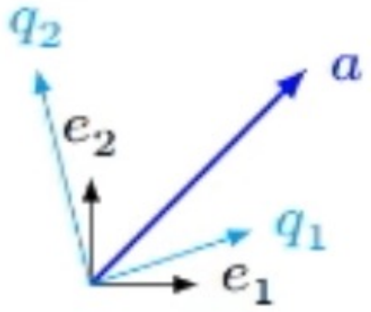
Eigenvectors and Eigenvalues

CE40282-1: Linear Algebra
Hamid R. Rabiee and Maryam Ramezani
Sharif University of Technology

Review



- n-vector a based on basis $\{e_1, \dots, e_n\}$

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

- n-vector a based on new basis $\{q, \dots, q_n\}$

$$a = \overline{a_1} q_1 + \overline{a_2} q_2 + \cdots + \overline{a_n} q_n = \underbrace{[q_1 \quad \dots \quad q_n]}_{Q} \begin{bmatrix} \overline{a_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{a_n} \end{bmatrix}$$

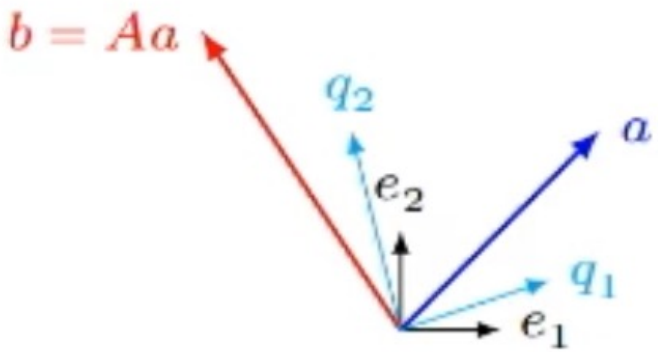
- Matrix Q is invertible.
- Any invertible matrix is a basic matrix.

Review

- A square matrix for a linear transform

$$A : n \times n \quad A : \mathbb{R}^n \rightarrow \mathbb{R}^n \implies Aa = b \quad a.b \in \mathbb{R}^n$$

$$\left.\begin{aligned} a &= Q\bar{a} \\ b &= Q\bar{b} \end{aligned}\right\} \implies AQ\bar{a} = Q\bar{b} \implies \underbrace{Q^{-1}AQ}_{\bar{A}}\,\bar{a} = \bar{b} \implies \bar{A}\bar{a} = \bar{b}$$



- Linear transform in new basis $\bar{A} = Q^{-1}AQ$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Similar Matrices

- Two n-by-n matrices A and B are called similar if there exists an invertible n-by-n matrix $\boldsymbol{Q}$ such that

$$A = Q^{-1}BQ$$

- A and B are similar if $QA = BQ$
- $A = Q^{-1}BQ \rightarrow B = QAQ^{-1}$
- Same determinant
- Inverse of A and B are similar (if exists)

Similarity Transformation

- We can use similarity transformation for changing the standard matrix of linear transformation

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Think!

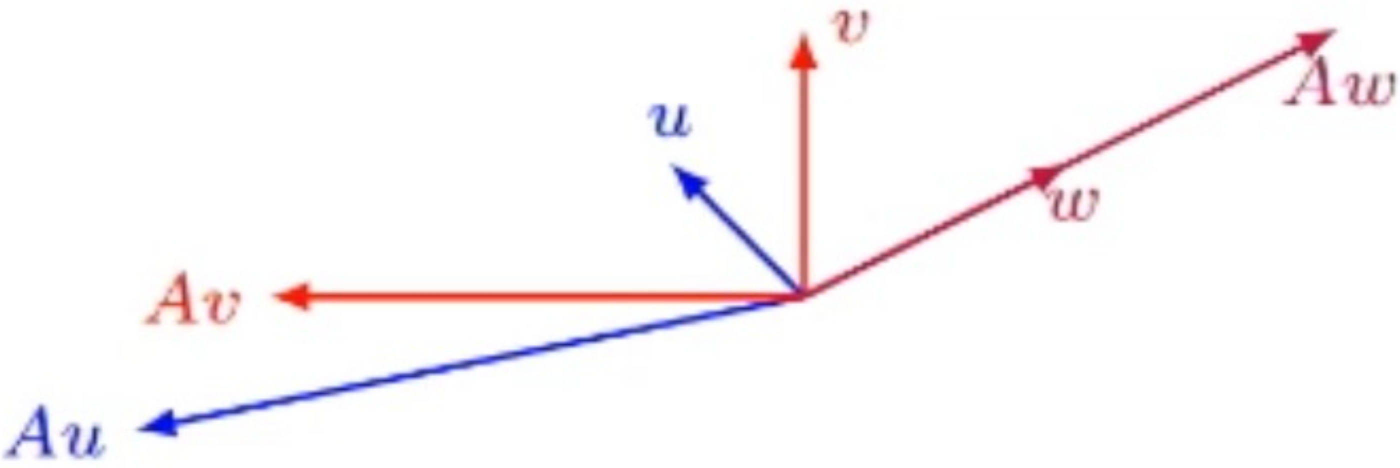
- Why trace is a similarity invariant?

