Notes for PHYS 27200 - Electric And Magnetic Interactions

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Course Introduction

This is a calculus-based physics course using concepts of electric and magnetic fields and an atomic description of matter to describe polarization, fields produced by charge distributions, potential, electrical circuits, magnetic forces, induction, and related topics, leading to Maxwell's equations and electromagnetic radiation and an introduction to waves and interference. 3-D graphical simulations and numerical problem solving by computer are employed throughout. For more information, consult the syllabus.

Equations

1. Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$
2. Electric field due to a point or charged sphere: $\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$
3. Force due to electric field: $\vec{F_2} = E_1 q_2$
4. Dipole moment between charges $-q$ and q separated by $\vec{s}$: $\vec{p} = q\vec{s}$
5. Electric field on dipole axis: $\frac{1}{4\pi\epsilon_0}\frac{-2sq}{r^3}\hat{p}$
6. Electric field on dipole bisecting plane: $\frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$
7. Electric field from point charge-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}$
8. Electric field from dipole-induced dipole: $\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{12\alpha p_1^2}{r^7}$
9. Electric field of a uniformly charged thin rod at a distance r from the midpoint, perpendicular to the rod: $\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r\sqrt{r^2+(L/2)^2}}\right]$
10. Electric field due to a charged ring, along the axis of the ring: $\frac{1}{4\pi\epsilon_0}\frac{xQ}{\left(x^2+r_1^2\right)^{\frac{3}{2}}}$
11. Electric field due to a charged disk, along the axis of the disk: $\frac{1}{2\epsilon_0}\left(\frac{Q}{\pi r_1^2}\right)x\left(\frac{1}{x}-\frac{1}{\sqrt{r_1^2+x^2}}\right)$
12. Electric field due to a uniformly charged insulating ball within the ball: $\frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}$
13. Electric potential in a uniform electric field: $\Delta V = -E\Delta x$
14. Work and potential: $W = q\Delta V$
15. Electric potential in a nonuniform electric field: $\Delta V = -\int_i^f \vec{E}\cdot\vec{l}$
16. Electric potential of a point charge, at a distance r: $\Delta V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$
17. Electric field within insulator: $E_{net} = \frac{E_{applied}}{K}$
18. Potential difference in capacitor with insulator: $\Delta V_{ins} = \frac{\Delta V_{vac}}{K}$
19. Charge on capacitor: $Q = \frac{\epsilon_0 AKV}{d}$
20. Energy density of electric field in capacitor: $\frac{U}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2$
21. Magnetic field at a distance $\vec{r}$ due to a charge q moving at velocity $\vec{v}$: $\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$
22. Magnetic force $\vec{F}$ on charge q moving at velocity $\vec{v}$ in magnetic field $\vec{B}$: $\vec{F} = q\vec{v}\times\vec{B}$

23. Magnitude of magentic force F on charge q moving at speed v in magnetic field of strength B with angle θ between magentic field vector and velocity: $F = qvB\sin(\theta)$

24. Electron current for n charges per unit volume: $i = nA\bar{v}$

25. Drift speed: $\bar{v} = uE$

26. Electric field within wire: $E = \frac{I}{\mu neA} = \frac{V}{L}$

27. Magnetic field of a straight wire a distance of r along the bisecting plane: $\frac{\mu}{4\pi}\frac{IL}{r\sqrt{r^2+(L/2)^2}}\hat{\theta}$

28. Magnetic field due to an infinite wire: $\frac{\mu_0}{4\pi}\frac{2I}{r}\hat{\theta}$

29. Magnetic field due to a loop: $\frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{(z^2+R^2)^{3/2}}$

30. Magnetic field far from a loop: $\frac{\mu_0}{4\pi}\frac{2\mu}{z^3}$

31. Magnetic field due to a coil of length L with N loops: $\frac{\mu_0 nI}{2}\frac{L}{\sqrt{(L/2)^2+R^2}}$

32. Current through a loop with height h moving out of a magnetic field B at speed v with resistance R: $I = \frac{vBh}{R}$

33. Force on said loop: IhB.

34. emf on loop of area A rotating at ω in magnetic field B: $\omega BA\sin(\omega t)$

35. Approximate magnetic field due to a coil: $\frac{\mu_0 N}{LI}$

36. Cyclotron frequency: $\omega = \frac{|q|B}{m}$

37. Magnetic dipole moment: $\mu = IA$

38. Radius of curvature in magnetic field: $r = \frac{mv}{qB}$

39. Ampere's Law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 NI$

40. Faraday's Law: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}dA$

41. Charge on charging capacitor in an RC circuit: $Q(t) = V_{emf}C(1 - e^{-t/RC})$

42. Charge on discharging capacitor in an RC circuit: $Q(t) = V_{emf}Ce^{-t/RC}$

43. Current through capacitor in an RC circuit: $I(t) = \frac{V_{emf}}{R}e^{-\frac{t}{RC}}$

44. Equivalent resistance for resistors in series: $R_{eq} = R_1 + R_2 + \cdots + R_n$.

45. Equivalent resistance for resistors in parallel: $\frac{1}{R_{eq}} = frac1R_1 + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$
46. Equivalent capacitance for capacitors in series: $\frac{1}{C_{eq}} = frac1C_1 + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$
47. Equivalent capacitance for capacitors in parallel: $C_{eq} = C_1 + C_2 + \cdots + C_n$

Electric charge

Electric Charge: Electric charge is an intrinsic characteristic of the fundamental particles that make up objects.
Conservation of charge: The net charge of a *closed system* never changes

Objects can have negative, zero, or postive charge. Charges are always multiples of the *elementary charge* $e = 1.60217662(63) \times 10^{-19}C$
Coulomb (C): One coulomb is the amount of charge that is transferred through the cross section of a wire in 1 second when there is a current of 1 ampere in the wire.

The charges of elementary particles are listed below.

It was mentioned in class that mass is likewise intrinsic. However, when discussing the Higgs field's role in giving particles mass, the distinction between mass as an intrinsic or emergent property becomes more nuanced. The mass of elementary particles like electrons and quarks is emergent in the sense that it arises from their interaction with the Higgs field, which itself is a fundamental aspect of the universe.

Particle	Charge (elementary charge, e)
Electron (e^-)	-1
Positron (e^+)	$+1$
Proton (p^+)	$+1$
Anti-proton (p^-)	-1
Neutron	0
Anti-neutron	0
Photon	0

Point Charge: A charged object whose radius is much smaller than the distance between itself and all other objects of interest.

The magnitude of electric force between two point charges is d irectly proportional to the magnitude of each charge and inversely proportional to the distance squared. Specifically:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

This is Coulomb's Law. Note the direction of the force changes with the sign of the charges involved. Like repels like and opposites attract.

Electric field

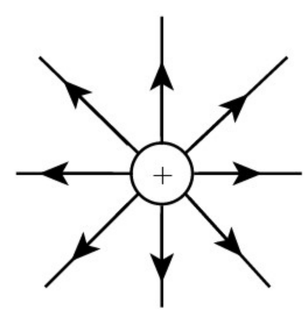


Figure 1: An Electric field coming from point charge. Notice how the densities of the lines vary with distance from the source.

Technically, fields can be the more general tensor, or even the fascinating and exotic spinor!

Consider a charged particle. We can represent its effect by drawing vectors that show the path a positively charged particle would follow if placed within its influence. These lines represent the electric field of the charged particle. The greater the density of the lines, the greater the strength of the electric field. Note that at the origin, the force is undefined (infinite), since $|r| = 0$.

There are many types of fields, which can be either scalar or vector. For example, a temperature map is a scalar field, since each point is associated with a scalar (the temperature at that point). A map of fluid velocity is a vector field, since each point is associated with a vector (the velocity of the fluid at that point). For a particle, the Electric field vector at any point is given by

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

The direction in which the field lines point depends on the sign of the charge. If the charge is negative, the field lines point in. If it is positive, the field lines point out. A useful mnemonic is to think of the charge as someone's STD test results. If it's negative, others will go for them and the lines point in. If it's positive, everyone will try to get away and the lines point out.

Consider the relative strengths of the electric and gravitational fields. The gravitational force is given by $F_g = G\frac{m_1 m_2}{r^2}\hat{r}$, with $m_{electron} = 9 \times 10^{-31}kg$ and $m_{proton} = 1.7 \times 10^{-27}kg$. If we consider a hydrogen atom, then $r = 5.3 \times 10^{-11}m$. With $G = 6.7 \times 10^{-11}$, we have

$$F_g = \frac{(1.7 \times 10^{-27})(9 \times 10^{-31})(6.7 \times 10^{-11})}{(5.3 \times 10^{-11})^2} \approx O(10^{-46})N$$

Now, the Electric force is given by $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. The charge of a proton and Electric are $q_1 \approx q_2 \approx 1.6 \times 10^{-19}C$. Ergo, since $\frac{1}{4\pi\epsilon_0} \approx 8.99 \times 10^9 \frac{Nm^2}{C^2}$,

$$F_e = \frac{(8.99 \times 10^9 \frac{Nm^2}{C^2})(1.60 \times 10^{-19}C)^2}{(5.3 \times 10^{-11}m)^2} \approx O(10^-17)N$$

This means that $\frac{F_e}{F_g} \approx 2.27 \times 10^{39}$, meaning the Electric force is much stronger for these masses and charges than gravity. On scales as large as humans and planets, gravity is the dominant force because gravity is strictly additive.

For sufficient distances, the Electric field of a uniformly charged spherical shell resembles the Electric field of a point charge.

