Notes for PHYS 27200 - Electric And Magnetic Interactions

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Contents

Course Introduction

This is a calculus-based physics course using concepts of electric and magnetic fields and an atomic description of matter to describe polarization, fields produced by charge distributions, potential, electrical circuits, magnetic forces, induction, and related topics, leading to Maxwell's equations and electromagnetic radiation and an introduction to waves and interference. 3-D graphical simulations and numerical problem solving by computer are employed throughout. For more information, consult the syllabus.

Equations

1. Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$
2. Electric field due to a point or charged sphere: $\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$
3. Force due to electric field: $\vec{F_2} = E_1 q_2$
4. Dipole moment between charges $-q$ and q separated by $\vec{s}$: $\vec{p} = q\vec{s}$
5. Electric field on dipole axis: $\frac{1}{4\pi\epsilon_0}\frac{-2sq}{r^3}\hat{p}$
6. Electric field on dipole bisecting plane: $\frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$
7. Electric field from point charge-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}$
8. Electric field from dipole-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{12\alpha p_1^2}{r^7}$
9. Drift speed: $\bar{v} = uE$
10. Electric field of a uniformly charged thin rod at a distance r from the midpoint, perpindicular to the rod: $\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r\sqrt{r^2+(L/2)^2}}\right]$
11. Electric field due to a charged ring, along the axis of the ring: $\frac{1}{4\pi\epsilon_0}\frac{xQ}{\left(x^2+r_1^2\right)^{\frac{3}{2}}}$
12. Electric field due to a charged disk, along the axis of the disk: $\frac{1}{2\epsilon_0}\left(\frac{Q}{\pi r_1^2}\right)x\left(\frac{1}{x}-\frac{1}{\sqrt{r_1^2+x^2}}\right)$
13. Electric field due to a uniformly charged insulating ball within the ball: $\frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}$
14. Electric potential in a uniform electric field: $\Delta V = -E\Delta x$
15. Work and potential: $W = q\Delta V$
16. Electric potential in a nonuniform electric field: $\Delta V = -\int_i^f \vec{E}\cdot\vec{l}$
17. Electric potential of a point charge, at a distance r: $\Delta V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$
18. Electric field within insulator: $E_{net} = \frac{E_{applied}}{K}$
19. Potential difference in capacitor with insulator: $\Delta V_{ins} = \frac{\Delta V_{vac}}{K}$
20. Charge on capacitor: $Q = \frac{\epsilon_0 AKV}{d}$
21. Energy density of electric field in capacitor: $\frac{U}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2$
22. Magnetic field at a distance $\vec{r}$ due to a charge q moving at velocity $\vec{v}$: $\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$

23. Magnetic force $\vec{F}$ on charge q moving at velocity $\vec{v}$ in magnetic field $\vec{B}$: $\vec{F} = q\vec{v} \times \vec{B}$

24. Magnitude of magentic force F on charge q moving at speed v in magnetic field of strength B with angle θ between magentic field vector and velocity: $F = qvB\sin(\theta)$

25. Electron current: $i = nA\bar{v}$

26. Magnetic field due to a loop: $\frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{(z+2+R^2)^{3/2}}$

Electric charge

Electric Charge: Electric charge is an intrinsic characteristic of the fundamental particles that make up objects.

Conservation of charge: The net charge of a *closed system* never changes

It was mentioned in class that mass is likewise intrinsic. However, when discussing the Higgs field's role in giving particles mass, the distinction between mass as an intrinsic or emergent property becomes more nuanced. The mass of elementary particles like electrons and quarks is emergent in the sense that it arises from their interaction with the Higgs field, which itself is a fundamental aspect of the universe.

Objects can have negative, zero, or postive charge. Charges are always multiples of the *elementary charge* $e = 1.60217662(63) \times 10^{-19}C$

Coulomb (C): One coulomb is the amount of charge that is transferred through the cross section of a wire in 1 second when there is a current of 1 ampere in the wire.

The charges of elementary particles are listed below.

Particle	Charge (elementary charge, e)
Electron (e^-)	-1
Positron (e^+)	$+1$
Proton (p^+)	$+1$
Anti-proton (p^-)	-1
Neutron	0
Anti-neutron	0
Photon	0

Point Charge: A charged object whose radius is much smaller than the distance between itself and all other objects of interest.

The magnitude of electric force between two point charges is d irectly proportional to the magnitude of each charge and inversely proportional to the distance squared. Specifically:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

This is Coulomb's Law. Note the direction of the force changes with the sign of the charges involved. Like repels like and opposites attract.

Electric field

Consider a charged particle. We can represent its effect by drawing vectors that show the path a positively charged particle would follow if placed within its influence. These lines represent the electric field of the charged particle. The greater the density of the lines, the greater the strength of the electric field. Note that at the origin, the force is undefined (infinite), since $|r| = 0$.

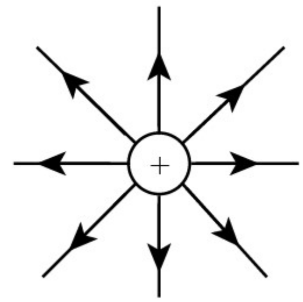


Figure 1: An Electric field coming from point charge. Notice how the densities of the lines vary with distance from the source.

There are many types of fields, which can be either scalar or vector. For example, a temperature map is a scalar field, since each point is associated with a scalar (the temperature at that point). A map of fluid velocity is a vector field, since each point is associated with a vector (the velocity of the fluid at that point). For a particle, the Electric field vector at any point is given by

Technically, fields can be the more general tensor, or even the fascinating and exotic spinor!