- Why rank is a similarity invariant?

Motivation

- $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$
$$v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \implies Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$
$$w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Definition

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

- An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.
- Example

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad , \lambda = 2.$$

  - Show that 7 is an eigenvalue of matrix A, and find the corresponding eigenvectors.

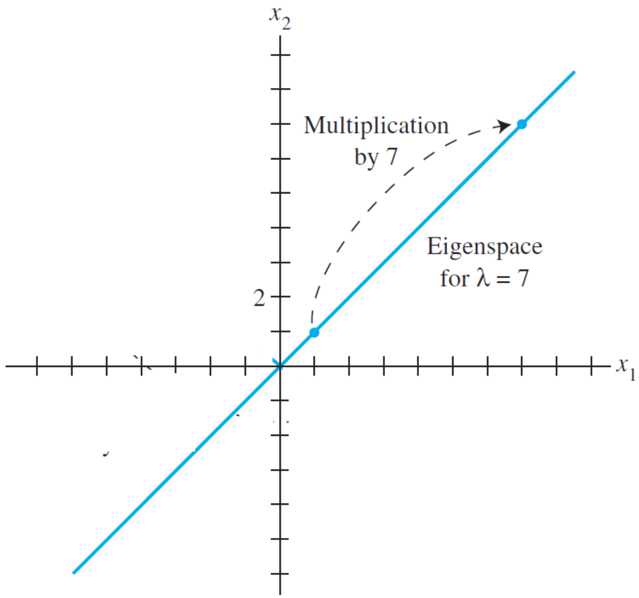
$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.



Characteristic Equation

- $Av = \lambda v \implies Av - \lambda v I = 0 \implies (A - \lambda I)v = 0 \quad v \neq 0$
- Characteristic equation $|A - \lambda I| = 0$
- Characteristic polynomial $|A - \lambda I|$ $\Delta_A(\lambda), \Delta(\lambda)$
  - Matrix $n \times n$ has …. eigenvalue

Characteristic Equation

- Example

  - The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

  - $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Matrix spectrum

- Set of all eigenvalues of matrix $\sigma(A)$
- Theorem: The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal
  - Proof?
- $. 0 \in \sigma(A) \Leftrightarrow |A| = 0$
- A is invertible if and only if ....
- 0 is an eigenvalue of A if and only if A is not invertible.

Similar Matrices

- Similar matrices has equal characteristic equation
  - vice versa?

- Example
  - $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}, \bar{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
  - $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvectors Linear Independence

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

- One way to prove the statement "If P then Q" is to show that P and the negation of Q leads to a contradiction

- Distinct eigenvalues -> eigenvectors are LI
- Duplicate eigenvalues -> ???
  - Example

Some notes

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

The number 0 is *not* an eigenvalue of A.

The determinant of A is *not* zero.

WARNINGS:

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

Example

- Find eigenvalues and eigenvectors?

