Notes for ECE 30200 - Probabilistic Methods in Electrical and Computer Engineering

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These are lecture notes for Fall 2025 ECE 30200 by professor Mary Comer at Purdue. Modify, use, and distribute as you please.

Contents

Modelling Random Experiments

Examples:

- Flip a coin
- Rolling dice
- Generate a bit from a random binary source
- Generate a sequence of n bits from a random binary source
- Count packets arriving at a router
- Measure the voltage at a point in a circuit

All of these examples could be at a fixed time, or over an interval of time.

There is a very precise framework that can be used to model any random experiment.
Overview:

- The outcome that occurs each time a random experiment is run is not known in advance, but **the set of all possible outcomes is assumed to be known**.
- Subsets of the set of all possible outcomes are called events.
- Probabilities are assigned to events, not outcomes, using a probability measure.

We need to use set theory to work with these sets.

Set Theory

A set is an **unordered** collection of elements denoted by $\{\}$.
Example:

$$\{1, 2, 3\} = \{3, 2, 1\} = \{2, 1, 3, 1\}$$

Notation:

- $w \in A$ means w is in set A
- $w \notin A$ means w is not in set A

There are two ways to specify a set:

1. Comma-separated list:
   - $A = \{1, 3, A, C\}$

- $A = \{x_1, x_2, \ldots, x_n\}$ (that is, n elements for some finite $n \geq 1$ and the ith element is x_i)
- $A = \{x_1, x_2, \ldots\}$ (an infinite number of elements)

2. Rule for membership or set builder notation:
   - $A = \{w \in \mathbb{Z} : 1 \leq w \leq 6\}$
   - Special notation for intervals $\in \mathbb{R}$:
     - (a, b) represents $\{x \in \mathbb{R} : a < x < b\}$
     - $[a, b]$ represents $\{x \in \mathbb{R} : a \leq x \leq b\}$

     Note: $(a, b) \subset \mathbb{R}$ is a set, but $(a, b) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is also a point on the Cartesian plane or an ordered pair.

Equal sets: sets A and B are equal if they contain exactly the same elements, and is denoted by $A = B$.

A ⊂ B: $(w \in A \implies w \in B) \implies (A \subset B)$ for sets A and B where $A \subset B$ means A is a subset of B.

Note: we will not distinguish between proper subsets:

$$(A \subset B) \& (A \neq B)$$

and subsets.

$$A = B \iff (A \subset B) \& (B \subset A)$$

Special Sets

The set with no elements is called the empty set or the null set. Denoted by $\emptyset$ or $\{\}$ which is not the same as $\{\emptyset\}$.

The set containing all possible elements of interest is called the universal set S. It is known as the sample space within probability, and will contain all possible outcomes of a random experiment.

Set Operations

- Intersect ($\cap$): $A \cap B = \{w \in S : (w \in A) \& (w \in B)\}$
- Union ($\cup$): $A \cap B = \{w \in S : (w \in A) | (w \in B)\}$
- Complement ($\square^c$, $\square'$, $\bar{\square}$): $A^c = \{w \in S : w \notin A\}$

- Disjoint: $A \cap B = \emptyset$
- *Difference*: $A - B = A \cap B^c$

Set Algebra

Set algebra provides us with a second method to prove that two sets A, B are equal.
It uses the fact that both union and intersection are commutative and associative to prove the statement.
The following are a few properties that come from commutativity and associativity:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$

$\cup$ is distributive over $\cap$ and vice versa:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Note: when translating between Set Theory and English

- $\cup =$ OR
- $\cap =$ AND

Venn Diagram

A Venn diagram is a graphical representation of a universal set and its subsets. For example:

Note: A Venn diagram does not represent arbitrary sets, and thus, is not a proof by itself.

Set Types

There are three kinds of sets:

1. Finite: a set is finite if it has a finite number of elements.
2. Countably Infinite: a set is countably infinite if it can be placed into one-to-one correspondance with the integers. Can be written as

$$A = \{x_1, x_2, \dots\} \text{ where } x_i \neq x_j \forall i \neq j$$

3. Uncountable: set that is neither countable nor finite.

> *Note*: For the context of this class, the uncountably infinite sets will only be $\mathbb{R}$ or intervals of the same.