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

The direction in which the field lines point depends on the sign of the charge. If the charge is negative, the field lines point in. If it is positive, the field lines point out. A useful mnemonic is to think of the charge as someone's STD test results. If it's negative, others will go for them and the lines point in. If it's positive, everyone will try to get away and the lines point out.

Consider the relative strengths of the electric and gravitational fields. The gravitational force is given by $F_g = G\frac{m_1 m_2}{r^2}\hat{r}$, with $m_{electron} = 9 \times 10^{-31} kg$ and $m_{proton} = 1.7 \times 10^{-27} kg$. If we consider a hydrogen atom, then $r = 5.3 \times 10^{-11} m$. With $G = 6.7 \times 10^{-11}$, we have

$$F_g = \frac{(1.7 \times 10^{-27})(9 \times 10^{-31})(6.7 \times 10^{-11})}{(5.3 \times 10^{-11})^2} \approx O(10^{-46})N$$

Now, the Electric force is given by $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. The charge of a proton and Electric are $q_1 \approx q_2 \approx 1.6 \times 10^{-19} C$. Ergo, since $\frac{1}{4\pi\epsilon_0} \approx 8.99 \times 10^9 \frac{Nm^2}{C^2}$,

$$F_e = \frac{(8.99 \times 10^9 \frac{Nm^2}{C^2})(1.60 \times 10^{-19} C)^2}{(5.3 \times 10^{-11} m)^2} \approx O(10^-17)N$$

This means that $\frac{F_e}{F_g} \approx 2.27 \times 10^{39}$, meaning the Electric force is much stronger for these masses and charges than gravity. On scales as large as humans and planets, gravity is the dominant force because gravity is strictly additive.

For sufficient distances, the Electric field of a uniformly charged spherical shell resembles the Electric field of a point charge.

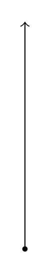


Figure 2: Notice how a circle resembles a point from a great distance.

That means for $r >> R$, $\vec{E_{sphere}} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$ This holds only for outside the sphere. Inside, it can be shown that the electric field is zero.

Superposition Principle: The net electric field at a location in space is a vector sum of the individual electric fields contributed by all charged particles located elsewhere.

To introduce systems with multiple sources of electric field lines, consider the particle pair known as a dipole. Dipoles consist of one negatively charged and one positively charged particle, like so:

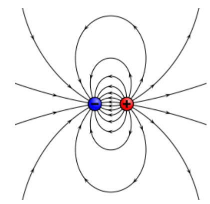


Figure 3: Two oppositely charged particles distanced from one another

Dipole Moment: The dipole moment is a way of expressing asymmetrical charge distribution. It is a vector quantity, i.e. it has magnitude as well as definite directions. The dipole moment is given by the expression $\vec{p} = q\vec{d}$, where q is the charge on one end of the dipole and $\vec{d}$ is the distance between dipoles.

On the axis of the dipole (i.e. the lines formed by the two particles), the electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2sq}{r^3}\hat{p}$$

where r is the distance from the point in consideration to the center of the dipole. On the bisecting plane (i.e., the plane exactly halfway from each point) the field is given by

$$\vec{E} = \frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$$

Where r is the distance from the point to either dipole. The force on a positive point charge q_1 a distance of d away from the dipole, aligned with the dipole, and on the side of the negative charge is given by

$$\vec{F} = q_1 \vec{E_{dipole}} = q_1(\frac{-1}{4\pi\epsilon_0}\frac{2qs}{d^3}, 0, 0)$$

Note that the field will be parallel to the axis of the dipole: this can simplfy vector calculations. Now consider a dipole in a uniform electric field, like so: The positive end will be pulled to the right

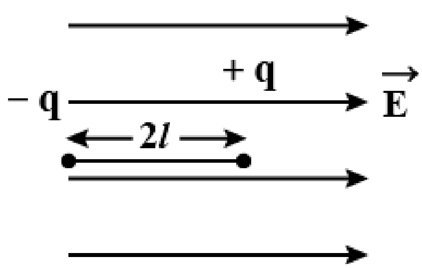


Figure 4: Dipole in uniform electric field

and the negative end to the left, exerting a torque given by $\vec{\tau} = \vec{p} \times \vec{E_{uniform}}$. Note that by the definition of $\times$, $\tau = pE_{uniform}\sin\theta$, where θ is the dipole's angle from horizontal. It can be shown that the potential energy of a dipole in a uniform Electric field is $U = -\vec{p} \cdot \vec{E_{uniform}}$ and similarly as before (but now with $\cdot$) $U = -pE_{uniform}\cos\theta$. Usefully, this means dipoles can be used to measure the direction of an electric field.

Throughout these examples, we have been assuming the associated speeds are much less than the speed of light. If the velocities approach a significant fraction of the speed of light Coulomb's law no longer holds, and we must account for relativity.

Conservation of Charge: Charge cannot be created nor destroyed, with the exception of electron-positron annihilation and other such quantum hijinks. We can use conservation of charge to predict the behavior of many systems. For example, consider tape pulled from a roll. You may have noticed when dangling strips of freshly-pulled tape they tend to drift towards nearby surfaces to stick and become tangled: this is because peeling a strip of tape off a roll strips electrons from the tape, resulting in a net positive charge because of conservation of charge. When this charge approaches a net neutral object, such as your hand, the electrons in the atoms of your hand are attracted to the positively charged tape and congregate closer to the tape. This results in a negative charge buildup near the tape. The positive tape is attracted to the negative charges, and so the tape moves towards your hand and becomes tangled and useless. The process of one charge inducing a charge on a neutral object occurs often enough for the phenomenon to be named. We call it **Polarization**: The process by which a dipole is formed in a neutral object by an electric field. The dipole moment is given by $\vec{p} = \alpha\vec{E}$, where α is a material-dependent constant called polarizability. Such a dipole is *induced*. Consider a point charge near a neutral atom. The point charge creates an electric field given by

Interestingly, in annihilation between positrons and electrons (or any other subatomic particles) the total energy and momentum of the initial pair are conserved in the process and distributed among a set of other particles in the final state (photons in the example of the electron-positron pair). Antiparticles have exactly opposite additive quantum numbers from particles, so the sums of all quantum numbers of such an original pair are zero. Hence any set of particles may be produced whose total quantum numbers are also zero as long as conservation of energy, conservation of momentum, and conservation of spin are obeyed.

