Notes for ECE 20002 - Electrical Engineering Fundamentals II

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These are lecture notes for spring 2024 ECE 20002 at Purdue. Modify, use, and distribute as you please.

Contents

Course Introduction

Continuation of Electrical Engineering Fundamentals I. The course addresses mathematical and computational foundations of circuit analysis (differential equations, Laplace Transform techniques) with a focus on application to linear circuits having variable behavior as a function of frequency, with emphasis on filtering. Variable frequency behavior is considered for applications of electronic components through single-transistor and operational amplifiers. The course ends with a consideration of how circuits behave and may be modeled for analysis at high frequencies.

Learning Objectives:

1. Analyze 2nd order linear circuits with sources and/or passive elements
2. Compute responses of linear circuits with and without initial conditions via one-sided Laplace transform techniques
3. Compute responses to linear circuits using transfer function and convolution techniques

4. Analyze and design transistor amplifiers at low, mid and high frequencies

ECE 20001 Review

Sinusoidal Signal (voltage and current) involve phasors, which bring complex numbers to the forefront. When in Sinusoidal Steady State (SSS):

$$Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = \frac{1}{j\omega C}$$

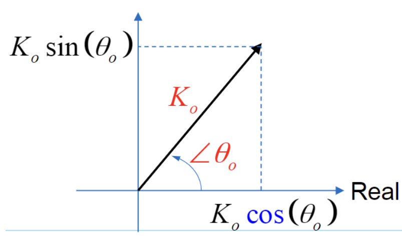
This can in turn be represented as as the following function:

$$x(t) = K_0 cos(\omega t + \theta_0)$$

Which can be transformed into the following form:

$$\begin{aligned} K_0 e^{j(\omega t + \theta_0)} &= K_0 e^{j\theta_0} \\ &= K_0(cos(\theta_0) + jsin(\theta_0)) \\ &= K_0 \underline{/\theta_0} \end{aligned}$$

This can be represented as the following in the cartesian plane:



Thus, these forms can be summed as following:

x(t)	**X**
$Kcos(\omega t)$	K
$Ksin(\omega t) = Kcos(\omega t)$	$-Kj$
$cos(\omega t) - sin(wt)$	$1 + j$
$acos(\omega t) + bsin(\omega t)$	$a - bj$

This is especially useful for circuit analysis methods such as KCL and KVL. The methods of conversion between polar and phasor are:

$$z = a + bj$$

$$z = \rho \underline{/\theta}$$

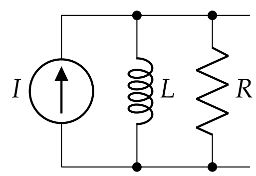
$$\rho = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \text{phase}(z) = tan^{-1}(\frac{b}{a})$$

These conversions and operations alongside KCL and KVL, can allow us to create a system of differential equations that will allow us to solve almost any circuit. However, we don't like ODEs, so we have developed methods to get around this.
We know that at SSS, the following equations are valid.

$$
\begin{aligned}
V_R &= RI_R \\
V_L &= j\omega L I_L \\
V_C &= \frac{1}{j\omega C} I_C \\
V_L(t) &= L\frac{di_L}{dt} && \implies i_L(t) = i_L(0) + \frac{1}{L}\int v_L \, dt \\
i_C(t) &= C\frac{dV_C}{dt} && \implies V_C(t) = V_C(0) + \frac{1}{C}\int i_C \, dt
\end{aligned}
$$

This can be used to solve most (to the knowledge of the student till this point) SSS circuits. Strategies have been developed to simplify these calculations. Some examples of this are current and voltage division, with the impedance Z replacing the linear R. Strategies like these allow the resolution of circuits like these while avoiding higher order differential equations.
Example:



$$
\begin{aligned}
I_R &= \frac{Z_L}{Z_R + Z_L} * I = \frac{j\omega L}{R + j\omega L} * I \\
I_L &= \frac{Z_R}{Z_R + Z_L} * I = \frac{R}{R + j\omega L} * I
\end{aligned}
$$

If this circuit had an additional energy storing component, such as a capacitor or an additional inductor, the complexity of current division does not increase by a lot, but the differential equation would go from being a first-order differential equation to a second-order differential equation. The complexity of this would be much higher than that of the first-order differential equation, and would continue increasing for each energy storing component that is added to the circuit.
Note: All of this is based on the assumption of **Sinusoidal Steady State**.

