Vector Space

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee   rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

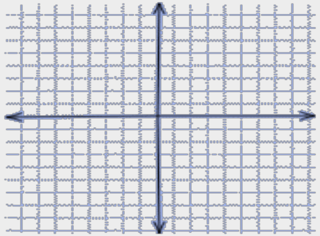
- Span – Linear Hull

Complex Number Review

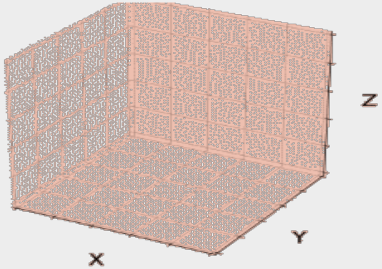
Definition

- A tuple is an ordered list of numbers.
- For example: $\begin{bmatrix} 1 \\ 2 \\ 32 \\ 10 \end{bmatrix}$ is a 4-tuple (a tuple with 4 elements).

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \ \ldots \right\}$$



$$\mathbb{R}^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \ \ldots \right\}$$



Review: Complex Numbers

Numbers:

- Real: Nearly any number you can think of is a Real Number!

1	12.38	−0.8625	3/4	√2	1998

- Imaginary: When squared give a negative result.

The “unit” imaginary number (like 1 for Real Numbers) is “i”, which is the square root of –1.

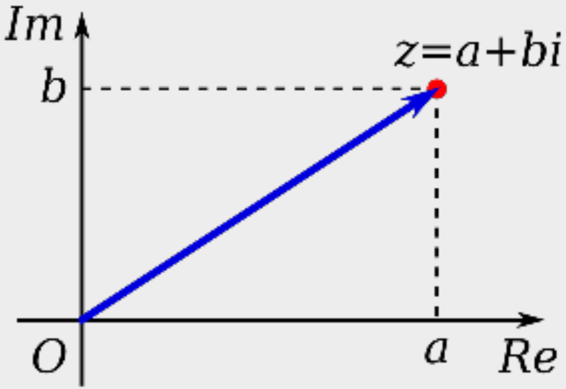
Examples of Imaginary Numbers:

3i	1.04i	−2.8i	3i/4	(√2)i	1998i

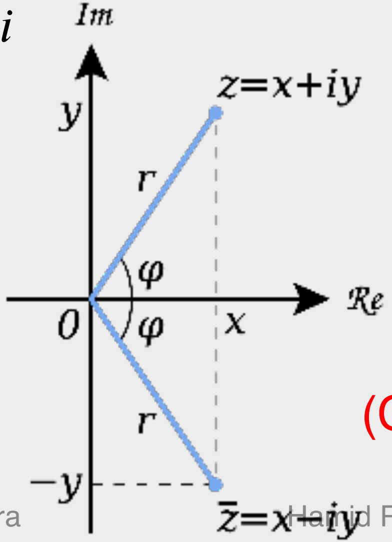
And we keep that little "i" there to remind us we need to multiply by √−1

Review: Complex Numbers

- $\mathbb{C}$ is a plane, where number $(a + bi)$ has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$
- Imaginary number: bi , $b \in R$



- Conjugate of $x + yi$ is noted by $x \,\overline{+}\, yi$:
  - $x - yi$

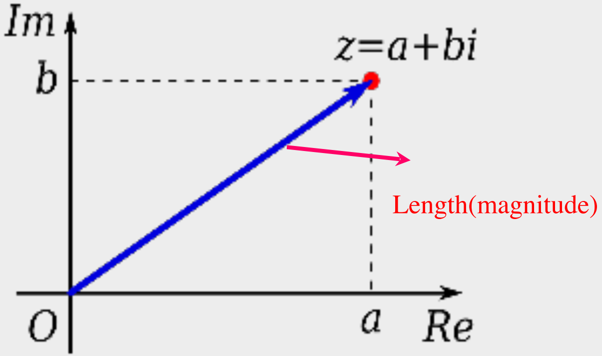


(Complex conjugate)

- Arithmetic with complex numbers $(a + bi)$:
  - $(a + bi) + (c + di)$
  - $(a + bi)(c + di)$
  - $\dfrac{a + bi}{c + di}$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

- Length (magnitude): $\left|\left|a + bi\right|\right|^2 = (a \overline{+} bi)(a + bi) = a^2 + b^2$
- Inner Product:
  - Real: $< x,\ y > \ = x_1y_1 + x_2y_2 + \ \dots + x_ny_n$
  - Complex: $< x,\ y > \ = \bar{x}_1y_1 + \bar{x}_2y_2 + \ \dots + \bar{x}_ny_n$



Extra resource:

If you want to learn more about complex numbers, this video is recommended!