That means for $r >> R$, $\vec{E_{sphere}} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$ This holds only for outside the sphere. Inside, it can be shown that the electric field is zero.

Figure 2: Notice how a circle resembles a point from a great distance.

Superposition Principle: The net electric field at a location in space is a vector sum of the individual electric fields contributed by all charged particles located elsewhere.

To introduce systems with multiple sources of electric field lines, consider the particle pair known as a dipole. Dipoles consist of one negatively charged and one positively charged particle, like so:

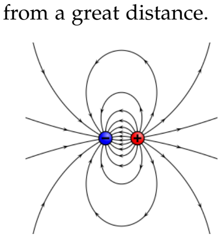


Figure 3: Two oppositely charged particles distanced from one another

Dipole Moment: The dipole moment is a way of expressing asymmetrical charge distribution. It is a vector quantity, i.e. it has magnitude as well as definite directions. The dipole moment is given by the expression $\vec{p} = q\vec{d}$, where q is the charge on one end of the dipole and $\vec{d}$ is the distance between dipoles.

On the axis of the dipole (i.e. the lines formed by the two particles), the electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2sq}{r^3}\hat{p}$$

where r is the distance from the point in consideration to the center of the dipole. On the bisecting plane (i.e., the plane exactly halfway from each point) the field is given by

$$\vec{E} = \frac{-1}{4\pi\epsilon_0}\frac{sq}{r^3}\hat{p}$$

Where r is the distance from the point to either dipole. The force on a positive point charge q_1 a distance of d away from the dipole, aligned with the dipole, and on the side of the negative charge is given by

$$\vec{F} = q_1\vec{E_{dipole}} = q_1(\frac{-1}{4\pi\epsilon_0}\frac{2qs}{d^3}, 0, 0)$$

Note that the field will be parallel to the axis of the dipole: this can simplfy vector calculations. Now consider a dipole in a uniform electric field, like so: The positive end will be pulled to the right

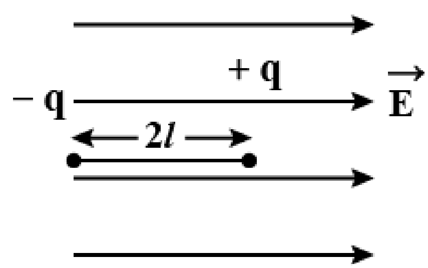


Figure 4: Dipole in uniform electric field

and the negative end to the left, exerting a torque given by $\vec{\tau} = \vec{p} \times \vec{E_{uniform}}$. Note that by the definition of $\times$, $\tau = pE_{uniform}\sin\theta$, where θ is the dipole's angle from horizontal. It can be shown that the potential energy of a dipole in a uniform Electric field is $U = -\vec{p} \cdot \vec{E_{uniform}}$ and similarly as before (but now with $\cdot$) $U = -pE_{uniform}\cos\theta$. Usefully, this means dipoles can be used to measure the direction of an electric field.

Interestingly, in annihilation between positrons and electrons (or any other subatomic particles) the total energy and momentum of the initial pair are conserved in the process and distributed among a set of other particles in the final state (photons in the example of the electron-positron pair). Antiparticles have exactly opposite additive quantum numbers from particles, so the sums of all quantum numbers of such an original pair are zero. Hence any set of particles may be produced whose total quantum numbers are also zero as long as conservation of energy, conservation of momentum, and conservation of spin are obeyed.

Throughout these examples, we have been assuming the associated speeds are much less than the speed of light. If the velocities approach a significant fraction of the speed of light Coulomb's law no longer holds, and we must account for relativity.

Conservation of Charge: Charge cannot be created nor destroyed, with the exception of electron-positron annihilation and other such quantum hijinks. We can use conservation of charge to predict the behavior of many systems. For example, consider tape pulled from a roll. You may have noticed when dangling strips of freshly-pulled tape they tend to drift towards nearby surfaces to stick and become tangled: this is because peeling a strip of tape off a roll strips electrons from the tape, resulting in a net positive charge because of conservation of charge. When this charge approaches a net neutral object, such as your hand, the electrons in the atoms of your hand are attracted to the positively charged tape and congregate closer to the tape. This results in a negative charge buildup near the tape. The positive tape is attracted to the negative charges, and so the tape moves towards your hand and becomes tangled and useless. The process of one charge inducing a charge on a neutral object occurs often enough for the phenomenon to be named. We call it **Polarization**: The process by which a dipole is formed in a neutral object by an electric field. The dipole moment is given by $\vec{p} = \alpha\vec{E}$, where α is a material-dependent constant called polarizability. Such a dipole is *induced*. Consider a point charge near a neutral atom. The point charge creates an electric field given by

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}$$

Inducing a dipole given by the expression

$$\vec{p} = \alpha\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{\alpha q_1}{r^2}\hat{r}$$

This dipole creates a field at the point charge of

$$\begin{aligned}\vec{E}_2 &= \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\alpha\vec{E}_1}{r^3}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\alpha}{r^3}(\frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r})\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q_1}{r^5}\hat{r}\end{aligned}$$

This formula is valid provided the electric field that induced the dipole is from a point charge. If the electric field is instead from, say, another (permanent) dipole then $\vec{p} = \alpha\vec{E}_1$ is still valid. However, in this case the formula for $\vec{E}_1$ will be given by the equation for the electric field along the axis of a dipole instead. Following this logic,

$$\begin{aligned}\vec{p} &= \alpha\vec{E}_1\\ &= \alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p}_1}{r^3}\end{aligned}$$

If we let $\vec{r'}$ be the location of the permanent dipole from the induced dipole, we have the electric field at the permanent dipole as

$$\begin{aligned}\vec{E}(\vec{r'}) &= \left(\frac{1}{4\pi\epsilon_0}\frac{2}{r^3}\right)\left(\alpha\frac{1}{4\pi\epsilon_0}\frac{2\vec{p}_1}{r'^3}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p}_1}{r^3r'^3}\end{aligned}$$

Calculating the force on the permanent dipole is slightly more complicated than multiplying by the charge of the dipole, since r' and the sign of q varies based on which end of the dipole we consider. To find the net force, we must find the force on each charge and sum them, like so:

$$\begin{aligned}F^+ &= q\vec{E}(r-\frac{s}{2})\\ F^- &= q\vec{E}(r+\frac{s}{2})\\ F_{net} &= F^+ + F^-\\ &= q\left[\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p}_1}{r^3(r+\frac{s}{2})^3} - \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{4\alpha\vec{p}_1}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left[\frac{4\alpha\vec{p}_1}{r^3(r+\frac{s}{2})^3} - \frac{4\alpha\vec{p}_1}{r^3(r-\frac{s}{2})^3}\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2(4\alpha\vec{p}_1)\left[\frac{1}{r^3(r+\frac{s}{2})^3} - \frac{1}{r^3(r-\frac{s}{2})^3}\right]\end{aligned}$$

With a bit more algebraic simplification and the assumption that $r >> s$, we can show that

$$\begin{aligned}F_{net} &= F^+ + F^-\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{4\alpha\vec{p}_1}{r^6}\right)\left[(1+\frac{3s}{2r}) - (1-\frac{3s}{2r})\right]\\ &= q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{4\alpha\vec{p}_1}{r^6}\right)\left[\frac{3s}{r}\right]\\ &\approx q\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{12\alpha\vec{p}_1 s}{r^7}\right)\\ &= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{12\alpha p_1^2}{r^7}\end{aligned}$$

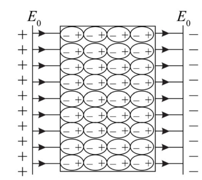
Insulators, conductors, and Van der Waals forces

On the topic of induced dipoles, consider how freely moving atoms in a substance interact. The electrons are dispersed in a cloud about the nucleus of each atom. These clouds can be thought of as constantly fluctuating, and occasionally these fluctuations will result in more electrons on one side than another. In this case there will be a small dipole. If this dipole approaches another atom (which may or may not have its own temporary dipole) the two will be attracted. The result of this interaction is that even neutral materials can be attracted to each other due to the fluctuating dipoles of its electron clouds. We call this phenomenon

Van der Waals forces: attraction and repulsions between atoms, molecules, as well as other intermolecular forces. Caused by correlations in the fluctuating polarizations of nearby particles (a consequence of quantum dynamics).

Insulator: An insulator is a material that does not easily allow electricity to pass through it. Inside an insulator the electrons are bound to their atoms, but they may still shift in response to an electric field (see fig. 5) This results in the insulator becoming polarized. Just as with most polarized objects, we can approximate its dipole moment with $\vec{p} = \alpha\vec{E}$ This relationship is only valid if the electrons are bound to the stationary constituent atoms. If this is not the case, then the a material in question is called a

Figure 5: Effect of electric field on insulator



Conductor: A conductor is a material that allows electricity to flow freely through it. For example, consider a liquid with charged atoms floating throughout. Here, when an electric field is applied, the charges each follow the electric field lines until they reach the edges of whatever container holds them. More commonly, we see conductors in the form of metals. The atomic structure of a metal allows electrons to move freely from atom to atom. Thus, within a metal, the electrons can freely move in whichever direction the electric field determines. "Freely" is used liberally here, since the electrons can collide with other electrons or defects in the metal and lose energy. The mobile electrons in the conductor will have a momentum and velocity given by

Such a liquid is called an *ionic solution*

Typically electrons are bound to the metal as a whole. However, if the electric field is strong enough, then air surrounding a metal can become ionized and allow electrons to flow freely through it, creating a spark.

$$
\begin{aligned}
\vec{\Delta} p &= \vec{F_{net}}\Delta t \\
&= q\vec{E_{net}} \\
&= -e\vec{E_{net}} \\
&\rightarrow \vec{\Delta} p = -e\vec{E_{net}}\Delta t \\
&= m_e v \\
&\rightarrow v = \frac{e\vec{E_{net}}\Delta t}{m_e}
\end{aligned}
$$

A good approximation for the average velocity of an electron is $\bar{v} = \mu E$. The constant μ is called the "mobility".

