Solutions and Conservation Laws of the Modified Burgers-KdV Equation

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by

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Dedication

To my mother, Francinah Mandundu, for her unwavering love and support, and to the entire Mandundu family for their constant encouragement.

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I would like to thank Dr L.D Moleleki for the support, guidance and patience in compiling this project. I would also like to thank him for his input and perseverance during the difficult times on this project.

Abstract

In this project, we solve the modified Burgers-KdV equation from the research paper titled ”Lie Symmetry-Based Analytical and Numerical Approach for the Modified Burgers-KdV Equation” by Vikas Kumar, Lakhveer Kaur, Ajar Kumar, and Mehmet Emir Koksal. The focus of this project is to apply the Lie point symmetries together with the simplest equation method to obtain the exact solution of the modified Burgers-KdV equation. Additionally, conservation laws are derived using the multiplier method.

Introduction

Symmetry is one of the most important concepts in the area of partial differential equations (PDEs) especially in integrable systems which exist infinitely many symmetries. The project investigate the invariance properties of the Korteweg-de Vries Burger's equation through Lie symmetry analysis. Furthermore, several reductions which resulted in group invariant solutions of the Korteweg-de Vries equation are obtained. Lastly multiplier method will be used to construct the conservation laws of the KdVB equation.

The detailed outline of the project is as follows.

In Chapter one we give a brief introduction to Lie group analysis of PDEs and useful formulas to be used throughout the project.

In Chapter two we use the heat equation to illustrate how we determine the Lie point symmetry of a partial differential equation. We also show how we can reduce a partial differential equation to an ordinary differential equation to obtain an invariant solution of the original equation.

In Chapter three we determine the Lie point symmetry of the modified Burgers Korteweg-de Vries (mBKdV) equation and use simplest equation method to obtain exact solutions. Lastly, we employ the multiplier method to find the conservation laws of the modified Burgers-KdV equation.

In Chapter four we summarise the work done in this project.

Chapter 1

Lie symmetry methods for differential equations

In this Chapter, we provide some basic methods of Lie symmetry analysis of differential equations including the algorithm to determine the Lie point symmetries of PDEs are given. The fundamental operators and their relations will also be presented.

1.1 Introduction

Sophus Lie (1842-1899) a Norwegian mathematician established the Lie's theorem in the nineteenth century. Lie's theorem is one the most significant and powerful techniques for obtaining a close-form solutions of differential equations. Lately, many good books have appeared in the literature in this field such as Ibragimov [1], Ovsianikov [2] , Olver [3], Bluman and Kumei [4], Stephani [5], Ibragimov [1, 6], Cantwell [7] and Mohamed [8].

Definitions and results given in this Chapter are taken from the books mentioned above.

1.2 Continuous one-parameter (local) Lie group

In this section a transformation will be understood to mean an invertible transformation, that is, a bijective map. Let t and x be two independent variables and u be a dependent variable. Consider a change of the variables t, x and u:

$$T_a : \bar{t} = f(t, x, u, a),\ \ \bar{x} = g(t, x, u, a),\ \ \bar{u} = h(t, x, u, a) \tag{1.1}$$

where a is a real parameter which continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f, g and h are differentiable functions.

Definition 1.1 A set G of transformations equation (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables t, x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b\, T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\, T_a = T_a\, T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\, T_{a^{-1}} = T_{a^{-1}}\, T_a = T_0$ (Inverse)

It follows from (i) that the associativity property is satisfied. Also, if the identity transformation occurs at $a = a_0 \neq 0$, i.e., $T_{a_0}$is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give T_0 as above. The group property (i) can be written as

$$\begin{aligned}
\bar{\bar{t}} &\equiv f(\bar{t}, \bar{x}, \bar{u}, b) = f(t, x, u, \phi(a, b)), \\
\bar{\bar{x}} &\equiv g(\bar{t}, \bar{x}, \bar{u}, b) = g(t, x, u, \phi(a, b)), \\
\bar{\bar{u}} &\equiv h(\bar{t}, \bar{x}, \bar{u}, b) = h(t, x, u, \phi(a, b)).
\end{aligned} \tag{1.2}$$

The function ϕ is termed as the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial\, \phi(s, b)}{\partial b} \right|_{b=0}.$$

Let us now give the definition of a symmetry group for PDEs by considering, for example, evolutionary equations of the second-order, namely

$$u_t = F(t, x, u, u_x, u_{xx}), \quad \frac{\partial F}{\partial u_{xx}} \neq 0. \tag{1.3}$$

Definition 1.2 (Symmetry group) A one-parameter group G of transformations equation (1.1) is called a symmetry group of equation (1.3) if equation (1.3) is form-invariant (has the same form) in the new variables $\bar{t}, \bar{x}$ and $\bar{u}$, i.e.,

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}), \tag{1.4}$$

where the function F is the same as in equation (1.3).

1.3 Infinitesimal transformations

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{t} \approx t + a\, \tau(t, x, u), \;\; \bar{x} \approx x + a\, \xi(t, x, u), \;\; \bar{u} \approx u + a\, \eta(t, x, u) \tag{1.5}$$

obtained from equation (1.1) by expanding the functions f, g and h into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f|_{a=0} = t, \quad g|_{a=0} = x, \quad h|_{a=0} = u\,.$$

Thus, we have

$$\tau(t, x, u) = \left. \frac{\partial f}{\partial a} \right|_{a=0}, \quad \xi(t, x, u) = \left. \frac{\partial g}{\partial a} \right|_{a=0}, \quad \eta(t, x, u) = \left. \frac{\partial h}{\partial a} \right|_{a=0}. \tag{1.6}$$

The vector (τ, ξ, η) with components equation (1.6) is the tangent vector at the point (t, x, u) to the surface curve described by the transformed points $(\bar{t}, \bar{x}, \bar{u})$, and is therefore called the *tangent vector field* of the group G.

One can now introduce the *symbol* of the infinitesimal transformations by writing equation (1.5) as

$$\bar{t} \approx (1 + a\,X)t, \quad \bar{x} \approx (1 + a\,X)x, \quad \bar{u} \approx (1 + a\,X)u\,,$$

where

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}\,. \tag{1.7}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by equation (1.3), we say that X is an *admitted operator* of equation (1.3) or X is an *infinitesimal symmetry* of equation (1.3).

1.4 Group invariants

Definition 1.3 A function $F(t,x,u)$ is called an *invariant of the group of transformation* equation (1.1) if

$$F(\bar{t},\bar{x},\bar{u}) \equiv F(f(t,x,u,a), g(t,x,u,a), h(t,x,u,a)) = F(t,x,u), \tag{1.8}$$

identically in t, x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(t,x,u)$ to be an invariant is that

$$X\,F \equiv \tau(t,x,u)\frac{\partial F}{\partial t} + \xi(t,x,u)\frac{\partial F}{\partial x} + \eta(t,x,u)\frac{\partial F}{\partial u} = 0\,. \tag{1.9}$$

It follows from the above theorem that every one-parameter group of point transformations equation (1.1) has two functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(t,x,u) = c_1, \;\; J_2(t,x,u) = c_2,$$

of the characteristic equations

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}.$$

Theorem 1.3 Given the infinitesimal transformation equation (1.5) or its symbol X, the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}) \tag{1.10}$$

subject to the initial conditions

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}\big|_{a=0} = x, \quad \bar{u}\big|_{a=0} = u\,.$$

1.5 Construction of a symmetry group

In this section we briefly describe the algorithm to determine a symmetry group for a given PDE. First we need to give some basic definitions.