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

Inducing a dipole given by the expression

$$\vec{p} = \alpha\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{\alpha q_1}{r^2}\hat{r}$$

This dipole creates a field at the point charge of

$$\begin{aligned}\vec{E}_2 &= \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\alpha\vec{E}_1}{r^3}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\alpha}{r^3}\left(\frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}\end{aligned}$$

This formula is valid provided the electric field that induced the dipole is from a point charge. If the electric field is instead from, say, another (permanent) dipole then $\vec{p} = \alpha\vec{E}_1$ is still valid. However, in this case the formula for $\vec{E_1}$ will be given by the equation for the electric field along the axis of a dipole instead. Following this logic,

$$\begin{aligned}\vec{p} &= \alpha\vec{E_1}\\ &= \alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p_1}}{r^3}\end{aligned}$$

If we let $\vec{r'}$ be the location of the permanent dipole from the induced dipole, we have the electric field at the permanent dipole as

$$\begin{aligned}\vec{E}(\vec{r'}) &= \left(\frac{1}{4\pi\epsilon_0}\frac{2}{r^3}\right)\left(\alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p_1}}{r'^3}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3r'^3}\end{aligned}$$

Calculating the force on the permanent dipole is slightly more complicated than multiplying by the charge of the dipole, since r' and the sign of q varies based on which end of the dipole we consider. To find the net force, we must find the force on each charge and sum them, like so:

$$\begin{aligned}F^+ &= q\vec{E}(r-\frac{s}{2})\\ F^- &= q\vec{E}(r+\frac{s}{2})\\ F_{net} &= F^+ + F^-\\ &= q\left[\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3(r+\frac{s}{2})^3} - \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p_1}}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left[\frac{4\alpha\vec{p_1}}{r^3(r+\frac{s}{2})^3} - \frac{4\alpha\vec{p_1}}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2(4\alpha\vec{p_1})\left[\frac{1}{r^3(r+\frac{s}{2})^3} - \frac{1}{r^3(r-\frac{s}{2})^3}\right]\end{aligned}$$

With a bit more algebraic simplification and the assumption that

$r >> s$, we can show that

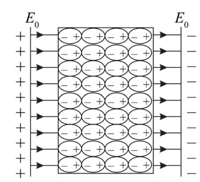
$$\begin{aligned}
F_{net} &= F^{+} + F^{-} \\
&= q\left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{4\alpha\vec{p}_1}{r^6}\right)\left[(1+\frac{3s}{2r}) - (1-\frac{3s}{2r})\right] \\
&= q\left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{4\alpha\vec{p}_1}{r^6}\right)\left[\frac{3s}{r}\right] \\
&\approx q\left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{12\alpha\vec{p}_1 s}{r^7}\right) \\
&= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{12\alpha p_1^2}{r^7}
\end{aligned}$$

Insulators, conductors, and Van der Waals forces

On the topic of induced dipoles, consider how freely moving atoms in a substance interact. The electrons are dispersed in a cloud about the nucleus of each atom. These clouds can be thought of as constantly fluctuating, and occasionally these fluctuations will result in more electrons on one side than another. In this case there will be a small dipole. If this dipole approaches another atom (which may or may not have its own temporary dipole) the two will be attracted. The result of this interaction is that even neutral materials can be attracted to each other due to the fluctuating dipoles of its electron clouds. We call this phenomenon

Van der Waals forces: attraction and repulsions between atoms, molecules, as well as other intermolecular forces. Caused by correlations in the fluctuating polarizations of nearby particles (a consequence of quantum dynamics).

Figure 5: Effect of electric field on insulator



Insulator: An insulator is a material that does not easily allow electricity to pass through it. Inside an insulator the electrons are bound to their atoms, but they may still shift in response to an electric field (see fig. 5) This results in the insulator becoming polarized. Just as with most polarized objects, we can approximate its dipole moment with $\vec{p} = \alpha\vec{E}$ This relationship is only valid if the electrons are bound to the stationary constituent atoms. If this is not the case, then the a material in question is called a

Conductor: A conductor is a material that allows electricity to flow freely through it. For example, consider a liquid with charged atoms floating throughout. Here, when an electric field is applied, the charges each follow the electric field lines until they reach the edges of whatever container holds them. More commonly, we see conductors in the form of metals. The atomic structure of a metal allows electrons to move freely from atom to atom. Thus, within a metal, the electrons can freely move in whichever direction the electric field determines. "Freely" is used liberally here, since the electrons can collide with other electrons or defects in the metal and lose energy. The mobile electrons in the conductor will have a momentum and velocity given

Such a liquid is called an *ionic solution*

Typically electrons are bound to the metal as a whole. However, if the electric field is strong enough, then air surrounding a metal can become ionized and allow electrons to flow freely through it, creating a spark.

by

$$\begin{aligned}\vec{\Delta p} &= \vec{F_{net}} \Delta t \\ &= q\vec{E_{net}} \\ &= -e\vec{E_{net}} \\ &\rightarrow \vec{\Delta p} = -e\vec{E_{net}} \Delta t \\ &= m_e v \\ &\rightarrow v = \frac{e\vec{E_{net}} \Delta t}{m_e}\end{aligned}$$

A good approximation for the average velocity of an electron is $\bar{v} = \mu E$. The constant μ is called the "mobility".

