Notes for ECE 20002 - Electrical Engineering Fundamentals II

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These are lecture notes for spring 2024 ECE 20002 at Purdue as taught by Professor Byunghoo Jung alongside recordings by Professor Michael Capano. Modify, use, and distribute as you please.

Contents

Course Introduction

Continuation of Electrical Engineering Fundamentals I. The course addresses mathematical and computational foundations of circuit analysis (differential equations, Laplace Transform techniques) with a focus on application to linear circuits having variable behavior as a function of frequency, with emphasis on filtering. Variable frequency behavior is considered for applications of electronic components through single-transistor and operational amplifiers. The course ends with a consideration of how circuits behave and may be modeled for analysis at high frequencies.

Learning Objectives:

1. Analyze 2nd order linear circuits with sources and/or passive elements

2. Compute responses of linear circuits with and without initial conditions via one-sided Laplace transform techniques

3. Compute responses to linear circuits using transfer function and convolution techniques

4. Analyze and design transistor amplifiers at low, mid and high
    frequencies

ECE 20001 Review

Sinusoidal Signal (voltage and current) involve phasors, which bring complex numbers to the forefront. When in Sinusoidal Steady State (SSS):

$$Z_R = R$$
$$Z_L = j\omega L$$
$$Z_C = \frac{1}{j\omega C}$$

This can in turn be represented as as the following function:

$$x(t) = K_0 cos(\omega t + \theta_0)$$

Which can be transformed into the following form:

$$K_0 e^{j(\omega t + \theta_0)} = K_0 e^{j\theta_0}$$
$$= K_0(cos(\theta_0) + jsin(\theta_0))$$
$$= K_0 \underline{/\theta_0}$$

This can be represented as the following in the cartesian plane:
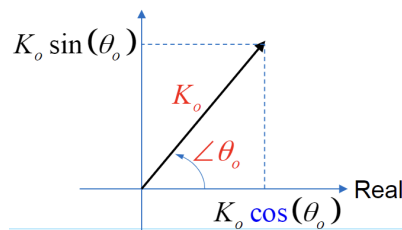
Thus, these forms can be summed as following:

x(t)	X
$K cos(\omega t)$	K
$K sin(\omega t) = K cos(\omega t)$	$-Kj$
$cos(\omega t) - sin(wt)$	$1 + j$
$a cos(\omega t) + b sin(\omega t)$	$a - bj$

This is especially useful for circuit analysis methods such as KCL and KVL. The methods of conversion between polar and phasor are:

$$z = a + bj$$
$$z = \rho \underline{/\theta}$$
$$\rho = |z| = \sqrt{a^2 + b^2}$$
$$\theta = \text{phase}(z) = tan^{-1}(\frac{b}{a})$$

These conversions and operations alongside KCL and KVL, can allow us to create a system of differential equations that will allow us to solve almost any circuit. However, we don't like ODEs, so we have developed methods to get around this.

We know that at SSS, the following equations are valid.

$$V_R = RI_R$$

$$V_L = j\omega L I_L$$

$$V_C = \frac{1}{j\omega C} I_C$$

$$V_L(t) = L\frac{di_L}{dt} \qquad \implies i_L(t) = i_L(0) + \frac{1}{L}\int v_L\,dt$$

$$i_C(t) = C\frac{dV_C}{dt} \qquad \implies V_C(t) = V_C(0) + \frac{1}{C}\int i_C\,dt$$

This can be used to solve most (to the knowledge of the student till this point) SSS circuits. Strategies have been developed to simplify these calculations. Some examples of this are current and voltage division, with the impedance Z replacing the linear R. Strategies like these allow the resolution of circuits like these while avoiding higher order differential equations.

Example:
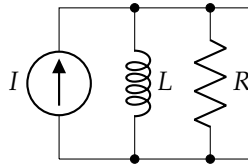
$$I_R = \frac{Z_L}{Z_R + Z_L} * I = \frac{j\omega L}{R + j\omega L} * I$$

$$I_L = \frac{Z_R}{Z_R + Z_L} * I = \frac{R}{R + j\omega L} * I$$

If this circuit had an additional energy storing component, such as a capacitor or an additional inductor, the complexity of current division does not increase by a lot, but the differential equation would go from being a first-order differential equation to a second-order differential equation. The complexity of this would be much higher than that of the first-order differential equation, and would continue increasing for each energy storing component that is added to the circuit.

Note: All of this is based on the assumption of **Sinusoidal Steady State**.

"We don't like differential equations." -Prof. Byunghoo Jung

Using Ordinary Differential Equations to solve RL and RC circuits

This section will provide a brief explanation of nonhomogenous differential equations and their applications to circuits (Completely ignoring the fact that we in fact do not like them, as repeatedly stated in the previous section).

Ordinary Differential Equations (ODE) Overview

Differential equations rely a lot on two properties. The invariability of the exponential function across derivatives, and the ability to express any function in terms of the exponential function itself. That is,

$$e^t = \frac{d}{dt}e^t = \frac{d^2}{dt^2}e^t \cdots$$

Let us take the following differential equation as an example:

$$y(t) = 6x(t) + 3\frac{d}{dt}x(t)$$

Because of the beauty (invariability) of the exponential function, we assume $x_h(t) = Ae^{\lambda t}$, and try to solve the homogenous equation $0 = 6x(t) + 3x'(t)$.

$$0 = 6x + 3x'$$
$$0 = 6Ae^{\lambda t} + 3A\lambda e^{\lambda t}$$
$$0 = 6 + 3\lambda$$
$$\lambda = -2$$
$$\implies x_h(t) = e^{-2t}$$

At this point, we will consider two different cases: $y(t) = 4e^{3t}$ and $y(t) = 3e^{-2t}$. We will find the particular solution $x_p(t)$ for each of these cases, as they have different methods of resolution.

$$4e^{3t} = 6x + 3x'$$
$$\text{Assume } x_p(t) = Be^{3t}$$
$$4e^{3t} = 6Be^{3t} + 9Be^{3t}$$
$$4 = 15B$$
$$B = \frac{15}{4}$$
$$\implies x_p(t) = \frac{15}{4}e^{3t}$$
$$\implies x(t) = x_h(t) + x_p(t) = Ae^{-2t} + \frac{15}{4}e^{3t}$$

Let us assume $y(0) = 4$

$$\implies 4 = A + \frac{15}{4}$$

$$\implies A = \frac{1}{4}$$

$$\implies x(t) = \frac{1}{4}e^{-2t} + \frac{15}{4}e^{3t}$$

On the other hand, if $y(t) = 3e^{-2t}$, since $x_p(t)$ and $x_h(t)$ cannot have the same form, we will need to make a few changes:

$$3e^{-2t} = 6x + 3x'$$

$$\text{Assume } x_p(t) = Bte^{-2t}$$

$$3e^{-2t} = 6Bte^{-2t} + 3Be^{-2t} - 6Bte^{-2t}$$

$$\implies 3 = 3B$$

$$\implies B = 1$$

$$\implies x_p(t) = te^{-2t}$$

Which can then be substituted into $x(t)$, allowing the differential equation to be fully solved by using the initial conditions.

Using ODEs in circuits

We will be solving the following two circuits (Note: you will never see something like this with a current source instead, because it would be too easy):
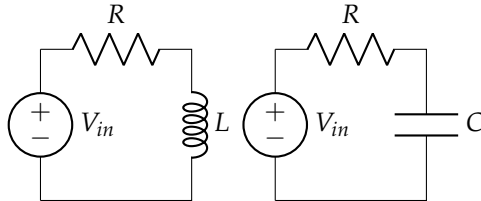
In the inductor circuit, if $i_L(0) = 0$, we know:

$$v_L = L\frac{di_L(t)}{dt}$$

$$v_L(t) = v_{in}(t) + Ri_L(t) \text{ because of KVL}$$

$$v_{in}(t) = L\frac{di_L(t)}{dt} + Ri_L(t) \text{ by substituting}$$

By dividing the timeline into two sections, $(-\infty, 0)$ and $(0, \infty)$, and solving for the second one we can find our desired values for anytime in the future. However, for this to be a viable option, we need to solve for a continuous variable, which is i for an inductor, and v for a capacitor.

By solving this equation as shown previously, and looking for i_L, we will find that $\lambda = -\frac{R}{L}$ which is in fact the inverse of the time constant for an RL circuit. Under the assumption of V_{in} being a constant, the particular solution will in fact be $i_p(t) = \frac{V_{in}}{R} = i_L(\infty)$. This results in the familiar equation:

$$i_L(t) = i_L(\infty) + (i_L(t_0) - i_L(\infty))e^{-\frac{R}{L}(t-t_0)}$$

The same is done for the capacitor circuit, but this time looking for v_C, which ends up resulting in the once again familiar equation:

$$v_C(t) = v_C(\infty) + (v_C(t_0) - v_C(\infty))e^{-\frac{t-t_0}{RC}}$$

Note: energy storing components (L & C) have memory but cannot consume power. R can consume power but has no memory.
Looking at this conceptually, the resistor is consuming energy at an exponential rate. This causes current, voltage, and all other signals to decay exponentially as well.
Since $P = I^2 R = \frac{V^2}{R}$, it makes sense for the time constant to be $\tau = \frac{L}{R} = RC$, for each of them, as a larger capacitor or inductor would store more energy, and the proportionality of the R to the power and current or voltage respectively. That is, if R is larger, the power consumption is larger for an RL and smaller for an RC.
Note: the particular solution of the ODE is a scaled version of the input.
The homogenous solution will be the same for all inputs, and of the form $Ae^{\frac{-t}{\tau}}$.
The particular solution represents the behavior of the analyzed parameter when everything has settled down, or, at infinity.
Let us look at the particular resolution with a forcing equation of the form $v_{in}(t) = A\sin(\omega t)u(t)$.

$$v_{C,p} = \frac{Z_C}{Z_R + Z_C}V_{in}$$

$$= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}(-Aj)$$

$$= \frac{1}{j\omega RC + 1}(-Aj)$$

Developing and simplifying that equation can lead to the form $A_1\cos(\omega t) + A_2\sin(\omega t)$.
These can then be substituted into the complete form, and, using the initial conditions, completely define the reactions. We will now analyze the following circuits, which resemble the previous ones, but use voltage instead, and have the components in parallel instead of series.
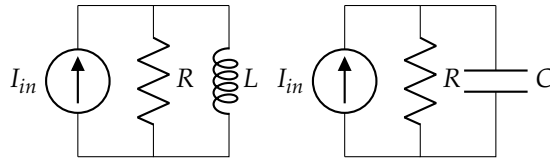
The only solution we will analyze in this document, is the RC circuit, but the RL circuit can be similarly solved.

$$i_{in}(t) = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

To solve this, we need to analyze the homogenous solution first, which is equivalent to there being no source in the circuit.

$$0 = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

$$0 = RC\frac{dv_C(t)}{dt} + v_C(t)$$

Assuming that $v_{C,h}(t) = Ae^{\lambda t}$, which results in:

$$0 = RC\lambda^1 + \lambda^0 = RC\lambda + 1$$

$$\lambda = -\frac{1}{RC} = -\frac{1}{\tau}$$

$$\implies v_{C,h} = Ae^{\frac{-t}{\tau}}$$

All signals in a circuit of this form will also decay exponentially, just like in the previous cases.