Now, consider a conductor with a net charge, such as a charged sphere. Inside the sphere like charges will repel each another and push to get as far away from one another as possible. This occurs when all the charges are on the surface of the conducting object, since anywhere else would be closer together and they cannot go outside the bounds of the object. This rearrangement has two effects. First, any charge in a conductor will be found on the surface. Second, the net electric field inside a conductor will be zero. An interesting result of electrons seeking to minimize repulsion inside a conductor is the *sharp point effect*. To visualize this effect, imagine a gymnasium full of students pretending to be electrons, staying as far away from others as possible. Anyone near the center of the crowd will feel badly pressed and will try to work there way towards the edge of the gym, where at least one side will no longer have fellow students milling about. The result? Most of the students will gravitate towards the edge of the gym and hover there, to take advantage of that lack of other students on the wall side of the gym. Now imagine a narrow corridor leading out of the gym. Even better! Students in that corridor will only have fellow students behind and in front of them. Now imagine the very end of that corridor, a sort of point. Even better! Now, the student who finds that spot will benefit from having only one student nearby. But somewhat ironically, that same effect will cause other students to pack themselves into the long, narrow corridor more tightly, since pretty much anywhere in the corridor makes them less exposed to the full set of students than being in the gym does. This effect makes edges, wires, and points more attractive to electrons, which similarly just don't want other electrons nearby.

To see why this must be the case, imagine if it were not. Then any charge within the conductor would be moved by the electric field, so we see that the case where no electric field is present is the only stable possibility. However, if electrons are moving, then there can be (and is) an electric field within the conductor. It is only when the conductor is in static equilibrium that there is no net electric field within.

Finding electric field

Now that we understand conceptually how charge behaves in a conductor, let's think about the electric field that charge creates. We can visualize any charged object as a collection of point charges in the shape of the object. If we'd like to find the net electric field of these charges, we simply use Coulomb's law and sum up each of their electric fields. To illustrate this approach, imagine a charged rod with length L and charge Q. We can approximate it as a bunch of charges in a row. For now, let's use ten, but recognize that the more charges we use the better our estimate will be. Each piece of the rod will have a charge of $\frac{Q}{10}$, so to find the electric field at a point we would need to calculate the vector between each piece and the point and use $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$. We would do this ten times and then sum the electric fields to get our approximate net field. Of course, perhaps we would like to find the exact electric field. To do this we would have to cut our object up into an infinite number of points, find the infinitesimal electric field due to each, and sum them up. Sound familiar?

To drive the point home, consider a vertical rod with a uniform charge of Q and length L. Let's take a point on the bisecting plane of this rod (the horizontal plane that cuts the rod in half). Say we view this point from the side and see that is has coordinates $(0, x)$. If we want to find the electric field due to a little bit of the rod at $\vec{x}$, we need to know the distance between them. If the bit is a distance of y up the rod, the distance between it and $\vec{x}$ will be $r = \sqrt{x^2 + y^2}$. The charge of this small piece will be $\frac{Q}{L}\Delta y$ (where Δy is the height of the peice), and $\hat{r}$ will be $\frac{(x,-y)}{\sqrt{x^2+y^2}}$. Therefore the electric field at $\vec{x}$ will be given by

$$\begin{aligned}\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2 + y^2}\frac{(x, -y)}{\sqrt{x^2 + y^2}}\end{aligned}$$

We can see by symmetry (isn't symmetry lovely?) that the contributions in the y direction will cancel, so we need only consider the x direction. Therefore,

$$\begin{aligned}\Delta\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{\frac{Q}{L}\Delta y}{x^2 + y^2}\frac{x}{\sqrt{x^2 + y^2}} \\ &= \frac{Q}{4\pi\epsilon_0 L}\frac{x\Delta y}{(x^2 + y^2)^{3/2}}\end{aligned}$$

Now comes the tricky part: adding these up. You have already likely guessed that we will need to integrate between the bottom and top of the rod, which corresponds to the integral from $-\frac{L}{2}$ to $\frac{L}{2}$. We have then

$$\vec{E} = \int_{-L/2}^{L/2}\frac{Q}{4\pi\epsilon_0 L}\frac{x}{(x^2 + y^2)^{3/2}}dy$$

I'll spare you the tedious integration and skip to the result:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{x\sqrt{x^2 + (L/2)^2}}\right]\hat{x}$$

The steps for finding the electric field due to a charged object in general are outlined below.

1. Cut up the charge distribution into pieces and draw $\Delta\vec{E}$
   - Divide the charge distribution into pieces whose field is known. In particular, very small pieces can be approximated by point

particles. You may also wish to break up a complex object into smaller objects whose electric field equations are already known.

- Pick a representative piece, and at the location of interest draw a vector $\Delta\vec{E}$ showing the contribution to the electric field of this representative piece. Drawing this vector helps you figure out the direction of the net field at the location of interest.

2. Write an expression for the electric field due to one piece

- Pick an origin for your coordinate system, and show it on your diagram. Draw the vector $\vec{r}$ from the source piece to the observation location. Write algebraic expressions for $\vec{r}$ and $\hat{r}$.
- Write an algebraic expression for the magnitude $|\Delta\vec{E}|$ contributed by the representative piece. Multiply by $\hat{r}$ to get the vector $\Delta\vec{E}$. You can break this up into x, y, and z components for integration. Once you do each expression should contain one or more "integration variables" ($\Delta x/\Delta y/\Delta z$ or $dx/dy/dz$) related to the coordinates of the piece. Write the amount of charge on the piece, Δq, in terms of your variables.

3. Sum the contributions of all the pieces

- The net field is the sum of the contributions of all the pieces. To write the sum as a definite integral, you must include limits given by the range of the integration variable. If the integral can be done symbolically, do it. If not, choose a finite number of pieces and do the sum with a calculator or a computer.

4. Check the result

- Check that the direction of the net field is qualitatively correct.
- Check the units of your result, which should be newtons per coulomb.
- Look at special cases. For example, if the net charge is nonzero, your result should reduce to the field of a point charge when you are very far away. For a numerical integration on a computer, check that the computation gives the correct numerical result for special cases that can be calculated by hand.

Let's apply these steps to some new problems. First, let's consider the electric field produced by a charged ring (fig. 6). We have a ring of radius a, with a charge of Q. The ring is centered on the xy plane and we are calculating the electric field on the z-axis.

1. Cut the charge distribution and find $\Delta\vec{E}$. Let's take a slice of this

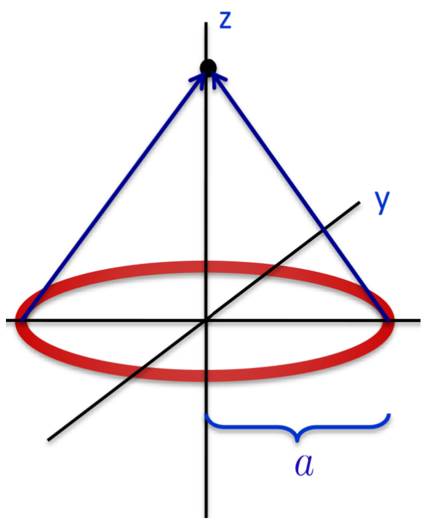


Figure 6: Charged ring

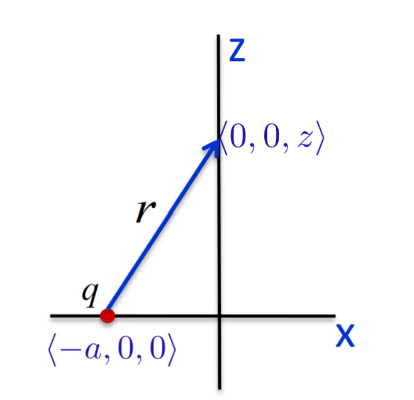


Figure 7: Charged ring slice

ring, like fig. 7. We can view the slice of the annulus here as a point charge a distance of $\sqrt{a^2+z^2}$ away from $(0,0,a)$. We find then that

$$\vec{r} = (a,0,z)$$
$$\hat{r} = \frac{(a,0,z)}{\sqrt{a^2+z^2}}$$

2. By symmetry, only the z components of each individual electric field contribute. Therefore,

$$\vec{E} = \sum \frac{1}{4\pi\epsilon_0} \frac{z\Delta q}{(a^2+z^2)^{3/2}}$$

3. Since the charge is uniformly distributed, we have that $\Delta q = \frac{Q\Delta\theta}{2\pi}$. Therefore our integral becomes

$$\begin{aligned}\vec{E} &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{z\frac{Q}{2\pi}}{(a^2+z^2)^{3/2}} d\theta \\ &= \frac{Q}{4\pi^2\epsilon_0} \frac{z}{(a^2+z^2)^{3/2}} \int_0^{2\pi} d\theta \\ &= \frac{1}{4\pi\epsilon_0} \frac{Qz}{(a^2+z^2)^{3/2}} \hat{z}\end{aligned}$$

4. This looks pretty good. It goes to zero as z goes to either zero or infinity, as it should.

Let's consider the electric field on the z-axis of a disk with radius r and charge Q now.

