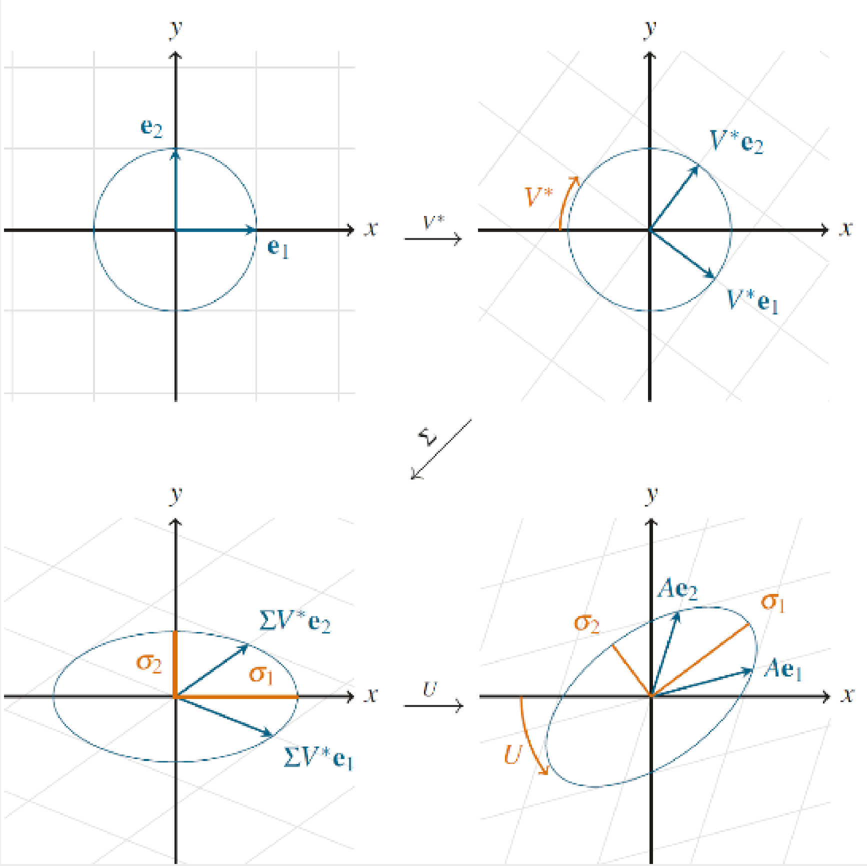
- If $m \neq n$ then A^*A, AA^* have different sizes, but they still have essentially the same eigenvalues—whichever one is larger just has some extra 0 eigenvalues.
- The same is actually true of AB and BA for any A and B.

- Proof SVD in another view!!


Diagonal Matrix: Stretching each axis differently
$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 2.0 \end{bmatrix}$

$$A = U\Sigma V^*$$

The product of a matrix's singular values equals the absolute value of its determinant



Determining the rank of a matrix

- Suppose A is a m x n rectangular matrix where m > n:

$$A = \begin{pmatrix} \vdots & \cdots & \vdots & \cdots & \vdots \\ u_1 & \cdots & u_n & \cdots & u_m \\ \vdots & \cdots & \vdots & \cdots & \vdots \end{pmatrix}_{m\times m} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} \cdots & v_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_n^T & \cdots \end{pmatrix}_{n\times n}$$

$$A = \begin{pmatrix} \vdots & \cdots & \vdots \\ u_1 & \cdots & u_n \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \cdots & \sigma_1 v_1^T & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \sigma_n v_n^T & \cdots \end{pmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

$A_1 = \sigma_1 u_1 v_1^T$ what is $\operatorname{rank}(A_1) = ?$

In general, $\operatorname{rank}(A_k) = k$

Conclusion

- Let $A \in \mathcal{M}_{m,n}$ be a matrix with rank(A) = r and the singular value decomposition $A = U\Sigma V^T$, where

$$U = [u_1 \mid u_2 \mid \ \dots \mid u_m] \text{ and } V = [v_1 \mid v_2 \mid \ \dots \mid v_n]$$