Now, consider a conductor with a net charge, such as a charged sphere. Inside the sphere like charges will repel each another and push to get as far away from one another as possible. This occurs when all the charges are on the surface of the conducting object, since anywhere else would be closer together and they cannot go outside the bounds of the object. This rearrangement has two effects. First, any charge in a conductor will be found on the surface. Second, the net electric field inside a conductor will be zero. An interesting result of electrons seeking to minimize repulsion inside a conductor is the *sharp point effect*. To visualize this effect, imagine a gymnasium full of students pretending to be electrons, staying as far away from others as possible. Anyone near the center of the crowd will feel badly pressed and will try to work there way towards the edge of the gym, where at least one side will no longer have fellow students milling about. The result? Most of the students will gravitate towards the edge of the gym and hover there, to take advantage of that lack of other students on the wall side of the gym. Now imagine a narrow corridor leading out of the gym. Even better! Students in that corridor will only have fellow students behind and in front of them. Now imagine the very end of that corridor, a sort of point. Even better! Now, the student who finds that spot will benefit from having only one student nearby. But somewhat ironically, that same effect will cause other students to pack themselves into the long, narrow corridor more tightly, since pretty much anywhere in the corridor makes them less exposed to the full set of students than being in the gym does. This effect makes edges, wires, and points more attractive to electrons, which similarly just don't want other electrons nearby.

To see why this must be the case, imagine if it were not. Then any charge within the conductor would be moved by the electric field, so we see that the case where no electric field is present is the only stable possibility. However, if electrons are moving, then there can be (and is) an electric field within the conductor. It is only when the conductor is in static equilibrium that there is no net electric field within.

Finding electric field

Now that we understand conceptually how charge behaves in a conductor, let's think about the electric field that charge creates. We can visualize any charged object as a collection of point charges in the shape of the object. If we'd like to find the net electric field of these charges, we simply use Coulomb's law and sum up each of their electric fields. To illustrate this approach, imagine a charged rod with length L and charge Q. We can approximate it as a bunch of charges in a row. For now, let's use ten, but recognize that the more charges we use the better our estimate will be. Each piece of the rod will have a charge of $\frac{Q}{10}$, so to find the electric field at a point we would need to calculate the vector between each piece and the point and use $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. We would do this ten times and then sum the electric fields to get our approximate net field. Of course, perhaps we would like to find the exact electric field. To do this we would have to cut our object up into an infinite number of points, find the infinitesimal electric field due to each, and sum them up. Sound familiar?

To drive the point home, consider a vertical rod with a uniform charge of Q and length L. Let's take a point on the bisecting plane of this rod (the horizontal plane that cuts the rod in half). Say we view this point from the side and see that is has coordinates $(0, x)$. If we want to find the electric field due to a little bit of the rod at $\vec{x}$, we need to know the distance between them. If the bit is a distance of y up the rod, the distance between it and $\vec{x}$ will be $r = \sqrt{x^2 + y^2}$. The charge of this small piece will be $\frac{Q}{L}\Delta y$ (where Δy is the height of the peice), and $\hat{r}$ will be $\frac{(x,-y)}{\sqrt{x^2+y^2}}$. Therefore the electric field at $\vec{x}$ will be given by

$$\begin{aligned}
\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2 + y^2}\frac{(x, -y)}{\sqrt{x^2 + y^2}}
\end{aligned}$$

We can see by symmetry (isn't symmetry lovely?) that the contributions in the y direction will cancel, so we need only consider the x direction. Therefore,

$$\begin{aligned}
\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2 + y^2}\frac{x}{\sqrt{x^2 + y^2}} \\
&= \frac{Q}{4\pi\epsilon_0 L}\frac{x\Delta y}{(x^2 + y^2)^{3/2}}
\end{aligned}$$

Now comes the tricky part: adding these up. You have already likely guessed that we will need to integrate between the bottom and top of the rod, which corresponds to the integral from $-\frac{L}{2}$ to $\frac{L}{2}$. We have then

$$\vec{E} = \int_{-L/2}^{L/2} \frac{Q}{4\pi\epsilon_0 L} \frac{x}{(x^2+y^2)^{3/2}} dy$$

I'll spare you the tedious integration and skip to the result:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{x\sqrt{x^2+(L/2)^2}} \right] \hat{x}$$

The steps for finding the electric field due to a charged object in general are outlined below.

1. Cut up the charge distribution into pieces and draw $\Delta\vec{E}$

   - Divide the charge distribution into pieces whose field is known. In particular, very small pieces can be approximated by point particles. You may also wish to break up a complex object into smaller objects whose electric field equations are already known.
   - Pick a representative piece, and at the location of interest draw a vector $\Delta\vec{E}$ showing the contribution to the electric field of this representative piece. Drawing this vector helps you figure out the direction of the net field at the location of interest.

2. Write an expression for the electric field due to one piece

   - Pick an origin for your coordinate system, and show it on your diagram. Draw the vector $\vec{r}$ from the source piece to the observation location. Write algebraic expressions for $\vec{r}$ and $\hat{r}$.
   - Write an algebraic expression for the magnitude $|\Delta\vec{E}|$ contributed by the representative piece. Multiply by $\hat{r}$ to get the vector $\Delta\vec{E}$. You can break this up into x, y, and z components for integration. Once you do each expression should contain one or more "integration variables" ($\Delta x/\Delta y/\Delta z$ or $dx/dy/dz$) related to the coordinates of the piece. Write the amount of charge on the piece, Δq, in terms of your variables.