1. Take some point on the disk. The calculation for the electric field due to this point is identical to the case of the ring. Therefore

$$\vec{r} = (a,0,z)$$
$$\hat{r} = \frac{(r,0,z)}{\sqrt{a^2+z^2}}$$

2. Now we have that $\Delta q = \frac{Q}{\pi r^2} da d\theta$. Therefore

$$\begin{aligned}\vec{E} &= \sum \frac{1}{4\pi\epsilon_0} \frac{z\Delta q}{(a^2+z^2)^{3/2}} \\ &= \sum \frac{1}{4\pi\epsilon_0} \frac{Q}{\pi r^2} \frac{z da d\theta}{(a^2+z^2)^{3/2}}\end{aligned}$$

3.

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{Qz}{\pi r^2} \int\int \frac{a}{(a^2+z^2)^{3/2}} da d\theta \\ &= \frac{1}{4\pi\epsilon_0} \frac{Qz}{\pi r^2} \int_0^{2\pi} \int_0^{r} \frac{a}{(a^2+z^2)^{3/2}} da d\theta \\ &= \frac{1}{2\epsilon_0} \frac{Qz}{\pi r^2} \left[\frac{1}{z} - \frac{1}{\sqrt{r^2+z^2}} \right] \hat{z}\end{aligned}$$

Say we have two conducting disks very near to one another, with opposite charges (this configuration is known as a *capacitor*). Our equation for the electric field due to a uniformly charged disk does not hold for a conductor, since the charges are free to locomote. The charges will be repelled to the edges of the disk and attracted to the negative charges in the other disk. This attraction means that the charge on the disks will be spread nearly uniformly on the inner surfaces of the disks and the field between the plates may be approximated as the field between two infinite plates, but what of the outside? Some charge will still be on the outer surface of these disks, and his charge will create an electric field outside of the capacitor. Not a strong field in comparison to the inner field, but still nonzero. To calculate it we can use the equation for electric field due to a uniformly charged disk,

$$\begin{aligned}\vec{E_{disk}} &= \frac{1}{2\epsilon_0}\left(\frac{Q}{\pi r_1^2}\right)x\left(\frac{1}{x}-\frac{1}{\sqrt{r_1^2+x^2}}\right)\\ &\approx \frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1-\frac{x}{2R}\right]\end{aligned}$$

The net field will be the sum of the fields from each disk. Ergo,

$$\begin{aligned}\vec{E} &= \vec{E}_1+\vec{E}_2\\ &\approx \frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1-\frac{x}{R}\right]+\frac{1}{2\epsilon_0}\frac{Q}{\pi R^2}\left[1-\frac{s-x}{R}\right]\\ &\approx -\frac{1}{2\epsilon_0}\frac{Qs}{\pi R^3}\end{aligned}$$

With that brief divagation out of the way, let us return our attention to charged spheres. Recall that the electric field outside of a charged sphere is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

provided $r > R$. If we have $r < R$, then we know there is no net electric field within the sphere. Imagine placing a proton inside a charged sphere, and now consider what the field inside will be. The answer depends on if the sphere is made of a conducting materal or not. If charges are free to move in the sphere, then they will migrate and exactly cancel out the electric field of the proton, since the electric field within a conductor is zero. If the sphere is instead made of an insulator, then the electric field inside due to the sphere will still be zero, but the proton will now contribute an Electric field of its own according to Coulomb's law. What if we have another, concentric insulating charged sphere within the first? Well, if we are within both, then the Electric field will still be zero due to the superposition

For a very loose explanation of why there is no electric field within a sphere, consider the field due to each bit of the sphere on a point inside. At any point you have some charges nearby and some far away, and if that were all to there story then the Electric field would point towards the center of the shell. However, you have more charges far away than close. It works out that the effect of this greater number of charges exactly cancels out the farther distance and there is no net field within the shell! This is an interesting proof if you care to work it out.

principle. If we are outside the bounds of the first but still within the second, then the inner sphere will have an electric field given by Coulomb's law, while the outer shell still contributes no net field. If we are outside both then it appears to use that there is a point with a charge that is the sum of the charges of each sphere, and the electric field will again be given by Coulomb's law. Well and good, but what if we have a insulating ball, a filled sphere? What is the electric field then? We know that it will be given by Coulomb's law outside of the ball, but on the inside it isn't zero anymore. A=If you picture the ball as a series of concentric shells then the shells radially past the inner point will contribute nothing, but you will still have a small sphere of charge radially inward with a net field. Let's find the field contributed by this sphere. First, cut up the charge distribution and find $\vec{E}$. The charge per volume will be

$$\frac{Q}{V} = \frac{Q}{4/3\pi R^3}$$

For a point a distance of r within the ball, we have

$$\frac{\Delta q}{4/3\pi r^3}$$

We know that the sphere is uniformly charged. That is, the charge density everywhere is the same. Thus,

$$\begin{aligned}\frac{\Delta q}{4/3\pi r^3} &= \frac{Q}{4/3\pi R^3} \\ \rightarrow \Delta q &= Q\frac{r^3}{R^3}\end{aligned}$$

So we now have that

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{Q\frac{r^3}{R^3}}{r^2} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}\end{aligned}$$

And we are done.

Electric potential

To begin this section, let's imagine what happens when we touch two conductors together. Say one has a charge of 6 nc and the other of 0 nC, but otherwise they are identical. What happens when the conductors are brought in contact? We intuit that the charge flows from one to the other until each has a charge of 3 nC, and this is indeed what would happen. Consider a similar situation, but now the conductor of 6 nC is much smaller than the neutral conductor. What happens when these two are brought in contact now?

As you ponder that, let's briefly review energy. The energy of a particle moved by a force $\vec{F}$ along a path $\vec{l}$ is given by

$$\Delta E_{particle} = W = \int \vec{F} d\vec{l}$$

If the forces involved are conservative, we also have that

$$\Delta U = -W_{internal}$$

Consider a particle between two charged infinite plates. Let's find the change in its potential energy as it moves from point A to point B. If the electric field strength is E N/C, then

$$\begin{aligned} \Delta U &= -W_{internal} \\ &= -\int qEd\vec{r} \\ &= -q \int dx \\ &= -qE\Delta x \\ &= -qV \end{aligned}$$

So the difference in potential energy of a particle inside a charged electric field is proportional to the strength of the field and the distance moved. We can classify the ability to have potential energy if a charge enters a system as the

Electric potential: the amount of work energy needed per unit of electric charge to move this charge from a reference point to the specific point in an electric field. Mathematically, the potential ΔV can be expressed as

$$\Delta V = -E\Delta x$$

V may seem like an odd choice of variable here, but electric potential is such an important concept that it gets its own units of Volts (which are equivalent to J/C). Despite the fancy name, we can think of electric potential the same way as gravitational or any other potential. Charges will move from high to low potential, just like a ball rolling down a hill is moving from high to low potential. Here the strength of the electric field is analogous to the height of the hill, and the charge is the ball. Just as with a hill and a ball, we also need two things in order to find electric potential energy. In the simplest case, imagine two charges near one another. The force of one on the other is simply

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

and as we have learned,

$$W = \int_a^b \vec{F} d\vec{x}.$$

The astute reader will notice that this equation is not very general, as it only applies when the charge moves in a straight line through a uniform electric field. In general,

$$\Delta V = -\int_i^f \vec{E} d\vec{l}$$

This integral is a *line integral*. It can be computed using multivariable calculus, but sometimes it simplifies to a form that makes this approach unecessary.

What's the potential energy of something floating around by itself in space? The question seems to make no sense, since potential energy is defined as a difference between two states. However, the potential at a single point is often defined as the potential relative to infinity, $V_A = V_A - V_\infty$. The electric potential is a scalar field, and has a value at every point in space.

So to find the work done as one particle goes from A to B, we have

$$\begin{aligned}
W &= \int_a^b \vec{F} dr \\
&= \int_a^b \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{r} dr \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \int_a^b \frac{dr}{r} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \left(\frac{-1}{r} \right) \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}$$

We can define the *electric potential energy* of a system of charges as the work needed to assemble the system by bringing each charge from infinity to its final position. In this case,

Electric potential energy is NOT *electric potential*. Electric potential energy has a dependency upon the charge of the object experiencing the electric field, electric potential is purely location dependent. Electric potential is the electric potential energy per charge.

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r},$$

where r is the distance between one charge and another. For a system with multiple particles, the total energy will simply be the sum of the energies of each individual charge with respect to each other.

Let's calculate the electric potential of three point charges A, B, and C. The distances are r_{12}, r_{13}, and r_{23}. The electric potential energy is the sum of the individual potential energies. That is,

$$\begin{aligned}
U &= \sum U_i \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}}
\end{aligned}$$

Now, you have probably noticed that, in general, things love moving from high potential to low potential. Lift something high up in the gravitational field, and it gains potential, let it go and it falls and loses potential. Gifted students and especially electric fields follow the same principle. The electric field points from the highest potential to the lowest potential, dictating how a charge will move if placed in a field. We can see this mathematically, since we know

$$\begin{aligned}
\Delta V &= -\int_i^f \vec{E} \cdot d\vec{l} \\
&\rightarrow dV = -\vec{E} \cdot d\vec{l}
\end{aligned}$$

Since $\vec{l}$ can be multidimensional, to find dV we'd need multivariable calculus. If we calculate the derivative what we would get is

$$\begin{aligned} dV &= -\vec{E} \cdot d\vec{l} \\ &\rightarrow \frac{dV}{d\vec{l}} = -\vec{E} \\ &= V\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \\ &= -\nabla V \end{aligned}$$

If you are familiar with gradients (∇) you will know this means the electric field points in the direction in which the potential decreases most rapidly. We also know the electric field always points in the direction of steepest descent of V and its magnitude is the slope. And just as with gravity, it can be shown that the electric potential between two points does not depend on the integration path. Any path will do. Moreover, any closed path has an electric potential of 0. In general, if you know electric field, you may integrate to find the electric potential between two points. If you know the potential, differentiate.

