

Notes for ECE 29595PD - Principles of Digital System Design

Shubham Saluja Kumar Agarwal

January 26, 2024

These are lecture notes for spring 2024 ECE 29595PD at Purdue as taught by Professor Irith Pomeranz. Modify, use, and distribute as you please.

Contents

<i>Course Introduction</i>	1
<i>Introduction</i>	3
<i>Digital Logic Signals</i>	3
<i>Logic Circuits</i>	4
<i>Gates</i>	5
<i>Timing</i>	5
<i>Software Aspects of Digital Design</i>	6
<i>Integrated Circuits</i>	7
<i>Logic Families</i>	7
<i>CMOS Logic Circuits</i>	7
<i>Preview of future topics</i>	9
<i>Number Systems and Codes</i>	10
<i>Adding and Subtracting</i>	11
<i>Using the complement to conduct operations</i>	12
<i>Binary Codes for Decimal Numbers</i>	13
<i>Switching Algebra and Combinational Logic</i>	15
<i>Axioms</i>	15
<i>Theorems</i>	15
<i>Connecting Switching Algebra and Combinational Logic</i>	17

Course Introduction

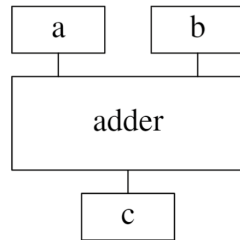
This course serves as an introduction to digital system design, with an emphasis on principles of digital hardware and embedded system design. It is an alternate class to ECE 27000.

Learning Outcomes:

1. Ability to analyze and design combinational logic circuits.
2. Ability to analyze and design sequential logic circuits.
3. Ability to analyze and design computer logic circuits.
4. Ability to realize, test, and debug practical digital circuits.

Introduction

Digital design entails creating hardware that can conduct an operation or set of operations within a computer system. For example, adding two numbers.



This hardware can add two numbers, that is, conduct the operation $c = a + b$. It could also perform $f = d + e$ or $i = g - h = g + (-h)$ as it is not restricted to the sole values of a and b as inputs. This process fits into the logic design and switching algebra portions of chip manufacturing.

The creation of systems like these is based on the fact that voltage and current are time-varying and can assume any value in a continuous range of real numbers, but are mapped to only two values.

Digital Logic Signals

A digital signal is modeled assuming that at anytime, it can have only one of two discrete values, which represent:

0	1
LOW	HIGH
FALSE	TRUE

This is called positive logic. It maps the infinite values of voltage and current to these two values. An example of this is CMOS 2-Volt logic:

0	1
0-0.5V	1.5-2.0V

These completely separated ranges of values allow for 0 and 1 to be completely separate, with noise and other possible errors being ignored. In this way, all physical values can be partitioned into the two values, though, for intents and purposes of digital system design, voltage is the most relevant.

Additionally, circuits known as buffer circuits can be used to restore logic values. That is, if a voltage is not sufficiently close to the values

of 0 or 2 (in the case of the CMOS), they can push these values far closer to the desired values, reducing the possibility of error.

Digital circuits have replaced analog circuits because they are far easier to design. They can be made using Hardware Description Languages (HDLs), softwares that are similar to programming languages. A logic value is called a binary digit, or bit. A set of n bits, represent 2^n values. For example:

	b_0	b_1
b_0	0	0
0	1	0
1	0	1
	1	1

Logic Circuits

At the highest level, a logic circuit is a *black box* with n inputs and m outputs. Only zeros and ones are required to represent inputs and outputs.

Combinational circuits are circuits that have outputs that solely depend on the inputs. It can be represented by a truth table.

An adder can be represented by a truth table in this manner.

x_1	x_2	x_3	z_2	z_1
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

On the other hand, a sequential circuit has memory. That is, the output is dependent on both the inputs and the current state of the circuit itself. This is representable through a state table.

input	present state	next state	output
0	00	00	0
1	00	01	0
0	01	01	0
1	01	10	0
0	10	10	0
1	10	00	1


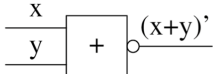
Gates

Basic digital devices are called gates. These implement basic logic functions. These are AND, OR, NOT. However, AND, NOT and OR, NOT are sufficient to make any combination, as the third gate can be formed as a combination of the other two.

