Notes for MA 26500 - Linear Algebra I

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These are lecture notes for spring 2024 MA 26500 at Purdue. Modify, use, and distribute as you please.

Contents

Course Introduction

This course serves as an introduction to the fundamental concepts and applications of linear algebra, a branch of mathematics that explores vector spaces, linear transformations, and systems of linear equations.

Linear Equations in Linear Algebra

A linear equation is an equation of the form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Where a_1, a_2... and b are given constants. Note that the exponents of all x terms is 1. Some examples are:

$$2x_1 + 3x_2 = 4$$

$$5x_1 + 6x_2 = 10$$

as opposed to

$$2x_1x_2 + x_3 = 9$$

$$2x_1 + \sqrt{x_2} = 8$$

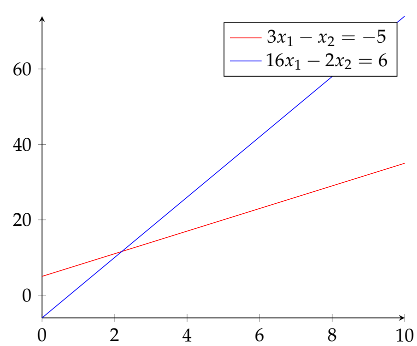
A system of equations is a collection of one or more linear equations involving the same variable set. An example of this would be:

$$\begin{cases} 3x_1 + 5x_2 + x_3 = 3 \\ 7x_1 - 2x_2 + 4x_3 = 4 \\ -6x_1 + 2x_3 = 2 \end{cases}$$

where the constant coefficient of x_2 in the third equation is 0.
A solution is a list of numbers $(s_1, s_2, ..., s_n)$ that makes each equation a true statement when we replace $x_1 = s_1$, $x_2 = s_2$, ... $x_n = s_n$. If we have a system of two linear equations, the solution will be the intersection of the two lines that define the equations on the cartesian plane. The system of equations

$$3x_1 - x_2 = 5$$
$$16x_1 - 2x_2 = 6$$

is mapped to the following graph:



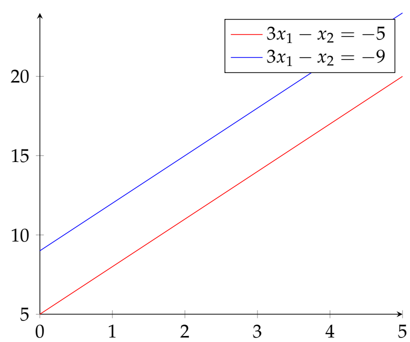
There are three kinds of systems:

1. One solution
2. Infinitely many solutions
3. No solutions

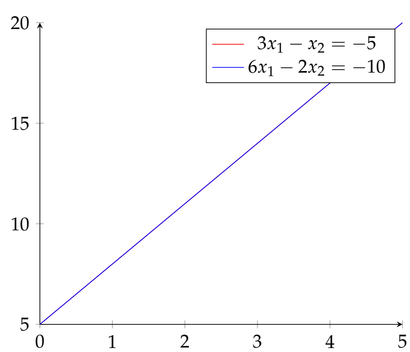
The above example has one solution. This is the intersection of the two lines. This could be further generalized to have far more dimensions as well as far more variables, but such systems are no longer representable in the number of dimensions we can perceive.

However, if the lines were parallel, the other two possibilities surge. If the equations are linearly dependent, there would be infinitely many solutions. However, if the value of b were different for the two, while all the variable coefficients were linearly dependent, it would have none.

The following is an example of no solution:



While this is infinitely many solutions:



Any system of equations has a matrix notation. For example, the above system can be represented as the following.

$$\begin{bmatrix} 3 & -1 & -5 \\ 16 & -2 & 6 \end{bmatrix}$$

Which has the coefficients of the x values as the first n values, and the value of b as the last one. This is called the augmented matrix of the system of equations. As opposed to:

$$\begin{bmatrix} 3 & -1 \\ 16 & -2 \end{bmatrix}$$

which is the coefficient matrix.
These systems can be solved using the three elementary row operations. This is called row reduction.
The three rules are:

1. Interchange: Exchange the positions of any two rows

2. Multiply: Multiply any row by a constant

3. Addition: Replace the value of a row, with its sum and that of one of the other rows multiplied by a scalar.

By using these three rules, one can reduce it to the following form:

$$\begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 1 & c_4 \end{bmatrix}$$

This will result in a trivially solvable one-solution system of equations. This is also known as a consistent system of equations. We can substitute the value of x_3, which is c_4, into the equation in the row above, and solve it, as it willnow be a single equation with a single variable. This can be done sequentially until all values are found.

Note: If the resulting form is instead

$$\begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 0 & c_4 \end{bmatrix}$$

with $c_4 \neq 0$, there would be no solutions. On the other hand, if $c_4 = 0$, there will be infinitely many solutions. Another point to note is that if the augmented matrices of two systems of equations are equivalent, that is, if you can transform one matrix into the other through elementary row operations, the systems of equations will have the same solutions.