



Matrix Inverse

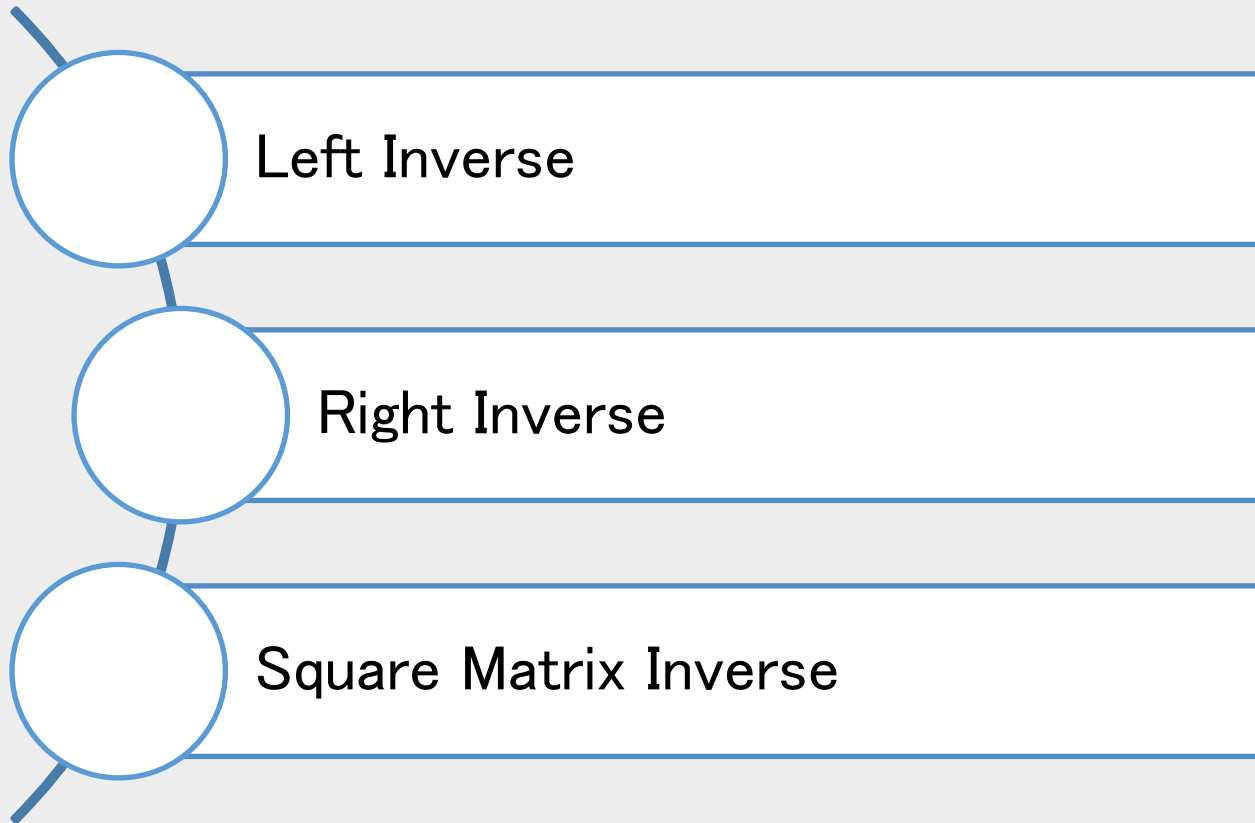
Linear Algebra

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Left Inverse



Definition

- A number x that satisfies $xa = 1$ is called the inverse of a
- Inverse (i.e., $\frac{1}{a}$) exists if and only if $a \neq 0$, and is unique
- A matrix X that satisfies $XA = I$ is called a left inverse of A
- If a left inverse exists we say that A is left-invertible
- $A: m \times n \Rightarrow I: n \times n \Rightarrow X: n \times m$

Example

The matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$

Has two different left inverses:

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix},$$

$$C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$



Method

- ❑ Suppose $Ax = b$, and A has a left inverse C
- ❑ Then $Cb = C(Ax) = (CA)x = Ix = x$
- ❑ So multiplying the right-hand side by a left inverse yields the solution



Note

- ❑ A non-zero column vector always has a left inverse.
- ❑ Left inverse is not unique.

