Eigenvalue – Eigenvector

Linear Algebra

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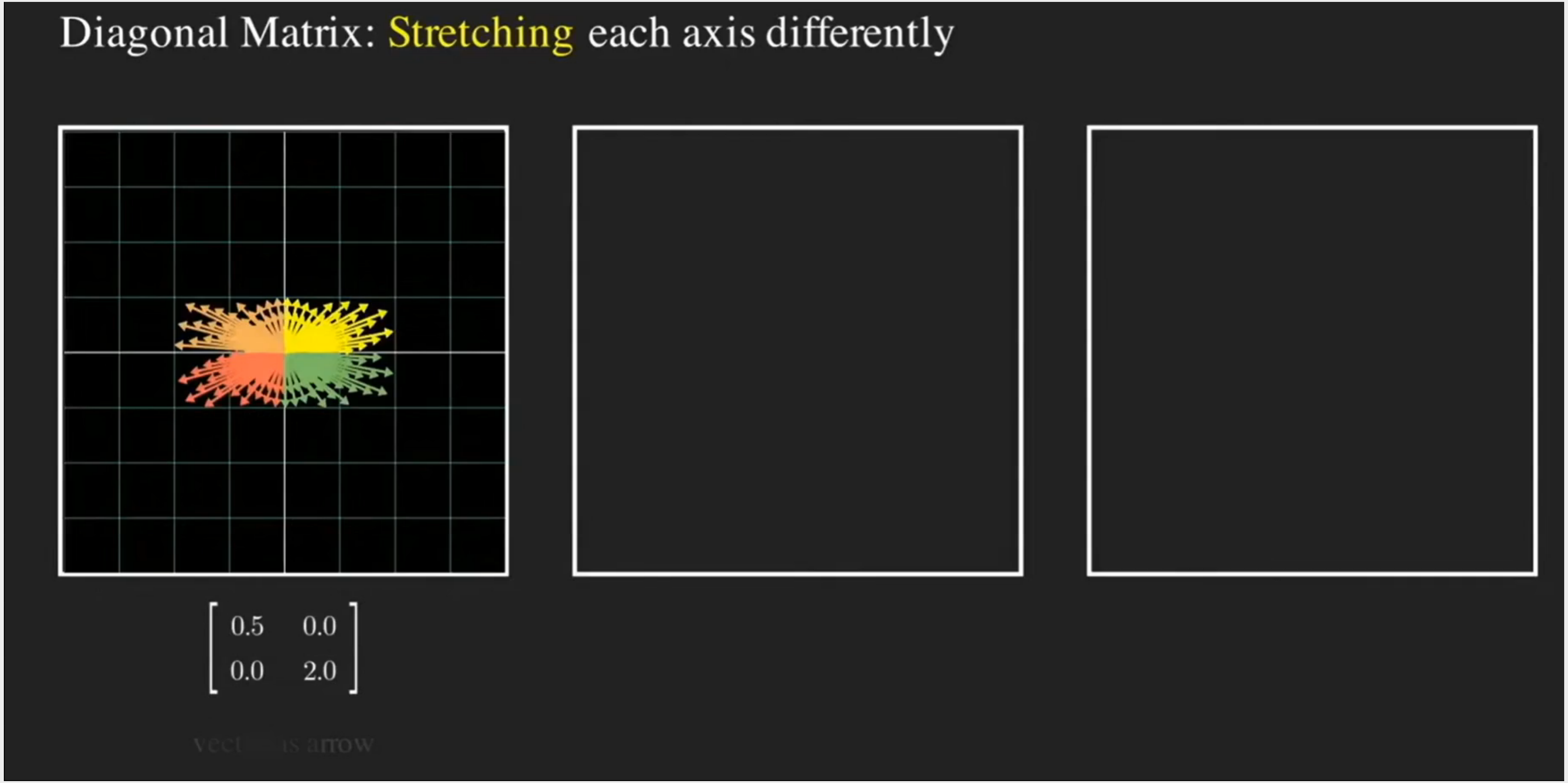
Review


Diagonal Matrix: Stretching each axis differently
0.5 0.0
0.0 2.0

Introduction

Motivation

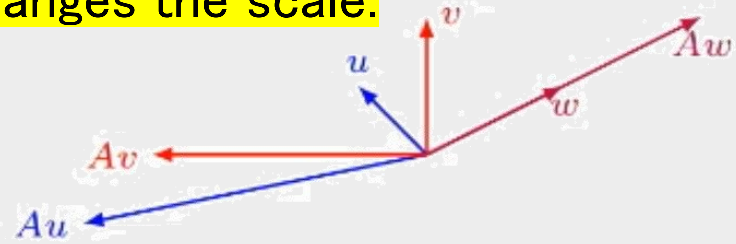
- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

- Vector “w” keeps the straight, but changes the scale.



Definition

An **eigenvector** of a square $n \times n$ matrix A is nonzero vector **x** such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $Ax = \lambda x$; such an x is called an *eigenvector corresponding to* λ.

- An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.

