Change of basis

Linear Algebra

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Overview

- Invertible Linear Maps
- Basis Review
- Change of Basis
- L(V) and Change of Basis
- L(V,W) and Change of Basis

Invertible Linear Maps

Invertible, Inverse

Definition

- A linear map $T \in L(V, W)$ is called invertible if there exists a linear map $S \in L(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.

- A linear map $S \in L(W, V)$ satisfying $ST = I$ and $TS = I$ is called an inverse of T (note that the first I is the identity operator on V and the second I is the identity operator on W).

Inverse is unique

Theorem

An invertible linear map has a unique inverse.

Definition

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Example

- Find the inverse of $T(x, y, z) = (-y, x, 4z)$

Invertibility

Theorem

A linear map is invertible if and only if it is injective and surjective.

Theorem

Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \Leftrightarrow T \text{ is injective} \Leftrightarrow T \text{ is surjective.}$$

Basis Review

Example

- Find the coordinate vector of $2 + 7x + x^2 \in \mathcal{P}^2$ with respect to the basis
B = $\{x + x^2, 1 + x^2, 1 + x\}$.

- If C = $\{1,\ x,\ x^2\}$ is the standard basis of $\mathcal{P}^2$ then we have
$[2 + 7x + x^2]_C = (2, 7, 1)$.

Solution

We want to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$2 + 7x + x^2 = c_1(x + x^2) + c_2(1 + x^2) + c_3(1 + x).$$

By matching coefficients of powers of x on the left–hand and right–hand sides above, we arrive at following system of linear equations:

$$\begin{aligned} c_2 + c_3 &= 2 \\ c_1 + c_3 &= 7 \\ c_1 + c_2 &= 1 \end{aligned}$$

This linear system has $c_1 = 3, c_2 = -2, c_3 = 4$ as its unique solution, so our desired coordinate vector is

$$[2 + 7x + x^2] = (c_1, c_2, c_3) = (3, -2, 4)$$

Change of Basis

Introduction to change of basis

- $B = \{v_1, \ldots, v_n\}$ are basis of $\mathbb{R}^n$.

- P = $[v_1 \;\; v_2 \;\; \ldots \;\; v_n]$

- $P[a]_B = a$

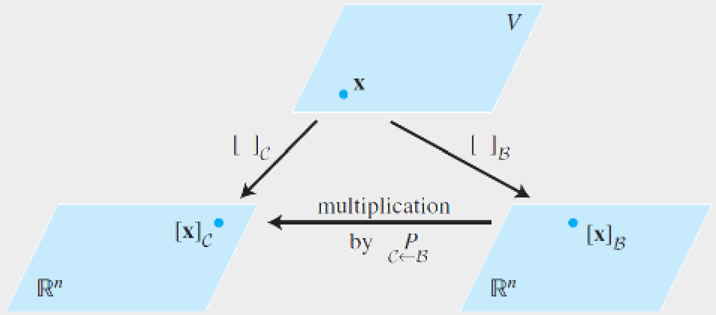
Theorem

Let B = $\{b_1, b_2, \dots, b_n\}$ and C = $\{c_1, c_2, \dots, c_n\}$ be basses of a vector space V. Then there is a unique n × n matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B}[x]_B$$

The columns of $P_{C \leftarrow B}$ are the C−coordinate vectors of the vectors in basis B.
That is ,

$$P_{C \leftarrow B} = [[b_1]_C \ \ [b_2]_C \ \ \dots \ \ [b_n]_C]$$



$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

$$P_\mathcal{B}[\mathbf{x}]_\mathcal{B} = \mathbf{x}, \quad P_\mathcal{C}[\mathbf{x}]_\mathcal{C} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_\mathcal{C} = P_\mathcal{C}^{-1}\mathbf{x}$$

$$[\mathbf{x}]_\mathcal{C} = P_\mathcal{C}^{-1}\mathbf{x} = P_\mathcal{C}^{-1}P_\mathcal{B}[\mathbf{x}]_\mathcal{B}$$

Change of basis

Example

Find the change-of-basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases

$$B = \{x + x^2, 1 + x^2, 1 + x\} \ and \ C = \{1, x, x^2\}$$

of $\mathcal{P}^2$. Then find the coordinate vector of $2 + 7x + x^2$ with respect to B.

Change of basis

Example

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for $\mathbb{R}^2$ given by B = $\{b_1, b_2\}$, C = $\{c_1, c_2\}$.

a. Find the change-of-coordinates matrix from C to B.

b. Find the change-of-coordinates matrix from B to C.