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left.\begin{matrix} \lambda_1 = 1 \\ (A - \lambda_1 I)q_1 = 0 \end{matrix}\right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \lambda_2 = 2 \\ (A - \lambda_2 I)q_2 = 0 \end{matrix}\right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Diagonalization

- With similarity transformation Q, matrix A changed to a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2)$
- Matrix A has n linear independent eigenvectors

$$Aq_1 = \lambda_1 q_1 = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad Aq_n = \lambda_n q_n = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} Aq_1 & Aq_2 & \cdots & Aq_n \end{bmatrix} = \underbrace{\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$A \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = Q\Lambda \implies AQ = Q\Lambda$$

$$\Lambda = Q^{-1}AQ \qquad A = Q\Lambda Q^{-1}$$

Example

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Diagonalizable

Definition

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

The columns of P is called an **eigenvector basis** of $\mathbb{R}^n$

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalization Process

- Find eigenvalues
- Find eigenvectors
- IF there is n linear independent eigenvectors, then matrix is diagonalizable.
- A similar transform Q can make matrix diagonalizable.
- Columns of Q are eigenvectors.
- Diagonal values of diagonal matrix is the eigenvalues (in the same order)

Non Diagonalizable Matrix

- Example $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$$\Delta(\lambda) = \lambda^2 - 4\lambda + 4 = 0 \implies \sigma(A) = \{2, 2\}$$

$$(A - 2I)\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0 \implies \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \alpha_1 = \alpha_2$$

  - Can write in form $\Lambda = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$?

$$Q\Lambda = AQ \Rightarrow [q_1 \quad q_2]\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = A\,[q_1 \quad q_2] \Rightarrow \begin{cases} Aq_1 = 2q_1 \\ Aq_2 = 2q_2 + q_1 \end{cases}$$

$$q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies (A - 2I)q_2 = q_1 \implies \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \beta_1 + \beta_2 = 1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Generalized Eigenvectors

Definition

If A is an $n \times n$ matrix, a **generalized eigenvector** of A corresponding to the eigenvalue λ is a nonzero vector $\mathbf{x}$ satisfying

$$(A - \lambda I)^p \, \mathbf{x} = \mathbf{0}$$

for some positive integer p. Equivalently, it is a nonzero element of the nullspace of $(A - \lambda I)^p$.

Example

- Eigenvectors are generalized eigenvectors with $p = 1$.
- In the previous example we saw that $\mathbf{v} = (1, 0)$ and $\mathbf{u} = (0, 1)$ are generalized eigenvectors for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \lambda = 1.$$

Jordan canonical form

- “most diagonal” representative from each family of similar matrices;

Jordan's theorem says that every square matrix A is similar to a Jordan matrix J, with Jordan blocks on the diagonal:

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_d \end{bmatrix}.$$

A Jordan block J_i has a repeated eigenvalue λ_i on the diagonal, zeros below the diagonal and in the upper right hand corner, and ones above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

Jordan canonical form

- Note
  - If A has n distinct eigenvalues, it is diagonalizable and its Jordan matrix is the diagonal matrix $J = \Lambda$.
  - If A has repeated eigenvalues and "missing" eigenvectors, then its Jordan matrix will have $n - d$ ones above the diagonal.
- Example: which are similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jordan canonical form

- Example: for 4*4 matrix with four duplicated eigenvalues
- linear independent eigenvectors
  - 4
  - 3
  - 2
  - 1

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Conclusion

- Every matrix can convert to Jordan form by a similar transform
- Diagonal form is a special case of Jordan form where all Jordan blocks are 1*1
- Number of Jordan block=number of linear independent eigenvectors
- If a matrix has n linear independent eigenvectors, it is diagonalizable.
- If matrix has one eigenvalue with m duplicates:
  - ?
  - ?
  - ?

Symmetric Matrix

Theorem

If A is symmetric, then any two eigenvectors from different eigenspace are **orthogonal**.

$$\left.\begin{array}{r} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \implies v_1^T v_2 = 0$$

Symmetric Matrix

- Eigenvalues of real symmetric matrix are real.
- If A is diagonalizable by an orthogonal matrix, then A is a symmetric matrix.
- A symmetric matrix is always diagonalizable.
- A similar transform that diagonalized the symmetric matrix is orthogonal.

$$Q^T Q = I \qquad A = Q\Lambda Q^T, \qquad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \lambda_i \in \mathbb{R}$$

Orthogonally Diagonalizable

Theorem

An $n \times n$ matrix A is **orthogonally diagonalizable** if and only if A is a symmetric matrix.

$$A = A^T \implies A = Q\Lambda Q^T,\ \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

$$A = A^T \impliedby A = Q\Lambda Q^T,\ \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

$$A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$$

Spectral Theorem

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

Gram matrix

- Eigenvalues are real.
- Eigenvalues are nonnegative.