Matrix Properties

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

- Solve systems of linear ordinary differential equations.

$$\frac{d}{dt}y(t) = Ay(t), \qquad y(0) = y_0$$

where A is a constant matrix, is given by

$$y(t) = e^{At}y_0$$

- Is a matrix function on square matrices (A) using Taylor series:

$$e^{A} = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

- Special Case: When A is Diagonal:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \Rightarrow e^{A} = \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{\beta} \end{bmatrix}$$

Matrix Operations Complexity

- $m \times n$ matrix stored A as $m \times n$ array of numbers (for sparse A, store only **nnz**(A) nonzero values)

- matrix addition, scalar-matrix multiplication cost m flops

- matrix-vector multiplication costs $m(2n - 1) \approx 2mn$ flops (for sparse A, around 2**nnz**(A) flops)

Transpose

- The **transpose** of a matrix results from "flipping" the rows and columns. Given a matrix $A \in \mathbb{R}^{m\times n}$, is the $m \times n$ matrix whose entries are given by

$$(A^T)_{ij} = A_{ji}$$

- Properties:
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - $(cA)^T = c(A^T)$
  - $(AB)^T = B^T A^T \ \rightarrow (A_1 A_2 A_3 \cdots A_n)^T = A_n^T \cdots A_3^T A_2^T A_1^T$

Conjugate Transpose (Adjoint)

$$A^* = A^H = (\bar{A})^T = \overline{A^T}$$

$$A = \begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix} \quad A^H = \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$$

- $(A+B)^H = A^H + B^H$ for any two matrices A and B of the same dimensions.
- $(zA)^H = \bar{z}A^H$ for any complex number z and any m-by-n matrix A.
- $(AB)^H = B^H A^H$ for any m-by-n matrix A and any n-by-p matrix B. Note that the order of the factors is reserved.
- $(A^H)^H = A$ for any m-by-n matrix A

For real matrices, the conjugate transpose is just the transpose, $A^H = A^T$.

Trace

- The **trace** of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted trA, is the sum of diagonal elements in the matrix:

$$trA = \sum_{i=1}^{n} A_{ii} \,,$$

$$Tr\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}$$

- The trace has following properties:
  - For $A \in \mathbb{R}^{n\times n}, trA = trA^T$.
  - For $A, B \in \mathbb{R}^{n\times n}, tr(A+B) = trA + trB$.
  - For $A \in \mathbb{R}^{n\times n}, t \in \mathbb{R}, tr(tA) = t\ trA$.
  - For A, B such that AB is square, $trAB = trBA$.
  - For A, B, C such that ABC is square, $\mathrm{tr}ABC = \mathrm{tr}BCA = \mathrm{tr}CAB$, and so on for the product of more matrices.

- Trace is a linear function on the matrix space. Why?

Example

Show that there do not exist matrices $A, B \in \mathcal{M}_n$ such that $AB - BA = I$.

Kronecker sum

- A and B are square matrices, the Kronecker sum is: (I_b identity matrix with size b*b)

$$A \oplus B = A \otimes I_b + I_a \otimes B$$

- Properties:

$$\exp(A) \otimes \exp(B) = \exp(A \oplus B)$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & b_{12} & a_{12} & 0 \\ b_{21} & a_{11}+b_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22}+b_{11} & b_{12} \\ 0 & a_{21} & b_{21} & a_{22}+b_{22} \end{bmatrix}.$$

- An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Note

If an elementary row operation is performed on an m x n matrix A, the resulting matrix can be written as EA, where m x n matrix E is created by performing the same row operation on I_m.

