Determinant

CE282: Linear Algebra

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Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$$

is called a **multilinear form** if, for each $1 \le j \le p$ and each $\mathrm{v}_1 \in \mathcal{V}_1, \mathrm{v}_2 \in \mathcal{V}_2, \dots, \mathrm{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathrm{v}) = f\left(\mathrm{v}_1, \dots, \mathrm{v}_{j-1}, \mathrm{v}, \mathrm{v}_{j+1}, \dots, \mathrm{v}_p\right) \qquad for\ all \ \ \mathrm{v} \in \mathcal{V}_j$$

is a linear form.

Definition

The determinant of a 2×2 matrix $A = [a_{ij}]$ is the number:

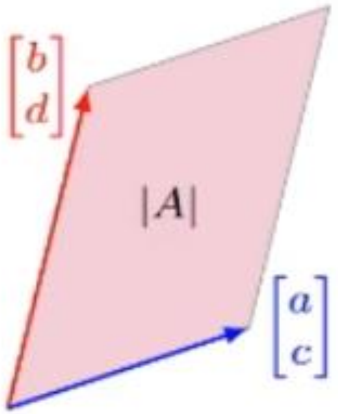
$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

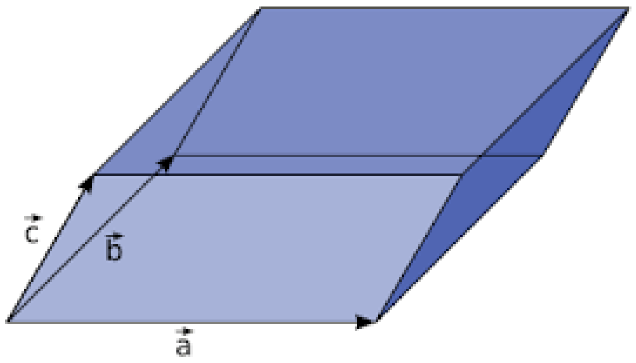
- A 2×2 matrix is invertible if and only if its determinant is nonzero.

- The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

Determinants as Area or Volume

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is det(A)
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is det(A)





Definition

Determinant is:

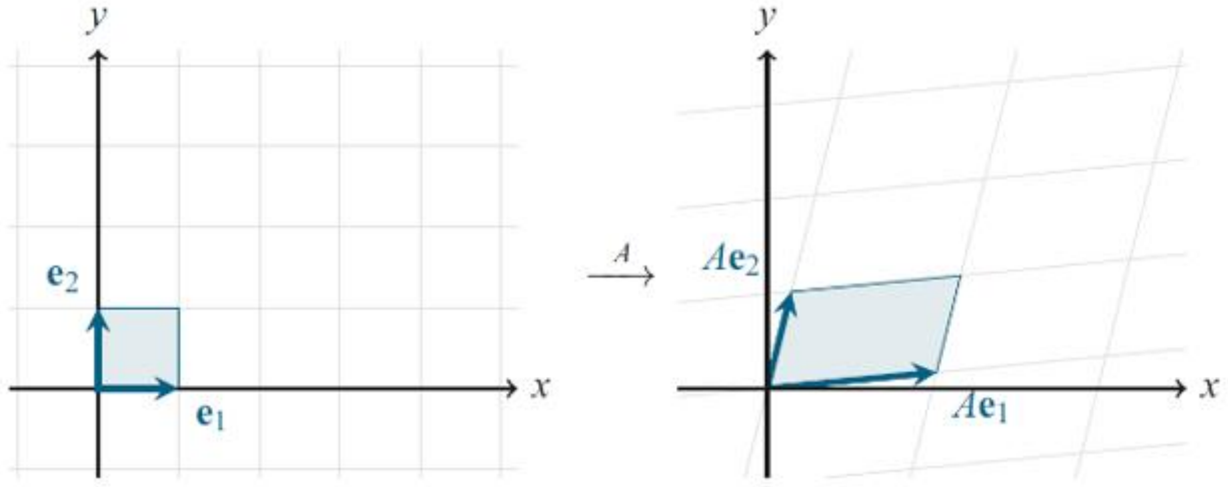
$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

$$\det(A) = |A|$$

- **multilinear function**
- **n – alternating**
- $\boldsymbol{\varphi(e_1, \dots, e_n) = 1}$

❑ The volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of standard unit parallelepiped is one.

$$\frac{\textit{volume of output region}}{\textit{volume of input region}}$$



A 2×2 matrix A stretches the unit square (with sides e_1 and e_2) into a parallelogram with sides Ae_1 and Ae_2 (the columns of A). The determinant of A is the area of this parallelogram.