Vector Operation

- Vector-Vector Addition
- Vector-Vector Subtraction
- Scalar-Vector Product
- Vector-Vector Products:
  - $\boldsymbol{x} \,.\, \boldsymbol{y}$ is called the inner product or dot product or scalar product of the vectors: $x^T y \ (y^T x)$
    - $< a, b >$ $\quad$ $< a | b >$ $\quad$ (a, b) $\quad$ $a \,.\, b$

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i .$$

  - Transpose of dot product:
    - $(a \,.\, b)^T = \left(a^T b\right)^T = \left(b^T a\right) = (b \,.\, a) = b^T a$
  - Length of vector

Dot Product Properties

- **Commutativity**
  - The order of the two vector arguments in the inner product does not matter.

$$a^T b = b^T a$$

- **Distributivity with vector addition**
  - The inner product can be distributed across vector addition.

$$(a+b)^T c = a^T c + b^T c$$

$$a^T (b+c) = a^T b + a^T c$$

- **Bilinear (linear in both a and b)**

$$a^T(\lambda b + \beta c) = \lambda a^T b + \beta a^T c$$

- **Positive Definite:**

$$(a \, . \, a) = a^T a \geq 0$$

  - 0 only if a itself is a zero vector $a = \mathbf{0}$

- **Associative**
  - Note: the associative law is that parentheses can be moved around, e.g., (x+y)+z = x+(y+z) and x(yz) = (xy)z

1) Associative property of the vector dot product with a scalar (scalar-vector multiplication embedded inside the dot product)

scalar

$$\gamma(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = (\gamma\mathbf{u}^{\mathrm{T}})\mathbf{v} = \mathbf{u}^{\mathrm{T}}(\gamma\mathbf{v}) = (\mathbf{u}^{\mathrm{T}}\mathbf{v})\gamma$$

$$= (\gamma \boldsymbol{u})^{T} \boldsymbol{v} = \gamma \boldsymbol{u}^{T} \boldsymbol{v}$$

Dot Product Properties

- Associative

2) Does vector dot product obey the associative property?

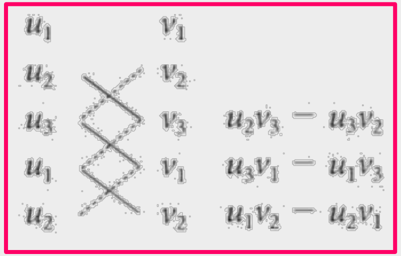
$$\underbrace{\mathbf{u}^{\mathrm{T}}(\mathbf{v}^{\mathrm{T}}\mathbf{w})}_{\substack{\text{vector-scalar product} \\ \text{row vector}}} = \underbrace{(\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{w}}_{\substack{\text{scalar-vector product} \\ \text{column vector}}}$$

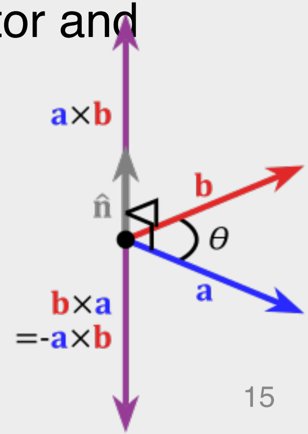
Cross product

- The cross product is defined only for two 3-element vectors, and the result is another 3-element vector. It is commonly indicated using a multiplication symbol (×).

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin(\theta_{ab}) \qquad \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

- It used often in geometry, for example to create a vector c that is orthogonal to the plane spanned by vectors a and b. It is also used in vector and multivariate calculus to compute surface integrals.





Vector Operations

- **Vector-Vector Products:**
  - Given two vectors $x \in R^m, y \in R^n$:
    - $x \otimes y = xy^T \in R^{m \times n}$ is called the outer product of the vectors: $\left(xy^T\right)_{ij} = x_i y_j$

$$xy^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

Example

- Represent $A \in R^{m \times n}$ with outer product of two vectors:

$$A = \begin{bmatrix} | & | & & | \\ x & x & \cdots & x \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m & \cdots & x_m \end{bmatrix}$$

- Properties:
  - $(u \otimes v)^T = (v \otimes \mathrm{u})$
  - $(v + w) \otimes u = v \otimes u + w \otimes u$
  - $u \otimes (v + w) = u \otimes v + u \otimes w$
  - $c(v \otimes u) = (cv) \otimes u = v \otimes (cu)$
  - $(u \, . \, v) = trace\left(u \bigotimes v\right) \quad (u, v \, \epsilon R^n)$
  - $(u \otimes v)w = (v \, . \, w)u$

- Vector-Vector Products:
  - Hadamard
  - Element-wise product

$$c = a \odot b = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{bmatrix}$$

- Hadamard product is used in image compression techniques such as JPEG. It is also known as Schur product

- Hadamard Product is used in LSTM (Long Short-Term Memory) cells of Recurrent Neural Networks (RNNs).