Take a momement to convince yourself of this by whatever means you find most useful. It can be shown both mathematically and intuitively.

If you have a shape, there are two ways to get the potential.

1. Break the object into point changes and add up the potential from each.

2. If you know the electric field, use $\Delta V = \int_a^b E \cdot dl$

Say we have a ring with radius R and charge Q. Using method 1, the potential due to each charge on the axis is $\Delta V_i = \frac{1}{4\pi\epsilon_0}\frac{\Delta q_i}{r}$. If we are a distance of z along the axis, $r = \sqrt{z^2 + R^2}$, and we have

$$V = \sum \frac{1}{4\pi\epsilon_0}\frac{\Delta q_i}{\sqrt{z^2 + R^2}}$$

$$\begin{aligned} V &= \sum \frac{1}{4\pi\epsilon_0}\frac{\Delta q_i}{\sqrt{z^2 + R^2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2 + R^2}}\sum \Delta q_i \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{z^2 + R^2}} \end{aligned}$$

Let's try to find the potential again, this time using method 2. We recall that

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qz}{(R^2 + z^2)^{3/2}}$$

So the potential will be

$$
\begin{aligned}
V &= -\int_{\infty}^{z} \frac{1}{4\pi\epsilon_0} \frac{Qz}{(R^2+z^2)^{3/2}} dz \\
&= -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^{z} \frac{z}{(R^2+z^2)^{3/2}} dz \\
&= \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{z^2+R^2}}
\end{aligned}
$$

Identical to the expression previously obtained, as it should be.

Allow me to now briefly remind the reader of the relationship between voltage and electric field.

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{l}$$

$$\vec{E} = -\left[\left(\frac{\partial}{\partial x} V \right) \hat{x} + \left(\frac{\partial}{\partial y} V \right) \hat{y} + \left(\frac{\partial}{\partial y} V \right) \hat{y} \right]$$

If you remember, a conductor inside a capacitor has no net electric field within its bounds. That means the electric field perpendicular to the surface is $\frac{\sigma}{\epsilon_0}$, while parallel to the surface the electric field is 0. That means that $V_f - V_i = 0$ on the surface, since any path on the surface has no electric field. We say that the surface of a conductor is *equipotential*; that is, every point has the same voltage. Of course, this is only true if the conductor is at equilibrium. We have no guarantee of charge distribution is the conductor is not in electrostatic equilibrium. The situation changes if we instead have an insulator between the two plates of a capacitor. The electric field within an insulator is not zero. The nonzero electric field induces dipoles within the molecules of the insulator, which then have an electric field of their own. It turns out that inside an insulator, the electric field is proportional to the applied electric field by a factor of $1/K$, where K is called the dielectric constant and is known for various materials. Let's see if we can calculate the potential between plates with an insulator between them. The problem: a capacitor with a 3mm gap has a potential difference of 6V. A disk of glass 1mm thick, with area the same as the metal plates has a dielectric constant of 2.5. The glass is inserted in the middle of the gap between the plates. What is now the potential difference between the plates? We know that without the insulator between the plates, the electric field will be.

$$
\begin{aligned}
E &= \frac{\Delta V}{\Delta x} \\
&= \frac{6V}{0.003m} \\
&= 200\frac{V}{m}
\end{aligned}
$$

Inside the glass,

$$\begin{aligned} E_{net} &= \frac{E}{K_glass} \\ &= \frac{2000}{2.5} \\ &= 800\frac{V}{m} \end{aligned}$$

The total voltage will be the sum of the voltage between the first plate and the glass, the two sides of the glass, and the glass and the second plate.

$$\begin{aligned} \Delta V &= V_1 + V_2 + V_3 \\ &= 2000(0.001) + 800(0.001) + 2000(0.001) \\ &= 4.8V \end{aligned}$$

And we are done.

Since the electric field can perform work, it must have energy associated with it. The energy depends on the electric field. Say you have two plates of charge Q a distance s apart, but instead of holding them there, you let one plate go to see how much energy you can get out of this setup. You know the force on one plate is

$$F = \frac{\sigma Q}{2\epsilon_0}.$$

Additionally, we recall that

$$Q = \int F dr.$$

Combining these two equations and noticing the bounds on the integral will be the distance between the plates, we have

$$\begin{aligned} W &= \int_0^s \frac{\sigma Q}{2\epsilon_0} dr \\ &= \frac{Q\sigma s}{2\epsilon_0} \\ &= \frac{Q^2 s}{2\epsilon_0 A} \\ &= \frac{1}{2}\epsilon_0 E^2 sA & &= -U \end{aligned}$$

We can define the energy density as the potential energy per volume. In this case,

$$\frac{U}{sA} = \frac{1}{2}E^2\epsilon_0 E^2$$

and for a general electric field,

$$\frac{U}{\text{volume}} = \frac{1}{2}E^2\epsilon_0 E^2$$

Magnetism

Electricity and magentism are fundementally intertwined in many ways. A moving electrically charged particle creates a magentic field. Here is some information about magnetism:

Note that this magentic field is different than the particle's electric field.

- The needle of a compass aligns with the direction of the net magnetic field at its location.
- An electric current is a continous flow of charge.
- The superposition principle is still valid for magentic fields, so to calculate the net magnetic field we can simply sum individual magentic fields.
- The units of magentic field are Teslas (T) or Gausses (G). $1G = 10^{-4}T$.
- Magentic fields are typically symbolized by the letter B.
- The earth has a magentic field approximately $.5G$ strong.

You have likely seen and played with magnets before, and already know some of their properties. You know that similar poles repel one another, and like poles attract. You know magents attract ferromagnetic materials (like iron, cobalt, or nickel). A magnet is any material or object that produces a magnetic field. A type of magnet you are likely familiar with is the bar magnet, which looks like fig. 8. The lines

This was confirmed in 1820 when a Danish physicist, Hans Christian Oersted, discovered if you held a compass near a live wire the needle deflected. Thus Oersted showed that moving electrons can create a magnetic field.

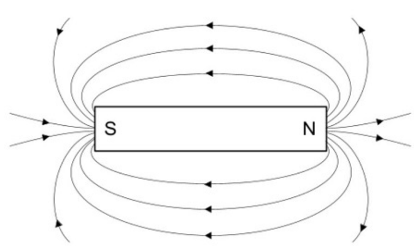


Figure 8: Bar magnet

emanating from the north pole and entering the south are magentic field lines.

I have previously said that a moving electric charge produces a magnetic field. The equation for the field a distance $\vec{r}$ from a charge q moving at velocity $\vec{v}$ is given by the following law, called the Biot-Savart law:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

Since current is just a bunch of moving charges flowing through a wire, then we should expect a current-carrying wire to have a magnetic field, and it does. The magnetic field looks like fig. 9. It is given

μ_0 is a constant, defined to be $4\pi \times 10^{-7}$. So $\frac{\mu_0}{4\pi} = 10^{-7}$

Recall that the magnitude of the cross product of two vectors A and B at an angle θ from one another is defined as

$$A \times B = |A||B| \sin\theta.$$

The full cross product can be found using the following formula:

$$A \times B = (a_2b_3 - a_3b_2)\hat{\imath} + (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

assuming a_i is the ith component of A and likewise for B.

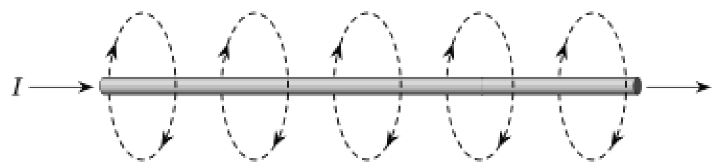


Figure 9: Magentic field of live wire

by the equation

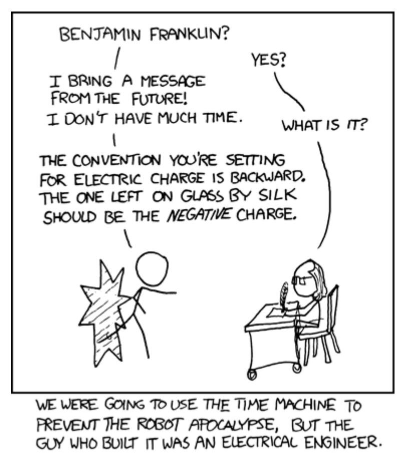
$$B = \frac{\mu_0 I}{2\pi r}$$

If you have calculated the magnitude of the magentic field and wish to find its direction, there is another right-hand-rule to follow, not to be confused with the right-hand-rule that allows one to find the direction of magentic field for a moving charge. Point your thumb along the direction of conventional current in the wire. Your fingers will curl in the direction of the magentic field.

Although we now know that the mobile charges in a wire are electrons, current is conventionally definded as following from flowing from the positive to negative terminal of a battery (that is, positive charges moving).
Despite this, we sometimes want to find the *electron current*, the number of electrons that flow through a given volume per second. Mathematically, the electron current i is given by

$$i = nA\bar{v},$$

where n is $\frac{electrons}{m^3}$, A is cross sectional area of the wire, and $\bar{v}$ is the average velocity of electrons in the wire. Electron current is what's physically happening, but since we're stuck with convention, let's try to think in conventional current. Electron current flowing one way is equivalent to conventional current flowing the other way.