Now we can solve for the particular solution of the ODE.

$$i_{in}(t) = C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t)$$

With a particular solution of the form $v_{C,p} = m$

$$k = C\frac{d(m)}{dt} + \frac{1}{R}m = \frac{1}{R}m$$

$$\implies m = kR$$

We can solve for inputs of constant, exponential, and sinusoidal inputs quite easily. If $i_{in}(t) = 5e^{-3t}$, and thus $v_{C,p} = me^{-3t}$ and $v_C(0^-) = 3$

$$i_{in}(t) = C\frac{dv_C}{dt} + \frac{1}{R}v_C$$

$$5e^{-3t} = -3Cme^{-3t} + \frac{1}{R}me^{-3t}$$

$$5 = \left(-3C + \frac{1}{R}\right)m$$

$$\implies m = \frac{5R}{-3RC + 1}$$

Substituting into the equation and using the initial value leads to the

following final form:

$$v_C(t) = v_{C,h}(t) + v_{C,p}(t)$$

$$= Ae^{-\frac{t}{\tau}} + \frac{5R}{-3RC+1}e^{-3t}$$

$$3 = A + \frac{5R}{-3RC+1}$$

$$\implies A = 3 - \frac{5R}{-3RC+1}$$

and the final form will be:

$$v_C(t) = \left(3 - \frac{5R}{-3RC+1}\right)e^{\frac{-t}{RC}} + \frac{5R}{-3RC+1}e^{-3t}$$

Similar solutions can be found for different inputs, and for RL circuits as well.

Using ODEs for LC circuits

Solving Second Order ODEs

We can start this by understanding the method of resolution of second order ODEs. For example:

$$3e^{-3t} = 2x(t) + 3\frac{dx(t)}{dt} + \frac{d^2x(t)}{dt^2}$$

$$x(0) = 2$$

$$x'(0) = 1$$

Which leads to the following resolution of the homogenous equation:

$$0 = \lambda^2 + 3\lambda + 2$$

$$0 = (\lambda + 1)(\lambda + 2)$$

$$\implies \lambda = -1, -2$$

$$\implies x_h(t) = A_1e^{-t} + A_2e^{-2t}$$

Now we solve the particular equation under the assumption $x_p(t) = ke^{-3t}$:

$$3e^{-3t} = 2ke^{-3t} + -9ke^{-3t} + 9ke^{-3t}$$

$$3 = 2k - 9k + 9k$$

$$\implies k = \frac{3}{2}$$

$$\implies x_p(t) = \frac{3}{2}e^{-3t}$$

Now, using $x(t) = x_p(t) + x_h(t)$:

$$x(t) = A_1 e^{-t} + A_2 e^{-2t} + \frac{3}{2} e^{-3t}$$

$$x(0) = 2$$

$$x'(0) = 1$$

$$2 = A_1 + A_2 + \frac{3}{2}$$

$$1 = -A_1 - 2A_2 - \frac{9}{2}$$

$$\implies A_2 = -6$$

$$\implies A_1 = \frac{13}{2}$$

$$\implies x(t) = -6e^{-t} + \frac{13}{2} e^{-2t} + \frac{3}{2} e^{-3t}$$

Had the forcing function $3e^{-3t}$ instead been of the form $B_1 e^{-4t} + B_2 e^{-3t}$, the particular solution would have needed to be of the form $k_1 e^{-4t} + k_2 e^{-3t}$.

Concept

Let us observe the following circuit:
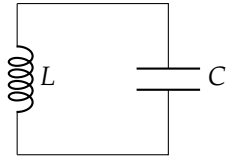
Both the inductor and the capacitor can store energy, and "send" energy to each other, creating oscillations, but maintain the total energy in the circuit continuous. Circuits of this form are called LC tanks.
Thus, the solution, as does the ideal pendulum, can be defined by sinusoidals.

Solving LC circuits

We will begin by solving the following circuit:

Which has a behavior defined by:

$$v_{in}(t) = v_L(t) + v_C(t)$$
$$i_L(t) = i_C(t)$$
$$v_L(t) = L\frac{di_L(t)}{dt}$$
$$i_C(t) = C\frac{dv_C(t)}{dt}$$
$$\implies v_L(t) = LC\frac{d^2v_C(t)}{dt^2}$$
$$\implies v_{in}(t) = LC\frac{d^2v_C(t)}{dt^2} + v_C(t)$$

This will now allow us to solve for $v_C(t)$.

$$v_{in}(t) = v_C(t) + LCv_C''(t)$$

Solving the homogenous equation:

$$0 = v_C(t) + LCv_C''(t)$$
$$\implies 0 = 1 + LC\lambda^2$$
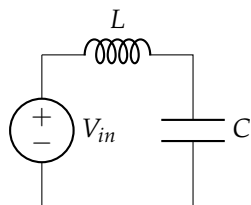$$\implies \lambda = \sqrt{\frac{-1}{LC}} = \pm j\sqrt{\frac{1}{LC}}$$

if $\omega_0 = \sqrt{\frac{1}{LC}}$

$$v_{C,h} = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$
$$= A_1(\cos(\omega_0 t) + j\sin(\omega_0 t)) + A_2(\cos(-\omega_0 t) + j\sin(-\omega_0 t))$$
$$= (A_1 + A_2)\cos(\omega_0 t) + j(A_1 - A_2)\sin(\omega_0 t)$$
$$= B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t)$$

Now we will search for the particular solution with an exponential forcing function, $V_{in} = Fe^{-\omega t}$

$$Fe^{-\omega t} = v_C(t) + LCv_C''(t)$$

With the particular solution of the form $v_{C,p} = Ae^{-\omega t}$

$$Fe^{-\omega t} = LC\omega^2 Ae^{-\omega t} + Ae^{-\omega t}$$
$$F = LC\omega^2 A + A$$
$$\implies A = \frac{F}{LC\omega^2 + 1}$$
$$\implies v_{C,p} = \frac{F}{LC\omega^2 + 1}e^{-\omega t}$$
$$\implies v_C = B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t) + \frac{F}{LC\omega^2 + 1}e^{-\omega t}$$

We can now solve for the complete equation if we have the initial conditions.

We could also have solved this for a forcing equation of the form
$v_{in}(t) = F_1 \cos(\omega t) + F_2 \sin(\omega t)$

$$F_1 \cos(\omega t) + F_2 \sin(\omega t) = v_C(t) + LC v_C''(t)$$
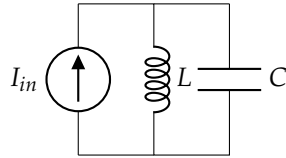$$v_{C,p} = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$
$$F_1 \cos(\omega t) + F_2 \sin(\omega t) = -LC\omega^2 A_1 \cos(\omega t) - LC\omega^2 A_2 \sin(\omega t) + A_1 \cos(\omega t) + A_2 \sin(\omega t)$$
$$= (-LC\omega^2 + 1)(A_1 \cos(\omega t) + A_2 \sin(\omega t))$$
$$\implies A_1 = \frac{F_1}{1 - LC\omega^2}$$
$$A_2 = \frac{F_2}{1 - LC\omega^2}$$

These values can then be substituted into the complete form and B_1
and B_2 can be found using the initial conditions.
We will now solve the following circuit:

In the same way as the last one, we get the differential equation,
which in this case is defined by:

$$i_{in} = LC \frac{d^2 i_L(t)}{dt^2} + i_L(t)$$

However, we can observe that the equation is of the same form as the
previous one, but with i replacing v. The solution will be of the exact
same form, and thus needs no additional resolution.
What is important to remember from this is the following:

$$v_{C,h}(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$$
$$= K \cos(\omega t + \theta)$$
$$i_{L,h}(t) = B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t)$$
$$= K \cos(\omega t + \theta)$$
$$\omega = \frac{1}{\sqrt{LC}}$$

If we were given initial conditions of the form $v_C(0) = k_1, i_L(0) = k_2$
instead of $v_C(0) = k_1, v_C'(0) = k_2$, we can use:

$$i_L(t) = i_C(t) = C \frac{dv_C(t)}{dt}$$

or

$$v_C(t) = v_L(t) = L \frac{di_L(t)}{dt}$$

to successfully derive the values of the missing constants.