Then

a. $\{u_1, u_2, \dots, u_r\}$ is an orthonormal basis of range(A),

b. $\{u_{r+1}, u_{r+2}, \dots, u_m\}$ is an orthonormal basis of null(A^*),

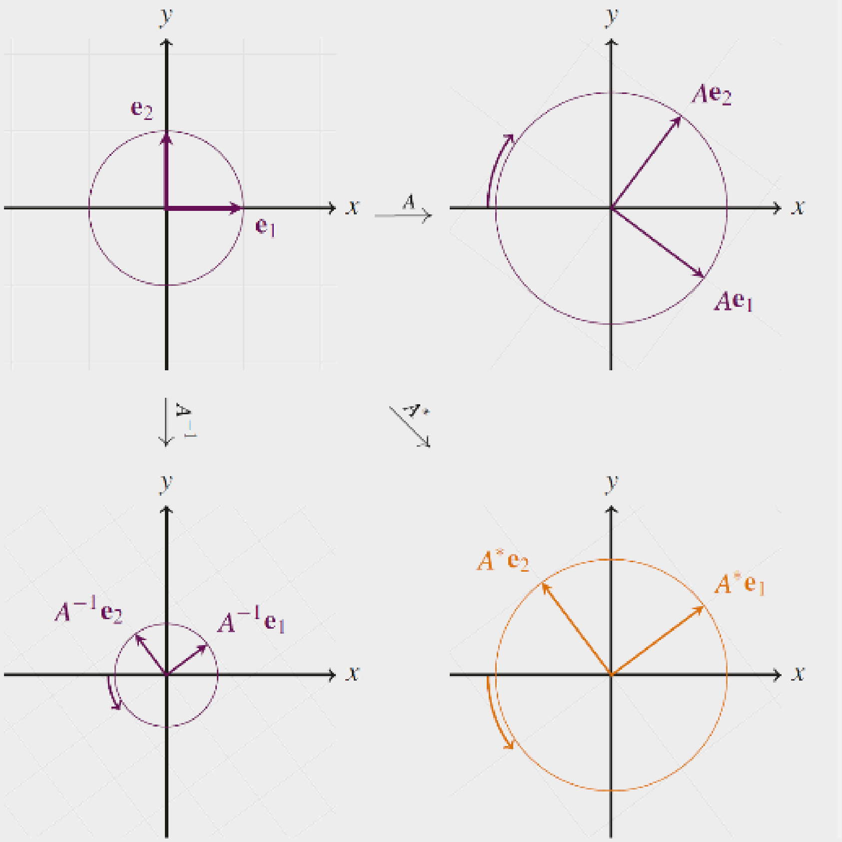
c. $\{v_1, v_2, \dots, v_r\}$ is an orthonormal basis of range(A^*), and

d. $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ is an orthonormal basis of null(A)

$$A = U\Sigma V^*$$

$$A^* = V\Sigma^* U^*$$

$$A^{-1} = V\Sigma^{-1} U^*$$



Applications

Orthogonal Rank-One Sum Decomposition

- Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $A \in \mathcal{M}_{m,n}(\mathbb{F})$ has $\operatorname{rank}(A) = r$. There exist orthonormal sets of vectors $\{u_j\}_{j=1}^{r} \subset \mathbb{F}^m$ and $\{v_j\}_{j=1}^{r} \subset \mathbb{F}^n$ such that