Each of the gates is represented symbolically like the following, and have their truth tables shown below:

AND			OR			NOT	
							
x	y	$x \wedge y$	x	y	$x \vee y$	x	$\neg x$
0	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0
1	0	0	1	0	1		
1	1	1	1	1	1		

This can be done for more inputs, by combining multiple gates. There are two more often used gates that can be created through a combination of these two.

NAND			NOR		
					
x	y	$\neg(x \wedge y)$	x	y	$\neg(x \vee y)$
0	0	1	0	0	1
0	1	1	0	1	0
1	0	1	1	0	0
1	1	0	1	1	0

It is a good exercise to see how two of the basic gates (AND, NOT or OR, NOT) can be used to create the rest, as it is possible to connect gates to form complex circuits and thus, complex functions.

Note: By convention, signal flows from left to right. Thus, output comes out the right and input goes in at left.

Timing

When the values of an input changes, it takes time for the outputs to change as well, which is known as *propagation delay*, which can be represented in a timing diagram. However, for most cases, it is possible to ignore these, as the behavior is well defined regardless of delay.

The following is a timing diagram:



Propagation delay, represented by *pd* in the timing diagram, is irrelevant if there are no changes, regardless of the fact that there is a new set of inputs. The values at the bottom of the diagram are the values that each of the elements hold after all the necessary transitions have happened for that set of inputs.

Transition time is the time it takes for a signal to change from 0 to 1, and is represented by "*rt*".

Software Aspects of Digital Design

Software is not technically essential to digital system design. People used to draw symbol by hand, and could make technically equivalent systems. However, the availability, utility, and ease of use of HDLs has made the use of software prevalent in the current technological landscape.

Electronic Design Automation tools are useful in improving designer productivity. The following are examples of these:

- Schematic entry: Allows for fully detailed digital diagrams to be created digitally.
- HDLs: Can be used to design anything from individual function modules to multichip digital systems. This course will involve extensive use of VHDL.
- HDL text editors, compilers, synthesizers: text editor to define, compiler to debug syntax, synthesizer to create corresponding circuit or chip.
- Simulator: It is virtually impossible to debug a synthesized chip, so simulators are used before synthesis. They allow for verification of functionality prior to the actual tedious process of synthesis. Among simulators, there are also PCB simulators, and FPGAs, which are like programmable chips. They are very important.

- Test benches: Designs are simulated and tested using these. They run a series of checks to ensure that nothing stopped working due to changes.
- Timing analyzers and verifiers: Correct timing of inputs and reactions are paramount in digital systems. This facet of simulation allows for the automation of timing functionality verification.
- Word Processors: Allow for pretty documentation to be created.

Software is important, but the understanding of what it actually does is even more so.

Integrated Circuits

A collection of gates on a single silicon chip is called an integrated circuit or IC.

An IC originates from a wafer, which is segmented into many identical ICs. The wafer is divided into rectangles, called *dies* which in turn have *pads* which allow for wire connections. Microscopic probes are used to debug the wafers, and defects are discarded, and the successes, cut out.

For this class, an IC, is a packaged die.

ICs were divided by size, small SSIs with 1 to 20 gates, medium MSIs with 20 to 200, large LSIs with upto 1000, and very large VLSIs from everything above that. The largest VLSIs now have over 10 billion transistors.

An IC process is the steps taken to create an IC, and are categorized by the transistor density within the chip.

Logic Families

There are many different ways of implementing logic circuits. Connectable logic circuits are of the same logic family. So, the technology controlling them is different. Chips need to be from the same family to be connected to each other. If they are from different families, they cannot be connected to each other.

For example TTL (Transistor-Transistor Logic) is a logic family based on bipolar junction transistors.

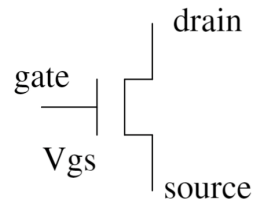
CMOS (Complementary Metal-Oxide Semiconductor) is based on MOSFETs (MOS Field-Effect Transistors). These are more commonly used, and the ones we will be analyzing in this class.