3. Sum the contributions of all the pieces

   - The net field is the sum of the contributions of all the pieces. To write the sum as a definite integral, you must include limits given by the range of the integration variable. If the integral can be done symbolically, do it. If not, choose a finite number of pieces and do the sum with a calculator or a computer.

4. Check the result

- Check that the direction of the net field is qualitatively correct.
- Check the units of your result, which should be newtons per coulomb.
- Look at special cases. For example, if the net charge is nonzero, your result should reduce to the field of a point charge when you are very far away. For a numerical integration on a computer, check that the computation gives the correct numerical result for special cases that can be calculated by hand.

Let's apply these steps to some new problems. First, let's consider the electric field produced by a charged ring (fig. 6). We have a ring of

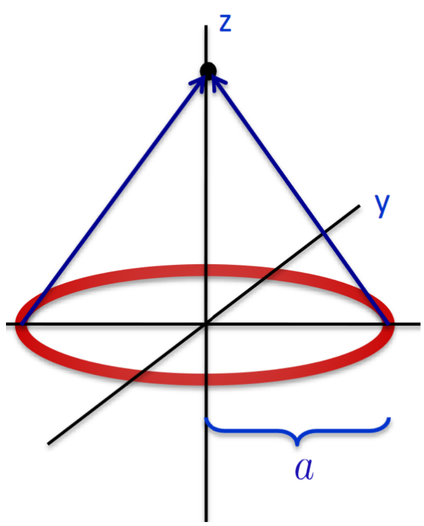


Figure 6: Charged ring

radius a, with a charge of Q. The ring is centered on the xy plane and we are calculating the electric field on the z-axis.

1. Cut the charge distribution and find $\Delta\vec{E}$. Let's take a slice of this

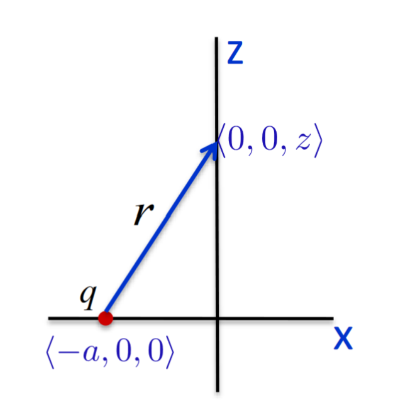


Figure 7: Charged ring slice

ring, like fig. 7. We can view the slice of the annulus here as a point charge a distance of $\sqrt{a^2+z^2}$ away from $(0,0,a)$. We find then that

$$\begin{aligned}\vec{r} &= (a,0,z)\\ \hat{r} &= \frac{(a,0,z)}{\sqrt{a^2+z^2}}\end{aligned}$$

2. By symmetry, only the z components of each individual electric field contribute. Therefore,

$$\vec{E} = \sum \frac{1}{4\pi\epsilon_0}\frac{z\Delta q}{(a^2+z^2)^{3/2}}$$

3. Since the charge is uniformly distributed, we have that $\Delta q = \frac{Q\Delta\theta}{2\pi}$. Therefore our integral becomes

$$\begin{aligned}\vec{E} &= \int_0^{2\pi}\frac{1}{4\pi\epsilon_0}\frac{z\frac{Q}{2\pi}}{(a^2+z^2)^{3/2}}d\theta\\ &= \frac{Q}{4\pi^2\epsilon_0}\frac{z}{(a^2+z^2)^{3/2}}\int_0^{2\pi}d\theta\\ &= \frac{1}{4\pi\epsilon_0}\frac{Qz}{(a^2+z^2)^{3/2}}\hat{z}\end{aligned}$$

4. This looks pretty good. It goes to zero as z goes to either zero or infinity, as it should.

Let's consider the electric field on the z-axis of a disk with radius r and charge Q now.

1. Take some point on the disk. The calculation for the electric field due to this point is identical to the case of the ring. Therefore

$$\begin{aligned}\vec{r} &= (a,0,z)\\ \hat{r} &= \frac{(r,0,z)}{\sqrt{a^2+z^2}}\end{aligned}$$

2. Now we have that $\Delta q = \frac{Q}{\pi r^2}dad\theta$. Therefore

$$\begin{aligned}\vec{E} &= \sum \frac{1}{4\pi\epsilon_0}\frac{z\Delta q}{(a^2+z^2)^{3/2}}\\ &= \sum \frac{1}{4\pi\epsilon_0}\frac{Q}{\pi r^2}\frac{zdad\theta}{(a^2+z^2)^{3/2}}\end{aligned}$$

3.

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{Qz}{\pi r^2}\int\int\frac{a}{(a^2+z^2)^{3/2}}dad\theta\\ &= \frac{1}{4\pi\epsilon_0}\frac{Qz}{\pi r^2}\int_0^{2\pi}\int_0^r\frac{a}{(a^2+z^2)^{3/2}}dad\theta\\ &= \frac{1}{2\epsilon_0}\frac{Qz}{\pi r^2}\left[\frac{1}{z}-\frac{1}{\sqrt{r^2+z^2}}\right]\hat{z}\end{aligned}$$

