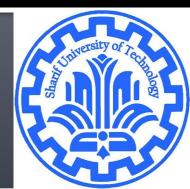
Matrix Inner Product and Norm

CE40282-1: Linear Algebra Hamid R. Rabiee and Maryam Ramezani Sharif University of Technology



Inner products

Definition 1 (Inner product). A function $\langle .,. \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an inner product if

1.
$$\langle x, x \rangle \ge 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0 \ (positivity)$$

2.
$$\langle x, y \rangle = \langle y, x \rangle$$
 (symmetry)

3.
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$
 (additivity)

4.
$$\langle rx, y \rangle = r \langle x, y \rangle$$
 for all $r \in \mathbb{R}$ (homogeneity)

 \blacksquare using properties (2) and (4) and again (2)

$$\langle x, ry \rangle = \langle ry, x \rangle = r \langle y, x \rangle = r \langle x, y \rangle$$

using properties (2), (3) and again (2).

$$\langle x, y + z \rangle = \langle y + z, x \rangle = \langle y, x \rangle + \langle z, x \rangle = \langle x, y \rangle + \langle x, z \rangle$$

Inner products

■ The standard inner product is

$$\langle x, y \rangle = x^T y = \sum x_i y_i, \ x, y \in \mathbb{R}^n.$$

The standard inner product between matrices is

$$\langle X, Y \rangle = \text{Tr}(X^T Y) = \sum_{i} \sum_{j} X_{ij} Y_{ij}$$

where $X, Y \in \mathbb{R}^{m \times n}$.

Example

$$p = 3 - x + 2x^2$$
 and $q = 4x + x^2$

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Norm

Definition (Norm). A function $f: \mathbb{R}^n \to \mathbb{R}$ is a norm if

1.
$$f(x) \ge 0$$
, $f(x) = 0 \Leftrightarrow x = 0$ (positivity)

2.
$$f(\alpha x) = |\alpha| f(x), \ \forall \alpha \in \mathbb{R} \ (homogeneity)$$

3.
$$f(x+y) \le f(x) + f(y)$$
 (triangle inequality)

"Entry-wise" matrix norms

$$\|A\|_{p,p} = \|\mathrm{vec}(A)\|_p = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p
ight)^{1/p}$$

- special case p=2 is the Frobenius norm, and $p=\infty$ yields the maximum norm. $\|A\|_E = \sqrt{\sum_{i=1}^n \sum_{j=1}^n (a_{ij})^2}$
 - Frobenius (Euclidean, Hilbert Schmidt) norm
 - invariant under rotations (unitary operations)

$$\|A\|_{ ext{F}} = \|AU\|_{ ext{F}} = \|UA\|_{ ext{F}} \ \sqrt{ ext{trace}(A^*A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)}.$$

$$\|A^*A\|_{\mathrm{F}} = \|AA^*\|_{\mathrm{F}} \le \|A\|_{\mathrm{F}}^2$$

The **max norm** is the elementwise norm with $p = q = \infty$:

$$\|A\|_{\max} = \max |a_{ij}|.$$

• sum-absolute-value norm: $|A||_{sav} = \sum_{i,j} |A_{i,j}|$

Singular value of A

Frobenius (Euclidean) norm

Let b_1, \ldots, b_n denote the columns of B. Then

$$||AB||_{HS}^2 = \sum_{i=1}^n ||Ab_i||^2 \le \sum_{i=1}^n ||A||^2 ||b_i||^2 = ||A||^2 ||B||_{HS}^2.$$

cauchy-Schwarz inequality

Matrix norms induced by vector norms

$$||A||_p = \max_{\vec{x} \neq \vec{0}} \frac{||A\vec{x}||_p}{||\vec{x}||_p} = \max_{||\vec{x}||_p = 1} ||A\vec{x}||_p$$

- Theorem: $||Ax|| \le ||A|| ||x||$ for all vectors ||x||
- Theorem: For all matrices $A, B, ||AB|| \le ||A|| ||B||$

Matrix norms induced by vector norms

 The norm of a matrix is a real number which is a measure of the magnitude of the matrix.

$$||A||_{1} = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

$$||A||_{\infty} = \max_{1 \le i \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

- spectral norm (I2) is the largest singular value (the square root of the largest eigenvalue of the matrix gram A
- Example

$$||A||_2 = \sqrt{\max\{\text{eigenvalue}(A^T A)\}} = \max\{\text{sing}(A)\}$$

$$B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

Norms Compare

The 2-norm (spectral norm) of a matrix is the greatest distortion of the unit circle/sphere/hyper-sphere. It corresponds to the largest singular value (or |eigenvalue| if the matrix is symmetric/hermitian).

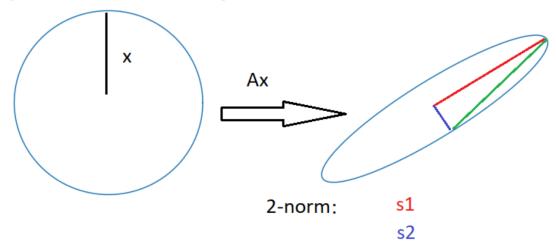
The Forbenius norm is the "diagonal" between all the singular values.

i.e.

$$||A||_2 = s_1 \;\; , \;\; ||A||_F = \sqrt{s_1^2 + s_2^2 + \ldots + s_r^2}$$

(r being the rank of A).

Here's a 2D version of it: x is any vector on the unit circle. Ax is the deformation of all those vectors. The length of the red line is the 2-norm (biggest singular value). And the length of the green line is the Forbenius norm (diagonal).



Forbenius norm: $sqrt(s1^2 + s2^2)$