RLC Circuits

The first circuit to be analyzed for this case is the parallel RLC circuit
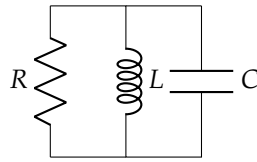
in which we will be analyzing the behavior of our components assuming there was a power source that has now been turned off.
It is important to note that as the value of the resistors' resistance increases, it's influence on the LC tank decreases, which will later help us visualize these circuits more conceptually.
The other circuit to be analyzed is the series RLC circuit:
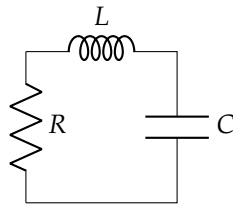
In this case, as the resistance increases, its damping effect on the LC tank increases.
Let us solve the series example, with a voltage source first.
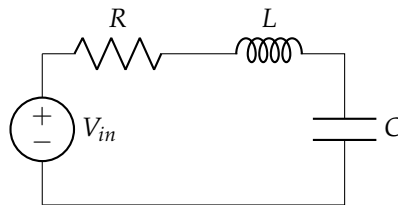
$$v_{in}(t) = v_R(t) + v_L(t) + v_C(t)$$

$$i_R(t) = i_L(t) = i_C(t)$$

$$v_{in}(t) = Ri_C(t) + L\frac{di_C(t)}{dt} + v_C(t)$$

$$\text{but, } i_C(t) = C\frac{dv_C(t)}{dt}$$

$$\implies v_{in}(t) = RC\frac{dv_C(t)}{dt} + LC\frac{d^2v_C(t)}{dt^2} + v_C(t)$$

$$\implies \frac{v_{in}(t)}{LC} = \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{d^2v_C(t)}{dt^2} + \frac{v_C(t)}{LC}$$

This can also be solved, but instead from the perspective of the current flowing through the inductor.

$$v_{in}(t) = v_R(t) + v_L(t) + v_C(t)$$

$$i_R(t) = i_L(t) = i_C(t)$$

$$v_{in}(t) = Ri_L(t) + L\frac{di_L(t)}{dt} + \frac{1}{C}\int i_L(t)dt$$

$$\frac{dv_{in}(t)}{dt} = R\frac{di_L(t)}{dt} + L\frac{d^2i_L(t)}{dt^2} + \frac{i_L(t)}{C}$$

$$\implies \frac{1}{L}\frac{dv_{in}(t)}{dt} = \frac{R}{L}\frac{di_L(t)}{dt} + \frac{d^2i_L(t)}{dt^2} + \frac{i_L(t)}{LC}$$

The same procedure can be followed for the parallel version, and the resulting ODEs are:

$$\frac{1}{C}\frac{di_{in}(t)}{dt} = \frac{d^2v_C(t)}{dt^2} + \frac{1}{RC}\frac{dv_C(t)}{dt} + \frac{v_C(t)}{LC}$$

and

$$\frac{i_{in}(t)}{LC} = \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{i_L(c)}{LC}$$
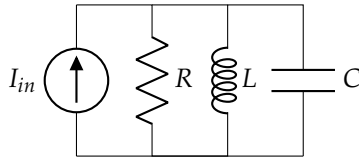
Notice how the damping component, that is, the first derivative, matches what we mentioned on the effect of the resistance on the LC tank, with it being inverse for parallel, and directly proportional for series.

So, in general, we can make a function that describes them in the following manner:

$$F = \frac{d^2x(t)}{dt^2} + \frac{1}{\tau}\frac{dx(t)}{dt} + \frac{x(t)}{LC}$$

where $\tau = RC$ for parallel, and $\tau = \frac{L}{R}$ for the series configuration.

NOTE: $\frac{1}{\tau} = Bw$, or the bandwidth of the circuit.
The quality factor Q of an RLC circuit is $\frac{\omega_0}{Bw}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the
natural frequency of the circuit.

So, we now solve the homogenous equation:

$$0 = \frac{d^2 x_h(t)}{dt^2} + \frac{1}{\tau}\frac{dx_h(t)}{dt} + \frac{x_h(t)}{LC}$$

As usual, $x_h(t) = Ae^{\lambda t}$

$$0 = \lambda^2 + \frac{\lambda}{\tau} + \frac{1}{LC}$$

$$\implies \lambda = \frac{-\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \frac{1}{LC}}$$

$$\implies \lambda_{parallel} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4RC^2} - \frac{1}{LC}} \text{ and}$$

$$\lambda_{series} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\implies x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

This divides the solution into three different cases:

- Overdamping: natural frequencies are real and distinct. That is $\frac{1}{4R^2C^2} > \frac{1}{LC}$ for parallel, and $\frac{R^2}{4L^2} > \frac{1}{LC}$ for series.
  Since all the elements are positive, the roots will both end up being negative.
  For this,

$$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

- Critical Damping: natural frequencies are real and equal. That is $\frac{1}{4R^2C^2} = \frac{1}{LC}$ for parallel, and $\frac{R^2}{4L^2} = \frac{1}{LC}$ for series.
  Both roots will be either $-\frac{1}{2RC}$ or $-\frac{R}{2L}$ depending on the configuration.
  This ends up behaving like explanation decay.
  In this case,

$$x_h(t) = A_1 e^{\lambda t} + A_2 t e^{\lambda t}$$

- Underdamping: natural frequencies are complex conjugates. That is $\frac{1}{4R^2C^2} < \frac{1}{LC}$ for parallel, and $\frac{R^2}{4L^2} < \frac{1}{LC}$ for series.
  This leads to λ being of the form $-\sigma_p \pm j\omega_d$, with $\sigma_p = \frac{1}{2\tau}$ the attenuation factor and ω_d the damped resonance frequency being the square root.
  Through a series of mathematical operations, and the use of Euler's equation, we can get

$$x_h(t) = e^{-\sigma_p t}(B_1 \cos(\omega_d t) + jB_2 \sin(\omega_d t)) = K\cos(\omega_d t + \theta)$$

This ends up behaving like a slowly shrinking sinusoidal wave.
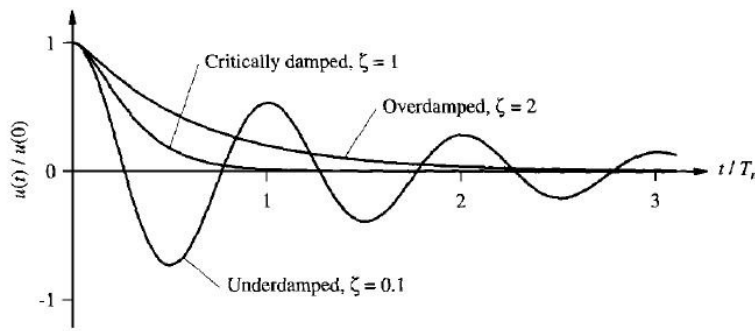
Let us look at some examples:

$$\lambda_1 = -1, \qquad \lambda_2 = -3 \qquad \Longrightarrow \quad V_{C,h} = A_1 e^{-t} + A_2 e^{-3t}$$

$$\lambda_1 = -1, \qquad \lambda_2 = -1 \qquad \Longrightarrow \quad V_{C,h} = A_1 e^{-t} + A_2 t e^{-t}$$

$$\lambda_1 = -1 + 2j, \quad \lambda_2 = -1 - 2j \quad \Longrightarrow \quad V_{C,h} = e^{-t}(A_1 \sin(2t) + A_2 \cos(2t))$$

To find the particular solution one could substitute into the ODE and use the same method as previous circuits.
However, there are some shortcuts:

- Constant input: the particular solution will be the input.

- Sinusoidal input: one can use SSS properties, and strategies like voltage division.

- Exponential input: no shortcut, use ODE.

Non-Ideal Elements

While we have been representing components as perfect inductors or capacitors, but these do not exist. If that were the case, inductors or capacitors would be used to store energy instead of batteries.
The non-ideal inductor, also known as a parasitic inductor, has a representation that takes into account the resistance of the inductor wire R_s, the capacitance between adjacent wires, and finally, the Ferrite Core loss, R_p, which consists of Eddy current loss and hysteresis loss.
Thus, a full representation of an inductor, in terms of ideal elements, would be:
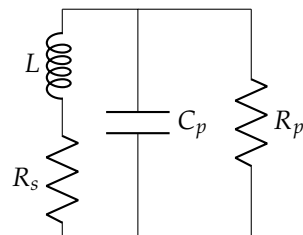
However, for this course, the effect of the capacitor and the ferrite core loss are negligible, allowing us to ignore them. If the frequency were closer to the self resonant frequencies of the inductor, C_p would become relevant.

The non-ideal capacitor has a wire inductance L_s, a wire resistance R_s, and a leakage, R_p, which allow us to model the capacitor as the following:
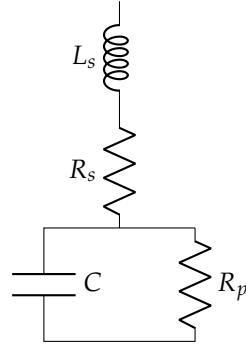
However, L_s and R_s are both negligible for this course's considerations, as they both have very small values.

These new representations of these components allow us to introduce a new topic, which is quality factor (Q).

$$Q(j\omega) = \left| \frac{\text{Reactance}}{\text{Resistance}} \right| = \left| \frac{\text{Im}(Z)}{\text{Re}(z)} \right|$$

$$\implies Q_L = \frac{\omega L}{R_s}, \quad Q_C = \omega R_p C$$

By using non-ideal representations of these elements within circuits, such as the RLC circuit, circuits are modified to result in resonant properties distinct to those found with ideal components.
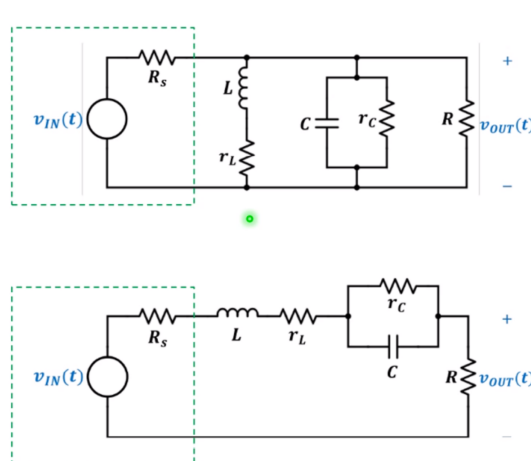
We will now learn to transform these configurations into more manageable and solvable configurations.

So, what we will try to do is to convert a series impedance and resistance configuration into a parallel one and vice-versa. That is, we want the parallel or series configuration with certain components to have the same impedance as the series or parallel configuration with the same components, but with different values.

$$
\begin{aligned}
Z(j\omega) = R_s + jX_s && Y(j\omega) = G_p + jB_p \\
= \frac{1}{G_p + jB_p} && = \frac{1}{R_s + jX_s} \\
G_p + jB_p = \frac{1}{R_s + jX_s} && R_s + jX_s = \frac{1}{G_p + jB_p} \\
= \frac{R_s - jX_s}{R_s^2 + X_s^2} && = \frac{G_p - jB_p}{G_p^2 + B_p^2} \\
\implies G_p(j\omega) = \frac{R_s}{R_s^2 + X_s^2}, && \implies R_s(j\omega) = \frac{G_p}{G_p^2 + B_p^2}, \\
B_p = \frac{-X_s}{R_s^2 + X_s^2} && X_s(j\omega) = \frac{-B_p}{G_p^2 + B_p^2}
\end{aligned}
$$

With Z being the impedance and Y being the admittance.
Now, through a few operations, we van find the values of L_p and R_p that would be equivalent to what we want.

$$
L_p = L_s \left(1 + \left(\frac{R_s}{\omega L_s} \right)^2 \right) = L_s \left(1 + \frac{1}{Q_L^2} \right)
$$

$$
R_p = R_s \left(1 + \left(\frac{\omega L_s}{R_s} \right)^2 \right) = R_s (1 + Q_L^2)
$$

At high Q:

$$
L_p \approx L_s
$$

$$
R_p \approx R_s Q_L^2
$$

This essentially means that at high Q, the inductor resembles an ideal inductor, as we would expect.
The same process can be conducted for capacitors:

$$
C_s = C_p \left(1 + \frac{1}{Q_c^2} \right)
$$

$$
R_s = \frac{R_p}{1 + Q_c^2}
$$

At high Q:

$$
C_s \approx C_p
$$

$$
R_s \approx \frac{R_p}{Q_c^2}
$$

These transformations should allow us to transform any non-ideal circuit into an equivalent circuit with ideal components, which we will be able to solve as shown in previous sections.