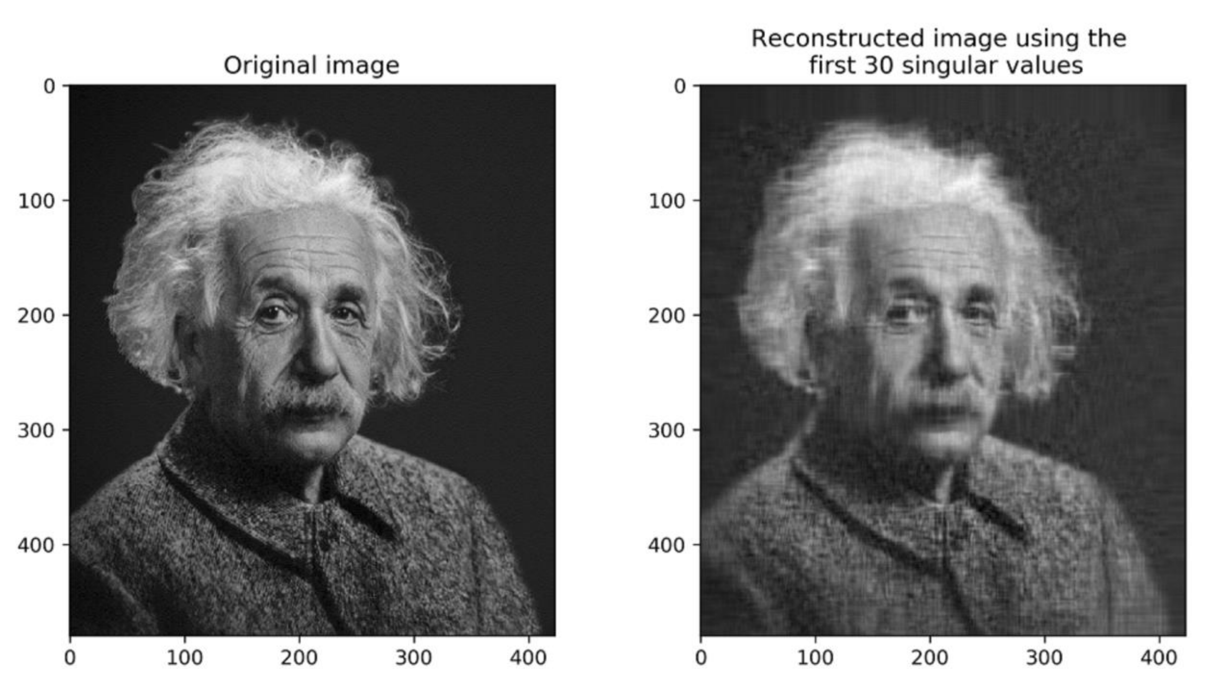
$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the non-zero singular values of A.

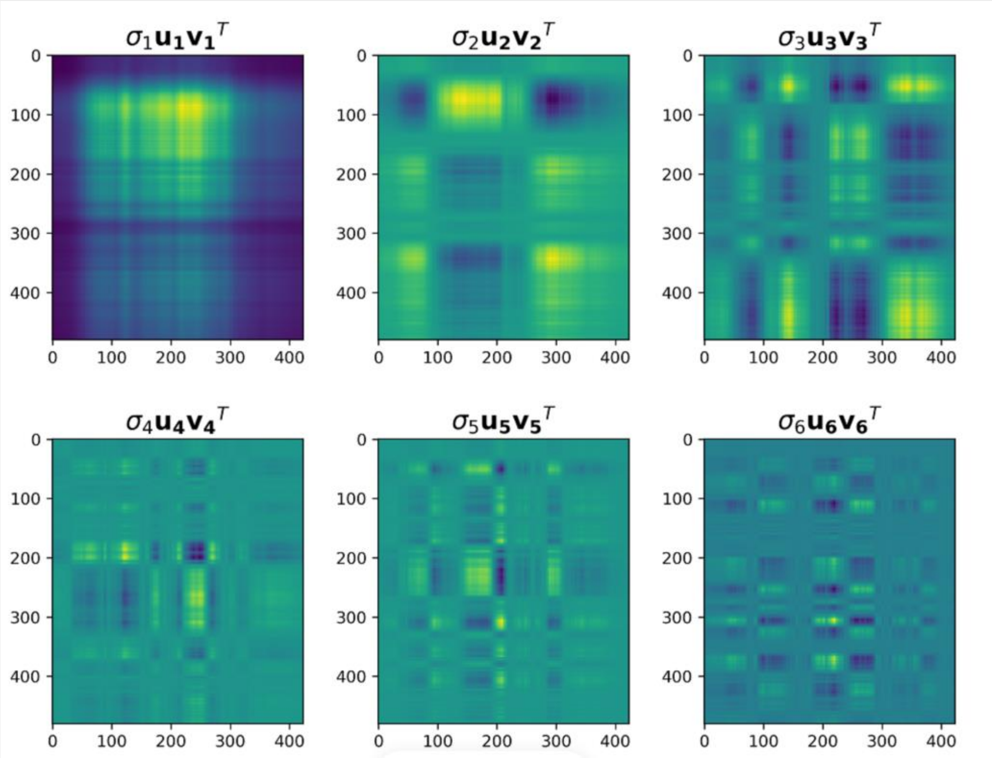
SVD and Matrix Similarity

- Suppose you want to find best rank−k approximation to **A**
  - Answer: set all but the largest k singular values to zero
- Can form compact representation by eliminating columns of **U** and **V** corresponding to zeroed Σ_i