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

we have

$$\begin{aligned}\phi\left(a_1, \ldots, a_n\right) &= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi\left(e_{j_1}, \ldots, e_{j_n}\right) \\ &= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) \right)\end{aligned}$$

n-alternating multilinear map

$$\phi(a_1, \ldots, a_n) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n})$$

$$= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \right)$$

$$= \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \operatorname{sgn}(\sigma) \phi(e_1, \ldots, e_n)$$

$$= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \right) \phi(e_1, \ldots, e_n)$$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Permutation and Transpositions

Definition

A permutation is even if it can be written as a product of an even number of transpositions, and odd if it can be written as an odd number of transpositions.

- Ref: https://www.ucl.ac.uk/~ucahmto/0007_2021/1-3-permutations.html

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Example

- 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = ?$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

Definition of Submatrix A_{ij}

Definition

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A

For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

A_{12} is

$$A_{12} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$

Recursive Definition of Determinant

Definition

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det(A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned}\det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(A_{1n}) \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det\left(A_{1j}\right)\end{aligned}$$

Recursive Definition of Determinant

- 2×2 matrix $\qquad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$= a \begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b \begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

Example

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

- 3×3 matrix $\quad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \quad |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

Recursive Definition of Determinant

Example

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$

Cofactor

Definition

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then

$$\det(A) \; = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Which is a cofactor expansion across the first row of A.

Cofactor Expansion

Important

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactor is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

Properties

- (1) If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

- Determinant of zero matrix is…

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

Properties

- (2) If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- (4) $\det(I) = 1$

(5) Row and Column Operations

- If a multiple of one row/column of A is added to another row/column to produce a matrix B, then $\det(A) = \det(B)$.

Proof?

Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

- (6) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is…

- U is unitary, so that $|\det(U)|=I$

Properties

- (7) If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$$

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = ka_{11}C_{11} + \cdots + ka_{1n}C_{1n} = k\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n|A_{n\times n}|$

Properties

- (8) If a row/column is multiple of another row/column then determinant is …..

Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero

- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

Note

Row operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$

b. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$

c. If one row of A is multiplied by k to produce B, then $\det(B) = k.\det(A)$

Echelon form

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Determinant of Transpose

Theorem

if A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$

Multiplicative Property

Theorem

if A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

Important

In general, $\det(A + B) \neq \det(A) + \det(B)$

- The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \Rightarrow \left|AA^{-1}\right| = |I| = 1 \Rightarrow |A|\left|A^{-1}\right| = 1 \Rightarrow \left|A^{-1}\right| = |A|^{-1}$$

- The determinant of orthogonal matrix is …

Determinant via QR Decomposition

Note

If $A \in \mathcal{M}_n$ has QR decomposition $A = UT$ with $U \in \mathcal{M}_n$ unitary and $T \in \mathcal{M}_n$ upper triangular, then

$$|\det(A)| = t_{1,1}.\, t_{2,2} \ldots t_{n,n}.$$

Example

$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ has QR decomposition $A = UT$ with $U = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Cramer's Rule

- $Ax = B$ and A is invertible $\qquad A = [a_1 \quad \dots \quad a_n] \qquad I = [e_1 \quad \dots \quad e_n]$

$$AI = A \implies A[e_1 \quad \dots \quad e_n] = [Ae_1 \quad \dots \quad Ae_n] = [a_1 \quad \dots \quad a_n]$$

$$\overbrace{[e_1 \quad e_2 \quad \dots \quad x \quad \cdots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \dots \quad Ax \quad \cdots \quad Ae_n]$$

$$= \underbrace{[a_1 \quad a_2 \quad \dots \quad b \quad \cdots \quad a_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

Note

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}, \qquad i = 1,2,\dots,n$$

Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \implies x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j-th column of A^{-1} is a vector x that satisfies $\quad Ax = e_j$

By Cramer's rule $\quad \{(\mathrm{i,j}) - \text{entry of } A^{-1}\} = x_i = \dfrac{|A_i(e_j)|}{|A|}$

$$|A_i(e_j)| = (-1)^{i+j}|A_{ji}|$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the adjugate (or classical adjoint) of A, denoted by adj A.

$$\begin{cases} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{cases} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A Formula for A^{-1}

Important

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

Transformations

Example

Show that the determinant, $det \colon \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ is not a linear transformation when $n \geq 2$

Note

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |\det A|.\{area\ of\ S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |\det A|.\{volume\ of\ S\}$$

Reference

- Chapter 3 Linear Algebra and Its Applications David C. Lay

- Nathaniel Johnston - Advanced Linear and Matrix Algebra-Springer (2021)