1.5.1 Prolongation of point transformations

Consider a second-order PDE

$$E(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = 0, \tag{1.11}$$

where t and x are two independent variables and u is a dependent variable. Let

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{1.12}$$

be the infinitesimal generator of the one-parameter group G of transformation equation (1.1). The *first prolongation* of X is denoted by $X^{[1]}$ and is defined by

$$X^{[1]} = X + \zeta_1(t, x, u, u_t, u_x)\frac{\partial}{\partial u_t} + \zeta_2(t, x, u, u_t, u_x)\frac{\partial}{\partial u_x},$$

where

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) \end{aligned}$$

and the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots\,, \tag{1.13}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots\,. \tag{1.14}$$

Likewise, the second prolongation of X, denoted by $X^{[2]}$, is given by

$$X^{[2]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}, \tag{1.15}$$

where

$$\begin{aligned}
\zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).
\end{aligned}$$

Using the definitions of D_t and D_x, one can write

$$\zeta_1 = \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u. \tag{1.16}$$

$$\zeta_2 = \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u. \tag{1.17}$$

$$\begin{aligned}
\zeta_{11} &= \eta_{tt} + 2u_t \eta_{tu} + u_{tt} \eta_u + u_t^2 \eta_{uu} - 2u_{tt} \tau_t - u_t \tau_{tt} - 2u_t^2 \tau_{tu} - 3u_t u_{tt} \tau_u - u_t^3 \tau_{uu} \\
&\quad -2u_{tx} \xi_t - u_x \xi_{tt} - 2u_t u_x \xi_{tu} - u_t^2 u_x \xi_{uu} - (u_x u_{tt} + 2u_t u_{tx}) \xi_u.
\end{aligned} \tag{1.18}$$

$$\begin{aligned}
\zeta_{12} &= \eta_{tx} + u_x \eta_{tu} + u_t \eta_{xu} + u_{xt} \eta_u + u_t u_x \eta_{uu} - u_{tx}(\tau_t + \xi_x) - u_t \tau_{tx} - u_{tt} \tau_x \\
&\quad -u_t u_x(\tau_{tu} + \xi_{xu}) - u_t^2 \tau_{xu} - (2u_t u_{tx} + u_x u_{tt}) \tau_u - u_t^2 u_x \tau_{uu} - u_x \xi_{tx} \\
&\quad -u_{xx} \xi_t - u_x^2 \xi_{tu} - (2u_x u_{tx} + u_t u_{xx}) \xi_u - u_t u_x^2 \xi_{uu}.
\end{aligned} \tag{1.19}$$

$$\begin{aligned}
\zeta_{22} &= \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + u_x^2 \eta_{uu} - 2u_{xx} \xi_x - u_x \xi_{xx} - 2u_x^2 \xi_{xu} - 3u_x u_{xx} \xi_u \\
&\quad -u_x^3 \xi_{uu} - 2u_{tx} \tau_x - u_t \tau_{xx} - 2u_t u_x \tau_{xu} - (u_t u_{xx} + 2u_x u_{tx}) \tau_u - u_t u_x^2 \tau_{uu}.
\end{aligned} \tag{1.20}$$

The third prolongation of X, denoted by $X^{[3]}$, is given by

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{222} \frac{\partial}{\partial u_{xxx}} + ..., \tag{1.21}$$

where

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \\
\zeta_{222} &= D_x(\zeta_{22}) - U_{txx} D_x(\tau) - U - xxx D_x(\xi),
\end{aligned}$$

Using the definitions of D_t and D_x, one can write

$$\zeta_1 = \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u. \tag{1.22}$$

$$\zeta_2 = \eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u. \tag{1.23}$$

$$\begin{aligned}\zeta_{11} = {} & \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + u_t^2\eta_{uu} - 2u_{tt}\tau_t - u_t\tau_{tt} - 2u_t^2\tau_{tu} - 3u_tu_{tt}\tau_u \\ & -u_t^3\tau_{uu} - 2u_{tx}\xi_t - u_x\xi_{tt} - 2u_tu_x\xi_{tu} - u_t^2u_x\xi_{uu} \\ & -(u_xu_{tt} + 2u_tu_{tx})\xi_u.\end{aligned} \tag{1.24}$$

$$\begin{aligned}\zeta_{12} = {} & \eta_{tx} + u_x\eta_{tu} + u_t\eta_{xu} + u_{xt}\eta_u + u_tu_x\eta_{uu} - u_{tx}(\tau_t + \xi_x) - u_t\tau_{tx} - u_{tt}\tau_x \\ & -u_tu_x(\tau_{tu} + \xi_{xu}) - u_t^2\tau_{xu} - (2u_tu_{tx} + u_xu_{tt})\tau_u - u_t^2u_x\tau_{uu} - u_x\xi_{tx} \\ & -u_{xx}\xi_t - u_x^2\xi_{tu} - (2u_xu_{tx} + u_tu_{xx})\xi_u - u_tu_x^2\xi_{uu}.\end{aligned} \tag{1.25}$$

$$\begin{aligned}\zeta_{22} = {} & \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} \\ & -3u_xu_{xx}\xi_u - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - 2u_tu_x\tau_{xu} \\ & -(u_tu_{xx} + 2u_xu_{tx})\tau_u - u_tu_x^2\tau_{uu}.\end{aligned} \tag{1.26}$$

$$\begin{aligned}\zeta_{222} = {} & \eta_{xxx} - 3u_tu_xu_{xx}\tau_{uu} + u_x^3\eta_{uuu} + 3u_x^2\eta_{uux} - u_t\tau_{xxx} - 3u_x^2\xi_{uux} \\ & -3u_x^3\xi_{uux} - u_x^4\xi_{uuu} - 3u_{xx}^2\xi_u + 3u_{xx}\eta_{ux} + u_{xxx}\eta_u - 3u_{txx}\tau_x \\ & -3u_{tx}\tau_{xx} - 3u_{xxx}\xi_x - 3u_{xx}\xi_{xx} - 3u_tu_x\tau_{uxx} - 3u_tu_{xx}\tau_{ux} \\ & -6u_{tx}u_x\tau_{ux} + 3u_{xx}u_x\eta_{uu} - 3u_{tx}u_x^2\tau_{uu} + 3u_x\eta_{uuu} - u_x\xi_{xxx} \\ & -3u_tu_x^2\tau_{uux} - u_tu_{xxx}\tau_u - 3u_{txx}u_x\tau_u - u_tu_x^3\tau_{uuu} - 4u_{xxx}u_x\xi_u \\ & -9u_xu_{xx}\xi_{ux} - 3u_{tx}u_{xx}\tau_u - 6u_x^2u_{xx}\xi_{uu}.\end{aligned} \tag{1.27}$$

1.5.2 Group admitted by a PDE

The operator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{1.28}$$

is said to be a (generator of) *point symmetry* of the second-order PDE

$$E(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}) = 0 \tag{1.29}$$

if

$$X^{[2]}(E) = 0 \tag{1.30}$$

whenever $E = 0$. This can also be written as (symmetry condition)

$$X^{[2]} E\Big|_{E=0} = 0, \tag{1.31}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.4 Equation (1.31) is called the *determining equation* of (1.29), because it determines all the infinitesimal symmetries of equation (1.29).