   Note: unlike the original components, their transformations are frequency dependent.

Another variable that may surge for non-ideal element questions is to find the self-resonant frequency of the circuit, which is $\omega_0 = \frac{1}{\sqrt{LC_s}}$ or $\omega_0 = \frac{1}{\sqrt{L_pC}}$ depending on the transformation.

This value can then be substituted into the equations for Q related terms, allowing for simplification, and thus the value of the transformed resistor.

Switched Circuits (Circuits with Switches)

The current through an inductor is and will be continuous regardless of whether the voltage across it is continuous. This is because of the property of the current through an inductor that defines it as:

$$i_L(t) = \int_{t_0}^{t} v_L(\tau)d\tau$$

The same happens for capacitors and voltage.

This is relevant because it is often important to know how long it takes for an RL or RC circuit to go from one state to another.

We know that

$$x(t) = x(\infty) + (x(t_0^+) - x(\infty))e^{\frac{-(t-t_0)}{\tau}}$$

What we want to discern is how long it will take for x to go from the value X_1 to the value X_2.

$$x(t_2) = x(\infty) + (x(t_1) - x(\infty))e^{\frac{-(t_2-t_1)}{\tau}}$$

$$x(t_2) - x(\infty) = (x(t_1) - x(\infty))e^{\frac{-(t_2-t_1)}{\tau}}$$

$$\frac{x(t_2) - x(\infty)}{x(t_1) - x(\infty)} = e^{\frac{-(t_2-t_1)}{\tau}}$$

$$\ln\left(\frac{x(t_2) - x(\infty)}{x(t_1) - x(\infty)}\right) = \frac{-(t_2-t_1)}{\tau}$$

$$\tau\ln\left(\frac{x(t_1) - x(\infty)}{x(t_2) - x(\infty)}\right) = t_2 - t_1$$

$$\tau\ln\left(\frac{X_1 - x(\infty)}{X_2 - x(\infty)}\right) = t_2 - t_1$$

If this is used on circuits in specific situations, we can discern at what times a solution is relevant or valid.

Another situation that might be faced is finding the equation describing a circuit that has had switches.

To do so, we would solve the circuit for each situation, or each time interval, as if there were no other switches.

To find the initial condition(s), one would need to find the equation for the previous time interval, find the value of X at the end of the time of said interval, and use that as the initial condition of the next time interval.

Convolution

We will visualize this as a circuit with an input $x(t)$ that is sent into a circuit function $h(t)$ that results in the output $y(t)$. However, $h(t)$ does not exist. But, convolution does. And this follows the property $y(t) = x(t) * h(t)$.

This method allows us to find the output of any-order circuits.

> "We hate convolution as much as we hate differential equations"
>   -Prof. Byunghoo Jung

Important Functions and Function Transformations

Functions

- Unit Step Function:

$$u(t) = \begin{cases} 0 & t < 0 \\ \text{undetermined} & t = 0 \\ 1 & t > 0 \end{cases}$$

  Note: for most intents and purposes, $u(0)$ can be considered to be 1 or irrelevant. However, non-continuous signals become undefined.

- Unit Function Shifted:

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases}$$

- Rectangular Function:

$$c(t) = u(t - t_1) - u(t - t_2) = \begin{cases} 0 & t < t_1 \\ 1 & t_1 \leq t < t_2 \\ 0 & t \geq t_2 \end{cases}$$

- Dirac Function:

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

$$\frac{du(t)}{dt} = \delta(t)$$

Transformations

- If a function $f(t)$ changes to $f(t-b)$ it will shift to the right, and if it changes to $f(t+b)$ it will shift to the left.

- If a function $f(t)$ changes to $f(at)$ with $|a| > 0$, it will expand along the x-axis, and if $|a| < 0$, it will contract. If a is negative, it will be reflected about the x-axis.

- If a function $f(t)$ changes to $cf(t)$, with $|c| > 1$, it will stretch along the y-axis, and if $|c| < 1$, it will contract. If c is negative, it will be reflected about the y-axis.

- If a function $f(t)$ changes to $f(t)+d$, with $d > 0$, it will shift up, and if $d < 0$, it will shift down.

Note: if multiple transformations are applied to a function, the order usually does not matter.

Graphical Convolution

The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

It has some other notations as well: $f(t) * g(t) = f * g(t) = (f * g)(t)$. Convolution can relate inputs to outputs using circuit models. That is, if we are given a highly complicated RLC circuit, we can relate the desired output to the input through convolution.
We will be analyzing the following two functions:
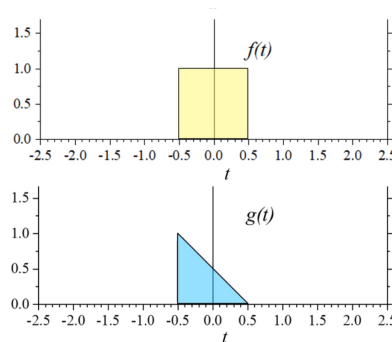
Both $f(t)$ and $g(t)$ are being analyzed respective to a reference time t. We will also define handedness as the side of t in which a function has a larger magnitude, should it not be symmetric. So, in this case $g(t)$ would be left-handed.
The values within the integral perform operations on the functions. That is, $f(t)$ becomes $f(\tau)$, while $g(t)$ becomes $g(-\tau)$ (changes handedness) and is moved along the τ axis from by some predefined value

t between $-\infty \rightarrow \infty$. The convolution is the integral of the product of these two functions.

So, while the two transformed functions do not overlap, the product is 0, but once a time t is analyzed at which they overlap, the integral begins to have values.

So, different predefined values of t convolute to different values. However, the convolution will maximize at the time in which they have the maximum intersection.

Actual Convolution (Mathematical and Continuous)

Looking at what happened above, we can observe that the output of convolution is a function of t, and has no effect of τ. That is:

$$f(t) * g(t) = Fn(t)$$

There is an important function within convolutions that we defined earlier. And it is important because it has the following property, also known as the shifting property:

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau = f(t)$$

and the sampling property:

$$f(t) * \delta(t_0) = \int_{-\infty}^{\infty} f(\tau)\delta(t_0 - \tau)d\tau = f(t_0)$$

That is, the δ function is the identity function for convolution.

This can be proven by writing and solving the following integral that is using the δ function and allowing us to control the limits of the integral:

$$x(t) * \delta(t - T) = \int_{(t-T)^-}^{(t-T)^+} x(\tau)\delta((t - T) - \tau)d\tau = x(t - T)$$

The above function is also known as time shifting.

We can analyze how convolution reacts to the unit step function:

$$f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)u(t - \tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau$$

However, for most functions $f(t)$, this is unbounded, and thus unusable. Instead, we convolute from a certain point of time, which is done like this:

$$y(t) = x(t)u(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(\tau)u(t - \tau)d\tau$$

$$= \int_{0}^{t} x(\tau)d\tau u(t)$$

These examples and uses have allowed us to get a basic understanding of convolution and its uses.

We will now learn some properties of convolution:

- Commutative Property: $f(t) * g(t) = g(t) * f(t)$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$$
$$\text{if } \gamma = t - \tau$$
$$= \int_{-\infty}^{\infty} f(t - \gamma)g(\gamma)d\gamma$$
$$= g(t) * f(t)$$

- Associative Property: $[f(t) * g(t)] * h(t) = f(t) * [g(t) * h(t)]$

- Distributive Property: $f(t) * [g(t) + h(t)] = f(t) * g(t) + f(t) * h(t)$

Convolution can also be used to create pulses through the convolution $x(t)u(t) * [u(t) - u(t - T)]$ which creates a pulse between $t - T$ and t.

Discrete Convolution

Convolution can also be done on a discrete domain, that is, on a domain that is non-continuous. In these domains, convolution is defined as:

$$f[n] * g[n] = \sum_{m=-\infty}^{\infty} f[m]g[n - m]$$

where $f[n] = [k_1, k_2, k_3, k_4, k_5]$ and $g[n] = [c_1, c_2, c_3, c_4]$ correspond to the values of these functions starting from 0 and moving forward on the integer domain. So, in this case, $f[1] = k_2$.

The convolution of these is similar to ordinary convolution, however, the "sliding" of $g[n]$ would be along integers and not the real numbers.

Discrete convolution is also commutative, associative, and distributive. Discrete convolution is used for computational calculations of convolution, with the domain including more numbers than just integers. It is considered that the appropriate spacing between two points should be at most $0.2 * (FWHM)$, where FWHM is the width of the formed curve at half the maximum height of the convolution. The extension of the tails, that is, the width of the domain should be about $20FWHM$ on each side of the maximum.

Let us take the following circuit:
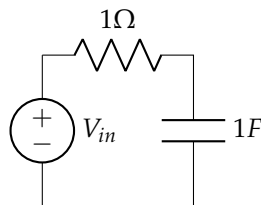
The describing function of this circuit will be

$$v_{out}(t) = v_{in}(t) - v_{in}(t)e^{-t}$$
$$\text{if } v_{in}(t) = u(t)$$
$$v_{out}(t) = (1 - e^{-t})u(t)$$
$$\text{since } \frac{d}{dt}u(t) = \delta(t)$$
$$\frac{d}{dt}(1 - e^{-t})u(t) = e^{-t}$$

The last equation represents the response of the circuit to an impulse. If we have one impulse, we can represent any input as a combination of step impulses. This would allow us to easily calculate the circuit's response and behavior computationally.