- Properties:
  - $a \odot b = b \odot a$
  - $a \odot \left(b \odot c\right) = \left(a \odot b\right) \odot c$
  - $a \odot (b + c) = a \odot b + a \odot c$
  - $(\theta a) \odot b = a \odot (\theta b) = \theta\left(a \odot b\right)$
  - $a \odot \mathbf{0} = \mathbf{0} \odot a = \mathbf{0}$

Binary Operation

Definition

- Any function from $A \times A \rightarrow A$ is a binary operation.

- Closure Law:
  - A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.

$$if\ a \in A,\ b \in A \rightarrow a * b \in A$$

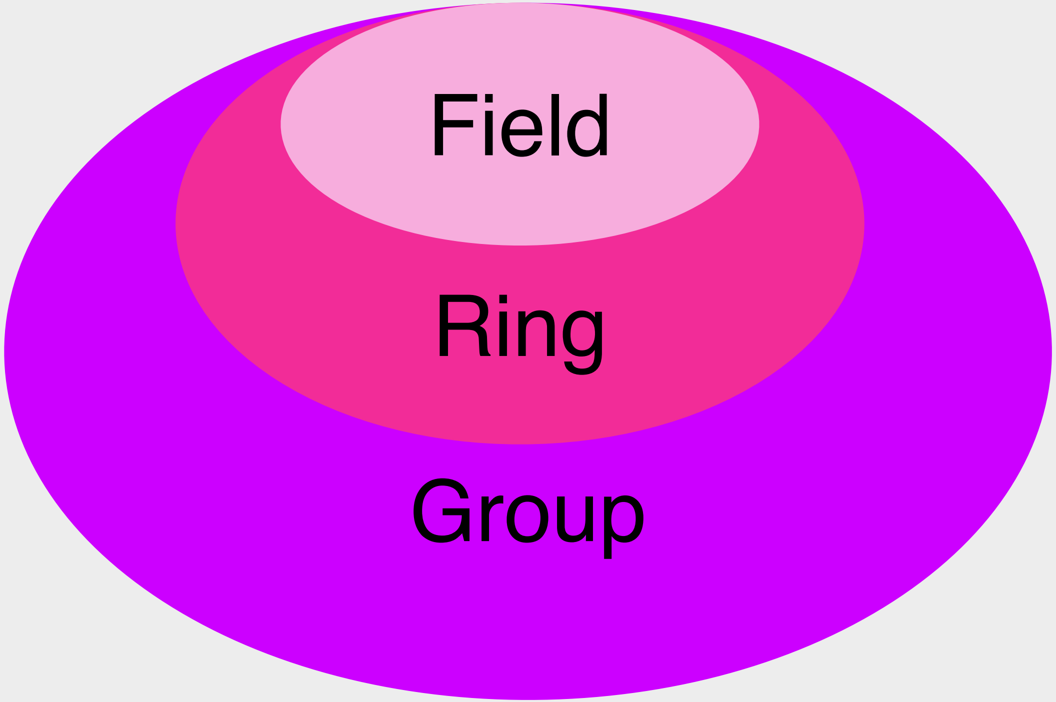
Binary Operations

Example

- Is “+” a binary operator on natural numbers?
- Is “x” a binary operator on natural numbers?
- Is “-” a binary operator on natural numbers?
- Is “/” a binary operator on natural numbers?

Group-Ring-Field

Group – Ring - Field



Definition

- A group G is a pair $(S, \circ)$, where S is a set and $\circ$ is a binary operation on S such that:
- $\circ$ is associative
- (Identity) There exists an element e $\in S$ such that:

$$e \circ a = a \circ e = a \qquad \forall a \in S$$

- (Inverses) For every $a \in S$ there is b $\in S$ such that:

$$a \circ b = b \circ a = e$$

If $\circ$ is commutative, then G is called a commutative group!