Example

- ❑ $A = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$
- ❑ Matrix with orthonormal columns

Definition

- ❑ Row vector does not have left inverse

$$A = [1 \quad 0 \quad 3]$$



Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof



Definition

- If A has a left inverse C then the columns of A are linearly independent
- We'll see later that the converse is also true, so:

A matrix is left-invertible if and only if its columns are linearly independent

- Matrix generalization of

A number is invertible if and only if it is nonzero

From Previous Theorem

Left-invertible matrices are all tall or square

- Wide matrix is not always left invertible
- Tall or square matrices can be left invertible

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Right Inverse



Definition

- A matrix X that satisfies $AX = I$ is a right inverse of A
- If a right inverse exists we say that A is right-invertible
- A is right-invertible if and only if A^T is left-invertible:

$$AX = I \Rightarrow (AX)^T = I \Rightarrow X^T A^T = I$$

- so we conclude:

A is right invertible if and only if its rows are linearly independent

- Right-invertible matrices are wide or square



Method

- ❑ Suppose A has a right inverse B
- ❑ Consider the (square or underdetermined) equations of $Ax = b$
- ❑ $x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- ❑ So $Ax = b$ has a solution for any b

Example

- ❑ Same A, B, C in last example.
- ❑ C^T and B^T are both right inverses of A^T
- ❑ Under-determined equations $A^T x = (1, 2)$ has (different) solutions.

$$B^T(1, 2) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right), \quad C^T(1, 2) = \left(0, \frac{1}{2}, -1\right)$$

there are many other solutions as well

Conclusion: Left and Right Inverse



Definition

Left-Invertible matrix: if X is a left inverse of A , then

$$Ax = b \Rightarrow x = XAx = Xb$$

There is at most one solution using X (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

Why “at most”??

$$XA = I$$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases}$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array} \right]$$



Note

- If the system of equations $Ax = b$ is consistent, and if a matrix B exists such that $BA = I$, then the system of equations has a unique solution, namely $x = Bb$.
- **Right-inversible matrix:** if X is a right inverse of A , then there is at least one solution ($x=Xb$):

$$x = Xb \Rightarrow Ax = AXb = b$$

- To pursue these ideas further, suppose that again we want to solve a system of linear equations, $Ax = b$. Assume now that we have another matrix, B , such that $AB = I$. Then we can write $A(Bb) = (AB)b = Ib = b$; hence Bb solves the equations $Ax = b$. This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether Bb is the only solution. There may be others.
- **Invertible matrix:** if A is invertible, then

$$Ax = b \Leftrightarrow x = A^{-1}b$$

There is a unique solution



- System of linear equations $Ax = b$:
 - A right inverse of A , say $AB = I$. Then Bb is a solution, as is verified by nothing $A(Bb) = (AB)b = Ib = b$.
 - A left inverse of A , say $CA = I$, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

Square Matrix Inverse



Definition

For $A \in M_{n \times n}$, if there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$, then:

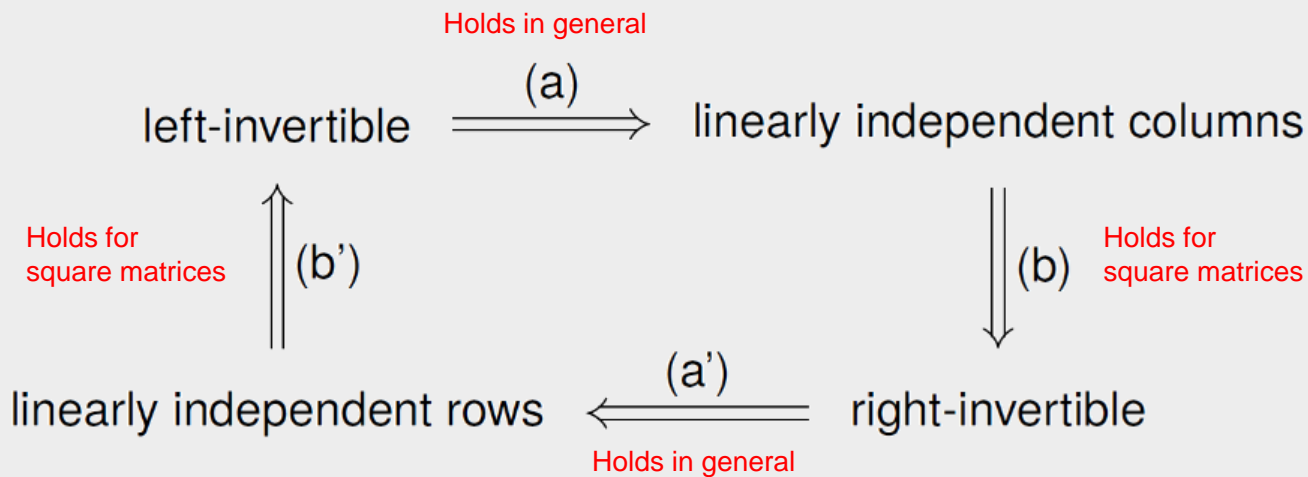
- A is invertible (or nonsingular)
- B is the inverse of A
- The inverse of A is denoted by $B = A^{-1}$

