Theorem (6). *The pivot columns of a matrix* $\mathbf{A}$ *form a basis for the column space* $Col(\mathbf{A})$.

Proof. The proof has two parts: show the pivot columns are linearly independent and show the pivot columns span the column space.

We use the reduced echelon matrix of $\mathbf{A}$ in the proof. We designate it by $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ have r pivot columns, then the pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$. For example, consider the reduced echelon matrix

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has pivot columns $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ in its first, third, and fourth columns. These are vectors in $\mathbb{R}^4$.

The solutions of the homogeneous equations $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$ are the same. Therefore, the relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same. (Any non-zero solution of these homogeneous equations gives a relation among these columns.)

Lemma 1. *(a) The pivot columns of* $\mathbf{B}$ *are linearly independent.*
(b) The pivot columns of $\mathbf{A}$ *are linearly independent.*

Proof. (a) The pivot columns of $\mathbf{B}$ are the standard vectors $\mathbf{e}^1, \ldots, \mathbf{e}^r$ in $\mathbb{R}^m$ that are linearly independent. (In the example of the matrix given above, $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ are linearly independent in $\mathbb{R}^4$.)

(b) The relations among the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same (as noted above), so the pivot columns of $\mathbf{A}$ are also linearly independent. □

Lemma 2. *The pivot columns of* $\mathbf{A}$ *span the column space of* $\mathbf{A}$

Proof. Let $\mathbf{b}^k$ be a non-pivot column of the reduced echelon matrix $\mathbf{B}$. Assume that there are j pivot columns to the left of $\mathbf{b}^k$ in $\mathbf{B}$. These pivot columns must be $\mathbf{e}^1, \ldots, \mathbf{e}^j$. The non-pivot column $\mathbf{b}^k$ can only have nonzero entries in the first j components (or it would be a pivot column) and so is a linear combination of $\mathbf{e}^1, \ldots, \mathbf{e}^j$.

Since $\mathbf{A}$ has the same relations among its columns as $\mathbf{B}$, its non-pivot column $\mathbf{a}^k$ is a linear combination of the j pivot columns to the left of it. By Theorem 5(a), the span of the pivot columns is the same as the span of all the columns. This shows that the pivot columns of $\mathbf{A}$ span the column space of $\mathbf{A}$. □

By Lemma 1(b), the pivot columns are linearly independent. By Lemma 2, the pivot columns span the column space of $\mathbf{A}$. Together, these two facts show that the pivot columns form a basis for the column space of $\mathbf{A}$. □

The row space

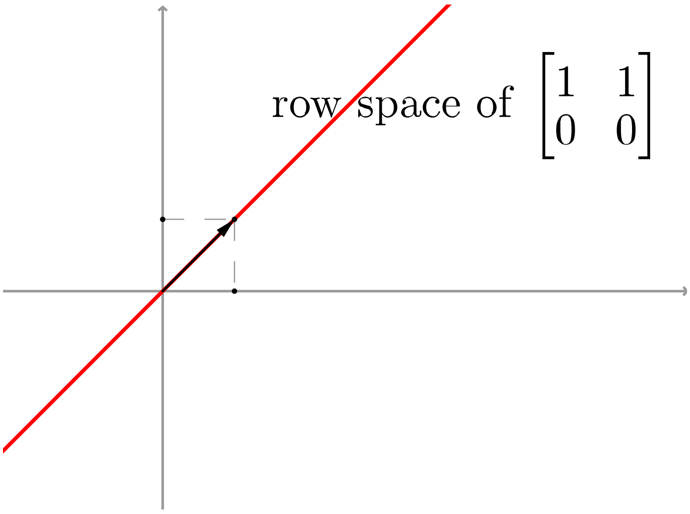
The *row space* of a matrix is the collection of all linear combinations of its rows.

Equivalently, the row space is the span of rows.

The elements of a row space are *row* vectors.

If a matrix has m columns, its row space is a subspace of (the row version of) $\mathbb{R}^m$.