CMOS Logic Circuits

A MOS transistor is modelable as a voltage controlled resistor. This means that the input voltage controls the resistance of the MOSFET,

allowing it to act as a kind of switch. This is because it has only two effective states: very high resistance and very low resistance.

This is an n-channel (NMOS) transistor:



When $V_{gs} = 0$, the resistance is very high.

When V_{gs} is increased, the resistance is very low.

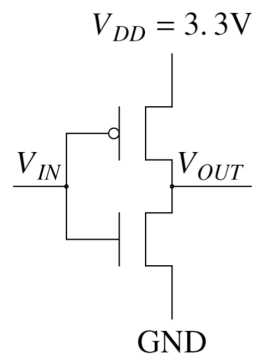
On the other hand, this is the PMOS transistor:



When $V_{sg} = 0$, the resistance is very high.

When V_{sg} is increased, the resistance is very low.

NMOS and PMOS together allow us to form CMOS logic. For example, this is a CMOS inverter, in which the output will be the opposite of the input.



If $V_{in} = 3.3V$, or high, the PMOS transistor be off, and the NMOS would be on. This would mean that the output and ground are shorted, causing the output to be low.

If $V_{in} = 0V$, or low, the PMOS transistor be on, and the NMOS would be off. This would mean that the output and V_{DD} are shorted, causing the output to be high.

This allows this to be act as an inverter.

Now we will observe the CMOS arrangement of a NAND gate.



If A and B are both HIGH, both PMOS are HIGH, and thus V_{DD} is shorted to Z.

If they are both LOW, the exact opposite happens, with Z shorting to ground.

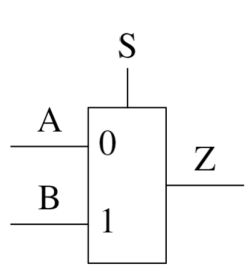
If A or B are HIGH, than one of the PMOS will be high, and one of the NMOS will be LOW, causing the output to be HIGH.

The NOR gate can be constructed easily from the same thought process, but NMOS in parallel, and PMOS in series.

These can be expanded to more inputs by adding more transistors.

Preview of future topics

The following is a multiplexer:



It outputs A when S is 0, and B when S is 1. It allows us to select between different functions, such as adding and subtracting.

Truth tables are another form of representation of a digital system. We will later learn how to analyze and derive equations from truth tables, as well as the gate implementation.

Note: CMOS logic doesn't have AND, OR gates, so the actual implementation of these two is NAND, NOR gates with an inverter.

Gates require 4 transistors, and inverters need two transistors. Verilog Structural model:

```
module Ch1mux_s(A,B,S,Z);
    input A,B,S;
    output Z;
    wire SN,ASN,SB;
    not U1(SN,S);
    and U2(ASN,A,SN);
    and U3(SB,B,S);
    or U4(Z,ASN,SB);
endmodule
```

Which is equivalent to:



Note: the order of these statements is irrelevant, they will result in the same digital system. Verilog Behavioral Model:

```
module Ch1mux_b(A,B,S,Z);
    input A,B,S; \\input declaration
    output reg Z; \\output declaration
    always @(A,B,S) \\if input changes, recompute
    if (S==1) Z=B;
    else Z=A; \\this is saying what the multiplexer above did
endmodule
```

Number Systems and Codes

A digital circuit processes binary digits in the form of bits.

It is important to see how these relate to real life values.

Numbers in the decimal system:

$$abcd = 1000a + 100b + 10c + d = 10^3a + 10^2b + 10^1c + 10^0d$$

The base, or radix is 10. For a general radix r , it would be:

$$abcd = r^3a + r^2b + r^1c + r^0d$$

For this course, we care about binary, or radix 2.

base 2	base 8	base 16
0	0	0
1	1	1
10	2	2
11	3	3
100	4	4
101	5	5
110	6	6
111	7	7
1000	10	8
1001	11	9
1010	12	A
1011	13	B
1100	14	C
1101	15	D
1110	16	E
1111	17	F

Occasionally, base 8 (octal) and base 16 (hexadecimal) are also relevant. To convert from octal to binary, replace every octal with three bits starting from the least significant bit. For hexadecimal, replace each digit with 4 bits.

The least significant bit (LSB) is the last bit, and the first is the most significant bit (MSB).

To convert from decimal to binary by dividing the number by 2, and assigning the remainder to the least significant unassigned bit. Do this continuously to the quotient until done.

Adding and Subtracting

To add and subtract in binary:

$$\begin{array}{r}
 1\ 0\ 1\ 1\ + \\
 1\ 0\ 1\ 0\ \hline
 1\ 0\ 1\ 0\ 1
 \end{array}
 \qquad
 \begin{array}{r}
 1\ 1\ 0\ 1\ 1\ - \\
 1\ 0\ 1\ 0\ \hline
 1\ 0\ 0\ 0\ 1
 \end{array}$$

To add signs to numbers, it has been standardized that the first bit defines the sign. If the leftmost bit is 0, the number is positive. and if it is 1, the number is negative.

This allows an n -bit signed integer to be from the range $-2^{n-1} + 1$ to $2^{n-1} - 1$.

To actually add and subtract numbers, we first check if the sign is the same, if it is, we add directly, and append the sign bit. If they have opposing signs, we find the larger number, subtract the smaller, and use the sign of the larger.

Using the complement to conduct operations

In the complement number system, we can add or subtract directly, the operations can be done directly. The complementation process is more complicated than sign checking, but it simplifies addition.

The 2-complement system is the difference between 2^n and the n -bit integer. To compute $-B$, we first need to calculate $2^n - B$, which we can compute as $(2^n - 1) - B + 1$, which is essentially inverting each bit, and adding 1 to the result. (Carries outside the bit length are ignored.)

The most significant bit of the complement acts as a sign bit. This can be seen in the following table: The magnitude of a number can be

number	binary	negative	binary
0	0000	-0	0000
1	0001	-1	1111
2	0010	-2	1110
3	0011	-3	1101
4	0100	-4	1100
5	0101	-5	1011
6	0110	-6	1010
7	0111	-7	1001
-	-	-8	1000

computed as for an unsigned number, except that the weight of the MSB is -2^{n-1} instead of 2^{n-1} . So, to subtract 2^n , replace the weight of the MSB with -2^{n-1} .

To add a bit to a complement number, we can simply duplicate the MSB at the beginning of the MSB. As proof of this actually working, we can calculate the complement of both the complement and the modified complement as uncomplemented numbers, and we can observe that they will be the same.

That is, the complement of 11001 and 111001 is the same number, the same happens for 011 and 0011.

We can also reverse this process if the digits corresponding to the removed bits of the given complement are all the same.

Now we will analyze how the complement can be used to add and subtract.

+6	0110	+
-3	1101	
<hr/>		
3	1	0011

This process may not work if overflow occurs. That is, if the number of necessary bits to represent the result of an operation are more than

the available bits. This can only occur if the numbers being operated on have the same sign.

To subtract two numbers, we instead use the following property:
 $X - Y = X + (-Y) = X + Y' + 1$ where Y' is the complement of Y .

$$\begin{array}{r}
 6 \quad 0110 \quad - \\
 2 \quad 0010 \\
 \hline
 \quad 0001 \\
 6 \quad 0110 \quad + \\
 2' \quad 1101 \\
 \hline
 1 \quad 0100 \\
 4 \quad 0100
 \end{array}$$

Binary Codes for Decimal Numbers

A set of n -bit strings in which different bit strings represent different elements of a set is called a code. A combination on these is a code word.

0 through 9 requires at least 4 bits, but there are many methods of doing so.

digit	BCD	2421	excess-3	1-out-of-10
0	0000	0000	0011	1000000000
1	0001	0001	0100	0100000000
2	0010	0010	0101	0010000000
3	0011	0011	0110	0001000000
4	0100	0100	0111	0000100000
5	0101	1011	1000	0000010000
6	0110	1100	1001	0000001000
7	0111	1101	1010	0000000100
8	1000	1110	1011	0000000010
9	1001	1111	1100	0000000001

Another very important representation is the gray code.

By partitioning a disk into eight regions, with n bits with sensors in each, any number between 0 and $2^n - 1$ can be represented.

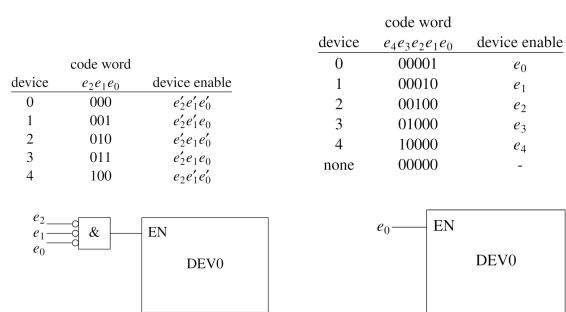


Note: if the disk stops between two sections, there is a possibility of error. Using standard binary representation will cause the error to be very large, as it can stop between 000 and 111, which would imply that any of the bits can have error, and thus any of the numbers between 0 and 7 can be represented. This is where the gray code comes in:

number	gray code
0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

This encoding allows there to be only one bit of change between any two consecutive digits, allowing us to reduce the margin of error. Codes can also be used to represent text, as the ASCII does, which uses a 7-bit code word.

The selection of code and code words is very important, as it can drastically change the complexity of the designed circuit:



Switching Algebra and Combinational Logic

A combinational circuit depends only on the inputs. We will begin by considering single output combinational circuits.

Switching algebra is a mathematical tool used for circuit design. It is a subset of boolean algebra. A variable is used to represent a signal.

Axioms

- (A1) $X = 0$ if $X \neq 1$
- (A1D) $X = 1$ if $X \neq 0$

There is a principle of duality in this axiom, as do most properties of switching algebra.

- (A2) $X = 0 \implies X' = 1$
- (A2D) $X = 1 \implies X' = 0$
- (A3) $0 \wedge 0 = 0$
- (A4) $1 \wedge 1 = 1$
- (A5) $0 \wedge 1 = 1 \wedge 0 = 0$
- (A6) $1 \vee 1 = 1$
- (A7) $0 \vee 0 = 0$
- (A8) $1 \vee 0 = 0 \vee 1 = 1$

AND has a higher precedence than OR in logical expressions"

$$W \wedge Y \vee Y \wedge Z = (W \wedge X) \vee (Y \wedge Z)$$

Theorems

There are 5 theorems that can be easily be proven through this:

1. $X \vee 0 = X$
2. $X \vee 1 = 1$
3. $X \vee X = X$
4. $(X')' = X$
5. $X \vee X' = 1$

Which have the following dualities:

1. $X \wedge 1 = X$

2. $X \wedge 0 = 0$
3. $X \wedge X = X$
4. No duality
5. $X \wedge X' = 0$

There are also two variable theorems:

- Commutativity:

1. $X \vee Y = Y \vee X$
2. $X \wedge Y = Y \wedge X$

- Associativity:

1. $(X \vee Y) \vee Z = X \vee (Y \vee Z)$
2. $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$

This allows for the representation of three or more input gates with ease.

- Distributivity:

1. $X \wedge Y \vee X \wedge Z = X \wedge (Y \vee Z)$
2. $(X \vee Y) \wedge (X \vee Z) = X \vee Y \wedge Z$

- Covering:

1. $X \vee X \wedge Y = X$
2. $X \wedge (X \vee Y) = X$

- Combining:

1. $X \wedge Y \vee X \wedge Y' = X$
2. $(X \vee Y) \wedge (X \vee Y') = X$

- Consensus:

1. $X \wedge Y \vee X' \wedge Z \vee Y \wedge Z = X \wedge Y \vee X' \wedge Z$
2. $(X \vee Y) \wedge (X' \vee Z) \wedge (Y \vee Z) = (X \vee Y) \wedge (X' \vee Z)$

All these theorems can be proven using perfect induction, that is, through building the complete truth table for the inputs, and showing that the output is always true.

Some of these can also be proven using the axioms and priorly proven theorems. *Note: The proof of one theorem, can be applied to the dual, but using the duals of the theorems used in the proof of the first.*

- Variable Theorems:

1. $X \vee X \vee X \dots \vee X = X$
 2. $X \wedge X \wedge X \dots \wedge X = X$
- De Morgan's Theorems:
 1. $(X_1 \wedge X_2 \wedge X_3 \dots \wedge X_n)' = X_1' \vee X_2' \vee X_3' \dots \vee X_n'$
 2. $(X_1 \vee X_2 \vee X_3 \dots \vee X_n)' = X_1' \wedge X_2' \wedge X_3' \dots \wedge X_n'$

Using De Morgan's Theorems allows us to change combinations of NANDs and NORs into more efficient NAND only or NOR only gate functions.

De Morgan's Theorem Generalization:

$$[F(X_1, X_2, \dots, X_n, \vee, \wedge)]' = [F(X_1', X_2', \dots, X_n', \wedge, \vee)]$$

- Shannon's Expansion theorems:
 1. $F(X_1, X_2, \dots, X_n) = X_1 \wedge F(1, X_2, \dots, X_n) \vee X_1' \wedge F(0, X_2, \dots, X_n)$
 2. $F(X_1, X_2, \dots, X_n) = X_1 \vee F(1, X_2, \dots, X_n) \wedge X_1' \vee F(0, X_2, \dots, X_n)$

These can be proven through finite induction.

Connecting Switching Algebra and Combinational Logic

We will start by defining some things:

- A *literal* is a variable that can be complemented or uncomplemented, for example X, X', Y, Y'
- A *product term* is a literal or a product (AND) of two or more literals, for example $X, X', X \wedge Y' \wedge Z$
- A *sum-of-products* is a sum (OR) of multiple products.
- A *sum term* is a single literal or the sum (OR) of multiple literals.
- A *product-of-sums* is a product of multiple sum terms.
- A *normal term* is a product or sum in which no variable appears more than once.
- An *n-variable minterm* is a normal product term with n literals. There are 2^n such terms. That is, each variable appears exactly once in the term, in either complemented or uncomplemented form. It is 1 in only one row of the truth table.
- An *n-variable maxterm* is a normal sum term with n literals. Once again, there are 2^n such terms. It is 0 in only one row of the truth table.

- A *canonical sum* of a function is the sum of the minterms corresponding to the rows where the truth-table of the function is 1.

	X	Y	Z	F	minterm	maxterm
0	0	0	0	$F(0,0,0)$	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	$F(0,0,1)$	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	$F(0,1,0)$	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	$F(0,1,1)$	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	$F(1,0,0)$	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	$F(1,0,1)$	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	$F(1,1,0)$	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	$F(1,1,1)$	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

This has another notation, in which each row that is true is added to the sum:

$$F = (1,0,0,1,1,1,0,1) = \sum_{X,Y,Z} (0,3,4,5,7)$$

- A *canonical product* of a function is the product of the maxterms corresponding to the truth table rows where the function is 0.

	X	Y	Z	F	minterm	maxterm
0	0	0	0	$F(0,0,0)$	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	$F(0,0,1)$	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	$F(0,1,0)$	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	$F(0,1,1)$	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	$F(1,0,0)$	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	$F(1,0,1)$	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	$F(1,1,0)$	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	$F(1,1,1)$	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

This also has another notation:

$$F = (1,0,0,1,1,1,0,1) = \prod_{X,Y,Z} (1,2,6)$$

Note: the union of the terms of the canonical product and the canonical sum, is the full truth table, or the numbers from $0 \rightarrow 2^n - 1$

A function can be written in Verilog by doing the following:

```
case {(X,Y,Z)}
  0,3,4,5,7:F=1;
  default:F=0;
endcase
```

Or it can be written like this:

```
case {(X,Y,Z)}
  1,2,6:F=;
  default:F=1;
endcase
```

Now that we have all the necessary definitions, we will go back to Shannon's Expansion Theorem.

$$F(X_1, X_2, \dots, X_n) = X_1 \wedge F(1, X_2, \dots, X_n) \vee X_1' \wedge F(0, X_2, \dots, X_n)$$

This theorem allows us to extract a variable from a function and divide it into two halves, in which the term is true or false, while the remains have a smaller minterm, maxterm truth table.