Impulse Response

We start this section by defining Linear Time Invariant (LTI) Systems, which are circuits that relate to inputs linearly and outside the domain or control of time. That is, the highest power of elements in the ODE that defines the circuit is 1.
For example:

$$y(t) = \frac{d^2x(t)}{dt^2}$$

is linear, but

$$y(t) = \frac{dx^2(t)}{dt}$$

is not.
The time invariance indicates a relation between input $x(t) \rightarrow y(t)$ output, and input $x(t - T) \rightarrow y(t - T)$ output.
Let us look at the following LTI system:
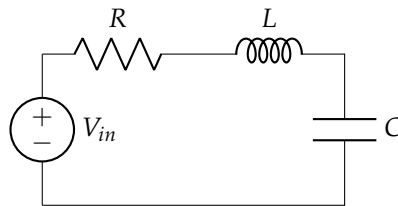
Let $h(t)$ be the convolution function of this LTI.

$$\implies x(t) * h(t) = y(t)$$
$$\text{but if } x(t) = \delta(t)$$
$$\delta(t) * h(t) = h(t) = y(t)$$

We will call this function $h(t)$ the impulse response. In essence, $h(t)$ is the output of the circuit when the input of the function is $\delta(t)$

If we were to know the magnitude of the input $x(t)$ at a given time, Δt, we could define the convolved function with $h(t)$ as $x(t) = |x(\Delta t)|\delta(t - \Delta t)$, which would in turn allow us to, through convolution, solve for the effect of the input on the output at this moment.

$$x(t) * h(t) = |x(\Delta t)|\delta(t - \Delta t) * h(t) = |x(\Delta t)|h(t - \Delta T)$$

We could then add all the effects across evenly spaced discrete times of the convolution and add them for their values at a given time t, allowing us to acquire a good estimate of the output at this time. This would end up looking like the following:

$$|x(0)|h(0) + |x(\Delta t)|h(t - \Delta t) + |x(2\Delta t)|h(t - 2\Delta t) + |x(3\Delta t)|h(t - 3\Delta t)\ldots$$

Let us apply this for a solved output to calculate the impulse response:

$$i(t) = (1 - e^{-10t})u(t)$$

$$\frac{d}{dt}u(t) = \delta(t)$$

$$\implies h(t) = \frac{d}{dt}(1 - e^{-10t})u(t)$$

$$= 10e^{-10t}u(t) + (1 - e^{-10t})\delta(t)$$

$$= 10e^{-10t}u(t) + (1 - e^{-10\times 0})\delta(0)$$

$$= 10e^{-10t}u(t) + 0\delta(t)$$

$$= 10e^{-10t}u(t)$$

Now let us assume, we changed the input to another, all we would need to do would be to convolve this new input with $h(t)$, and we would get the output $y(t)$. This output will in fact contain both the homogenous and particular solutions of the system.
NOTE: $h(t)$ is the derivative of the step response for an LTI, which is of the form $f(t)u(t)$, with f(t) being the defining equation of the system.
The above note leads to the following two steps to find the impulse response:

1. Find unit step response of the function by setting the input to be $u(t)$

2. $h(t) = \frac{d}{dt}(\text{unit step response})$

Laplace Transformations

"This is the single most important topic in the whole course ... it will also make you mad and depressed." -Prof. Byunghoo Jung

Introduction to Laplace Transformations

All circuit analysis till now has been in the time domain. We will now move from the time domain to the frequency domain. This will in turn simplify circuit analysis.

This is done using Laplace Transformations. Let

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)d\lambda$$
$$\implies s = \sigma + j\omega$$

with σ and ω being the real and imaginary parts of the frequency. s is a complex number, and thus to completely graphically represent it, we need two different 3D plots. One has the axis $[Re(s), Im(s), |F(s)|]$, and the other $[Re(s), Im(s), \underline{/F(s)}]$
Keep in mind:

$$Ae^{j(\omega t + \phi)} = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$$

as we continue this section.
We will define a one-sided Laplace transform as the following:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Note: as a fun fact, Fourier transforms, which are covered in higher level courses, are a subset of Laplace transforms.
The transformations form the following table:

- $\mathcal{L}\{Af(t)\} = AF(s)$

- $\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$

- $\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

- $\mathcal{L}\{t\} = \frac{1}{s^2}$

- $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$

- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

- $\mathcal{L}\{f(t - t_0)u(t - t_0)\} = e^{-st_0}F(s)$

- $\mathcal{L}\{\frac{t^n}{n!}\} = \frac{1}{s^{n+1}}$

- $\mathcal{L}\{\delta(t)\} = 1 \leftarrow$ **Important**

- $\mathcal{L}\{u(t)\} = \frac{1}{s}$ with $Re\{s\} \neq 0 \leftarrow$ **Important**

- $\mathcal{L}\{t(u(t))\} = \frac{1}{s^2}$ with $Re\{s\} \neq 0 \leftarrow$ **Important**

- $\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$ with $Re\{s + a\} > 0 \leftarrow$ **Important**

- $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

- $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$ with $Re\{s\} > 0$

- $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$ with $Re\{s\} > 0$

- $\mathcal{L}\{\cos(\omega t) + \sin(\omega t)\} = \frac{s+\omega}{s^2+\omega^2}$ with $Re\{s\} > 0$

- $\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$ with $F(s)$ being the Laplace transform of $f(t)$

- $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0^-) - f'(0^-)$ with $F(s)$ being the Laplace transform of $f(t)$

- $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - \sum_{k=0}^{n-1}(s^{n-1-k}\frac{d^kf(0^-)}{dt^k})$ with $F(s)$ being the Laplace transform of $f(t)$

- $\mathcal{L}\{\int_{-\infty}^{t}f(\tau)d\tau\} = \frac{1}{s}\int_{-\infty}^{0}f(\tau)d\tau + \frac{1}{s}F(s)$

The beauty of these, and their applications in this course come from the property $f(t) * g(t) = F(s)G(s)$.

Note: For most circuits covered in this course, the order of the numerator is lower than the order of the denominator. This will allow us to conduct inverse Laplace Transforms more easily.

Later in this course we will learn to find $H(s)$ of a system using KVL and KCL without going through ODEs.

Properties of Laplace Transforms

The Laplace transform of the first derivative can be used to prove the following property, also known as the Initial-Value Theorem:

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

It also proves the Final-Value Theorem:

$$\lim_{t \to \infty} = \lim_{s \to 0} sF(s)$$

It is important to notice that there are limitations of the value of s in the Final-Value theorem.

Inverse Laplace Transform

Note: the lecturer and videos gave different processes, if you have a better explanation, please fix this.

The inverse Laplace transform is defined as the following:

$$f(t) = \frac{1}{j2\pi} \oint_{\Gamma} F(s)e^{st}ds$$

However, this is quite complicated to solve, and it may even be completely impossible. So, let

$$X(s) = \frac{P_1(s)}{Q_1(s)}$$

$$H(s) = \frac{P_2(s)}{Q_2(s)}$$

$$\implies Y(s) = \frac{P_1(s)P_2(s)}{Q_1(s)Q_2(s)}$$

$$= \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

$$= \frac{K_1}{(s + p_1)^{r_1}} + \frac{K_2}{(s + p_2)^{r_2}} + \cdots + \frac{K_n}{(s + p_n)^{r_n}}$$

Rewriting Y as partial fractions allows us to use out look-up table to go back into the time domain.

Let us take a look at each of three cases separately:

- Simple Real Poles:

$$F(s) = \frac{P(s)}{(s + p_1)(s + p_2)Q_1(s)} = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \frac{P_1(s)}{Q_1(s)}$$

$$K_a = [F(s)(s + p_1)]_{s=-p_1} = \left[\frac{P(s)}{(s + p_1)(s + p_2)Q_1(s)}(s + p_1) \right]_{s=-p_1}$$

$$= \left[\frac{K_1}{s + p_1}(s + p_1) + \frac{K_2}{s + p_2}(s + p_1) + \frac{P_1(s)}{Q_1(s)}(s + p_1) \right]$$

In essence, we multiply by each denominator, set the value $s = -p_n$, making $s + p_n = 0$, leaving only the coefficient on the right side, as everything else is multiplied by 0. The other side is then evaluated to get the value of the coefficient.

- Complex Conjugate Poles:

$$F(s) = \frac{P(S)}{(s + \alpha - j\beta)(s + \alpha + j\beta)Q_1(s)}$$

$$= \frac{K_1}{s + \alpha - j\beta} + \frac{K_2}{s + \alpha + j\beta} + \frac{P_1(s)}{Q_1(s)}$$

$$= \frac{sA_1 + A_2}{(s + \sigma)^2 + \omega^2} + \frac{P_1(s)}{Q_1(s)} \text{ with } A_1 = 2K_1, A_2 = 2\sigma K_1 + 2\omega K_2$$

We can then solve for that using the partial fraction method. However, the bottom may have more complex terms, which would end up complicating the function. This uses:

$$\mathcal{L}^{-1}\left(\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta} \right) = 2|K|e^{-\alpha t}\cos(\beta t + \theta)$$

- Repeated, Real Poles:

$$F(s) = \frac{P(S)}{(s+p_1)(s+p_1)Q_1(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{(s+p_1)^2} + \frac{P_1(s)}{Q_1(s)}$$

However, for a more general solution, the following property can be used:

$$K_{1i} = \frac{1}{(r-i)!}\frac{d^{r-i}}{ds^{r-i}}\left[(s+p_1)^r\frac{P(s)}{Q(s)}\right]$$

For all except term with quadratic denominator, follow same procedure as in Simple Real Poles. For the quadratic term, multiply the equation by $(s-p_1)^2$, and set $s = p_1$, this will cause all terms to drop out, except K_2, which we will then be able to evaluate.
If there are more than two repeated roots, multiply by the highest exponent, and make everything fall out. Solve as mentioned above. For the rest of the coefficients, a system of equations will have to be solved.

Once everything is in partial fraction form, we can easily use the lookup table to take it back to the time-domain.

Circuit Analysis in Frequency Domain

Impedance, Admittance, and Zero-State Response

"You will now begin to understand the beauty of frequency domain analysis" - Prof. Byunghoo Jung

We will now analyze the behavior of circuit parameters in the frequency domain.

