Inner Product Space

Linear Algebra

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Overview

- Linear Form
- Bilinear Form
- Inner Product

Linear Form

What are Linear Functions?

- $f: R^n \to R$ means that f is a function that maps real $n-$vectors to real numbers
- $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- $f(x) = (x_1, x_2, \ldots, x_n)$: argument

Definition

A function $f: R^n \to R$ is linear if it satisfies the following two properties:

- Additivity: For any $n-$vector x and y, $f(x + y) = f(x) + f(y)$
- Homogeneity: For any $n-$vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$

Superposition property:

Definition

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Note

- A function that satisfies the superposition property is called linear

Homogeneity and Additivity

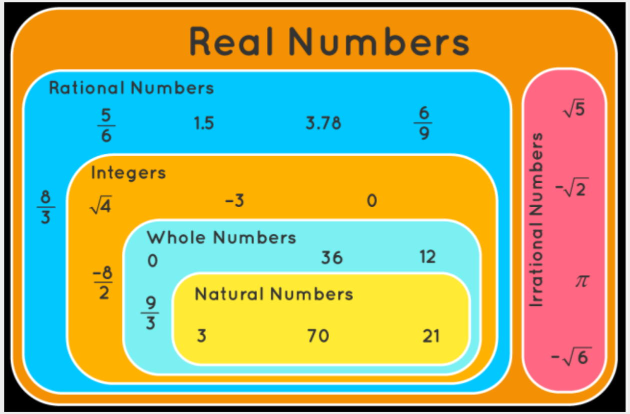
Definition

- Additivity:

For any n−vector x and y, $f(x+y) = f(x) + f(y)$

- Homogeneity:

For any n−vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$



Counterexample:
$f\left(a+\sqrt{5}\,b\right) \rightarrow a+b+\sqrt{5}\,b$

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

Inner product is Linear Function?

Theorem

A function defined as the inner product of its argument with some fixed vector is linear.

Proof?

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

What are Linear Functions?

Theorem

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof?

What are Linear Functions?

Theorem

The representation of a linear function f as $f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.

Proof?

Linear Form Examples

Example

- Is average a linear function?
- Is maximum a linear function?

Bilinear Form

Definition

Suppose V and W are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \rightarrow \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and
   ii. $f(c\mathbf{v_1}, \mathbf{w}) = cf(\mathbf{v_1}, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is linear in its second argument:
   i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and
   ii. $f(\mathbf{v}, c\mathbf{w_1}) = cf(\mathbf{v}, \mathbf{w_1})$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Note

Let V be a vector space over a field $\mathbb{F}$. Then the **dual** of V, denoted by V^*, is the vector space consisting of all linear forms on V.

Example

Let V be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \rightarrow \mathbb{F}$ defined by

$$g(f, \mathbf{v}) = f(\mathbf{v}) \text{ for all } f \in V^*, \mathbf{v} \in V$$

is a bilinear form.

Positive Definite Bilinear Form

Definition 3.5 – Positive definite

A bilinear form $\langle\,,\rangle$ on a real vector space V is positive definite, if

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0 \quad \text{for all } \boldsymbol{v} \neq 0.$$

Example

- Bilinear form: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$

- Bilinear form: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$

Symmetric Bilinear Form

Definition 3.6 – Symmetric

A bilinear form $\langle\,,\rangle$ on a real vector space V is called symmetric, if

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in V.$$

Bilinear forms over a complex vector space

Bilinear forms on $\mathbb{R}^n$	**Bilinear forms on $\mathbb{C}^n$**
Linear in the first variable	Conjugate linear in the first variable
$\langle \boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}\rangle = \langle \boldsymbol{u},\boldsymbol{w}\rangle + \langle \boldsymbol{v},\boldsymbol{w}\rangle$	$\langle \boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}\rangle = \langle \boldsymbol{u},\boldsymbol{w}\rangle + \langle \boldsymbol{v},\boldsymbol{w}\rangle$
$\langle \lambda \boldsymbol{u},\boldsymbol{v}\rangle = \lambda \langle \boldsymbol{u},\boldsymbol{v}\rangle$	$\langle \lambda \boldsymbol{u},\boldsymbol{v}\rangle = \overline{\lambda} \langle \boldsymbol{u},\boldsymbol{v}\rangle$
Linear in the second variable	Linear in the second variable

Inner product

Inner products over real vector space

- An inner product is a positive-definite symmetric bilinear form.
- An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle v, v \rangle \geq 0$ for all $v \in V$.
2. $\langle v, v \rangle = 0$ if and only if $v = 0$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle w, v \rangle = \langle v, w \rangle$.

Definition (Inner Product)

- A function $\langle \cdot,\cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product if
  1. $\langle x, x \rangle \geq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0$ (positivity)
  2. $\langle x, y \rangle = \langle y, x \rangle$ (symmetry)
  3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (additivity)
  4. $\langle rx, y \rangle = r\langle x, y \rangle$ for all $r \in \mathbb{R}$ (homogeneity)

- Using properties (2) and (4) and again (2)

$$\langle x, ry \rangle = \langle ry, x \rangle = r\langle y, x \rangle = r\langle x, y \rangle$$

- Using properties (2), (3) and again (2)

$$\langle x, y + z \rangle = \langle y + z, x \rangle = \langle y, x \rangle + \langle z, x \rangle = \langle x, y \rangle + \langle x, z \rangle$$

Inner Products

Definition

- The standard inner product is:

$$\langle x, y \rangle = x^T y = \sum x_i y_i \,, \qquad x, y \in \mathbb{R}^n$$

- The standard inner product between matrices is:($X, Y \in \mathbb{R}^{m \times n}$)

$$\langle X, Y \rangle = Tr(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

Example

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Definition

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Then an **inner product** on V is a function $\langle \cdot,\cdot \rangle : V \times V \ \rightarrow \ \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \ \in V$:

a) $\langle \mathbf{v}, \mathbf{w} \rangle = \ \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

$b)$ $\langle \mathbf{v}+c\mathbf{x},\mathbf{w}\rangle=\langle \mathbf{v},\mathbf{w}\rangle+c\langle \mathbf{x},\mathbf{w}\rangle$ (linearity)

$c)$ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

Example

Show that the function $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \boldsymbol{v}\mathbf{w}^* = \sum_{i=1}^{n} v_i \overline{w_i} \quad \text{for all} \qquad \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$$

is an inner product on $\mathbb{C}^n$.

Example

Let $a < b$ be real numbers and let $C[a, b]$ be the vector space of continuous functions on the real interval $[a, b]$.
Show that the function $\langle\cdot,\cdot\rangle : C[a, b] \times C[a, b] \to \mathbb{R}$ defined by

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx \quad \text{for all} \quad f, g \in C[a, b]$$

is and inner product on $C[a, b]$.

Inner product on Polynomials

Example

Find $\langle p, q \rangle$ which $p(x) = 3 - x + 2x^2$ and $q(x) = 4x + x^2$ on [0,1].