So far, we have only examined magnetic fields due to moving charges. How can we explain the magnetic field of permanent magnets? What charge moves in a stationary object? The answer is the electrons. Within the magnet, electrons are moving about their nuclei and spinning on an axis. Both of these movements contribute to create a magnetic dipole. If the magnetic fields from other electrons don't cancel out, then there will be a net magnetic field on the atom! In order for the magnetic fields of the electrons to not cancel, the atom must half a half-full outer shell. When the atomic magnetic dipole moments align, the material is known as a ferromagnet and will have a net magnetic field. Eletron magnetic dipole is a fundementally quantum mechanical phenomenon, but in class we treat atoms like a classical system, where the electron orbits in a loop around the

The magnetic field of permanent magnets can be destroyed by heating the magnet. The temperature below which atomic magnets align is called the "Curie Temperature," and is a phase transition of the electrons inside a material! In the case of iron, it's approximately $1000K^\circ$

nucleus. From this we derive the equation for magnetic dipole of an atom as

$$\mu = \frac{1}{2} eRv.$$

To treat it properly as a quantum system, we would need to solve the Schrodinger wave equation for a hydrogen atom. This is a bit above our pay grade, but it turns out that the classical approximation is not bad.

For our purposes, there are three kinds of magnetism

1. Ferrmagnetism, the permanent magnetism explained by the magnetic moments of atoms within the material.

2. Paramagnetism, the kind of magnetism that occurs when the magnetic moments in a material only align in response to an external magnetic field. This is why certain metals can be picked up despite having no magnetic field of their own.

3. Diamagnetism, where the magnetic moments in the material align opposite to the applied magentic field. This causes diamagnetic objects to be repelled by a magnetic field.

Circuits

A *circuit* is a complete circular path that electricity flows through. Usually, the current is flowing through a wire. There are three states a circuit can be in.

1. Equilibrium: no current flows. The average drift velocity of the electrons is 0.

2. Transient: when the current is changing, usually after elements are hooked up.

3. Steady-state: current is flowing. The average drift velocity is nonzero, and there is no change in the deposits of excess charge on the wire anywhere.

Equilibrium is rather boring, so let's focus on transient and steady-state. Imagine connecting a battery to a loop of wire. Just after the circuit is connected, the circuit is in transience. There is a disturbance in the previous (equilibrium) electric field right next to the battery. At the speed of light, the region next to the disturbance updates its electric field in response, and the next region updates, and so on until steady state is achieved.

Importantly, when a circuit is in steady-state, the current at the start equals the current at the end. Since current is just the flow of electrons, the current needs to be conserved in the same way charge is conserved. I.e. current in is current out, current can't be "used up", so on. You may wonder how a current lights a lightbulb if it isn't used up - wouldn't that violate conservation of energy? The answer is no, whatever is driving the current powers the lightbulb. Think of it like water flowing and turning a turbine that lights the bulb. Water in is water out, but the energy comes from whatever propelled the water through. In the case of water and a turbine the energy comes from kinetic energy and is transferred to electric energy, in the case of the lightbulb the energy usually comes from stored potential in a battery. As the electrons collide with the atoms in the bulb filament, light and heat are produced. In order to keep shoving these electrons along there needs to be an electric field within the wire. This field is produced by an uneven distribution of charge on the wire: electrons piled up at one terminal repel other electrons to the other terminal, like in figs. 10 and 11 This means that in a circuit, there is a pile of

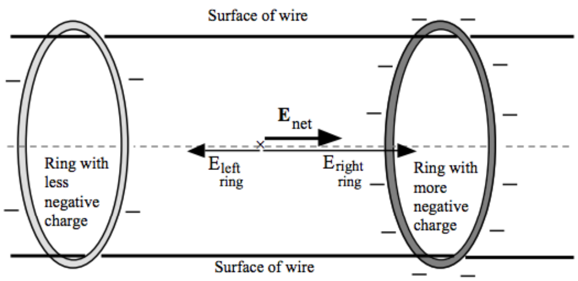


Figure 10: Electric field in circuit

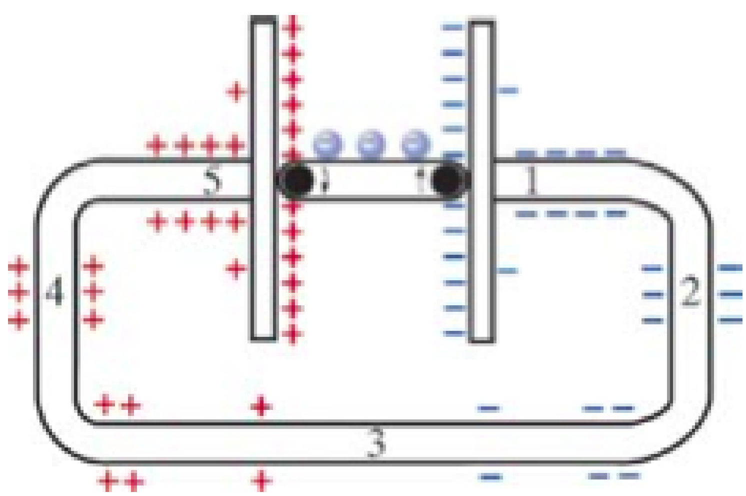


Figure 11: Charge distribution in a wire

negative charge where the electrons are coming out, and a pile of positive charge at the other end.

The analogy of water continues to be useful when you consider a wire with variable cross-sectional area. Just as water rushing through a pipe picks up speed when the pipe narrows (think of putting your thumb over the end of a hose), the drift velocity increases when the wire narrows. The electric field and drift velocity both increase in order to keep the current constant, according to the equation

Note that the current is still constant provided the circuit is in steady-state, although the drift velocity can change.

$$I = |q|nAu|E|$$

where $|q|$ is the charge on an electron, n is the number of electrons per volume of material, A is the cross-sectional area in consideration, u is the electron mobility (a constant that determines how quickly an electron can move through a material), and E is the strength of the electric field at that point. The electric field at every point in the wire is actually parallel to the wire at that point (see fig. 12)

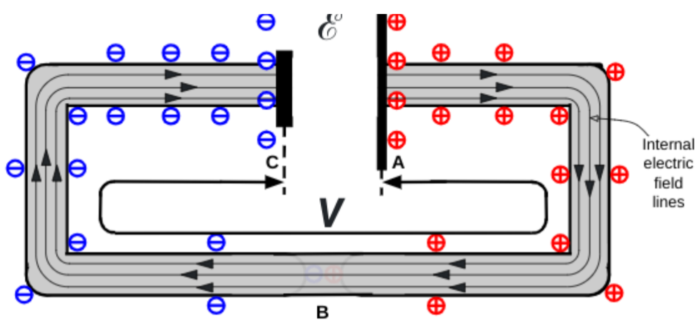


Figure 12: Electric field within a wire

We usually make the assumption that wires are ideal and allow current to flow freely. Elements for which this is not the case are called *resistors*.

Resistor: a part of the circuit that resists the passage of electrons. Placing a resistor in a circuit reduces the overall electric field of the circuit, but within the resistor, the electric field is higher in order to force eletrons along and maintain a constant current.

Now, as you are familiar with voltage and the general concept of potential, this next part should be no surprise to you. We are now going to discuss *Kirchhoff's loop rule*. The rule states that the total change in electric potential as you move around a circuit is zero. That is sensible, since no matter where you go in an electric field, if you return to the same point you will have the same potential regardless of what you did between leaving and coming back. Mathematically, we express this as

This is analogous to gravitational potential and height. If you climb a tree, then jump down, then run in circles, and climb back up the tree, you end with the same potential as when you started.

$$\sum_{i=0}^{n} \Delta V_i = 0.$$

You can pick any closed loop to use the loop rule, and as long as you end where you started the rule will be valid.

Assuming there isn't anything within your circuit adding to electric potential, the only gain in potential difference will be across the battery. Imagine a battery as a capacitor, where chemical reactions constantly replenish the electrons that flow from the negative plate. The change in voltage across the battery of length s is simply

$$|\Delta V| = Es = \frac{Fs}{e}.$$

This quantity, the energy input per unit charge, is called the *electromotive force* (emf). The emf is the function of a battery to maintain a potential difference between its terminals. The emf is measured in volts, but the energy input isn't electric. The energy input is from however the battery works. It's most commonly chemical energy, but can be nuclear, gravitational, kinetic, or any other flavor.

If you have two batteries in series, then the potential difference across them is $2emf$. The electric field is doubled everywhere in the circuit, the drift speed is doubled, and the current is doubled. Any element that uses power in the circuit (such as a lightbulb) will have its power absorbtion quadrupled according the the equation for power, $P = I^2R$.