- Resistors:

$$\mathcal{L}\{v_R(t) = Ri_R(t)\} = V_R(s) = RI_R(s)$$
$$\mathcal{L}\{i_R(t) = Gv_R(t)\} = I_R(s) = GV_R(s)$$

- Inductors:

$$\mathcal{L}\left\{v_L(t) = L\frac{di_L(t)}{dt}\right\} = V_L(s) = L(sI_L(s) - i_L(0))$$

$$\mathcal{L}\left\{i_L(t) = \frac{1}{L}\int_{-\infty}^{t}v_L(\tau)d\tau\right\} = I_L(s) = \frac{1}{L}\left(\frac{V_L(s)}{s} + \frac{\int_{-\infty}^{0}v_L(\tau)d\tau}{s}\right)$$

- Capacitors:

$$\mathcal{L}\left\{i_C(t) = L\frac{dv_C(t)}{dt}\right\} = I_C(s) = C(sV_C(s) - v_C(0))$$

$$\mathcal{L}\left\{v_C(t) = \frac{1}{C}\int_{-\infty}^{t}i_C(\tau)d\tau\right\} = V_C(s) = \frac{1}{C}\left(\frac{I_C(s)}{s} + \frac{\int_{-\infty}^{0}I_C(\tau)d\tau}{s}\right)$$

Zero-State Response: Response of a circuit when all its initial conditions are 0.

$$V_R(s) = RI_R(s)$$
$$V_L(s) = sLi_L(s)$$
$$V_C(s) = \frac{I_C(s)}{sC}$$

Once these terms are defined, we can also define impedance, which is the equivalent of resistance, but in the frequency domain:

$$Z(s) = \frac{V(s)}{I(s)}$$

Which means,

$$Z_R = R$$
$$Z_L = sL$$
$$Z_C = \frac{1}{sC}$$

On the other hand, we will define admittance as

$$Y(s) = \frac{I(s)}{V(s)}$$

Which leads to:

$$Y_R = \frac{1}{R} = G$$
$$Y_L = \frac{1}{sL}$$
$$Y_C = sC$$

These terms can be used to turn RC, RL, LC and RLC circuits into circuits equivalent to solely Z circuits. These circuits can then be solved using strategies like current division, voltage division, among others learned in earlier courses.
We have successfully reduced a linear circuit with inductors and capacitors which initially needed differential equations, into a circuit that is equivalent to one that only has resistances.

Initial Conditions in Laplace Transform

We will now analyze what happens when the initial conditions are non-zero. Let us take the following circuit as an example:
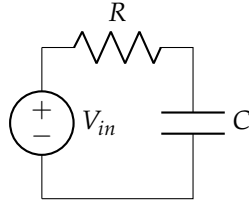
$$V_{in}(s) = V_R(s) + V_C(s)$$
$$V_{in}(s) = RI_C(s) + V_C(s)$$
$$\implies V_{in}(s) = RC[sV_C(s) - v_C(0^-)] + V_C(s)$$

by substituting the values of the initial condition $v_C(0^-)$, we can obtain an equation solely in terms of s and $V_C(s)$.
This means we can isolate $V_C(s)$:

$$V_C(s) = \frac{V_{in}(s) + RCv_C(0^-)}{RCs + 1}$$

This can then be decomposed through partial fractions, and returned to the time domain.
This has successfully allowed us to completely solve a circuit with initial conditions.
However, there is more depth to this. What we just did was find an expression for $V_C(s)$, but this is not always trivially possible, for example, if the circuit were to have initial current conditions instead of initial voltage conditions, but still require to calculate for the voltage.
Let us say we instead wanted $i_C(t) = C\frac{dv_C(t)}{dt}$.

$$I_C(s) = C[sV_C(s) - v_C(0^-)]$$

Notice how this has two components, one related to the input, and one related to the initial condition. The same would happen if we had the current and wanted to derive the voltage from it, that is, we would have two components.
Any combination, that is, if we have current and want current, have current and want voltage, have voltage and want voltage, or have voltage and want current, can be divided into these two kinds of initial conditions. The initial conditions can then be modeled as either current sources or voltages and added to the circuit to be analyzed.
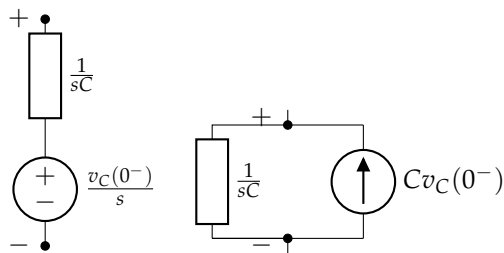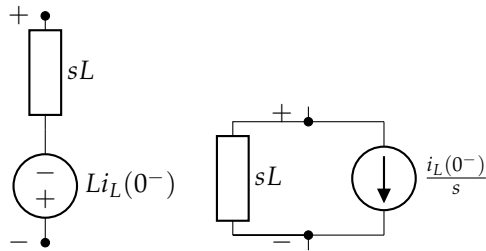The circuit shown above can turn into:
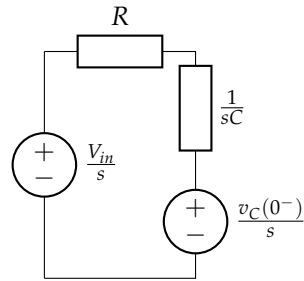
This can then be solved through superposition.

Other models, based on the same principle exist for each of the components, and can be used based on the analyzed configuration. The models for capacitors are:

The models for inductors are:

and they can be placed in the circuit instead of the component with initial conditions to model the whole component, and should be used based on the configuration of the circuit and what it is that will make the circuit easiest to solve. So, in general, for series configurations, you would use the series models, and for parallel configurations, you would use the parallel configuration.

Once this model has been placed, we can, as mentioned before, solve through superposition. The superposition part that represents the source is called the zero-state response. The part that takes into account the initial conditions is called the zero-input response. The addition of these two is the complete response.

Decomposition of Complete Response

A complete response is that which includes responses from all inputs.

It is composed of Zero-Input Response (ZIR), which is the response to initial conditions with all inputs deactivated, and Zero-State Response (ZSR), which is the response when initial conditions are set to 0.

There are four labels that can be placed on a response: responses:

Forced Response: response from all components not associated with the roots of the characteristic equation.

Natural Response: response from all components associated with the roots of the characteristic equation.

Transient Response: response that either increases or decays with time, so, not constant or sinusoidal.

SSS response: sinusoidal response.

The hidden output that comes from an initial condition is always homogenous, while the one coming from the input has a homogenous part and a particular part.

The steps to solve a circuit are the following:

1. Transform the circuit into the Laplace domain while accounting for the initial condition created by the energy storing component(s). If it is a parallel circuit, use parallel transformation, and if it is a series circuit, use the series transformation.

2. Turn of all initial conditions. If there is more than one input, account for that by turning off all but one.

3. Solve the circuit based on this input, for the relevant parameter of the relevant component.

4. Cycle through all inputs and initial conditions and add their responses. The sum of all the input responses will be the ZSR, and the sum of all the initial condition responses will be ZIR.

5. If it is necessary to calculate the voltage and what has been calculated till now is the current, perform relevant operations, that is multiply or divide by impedance of component as needed.

6. Transform back to time domain.

Note how there will likely be some components in the equation with sinusoidal responses and some with transient response.

By following these steps, the whole circuit, regardless of the number of components and inputs, can be solved.

Transfer Function

We know that

$$y(t) = x(t) * h(t)$$
$$\implies Y(s) = X(s)H(s)$$
$$\implies H(s) = \frac{Y(s)}{X(s)}$$

This $H(s)$ is the transfer function. This works under the premise of no
independent sources and the circuit being at rest.
In essence, it can be found by transforming all components to the
frequency domain, and finding the proportion between the input
and the output. This can then be used for any other inputs by simple
multiplication with the transformation of said input. This value can
then be returned to the time domain, effectively finding the output for
the aforementioned voltage.

## Complex Plane and Stability

Any transfer function $H(s)$ can be represented as:

$$H(S) = K \frac{(s - z_1)^{q_1} (s - z_2)^{q_2} (s - z_3)^{q_3} \cdots (s - z_m)^{q_m}}{(s - p_1)^{r_1} (s - p_2)^{r_2} (s - p_3)^{r_3} \cdots (s - p_n)^{r_n}}$$

with p_n being the pole and z_m the zeros. K is the gain constant. The
exponents represent repetitions.
Complex plain analysis is analyzed by setting $s = \sigma + j\omega$. By setting
s to this value, we can find the magnitude of the transfer function at
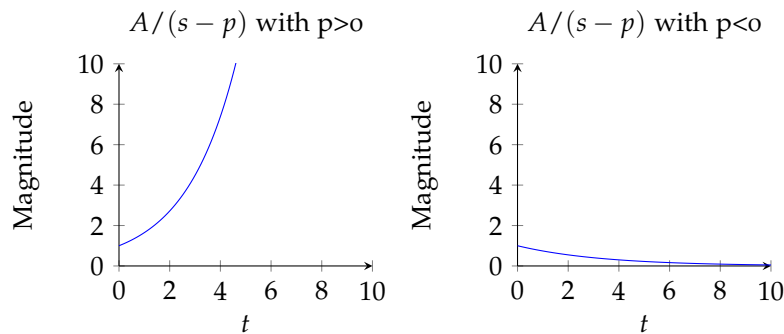any point in the complex plane.
If we were to plot σ v. $j\omega$, we can observe different behaviors at differ-
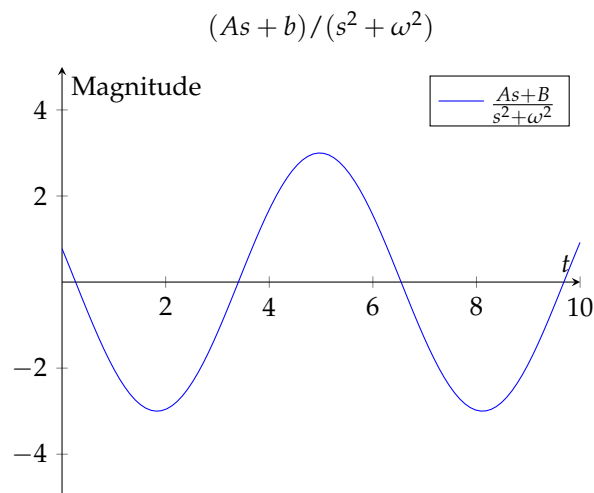ent coordinate points of the plane.
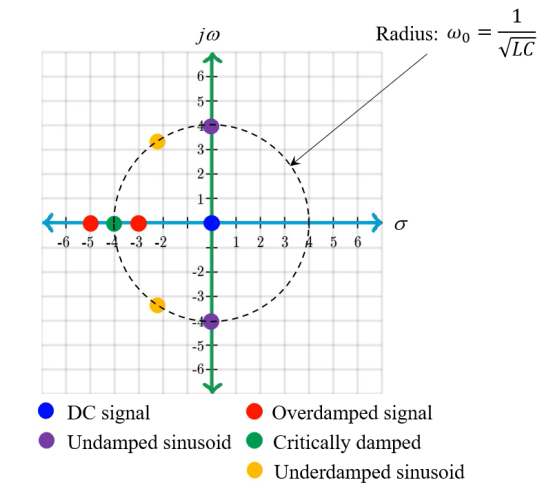
Poles represent the natural response of the circuit. If the order of a pole is higher than 1, the term will be multiplied by t^n.