A square matrix that does not have an inverse is called non-invertible (or singular)

For a square matrix left and right inverse are the same. Rows and columns are linear independent.

Theorem

The inverse of a matrix is unique





Method

- ❑ Let A be a $n \times n$ matrix:
 - ❑ Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
 - ❑ Compute the reduced echelon form of $[A : I_n]$.
- ❑ If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A .
- ❑ If the reduced echelon form is not the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n then A has no inverse.

$$[A \mid I] \text{ Gauss–Jordan elimination } [I \mid A^{-1}]$$

Important

An $n \times n$ matrix is invertible if and only if its reduced echelon form is I_n .

A is row equivalent to I_n



Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \Rightarrow \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad (1)$$
$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases} \quad (2)$$

This two system of linear equations have the same coefficient matrix, which is exactly the matrix A





Rest of The Example

Using Gauss-Jordan Elimination on the matrix A with the same row operations

$$\begin{aligned}
 (1) &\Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow x_{11} = -3, x_{21} = 1 \\
 (2) &\Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow x_{12} = -4, x_{22} = 1
 \end{aligned}$$

Thus $X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Solution for $\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$ Solution for $\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array} \right] \xrightarrow{\text{Gauss-Jordan elimination}} \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array} \right]$$

A I I A^{-1}



Definition

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- A^k is invertible and $(A^k)^{-1} = A^{-k} = (A^{-1})^k$
- cA is invertible if $c \neq 0$ and $(cA)^{-1} = \frac{1}{c}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Theorem

If A and B are invertible matrices of order n , then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1 A_2 A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1} A_2^{-1} A_1^{-1}$$



Theorem

Let $Ax = b$ be a system of n linear equations in n variable.
If A^{-1} exists, the solution is unique and is given by $x = A^{-1}b$



Theorem

The solution set K of any system $Ax=b$ of m linear in n unknowns is, s is a particular solution:

$$K = s + \text{Null}(T_A)$$

Theorem

Let $Ax = b$ be a system of n linear equations in n variable.

The system has exactly one solution $A^{-1}b$ if and only if A is invertible.



Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Note

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$.

2×2 matrix A is invertible if and only if $\det A \neq 0$.



Definition

Each Elementary Matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I .

Example

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$



Method

- ❑ Suppose A is invertible
- ❑ For any b , $Ax = b$ has the unique solution
$$x = A^{-1}b$$
- ❑ Matrix generalization of simple scalar equation $ax = b$ having solution $x = \left(\frac{1}{a}\right)b$ (for $a \neq 0$)
- ❑ Simple-looking formula $x = A^{-1}b$ is basis for many applications



Conclusion

The following are equivalent for a square matrix A :

- ☐ A is invertible
- ☐ Columns of A are linearly independent
- ☐ Rows of A are linearly independent
- ☐ A has a left inverse
- ☐ A has a right inverse

$$\text{row rank}(A) = \text{col rank}(A) = n$$

If any of these hold, all others do



Examples

□ $I^{-1} = I$

□ If Q is orthogonal, i.e. , square with $Q^T Q = I$, then $Q^{-1} = Q^T$

□ 2×2 matrix A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

- You need to know this formula
- There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)

□ Consider matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

➤ A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

- Verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)
- We'll soon see how to compute the inverse



Properties

- $(AB)^{-1} = B^{-1}A^{-1}$
- If A is nonsingular, then A^T is nonsingular
 $(A^T)^{-1} = (A^{-1})^T$ (sometimes denoted A^{-T})
- **Negative matrix powers:** $(A^{-1})^k$ is denoted by A^{-k}
- With $A^0 = I$, Identity $A^k A^l = A^{k+l}$ holds for any integers k, l



Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??



Why Matrix of Change of Basis is invertible?