Example

$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$

Show that 7 is an eigenvalue of matrix A, and find the corresponding eigenvectors.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

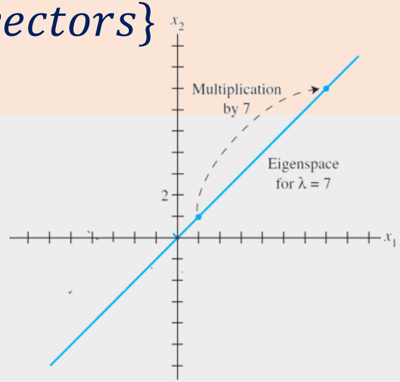
Note

λ is an eigenvalue of an $n \times n$ matrix:

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace* of $\mathbb{R}^n$and is called the **eigenspace** of A corresponding to λ . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Eigenspace: A vector space formed by eigenvectors corresponding to the same eigenvalue and the origin point. $span\{corresponding\ eigenvectors\}$



Note

- $Av = \lambda v \Rightarrow Av - \lambda vI = 0 \Rightarrow (\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{v} = \mathbf{0} \;\; \boldsymbol{v} \neq \mathbf{0}$
  - $\boldsymbol{v} \in \boldsymbol{N}(\boldsymbol{A} - \boldsymbol{\lambda I})$
  - $\boldsymbol{A} - \boldsymbol{\lambda I}$ **must be singular.**
  - Proof that for finding the eigenvalue we should solve the determinate zero equation. Look at nullspace, rank and nullity theorem, singular matrix, and det zero!
- Characteristic polynomial $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I})$
- Characteristic equation $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}) = \mathbf{0}$
- If λ is an eigenvalue of A, then the subspace $E_\lambda = \{v \mid Av = \lambda v\}$ is called the eigenspace of A associated with λ. (This subspace contains all the eigenvectors with eigenvalue λ, and also the zero vector.)
- Set of all eigenvalues of matrix is $\sigma(A)$ *named spectrum of a matrix*

Note

- Instead of $\det(A - \lambda I)$, we will compute $\mathbf{det}(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A})$. Why?
  - $\det(A - \lambda I) = (-1)^{\mathrm{n}}\det(\lambda I - \mathrm{A})$
  - Matrix $n \times n$ has ⋯⋯ eigenvalues.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)\, X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example

Example

Find eigenvalues and eigenvectors, eigenspace (E), and ***spectrum*** **of matrix** $\boldsymbol{A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}}$**:**

$$\det(A-\lambda \mathrm{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^3 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{matrix} \lambda_1 = 1 \\ (A-\lambda_1 I)q_1 = 0 \end{matrix}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \lambda_2 = 2 \\ (A-\lambda_2 I)q_2 = 0 \end{matrix}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues={1,2}

Eigenvectors=$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$E_1(A) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ $E_2(A) = span\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$

$\sigma(A)$={1,2}

$$\boldsymbol{AV = \Lambda V \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}}$$

Eigenvalues

Theorem

To have (1) scalar for largest degree instead of $|\boldsymbol{A} - \boldsymbol{\lambda}\mathbf{I}|$, consider $|\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{A}|$

$$\boldsymbol{f}(\boldsymbol{\lambda}) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$ Proof?

- **The n roots of this polynomial are eigenvalues!**
  - $\boldsymbol{f}(\boldsymbol{\lambda}) = (\boldsymbol{\lambda} - \boldsymbol{\lambda}_1)(\boldsymbol{\lambda} - \boldsymbol{\lambda}_2)\dots(\boldsymbol{\lambda} - \boldsymbol{\lambda}_n)$
- What is c_{n-1}?
  - $c_{n-1} = -trace(A)$
- What is c_0?
  - $c_0 = -\det(A)$

Theorem

If A is an n × n matrix, then the sum of the n eigenvalues of A is the trace of A.
(coefficient c_{n-1} in expanded characteristic equation)

Other view: $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

Proof?

Theorem

If A is an n × n matrix, then the product of the n eigenvalues is the determinant of A.
(coefficient c_0 in expanded characteristic equation)

Proof?

Theorem

$0 \in \sigma(A) \Leftrightarrow |A| = 0$

Proof?

Conclusion: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- ❑ The number 0 is not an eigenvalue of A .
- ❑ The determinant of A is not zero.

An Important Theorem!

Theorem

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal. The eigenvectors are e_is.

Proof?

Real Eigenvalues of different matrices

- Projection matrix
  - 0 , 1
  - If rank(P)=r with n columns, what are the repetition of the eigenvalues?
    - 0: n−r   1:r
- Reflection matrix
  - 1 , −1
- Permutation matrix
  - 1 , −1

Example

Find the eigenvalues with their repetition and eigenvectors:

- ❑ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- ❑ The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.
- ❑ $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$
- ❑ $C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$
- ❑ $D = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Eigenvalues of matrix products

Theorem

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA.

Proof?

Why Diagonalization?

Conclusion from pervious theorems

- Theorem "The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal." can leads to if we have matrix A and B that $D = B^{-1}AB$ be a diagonal matrix:

$$\det(\lambda I - A) = \det(\lambda I - B^{-1}AB)$$

Proof?