To this point, we have been approximating our batteries as capacitors, those plates of charge separated by a distance. Imagine you have a battery hooked up to a wire with two plates on each end, like in fig. 13. The symbol for capacitors is shown in fig. 14 Capacitors, as many

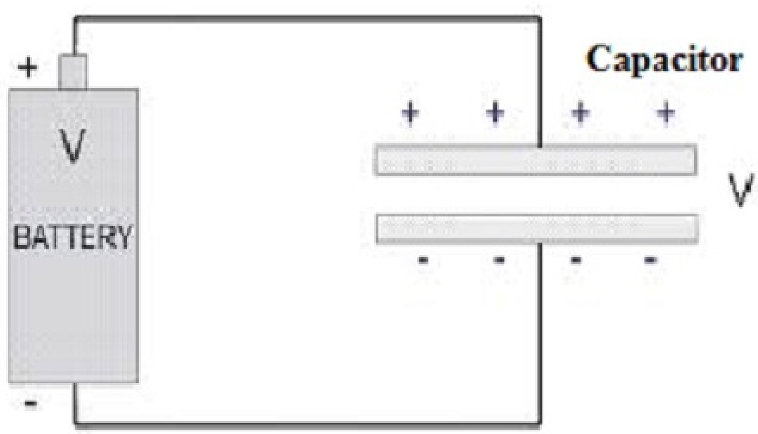


Figure 13: Capacitor and battery

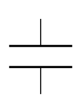


Figure 14: Capacitor symbol

ECE students are aware, are immensely useful elements. They can be used to store energy, stabilize signals, and detonators for explosives.

If you imagine hooking up an uncharged capacitor to a battery, the voltage will look like fig. 15. As more and more charge piles up on

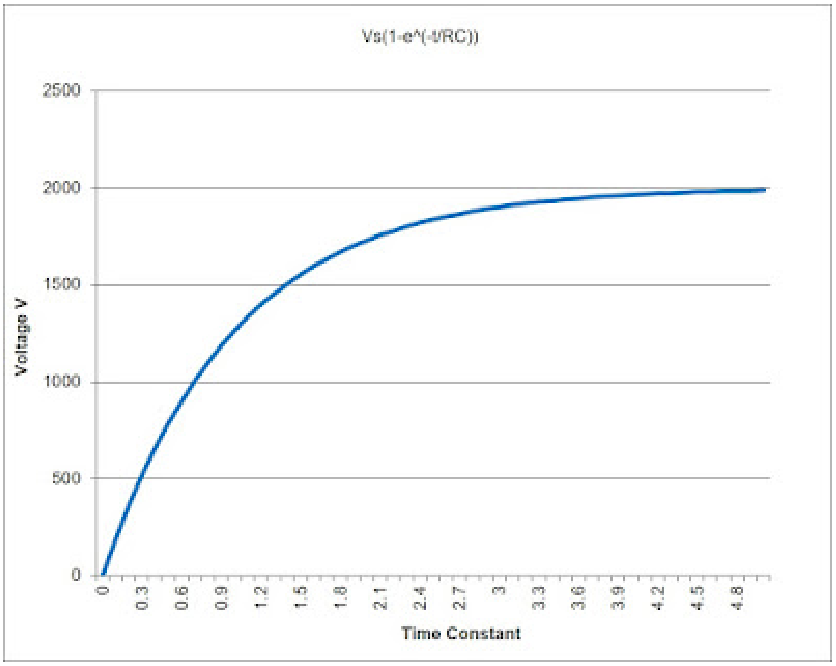


Figure 15: Capacitor charging

the plates of the capcitor, the incoming charge is gradually reduced until current flow essentially stops. Likewise, when discharging a capacitor, the current will initially be large as the charges piled up on the capacitor, but as charges on the plates flow around the wire and equalize, the current will gradually slow and eventually stop. The graph of voltage across a capacitor as it discharges looks like fig. 16.

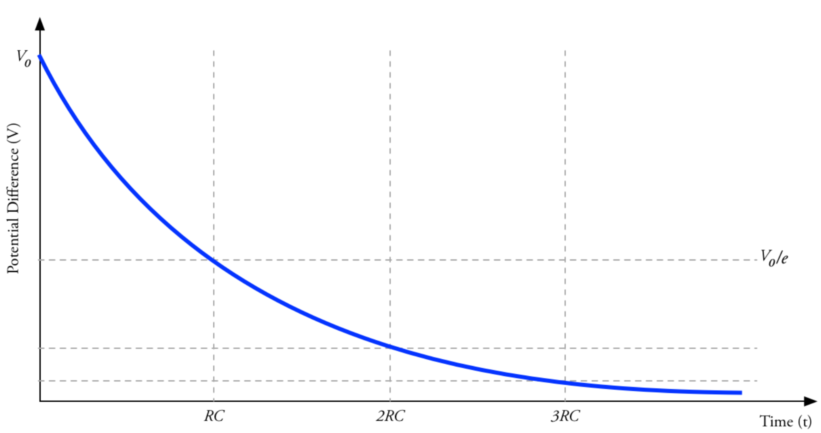


Figure 16: Capacitor discharging

The relationship between the amount of charge piled up on a plate of the capacitor and the voltage of the capacitor is

$$Q = C|V|,$$

where C is known as the capacitance and is a constant specific to a certain capacitor. The capacitance is found via the formula

$$C = \frac{\epsilon_0 A}{s},$$

where A is the area of a plate and s is the distance between plates. If capacitors are connected in parallel, you can imagine them as just one big capacitor with an area that is the sum of the individual areas and charge that is the sum of the charges. Thus the capacitance of parallel capacitors is simply the sum of the individual capacitances, or mathematically,

$$C_{total} = C_1 + C_2 + ...C_n = \sum_{i=1}^{n} C_i.$$

If we instead have capacitors in series, i.e. hooked up one after the other, we know that the voltage drop across the total will be equal to the sum of voltage drops across each capacitor. That is,

$$\begin{aligned} V_{total} &= V_1 + V_2 + \ldots V_n \\ \frac{Q}{C_{total}} &= \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n} \\ \frac{1}{C_{total}} &= \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} \end{aligned}$$

In these cases we have been assuming a vacuum between the plates of the capacitor. If instead there is some insulating medium, the electric field within the plates of the capacitor will be

$$E = \frac{Q}{KA\epsilon_0}.$$

That implies

$$C = K\frac{\epsilon_0 A}{s},$$

which tells us that the higher K (the dielectric constant) is between the plates of a capacitor, the higher the capacitance. Manufacturers use this fact in making capacitors by putting a highly insulating material between the plates, which are then rolled together into a cylinder like in fig. 17

As I was typing these notes, my friend asked an excellent question about Kirchhoff's current law and capacitors. Since charge in piling up on the surfaces of the capacitor, the current never has a chance to flow across the capacitor and Kirchhoff's current law is violated. Kirchhoff's current law only holds for steady-state, and since the current is always changing across a capacitor, then we can't use Kirchhoff's current law! However, any negative charge that accumulates on one plate will push away charge on the other plate, so current entering one plate of a capacitor equals current leaving the other plate.

We can define the current density $\vec{J}$ as

$$\begin{aligned} \vec{J} = \vec{J} &= \frac{\vec{I}}{A} \\ &= \frac{A|q|nu\vec{E}}{A} \\ &= |q|nu\vec{E} \\ &= \sigma\vec{E} \end{aligned}$$

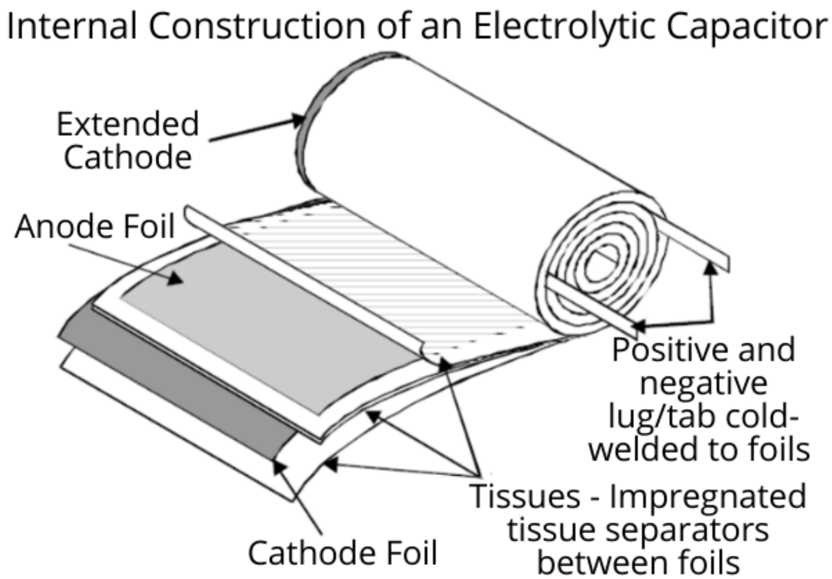


Figure 17: Real capacitor

where σ is known as the conductivity and is intrinsic to the material. We can show that the equation $\vec{J} = \sigma\vec{E}$ is equivalent to Ohm's law $\Delta V = IR$. In a resistor of length L,

$$\begin{aligned}\Delta V &= -\int \vec{E} \cdot d\vec{L} \\ &= EL\end{aligned}$$

So the current through the wire is

$$\begin{aligned}I &= \frac{\sigma AEL}{L} \\ &= \frac{\sigma A}{L}\Delta V \\ &\rightarrow V = I\frac{L}{\sigma A} \\ &= IR\end{aligned}$$

There are several useful facts that arise out of this relationship. We can see that for a resistor, $R = \frac{L}{\sigma L}$. Since the current through the resistor is $I = \frac{V}{R}$, increasing the length of the resistor decreases the current proportionally, while doubling the corss-sectional area of the resistor halves the resistance. This will be useful for both exams and homework.

Interesting fact, if the conductivity remains constant regardless of how much current flows, then we call the material "ohmic". Resistors are pretty ohmic, while capacitors are not.