The theorem below enables us to construct some solutions of equation (1.29) from known one.

Theorem 1.4 A symmetry of equation (1.29) transforms any solution of equation (1.29) into another solution of the same equation.

Proof: It follows from the fact that a symmetry of an equation leaves invariant that equation.

1.6 Lie algebras

Let X_1 and X_2 be any two operators defined by

$$X_1 = \tau_1(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \eta_1(t,x,u)\frac{\partial}{\partial u}$$

and

$$X_2 = \tau_2(t,x,u)\frac{\partial}{\partial t} + \xi_2(t,x,u)\frac{\partial}{\partial x} + \eta_2(t,x,u)\frac{\partial}{\partial u}.$$

Definition 1.5 (Commutator) The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by the formula $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.6 (Lie algebra) A Lie algebra is a vector space L of operators such that, for all $X_1, X_2 \in L$, the commutator $[X_1, X_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \; [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Theorem 1.5 The set of all solutions of any determining equation forms a Lie algebra.

1.7 Fundamental operators and their relations

In this section we will briefly give some definitions and notations which we will utilize later in subsection (4.1).

Let us consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{1.32}$$

with $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ representing the repertoires of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$ respectively, where the *total derivative operator* with respect to x^i is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \tag{1.33}$$

Consider the *Euler-Lagrange operator*, viz.,

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \qquad \alpha = 1, \dots, m. \tag{1.34}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is called a *conserved vector* of (1.32) if

$$D_i T^i|_{(1.32)} = 0, \tag{1.35}$$

holds. Equation (1.35) defines a *local conservation law* of system (1.32). A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.36}$$

hold identically. We will consider multipliers of the first order, namely, $\Lambda_\alpha = \Lambda_\alpha(t, x, y, u, u_t, u_x, u_y)$. The determining equation for the multiplier Λ_α is given by

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.37}$$

Once the multipliers are derived, the homotopy formula [9] will be used to construct the associated conserved vectors.

1.8 Conclusion

In this Chapter we gave a brief introduction to the Lie group analysis of PDEs and presented some results which will be used throughout this work. We also gave the algorithm to determine the Lie point symmetries of PDEs. Fundamental operators and their relations to find the conservation laws are also presented.

Chapter 2

Lie group analysis of the heat equation

We first consider an example of a *linear* PDE, namely *heat equation* and calculate its symmetry Lie algebra. We also find invariant solutions under certain symmetry generators of the one dimensional heat equation.

2.1 Lie symmetries of heat equation

Example 2.1 (see e.g., [3], [8])
Let us determine the Lie point symmetries of the one-dimensional heat equation

$$u_t - u_{xx} = 0, \tag{2.1}$$

in which the dependent variable is u and independent variables are t and x. This equation admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{2.2}$$

if and only if

$$X^{[2]}(u_t - u_{xx})\big|_{(2.1)} = 0. \tag{2.3}$$

Using the definition of $X^{[2]}$ from chapter one, we obtain

$$\Big[\tau(t,x,u)\frac{\partial}{\partial t}+\xi(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}+\zeta_1\frac{\partial}{\partial u_t}+\zeta_2\frac{\partial}{\partial u_x}+\zeta_{11}\frac{\partial}{\partial u_{tt}}$$
$$+\zeta_{12}\frac{\partial}{\partial u_{tx}}+\zeta_{22}\frac{\partial}{\partial_{xx}}\Big]\Big(u_t-u_{xx}\Big)\Big|_{u_t=u_{xx}}=0,$$

and this gives

$$\zeta_1-\zeta_{22}\Big|_{u_t=u_{xx}}=0, \tag{2.4}$$

where ζ_1 and ζ_{22} are given by equations (1.22) and (1.26) respectively. Substituting the values of ζ_1 and ζ_{22} in equation (2.4) and replacing u_t by u_{xx}, we obtain

$$\eta_t+u_{xx}(\eta_u-\tau_t)-u_x\xi_t-u_xu_{xx}\xi_u-u_{xx}^2\tau_u-\Big[\eta_{xx}+u_x(2\eta_{xu}-\xi_{xx})-u_{xx}\tau_{xx}$$
$$+u_x^2(\eta_{uu}-2\xi_{xu})-2u_{xx}u_x\tau_{xu}-u_x^3\xi_{uu}-u_x^2u_{xx}\tau_{uu}+(\eta_u-2\xi_x)u_{xx}-2u_{tx}\tau_x$$
$$-3u_xu_{xx}\xi_u-u_{xx}^2\tau_u-2\tau_uu_xu_{tx}\Big]=0. \tag{2.5}$$

Since τ, ξ and η depend only on t, x and u and are independent of the derivatives of u, the coefficients of like derivatives of u can be equated to yield an over determined system of linear PDEs:

$$u_xu_{tx} \;:\; 2\tau_u=0, \tag{2.6}$$
$$u_{tx} \;:\; 2\tau_x=0, \tag{2.7}$$
$$u_x^2u_{xx} \;:\; \tau_{uu}=0, \tag{2.8}$$
$$u_xu_{xx} \;:\; -\xi_u+2\tau_{xu}+3\xi_u=0, \tag{2.9}$$
$$u_{xx} \;:\; \eta_u-\tau_t+\tau_{xx}-\eta_u+2\xi_x=0, \tag{2.10}$$
$$u_x^3 \;:\; \xi_{uu}=0, \tag{2.11}$$
$$u_x^2 \;:\; \eta_{uu}-2\xi_{xu}=0, \tag{2.12}$$
$$u_x \;:\; \xi_t+2\eta_{xu}-\xi_{xx}=0, \tag{2.13}$$
$$1 \;:\; \eta_t-\eta_{xx}=0. \tag{2.14}$$

From equations (2.6) and (2.7), we obtain

$$\tau\equiv\tau(t)=a(t), \tag{2.15}$$

where $a(t)$ is an arbitrary function of t. Equation (2.8) is also satisfied. Substituting the value of τ in equation (2.9) and integrating with respect to u we get

$$\xi = b(t,x),$$

where $b(t,x)$ is an arbitrary function of t and x. Then equation (2.11) is satisfied too. Now, substituting the values of τ and ξ in equation(2.10) and integrating with respect to x ,yields

$$b(t,x) = \frac{1}{2}\,a'(t)x + c(t) \tag{2.16}$$

where $c(t)$ is an arbitrary function of t and so

$$\xi = \frac{1}{2}\,a'(t)x + c(t). \tag{2.17}$$

After substitution the value of ξ in equation (2.12) and integrating with respect to u, we have

$$\eta = d(t,x)u + \alpha(t,x) \tag{2.18}$$

where $d(t,x)$ and $\alpha(t,x)$ are arbitrary functions of t and x. It follows from substitution equations (2.17) and (2.18) in (2.13) that

$$d = -\frac{1}{8}\;a''(t)x^2 - \frac{1}{2}\;c'(t)x + f(t)$$

where $f(t)$ is an arbitrary function of t and so

$$\eta = -\frac{1}{8}\;a''(t)ux^2 - \frac{1}{2}\;c'(t)ux + uf(t) + \alpha(t,x)$$

Substituting the values of η_t and η_{xx} in equation (2.14), yields

$$-\frac{1}{8}\;a'''(t)x^2u - \frac{1}{2}\;c''(t)xu + f'(t)u + \alpha_t + \frac{1}{4}\;a'''(t)u + \alpha_{xx} = 0.$$