"We don't like differential equations." -Prof. Byunghoo Jung

Using Ordinary Differential Equations to solve RL and RC circuits

This section will provide a brief explanation of nonhomogenous differential equations and their applications to circuits (Completely ignoring the fact that we in fact do not like them, as repeatedly stated in the previous section).

Ordinary Differential Equations (ODE) Overview

Differential equations rely a lot on two properties. The invariability of the exponential function across derivatives, and the ability to express any function in terms of the exponential function itself. That is,

$$e^t = \frac{d}{dt}e^t = \frac{d^2}{dt^2}e^t \cdots$$

Let us take the following differential equation as an example:

$$y(t) = 6x(t) + 3\frac{d}{dt}x(t)$$

Because of the beauty (invariability) of the exponential function, we assume $x_h(t) = Ae^{\lambda t}$, and try to solve the homogenous equation $0 = 6x(t) + 3x'(t)$.

$$\begin{aligned} 0 &= 6x + 3x' \\ 0 &= 6Ae^{\lambda t} + 3A\lambda e^{\lambda t} \\ 0 &= 6 + 3\lambda \\ \lambda &= -2 \\ \implies x_h(t) &= e^{-2t} \end{aligned}$$

At this point, we will consider two different cases: $y(t) = 4e^{3t}$ and $y(t) = 3e^{-2t}$. We will find the particular solution $x_p(t)$ for each of these cases, as they have different methods of resolution.

$$\begin{aligned} 4e^{3t} &= 6x + 3x' \\ \text{Assume } x_p(t) &= Be^{3t} \\ 4e^{3t} &= 6Be^{3t} + 9Be^{3t} \\ 4 &= 15B \\ B &= \frac{15}{4} \\ \implies x_p(t) &= \frac{15}{4}e^{3t} \\ \implies x(t) &= x_h(t) + x_p(t) = Ae^{-2t} + \frac{15}{4}e^{3t} \end{aligned}$$

Let us assume $y(0) = 4$

$$\begin{aligned}
\implies 4 &= A + \frac{15}{4} \\
\implies A &= \frac{1}{4} \\
\implies x(t) &= \frac{1}{4}e^{-2t} + \frac{15}{4}e^{3t}
\end{aligned}$$

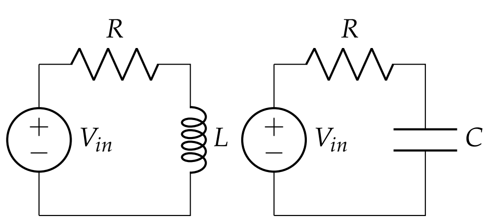
On the other hand, if $y(t) = 3e^{-2t}$, since $x_p(t)$ and $x_h(t)$ cannot have the same form, we will need to make a few changes:

$$\begin{aligned}
3e^{-2t} &= 6x + 3x' \\
\text{Assume } x_p(t) &= Bte^{-2t} \\
3e^{-2t} &= 6Bte^{-2t} + 3Be^{-2t} - 6Bte^{-2t} \\
\implies 3 &= 3B \\
\implies B &= 1 \\
\implies x_p(t) &= te^{-2t}
\end{aligned}$$

Which can then be substituted into $x(t)$, allowing the differential equation to be fully solved by using the initial conditions.

Using ODEs in circuits

We will be solving the following two circuits (Note: you will never see something like this with a current source instead, because it would be too easy):



In the inductor circuit, if $i_L(0) = 0$, we know:

$$\begin{aligned}
v_L &= L\frac{di_L(t)}{dt} \\
v_L(t) &= v_{in}(t) + Ri_L(t) \text{ because of KVL} \\
v_{in}(t) &= L\frac{di_L(t)}{dt} + Ri_L(t) \text{ by substituting}
\end{aligned}$$

By dividing the timeline into two sections, $(-\infty, 0)$ and $(0, \infty)$, and solving for the second one we can find our desired values for anytime in the future. However, for this to be a viable option, we need to solve for a continuous variable, which is i for an inductor, and v for a capacitor.