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Useful Matrices

- An $m \times n$ matrix is
  - **Tall** $m > n$
  - **Wide** $n > m$
  - **Square** $m = n$
- Main diagonal of matrix

$$A_{n\times n} = \begin{bmatrix} \diagdown \end{bmatrix} \qquad a_{11}, a_{22}, \dots, a_{nn}$$

- Anti diagonal of matrix

$$A_{n\times n} = \begin{bmatrix} \diagup \end{bmatrix} \qquad a_{1,n}, a_{2,n-1}, \dots, a_{n,1}$$

Useful Matrices

- Identity matrix

  $I \in \mathbb{R}^{n\times n}$, is a square matrix with ones on the diagonal and zeros everywhere else. That is, $I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

  It has the property that for all $A \in \mathbb{R}^{m\times n}$, $\quad AI = A = IA$.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow I_n = [e_1 \quad e_2 \quad e_3]$$

- Diagonal matrix

  a matrix where all non-diagonal elements are 0. $D = diag(d_1, \dots, d_n)$,

  with $D_{ij} = \begin{cases} d_{ij} & i = j \\ 0 & i \neq j \end{cases}$.

$$\mathrm{A} = \mathrm{diag}(\mathrm{a}_1, \dots, \mathrm{a}_\mathrm{m}) = \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & a_i & \vdots \\ 0 & \dots & a_m \end{bmatrix}$$

  Clearly, $I = diag(1, 1, \dots, 1)$.

- Scalar matrix A special kind of diagonal matrix in which all diagonal elements are the same

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Useful Matrices

- A square matrix A over R is called:
  - symmetric if $A^T = A$
  - skew−symmetric if $A^T = -A$ (Good Property??)
  - $A^T A$ must be symmetric (*A with any size, it is not necessary for A to be a square matrix*)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

- A is orthogonal if $AA^T = A^T A = I$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

Example

The matrix exponential of a skew−symmetric matrix is an orthogonal matrix?

Hermitian Matrix

- Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own **conjugate transpose**

$$A\ Hermitian \quad \Longleftrightarrow \quad A = A^H$$

- **conjugate transpose**

$$\boldsymbol{A^H = A^* = (\bar{A})^T}$$

Unitary matrix

- $U^*U = UU^* = UU^{-1} = I$

Note

If U is a square, complex matrix, then the following conditions are equivalent:

1. U is unitary.
2. U^* is unitary.
3. U is invertible with $U^{-1} = U^*$.
4. The columns of U form an orthonormal basis of $\mathbb{C}^n$ with respect to usual inner product. In other words, $U^*U = 1$.
5. The rows of U form an orthonormal basis of $\mathbb{C}^n$ with respect to usual inner product. In other words, $UU^* = 1$.

Normal Matrix

- A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is called **normal** if $A^*A = AA^*$

- A normal and upper triangle matrix is a diagonal matrix.

Useful Matrices

- Submatrix of matrix: A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[1], [2], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \quad 5], \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, A. \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

❑ Zero or null Matrix

If $A \in M_{m\times m}$, and c is a scalar,

then (1) $A + 0_{m\times n} = A$

※ So, $\mathbf{0}_{m\times n}$ is also called the additive identity for the set of all $m\times n$ matrices

(2) $A + (-A) = 0_{m\times n}$

※ Thus , $-A$ is called the additive inverse of A

(3) $cA = 0_{m\times n} \Rightarrow\ c = 0$ or $A = 0_{m\times n}$

All above properties are very similar to the counterpart properties for the real number 0

Useful Matrices

- **Block Matrix** whose entries are matrices, such as

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \to submatrix\ or\ block\ of\ A$$

$$B = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}, C = [-1], D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix}, E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

then

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}$$

- Matrices in each block row must have same height (row dimension)
- Matrices in each block column must have same width (column dimension)
- Note: A is not a square matrix but it is a block square matrix

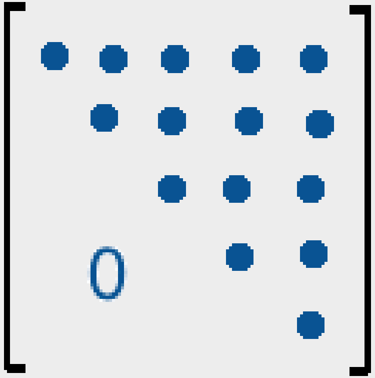
Block Matrix

- Transpose of block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$
- Multiplication

