Matrix Inverse

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

- Left Inverse
- Right Inverse
- Square Matrix Inverse

Left Inverse

Left Inverse

Definition

- A number x that satisfies $xa = 1$ is called the inverse of a
- Inverse (i.e., $\frac{1}{a}$) exists if and only if $a \neq 0$, and is unique
- A matrix X that satisfies $XA = I$ is called a left inverse of A
- If a left inverse exists we say that A is left-invertible
- $A: m \times n \Longrightarrow I: n \times n \Longrightarrow X: n \times m$

Example

The matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$

Has two different left inverses:

$$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \qquad C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

Solving linear equations with a left inverse

Method

- Suppose $Ax = b$, and A has a left inverse C
- Then $Cb = C(Ax) = (CA)x = Ix = x$
- So multiplying the right-hand side by a left inverse yields the solution

Left inverse of vector

Note

- ❑ A non-zero column vector always has a left inverse.
- ❑ Left inverse is not unique.

Example

- ❑ $A = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$
- ❑ Matrix with orthonormal columns

Definition

- ❑ Row vector does not have left inverse

$$A = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$$

Left inverse and column independence

Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof

Left inverse and column independence

Definition

- If A has a left inverse C then the columns of A are linearly independent
- We'll see later that the converse is also true, so:

A matrix is left-invertible if and only if its columns are linearly independent

- Matrix generalization of

A number is invertible if and only if it is nonzero

From Previous Theorem

Left-invertible matrices are all tall or square

- Wide matrix is not always left invertible
- Tall or square matrices can be left invertible

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Right Inverse

Right inverses

Definition

- A matrix X that satisfies $AX = I$ is a right inverse of A
- If a right inverse exists we say that A is right-invertible
- A is right-invertible if and only if A^T is left-invertible:

$$AX = I \Longrightarrow (AX)^T = I \Longrightarrow X^T A^T = I$$

- so we conclude:

A is right invertible if and only if its rows are linearly independent

- Right-invertible matrices are wide or square

Solving linear equations with a right inverse

Method

- Suppose A has a right inverse B
- Consider the (square or underdetermined) equations of $Ax = b$
- $x = Bb$ is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- So $Ax = b$ has a solution for any b

Example

- Same A, B, C in last example.
- C^T and B^T are both right inverses of A^T
- Under-determined equations $A^T x = (1,2)$ has (different) solutions.

$$B^T(1,2) = (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}), \quad C^T(1,2) = \left(0, \tfrac{1}{2}, -1\right)$$

there are many other solutions as well

Conclusion: Left and Right Inverse

Linear equations and matrix inverse

Definition

Left-Invertible matrix: if X is a left inverse of A, then

$$Ax = b \Longrightarrow x = XAx = Xb$$

There is at most one solution using X (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

Why "at most"??

$XA = I$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases} \qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Linear equations and matrix inverse

Note

- If the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, and if a matrix $\boldsymbol{B}$ exists such that $\boldsymbol{BA} = \boldsymbol{I}$, then the system of equations has a unique solution, namely $\boldsymbol{x} = \boldsymbol{Bb}$.
- **Right-inversible matrix:** if X is a right inverse of A, then there is at least one solution (x=Xb):

$$x = Xb \implies Ax = AXb = b$$

- To pursue these ides further, suppose that again we want to solve a system of linear equations, $\boldsymbol{Ax} = \boldsymbol{b}$. Assume now that we have another matrix, $\boldsymbol{B}$, such that $\boldsymbol{AB} = \boldsymbol{I}$. Then we can write $\boldsymbol{A}(\boldsymbol{Bb}) = (\boldsymbol{AB})\boldsymbol{b} = \boldsymbol{Ib} = \boldsymbol{b}$; hence $\boldsymbol{Bb}$ solves the equations $\boldsymbol{Ax} = \boldsymbol{b}$. This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether $\boldsymbol{Bb}$ is the only solution. There may be others.
- **Invertible matrix:** if A is invertible, then

$$\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

  There is a unique solution

Conclusion

- System of linear equations $Ax = b$:
  - A right inverse of A, say $AB = I$. Then Bb is a solution, as is verified by nothing $A(Bb) = (AB)b = Ib = b$.
  - A left inverse of A, say $\mathrm{CA} = I$, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

Square Matrix Inverse

Inverse

Definition

For $A \in M_{n\times n}$, if there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$, then:

- ❑ A is invertible (or nonsingular)
- ❑ B is the inverse of A
- ❑ The inverse of A is denoted by $B = A^{-1}$

A square matrix that does not have an inverse is called non-invertible (or singular)

For a square matrix left and right inverse are the same. Rows and columns are linear independent.

