

# Matrix Inverse

## Linear Algebra

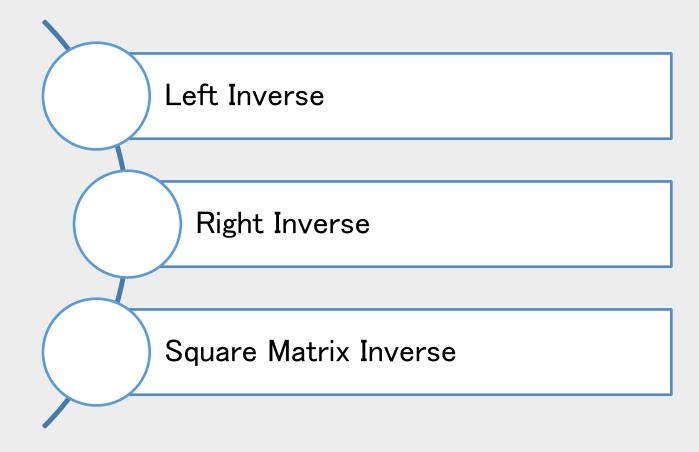
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## Overview





# Left Inverse

## Left Inverse



## Definition

- $\square$  A number x that satisfies xa = 1 is called the inverse of a
- $\square$  Inverse (i.e.,  $\frac{1}{a}$ ) exists if and only if  $a \neq 0$ , and is unique
- $\square$  A matrix X that satisfies XA = I is called a left inverse of A
- $\Box$  If a left inverse exists we say that A is left-invertible
- $\square$   $A: m \times n \Rightarrow I: n \times n \Rightarrow X: n \times m$

## Example

The matrix 
$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

Has two different left inverses:

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix},$$

$$C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

# Solving linear equations with a left inverse



## Method

- $\square$  Suppose Ax = b, and A has a left inverse C
- $\Box$  Then Cb = C(Ax) = (CA)x = Ix = x
- ☐ So multiplying the right-hand side by a left inverse yields the solution

## Left inverse of vector



## Note

- ☐ A non-zero column vector always has a left inverse.
- ☐ Left inverse is not unique.

## Example

- $\square \ A = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$
- Matrix with orthonormal columns

## Definition

☐ Row vector does not have left inverse

$$A = [1 \ 0 \ 3]$$

# Left inverse and column independence



### Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof

## Left inverse and column independence



#### **Definition**

- $\square$  If A has a left inverse C then the columns of A are linearly independent
- ☐ We'll see later that the converse is also true, so:

A matrix is left-invertible if and only if its columns are linearly independent

☐ Matrix generalization of

A number is invertible if and only if it is nonzero

#### From Previous Theorem

Left-invertible matrices are all tall or square

- ☐ Wide matrix is not always left invertible
- ☐ Tall or square matrices can be left invertible

## Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

# Right Inverse

## Right inverses



#### **Definition**

- $\square$  A matrix X that satisfies AX = I is a right inverse of A
- ☐ If a right inverse exists we say that A is right-invertible
- $\square A$  is right-invertible if and only if  $A^T$  is left-invertible:

$$AX = I \Longrightarrow (AX)^T = I \Longrightarrow X^T A^T = I$$

□ so we conclude:

A is right invertible if and only if its rows are linearly independent

☐ Right-invertible matrices are wide or square

# Solving linear equations with a right inverse



#### Method

- $\square$  Suppose A has a right inverse B
- $\Box$  Consider the (square or underdetermined) equations of Ax = b
- $\square x = Bb$  is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

 $\Box$  So Ax = b has a solution for any b

## Example

- $\square$  Same A, B, C in last example.
- $\square$   $C^T$  and  $B^T$  are both right inverses of  $A^T$
- $\Box$  Under-determined equations  $A^Tx=(1,2)$  has (different) solutions.

$$B^{T}(1,2) = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}), \quad C^{T}(1,2) = (0, \frac{1}{2}, -1)$$

there are many other solutions as well

# Conclusion: Left and Right Inverse

## Linear equations and matrix inverse



#### Definition

**Left-Invertible matrix**: if *X* is a left inverse of *A*, then

$$Ax = b \Longrightarrow x = XAx = Xb$$

There is at most one solution using X (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

## Why "at most"??

$$XA = I$$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases} \qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1\\ 0 & -1\\ 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & | & -4 \\ 0 & -1 & | & 3 \\ 2 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & 1 \end{bmatrix}$$

## Linear equations and matrix inverse



#### Note

- $\square$  If the system of equations Ax = b is consistent, and if a matrix B exists such that BA = I, then the system of equations has a unique solution, namely x = Bb.
- $\square$  Right-inversible matrix: if X is a right inverse of A, then there is at least one solution (x=Xb):

$$x = Xb \implies Ax = AXb = b$$

- To pursue these ides further, suppose that again we want to solve a system of linear equations, Ax = b. Assume now that we have another matrix, B, such that AB = I. Then we can write A(Bb) = (AB)b = Ib = b; hence Bb solves the equations Ax = b. This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether Bb is the only solution. There may be others.
- ☐ Invertible matrix: if A is invertible, then

$$Ax = b \iff x = A^{-1}b$$

There is a unique solution

## Conclusion



 $\Box$  System of linear equations Ax = b:

- $\circ$  A right inverse of A, say AB = I. Then Bb is a solution, as is verified by nothing A(Bb) = (AB)b = Ib = b.
- $_{\circ}$  A left inverse of A, say CA = I, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

# Square Matrix Inverse

## Inverse



#### **Definition**

For  $A \in M_{n \times n}$ , if there exists a matrix  $B \in M_{n \times n}$  such that  $AB = BA = I_n$ , then:

- ☐ A is invertible (or nonsingular)
- ☐ B is the inverse of A
- $\Box$  The inverse of A is denoted by  $B = A^{-1}$

A square matrix that does not have an inverse is called non-invertible (or singular)

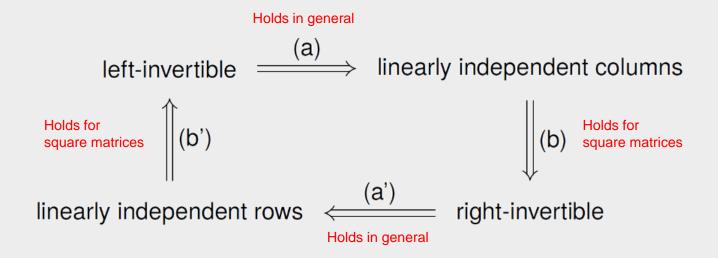
For a square matrix left and right inverse are the same. Rows and columns are linear independent.

## **Theorem**

The inverse of a matrix is unique

## Invertible Matrices





## Gauss-Jordan Elimination for finding the Inverse of a matrix



#### Method

- $\square$  Let A be a  $n \times n$  matrix:
  - $\square$  Adjoin the identity  $n \times n$  matrix  $I_n$  to A to form the matrix  $[A:I_n]$ .
  - $\square$  Compute the reduced echelon form of  $[A:I_n]$ .
- $\square$  If the reduced echelon form is of the type  $[I_n:B]$ , then B is the inverse of A.
- $\square$  If the reduced echelon form is not the type  $[I_n:B]$ , in that the first  $n\times n$  submatrix is not  $I_n$  then A has no inverse.

 $[A \mid I]$  Gauss—Jordan elimination  $[I \mid A^{-1}]$ 

### **I**mportant

An  $n \times n$  matrix is invertible if and only if its reduced echelon form is  $I_n$ .

# Inverse (Example)



### Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \implies \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases}$$
(1)  
$$x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1$$
(2)

This two system of linear equations have the same coefficient matrix, which is exactly the matrix  $\boldsymbol{A}$ 

# Inverse (Example)



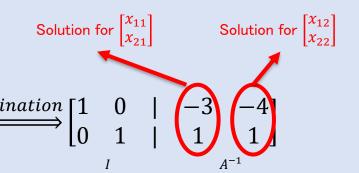
### Rest of The Example

$$(1) \Rightarrow \begin{bmatrix} 1 & 4 & | & 1 \\ -1 & -3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \begin{bmatrix} 1 & 4 & | & 0 \\ -1 & -3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -4 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus 
$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ -1 & -3 & | & 0 & 1 \end{bmatrix} \xrightarrow{Guass-Jordan\ elimination} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Using Gauss-Jordan Elimination on the

## Inverse



#### **Definition**

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):

- $\square$   $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
- $\square$   $A^k$  is invertible and  $(A^k)^{-1} = A^{-k} = (A^{-1})^k$
- $\Box$  cA is invertible if  $c \neq 0$  and  $(cA)^{-1} = \frac{1}{c}A^{-1}$  $\Box$   $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$

#### Theorem

If A and B are invertible matrices of order n, then AB is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$ 

$$(A_1 A_2 A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1} A_2^{-1} A_1^{-1}$$

## Inverse



## Theorem

Let AX = B be a system of n linear equations in n variable. If  $A^{-1}$  exists, the solution is unique and is given by  $X = A^{-1}B$ 

## Invertible Matrix



#### **Definition**

Let 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. If  $ad - bc \neq 0$ , then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If ad - bc = 0, then A is not invertible

### Note

Let 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. det  $A = ad - bc$ .

 $2 \times 2$  matrix A is invertible if and only if  $\det A \neq 0$ .

## Elementary Matrices



#### Definition

Each Elementary Matrix is E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

### Example

Find the inverse of 
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

# Solving square systems of linear equations



### Method

- $\square$  Suppose *A* is invertible
- $\Box$  For any b, Ax = b has the unique solution

$$x = A^{-1}b$$

- ☐ Matrix generalization of simple scalar equation ax = b having solution  $x = \left(\frac{1}{a}\right)b$  (for  $a \neq 0$ )
- $\square$  Simple-looking formula  $x = A^{-1}b$  is basis for many applications

# Invertible (Nonsingular) matrices



#### Conclusion

The following are equivalent for a square matrix A:

- ☐ A is invertible
- $\square$  Columns of A are linearly independent
- $\square$  Rows of A are linearly independent
- $\square A$  has a left inverse
- $\square A$  has a right inverse

$$row \, rank(A) = col \, rank(A) = n$$

If any of these hold, all others do

## Invertible matrices



### Examples

- $\Box I^{-1} = I$
- $\square$  If Q is orthogonal, i.e., square with  $Q^TQ = I$ , then  $Q^{-1} = Q^T$
- $\square$  2 × 2 matrix A is invertible if and only if  $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

- You need to know this formula
- There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)
- ☐ Consider matrix  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$ 
  - > A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

- $\triangleright$  Verified by checking  $AA^{-1} = I$  (or  $A^{-1}A = I$ )
- > We'll soon see how to compute the inverse

## **Properties**



## **Properties**

- $\Box (AB)^{-1} = B^{-1}A^{-1}$
- ☐ If A is nonsingular, then  $A^T$  is nonsingular  $(A^T)^{-1} = (A^{-1})^T$  (sometimes denoted  $A^{-T}$ )
- $\square$  Negative matrix powers:  $(A^{-1})^k$  is denoted by  $A^{-k}$
- $\square$  With  $A^0 = I$ , Identity  $A^k A^l = A^{k+l}$  holds for any integers k, l

# Triangular matrices



Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??

# Question?



Why Matrix of Change of Basis is invertible?