Notes for ECE 30200 - Probabilistic Methods in Electrical and Computer Engineering

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These are lecture notes for Fall 2025 ECE 30200 by professor Mary Comer at Purdue. Modify, use, and distribute as you please.

Contents

Modelling Random Experiments

Examples:

- Flip a coin
- Rolling dice
- Generate a bit from a random binary source
- Generate a sequence of n bits from a random binary source
- Count packets arriving at a router
- Measure the voltage at a point in a circuit

All of these examples could be at a fixed time, or over an interval of time.

There is a very precise framework that can be used to model any random experiment.
Overview:

- The outcome that occurs each time a random experiment is run is not known in advance, but **the set of all possible outcomes is assumed to be known**.
- Subsets of the set of all possible outcomes are called events.
- Probabilities are assigned to events, not outcomes, using a probability measure.

We need to use set theory to work with these sets.

Set Theory

A set is an **unordered** collection of elements denoted by $\{\}$.
Example:

$$\{1, 2, 3\} = \{3, 2, 1\} = \{2, 1, 3, 1\}$$

Notation:

- $w \in A$ means w is in set A
- $w \notin A$ means w is not in set A

There are two ways to specify a set:

1. Comma-separated list:
    - $A = \{1, 3, A, C\}$

- $A = \{x_1, x_2, \ldots, x_n\}$ (that is, n elements for some finite $n \geq 1$ and the ith element is x_i)
- $A = \{x_1, x_2, \ldots\}$ (an infinite number of elements)

2. Rule for membership or set builder notation:
   - $A = \{w \in \mathbb{Z} : 1 \leq w \leq 6\}$
   - Special notation for intervals $\in \mathbb{R}$:
     - (a, b) represents $\{x \in \mathbb{R} : a < x < b\}$
     - $[a, b]$ represents $\{x \in \mathbb{R} : a \leq x \leq b\}$

     Note: $(a, b) \subset \mathbb{R}$ is a set, but $(a, b) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is also a point on the Cartesian plane or an ordered pair.

Equal sets: sets A and B are equal if they contain exactly the same elements, and is denoted by $A = B$.

A $\subset$ B: $(w \in A \implies w \in B) \implies (A \subset B)$ for sets A and B where $A \subset B$ means A is a subset of B.

Note: we will not distinguish between proper subsets:

$$(A \subset B)\&(A \neq B)$$

and subsets.

$$A = B \iff (A \subset B)\&(B \subset A)$$

Special Sets

The set with no elements is called the empty set or the null set. Denoted by $\emptyset$ or $\{\}$ which is not the same as $\{\emptyset\}$.

The set containing all possible elements of interest is called the universal set S. It is known as the sample space within probability, and will contain all possible outcomes of a random experiment.

Set Operations

- Intersect ($\cap$): $A \cap B = \{w \in S : (w \in A)\&(w \in B)\}$
- Union ($\cup$): $A \cap B = \{w \in S : (w \in A)|(w \in B)\}$
- Complement ($\square^c$, $\square'$, $\bar{\square}$): $A^c = \{w \in S : w \notin A\}$

- Disjoint: $A \cap B = \emptyset$
- *Difference*: $A - B = A \cap B^c$

Set Algebra

Set algebra provides us with a second method to prove that two sets A, B are equal.
It uses the fact that both union and intersection are commutative and associative to prove the statement.
The following are a few properties that come from commutativity and associativity:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$

$\cup$ is distributive over $\cap$ and vice versa:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Note: when translating between Set Theory and English

- $\cup$ = OR
- $\cap$ = AND

Venn Diagram

A Venn diagram is a graphical representation of a universal set and its subsets. For example:

Note: A Venn diagram does not represent arbitrary sets, and thus, is not a proof by itself.

Set Types

There are three kinds of sets:

1. Finite: a set is finite if it has a finite number of elements.
2. Countably Infinite: a set is countably infinite if it can be placed into one-to-one correspondance with the integers. Can be written as

$$A = \{x_1, x_2, \dots\} \text{ where } x_i \neq x_j \forall i \neq j$$

3. Uncountable: set that is neither countable nor finite.

> *Note*: For the context of this class, the uncountably infinite sets will only be $\mathbb{R}$ or intervals of the same.

They cannot be written as $\{x_1, x_2, \dots\}$

Collections of Sets

- Finite Collections:

$$A_1, A_2, \dots, A_n$$

  where $A_i \subset S$

- Countably Infinite Collection:

$$A_1, A_2, \dots$$

  where $A_i \subset S$

- Uncountable Collection: out of bounds for this class!

> **Union of collection**: $\bigcup_i^{\infty,n} A_i = \{w \in S \colon \exists i | w \in A_i\}$

> **Intersection of collection**: $\bigcap_i^{\infty,n} A_i = \{w \in S \colon w \in A_i \forall i\}$

Modelling Random Experiments Pt.2

We use a probability space to model (almost) any random experiment. Three parts make up a probability space:

1. Sample Space (S): nonempty set of elements called outcomes. Each time an experiment is run, exactly one outcome occurs.
2.