Say we have two conducting disks very near to one another, with opposite charges (this configuration is known as a *capacitor*). Our equation for the electric field due to a uniformly charged disk does not hold for a conductor, since the charges are free to locomote. The charges will be repelled to the edges of the disk and attracted to the negative charges in the other disk. This attraction means that the charge on the disks will be spread nearly uniformly on the inner surfaces of the disks and the field between the plates may be approximated as the field between two infinite plates, but what of the outside? Some charge will still be on the outer surface of these disks, and his charge will create an electric field outside of the capacitor. Not a strong field in comparison to the inner field, but still nonzero. To calculate it we can use the equation for electric field due to a uniformly charged disk,

$$\begin{aligned} \vec{E_{disk}} &= \frac{1}{2\epsilon_0}\left(\frac{Q}{\pi r_1^2}\right)x\left(\frac{1}{x} - \frac{1}{\sqrt{r_1^2 + x^2}}\right) \\ &\approx \frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1 - \frac{x}{2R}\right] \end{aligned}$$

The net field will be the sum of the fields from each disk. Ergo,

$$\begin{aligned} \vec{E} &= \vec{E}_1 + \vec{E}_2 \\ &\approx \frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1 - \frac{x}{R}\right] + \frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1 - \frac{s - x}{R}\right] \\ &\approx -\frac{1}{2\epsilon_0}\frac{Qs}{\pi R^3} \end{aligned}$$

With that brief divagation out of the way, let us return our attention to charged spheres. Recall that the electric field outside of a charged sphere is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

provided $r > R$. If we have $r < R$, then we know there is no net electric field within the sphere. Imagine placing a proton inside a charged sphere, and now consider what the field inside will be. The answer depends on if the sphere is made of a conducting materal or not. If charges are free to move in the sphere, then they will migrate and exactly cancel out the electric field of the proton, since the electric field within a conductor is zero. If the sphere is instead made of an insulator, then the electric field inside due to the sphere will still be zero, but the proton will now contribute an Electric field of its own according to Coulomb's law. What if we have another, concentric insulating charged sphere within the first? Well, if we are within both, then the Electric field will still be zero due to the superposition

For a very loose explanation of why there is no electric field within a sphere, consider the field due to each bit of the sphere on a point inside. At any point you have some charges nearby and some far away, and if that were all to there story then the Electric field would point towards the center of the shell. However, you have more charges far away than close. It works out that the effect of this greater number of charges exactly cancels out the farther distance and there is no net field within the shell! This is an interesting proof if you care to work it out.

principle. If we are outside the bounds of the first but still within the second, then the inner sphere will have an electric field given by Coulomb's law, while the outer shell still contributes no net field. If we are outside both then it appears to use that there is a point with a charge that is the sum of the charges of each sphere, and the electric field will again be given by Coulomb's law. Well and good, but what if we have a insulating ball, a filled sphere? What is the electric field then? We know that it will be given by Coulomb's law outside of the ball, but on the inside it isn't zero anymore. A=If you picture the ball as a series of concentric shells then the shells radially past the inner point will contribute nothing, but you will still have a small sphere of charge radially inward with a net field. Let's find the field contributed by this sphere. First, cut up the charge distribution and find $\vec{E}$. The charge per volume will be

$$\frac{Q}{V} = \frac{Q}{4/3\pi R^3}$$

For a point a distance of r within the ball, we have

$$\frac{\Delta q}{4/3\pi r^3}$$

We know that the sphere is uniformly charged. That is, the charge density everywhere is the same. Thus,

$$\frac{\Delta q}{4/3\pi r^3} = \frac{Q}{4/3\pi R^3}$$
$$\rightarrow \Delta q = Q\frac{r^3}{R^3}$$

So we now have that

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\frac{r^3}{R^3}}{r^2}$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}$$

And we are done.

Electric potential

To begin this section, let's imagine what happens when we touch two conductors together. Say one has a charge of 6 nc and the other of 0 nC, but otherwise they are identical. What happens when the conductors are brought in contact? We intuit that the charge flows from one to the other until each has a charge of 3 nC, and this is indeed what would happen. Consider a similar situation, but now the conductor of 6 nC is much smaller than the neutral conductor. What happens when these two are brought in contact now?

As you ponder that, let's briefly review energy. The energy of a particle moved by a force $\vec{F}$ along a path $\vec{l}$ is given by

$$\Delta E_{particle} = W = \int \vec{F} d\vec{l}$$

If the forces involved are conservative, we also have that

$$\Delta U = -W_{internal}$$

Consider a particle between two charged infinite plates. Let's find the change in its potential energy as it moves from point A to point B. If the electric field strength is E N/C, then

$$\begin{aligned}
\Delta U &= -W_{internal} \\
&= -\int qEd\vec{r} \\
&= -q\int dx \\
&= -qE\Delta x \\
&= -qV
\end{aligned}$$

So the difference in potential energy of a particle inside a charged electric field is proportional to the strength of the field and the distance moved. We can classify the ability to have potential energy if a charge enters a system as the

Electric potential: the amount of work energy needed per unit of electric charge to move this charge from a reference point to the specific point in an electric field. Mathematically, the potential ΔV can be expressed as

$$\Delta V = -E\Delta x$$

V may seem like an odd choice of variable here, but electric potential is such an important concept that it gets its own units of Volts (which are equivalent to J/C). Despite the fancy name, we can think of electric potential the same way as gravitational or any other potential. Charges will move from high to low potential, just like a ball rolling down a hill is moving from high to low potential. Here the strength of

The astute reader will notice that this equation is not very general, as it only applies when the charge moves in a straight line through a uniform electric field. In general,

$$\Delta V = -\int_i^f \vec{E} d\vec{l}$$

This integral is a *line integral*. It can be computed using multivariable calculus, but sometimes it simplifies to a form that makes this approach unecessary.

the electric field is analogous to the height of the hill, and the charge is the ball. Just as with a hill and a ball, we also need two things in order to find electric potential energy. In the simplest case, imagine two charges near one another. The force of one on the other is simply

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

and as we have learned,

$$W = \int_a^b \vec{F} d\vec{x}.$$

So to find the work done as one particle goes from A to B, we have

$$\begin{aligned} W &= \int_a^b \vec{F} dr \\ &= \int_a^b \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r} dr \\ &= \frac{q_1 q_2}{4\pi\epsilon_0}\int_a^b \frac{dr}{r} \\ &= \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{-1}{r}\right) \\ &= \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \end{aligned}$$

We can define the *electric potential energy* of a system of charges as the work needed to assemble the system by bringing each charge from infinity to its final position. In this case,

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r},$$

where r is the distance between one charge and another. For a system with multiple particles, the total energy will simply be the sum of the energies of each individual charge with respect to each other.