Splitting the above equation on u, we obtain

$$u \;:\; -\frac{1}{8}a'''(t)x^2 - \frac{1}{2}c''(t)x + f'(t) + \frac{1}{4}a''(t) = 0, \tag{2.19}$$

$$1 \;:\; \alpha_t - \alpha_{xx} = 0. \tag{2.20}$$

Splitting (2.19) with respect to powers of x, we get

$$x^2 \quad : \quad a'''(t) = 0, \tag{2.21}$$

$$x^1 \quad : \quad c''(t) = 0, \tag{2.22}$$

$$1 \quad : \quad f'(t) + \frac{1}{4}a''(t) = 0. \tag{2.23}$$

Integrating equations (2.21) and (2.22) with respect to t, yields respectively

$$a(t) = \frac{1}{2}A_1 t^2 + A_2 t + A_3, \tag{2.24}$$

$$c(t) = A_4 t + A_5 \tag{2.25}$$

where A_1, A_2, A_3, A_4 and A_5 are arbitrary constants. It follows from equations (2.23) and (2.24) that

$$f(t) = -\frac{1}{4}A_1 t + A_6$$

where A_6 is an arbitrary constant. Hence the general solution of the system of equations (2.6)−(2.14) is

$$\begin{aligned}
\tau &= \frac{1}{2}A_1 t^2 + A_2 t + A_3, \\
\xi &= \frac{1}{2}(A_1 t + A_2)x + A_4 t + A_5, \\
\eta &= -\frac{1}{8}A_1 x^2 u - \frac{1}{2}A_4 x u - (\frac{1}{4}A_1 t - A_6)u + \alpha(t,x),
\end{aligned}$$

where the A_s are constants and $\alpha(t,x)$ satisfies $\alpha_t = \alpha_{xx}$.

Thus the Lie point symmetries of the heat equation are given by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= u\frac{\partial}{\partial u}, \\
X_4 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\
X_5 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \\
X_6 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 u + 2tu)\frac{\partial}{\partial u}, \\
X_\beta &= \alpha(t,x)\frac{\partial}{\partial u}.
\end{aligned}$$

which generates a Lie algebra of infinite dimension. We now calculate the commutation relations for all the symmetry generators. We first compute $[X_4, X_1]$. By the definition of the Lie bracket, we have

$$
\begin{aligned}
[X_4, X_1] &= X_4X_1 - X_1X_4 \\
&= \left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right) \\
&= -2X_1
\end{aligned}
$$

Proceeding in a similar manner we can compute other commutation relations. In a table form these commutation relations can be written as:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_α
X_1	0	0	0	$2X_1$	$2X_2$	$4X_4 - 2X_3$	X_{α_t}
X_2	0	0	0	X_2	$-X_3$	$2X_5$	X_{α_x}
X_3	0	0	0	0	0	0	$-X_\alpha$
X_4	$-2X_1$	$-X_2$	0	0	X_5	$2X_6$	$X_{\alpha'}$
X_5	$-2X_2$	X_3	0	$-X_5$	0	0	$X_{\alpha''}$
X_6	$2X_3 - 4X_4$	$-2X_5$	0	$-2X_6$	0	0	$X_{\alpha'''}$
X_α	$-X_{\alpha_t}$	$-X_{\alpha_x}$	X_α	$-X_{\alpha'}$	$-X_{\alpha''}$	$-X_{\alpha'''}$	0

The values of α', α'' and α''' in the table above are given by,

$$
\begin{aligned}
\alpha' &= x\alpha_x + 2t\alpha_t, \\
\alpha'' &= 2t\alpha_x + x\alpha, \\
\alpha''' &= 4tx\alpha_x + 4t^2\alpha_t + (x^2 + 2t)\alpha.
\end{aligned}
$$

2.1.1 one parameter group

The one-parameter group can be obtained using the Lie equations

$$
\begin{aligned}
\frac{d\bar{t}}{da} &= \xi^1(t, x, u), \quad \bar{t}|_{a=0} = t, \\
\frac{d\bar{x}}{da} &= \xi^2(t, x, u), \quad \bar{x}|_{a=0} = x, \\
\frac{d\bar{u}}{da} &= \eta(t, x, u), \quad \bar{u}|_{a=0} = u.
\end{aligned}
$$

We now compute the one-parameter groups. For each X_i, let T_{a_i} be the corresponding group. Let us first calculate the one-parameter group corresponding to infinitesimal generator X_4, namely

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Using Lie equations, we have

$$\frac{d\bar{x}}{da} = \bar{x}, \quad \bar{x}|_{a=0} = x, \quad \frac{d\bar{t}}{da} = \bar{2}t, \quad \bar{t}|_{a=0} = t.$$

Solving the above equations, we get

$$\bar{x} = xe^{a}, \quad \bar{t} = te^{2a}.$$

Thus the one-parameter group T_{a_4} corresponding to the operator X_4 is given by

$$T_{a_4} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{2a_4}, xe^{a_4}, u).$$

If we continue in the same manner as above, we get the following one-parameter groups:

$$\begin{aligned}
T_{a_1} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u), \\
T_{a_2} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u), \\
T_{a_3} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, ue^{a_3}), \\
T_{a_5} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + 2a_5t, ue^{-a_5x - a_5^2t}), \\
T_{a_6} &: (\bar{x}, \bar{t}, \bar{u}) \longrightarrow \left(\frac{t}{1 - 4\alpha ta_6}, \frac{x}{1 - 4\alpha ta_6}, u\sqrt{(1 - 4\alpha ta_6}e^{\frac{-a_6x^2}{1-4a_6t}}\right) \\
T_{\alpha} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, u + a\alpha(t, x)).
\end{aligned}$$

2.2 The use of symmetry transformations

In this section we make use of the symmetries calculated in the previous section to obtain special exact solutions for the heat equation. The Lie group analysis supply us with two basic ways for constructing exact solutions of PDEs: group transformations of known solutions and construction of group invariant solutions. These methods are described in detail by means of examples.

If $\bar{u} = h(\bar{t}, \bar{x})$ is a solution of equation (2.1), then so is

$$\phi(t, x, u, a) = h(f_1(t, x, u, a), f_2(t, x, u, a))$$

or in solved form w.r.t $u : u = H_a(t, x)$ is a one-parameter family of solutions. For

$$T_{a_1} : \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u,$$

if $\bar{u} = h(t, x)$ is a solution, then

$$u = h(t + a_1, x).$$

We now write down the generated solutions for the other cases:

$$\begin{aligned}
T_{a_2} : u &= h(t, x + a_2), \\
T_{a_3} : u &= h(t, x)e^{-a_3}, \\
T_{a_4} : u &= h(te^{2a_4}, xe^{a_4}) \\
T_{a_5} : u &= h(t, x + 2a_5 t)e^{(a_5 x - a_5^2 t)}, \\
T_{a_6} : u &= h\left(\frac{t}{1 - 4\alpha t a_6}, \frac{x}{1 - 4\alpha t a_6}\right)\frac{1}{\sqrt{(1 - 4\alpha t a_6)}} e^{\frac{a x^2}{1 - 4a_6 t}} \\
T_\alpha : u &= h(t, x) - a\alpha(t, x).
\end{aligned}$$

2.3 Group invariant solutions

A group-invariant solution with respect to a subgroup of the symmetry group is an exact solution which is unchanged by all the transformations of the subgroup. Invariant solutions are expressed in terms of invariant of the subgroup. The number of independent variables in the reduced system is fewer than the original system. Thus, if an equation which its invariant solutions with respect to G is obtained by solution of the reduced ODEs.