Ring

Definition

- A ring R is a set together with two binary operations + and *, satisfying the following properties:

1. (R,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. * is associative
3. The distributive laws hold in R: (Multiplication is distributive over addition)

$$(a+b)*c=(a*c)+(b*c)$$

$$a*(b+c)=(a*b)+(a*c)$$

Fields

Definition

- A field F is a set together with two binary operations + and *, satisfying the following properties:

1. (F,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. (F-{0},*) is a commutative group
3. The distributive law holds in F:

$$(a + b) * c = (a * c) + (b * c)$$

$$a * (b + c) = (a * b) + (a * c)$$

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

Example

(R; +, .) and (Q; +, .) serve as examples of fields.
(Z; +, .) is an example of a ring which is not a field!

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

$\forall\ \mathbf{a}, \mathbf{b}, \mathbf{c}\ \in \boldsymbol{F}$

Properties	Binary Operations	
	Addition (+)	Multiplication (.)
Closure(بسته بودن)		
Associative (شرکت‌پذیری)		
Commutative (جابه‌جایی‌پذیری)		
Existence of identity		
Existence of inverse: For each in F there must exist in		F
Multiplication is distributive over addition		

Example

Set $B = \{0, 1\}$ under following operations is a field?

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Fields

Example

Which are fields? (two binary operations + , *)

$\mathbb{R}$

$\mathbb{C}$

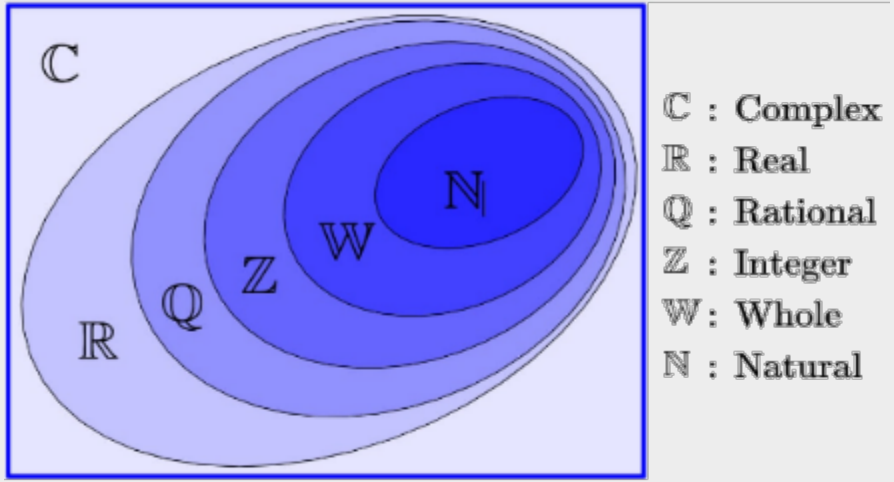
$\mathbb{Q}$

$\mathbb{Z}$

$\mathbb{W}$

$\mathbb{N}$

$\mathbb{R}^{2\times2}$



Vector Space

Vector Space

- Building blocks of linear algebra.

- A non-empty set V with field F (most of time R or C) forms a vector space with two operations:

  1. $+$: Binary operation on V which is $V \times V \rightarrow V$
  2. $.$: $F \times V \rightarrow V$

Note

In our course, by default, field is R (real numbers).

Vector Space

Definition

A vector space over a field F is the set V equipped with two operations: $(V, F, +, .)$

i. Vector addition: denoted by "+" adds two elements $x, \ y \in V$ to produce another element $x + y \in V$

ii. Scalar multiplication: denoted by "." multiplies a vector $x \in V$ with a scalar $\alpha \in F$ to produce another vector $\alpha . x \in V$. We usually omit the "." and simply write this vector as αx

- Addition of vector space $(x + y)$
  - Commutative $x + y = y + x \ \forall x, y \in V$
  - Associative $(x + y) + z = x + (y + z) \ \forall x, y, z \in V$
  - Additive identity $\exists \mathbf{0} \in V$ such that $x + \mathbf{0} = x, \ \forall x \in V$
  - Additive inverse $\exists(-x) \in V$ such that $x + (-x) = 0, \ \forall x \in V$

- Action of the scalars field on the vector space (αx)

  - Associative $\quad \alpha(\beta x) = (\alpha\beta)x \quad \forall \alpha, \beta \in F; \forall x \in V$

  - Distributive over ……

    scalar addition: $(\alpha + \beta)x = \alpha x + \beta x$

    $\forall \alpha, \beta \in F; \forall x \in V$

    vector addition: $\alpha(x + y) = \alpha x + \alpha y$

    $\forall \alpha \in F; \forall x, y \in V$

  - Scalar identity $\quad 1x = x \quad \forall x \in V$

Example

Let V be the set of all real numbers with the operations $u \oplus v = u - v$, $\oplus$ is an ordinary subtraction) and $c \boxdot u = cu$ ($\boxdot$ is an ordinary multiplication).
Is V a vector space? If it's not, which properties fail to hold?