Change of basis

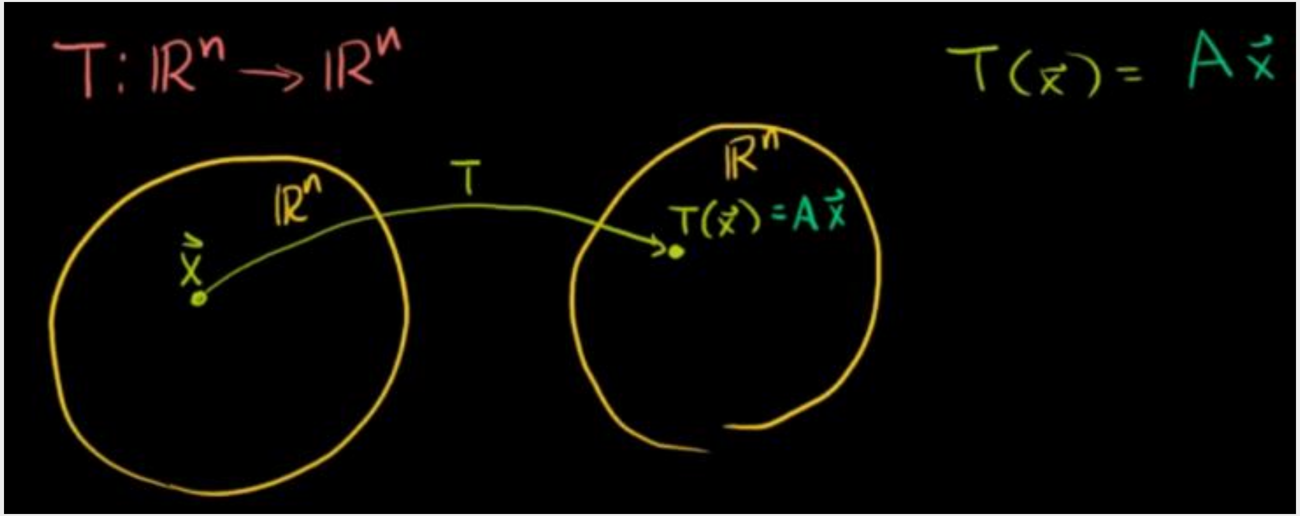
Example

Find the change-of-basis matrix $P_{C \leftarrow B}$, where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \text{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

L(V)
and Change of Basis

Transformation with change of basis



- B = $\{v_1, v_2, \dots, v_n\}$ *are basis of* $\mathbb{R}^n$.
- P = $[v_1 \quad v_2 \quad \dots \quad v_n]$
- $[T(x)]_B = P^{-1}AP[x]_B$

Change of basis

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[A]_B}"] \arrow[d, "P"'] & \mathbb{R}^n \\ \mathbb{R}^n \arrow[r, "{[A]_E}"'] & \mathbb{R}^n \arrow[u, "P^{-1}"'] \end{tikzcd}$$

$$[A]_B = P^{-1}[A]_E P$$

L(V,W)
and Change of Basis

Matrix representation of linear function

A linear transformation which looks complex with respect to one basis can become much easier to understand when you choose the correct basis.

Important

Let T: $V \rightarrow W$ be a linear function and u = $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in V$ where $E = \{e_1, \dots, e_n\}, B = \{b_1, .., b_m\}$ are basis of V, W.

$$u = c_1 e_1 + \cdots + c_n e_n \quad -> T(u) = c_1 T(e_1) + \cdots + c_n T(e_n)$$

$$T(u) = d_1 b_1 + \cdots + d_m b_m$$

$$[T(u)]_B = [[T(e_1)]_B, \dots, [T(e_n)]_B][T(u)]_E$$

Example

We have B = $\{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $\mathcal{P}_3(x)$ and $\mathcal{P}_2(x)$, respectively. Find the matrix of transformation T: $P_3(x) \to P_2(x)$.

Solution

Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathcal{P}^3(\mathrm{x})$, we have

$$\left[\frac{d}{dt}\right]_{\{B,B'\}} = \left[\frac{d}{dt}(x^3) \quad \frac{d}{dt}(x^2) \quad \frac{d}{dt}(x) \quad \frac{d}{dt}(1)\right]$$
$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Isomorphisms

Definition

Suppose V and W are vector spaces over the same field. We say that V and W are **isomorphic**, denoted by $V \cong W$, if there exists an invertible linear transformation T: $V \to W$(called an **isomorphism** from V to W).

- If T: $V \to W$ is an isomorphism then so is $T^{-1}: W \to V$.
- If T: $V \to W$ and S: $W \to X$ are isomorphism then so is $S \circ T: V \to X$.

  in particular, if $V \cong W$ and W $\cong X$ then $V \cong X$.

Theorem

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

Isomorphisms

Example

Show that the vector space V = span(e^x, xe^x, x^2e^x) and $\mathbb{R}^3$ are isomorphic.

The standard way to show that two space are isomorphic is to construct an isomorphism between them. To this end, consider the linear transformation T: $\mathbb{R}^3 \to V$ defined by

$$T(a,b,c) = ae^x + bxe^x + cx^2e^x.$$

It is straightforward to show that this function is linear transformation, so we just need to convince ourselves that it is invertible. We can construct the standard matrix $[T]_{B\leftarrow E}$, where $E = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$:

$$[T]_{B\leftarrow E} = \big[[T(1,0,0)]_B, [T(0,1,0)]_B, [T(0,0,1)]_B\big]$$

$$= \big[[e^x]_B, [xe^x]_B, [x^2e^x]_B\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $[T]_{B\leftarrow E}$ is clearly invertible (the identity matrix is its own inverse), T is invertible too and is thus an isomorphism.

References

- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra David Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares