Notes for PHYS 27200 - Electric And Magnetic Interactions

Ezekiel Ulrich

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Contents

Course Introduction

This is a calculus-based physics course using concepts of electric and magnetic fields and an atomic description of matter to describe polarization, fields produced by charge distributions, potential, electrical circuits, magnetic forces, induction, and related topics, leading to Maxwell's equations and electromagnetic radiation and an introduction to waves and interference. 3-D graphical simulations and numerical problem solving by computer are employed throughout. For more information, consult the syllabus.

Equations

1. Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$
2. Electric field due to a point or charged sphere: $\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$
3. Force due to electric field: $\vec{F_2} = E_1 q_2$
4. Dipole moment between charges $-q$ and q separated by $\vec{s}$: $\vec{p} = q\vec{s}$
5. Electric field on dipole axis: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{-2sq}{r^3}\hat{p}$
6. Electric field on dipole bisecting plane: $\vec{E} = \frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$
7. Electric field from point charge-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}$

8. Electric field from dipole-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{12\alpha p_1^2}{r^7}$
9. Drift speed: $\bar{v} = uE$
10. Electric field of a uniformly charged thin rod at a distance r from the midpoint, perpindicular to the rod: $E = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r\sqrt{r^2+(L/2)^2}}\right]$

Electric charge

Electric Charge: Electric charge is an intrinsic characteristic of the fundamental particles that make up objects.
Conservation of charge: The net charge of a *closed system* never changes

Objects can have negative, zero, or postive charge. Charges are always multiples of the *elementary charge* $e = 1.60217662(63) \times 10^{-19} C$
Coulomb (C): One coulomb is the amount of charge that is transferred through the cross section of a wire in 1 second when there is a current of 1 ampere in the wire.

The charges of elementary particles are listed below.

It was mentioned in class that mass is likewise intrinsic. However, when discussing the Higgs field's role in giving particles mass, the distinction between mass as an intrinsic or emergent property becomes more nuanced. The mass of elementary particles like electrons and quarks is emergent in the sense that it arises from their interaction with the Higgs field, which itself is a fundamental aspect of the universe.

Particle	Charge (elementary charge, e)
Electron (e^-)	-1
Positron (e^+)	$+1$
Proton (p^+)	$+1$
Anti-proton (p^-)	-1
Neutron	0
Anti-neutron	0
Photon	0

Point Charge: A charged object whose radius is much smaller than the distance between itself and all other objects of interest.

The magnitude of electric force between two point charges is d irectly proportional to the magnitude of each charge and inversely proportional to the distance squared. Specifically:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

This is Coulomb's Law. Note the direction of the force changes with the sign of the charges involved. Like repels like and opposites attract.

Electric field

Consider a charged particle. We can represent its effect by drawing vectors that show the path a positively charged particle would follow if placed within its influence. These lines represent the electric field of the charged particle. The greater the density of the lines, the greater the strength of the electric field. Note that at the origin, the force is undefined (infinite), since $|r| = 0$.

There are many types of fields, which can be either scalar or vector. For example, a temperature map is a scalar field, since each point

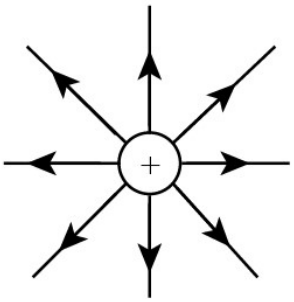


Figure 1: An eletric field coming from point charge. Notice how the densities of the lines vary with distance from the source.
Technically, fields can be the more general tensor, or even the fascinating and exotic spinor!

is associated with a scalar (the temperature at that point). A map of fluid velocity is a vector field, since each point is associated with a vector (the velocity of the fluid at that point). For a particle, the eletric field vector at any point is given by

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

The direction in which the field lines point depends on the sign of the charge. If the charge is negative, the field lines point in. If it is positive, the field lines point out. A useful mnemonic is to think of the charge as someone's STD test results. If it's negative, others will go for them and the lines point in. If it's positive, everyone will try to get away and the lines point out.

Consider the relative strengths of the electric and gravitational fields. The gravitational force is given by $F_g = G\frac{m_1 m_2}{r^2}\hat{r}$, with $m_{electron} = 9 \times 10^{-31} kg$ and $m_{proton} = 1.7 \times 10^{-27} kg$. If we consider a hydrogen atom, then $r = 5.3 \times 10^{-11} m$. With $G = 6.7 \times 10^{-11}$, we have

$$F_g = \frac{(1.7 \times 10^{-27})(9 \times 10^{-31})(6.7 \times 10^{-11})}{(5.3 \times 10^{-11})^2} \approx O(10^{-46})N$$

Now, the eletric force is given by $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. The charge of a proton and eletric are $q_1 \approx q_2 \approx 1.6 \times 10^{-19} C$. Ergo, since $\frac{1}{4\pi\epsilon_0} \approx 8.99 \times 10^9 \frac{Nm^2}{C^2}$,

$$F_e = \frac{(8.99 \times 10^9 \frac{Nm^2}{C^2})(1.60 \times 10^{-19} C)^2}{(5.3 \times 10^{-11} m)^2} \approx O(10^-17)N$$

This means that $\frac{F_e}{F_g} \approx 2.27 \times 10^{39}$, meaning the eletric force is much stronger for these masses and charges than gravity. On scales as large as humans and planets, gravity is the dominant force because gravity is strictly additive.

For sufficient distances, the eletric field of a uniformly charged spherical shell resembles the eletric field of a point charge.

That means for $r >> R$, $\vec{E_{sphere}} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$ This holds only for outside the sphere. Inside, it can be shown that the electric field is zero.

Superposition Principle: The net electric field at a location in space is a vector sum of the individual electric fields contributed by all charged particles located elsewhere.

To introduce systems with multiple sources of electric field lines, consider the particle pair known as a dipole. Dipoles consist of one negatively charged and one positively charged particle, like so:

Dipole Moment: The dipole moment is a way of expressing asymmetrical charge distribution. It is a vector quantity, i.e. it has magnitude as

Figure 2: Notice how a circle resembles a point from a great distance.

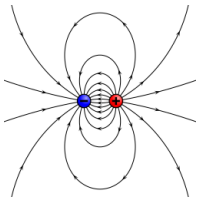


Figure 3: Two oppositely charged particles distanced from one another

well as definite directions. The dipole moment is given by the expression $\vec{p} = q\vec{d}$, where q is the charge on one end of the dipole and $\vec{d}$ is the distance between dipoles.

On the axis of the dipole (i.e. the lines formed by the two particles), the electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2sq}{r^3}\hat{p}$$

where r is the distance from the point in consideration to the center of the dipole. On the bisecting plane (i.e., the plane exactly halfway from each point) the field is given by

$$\vec{E} = \frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$$

Where r is the distance from the point to either dipole. The force on a positive point charge q_1 a distance of d away from the dipole, aligned with the dipole, and on the side of the negative charge is given by

$$\vec{F} = q_1\vec{E_{dipole}} = q_1(\frac{-1}{4\pi\epsilon_0}\frac{2qs}{d^3}, 0, 0)$$

Note that the field will be parallel to the axis of the dipole: this can simplfy vector calculations. Now consider a dipole in a uniform electric field, like so: The positive end will be pulled to the right

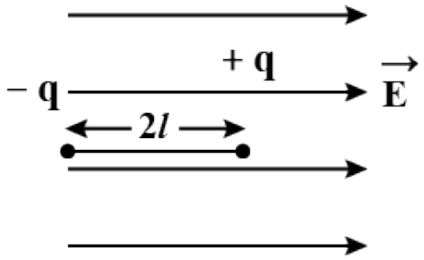


Figure 4: Dipole in uniform electric field

and the negative end to the left, exerting a torque given by $\vec{\tau} = \vec{p} \times \vec{E_{uniform}}$. Note that by the definition of $\times$, $\tau = pE_{uniform}\sin\theta$, where θ is the dipole's angle from horizontal. It can be shown that the potential energy of a dipole in a uniform eletric field is $U = -\vec{p} \cdot \vec{E_{uniform}}$ and similarly as before (but now with $\cdot$) $U = -pE_{uniform}\cos\theta$. Usefully, this means dipoles can be used to measure the direction of an electric field.

Throughout these examples, we have been assuming the associated speeds are much less than the speed of light. If the velocities approach a significant fraction of the speed of light Coulomb's law no longer holds, and we must account for relativity.

Conservation of Charge: Charge cannot be created nor destroyed, with the exception of electron-positron annihilation and other such quantum hijinks. We can use conservation of charge to predict the

Interestingly, in annihilation between positrons and electrons (or any other subatomic particles) the total energy and momentum of the initial pair are conserved in the process and distributed among a set of other particles in the final state (photons in the example of the electron-positron pair). Antiparticles have exactly opposite additive quantum numbers from particles, so the sums of all quantum numbers of such an original pair are zero. Hence any set of particles may be produced whose total quantum numbers are also zero as long as conservation of energy, conservation of momentum, and conservation of spin are obeyed.

behavior of many systems. For example, consider tape pulled from a roll. You may have noticed when dangling strips of freshly-pulled tape they tend to drift towards nearby surfaces to stick and become tangled: this is because peeling a strip of tape off a roll strips electrons from the tape, resulting in a net positive charge because of conservation of charge. When this charge approaches a net neutral object, such as your hand, the electrons in the atoms of your hand are attracted to the positively charged tape and congregate closer to the tape. This results in a negative charge buildup near the tape. The positive tape is attracted to the negative charges, and so the tape moves towards your hand and becomes tangled and useless. The process of one charge inducing a charge on a neutral object occurs often enough for the phenomenon to be named. We call it **Polarization**: The process by which a dipole is formed in a neutral object by an electric field. The dipole moment is given by $\vec{p} = \alpha\vec{E}$, where α is a material-dependent constant called polarizability. Such a dipole is *induced*. Consider a point charge near a neutral atom. The point charge creates an electric field given by

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

Inducing a dipole given by the expression

$$\vec{p} = \alpha\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{\alpha q_1}{r^2}\hat{r}$$

This dipole creates a field at the point charge of

$$\begin{aligned}
\vec{E_2} &= \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\alpha\vec{E_1}}{r^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\alpha}{r^3}\left(\frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}\right) \\
&= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}
\end{aligned}$$

This formula is valid provided the electric field that induced the dipole is from a point charge. If the electric field is instead from, say, another (permanent) dipole then $\vec{p} = \alpha\vec{E_1}$ is still valid. However, in this case the formula for $\vec{E_1}$ will be given by the equation for the electric field along the axis of a dipole instead. Following this logic,

$$\begin{aligned}
\vec{p} &= \alpha\vec{E_1} \\
&= \alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p_1}}{r^3}
\end{aligned}$$

If we let $\vec{r'}$ be the location of the permanent dipole from the induced dipole, we have the electric field at the permanent dipole as

$$\begin{aligned}\vec{E}(\vec{r'}) &= \left(\frac{1}{4\pi\epsilon_0}\frac{2}{r^3}\right)\left(\alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p_1}}{r'^3}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3r'^3}\end{aligned}$$

Calculating the force on the permanent dipole is slightly more complicated than multiplying by the charge of the dipole, since r' and the sign of q varies based on which end of the dipole we consider. To find the net force, we must find the force on each charge and sum them, like so:

$$\begin{aligned}F^+ &= q\vec{E}(r-\frac{s}{2})\\ F^- &= q\vec{E}(r+\frac{s}{2})\\ F_{net} &= F^+ + F^-\\ &= q\left[\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3(r+\frac{s}{2})^3} - \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left[\frac{4\alpha\vec{p_1}}{r^3(r+\frac{s}{2})^3} - \frac{4\alpha\vec{p_1}}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2(4\alpha\vec{p_1})\left[\frac{1}{r^3(r+\frac{s}{2})^3} - \frac{1}{r^3(r-\frac{s}{2})^3}\right]\end{aligned}$$

With a bit more algebraic simplification and the assumption that $r >> s$, we can show that

$$\begin{aligned}F_{net} &= F^+ + F^-\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{4\alpha\vec{p_1}}{r^6}\right)\left[(1+\frac{3s}{2r}) - (1-\frac{3s}{2r})\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{4\alpha\vec{p_1}}{r^6}\right)\left[\frac{3s}{r}\right]\\ &\approx q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{12\alpha\vec{p_1}s}{r^7}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{12\alpha p_1^2}{r^7}\end{aligned}$$