Theorem

The inverse of a matrix is unique

Invertible Matrices

$$
\begin{tikzcd}
\text{left-invertible} \arrow[r, Rightarrow, "\text{(a)}", "\text{Holds in general}"'] & \text{linearly independent columns} \arrow[d, Rightarrow, "\text{(b)}", "\text{Holds for square matrices}"'] \\
\text{linearly independent rows} \arrow[u, Rightarrow, "\text{(b')}"', "\text{Holds for square matrices}"] & \text{right-invertible} \arrow[l, Rightarrow, "\text{(a')}"', "\text{Holds in general}"]
\end{tikzcd}
$$

Method

- Let A be a $n \times n$ matrix:
  - Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
  - Compute the reduced echelon form of $[A : I_n]$.
- If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.
- If the reduced echelon form is not the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n then A has no inverse.

$$[A \quad | \quad I] \text{ Gauss—Jordan elimination } [I \quad | \quad A^{-1}]$$

Important

An $n \times n$ matrix is invertible if and only if its reduced echelon form is I_n.

A is row equivalent to I_n

Inverse (Example)

Example

Find inverse of the following matrix using Gauss–Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \Rightarrow \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad (1)$$
$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases} \quad (2)$$

This two system of linear equations have the same coefficient matrix, which is exactly the matrix A

Inverse (Example)

Rest of The Example

Using Gauss–Jordan Elimination on the matrix A with the same row operations

$$(1) \Rightarrow \begin{bmatrix} 1 & 4 & | & 1 \\ -1 & -3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \begin{bmatrix} 1 & 4 & | & 0 \\ -1 & -3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -4 \\ 0 & 1 & | & 1 \end{bmatrix} \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus $X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Solution for $\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$ Solution for $\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}}_{A} \begin{matrix} | \\ | \end{matrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \xrightarrow{Guass-Jordan\ elimination} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \begin{matrix} | \\ | \end{matrix} \underbrace{\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}}_{A^{-1}}$$

Inverse

Definition

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- A^k is invertible and $\left(A^k\right)^{-1} = A^{-k} = (A^{-1})^k$
- cA is invertible if $c \neq 0$ and $(cA)^{-1} = \frac{1}{c}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Theorem

If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1A_2A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

Inverse

Theorem

Let $AX = B$ be a system of n linear equations in n variable.
If A^{-1} exists, the solution is unique and is given by $X = A^{-1}B$

Invertible Matrix

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Note

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$.

2×2 matrix A is invertible if and only if $\det A \neq 0$.

Elementary Matrices

Definition

Each Elementary Matrix is E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Solving square systems of linear equations

Method

- Suppose A is invertible
- For any b, $Ax = b$ has the unique solution

$$x = A^{-1}b$$

- Matrix generalization of simple scalar equation $ax = b$ having solution $x = \left(\frac{1}{a}\right)b$ (for $a \neq 0$)
- Simple-looking formula $x = A^{-1}b$ is basis for many applications

Invertible (Nonsingular) matrices

Conclusion

The following are equivalent for a square matrix A:

- A is invertible
- Columns of A are linearly independent
- Rows of A are linearly independent
- A has a left inverse
- A has a right inverse

$$row\, rank(A) = col\, rank(A) = n$$

If any of these hold, all others do

Examples

- ❑ $I^{-1} = I$
- ❑ **If Q is orthogonal, i.e. , square with $Q^TQ = I$, then $Q^{-1} = Q^T$**
- ❑ 2×2 matrix A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

  - You need to know this formula
  - There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)

- ❑ Consider matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$
  - ➢ A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

  - ➢ Verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)
  - ➢ We'll soon see how to compute the inverse

Properties

- ❑ $(AB)^{-1} = B^{-1}A^{-1}$
- ❑ If A is nonsingular, then A^T is nonsingular

$$(A^T)^{-1} = (A^{-1})^T \text{ (sometimes denoted } A^{-T}\text{)}$$

- ❑ Negative matrix powers: $(A^{-1})^k$ is denoted by A^{-k}
- ❑ With $A^0 = I$, Identity $A^k A^l = A^{k+l}$ holds for any integers k, l

Triangular matrices

Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??

Question?

Why Matrix of Change of Basis is invertible?