Considering the Heat equation,let G be a one-parameter symmetry group of equation

(2.1). A solution $u = u(t,x)$ of equation (2.1) is invariant under G with generator X if

$$\eta - \xi^1 u_t - \xi^2 u_x = 0,$$

whenever $u = u(t,x)$. This is the invariant surface condition the solution of which provides the form of the invariant solution to the equation.

Let us now illustrate the above method by considering the heat equation which consist six-parameter group of symmetries and infinite-dimensional subgroup .We will construct invariant solutions under the operators X_5 and form the combination of X_1 and X_2.

Traveling wave solutions

Consider the following linear combination of the translation operators X_1 and X_2:

$$c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

The characteristic equations are

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{0}.$$

Thus, one invariant is $I_1 = u$. The other is obtained from the equation

$$\frac{dx}{c} = \frac{dt}{1}$$

and is given by $I_2 = x - ct$.

The invariant solution can be written as $I_1 = h(I_2)$, i.e.,

$$u = h(x - ct), \tag{2.19}$$

where h is an arbitrary function of its argument. Differentiation of u with respect to x and t, gives us

$$u_t = -c\,h', \quad u_x = h', \quad u_{xx} = h''.$$

Substituting these expressions into equation (2.1), we obtain the reduced ODE

$$h'' + ch' = 0, \tag{2.20}$$

which is a second-order ODE with constant coefficients. Its general solution is

$$h(I_2) = A_1 e^{-cI_2} + A_2 = A_1 e^{-c(x-ct)} + A_2, \tag{2.21}$$

where A_1 and A_2 are arbitrary constants of integration. Thus the most general traveling wave solution to the heat equation is of the form

$$u(t, x) = A_1 e^{-c(x-ct)} + A_2. \tag{2.22}$$

Galilean-invariant solutions

As a second example let us construct an invariant solution under operator X_5, namely

$$X_5 \quad = \quad 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}.$$

The characteristic equations are

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{du}{-xu}.$$

Thus, one invariant is $I_1 = t$. The other is obtained from the equation

$$\frac{dx}{2t} = \frac{du}{-xu}$$

and is given by $I_2 = ue^{\frac{x^2}{4t}}$.

Consequently, the invariant solution is $I_2 = h(I_1)$, i.e.

$$u = e^{-\frac{x^2}{4t}} h(t). \tag{2.23}$$

Then

$$\begin{aligned} u_t &= \left(\frac{x^2}{4t^2}h + h'\right)e^{-\frac{x^2}{4t}}, \\ u_{xx} &= \left(-\frac{1}{2t} + \frac{x^2}{4t^2}\right)he^{-\frac{x^2}{4t}}. \end{aligned}$$

Substitution of the above values of u_t and u_{xx} in equation (2.1), gives us the first-order ODE

$$h' = -\frac{1}{2t}h \tag{2.24}$$

which on integration gives

$$h(t) = \frac{C}{\sqrt{t}}. \tag{2.25}$$

Hence the most general Galilean-invariant solution is

$$u(t, x) = \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}}. \tag{2.26}$$

2.4 Conclusion

In this Chapter we have looked at an example of a linear PDE heat equation. We calculated its Lie point symmetries, constructed solutions under transformation groups on known solutions and also obtained invariant solution under certain symmetry generators.

Chapter 3

Solutions and Conservation Laws of the Modified Burgers-KdV Equation

3.1 Introduction

The Korteweg-de Vries-Burgers (KdVB) equation, introduced by Jan Burgers in 1969, extends the classic KdV equation by adding a dissipative term to model the formation of shock waves [9]. The original KdV equation, discovered by Joseph Boussinesq in 1877 and later rediscovered by Korteweg and de Vries in 1895, describes the motion of weakly nonlinear waves in dispersive media, like shallow water waves. The KdVB equation builds on this by accounting for viscosity and dissipation, making it relevant in a wide range of physical systems, such as fluid dynamics, plasma physics, and optics [10]. The modified Burgers-KdV equation, incorporates both the nonlinearity from Burgers' equation and the dispersive nature of the KdV equation [11]. It is useful for studying wave behavior in more complex systems where both dissipation and dispersion play significant roles.

In this chapter, we compute the Lie point symmetries of the modified-Burgers KdV equation (3.1) and then perform symmetry reductions, which will lead to exact solutions in certain cases. Furthermore, we derive conservation laws for the modified Burgers-KdV equation (3.1) using the multiplier method

3.2 Lie point Symmetries

Consider the modified Burgers-KdV equation (mBKdV) given by

$$u_t + \ pu^2 u_x + \ qu_{xx} - \ ru_{xxx} = 0. \tag{3.1}$$

where p , q and r are real constants. Equation (3.1) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u},$$

if and only if

$$X^{[3]}(u_t + \ pu^2 u_x + \ qu_{xx} - \ ru_{xxx})\big|_{(3.1)} = 0.$$

Using the definition of $X^{[3]}$ from Chapter one, we obtain

$$(\ 2puu_x\eta + \zeta_1 + \ pu^2\zeta_2 + \ q\zeta_{22} + \ (-r)\zeta_{222})|_{(3.1)} = 0, \tag{3.2}$$

where ζ_1, ζ_2, ζ_{22} and ζ_{222} are given by equations (1.22), (1.23), (1.26) and (1.27) respectively. Substituting the values of ζ_1, ζ_2, ζ_{22} and ζ_{222} into the above equation u_{xxx} replaced by $r^{-1}\,(\ u_t - \ pu^2u_x + \ quu_{xx})$, we obtain