By solving this equation as shown previously, and looking for i_L, we will find that $\lambda = -\frac{R}{L}$ which is in fact the inverse of the time constant for an RL circuit. Under the assumption of V_{in} being a constant, the particular solution will in fact be $i_p(t) = \frac{V_{in}}{R} = i_L(\infty)$. This results in the familiar equation:

$$i_L(t) = i_L(\infty) + (i_L(t_0) - i_L(\infty))e^{-\frac{R}{L}(t-t_0)}$$

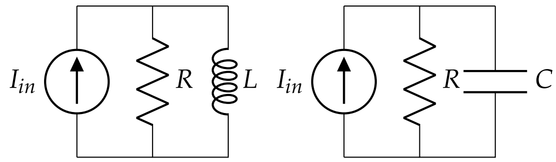
The same is done for the capacitor circuit, but this time looking for v_C, which ends up resulting in the once again familiar equation:

$$v_C(t) = v_C(\infty) + (v_C(t_0) - v_C(\infty))e^{-\frac{t-t_0}{RC}}$$

Note: energy storing components (L & C) have memory but cannot consume power. R can consume power but has no memory.
Looking at this conceptually, the resistor is consuming energy at an exponential rate. This causes current, voltage, and all other signals to decay exponentially as well.
Since $P = I^2R = \frac{V^2}{R}$, it makes sense for the time constant to be $\tau = \frac{L}{R} = RC$, for each of them, as a larger capacitoror inductor would store more energy, and the proportionality of the R to the power and current or voltage respectively. That is, if R is larger, the power consumption is larger for an RL and smaller for an RC.
Note: the particular solution of the ODE is a scaled version of the input.
The homogenous solution will be the same for all inputs, and of the form $Ae^{\frac{-t}{\tau}}$.
The particular solution represents the behavior of the analyzed parameter when everything has settled down, or, at infinity.
Let us look at the particular resolution with a forcing equation of the form $v_{in}(t) = A\sin(\omega t)u(t)$.

$$\begin{aligned} v_{C,p} &= \frac{Z_C}{Z_R + Z_C} V_{in} \\ &= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}(-Aj) \\ &= \frac{1}{j\omega RC + 1}(-Aj) \end{aligned}$$

Deveoping and simifying that equation can lead to the form $A_1\cos(\omega t) + A_2\sin(\omega t)$.
These can then be substituted into teh complete form, and, using the initial conditions, completely define the reactions. We will now analyze the following circuits, which resemble the previous ones, but use voltage instead, and have teh components in parallel instead of series.



The only solution we will analyze in this document, is the RC circuit, but the RL circuit can be similarly solved.

$$i_{in}(t) = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

To solve this, we need to analyze the homogenous solution first,

which is equivalent to there being no source in the circuit.

$$0 = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

$$0 = RC\frac{dv_C(t)}{dt} + v_C(t)$$

Assuming that $v_{C,h}(t) = Ae^{\lambda t}$, which results in:

$$0 = RC\lambda^1 + \lambda^0 = RC\lambda + 1$$

$$\lambda = -\frac{1}{RC} = -\frac{1}{\tau}$$

$$\implies v_{C,h} = Ae^{\frac{-t}{\tau}}$$

All signals in a circuit of this form will also decay exponentially, just like in the previous cases.
Now we can solve for the particular solution of the ODE.