A/s

A/s^k
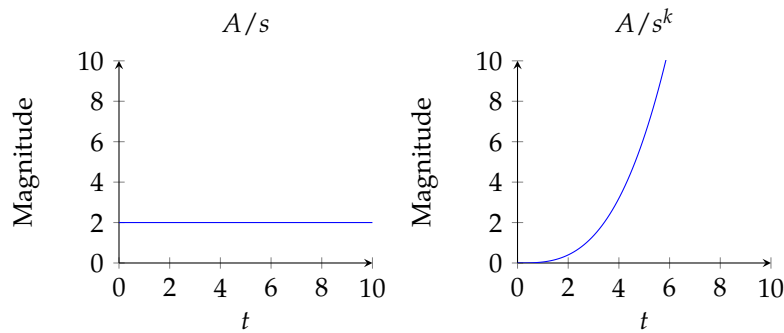
If $p > 0$ (unstable) and $p < 0$ (stable) respectively:

$A/(s-p)$ with p>0

$A/(s-p)$ with p<0
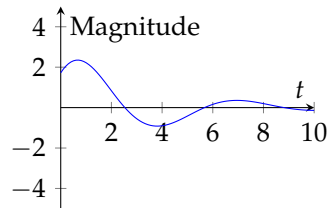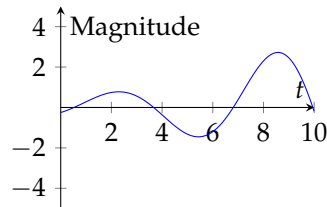
Undamped sinusoid (marginally stable):

$$(As+b)/(s^2+\omega^2)$$

If $a > 0$ (stable) and $a < 0$ (unstable) respectively:

$(As + b)/((s + a)^2 + \omega^2)$ with a>o

$(As + b)/((s + a)^2 + \omega^2)$ with a<o

We will only be considering stable and marginally stable inputs and outputs for this course.

The zeros, on the other hand allow us to know if the excitation will be cancelled out. That is, if the input exponent is the same value as that which will cause a o in the transfer function, there will be no forced response term.

If the input is bounded, the Forced Response must also be bounded. Thus, the unstability can only come from the poles from the transfer function.

A system will be unstable if the pole, when plotted, is in the first or fourth quadrant.

A passive circuit can only ever be stable.

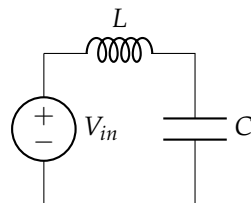## *Resonance and 2nd Order Systems*

For a system to resonate, it needs at least two energy storing elements. It must also be an Underdamped System.

The most commonly observed system is an LC circuit.

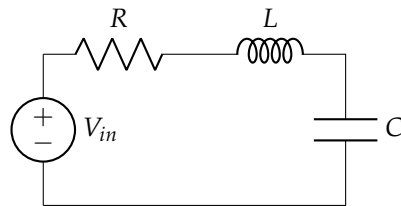We will analyze the following circuit:

In the example above:

$$H_1(s) = \frac{I_L(s)}{V_{IN}(s)} = \frac{1}{Z(s)} = \frac{\frac{s}{L}}{s^2 + \frac{1}{LC}}$$

$$= \frac{\frac{j\omega}{L}}{-\omega^2 + \frac{1}{LC}} \text{ at SSS}$$

$$[H_1(j\omega)]_{\omega = \frac{1}{\sqrt{LC}}} = \infty$$

The impedance at the given frequency is 0, which causes the current to go to infinity.

If a circuit, as seen by a source has 0 impedance, it will be in a resonant state.

We will now look at an RLC circuit, which is more realistic than the previous case:

When provided with a resonant frequency, that is, $\omega = \frac{1}{\sqrt{LC}}$, the capacitor and inductor will behave like a short, causing the total impedance of the circuit to be R.

Note that the voltage drop across the capacitor and inductor is not in fact zero. The voltages across the inductor and capacitor are actually:

$$V_L = \frac{-1}{j\omega_0 RC}$$

$$V_C = \frac{1}{j\omega_0 RC}$$

$$\implies V_L + V_C = 0$$

The voltages across them are not necessarily smaller than the input, and thus they can be used to amplify voltages.

In resonance, the capacitor and inductor voltages will be exactly 180 degrees out of phase.

The voltage they output is in fact known as the voltage response, and it is proportional to the resistance of the circuit, because the resistor is the only passive component that has a completely real impedance.

Bandwidth(B_ω): The difference between the two frequencies at which the voltage response of the circuit is $\frac{1}{\sqrt{2}}$ of the voltage response at the resonant frequency.

Let us instead look at the following circuit:
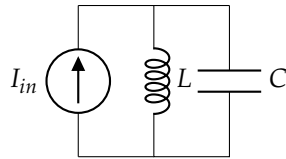
At resonant frequency, the impedance becomes infinite, or an open circuit. That is, the voltage across them becomes ∞. If it were instead RLC:
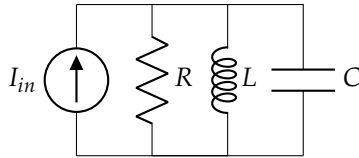
The current through the resistor will be the input, however, currents through the capacitor and inductor will not be null. Once again, they will be equal in magnitude but with opposite signs:

$$I_L = \frac{1}{j\omega_0 \frac{L}{R}}$$

$$I_C = \frac{-1}{j\omega_0 \frac{L}{R}}$$

$$I_L + I_C = 0$$

Once again, this can be used as an amplifier, this time for current. Resonant frequency (ω_r) can be defined as the frequency at which the impedance of a circuit becomes purely real.

It is directly related to the natural frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, and the damped resonance frequency ω_d.

Note: ω_d will always be smaller that $\frac{1}{\sqrt{LC}}$.

If we were to have a circuit that was not purely parallel nor series, we would have to refer to the definition of resonant frequency.

That is, the steps to solving for resonant frequency in a circuit are:

1. Turn everything into frequency domain, using Laplace transform

2. Substitute $s = j\omega$

3. Calculate impedance from perspective of voltage or current supply, and set all supplies to zeor.

4. Make denominator of resulting expression purely real by multiplying by conjugate.

5. Group all imaginary terms (and take out as many common factors as possible)

6. Equate sum of imaginary terms to zero and solve for ω.

Note: SSS analysis is not the same as s-domain analysis. It only gives us the particular solution.

Frequency Response

We know that:

$$|V_{in}(j\omega)| = |Z_{in}(j\omega)||I_{in}(j\omega)|$$
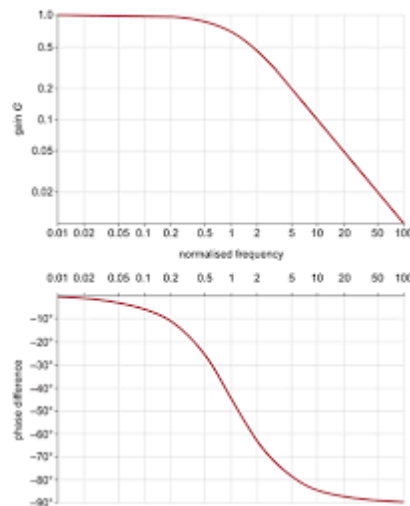
$$H(j\omega) = \frac{V_{OUT}}{V_{IN}}$$

$$\text{if Input} = v_{IN}(t) = A_{IN}\cos(\omega t + \theta_{IN})$$

$$|H(j\omega)| = \frac{A_{OUT}}{A_{IN}},$$

$$\underline{/H(j\omega)} = \frac{\underline{/A_{IN}}}{\underline{/A_{OUT}}}$$

The Bode plot allows us to move between a time-variable function and complex variable function.
They look like the following:

Note: One can create a filter, and, depending on what it is connected to, it will act as a low pass filter or a high pass filter.
We can also define gain as the following:

$$|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)|$$

This means that the point at which the dB gain is -3, is the same as the limits of the bandwidth mentioned in the previous section. This frequency is known as the half-power frequency.
Note: $a + jb = \sqrt{a^2 + b^2}\underline{/\tan^{-1}(\frac{b}{a})}$.
From a Bode plot, one can find the magnitude and phase of the output as long as one knows the input.
For intents and purposes of this course, we will use dB and rad/sec.
The pole can be found from the dB based Bode plot as the point at which the slope begins to change, or the point at which the gain is

3dB lower than the pass band. This is only true if there are no other nearby poles and the pole itself is real.

The pole will also be the point at which the phase shift is 45 degrees less than the initial phase. A zero will be at 45 degrees higher than the initial. The slope of the phase plot will be shifted by 45 degrees per decade.

NEED TO ADD PLOTS AND BETTER EXPLANATION OF POLES AND THEIR EFFECT ON BODE PLOTS.

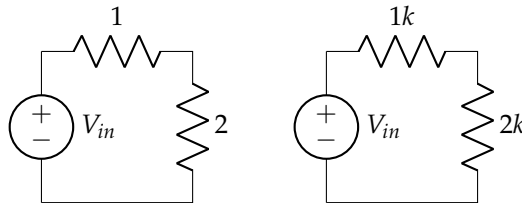The Bode plot is also represented through partial fractions:

$$H(s) = \frac{1}{as+b} = \frac{1}{a(s+b/a)} = \frac{1}{a}\frac{1}{s+p}$$

Circuit Scaling

There are two kinds of scaling.

Magnitude Scaling

It is that which scales the values of the components in the circuit. For example, the transfer function of the following two circuits is the same:

In essence, scaling is to multiply the impedances of the components by the same factor.

$$R \rightarrow K_m R$$
$$L \rightarrow K_m L$$
$$C \rightarrow \frac{C}{K_m}$$

If all components in the circuit are scaled as shown above, the poles and zeros will remain the same. However, they could modify the magnitude of the transfer function.