$$
\begin{aligned}
&2puu_x\eta+\eta_t+u_t\eta_u-u_t\tau_t-u_t^2\tau_u-u_x\xi_t-u_tu_x\xi_u+pu^2\eta_x+pu^2u_x\eta_u-pu^2u_t\tau_x-pu^2u_tu_x\tau_u\\
&-pu^2u_x\xi_x-pu^2u_x^2\xi_u+q\eta_{xx}+2qu_x\eta_{xu}+qu_{xx}\eta_u+qu_x^2\eta_{uu}-2u_{xx}-2qu_{xx}\xi_x-qu_x\xi_{xx}\\
&-2qu_x^2\xi_{xu}-3qu_xu_{xx}\xi_u-qu_x^3\xi_{uu}-2qu_{tx}\tau_x-qu_t\tau_{xx}-2qu_tu_x\tau_{ux}-qu_tu_{xx}\tau_u-2qu_xu_{tx}\tau_u\\
&-qu_tu_x^2\tau_{uu}-r\eta_{xxx}+3ru_tu_xu_{xx}\tau_{uu}-ru_x^3\eta_{uux}+ru_t\tau_{xxx}+3ru_x^2\xi_{uux}+3ru_x^3\xi_{uux}+ru_x^4\xi_{uuu}\\
&+3ru_{xx}^2\xi_u-3ru_{xx}\eta_{ux}-u_t\eta_u-pu^2u_x\eta_u-qu_{xx}\eta_u+3ru_{txx}\tau_x+3ru_{tx}\tau_{xx}+3u_t\xi_x+3pu^2u_x\xi_x\\
&+3pu^2u_x\xi_x+3qu_{xx}\xi_x+3ru_{xx}\xi_{xx}+3ru_tu_x\tau_{uxx}+3ru_tu_{xx}\tau_{ux}+6ru_{tx}u_x\tau_{ux}-3ru_{xx}u_x\eta_{uu}\\
&+3ru_{tx}u_x^2\tau_{uu}-3ru_x\eta_{uxx}+ru_x\xi_{xxx}+3ru_tu_x^2\tau_{uux}+u_t^2\tau_u+pu^2u_xu_t\tau_u+qu_{xx}u_t\tau_u+3ru_{txx}u_x\tau_u\\
&+ru_tu_x^3\tau_{uuu}-4u_tu_x\xi_u+4pu^2u_x^2\xi_u+4qu_{xx}u_x\xi_u+9ru_xu_{xx}\xi_{ux}+3ru_{tx}u_{xx}\tau_u+6u_x^2u_{xx}\xi_{uu}=0.
\end{aligned}
\tag{3.3}
$$

Since ξ, τ and η depend only on x, t and u, then by treating the variables x, t, u, u_x, u_t, u_{xx}, u_{xt} and u_{txx} as independent variables, we obtain the following determining equations:

$$
\begin{aligned}
u_{txx} &: \quad \tau_x = 0, && (3.4)\\
u_{tx}u_{xx} &: \quad \tau_u = 0, && (3.5)\\
u_xu_t &: \quad \xi_u = 0, && (3.6)\\
u_{xx}u_x &: \quad \eta_{uu} = 0, && (3.7)\\
u_t &: \quad 3\xi_x - \tau_t = 0, && (3.8)\\
u_{xx} &: \quad q\xi_x - 3r\eta_{ux} + 3r\xi_{xx} = 0, && (3.9)\\
u_x &: \quad 2pu\eta - \xi_t + 2pu^2\xi_x + 2q\eta_{xu} - 3r\eta_{uxx} - r\xi_{xxx} = 0, && (3.10)\\
Rest &: \quad p\eta_t - pu^2\eta_x + q\eta_{xx} + r\eta_{xxx} = 0. && (3.11)
\end{aligned}
$$

From equation (3.4) and (3.5), we obtain

$$\tau = a(t), \tag{3.12}$$

where $a(t)$ is an arbitrary function of t. Solving equation (3.6), we obtain

$$\xi = b(t,x), \tag{3.13}$$

where $b(t,x)$ is an arbitrary function of x and t. Solving equation (3.7) we get

$$\eta = c(t,x)u + d(t,x), \tag{3.14}$$

where $c(t,x)$ and $d(t,x)$ are arbitrary functions of x and t. Substituting equation (3.12) and (3.13) into equation (3.8) yields

$$3b_x(t,x) - a'(t) = 0. \tag{3.15}$$

Integrating equation (3.15), we have

$$b(t,x) = \frac{1}{3}a'(t)x + e(t), \tag{3.16}$$

where $e(t)$ is an arbitrary function of t. Substituting equation (3.16) into (3.13) gives

$$\xi(t,x) = \frac{1}{3}a'(t)x + e(t). \tag{3.17}$$

Substituting equation (3.14) and (3.17) into (3.9) gives

$$\frac{1}{3}\ qa'(t) - 3rc_x = 0. \tag{3.18}$$

When integrating (3.18) we get

$$c(x,t) = \frac{1}{9}\frac{q}{r}a'(t)x + f(t), \tag{3.19}$$

where $f(t)$ is an arbitrary function of t. Hence equation (3.14) becomes,

$$\eta = \left(\frac{1}{9}\frac{q}{r}a'(t)x + f(t)\right)u + d(t,x). \tag{3.20}$$

Substituting the expressions of ξ and η into equation (3.10) gives

$$\frac{2}{9}\frac{pq}{r}a'(t)xu^2 + 2pf(t)u^2 + 2pud(t,x) - \frac{1}{3}a''(t)x - e'(t) + \frac{2}{3}u^2a'(t) + \frac{2}{9}\frac{q^2}{r}a'(t) = 0. \tag{3.21}$$

Splitting on u, we get

$$\begin{aligned} u^2 &: \frac{2}{3}\ pa'(t) + \frac{2}{9}\frac{pq}{r}a'(t)x + 2pf(t) = 0, && (3.22)\\ u &: \mathrm{d}(t,x) = 0, && (3.23)\\ Rest &: -\frac{1}{3}a''(t)x - e'(t) - \frac{2}{9}\frac{q^2}{r}a'(t) = 0. && (3.24) \end{aligned}$$

Splitting on x from equation (3.22) yield,

$$
\begin{aligned}
x \quad &: \quad a'(t) = 0, && (3.25)\\
Rest \quad &: \quad f(t) + \frac{2}{3}\, pa'(t) = 0. && (3.26)
\end{aligned}
$$

Now, equation (3.24) becomes,

$$e'(t) = 0, \tag{3.27}$$

and also updating (3.26) gives

$$f(t) = 0. \tag{3.28}$$

Substituting equation (3.23), (3.25) and equation (3.28) into equation (3.20) we get,

$$\eta = 0. \tag{3.29}$$

Therefore equation (3.11) is satisfied by (3.29). Intergrating equation (3.25) and (3.27) we get,

$$
\begin{aligned}
a(t) &= A_1, && (3.30)\\
e(t) &= A_2, && (3.31)
\end{aligned}
$$

where A_2 and A_3 are also arbitrary constants of intergration. Therefore,

$$
\begin{aligned}
\tau &= A_1,\\
\xi &= A_2,\\
\eta &= 0.
\end{aligned}
$$

Thus the Lie point symmetries of the mBKdV equation are given by

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x}.
\end{aligned}
$$

3.3 Symmetry algebra and symmetry groups

3.3.1 One-parameter groups G_i generated by symmetries

The corresponding one-parameter groups can be obtained using the Lie equations

$$\begin{aligned}\frac{d\bar{t}}{da} &= \tau(t,x,u), \quad \bar{t}|_{a=0} = t,\\ \frac{d\bar{x}}{da} &= \xi(t,x,u), \quad \bar{x}|_{a=0} = x,\\ \frac{d\bar{u}}{da} &= \eta(t,x,u), \quad \bar{u}|_{a=0} = u.\end{aligned}$$

We now compute the one-parameter groups G_i generated by the X_i. For each X_i, let G_i be the corresponding group. Let us first calculate the one-parameter group G_3 corresponding to the infinitesimal generator X_1, namely

$$X_1 = \frac{\partial}{\partial t},$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 1,\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = 0,\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = 0,\ \bar{u}|_{a=0} = u.$$

