Determinant

CE40282-1: Linear Algebra
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Multilinear Forms

Suppose $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function $f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$ is called a **multilinear form** if, for each $1 \leq j \leq p$ and each $\mathbf{v}_1 \in \mathcal{V}_1, \mathbf{v}_2 \in \mathcal{V}_2, \ldots, \mathbf{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathbf{v}) = f(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_p) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}_j$$

is a linear form.

Recall

- **Definition**

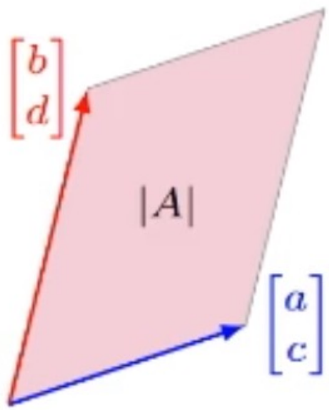
  The **determinant** of a 2×2 matrix $A = [a_{ij}]$ is the number
  $$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

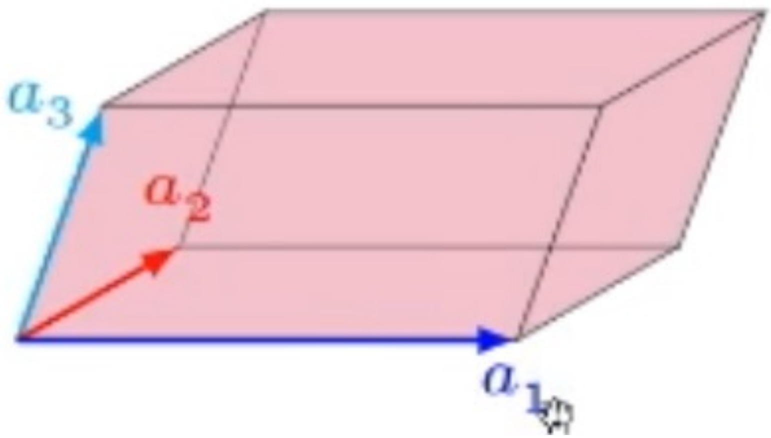
  - A 2×2 matrix is invertible if and only if its determinant is nonzero.

- The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

Determinants as Area or Volume

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $\det A$.
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $\det A$.





Geometric interpretation

- $$\frac{\text{volume of output region}}{\text{volume of input region}}.$$

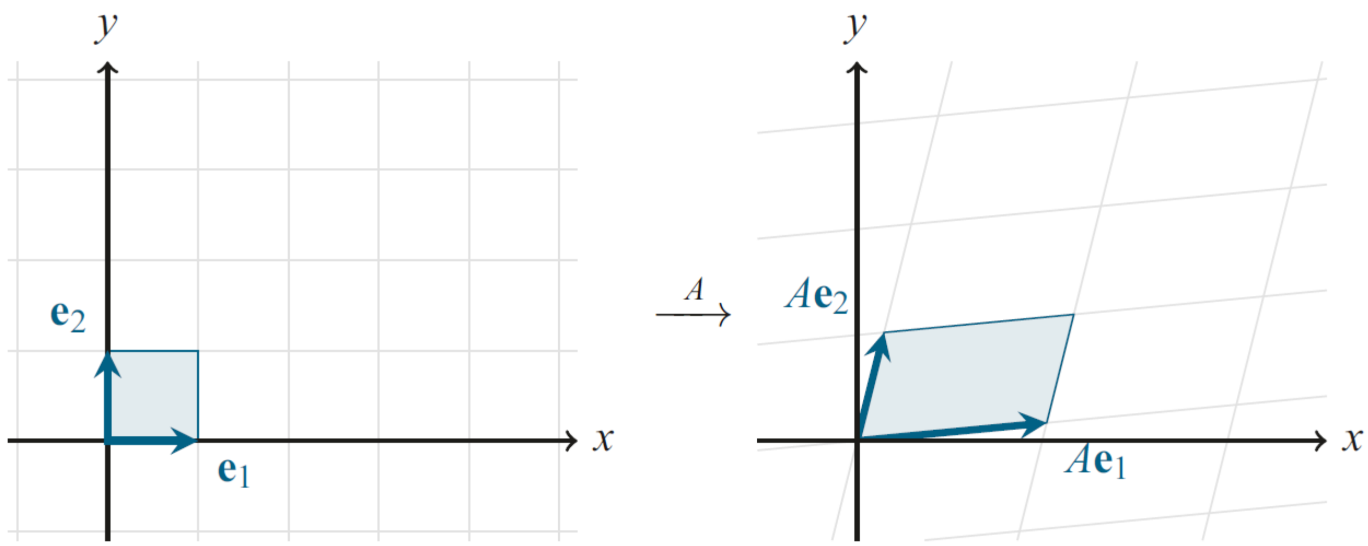


Figure A.3: A 2×2 matrix A stretches the unit square (with sides $\mathbf{e}_1$ and $\mathbf{e}_2$) into a parallelogram with sides $A\mathbf{e}_1$ and $A\mathbf{e}_2$ (the columns of A). The determinant of A is the area of this parallelogram.