Let us consider the following kinds of transfer functions:

$$H = \frac{V_o}{V_{in}} \to_{K_m} \quad H$$

$$H = \frac{I_o}{I_{in}} \to_{K_m} \quad H$$

$$H = \frac{V_o}{I_{in}} \to_{K_m} \quad K_m H$$

$$H = \frac{I_o}{V_{in}} \to_{K_m} \quad \frac{H}{K_m}$$

So, the poles will remain the same, but the curve will move up or down.

To find the passband gain, one needs to discern whether this is a low pass filter or a high pass filter. If it is a low pass filter, evaluate the gain of the transfer function at $s = 0$, while if it is a high pass filter, evaluate the gain at $s = \infty$ and do the same.

The passband gain will change for the bottom two scalings shown above.

Frequency Scaling

Let us take the following Bode plot:

TO ADD: BODE PLOT!!!

If we wanted to shift all zeros and poles within it
A resistive circuit cannot have frequency scaling.
Note that for RC and RL circuits, the poles are at $\frac{1}{\tau}$.
To reliably scale it, one cannot change the resistance, as it would behave differently for RL and RC. Instead, we scale L and C:

$$L \to_{K_f} \frac{L}{K_f}$$

$$C \to_{K_f} \frac{C}{K_f}$$

Combined Scaling

If we wanted to scale both at the same time, the following transformations would be necessary:

$$R \to K_m R$$

$$C \to \frac{C}{K_m K_f}$$

$$L \to \frac{K_m L}{K_f}$$

If the input and output are different (in terms of units), the passband gain will change, despite these transformations. This is the same as what is shown in the magnitude scaling section.

Bandpass Response of 2nd order systems

Let us consider a parallel RLC circuit with a sinusoidal current input. The transfer function will be:

$$H(s) = \frac{V_o}{I_{in}} = Z_{total}(s) = \left(\frac{1}{R_p} + \frac{1}{sL_p} + sC_p \right)^{-1} = \frac{L_p s}{L_p C_p s^2 + \frac{L_p}{R_p} s + 1}$$

If we were to solve for its poles, it would behave as a bandpass filter, of different kinds based on the sign of $\left(\frac{L_p}{R_p} \right)^2 - 4L_p C_p$. It would be overdamped if positive, critically damped if zero, and underdamped if negative.

Commonly used bandpass filters are those with a narrow peak (high quality factor), which usually ends up falling under the underdamped category.

At resonance, which is where the Bode plot will peak, the plot will be at $20 \log(R)$.

Bases on which transfer function is being measured, it will act as a band pass for the resistor, as a high pass for the capacitor, and as a low pass for an inductor.

- H_m: The maximum of the transfer function.

$$H_m = R_p$$

- ω_m: Frequency at which the maximum is achieved.

$$\omega_m = \frac{1}{\sqrt{L_p C_p}}$$

- ω_L: lower half-frequency of the bandpass characteristic, that is, the point at which the gain is 3dB less than the peak, and the frequency is lower than ω_m.

- ω_H: upper half-frequency of the bandpass characteristic, that is, the point at which the gain is 3dB less than the peak, and the frequency is higher than ω_m.

$$|H(j\omega_L, j\omega_H)| = \frac{\frac{\omega}{C_p}}{\sqrt{\left(\frac{1}{L_p C_p} - \omega^2 \right)^2 + \left(\frac{\omega}{R_p C_p} \right)^2}} = \frac{R_p}{\sqrt{2}}$$

- B_ω: difference between half-frequencies.

$$B_\omega = \omega_H - \omega_L$$

- Q: proportion between maximum frequency and bandpass.

$$Q = \frac{\omega_m}{B_\omega}$$

So, for the previous configuration, that is, parallel RLC:

$$Q = R_p \sqrt{\frac{C_p}{L_p}} = \omega_o R_p C_p$$

$$B_\omega = \frac{1}{R_p C_p}$$

The same principle applies to the series configuration:

$$Q = \frac{1}{R_s} \sqrt{\frac{L_s}{C_s}} = \omega_o \frac{L_s}{R_s}$$

$$B_\omega = \frac{R_s}{L_s}$$

Designing Filters

To know if a circuit is Low Pass, Band Pass or High Pass, consider the circuit at $\omega = 0$ and at $\omega = \infty$. Replace all components with their impedances at these frequencies, and analyze the output.
A low pass filter is defined by the following:

$$\lim_{\omega \to \infty} |H(j\omega)| = 0$$

$$\lim_{\omega \to 0} |H(j\omega)| \neq 0$$

The passband of a circuit is the frequency range at which you want the transfer value to be above a predefined H_{max}. The stopband is the frequency range in which the transfer value is smaller than a predefined H_{min}.
Attenuation: $A(j\omega) = \frac{1}{H(s)}$.

Butterworth Low-Pass Filters

Butterworth filters are those which have a frequency response magnitude that looks like the following:

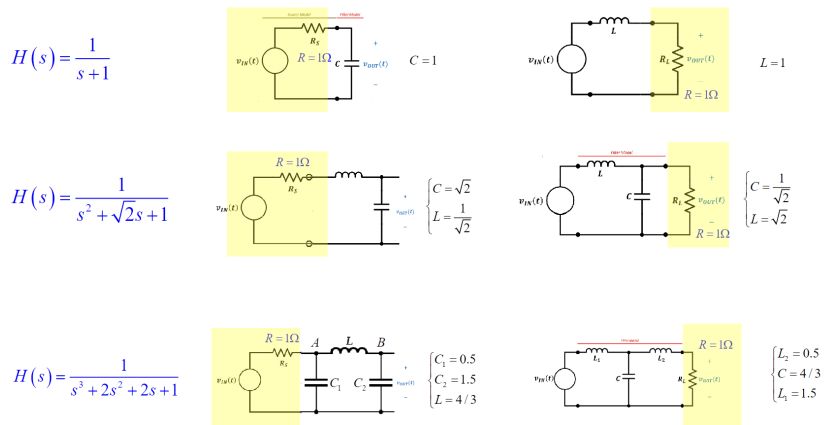$$H(j\omega) = \frac{|K|}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

where ω_c is the cutoff frequency, and n is the order of the circuit.
So, the steps are:

1. Find filter order.

$$n_{min} \geq \frac{\frac{1}{2}\log_{10}\left(\frac{10^{0.1A_{min}}-1}{10^{0.1A_{max}}-1}\right)}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}$$

2. Choose the correct normalized function.

3. Choose the corresponding circuit based on whether the load side or source side resistance is known.

$H(s) = \dfrac{1}{s+1}$

$C = 1$

$L = 1$

$H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

$\begin{cases} C = \sqrt{2} \\ L = \dfrac{1}{\sqrt{2}} \end{cases}$

$\begin{cases} C = \dfrac{1}{\sqrt{2}} \\ L = \sqrt{2} \end{cases}$

$H(s) = \dfrac{1}{s^3 + 2s^2 + 2s + 1}$

$\begin{cases} C_1 = 0.5 \\ C_2 = 1.5 \\ L = 4/3 \end{cases}$

$\begin{cases} L_2 = 0.5 \\ C = 4/3 \\ L_1 = 1.5 \end{cases}$

4. Adjust frequency scale (ω_c) if $A_{max} \neq 3dB$. That is, scale so that $\omega_p = 1$ fits the specs using the following equation:

$$\frac{\omega_p}{(10^{0.1A_{max}} - 1)^{\frac{1}{2n}}} \leq \omega_c \leq \frac{\omega_s}{(10^{0.1A_{min}} - 1)^{\frac{1}{2n}}}$$

5. Frequency and magnitude scale to the right value of the existing components and specs.
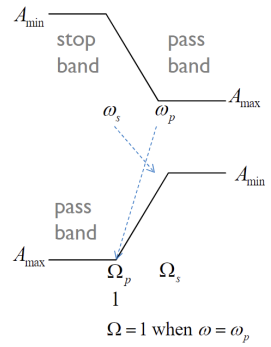
Butterworth High-Pass Filters

Create a High-Pass filter by inverting a Low-Pass filter.
That is, instead of the specs causing $\omega_s < \omega < \omega_p$, they should cause $\omega_p < \omega < \omega_s$.
To do so, invert the high-pass filters attenuation graph, making a graph of Ω, which is the equivalent of a low-pass filter graph.
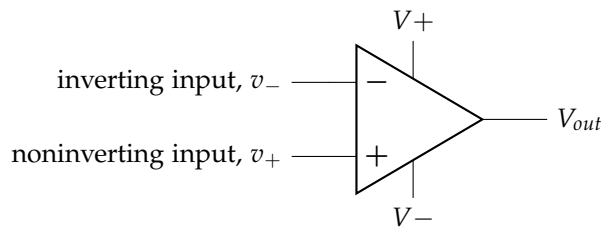
$\Omega = 1$ when $\omega = \omega_p$

where:

$$\Omega_p = \frac{\omega_p}{\omega_p} = 1$$
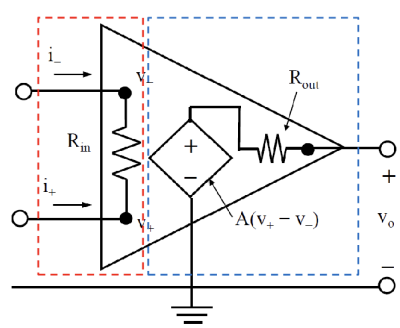
$$\Omega_s = \frac{\omega_p}{\omega_s} > 1$$

After this, follow the same steps as shown for Low-Pass filters, with Ω replacing all ω's. Once the first four steps are completed, convert back into a high pass filters by replacing all capacitors with inductors and all inductors with capacitors.

Op-Amps

The following is an Operational Amplifier:

and this is how it is modeled:

An ideal op-amp would have infinite input resistance, zero output resistance, and infinite voltage gain.

Note: for intents and purposes of this course, all op-amps are ideal.
Steps to follow:

1. Virtual Short Circuit to find v_-. That is, find v_+, and since $v_- = v_+$, you have found both.

2. Calculate the input path current:
$$I_I = \frac{V_I - V_-}{Z_I}$$

3. Calculate the feedback path current:
$$I_f = \frac{V_- V_o}{Z_f}$$

4. Equate to calculate gain:
$$I_I = I_f \implies gain = \frac{V_o}{V_I}$$