Solving the above equations we get

$$\bar{t} = t + a,\ \bar{x} = x,\ \bar{u} = u.$$

Thus, the one-parameter group G_3 corresponding to the operator X_3 is given by

$$G_1 : (\bar{t},\bar{x},\bar{u}) \longrightarrow (t + a_1, x, u).$$

If we continue in the same manner as above, we can obtain the other one-parameter groups. Thus, we have

$$\begin{aligned}G_1 &: (\bar{t},\bar{x},\bar{u}) \longrightarrow (t + a_1, x, u),\\ G_2 &: (\bar{t},\bar{x},\bar{u}) \longrightarrow (t, x + a_2, u),\end{aligned}$$

3.4 Group Invariant solutions

In this section we will construct invariant solutions using the symmetries X_1 and X_2. Consider a Lie point symmetry generator

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,,u)\frac{\partial}{\partial u}$$

of the modified-Burgers KdV equation

$$u_t + \ pu^2 u_x + \ qu_{xx} - \ ru_{xxx} = 0.$$

The invariant solutions under the one-parameter group generated by X will be obtained as follows: We calculate two linearly independent invariants

$$J_1 = \alpha(x,t) \, \text{and} \, J_2 = \beta(x,t)$$

by solving the first order quasi-linear PDE

$$X(J) \equiv \tau(x,t,u)\frac{\partial J}{\partial t} + \xi(x,t,u)\frac{\partial J}{\partial x} + \eta(x,t,u)\frac{\partial J}{\partial u} = 0,$$

or its characteristic equations

$$\frac{dt}{\tau(x,t,u)} = \frac{dx}{\xi(x,t,u)} = \frac{du}{\eta(x,t,u)}.$$

Then we write one of the invariants as a function of the other, i.e

$$J_2 = f(J_1).$$

and solve for u. Substitute the expression of u into the mBKdV equation and this will yield an ordinary differential equation (ODE) for the unknown function f.
We now use the above method to find group-invariant solutions for the modified Burgers-KdV equation (3.1).

Case 1: We firstly consider the symmetry operator

$$X_1 = \frac{\partial}{\partial t},$$

The characteristic equations associated with the operator X_1 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}$$

which gives two invariant solutions $J_1 = x$ and $J_2 = u$. The group-invariant solution is given by $J_2 = f(J_1)$, this implies that $u = f(x)$, where f is an arbitrary function. Substituting the expression of u into the mBKdV equation (3.1), we obtain a third-order ODE

$$pf^2(x)f'(x) + qf''(x) - rf'''(x) = 0. \tag{3.32}$$

Case 2: We now consider the second symmetry operator

$$X_2 = \frac{\partial}{\partial x}.$$

The characteristic equations associated with the operator X_2 are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}.$$

Thus we get the following two invaruiants: $J_1 = t$ and $J_2 = u$. Hence the the group-invariant solution can be written as $J_2 = f(J_1)$, this implies that $u = f(t)$, where f is an arbitrary function. Substitute the expression of u into the mBKdV equation and this will yield an ordinary differential equation (ODE) for the unknown function f given as

$$f'(t) = 0. \tag{3.33}$$

Integrating the above equation yields

$$f(t) = C, \tag{3.34}$$

where C is an arbitrary constant of integration. Thus the group-invariant solution for (3.1) under X_1 is $u(t, x) = C$

Case 3: Travelling wave solution

We consider the symmetry operator (Combination of symmetries, X_1 and X_2)

$$X_1 + cX_2 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x},$$

where c is a constant. The associated characteristic equations to $X_1 + cX_2$ are:

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0},$$

which yields the two invariants $J_1 = x - ct$ and $J_2 = u$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function. This implies that

$$u(t,x) = f(z), \quad z = x - ct \tag{3.35}$$

Substituting this expression of u into the mBKdV equation we obtain the following ODE

$$-cf' + pf^2 f' + qf'' - rf''' = 0. \tag{3.36}$$

3.4.1 Exact solutions of (3.1) using simplest equation method

In this section we employ the simplest equation method [12,13] to solve the nonlinear ODE (3.36). This will then give us the exact solutions for our modified Burgers-KdV equation (3.1). The simplest equations that we will use in our work are the Bernoulli and Riccati equations.

Here we first present the simplest equation method and consider the solutions of (3.36) in the form

$$F(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{3.37}$$

where $G(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

The Bernoulli equation

$$G'(z) = aG(z) + bG^2(z), \tag{3.38}$$

where a and b are arbitrary constants has the general solution given by

$$G(z) = a\left\{\frac{\cosh[a(z+B)] + \sinh[a(z+B)]}{1 - b\cosh[a(z+B)] - b\sinh[a(z+B)]}\right\}.$$

The Riccati equation considered in this work is

$$G'(z) = bG^2(z) + aG(z) + d, \tag{3.39}$$

where a, b and d are arbitrary constants. Its solutions are

$$G(z) = -\frac{a}{2b} - \frac{\theta}{2b} \tanh\left[\frac{1}{2}\theta(z+B)\right]$$

and

$$G(z) = -\frac{a}{2b} - \frac{\theta}{2b} \tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{B\cosh\left(\frac{\theta z}{2}\right) - \frac{2b}{\theta}\sinh\left(\frac{\theta z}{2}\right)},$$

with $\theta^2 = a^2 - 4bd$ and B is an arbitrary constant of integration.

Solutions of (3.1) using the Bernoulli equation as the simplest equation

The balancing procedure gives $M = 1$ so the solutions of (3.36) are of the form

$$F(z) = A_0 + A_1 G. \tag{3.40}$$

Inserting (3.40) into (3.36) and using the Bernoulli equation (3.38) and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of four equations in terms of A_0, A_1, namely

$$\begin{aligned}
-6b^3rA_1 + bpA_1^3 &= 0,\\
-12ab^2rA_1 + apA_1^3 + 2bpA_0A_1^2 + 2b^2qA_1 &= 0,\\
-a^3rA_1 + apA_0^2A_1 + a^2qA_1 - acA_1 &= 0,\\
-7a^2brA_1 + 2apA_0A_1^2 + bpA_0^2A_1 + 3abqA_1 - bcA_1 &= 0.
\end{aligned}$$

With the aid of Mathematica, solving the above system of algebraic equations, one possible solution for c, A_0, A_1 is

$$\begin{aligned}
A_0 &= \frac{3ar - q}{\sqrt{6rp}},\\
A_1 &= b\sqrt{\frac{6r}{p}},\\
c &= \frac{3a^2r^2 + q^2}{6r}.
\end{aligned}$$

Thus, reverting back to the original variables, a solution of (3.1) is

$$u(t,x) = A_0 + A_1 a\left\{\frac{\cosh[a(z+B)] + \sinh[a(z+B)]}{1 - b\cosh[a(z+B)] - b\sinh[a(z+B)]}\right\}, \tag{3.41}$$

where $z = x - ct$ and B is an arbitrary constant of integration.