Example: Fields are R in this example:

- The n-tuple space,
- The space of m x n matrices
- The space of functions:

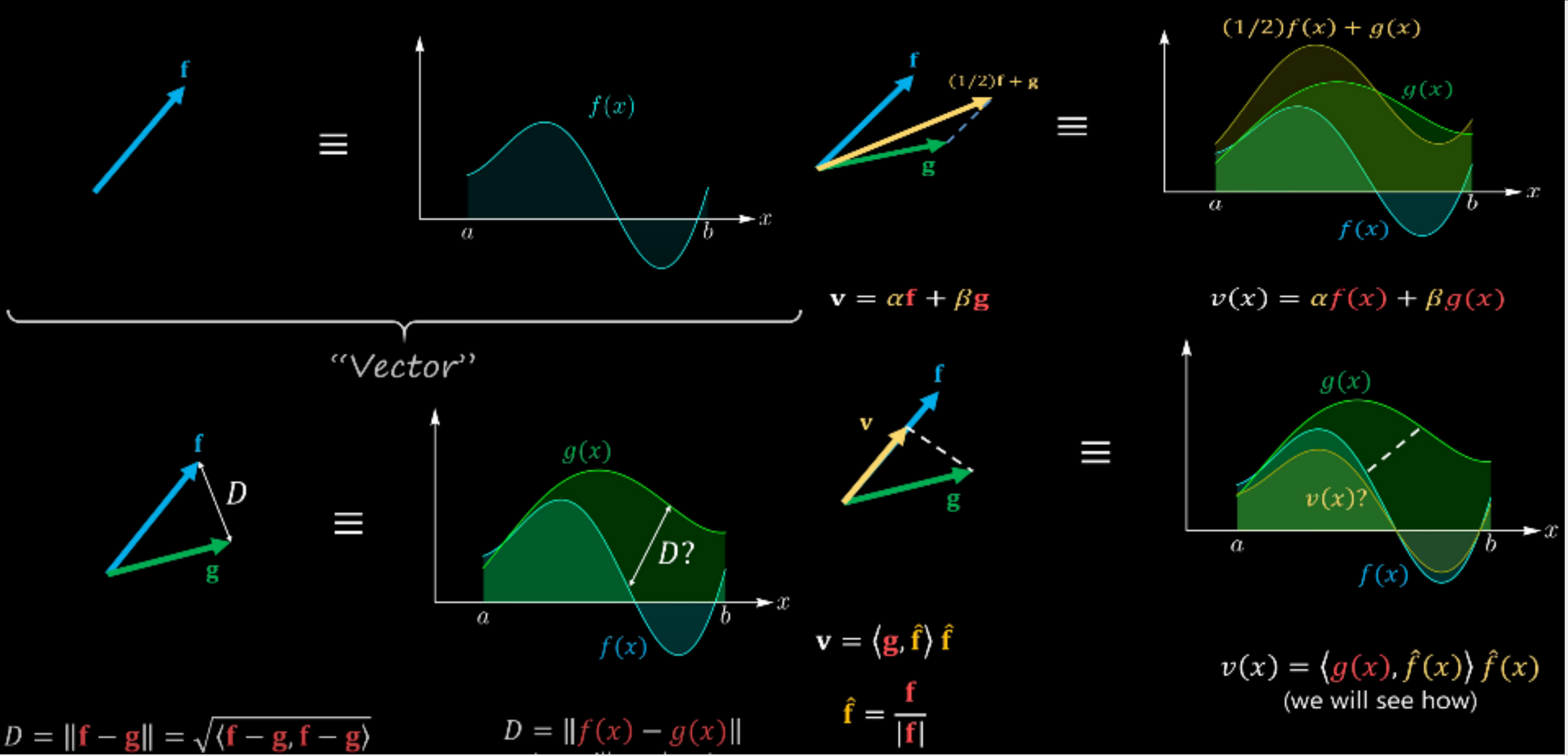
  (f + g)(x) = f(x) + g(x)   and   (cf)(x) = cf(x)

  f(t) = 1 + sin(2t)   and g(t) = 2 + 0.5t

- The space of polynomial functions over a field f(x):

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \ \dots + a_n t^n$$

Vector Space



Vector Space of functions

- Function addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x) \quad and\ (af)(x) = af(x)$$

Non-empty set X and any field F

$F^x = \{f: X \to F\}$

Example

- Set of all polynomials with real coefficients
- Set of all real-valued continuous function on [0,1]
- Set of all real-valued function that are differentiable on [0,1]

$P_n(\mathbb{R})$: Polynomials with max degree (n)

- Vector addition
- Scalar multiplication
- And other 8 properties!