Insulators, conductors, and Van der Waals forces

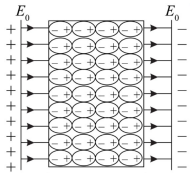
On the topic of induced dipoles, consider how freely moving atoms in a substance interact. The electrons are dispersed in a cloud about the

nucleus of each atom. These clouds can be thought of as constantly fluctuating, and occasionally these fluctuations will result in more electrons on one side than another. In this case there will be a small dipole. If this dipole approaches another atom (which may or may not have its own temporary dipole) the two will be attracted. The result of this interaction is that even neutral materials can be attracted to each other due to the fluctuating dipoles of its electron clouds. We call this phenomenon

Van der Waals forces: attraction and repulsions between atoms, molecules, as well as other intermolecular forces. Caused by correlations in the fluctuating polarizations of nearby particles (a consequence of quantum dynamics).

Insulator: An insulator is a material that does not easily allow electricity to pass through it. Inside an insulator the electrons are bound to their atoms, but they may still shift in response to an electric field (see fig. 5) This results in the insulator becoming polarized. Just as with most polarized objects, we can approximate its dipole moment with $\vec{p} = \alpha\vec{E}$ This relationship is only valid if the electrons are bound to the stationary constituent atoms. If this is not the case, then the a material in question is called a

Figure 5: Effect of electric field on insulator



Conductor: A conductor is a material that allows electricity to flow freely through it. For example, consider a liquid with charged atoms floating throughout. Here, when an electric field is applied, the charges each follow the electric field lines until they reach the edges of whatever container holds them. More commonly, we see conductors in the form of metals. The atomic structure of a metal allows electrons to move freely from atom to atom. Thus, within a metal, the electrons can freely move in whichever direction the electric field determines. "Freely" is used liberally here, since the electrons can collide with other electrons or defects in the metal and lose energy. The mobile electrons in the conductor will have a momentum and velocity given by

Such a liquid is called an *ionic solution*

Typically electrons are bound to the metal as a whole. However, if the electric field is strong enough, then air surrounding a metal can become ionized and allow electrons to flow freely through it, creating a spark.

$$\begin{aligned}
\vec{\Delta p} &= \vec{F_{net}}\Delta t \\
&= q\vec{E_{net}} \\
&= -e\vec{E_{net}} \\
&\rightarrow \vec{\Delta p} = -e\vec{E_{net}}\Delta t \\
&= m_e v \\
&\rightarrow v = \frac{e\vec{E_{net}}\Delta t}{m_e}
\end{aligned}$$

A good approximation for the average velocity of an electron is $\bar{v} = \mu E$. The constant μ is called the "mobility".

Now, consider a conductor with a net charge, such as a charged sphere. Inside the sphere like charges will repel each another and push to get as far away from one another as possible. This occurs when all the charges are on the surface of the conducting object, since anywhere else would be closer together and they cannot go outside the bounds of the object. This rearrangement has two effects. First, any charge in a conductor will be found on the surface. Second, the net electric field inside a conductor will be zero. An interesting result of electrons seeking to minimize repulsion inside a conductor is the *sharp point effect*. To visualize this effect, imagine a gymnasium full of students pretending to be electrons, staying as far away from others as possible. Anyone near the center of the crowd will feel badly pressed and will try to work there way towards the edge of the gym, where at least one side will no longer have fellow students milling about. The result? Most of the students will gravitate towards the edge of the gym and hover there, to take advantage of that lack of other students on the wall side of the gym. Now imagine a narrow corridor leading out of the gym. Even better! Students in that corridor will only have fellow students behind and in front of them. Now imagine the very end of that corridor, a sort of point. Even better! Now, the student who finds that spot will benefit from having only one student nearby. But somewhat ironically, that same effect will cause other students to pack themselves into the long, narrow corridor more tightly, since pretty much anywhere in the corridor makes them less exposed to the full set of students than being in the gym does. This effect makes edges, wires, and points more attractive to electrons, which similarly just don't want other electrons nearby.

To see why this must be the case, imagine if it were not. Then any charge within the conductor would be moved by the electric field, so we see that the case where no electric field is present is the only stable possibility. However, if electrons are moving, then there can be (and is) an electric field within the conductor. It is only when the conductor is in static equilibrium that there is no net electric field within.