Let's calculate the electric potential of three point charges A, B, and C. The distances are r_{12}, r_{13}, and r_{23}. The electric potential energy is the sum of the individual potential energies. That is,

$$\begin{aligned} U &= \sum U_i \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} \end{aligned}$$

Now, you have probably noticed that, in general, things love moving from high potential to low potential. Lift something high up in the gravitational field, and it gains potential, let it go and it falls and loses potential. Gifted students and especially electric fields follow the same principle. The electric field points from the highest potential to

What's the potential energy of something floating around by itself in space? The question seems to make no sense, since potential energy is defined as a difference between two states. However, the potential at a single point is often defined as the potential relative to infinity, $V_A = V_A - V_\infty$. The electric potential is a scalar field, and has a value at every point in space.

Electric potential energy is NOT *electric potential*. Electric potential energy has a dependency upon the charge of the object experiencing the electric field, electric potential is purely location dependent. Electric potential is the electric potential energy per charge.

the lowest potential, dictating how a charge will move if placed in a field. We can see this mathematically, since we know

$$\begin{aligned}\Delta V &= -\int_i^f \vec{E} \cdot d\vec{l} \\ &\rightarrow dV = -\vec{E} \cdot d\vec{l}\end{aligned}$$

Since $\vec{l}$ can be multidimensional, to find dV we'd need multivariable calculus. If we calculate the derivative what we would get is

$$\begin{aligned}dV &= -\vec{E} \cdot d\vec{l} \\ &\rightarrow \frac{dV}{d\vec{l}} = -\vec{E} \\ &= V\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \\ &= -\nabla V\end{aligned}$$

If you are familiar with gradients (∇) you will know this means the electric field points in the direction in which the potential decreases most rapidly. We also know the electric field always points in the direction of steepest descent of V and its magnitude is the slope. And just as with gravity, it can be shown that the electric potential between two points does not depend on the integration path. Any path will do. Moreover, any closed path has an electric potential of 0. In general, if you know electric field, you may integrate to find the electric potential between two points. If you know the potential, differentiate.

Take a momement to convince yourself of this by whatever means you find most useful. It can be shown both mathematically and intuitively.

If you have a shape, there are two ways to get the potential.

1. Break the object into point changes and add up the potential from each.
2. If you know the electric field, use $\Delta V = \int_a^b E \cdot dl$

Say we have a ring with radius R and charge Q. Using method 1, the potential due to each charge on the axis is $\Delta V_i = \frac{1}{4\pi\epsilon_0}\frac{\Delta q_i}{r}$. If we are a distance of z along the axis, $r = \sqrt{z^2 + R^2}$, and we have

$$V = \sum \frac{1}{4\pi\epsilon_0} \frac{\Delta q_i}{\sqrt{z^2 + R^2}}$$

$$\begin{aligned}V &= \sum \frac{1}{4\pi\epsilon_0} \frac{\Delta q_i}{\sqrt{z^2 + R^2}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2 + R^2}} \sum \Delta q_i \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{z^2 + R^2}}\end{aligned}$$

Let's try to find the potential again, this time using method 2. We recall that

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}$$

So the potential will be

$$\begin{aligned} V &= -\int_{\infty}^{z}\frac{1}{4\pi\epsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}dz \\ &= -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{z}\frac{z}{(R^2+z^2)^{3/2}}dz \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{z^2+R^2}} \end{aligned}$$

Identical to the expression previously obtained, as it should be.

Allow me to now briefly remind the reader of the relationship between voltage and electric field.

$$\Delta V = -\int_i^f \vec{E}\cdot d\vec{l}$$

$$\vec{E} = -\left[\left(\frac{\partial}{\partial x}V\right)\hat{x} + \left(\frac{\partial}{\partial y}V\right)\hat{y} + \left(\frac{\partial}{\partial y}V\right)\hat{y}\right]$$

If you remember, a conductor inside a capacitor has no net electric field within its bounds. That means the electric field perpindicular to the surface is $\frac{\sigma}{\epsilon_0}$, while parallel to the surface the electric field is 0. That means that $V_f - V_i = 0$ on the surface, since any path on the surface has no electric field. We say that the surface of a conductor is *equipotential*; that is, every point has the same voltage. Of course, this is only true if the conductor is at equilibrium. We have no guarantee of charge distribution is the conductor is not in electrostatic equilibrium. The situation changes if we instead have an insulator between the two plates of a capacitor. The electric field within an insulator is not zero. The nonzero electric field induces dipoles within the molecules of the insulator, which then have an electric field of their own. It turns out that inside an insulator, the electric field is proportional to the applied electric field by a factor of $1/K$, where K is called the dielectric constant and is known for various materials. Let's see if we can calculate the potential between plates with an insulator between them. The problem: a capacitor with a 3mm gap has a potential difference of 6V. A disk of glass 1mm thick, with area the same as the metal plates has a dielectric constant of 2.5. The glass is inserted in the middle of the gap between the plates. What is now the potential difference between the plates? We know that without the insulator between the plates, the electric field will be.

$$\begin{aligned} E &= \frac{\Delta V}{\Delta x} \\ &= \frac{6V}{0.003m} \\ &= 200\frac{V}{m} \end{aligned}$$

Inside the glass,

$$\begin{aligned} E_{net} &= \frac{E}{K_glass} \\ &= \frac{2000}{2.5} \\ &= 800\frac{V}{m} \end{aligned}$$

The total voltage will be the sum of the voltage between the first plate and the glass, the two sides of the glass, and the glass and the second plate.

$$\begin{aligned} \Delta V &= V_1 + V_2 + V_3 \\ &= 2000(0.001) + 800(0.001) + 2000(0.001) \\ &= 4.8V \end{aligned}$$

And we are done.

