Stiff Differential Equations Supplementary Material

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Recall equation (1) from the report.

$$\frac{dx}{dt} = -30x \tag{1}$$

with initial condition $x(0) = 1$

To solve such an equation, we can use the method of **variable separable**.

Multiplying both sides of 1 by dt and dividing by x yields.

$$\frac{dx}{x} = -30dt$$

Integrating over both sides then gives us that:

$$ln(x) = -30t + c \implies x(t) = e^{-30t+c} = x(t) = Ce^{-30t}$$

Using our initial condition, we get that.

$$x(0) = 1 = Ce^{-30(0)} = C$$

Hence, we get equation (3) from the report ie:

$$x(t) = e^{-30t} \tag{2}$$

Now, recall the more complicated equation (2) from the report.

$$\frac{dx}{dt} = -20x + 20\sin(\omega t) + \omega\cos(\omega t) \tag{3}$$

To solve this equation we will use the method of the **Integrating Factor** Firstly, Rearranging equation (2)gives us:

$$\frac{dx}{dt} + 20x = 20\sin(\omega t) + \omega\cos(\omega t)$$

Let $\mu(t) = e^{\int 20dt}$ and multiply both sides by $\mu(t)$ gives:

$$e^{20t}\frac{dx}{dt} + 20e^{20t}x = -e^{20t}(-20\sin(\omega t) - \omega\cos(\omega t))$$

Since $20e^{20t} = \frac{d}{dt}e^{20t}$, we get that:

$$e^{20t}\frac{dx}{dt} + \frac{d}{dt}(e^{20t})x = -e^{20t}(-20\sin(\omega t) - \omega\cos(\omega t))$$

Applying the reverse product rule, we get:

$$\frac{d}{dt}(xe^{20t}) = -e^{20t}(-20\sin(\omega t) - \omega\cos(\omega t))$$

Integrating both sides with respect to t, we get that:

$$x(t)e^{20t} = e^{20t}\sin(\omega t) + C$$

Finally, dividing booth sides by $\mu(t)$ we get:

$$x(t) = \sin(\omega t) + Ce^{-20t}$$

Lastly, using are initial conditions $x(0) = 1$, we get that $C = 1$ and hence we obtain equation (4):

$$x(t) = \sin(\omega t) + e^{-20t} \tag{4}$$

Below, we can see the two plots for equation 3 and 4.

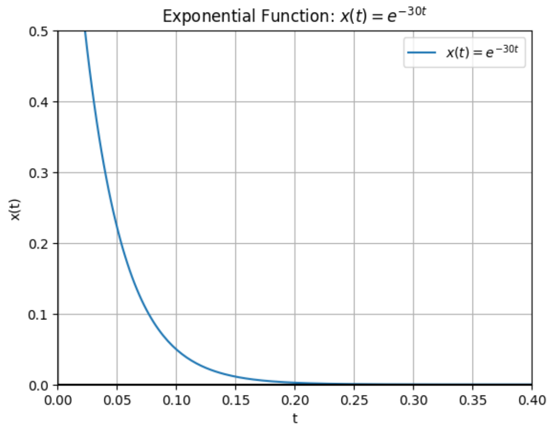

Figure 1: The plot above shows the solution to equation 1.

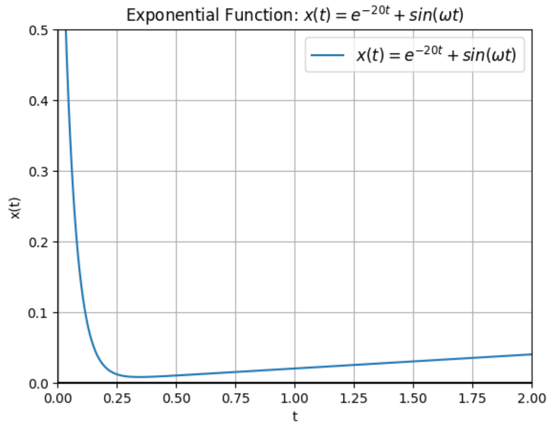

Figure 2: The plot above shows the solution to equation 2

The formula for the **Runge-Kutta** is stated below

$$y(x+h) \approx y(x) + \frac{1}{6}(k_1 + 2k_2 + 2_3 + k_4) \tag{5}$$

Where,

$$k_1 = hf(x, y)$$
$$k_2 = hf(x + \frac{1}{2}h, y + \frac{1}{2}k_1)$$
$$k_3 = hf(x + \frac{1}{2}h, y + \frac{1}{2}k_2)$$
$$k_4 = hf(x + h, y + k_1)$$