EXAMPLE The row space of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ consists of all linear combinations of the form $c_1 \begin{bmatrix} 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \end{bmatrix}$. It is thus the span of $\begin{bmatrix} 1 & 1 \end{bmatrix}$. If we identify the row vectors $\begin{bmatrix} x & y \end{bmatrix}$ with points (x, y), we may picture this span as follows:



Elementary row operations **do not** alter the row space.

Thus a matrix and its echelon form have the same row space.

The pivot rows of an echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

The pivot rows of an echelon form span the row space of the original matrix.

The dimension of the row space is given by the number of pivot rows.

This dimension does not exceed the total row count.

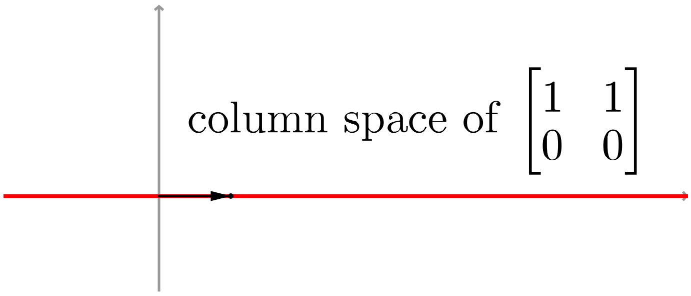
The column space

The *column space* of a matrix is the collection of all linear combinations of its columns.

It is the span of columns, the range of the linear transformation carried out by the matrix.

If a matrix has n rows, its column space is a subspace of $\mathbb{R}^n$.

EXAMPLE The column space of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ consists of all linear combinations of the form $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; it is the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If we identify the column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with points (x, y), we may picture this span as follows:



Elementary row operations **affect** the column space.

So, generally, a matrix and its echelon form have different column spaces.

However, since the row operations preserve the linear relations between columns,

the columns of an echelon form and the original columns obey the *same* relations.

The pivot columns of a reduced row-echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

The pivot columns of a reduced row-echelon form **span** its column space.

So the pivot columns of a matrix are linearly independent and span its column space.

The dimension of the column space is given by the number of pivot columns.

This dimension does not exceed the total column count.

The null space

The *null space* (or *kernel*) of a matrix A consists of all vectors x such that $Ax = 0$.

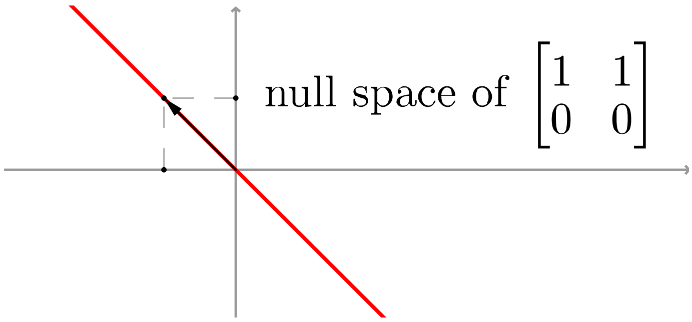
It is the preimage of the zero vector under the transformation carried out by A.

If A has m columns, its null space is a subspace of $\mathbb{R}^m$.

If the columns are linearly independent, the null space consists of just the zero vector.

EXAMPLE Every solution to $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ is of the form $\begin{bmatrix}-y\\y\end{bmatrix}$ for some y.

So the null space of $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ consists of all multiples of $\begin{bmatrix}-1\\1\end{bmatrix}$; it is the span of $\begin{bmatrix}-1\\1\end{bmatrix}$.



Elementary row operations preserve the null space.

A matrix and its echelon form have the same null space.

Consider the general solution to $Ax = 0$ expressed in terms of free variables.

Setting all but one free variable to zero, we obtain a vector in the null space of A.

The vectors obtained in this way are linearly independent and span the null space.

The dimension of the null space is given by the number of free variables,

which is the same as the number of non-pivot columns.

Conclusions

For any matrix, the row space dimension agrees with the column space dimension.

Both are given by the number of pivots. This is the *rank* of a matrix.

The null space dimension and the rank add up to the total number of columns.