Example

Which are vector spaces?

- Set $\mathbb{R}^n$ over $\mathbb{R}$
- Set $\mathbb{C}$ over $\mathbb{R}$
- Set $\mathbb{R}$ over $\mathbb{C}$
- Set $\mathbb{Z}$ over $\mathbb{R}$
- Set of all polynomials with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- Set of all polynomials of degree at most n with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- Matrix: $M_{m,n}(\mathbb{R})$ over $\mathbb{R}$
- Function: $f(x): x \longrightarrow \mathbb{R}$ over $\mathbb{R}$

Conclusion

The operations on field F are:

- $+ : F \times F \rightarrow F$
- $\times : F \times F \rightarrow F$

The operations on a vector space V over a field F are:

- $+ : V \times V \rightarrow V$
- $. : F \times V \rightarrow V$

Linear Combination

- The linear combinations of m vectors $a_1, \ldots a_m,$ each with size n is:

$$\beta_1 a_1 + \ldots + \beta_m a_m$$

where $\beta_1, \ldots, \beta_m$ are scalars and called the coefficients of the linear combination

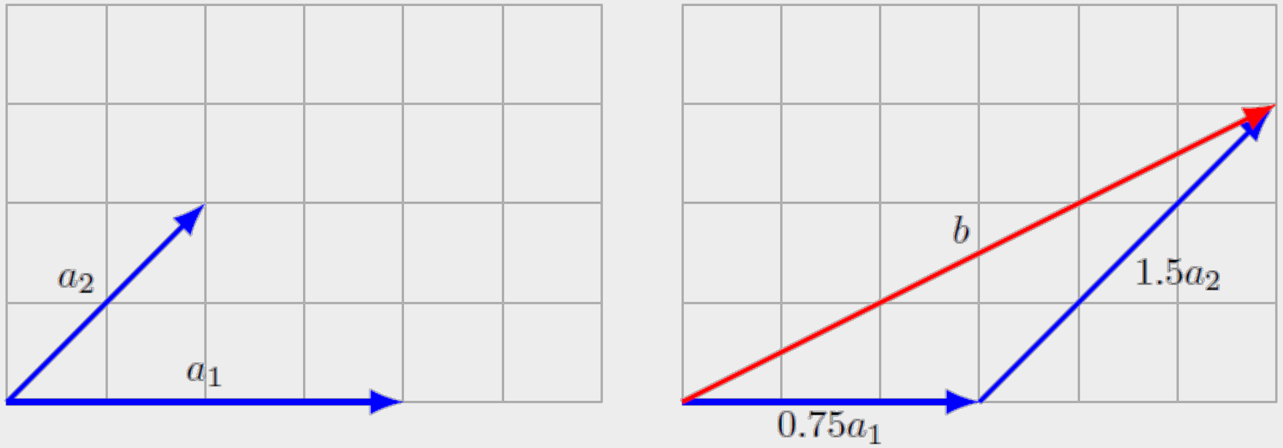
- Coordinates: We can write any n-vector b as a linear combination of the standard unit vectors, as:

$$b = b_1 e_1 + \ldots + b_n e_n$$

  - Example: What are the coefficients and combination for this vector?

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

Linear Combinations



Left. Two 2-vectors a_1 and a_2. *Right.* The linear combination $b = 0.75a_1 + 1.5a_2$

Special Linear Combinations

- ❑ Sum of vectors
- ❑ Average of vectors

Span – Linear Hull

Definition

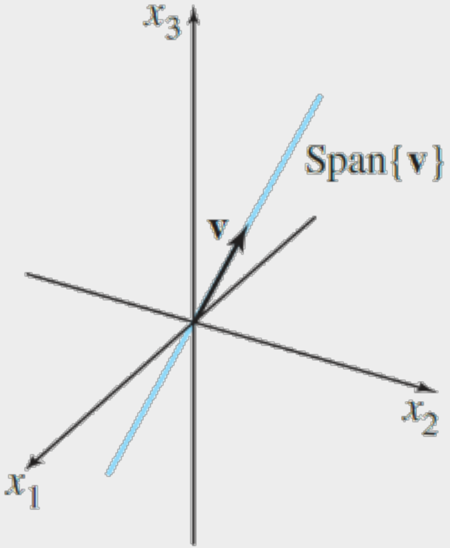
If $v_1,\ v_2,\ v_3,\ \dots,\ v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1\ ,\ v_2,\ \dots,\ v_p$ is denoted by Span $\{v_1\ ,\ v_2,\ \dots,\ v_p\ \}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $v_1\ ,\ v_2,\ \dots,\ v_p$.