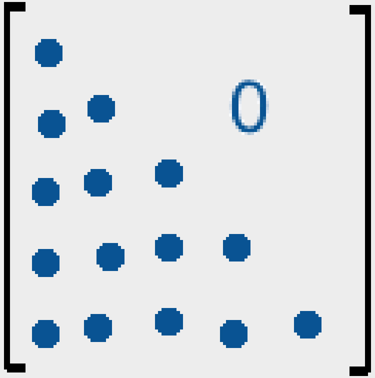
$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ \hline 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ \hline -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ \hline -6 & 2 \\ 2 & 1 \end{array}\right]$$

- **Triangular matrix**
  - Upper triangular $a_{ij} = 0,\ i > j$
  - Lower triangular $a_{ij} = 0,\ i < j$



Upper Triangular Matrix



Lower Triangular Matrix

Useful Matrices

- **Sparse matrix**
  - Density of matrix $A_{m\times n}$
  - Density of identity matrix?
  - Sparse matrix has low density

$$1 \geq \frac{nnz(A)}{mn}$$

Useful Matrices

- Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = \begin{cases} 1 & if\ i = j+1 \\ 0 & other \end{cases}$. Then $A^n = 0$ and $A^k \neq 0$ for $1 \leq k \leq n-1$
- Nilpotent: A for which a positive integer p exists such that $A^p = 0$.
- Order of nilpotency (degree , index): Least positive integer p for which $A^p = 0$ is called the.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} B^2 = \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Useful Matrices

- **Idempotent**: satisfy the condition that $A^2 = A$

Example

2 x 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

3 x 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Note

If a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is idempotent, then

- $a = a^2 + bc$,
- $b = ab + bd$, implying $b(1 - a - d) = 0$ so $d = 1 - a$,
- $c = ca +$ cd, implying $c(1 - a - d) = 0$ so $d = 1 - a$,
- d= $d^2 + bc$

Useful Matrices

- Toeplitz: diagonal-constant matrix: values on diagonals are equal
- A Toeplitz matrix is not necessarily square.

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix} \quad A_{i,j} = A_{i+1,j+1} = a_{i-j}$$

$$T(b) = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & b_2 & b_1 & 0 \\ 0 & b_3 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$$

Permutation Matrix

- A square $n \times n$ matrix (P) obtained by rearranging the rows of I_n
- Permutation matrix is orthogonal $(PP^T = I)$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - How many possible permutation matrix?
  - A product of permutation matrices is again a permutation matrix
  - Some power of a permutation matrix is identity. Why? $(e.g: p^3 = I)$
  - The inverse of a permutation matrix is again a permutation matrix

Permutation Matrix Application

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Interchange the columns of matrix B: $P_{ij} = 1$ column i is moved to column j

$$BP = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 3 & 0 \\ 7 & 6 & 5 \end{bmatrix}$$

- Interchange the rows of matrix B: $P_{ij} = 1$ row j is moved to row i

$$PB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 0 & 3 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

Vec Operator

- The vec-operator applied on a matrix A stacks the columns into a vector

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad vec(A) = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{12} \\ A_{22} \end{bmatrix}$$

- Properties:

$$vec(AXB) = (B^T \otimes A)vec(X)$$
$$Tr(A^T B) = vec(A)^T vec(B)$$
$$vec(A + B) = vec(A) + vec(B)$$
$$vec(\alpha A) = \alpha . vec(A)$$
$$a^T XBX^T c = vec(X)^T (B \otimes ca^T) vec(X)$$

Conclusion

Real Case	Complex Case
$u.v = u^T v = v^T u$	$u.v = v^* u$
Transpose $(\,)^T$	Conjugate transpose $(\,)^*$
Orthogonal matrix $AA^T = I$	Unitary matrix $UU^* = I$
Symmetric matrix $A = A^T$	Hermitian matrix $H = H^*$