Matrix Rank

Linear Algebra

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Overview

- Row and Column Space
- Basis of Row and Column Space
- Rank

Row and Column Space

Review: Matrix-Vector Multiplication

- If we write A by rows, then we can express Ax as,

$$A \in \mathbb{R}^{m\times n} \qquad y = Ax = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}. \qquad a_i^T x = \sum_{j=1}^{n} a_{ij} x$$

- If we write A by columns, then we have:

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1]x_1 + [a_2]x_2 + \cdots + [a_n]x_n \,.$$

  - y is a **linear combination** of the columns A.

columns of A are linearly independent if Ax = 0 implies x = 0

It is also possible to multiply on the left by a row vector.

- If we write A by columns, then we can express $x^T A$ as,

$$y^T = x^T A = x^T \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x^T a_1 & x^T a_2 & \cdots & x^T a_n \end{bmatrix}$$

- expressing A in terms of rows we have:

$$y^T = x^T A = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

$$= x_1 \begin{bmatrix} - & a_1^T & - \end{bmatrix} + x_2 \begin{bmatrix} - & a_2^T & - \end{bmatrix} + \cdots + x_m \begin{bmatrix} - & a_m^T & - \end{bmatrix}$$

- y^T is a linear combination of the rows of A.

1. As a set of vector-vector products

$$C = AB = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

2. As a sum of outer products

$$C = AB = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^{n} a_i b_i^T$$

Row and Column Space

Definition

Let A be a $m \times n$ matrix. Then the **column space** *of A is $\mathcal{C}(A)$ is*

$$\mathcal{C}(A) := \{Ax : x \in \mathbb{R}^n\}$$

and the **row space** *of A is*

$$\mathcal{R}(A) := \{y^T A : y \in \mathbb{R}^m\}.$$

Row Space

❑ Its think about the following facts and proof them:

The *row space* of a matrix is the collection of all linear combinations of its rows.

Equivalently, the row space is the span of rows.

The elements of a row space are *row* vectors.

If a matrix has m columns, its row space is a subspace of (the row version of) $\mathbb{R}^m$

Elementary row operations **do not** alter the row space.

Thus a matrix and its echelon form have the same row space.

The pivot rows of an echelon form are **linearly independent**.

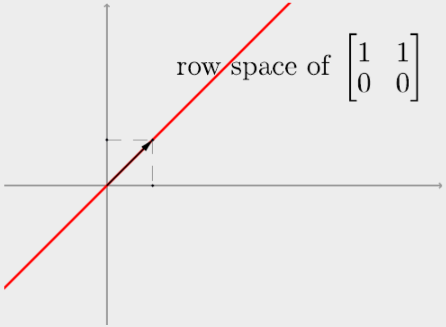
$$\begin{bmatrix} 1 & * & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

The pivot rows of an echelon form span the row space of the original matrix.

The dimension of the row space is given by the number of pivot rows.

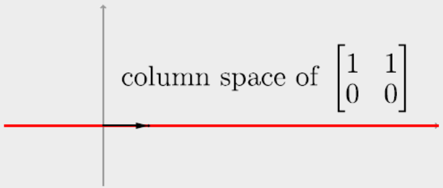
This dimension does not exceed the total row count.

$$c_1 \begin{bmatrix} 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \end{bmatrix}$$



Column Space

❑ Its think about the following facts and proof them:



$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The *column space* of a matrix is the collection of all linear combinations of its columns.

It is the span of columns, the range of the linear transformation carried out by the matrix.

If a matrix has n rows, its column space is a subspace of $\mathbb{R}^n$.

Elementary row operations **affect** the column space.

So, generally, a matrix and its echelon form have different column spaces.

However, since the row operations preserve the linear relations between columns,

the columns of an echelon form and the original columns obey the *same* relations.

Example in next page!

The pivot columns of a reduced row-echelon form are **linearly independent**.

$$\begin{bmatrix} 1 & * & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

The pivot columns of a reduced row-echelon form **span** its column space.

So the pivot columns of a matrix are linearly independent and span its column space.

The dimension of the column space is given by the number of pivot columns.

This dimension does not exceed the total column count.

