Notes for MA 26500 - Linear Algebra I

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Course Introduction

This course serves as an introduction to the fundamental concepts and applications of linear algebra, a branch of mathematics that explores vector spaces, linear transformations, and systems of linear equations.

Systems of Equations in Linear Algebra

A linear equation is an equation of the form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Where a_1, a_2... and b are given constants. Note that the exponents of all x terms is 1. Some examples are:

$$2x_1 + 3x_2 = 4$$

$$5x_1 + 6x_2 = 10$$

as opposed to

$$2x_1x_2 + x_3 = 9$$

$$2x_1 + \sqrt{x_2} = 8$$

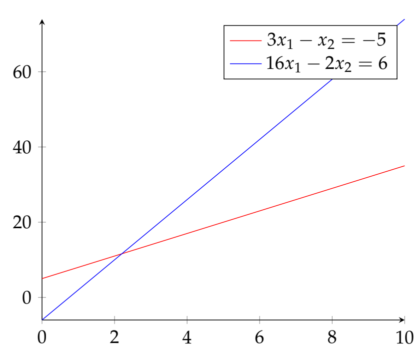
A system of equations is a collection of one or more linear equations involving the same variable set. An example of this would be:

$$\begin{cases} 3x_1 + 5x_2 + x_3 = 3 \\ 7x_1 - 2x_2 + 4x_3 = 4 \\ -6x_1 + 2x_3 = 2 \end{cases}$$

where the constant coefficient of x_2 in the third equation is 0.
A solution is a list of numbers $(s_1, s_2, ..., s_n)$ that makes each equation a true statement when we replace $x_1 = s_1$, $x_2 = s_2$, ... $x_n = s_n$. If we have a system of two linear equations, the solution will be the intersection of the two lines that define the equations on the cartesian plane. The system of equations

$$3x_1 - x_2 = 5$$
$$16x_1 - 2x_2 = 6$$

is mapped to the following graph:



There are three kinds of systems:

1. One solution

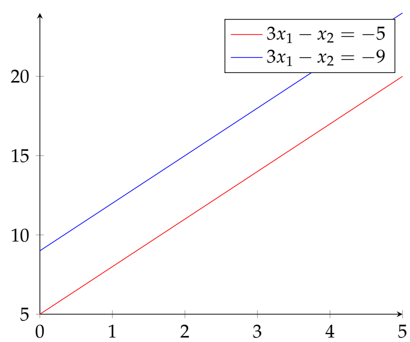
2. Infinitely many solutions

3. No solutions

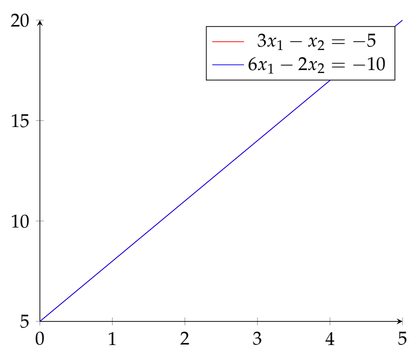
The above example has one solution. This is the intersection of the two lines. This could be further generalized to have far more dimensions as well as far more variables, but such systems are no longer representable in the number of dimensions we can perceive.

However, if the lines were parallel, the other two possibilities surge. If the equations are linearly dependent, there would be infinitely many solutions. However, if the value of b were different for the two, while all the variable coefficients were linearly dependent, it would have none.

The following is an example of no solution:



While this is infinitely many solutions:



Any system of equations has a matrix notation. For example, the above system can be represented as the following.

$$\begin{bmatrix} 3 & -1 & -5 \\ 16 & -2 & 6 \end{bmatrix}$$

Which has the coefficients of the x values as the first n values, and the value of b as the last one. This is called the augmented matrix of the system of equations. As opposed to:

$$\begin{bmatrix} 3 & -1 \\ 16 & -2 \end{bmatrix}$$

which is the coefficient matrix.
These systems can be solved using the three elementary row operations. This is called row reduction.
The three rules are:

1. Interchange: Exchange the positions of any two rows

2. Multiply: Multiply any row by a constant

3. Addition: Replace the value of a row, with its sum and that of one of the other rows multiplied by a scalar.

By using these three rules, one can reduce it to the following form:

$$\begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 1 & c_4 \end{bmatrix}$$

This will result in a trivially solvable unique-solution system of equations. This is also known as a **consistent** system of equations. We can substitute the value of x_3, which is c_4, into the equation in the row above, and solve it, as it willnow be a single equation with a single variable. This can be done sequentially until all values are found.

Note: If the resulting form is instead

$$\begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 0 & c_4 \end{bmatrix}$$

with $c_4 \neq 0$, there would be no solutions, also known as an **inconsistent** system. On the other hand, if $c_4 = 0$, there will be infinitely many solutions, but it will still be considered a **consistent** system. Another point to note is that if the augmented matrices of two systems of equations are equivalent, that is, if you can transform one matrix into the other through elementary row operations, the systems of equations will have the same solutions.

Row Reduction and Echelon Forms

The leading entry of a row refers to the left-most non zero element in a row. Thus, the leading entries in the following matrix are:

$$\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 0 & \mathbf{6} & 9 & 6 \\ 0 & 0 & \mathbf{8} & 7 \end{bmatrix}$$

A matrix is in reduced echelon form (REF) if it has the following three properties:

1. All non-zero rows are above any rows of all zeros.
2. Each leading entry in a row is in a column to the right of the leading entry in the column above it.
3. All entries below a leading entry are zeros.