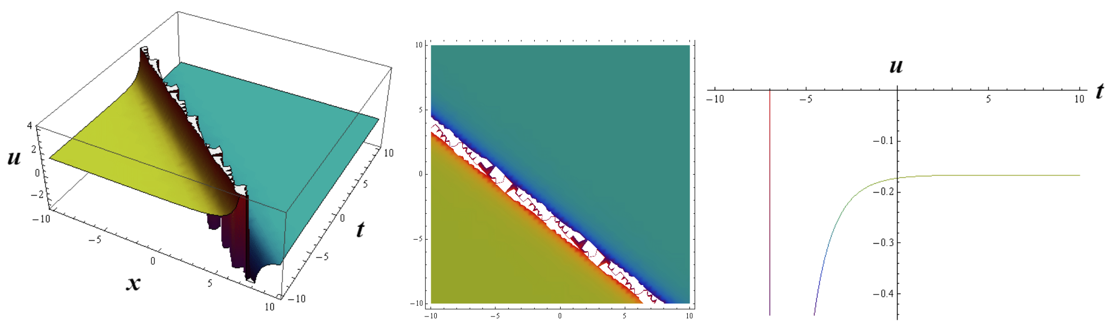

Figure 3.1: Profile of solution (3.41)

The solution is expressed using hyperbolic functions *cosh* and *sinh*, suggesting a soliton-like wave that maintains its shape while traveling at a constant velocity. The solution depends on a traveling wave coordinate $z = x - ct$ and is influenced by constants related to boundary conditions. The visualizations in Figure 3.1 illustrate the solution's evolution. The 3D surface plot shows how the wave propagates over time and space, resembling a localized structure. The contour plot highlights the wave's spatial progression, and the time evolution graph depicts how the wave changes at a fixed point. These plots capture the wave's nonlinear and stable behavior.

Solutions of (3.1) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 1$ so the solution of (3.36) take the form

$$F(z) = A_0 + A_1 G. \tag{3.42}$$

Inserting (3.42) into (3.36) and making use of the Riccati equation (3.39), we obtain algebraic system of equations in terms of A_0, A_1 and c by equating the coefficients of powers of G^i to zero. The resulting algebraic equations are

$$\begin{aligned}
-6b^3rA_1 + bpA_1^3 &= 0,\\
-12ab^2rA_1 + apA_1^3 + 2bpA_0A_1^2 + 2b^2qA_1 &= 0,\\
-a^2drA_1 - 2bd^2rA_1 + dpA_0^2A_1 + adqA_1 - cdA_1 &= 0,\\
-7a^2brA_1 + 2apA_0A_1^2 - 8b^2drA_1 + bpA_0^2A_1 + dpA_1^3 + 3abqA_1 - bcA_1 &= 0,\\
-a^3rA_1 - 8abdrA_1 + apA_0^2A_1 + 2dpA_0A_1^2 + a^2qA_1 + 2bdqA_1 - acA_1 &= 0.
\end{aligned}$$

Solving the above equations, we get

$$\begin{aligned}
A_0 &= \frac{3ar - q}{\sqrt{6rp}},\\
A_1 &= b\sqrt{\frac{6r}{p}},\\
c &= \frac{3a^2r^2 + q^2 - 12r^2bd}{6r},
\end{aligned}$$

and consequently, the solution of (3.1) is given as

$$u(t,x) = A_0 + A_1\left\{-\frac{a}{2b} - \frac{\theta}{2b}\tanh\left[\frac{1}{2}\theta(z+B)\right]\right\}. \tag{3.43}$$

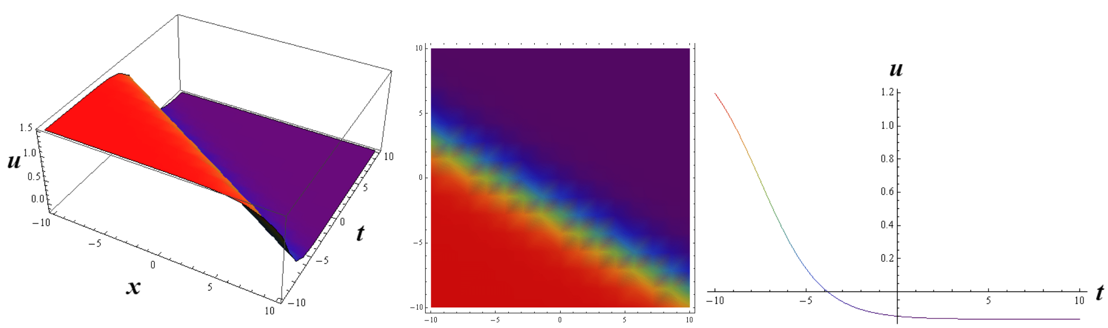

Figure 3.2: Profile of solution (3.43)

The solution in equation (3.43) describes a traveling wave of the form $u(t,x)$, involving a *tanh* (hyperbolic tangent) function, indicating a smooth, localized wave with a sharp transition, characteristic of soliton-like behavior. The wave maintains its shape as it moves through space at a constant speed. In Figure 3.2, the 3D surface plot shows the wave's stable propagation, the contour plot highlights its sharp spatial transition, and the time evolution graph demonstrates the consistent wave profile at a fixed point. These visualizations confirm the wave's stable, nonlinear nature.

3.4.2 Conservation laws using the multiplier approach of equation (3.1)

This section aims to construct conservation laws of the modified Burgers-KdV equation (mBKdV) (3.1) via the multiplier method. [14, 15]. A conservation law of the modified-Burgers KdV equation (3.1) is a total space-time divergence expression that vanishes on the solution space ε of equation (3.1),

$$D_i T^i|_{\varepsilon} = 0, \tag{3.44}$$

where D_i is the total derivative operator and T^i is a conserved vector for $i = 1, 2, 3, 4$. The zeros-order multipliers for the modified Burgers-KdV equation (3.1) will be determined by implementing the multiplier method. See for example [14, 15] and references therein.

The zeroth-order multiplier for the modified Burgers-KdV equation (3.1) is

$$\Lambda = 1, \tag{3.45}$$

Corresponding to the above multiplier the associated *low-order* conservation laws for equation (3.1) are:

$$\begin{aligned} T_1^t &= u, \\ T_1^x &= \frac{1}{3} p u^3 + q u_x - r u_{xx}, \end{aligned}$$

respectively.

3.4.3 Conclusion

In this chapter we used the Lie point symmetries to find the symmetries of modified Burgers-KdV equation. We also used the simplest equation method to obtain the exact solutions of mBKdV. Plots and analysis of our solutions were also presented. Lastly, conservation laws were also derived using multiplier methods.

Chapter 4

Conclusion and Discussions

In this project we first reviewed some of the basic results of the Lie group analysis of PDEs. In particular, we provided the algorithm to calculate the Lie point symmetries of PDEs.Fundamental operators of calculating the conservation laws and their relations were also provided.

In chapter two we considered a linear PDE namely, heat equation and calculated its symmetry Lie algebra. Invariant solutions of the heat equation were found under certain symmetry generators. We also noted that Lie group analysis provides us with two basic ways for constructing exact solutions of PDEs namely, application of transformation groups on known solutions and construction of invariant solutions Both methods were described in detail and specific examples were considered to illustrate these methods.

In chapter three we considered the main problem of our project. That is, we used Lie symmetry techniques to find the Lie point symmetries and exact solutions of the modified Burgers Korteweg-de Vries equation (mBKdV). The conservation laws of mBKdV equation using the multiplier method were also derived. Finally, in Chapter four we summarized the work done in the project.

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