$$i_{in}(t) = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

With a particular solution of the form $v_{C,p} = m$

$$k = C\frac{d(m)}{dt} + \frac{1}{R}m = \frac{1}{R}m$$

$$\implies m = kR$$

We can solve for inputs of constant, exponential, and sinusoidal inputs quite easily. If $i_{in}(t) = 5e^{-3t}$, and thus $v_{C,p} = me^{-3t}$ and $v_C(0^-) = 3$

$$i_{in}(t) = C\frac{dv_C}{dt} + \frac{1}{R}v_C$$

$$5e^{-3t} = -3Cme^{-3t} + \frac{1}{R}me^{-3t}$$

$$5 = \left(-3C + \frac{1}{R}\right)m$$

$$\implies m = \frac{5R}{-3RC + 1}$$

Substituting into the equation and using the initial value leads to the following final form:

$$\begin{aligned}
v_C(t) &= v_{C,h}(t) + v_{C,p}(t) \\
&= Ae^{-\frac{t}{\tau}} + \frac{5R}{-3RC+1}e^{-3t} \\
3 &= A + \frac{5R}{-3RC+1} \\
\implies A &= 3 - \frac{5R}{-3RC+1}
\end{aligned}$$

and the final form will be:

$$v_C(t) = \left(3 - \frac{5R}{-3RC+1}\right)e^{\frac{-t}{RC}} + \frac{5R}{-3RC+1}e^{-3t}$$

Similar solutions can be found for different inputs, and for RL circuits as well.

Using ODEs for LC circuits

Solving Second Order ODEs

We can start this by understanding the method of resolution of second order ODEs. For example:

$$
\begin{aligned}
3e^{-3t} &= 2x(t) + 3\frac{dx(t)}{dt} + \frac{d^2x(t)}{dt^2} \\
x(0) &= 2 \\
x'(0) &= 1
\end{aligned}
$$

Which leads to the following resolution of the homogenous equation:

$$
\begin{aligned}
0 &= \lambda^2 + 3\lambda + 2 \\
0 &= (\lambda+1)(\lambda+2) \\
\implies \lambda &= -1, -2 \\
\implies x_h(t) &= A_1 e^{-t} + A_2 e^{-2t}
\end{aligned}
$$

Now we solve the particular equation under the assumption $x_p(t) = ke^{-3t}$:

$$
\begin{aligned}
3e^{-3t} &= 2ke^{-3t} + -9ke^{-3t} + 9ke^{-3t} \\
3 &= 2k - 9k + 9k \\
\implies k &= \frac{3}{2} \\
\implies x_p(t) &= \frac{3}{2}e^{-3t}
\end{aligned}
$$

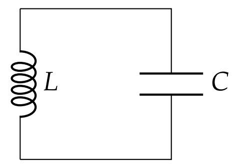
Now, using $x(t) = x_p(t) + x_h(t)$:

$$
\begin{aligned}
x(t) &= A_1 e^{-t} + A_2 e^{-2t} + \frac{3}{2}e^{-3t} \\
x(0) &= 2 \\
x'(0) &= 1 \\
2 &= A_1 + A_2 + \frac{3}{2} \\
1 &= -A_1 - 2A_2 - \frac{9}{2} \\
\implies A_2 &= -6 \\
\implies A_1 &= \frac{13}{2} \\
\implies x(t) &= -6e^{-t} + \frac{13}{2}e^{-2t} + \frac{3}{2}e^{-3t}
\end{aligned}
$$

Had the forcing function $3e^{-3t}$ instead been of the form $B_1e^{-4t} + B_2e^{-3t}$, the particular solution would have needed to be of the form $k_1e^{-4t} + k_2e^{-3t}$.

Concept

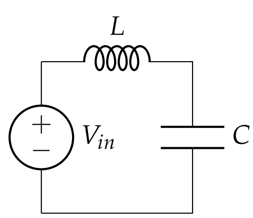
Let us observe the following circuit:



Both the inductor and the capacitor can store energy, and "send" energy to each other, creating oscillations, but maintain the total energy in the circuit continuous. Circuits of this form are called LC tanks.
Thus, the solution, as does the ideal pendulum, can be defined by sinusoidals.