   If it has the following two properties, it will be a row reduced echelon form (RREF):

4. The leading entry in each non-zero row is 1.
5. Each leading 1 is the only non-zero element in its column.

Example of echelon form:

$$\begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 3 & 9 & 6 \\ 0 & 0 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of REF:

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem: Each matrix is equivalent to one and only one RREF.

A pivot position is a location in a matrix that corresponds to a leading 1 in the RREF of a matrix.

A pivot column is a column that contains a pivot position.

Tip: if there is a 1 in the first positions of a row, when the matrix is first "observed", it should be moved to the top of the matrix, as it will most probably make all subsequent computations much easier.

Row reduction algorithm:

1. Begin with the leftmost non-zero position, and make it the first pivot position.

2. Make all positions under the selected pivot position 0.

3. Interchange columns to make sure that the rows with the most leading zeros are closest to the bottom

4. Cover the first column and row, and repeat steps 1 to 3 on the resulting matrix.

5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.

The solution of the RREF is the solution of the system of equations that has an augmented matrix with the aforementioned RREF.

Another thing we can note is that all pivot positions refer to basic variables. If there are any non-pivot columns, the variable referring to these columns will be free.

For example a 4x7 matrix is underconstrained and with free variables.

Vector Equations

A matrix with only one column is called a column vector or vector.
For example:

$$u = \begin{bmatrix} 3 \\ 8 \\ 12 \end{bmatrix} \text{ or } w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Two vectors are equal if and only if their corresponding terms are equal. There are some basic operations that can be conducted with them.
Addition between two vectors of the same size:

$$u = \begin{bmatrix} 3 \\ 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 9 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+9 \\ 8+4 \\ 12+5 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 17 \end{bmatrix}$$

Scalar Multiplication:

$$u = 3 * \begin{bmatrix} 3 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 3*3 \\ 8*3 \\ 12*3 \end{bmatrix} = \begin{bmatrix} 9 \\ 24 \\ 36 \end{bmatrix}$$

Note: these operations both have graphical representations. Addition is addition of two vectors, as is done by the parallelogram method, and scalar multiplication is equivalent to proportionally scaling a vector.

A line can be defined as the set of all scalar multiples of a vector. Algebraic properties of a vector in $\mathbb{R}^n$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = cd(\mathbf{u})$
8. $1\mathbf{u} = \mathbf{u}$

Linear Combinations

A vector is a linear combination of other vectors if:

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} \dots c_n\mathbf{u_n} = \mathbf{v}$$

For any set of vectors, there is an infinite amount of linear combinations, as the values of the constants they are multiplied with are not restrained by anything and they cound be a part of any domain, be it $\mathbb{R}, \mathbb{I}$, or $\mathbb{C}$.
To find if a vector is a linear combination of other matrices, we can create an augmented matrix with the desired vector as the constant column, and all relevant vectors, as the elements of the coeffiecient matrix part of the augmented matrix.
Should there be a free variable in the resolution of teh augmented matrix, there will be infinitely many linear combinations that fulfill the condition we desire.
The $span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots, \mathbf{v_n})$ is the set of all linear combinations of these vectors. So , if asked whether a vector is in the span of other vectors, it is equivalent ot asking if there exists a linear combination of vectors that results in the desired vector.

Matrix Equation $\boldsymbol{Ax} = \boldsymbol{b}$

Let $\mathbf{A}$ be a $m \times n$ matrix. The product of $\mathbf{A}$ and $\mathbf{x}$ will be the linear combination of the columns of $\mathbf{A}$ using the entries of $\mathbf{x}$ as weights.

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}$$

Any linear combination can be written in the form $\mathbf{Ax}$, and, if it is a system of linear equations, it can be written as $\mathbf{Ax} = \mathbf{b}$.
The matrix equation $\mathbf{Ax} = \mathbf{b}$ has the same solution as $x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}$ which in turn has the same solution set as the augmented matrix $\begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} & \mathbf{b} \end{bmatrix}$.
In essence, what we are searching for is the value of $\mathbf{x}$.
Existence of solutions:
The equation $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of $\mathbf{A}$.
The following statements are equivalent:

- For each $\mathbf{b}$ in $\mathbb{R}^n$ the equation $\mathbf{Ax} = \mathbf{b}$ has a solution
- The equation $\mathbf{Ax} = \mathbf{b}$ has a solution
- Each $\mathbf{b}$ in $\mathbb{R}^n$ is a linear combination of the columns of $\mathbf{A}$
- $\mathbf{A}$ has a pivot position in every row.

Row-Vector Rule for Computing $\mathbf{Ax}$:
The valid product of two matrices is the sum of the linear combination of each row with their respective terms in the $\mathbf{x}$ matrix.

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 5 & 1 & 4 & 6 \\ 6 & 9 & 1 & 7 \\ 0 & 7 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1x_1 + 3x_2 + 3x_3 + 8x_4 \\ 5x_1 + 1x_2 + 4x_3 + 6x_4 \\ 6x_1 + 9x_2 + 1x_3 + 7x_4 \\ 0x_1 + 7x_2 + 4x_3 + 3x_4 \end{bmatrix}$$

If $\mathbf{A}$ is an $m \times n$ matrix:

- $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$
- $\mathbf{A}(\mathbf{u}c) = c(\mathbf{Au})$