Definition

Two n-by-n matrices A and B are called similar if there exists an <u>invertible n-by-n matrix Q</u>

such that

$$A = Q^{-1}BQ$$

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix D: $\mathrm{D} = Q^{-1}AQ$,
that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

Similarity

Relation between similar matrix and change of basis!

Note

- A square matrix for a linear transform

$$A: n \times n \qquad T: R^n \to R^n \quad \Rightarrow \quad \boldsymbol{Aa = b} \qquad a, b \in R^n$$

$$\left.\begin{array}{l} a = Q\bar{a} \\ \mathrm{b} = Q\bar{b} \end{array}\right\} \Rightarrow \mathrm{AQ}\bar{a} = Q\bar{b} \Rightarrow \underbrace{Q^{-1}AQ}_{\bar{A}}\,\bar{a} = \bar{b} \;\Rightarrow\; \bar{A}\bar{a} = \bar{b}$$

- Linear transform in new basis $\bar{A} = Q^{-1}AQ$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Think!

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

- A matrix is a similarity invariant, meaning it remains unchanged under a similarity transformation.

- Why trace is a similarity invariant?
- Why rank is a similarity invariant?

Theorem

- Similar matrices have:
  - same determinant
  - equal characteristic equations
  - same trace
  - same rank
  - inverse of A and B are similar (if exists)

Proof?

Example

Find the similarity matrix of A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\boldsymbol{i} & \boldsymbol{i} \end{bmatrix}$$

Diagonalization

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

- The columns of Q is called an eigenvector basis of $\mathbb{R}^n$.

Corollary

- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalisable and Non-Diagonalisable Matrices

- Distinct eigenvalues -> eigenvectors are Linear Independent
- Duplicate eigenvalues -> 🧐 🧐

- Not all matrices are diagonalizable.
  - Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The diagonalizing matrix S is not unique.

- For matrix $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$AS = SD$$

$$\begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & -4 & 3 \\ 0 & 6 & -9 \end{pmatrix}$$

Diagonal Matrix

S is not invertible!

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

So what's going on here?

Diagonal Matrix

R is invertible!

- Details for matrix A:

(i) For the eigenvalue -3, we have

$$\begin{pmatrix} 3 & -6 & -4 \\ 5 & -8 & -6 \\ -6 & 9 & 7 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) For the repeated eigenvalue -2, we have

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which equally straightforwardly gives the eigenvector

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Details for matrix B:

(i) For the eigenvalue -3, we have

$$\begin{pmatrix} 7 & 8 & -2 \\ -3 & -3 & 1 \\ 9 & 12 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which, as before, straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) This time, for the repeated eigenvalue -2, we have

$$\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, here things are different, because all three of the rows of this matrix may be reduced to the equation

$$3X + 4Y - Z = 0.$$

❑ Details for matrix B:

This represents a **plane** in 3D space, and any vector in this plane is an eigenvector. We may therefore form our diagonalising matrix S out of

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

together with any two non-parallel vectors of the form

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

that satisfy

$$3X + 4Y - Z = 0;$$

that is, that are perpendicular to the vector

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Both of the choices

$$S = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$$

will work fine, as will infinitely many others.

Diagonalisable and Non-Diagonalisable Matrices

- General considerations

1. In general, any n by n matrix whose eigenvalues are distinct can be diagonalised.
2. If there is a repeated eigenvalue, whether or not the matrix can be diagonalised depends on the eigenvectors.
   (i) If there $k<n$ eigenvectors (up to multiplication by a constant), then the matrix cannot be diagonalised.
   (ii) If the unique eigenvalue corresponds to an eigenvector e, but the repeated eigenvalue corresponds to an entire plane, then the matrix can be diagonalised, using e together with any two vectors that lie in the plane.
3. If all n eigenvalues are repeated, then things are much more straightforward: the matrix can't be diagonalised unless it's already diagonal.

Example

Find A^n?

Conclusion

Another Notation

- With similarity transformation Q, matrix A changed to a diagonal matrix $diag(\lambda_1, \lambda_2)$
- Matrix A has n linear independent eigenvectors

- $Aq_1 = \lambda_1 q_1 = [q_1\ q_2\ \cdots\ q_n]\begin{bmatrix}\lambda_1\\ 0\\ \vdots\\ 0\end{bmatrix} \cdots Aq_n = \lambda_n q_n = [q_1\ q_2\ \cdots\ q_n]\begin{bmatrix}0\\ 0\\ \vdots\\ \lambda_n\end{bmatrix}$

- $[Aq_1\ \ Aq_2\ \ \cdots\ \ Aq_n] = \underbrace{[q_1\ \ q_2\ \cdots\ \ q_n]}_{Q}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_n\end{bmatrix}}_{\Lambda}$

- $A\,[q_1\ q_2\ \cdots\ \ q_n] = Q\Lambda \Longrightarrow AQ = Q\Lambda$
- $\Lambda = Q^{-1}AQ^{T}$
- $A = Q\Lambda Q^{-1}$