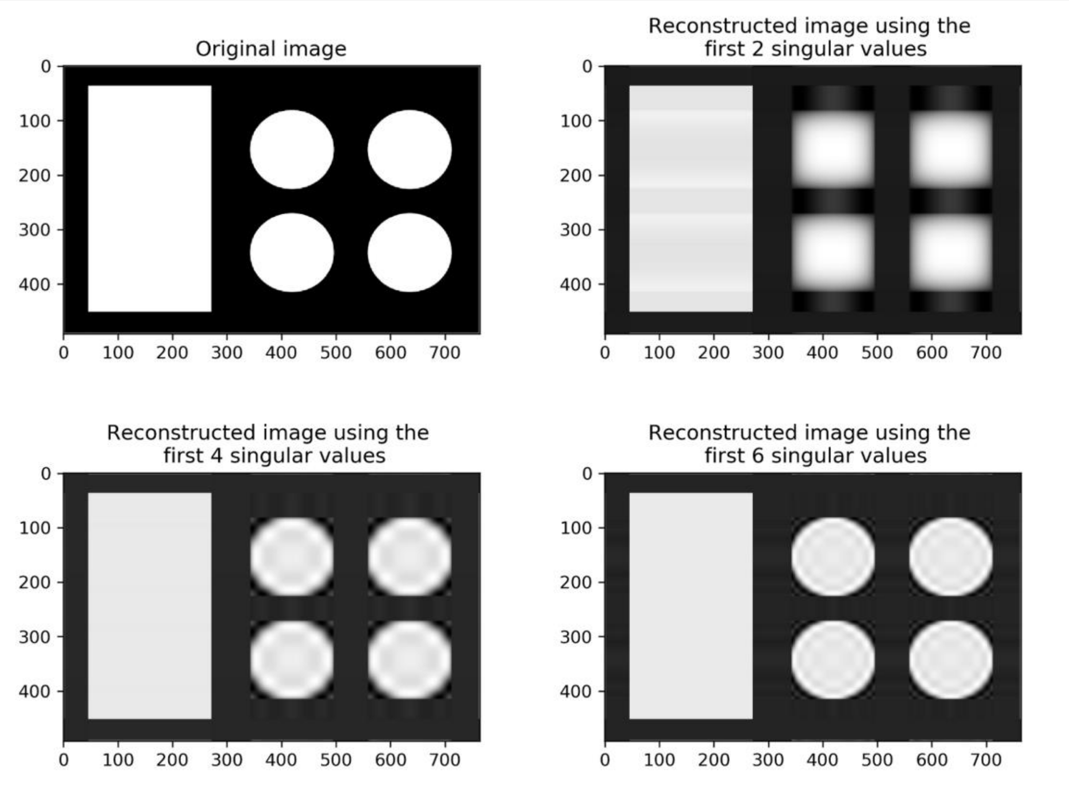
Application: Dimensionality Reduction



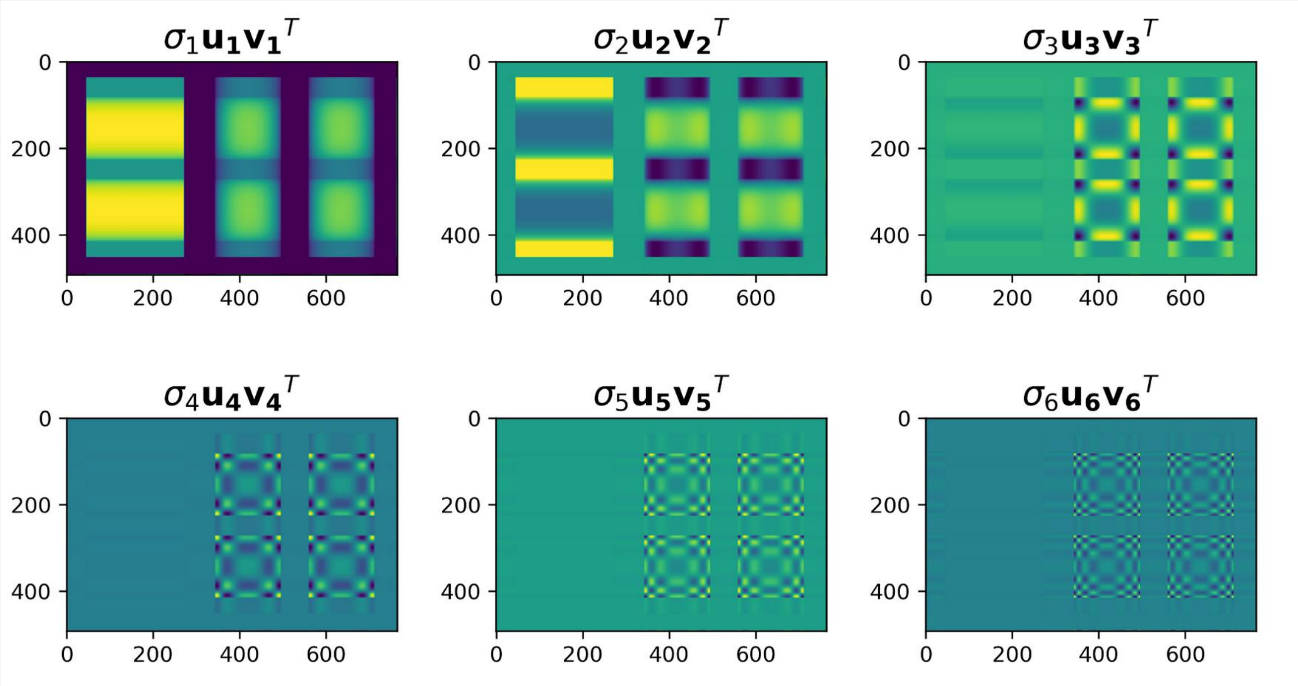
Application: Dimensionality Reduction



Application: Dimensionality Reduction



Application: Dimensionality Reduction



Image

SVD and Inverses

- $A^{-1} = V\Sigma^{-1}U^T$
- This fails when some Σ_i are 0
  - It's *supposed* to fail – singular matrix
- Pseudoinverse: if $\Sigma_i = 0$, set $\frac{1}{\Sigma_i}$ to 0 (!)
  - "Closest" matrix to inverse
  - Defined for all (even non-square, singular, etc.) matrices
  - Equal to $(A^T A)^{-1} A^T$ if $A^T A$ invertible

Pseudo Inverse

- Problem:
  if A is rank−deficient, Σ is not invertible.
- How to fix it:
  Define the Pseudo Inverse
- Pseudo Inverse of a diagonal matrix:

$$(\Sigma^+)_i = \begin{cases} \dfrac{1}{\sigma_i}, & if\ \sigma_i \neq 0 \\ 0, & if\, \sigma_i = 0 \end{cases}$$

- Pseudo Inverse of a matrix A:

$$A^+ = V\Sigma^+U^T$$

Pseudo Inverse

- If a matrix A has the singular value decomposition
$$A = UWV^T$$

then the pseudo-inverse or Moore-Penrose inverse of A is
$$A^+ = V^T W^{-1} U$$

- If A is 'tall' (m > n) and has full rank
$$A^+ = (A^T A)^{-1} A^T$$
(it gives the least-squares solution $x_{lsq} = A^+ b$)
- If A is 'short' (n > m) and has full rank
$$A^+ = A^T (AA^T)^{-1}$$
(it gives the least-norm solution $x_{l-n} = A^+ b$)

- In general, $x_{pinv} = A^+ b$ is the minimum-norm, least-square solution.

SVD and Matrix Similarity

- One common definition for the norm of a matrix is the Frobenius norm:

$$\|A\|_F = \sum_i \sum_j a_{ij}^2$$

- Frobenius norm can be computed from SVD

$$\|A\|_{\mathrm{F}} = \sum_i \Sigma_i{}^2$$

- So changes to a matrix can be evaluated by looking at changes to singular values