They cannot be written as $\{x_1, x_2, \dots\}$

Collections of Sets

- Finite Collections:

$$A_1, A_2, \dots, A_n$$

where $A_i \subset S$

- Countably Infinite Collection:

$$A_1, A_2, \dots$$

where $A_i \subset S$

- Uncountable Collection: out of bounds for this class!

> *Note*: All the collections mentioned above are denoted by $\{A_i\}$.

Union of collection: $\bigcup_i^{\infty,n} A_i = \{w \in S \colon \exists i | w \in A_i\}$

Intersection of collection: $\bigcap_i^{\infty,n} A_i = \{w \in S \colon w \in A_i \forall i\}$

Modelling Random Experiments Pt.2

We use a probability space to model (almost) any random experiment. Three parts make up a probability space:

1. Sample Space ($\mathcal{S}$): nonempty set of elements called outcomes. Each time an experiment is run, exactly one outcome occurs.

Note: If the outcome is outside $\mathcal{S}$, either the sample space is incomplete, or the case is enough of an outlier to be ignored.

The sample space will be a set that falls under one of the categories that were mentioned before.
For infinite sets, the probability cannot be treated as simply the number of times an outcome occurs over the number of possible outcomes.

2. Event Space ($\mathcal{F}$): collection of all events (collection of all possible subsets, also known as the power set) in the random experiment.
An event is a set of outcomes to which a probability is assigned.
An event is a subset of $\mathcal{S}$.
An event either occurs or does not occur every time an experiment is run. We can define the occurence of an event as the presence of the occuring outcome within the event.

Note: More than one event can occur in a single experiment. That is, an outcome can be in multiple events at once.

Note: It can be assumed that any subset of $\mathcal{S}$ is a valid event for the context of this class. This is not strictly true in practice, but it rarely matters.

Elementary Event: and event containing only one outcome.

In the case of uncountably infinite sets, $\mathcal{F}$ cannot be explicitly written. However, certain specific events of interest can be defined and analyzed.

3. Probability Function or Probability Measure (P): a set function that assigns probabilities to events.

$$P : \mathcal{F} \to \mathbb{R}$$

That is, the probability measure maps the event space to the real numbers. These valid functions are restricted by probability axioms:

Axiom: a fundamental rl or assumption that can be used to prove all other properties of the relevant math. They are not proven, but stated.

(a) $P(A) \geq 0 \; \forall A \in \mathcal{F}$

(b) $P(S) = 1 \; \forall \; S$

(c) $\forall \; A, B : A \cap B = \{\} \implies P(A \cup B) = P(A) + P(B)$

(d) Same as previous but for countably infinite collection.

> *Note*: The reason axiom 3 exists is because it proves axiom 4 for a finite collection. It does not, however, prove it for infinite n.

> *Note*: Axiom (c) is not truly necessary.

The axioms can be used to prove properties such as:

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A) \; \forall \; A \in \mathcal{F}$
- $\forall \; A, B \in \mathcal{F} | P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $A \subset B \implies P(A) \leq P(B)$

Probability

There are several ways to define probabilities.

> **Counting Approach Probability**: $P(A) = \frac{N_A}{N}$ where N_A the number of times event A occurs in all instances of the experiment.

> **Relative Approach Probability**: $P(A) = \lim_{n \to \infty} \frac{n_A}{n}$ where n_A is the number of times event occurred during the experiments.

Both of these approaches satisfy the axioms we defined before.

> **Conditional Probability**: for any two events A, B with $P(B) \neq 0$, the conditional probability of A given B is:
>
> $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The previous definition leads to Bayes Theorem, which states:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) \text{ if } P(A), P(B) \neq 0$$
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

In case we do not have enough information to get three of the variables in Bayes Theorem, we can use the Total Probability Law (TPL):

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \text{ if } P(A_i) \neq 0$$