Example

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 1 & 3 & 5 & 8 & 11 \\ 4 & 10 & 16 & 23 & 30 \end{pmatrix} \qquad B_{\text{rref}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= x_3 - x_5, \\ x_2 &= -2x_3 + 2x_5, \\ x_4 &= -2x_5. \end{aligned} \qquad \mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

The column space of B is 3-dimensional, and that a basis is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 8 \\ 23 \end{bmatrix} \right\}$

Note that we do not use the columns of B_{rref}! We use the columns of B.

Basis of Row and Column Space

Theorem

If two matrices $A \, and \, B$ are row-equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows (pivot rows) of B form a basis for the row space of A as well as for that of B.

Review (Reduced Echelon Form (RREF))

Definition

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
  1. The leading entry in each non-zero row is 1.
  2. Each leading 1 is the only non-zero entry in its columns.
  3. The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
  4. Any row containing only 0's is at the bottom.

$$\underbrace{\begin{array}{cccc} e_1 & e_2 & e_3 & \\ \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \begin{matrix}29\\16\\3\end{matrix} \end{array}}_{\text{Reduced Echelon form}}$$

Row and Column Space

Theorem

The pivot columns of a matrix A form a basis for $Col(A)$

- Lemma1: The pivot columns of A are linearly independent
- Lemma 2. The pivot columns of A span the column space of A
- From first lectures we know that “The span of the pivot columns is the same as the span of all the columns”

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Find:
- ❑ Row Basis
- ❑ Column Basis
- ❑ $\dim(Row(A))$
- ❑ $\dim(Col(A))$
- ❑ $\dim(Null(A))$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = pivot columns

Rank

Definition

- We call $\dim(R(A))$ the row rank of A and $\dim(C(A))$ the column rank of A.
- We refer to a basis of $C(A)$ consisting of columns of A as a column basis. A row basis is defined similarly.

Rank of Matrix

Definition

- ❑ The number of linearly independent rows or columns in the matrix
- ❑ Dimension of the row (column) space
- ❑ Number of nonzero rows of the matrix in row echelon form (Ref)

Note

The dimension of the Column Space of A and $rref(A)$ is the same.

Example 1

If Columns of matrix A are linearly independent:

$$\begin{aligned} nullity(A) &= ? \\ colrank(A) &= ? \end{aligned}$$

Example 2

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$nullity(A) = 2, colRank(A) = 2$$

Conclusion

Note

The dimension of $Null(A)$ is the number of free variables in the equation $Ax = 0$, and the dimension of $Col(A)$ is the number of pivot columns in A.

Example

Back to Example!

Find the dimension of the Null Space and the Column Space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(row reduce the Augmented Matrix $[A\ 0]$ to echelon form)

Rank–Nullity Theorem

$\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W} \quad \operatorname{Dim}(V) = Nullity(T) + Dim(range(T))$

Important Note!!! (WHY?)

Column Space (A) = Range(T)

Dim(range(T)) = ColumnRank(A)

Theorem

- ❑ $Nullity(A) \ + \ ColRank(A) \ = \ n$
- ❑ $Dim(Null(A)) \ + \ Dim(Range(A)) \ = \ n$

$$\{ number\ of\ pivot\ columns\} + \{number\ of\ non - pivot\ columns\} = \{number\ of\ columns\}$$

Theorem

- Let A denote an m × n matrix of rank r. Then:

  the n−r basic solutions to the system Ax =0 provided by the gaussian algorithm are a basis of null (A), so dim[null (A)] = n−r.

Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, Ax = \begin{bmatrix} x_2 + x_3 + 2x_4 \\ x_1 + 2x_3 + x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$nullity(A) = 2, colRank(A) = 2$ Nullspace(A)= $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

RMRT

Theorem (RMRT)

(Rank of a matrix is equal to the rank of its transpose)

Suppose A is an $m \times n$ matrix.

Then col$rank(A) = colrank\ (A^T)$

Proof?

Lemma

- ❑ $Ax = 0 \quad \leftrightarrow \quad Ax = 0$
- ❑ $\mathrm{ColRank}\left(\mathrm{A}^{\mathrm{T}}\mathrm{A}\right) = \mathrm{ColRank}(\mathrm{A})$
- ❑ $\mathrm{ColRank}(\mathrm{A}) = \mathrm{ColRank}(\mathrm{A}^{\mathrm{T}})$

Proof?

Rank Theorem

Theorem

- $ColRank(A) = RowRank(A)$
- In general, It's called rank of matrix ($rank(A)$)

Proof?