That is, Span$\{v_1\ ,\ v_2,\ \dots,\ v_p\}$ is the collection of all vectors that can be written in the form:

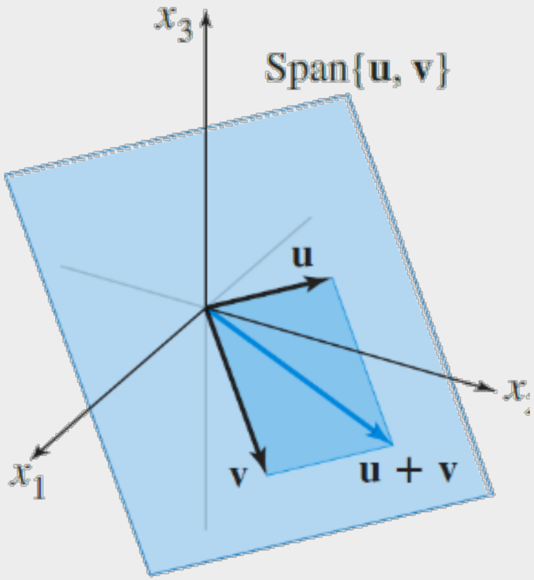
$$c_1v_1 + c_2v_2 + \ \dots + c_pv_p$$

with $c_1,\ c_2,\ \dots,\ c_p$ being scalars.

v and u are non-zero vectors in $\mathbb{R}^3$ where v is not a multiple of u

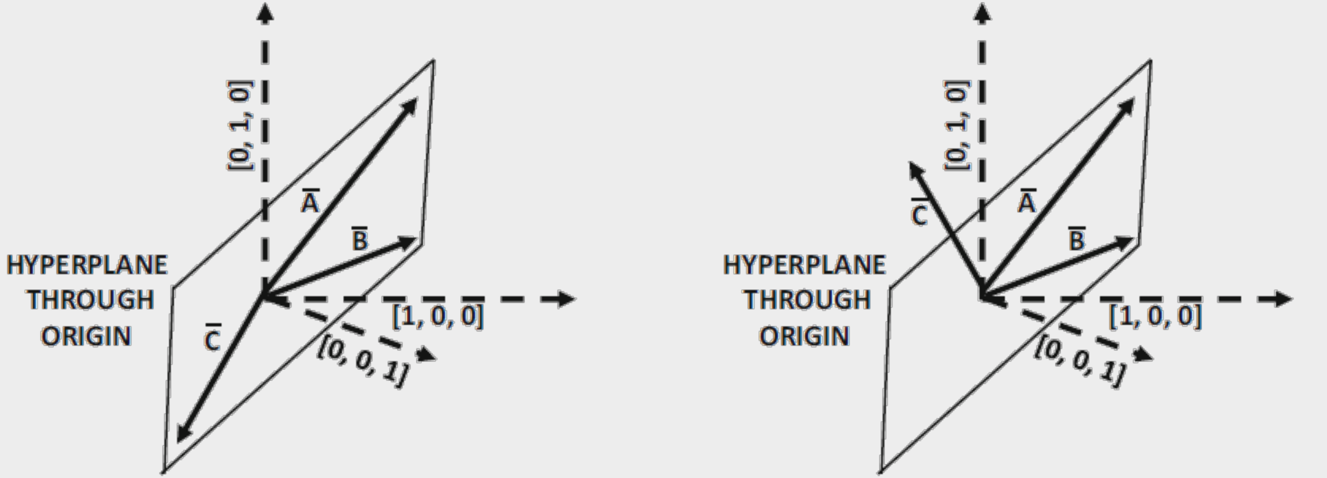


Span {**v**} as a
line through the origin.



Span {**u**, **v**} as a
plane through the origin.