Solving LC circuits

We will begin by solving the following circuit:



Which has a behavior defined by:

$$
\begin{aligned}
v_{in}(t) &= v_L(t) + v_C(t) \\
i_L(t) &= i_C(t) \\
v_L(t) &= L\frac{di_L(t)}{dt} \\
i_C(t) &= C\frac{dv_C(t)}{dt} \\
\implies v_L(t) &= LC\frac{d^2v_C(t)}{dt^2} \\
\implies v_{in}(t) &= LC\frac{d^2v_C(t)}{dt^2} + v_C(t)
\end{aligned}
$$

This will now allow us to solve for $v_C(t)$.

$$
\begin{aligned}
v_{in}(t) &= v_C(t) + LCv_C''(t) \\
&\text{Solving the homogenous equation:} \\
0 &= v_C(t) + LCv_C''(t) \\
\implies 0 &= 1 + LC\lambda^2 \\
\implies \lambda &= \sqrt{\frac{-1}{LC}} = \pm j\sqrt{\frac{1}{LC}} \\
\text{if } \omega_0 &= \sqrt{\frac{1}{LC}} \\
v_{C,h} &= A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} \\
&= A_1(\cos(\omega_0 t) + j\sin(\omega_0 t)) + A_2(\cos(-\omega_0 t) + j\sin(-\omega_0 t)) \\
&= (A_1 + A_2)\cos(\omega_0 t) + j(A_1 - A_2)\sin(\omega_0 t) \\
&= B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t)
\end{aligned}
$$

Now we will search for the particular solution with an exponential forcing function, $V_{in} = Fe^{-\omega t}$

$$Fe^{-}\omega t = v_C(t) + LCv_C''(t)$$

With the particular solution of the form $v_{C,p} = Ae^{-\omega t}$

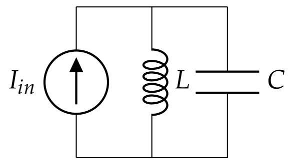
$$
\begin{aligned}
Fe^{-\omega t} &= LC\omega^2 Ae^{-\omega t} + Ae^{-\omega t} \\
F &= LC\omega^2 A + A \\
\implies A &= \frac{F}{LC\omega^2 + 1} \\
\implies v_{C,p} &= \frac{F}{LC\omega^2 + 1} e^{-\omega t} \\
\implies v_C &= B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t) + \frac{F}{LC\omega^2 + 1} e^{-\omega t}
\end{aligned}
$$

We can now solve for the complete equation if we have the initial conditions.
We could also have solved this for a forcing equation of the form $v_{in}(t) = F_1\cos(\omega t) + F_2\sin(\omega t)$

$$
\begin{aligned}
F_1\cos(\omega t) + F_2\sin(\omega t) &= v_C(t) + LCv_C''(t) \\
v_{C,p} &= A_1\cos(\omega t) + A_2\sin(\omega t) \\
F_1\cos(\omega t) + F_2\sin(\omega t) &= -LC\omega^2 A_1\cos(\omega t) - LC\omega^2 A_2\sin(\omega t) + A_1\cos(\omega t) + A_2\sin(\omega t) \\
&= (-LC\omega^2 + 1)(A_1\cos(\omega t) + A_2\sin(\omega t)) \\
\implies A_1 &= \frac{F_1}{1 - LC\omega^2} \\
A_2 &= \frac{F_2}{1 - LC\omega^2}
\end{aligned}
$$

These values can then be subtituted into the complete form and B_1 and B_2 can be found using the initial conditions.

We will now solve the following circuit:



In the same way as the last one, we get the differential equation, which in this case is defined by:

$$i_{in} = LC\frac{d^2 i_L(t)}{dt^2} + i_L(t)$$

However, we can observe that the equation is of the same form as the previous one, but with i replacing v. The solution will be of the exact same form, and thus needs no additional resolution.
What is important to remember from this is the following:

$$\begin{aligned} v_{C,h}(t) &= A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t) \\ &= K \cos(\omega t + \theta) \\ i_{L,h}(t) &= B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t) \\ &= K \cos(\omega t + \theta) \\ \omega &= \frac{1}{\sqrt{LC}} \end{aligned}$$

If they were to give initial conditions of the form $v_C(0) = k_1, i_L(0) = k_2$ instead of $v_C(0) = k_1, v'_C(0) = k_2$, we can use:

$$i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt}$$

or

$$v_C(t) = v_L(t) = L\frac{di_L(t)}{dt}$$

to succesfully derive the values of the missing constants.