Finding electric field

Now that we understand conceptually how charge behaves in a conductor, let's think about the electric field that charge creates. We can visualize any charged object as a collection of point charges in the shape of the object. If we'd like to find the net electric field of these charges, we simply use Coulomb's law and sum up each of their electric fields. To illustrate this approach, imagine a charged rod with length L and charge Q. We can approximate it as a bunch of charges in a row. For now, let's use ten, but recognize that the more charges we use the better our estimate will be. Each piece of the rod will have a charge of $\frac{Q}{10}$, so to find the electric field at a point we would need to calculate the vector between each piece and the point and use $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. We would do this ten times and then sum the electric fields to get our approximate net field. Of course, perhaps we would like to find the exact electric field. To do this we would have to cut

our object up into an infinite number of points, find the infinitesimal electric field due to each, and sum them up. Sound familiar?

To drive the point home, consider a vertical rod with a uniform charge of Q and length L. Let's take a point on the bisecting plane of this rod (the horizontal plane that cuts the rod in half). Say we view this point from the side and see that is has coordinates $(0, x)$. If we want to find the electric field due to a little bit of the rod at $\vec{x}$, we need to know the distance between them. If the bit is a distance of y up the rod, the distance between it and $\vec{x}$ will be $r = \sqrt{x^2 + y^2}$. The charge of this small piece will be $\frac{Q}{L}\Delta y$ (where Δy is the height of the peice), and $\hat{r}$ will be $\frac{(x,-y)}{\sqrt{x^2+y^2}}$. Therefore the electric field at $\vec{x}$ will be given by

$$\begin{aligned}\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2+y^2}\frac{(x,-y)}{\sqrt{x^2+y^2}}\end{aligned}$$

We can see by symmetry (isn't symmetry lovely?) that the contributions in the y direction will cancel, so we need only consider the x direction. Therefore,

$$\begin{aligned}\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} \\ &= \frac{Q}{4\pi\epsilon_0 L}\frac{x\Delta y}{(x^2+y^2)^{3/2}}\end{aligned}$$

Now comes the tricky part: adding these up. You have already likely guessed that we will need to integrate between the bottom and top of the rod, which corresponds to the integral from $-\frac{L}{2}$ to $\frac{L}{2}$. We have then

$$\vec{E} = \int_{-L/2}^{L/2}\frac{Q}{4\pi\epsilon_0 L}\frac{x}{(x^2+y^2)^{3/2}}dy$$

I'll spare you the tedious integration and skip to the result:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{x\sqrt{x^2+(L/2)^2}}\right]\hat{x}$$

The steps for finding the electric field due to a charged object in general are outlined below.

1. Cut up the charge distribution into pieces and draw $\Delta\vec{E}$
   - Divide the charge distribution into pieces whose field is known. In particular, very small pieces can be approximated by point

particles. You may also wish to break up a complex object into smaller objects whose electric field equations are already known.

- Pick a representative piece, and at the location of interest draw a vector $\Delta\vec{E}$ showing the contribution to the electric field of this representative piece. Drawing this vector helps you figure out the direction of the net field at the location of interest.

2. Write an expression for the electric field due to one piece

   - Pick an origin for your coordinate system, and show it on your diagram. Draw the vector $\vec{r}$ from the source piece to the observation location. Write algebraic expressions for $\vec{r}$ and $\hat{r}$.
   - Write an algebraic expression for the magnitude $|\Delta\vec{E}|$ contributed by the representative piece. Multiply by $\hat{r}$ to get the vector $\Delta\vec{E}$. You can break this up into x, y, and z components for integration. Once you do each expression should contain one or more "integration variables" ($\Delta x/\Delta y/\Delta z$ or $dx/dy/dz$) related to the coordinates of the piece. Write the amount of charge on the piece, Δq, in terms of your variables.

3. Sum the contributions of all the pieces

   - The net field is the sum of the contributions of all the pieces. To write the sum as a definite integral, you must include limits given by the range of the integration variable. If the integral can be done symbolically, do it. If not, choose a finite number of pieces and do the sum with a calculator or a computer.

4. Check the result

   - Check that the direction of the net field is qualitatively correct.
   - Check the units of your result, which should be newtons per coulomb.
   - Look at special cases. For example, if the net charge is nonzero, your result should reduce to the field of a point charge when you are very far away. For a numerical integration on a computer, check that the computation gives the correct numerical result for special cases that can be calculated by hand.