Span Geometry

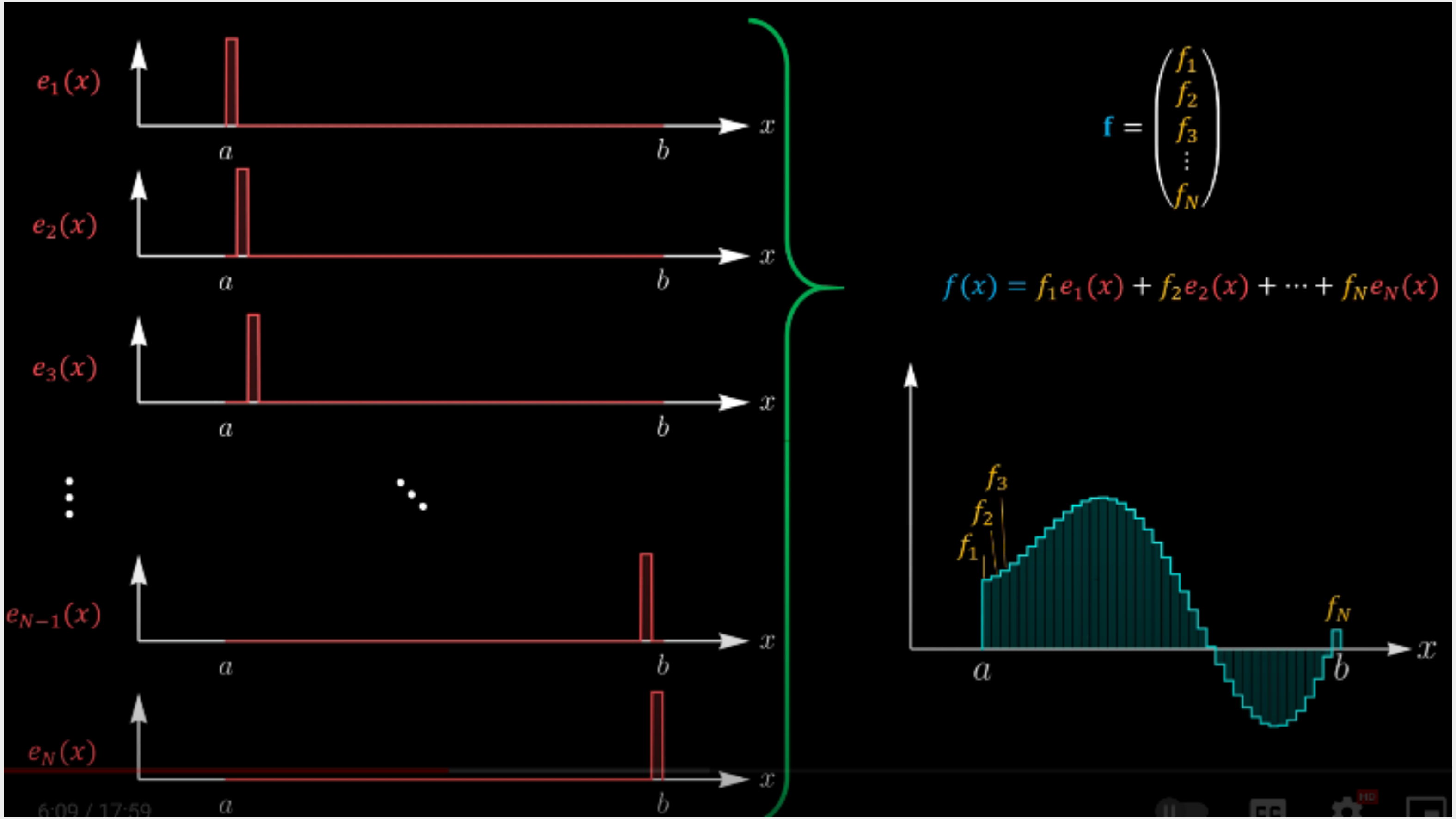


(a) $\text{Span}(\{\overline{A}, \overline{B}\}) = \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) = \text{All vectors on hyperplane}$

(b) $\text{Span}(\{\overline{A}, \overline{B}\}) \neq \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) = \text{All vectors in } \mathcal{R}^3$

Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set

Span or linear hull



Span or linear hull

Example

- Is vector b in Span $\{v_1, v_2, \ldots, v_p\}$
- Is vector v_3 in Span $\{v_1, v_2, \ldots, v_p\}$
- Is vector 0 in Span $\{v_1, v_2, \ldots, v_p\}$
- Span of polynomials: $\{(1+x), (1-x), x^2\}$?
- Is b in Span $\{a_1, a_2\}$?

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Conclusion

- Vector-Vector Operations
- Binary operations
- Field
- Vector space
- Linear combination and introduction to affine combination
- Span of vectors (linear hull)

References

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- Online Courses!
- Chapter 4 of Elementary Linear Algebra with Applications
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