Since the electric field can perform work, it must have energy associated with it. The energy depends on the electric field. Say you have two plates of charge Q a distance s apart, but instead of holding them there, you let one plate go to see how much energy you can get out of this setup. You know the force on one plate is

$$F = \frac{\sigma Q}{2\epsilon_0}.$$

Additionally, we recall that

$$Q = \int F dr.$$

Combining these two equations and noticing the bounds on the integral will be the distance between the plates, we have

$$\begin{aligned} W &= \int_0^s \frac{\sigma Q}{2\epsilon_0} dr \\ &= \frac{Q\sigma s}{2\epsilon_0} \\ &= \frac{Q^2 s}{2\epsilon_0 A} \\ &= \frac{1}{2}\epsilon_0 E^2 s A & &= -U \end{aligned}$$

We can define the energy density as the potential energy per volume.
In this case,

$$\frac{U}{sA} = \frac{1}{2}E^2\epsilon_0 E^2$$

and for a general electric field,

$$\frac{U}{\text{volume}} = \frac{1}{2}E^2\epsilon_0 E^2$$

Magnetism

Electricity and magentism are fundementally intertwined in many ways. A moving electrically charged particle creates a magentic field. Here is some information about magnetism:

Note that this magentic field is different than the particle's electric field.

- The needle of a compass aligns with the direction of the net magnetic field at its location.
- An electric current is a continous flow of charge.
- The superposition principle is still valid for magentic fields, so to calculate the net magnetic field we can simply sum individual magentic fields.
- The units of magentic field are Teslas (T) or Gausses (G). $1G = 10^{-4}T$.
- Magentic fields are typically symbolized by the letter B.
- The earth has a magentic field approximately $.5G$ strong.

You have likely seen and played with magnets before, and already know some of their properties. You know that similar poles repel one another, and like poles attract. You know magents attract ferromagnetic materials (like iron, cobalt, or nickel). A magnet is any material or object that produces a magnetic field. A type of magnet you are likely familiar with is the bar magnet, which looks like fig. 8. The lines

This was confirmed in 1820 when a Danish physicist, Hans Christian Oersted, discovered if you held a compass near a live wire the needle deflected. Thus Oersted showed that moving electrons can create a magnetic field.

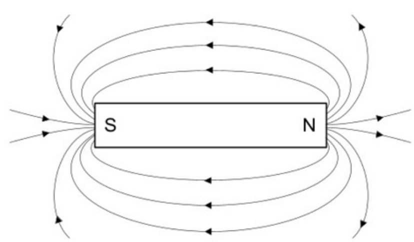


Figure 8: Bar magnet

emanating from the north pole and entering the south are magentic field lines.

I have previously said that a moving eletric charge produces a magnetic field. The equation for the field a distance $\vec{r}$ from a charge q moving at velocity $\vec{v}$ is given by the following law, called the Biot-Savart law:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}$$

Since current is just a bunch of moving charges flowing through a wire, then we should expect a current-carrying wire to have a magnetic field, and it does. The magnetic field looks like fig. 9. It is given

μ_0 is a constant, defined to be $4\pi \times 10^{-7}$. So $\frac{\mu_0}{4\pi} = 10^{-7}$

Recall that the magnitude of the cross product of two vectors A and B at an angle θ from one another is defined as

$$A \times B = |A||B| \sin\theta.$$

The full cross product can be found using the following formula:

$$A \times B = (a_2b_3 - a_3b_2)\hat{\imath} + (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

assuming a_i is the ith component of A and likewise for B.

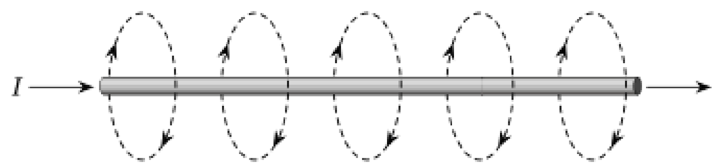


Figure 9: Magentic field of live wire

by the equation

$$B = \frac{\mu_0 I}{2\pi r}$$

If you have calculated the magnitude of the magentic field and wish to find its direction, there is another right-hand-rule to follow, not to be confused with the right-hand-rule that allows one to find the direction of magentic field for a moving charge. Point your thumb along the direction of conventional current in the wire. Your fingers will curl in the direction of the magentic field.

Although we now know that the mobile charges in a wire are electrons, current is conventionally definded as following from flowing from the positive to negative terminal of a battery (that is, positive charges moving).
Despite this, we sometimes want to find the *electron current*, the number of electrons that flow through a given volume per second. Mathematically, the electron current i is given by

$$i = nA\bar{v},$$

where n is $\frac{electrons}{m^3}$, A is cross sectional area of the wire, and $\bar{v}$ is the average velocity of electrons in the wire. Electron current is what's physically happening, but since we're stuck with convention, let's try to think in conventional current. Electron current flowing one way is equivalent to conventional current flowing the other way.