Definition of Submatrix Aij

- **Definition**

  For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A

For instance, if

$$A = \begin{pmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{pmatrix}$$

Recursive Definition of Determinant

The determinant of an $n \times n$ matrix $A = [a_{ij}$ is the sum of n terms of the form $\pm a_{1j} \text{det} A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \cdots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \text{det} A &= a_{11} \text{det} A_{11} - a_{12} \text{det} A_{12} + \cdots \\ &\qquad + (-1)^{1+n} a_{1n} \text{det} A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \text{det} A_{1j} \end{aligned}$$

Recursive Definition of Determinant

- 2×2 matrix

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$= a \begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b \begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

- 3×3 matrix

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$

Cofactor

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n},$$

which is a **cofactor expansion across the first row** of A.

Cofactor Expansion

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion **across any row** or **down any column**. The expansion across the ith row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor Expansion

- Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

Properties

- If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

  - Determinant of zero matrix is:

Properties

- If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is:

Properties

- U is unitary, so that $|\det(U)| = I$

Properties

- If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$$

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = ka_{11}C_{11} + \cdots + ka_{1n}C_{1n} = k\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n |A_{n\times n}|$
- If a row/column is multiple of another row/column then determinant is .....

Properties

- Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then detB = detA.
  - Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

Properties

- If columns/rows of matrix are linear dependent if and only if its determinant is zero
  - Proof?

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Echelon form

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Determinant of Transpose

Theorem

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

- **Theorem**
  If A and B are $n \times n$ matrices, then
  $\det AB = \det A \det B$.

- **Warning**
  In general, $\det(A + B) \neq \det A + \det B$.

- The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \implies |AA^{-1}| = |I| = 1 \implies |A||A^{-1}| = 1 \implies |A^{-1}| = |A|^{-1}$$

- The determinant of orthogonal matrix is ……

Determinant via QR Decomposition

- If $A \in \mathcal{M}_n$ has QR decomposition $A = UT$ with $U \in \mathcal{M}_n$ unitary and $T \in \mathcal{M}_n$ upper triangular, then

$$|\det(A)| = t_{1,1} \cdot t_{2,2} \cdots t_{n,n}.$$

Cramer's Rule

- Ax=b and A is invertible

$$A = [a_1 \quad \cdots \quad a_n] \qquad I = [e_1 \quad \cdots \quad e_n]$$

$$AI = A \implies A[e_1 \quad \cdots \quad e_n] = [Ae_1 \quad \cdots \quad Ae_n] = [a_1 \quad \cdots \quad a_n]$$

$$A\overbrace{[e_1 \quad e_2 \quad \cdots \quad x \quad \cdots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \cdots \quad Ax \quad \cdots \quad Ae_n]$$
$$= \underbrace{[a_1 \quad a_2 \quad \cdots \quad b \quad \cdots \quad a_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

Cramer's Rule

- **Cramer's Rule**

  Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

  $$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n$$

- Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \implies x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j-th column of A^{-1} is a vector x that satisfies

$$Ax = e_j$$

By Cramer's rule

$$\{(i,j) - \text{entry of } A^{-1}\} = x_i = \frac{\det A_i(e_j)}{\det A}$$

$$\det A_i(e_j) = (-1)^{i+j} \det A_{ji}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the **adjugate** (or **classical adjoint**) of A, denoted by adj$A$.

A Formula for A^{-1}

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\text{adj}A$$

$$\left.\begin{array}{l} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{array}\right\} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant via QR Decomposition

If $A \in \mathcal{M}_n$ has QR decomposition $A = UT$ with $U \in \mathcal{M}_n$ unitary and $\dot{T} \in \mathcal{M}_n$ upper triangular, then

$$|\det(A)| = t_{1,1} \cdot t_{2,2} \cdots t_{n,n}.$$

- Example

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

ıas QR decomposition $A = UT$ with

$$U = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Transformation

Show that the determinant, $\det : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$, is not a linear transformation when $n \geq 2$.

Transformations

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

Resource

- Chapter 3 Linear Algebra and Its Applications David C. Lay
- Nathaniel Johnston - Advanced Linear and Matrix Algebra-Springer (2021)