Now, let's turn our attention to circuits with a resistor and capacitor in series, as shown in fig. 18. Such a circuit is known as an *RC circuit*, for the resistor and capacitor within it. Let's recall that, for capacitors, $Q = CV$. It's also useful to remember the definition of

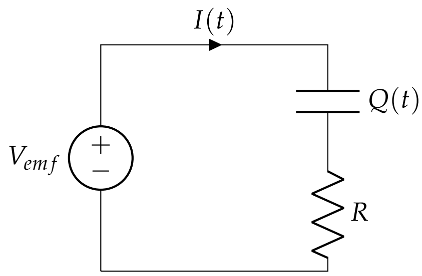


Figure 18: RC circuit

current, $I = \frac{dQ}{dt}$. Let's now apply Kirchhoff's loop rule to the circuit as a whole.

$$\begin{aligned} V_{emf} &= V_C + IR \\ &= \frac{Q}{C} + IR \\ &= \frac{Q}{C} + \frac{dQ}{dt}R \end{aligned}$$

This is a first order differential equation. We can find the solution by assuming $Q(t) = ae^{bt}$. Solving the differential equation yields

$$Q(t) = V_{emf}C(1 - e^{-t/RC}).$$

If we wish to have the current, we can simply take the derivative of the above and find that

$$I(t) = \frac{V_{emf}}{R}e^{-t/RC}.$$

Let's now consider a discharging RC circuit. After the capcitor has stored up all the voltage, we disconnect the battery and let the capacitor spill out its charge. The circuit diagram in this case looks like fig 19. In this case, Kirchhoff's loop rule yields

For some reason, physicists find it necessary to define the time constant $\tau = RC$. This unremarkable variable makes up a disproportionate quantity of exam questions, so it is worth your time to study it if you care about grades.

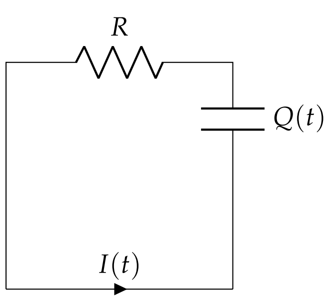


Figure 19: Discharging RC circuit

$$\begin{aligned} 0 &= I(t)R - \frac{Q(t)}{C} \\ &= \frac{dQ(t)}{dt} - \frac{Q(t)}{C} \end{aligned}$$

The solution of which is

$$Q(t) = V_{emf} C e^{-t/RC}.$$

Again, differentiating yields

$$I(t) = \frac{V_{emf}}{R} e^{-t/RC}.$$

Now, let's examine how to find the equivalent resistance of resistors in series and parallel. First, let's define each of these terms.

Series One after the other.

Parallel Share two nodes, i.e. next to each other.

If resistors are in series, the current through each is the same. If they are in parallel, the voltage across each is the same. Here is how we find the combined resistance if resistors $R_1, R_2, \ldots R_n$ are in series:

$$R_{eq} = R_1 + R_2 + \ldots R_N.$$

If they are in parallel, the equivalent resistance is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}.$$

If we have a circuit with N nodes and B branches, then the number of independent nodes is $N - 1$ and the number of independent loops is $B - N + 1$. In fig. 20 there are two nodes (intersections of wires) and three branches (wires between two nodes). Ergo, we will have

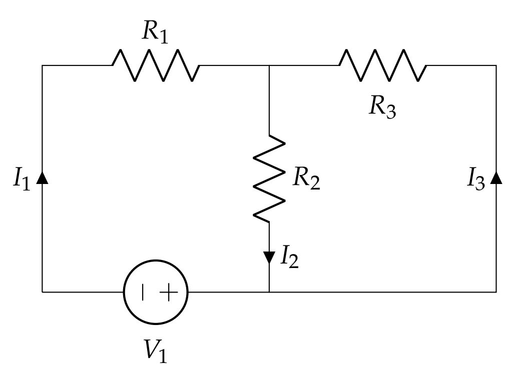


Figure 20: Nodes and branches

$2 - 1 = 1$ independent nodes and $3 - 2 + 1 = 2$ independent loops. This will give us a total of $1 + 2$ equations, which will be enough to solve our circuit (i.e. find all currents). These equations are

$$\begin{aligned} I_2 &= I_1 + I_3 \\ 5V - I_1 R_1 - I_2 R_2 &= 0 \\ I_2 R_2 - I_3 R_3 &= 0 \end{aligned}$$

It is useful to know how to solve systems of linear equations with linear algebra.

Let's now consider capacitors in series and parallel. For capacitors in parallel, the equivalent capacitance is simply the sum of the individual capacitances:

$$C_{eq} = C_1 + C_2 + \cdots + C_n.$$

If they are in series instead,

$$\frac{1}{C_{eq}} = frac1C_1 + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.$$

A term that often arises when discussing circuits is *power*. Power is defined as $P = I\Delta V$. For resistors, $P = I\Delta V = I \times IR = I^2 R = \frac{V^2}{R}$. That means that if you have lightbulbs in series (so their current is the same). Then the one that dissipates the most power (and is therefore the brightest) will be the one with the greatest resistance. If you have lightbulbs in parallel and a voltage source V, then the power dissipated by each will be

$$P_1 = \frac{V^2}{R_1}$$
$$P_2 = \frac{V^2}{R_2}$$
$$\ldots$$
$$P_n = \frac{V}{R_n}.$$

This means that the one with the smallest resistance will dissipate the most power.

Let's now consider *ammeters*, *voltmeters*, and *ohmmeters*.

Ammeter measures current (Amps)

Voltmeter measures voltage difference (Volts)

Ohmmeter measures resistance (Ohms)

An ammeter is inserted into a circuit in series with the circuit element whose current you want to measure. It has a low resistance so it doesn't impede the current. A voltmeter is connected in parallel with the element. It has a very high internal resistance so it doesn't affect the current much (it still needs to take some current, since it works by measuring the voltage drop across the internal resistor). An ohmemter is connected in series and contains a small voltage source. It calculates the resistance by taking the value of the voltage and dividing it by the current observed in the contained ammeter.

All of these are ideal components, but in reality, ammeters have internal resistance just as batteries and wires do.

Magnetic force

The force F on a charge q moving with velocity $\vec{v}$ through a region of space with electric field $\vec{E}$ and magnetic field B is given by

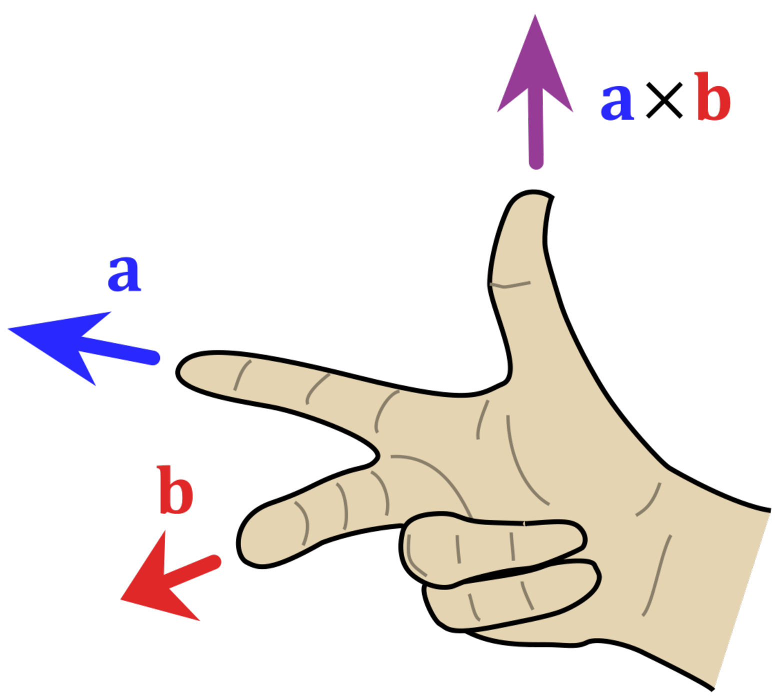
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$

We can isolate the magnetic term and find that the force of a magentic field on a moving charge is

$$\vec{F}_B = q\vec{V} \times \vec{B}$$
$$|vecF_B| = |q||\vec{V}||\vec{B}|\sin(\theta)$$

Computing the cross product is tedious, so here's a *handy* shortcut to determining direction. If you point your pointer finger in the direction the positive charge is moving, and then your middle finger in the direction of the magnetic field, your thumb points in the direction of the magnetic force pushing on the moving charge. This is known as the *right hand rule*. The right hand rule immediately gives you the direction if the charge in question is positive. If the charge is negative, the direction is opposite to the direction given by the right hand rule.

Figure 21: Right hand rule



The magnetic force on a charged object that moves in a magnetic field does not do any work, because it's perpendicular to $\vec{v}$. Therefore the magnetic force cannot change the magnitude of the velocity of a charged object, but can change the direction of motion. We can find the momentum of a moving particle in a magnetic field, which actually turns out to be quite useful. If the particle moves in a circle of radius R, then

The magentic force is always perpendicular to velocity, and can only change the direction, not the magnitude. Sound familiar? This is what centrifugal force does. Indeed, magnetism causes a charge to move in a circle if there is a constant magnetic field perpendicular to the velocity.

$$\begin{aligned}
\vec{r} &= R\hat{r} \\
\vec{v} &= R\omega\hat{\theta} \\
\vec{p} &= \gamma m\vec{v}, \text{ where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\
\frac{d\vec{p}}{dt} &= -\gamma mR\omega^2\hat{r} \\
|\frac{d\vec{p}}{dt}| &= \gamma mR\omega^2 \\
&= p\omega \\
&= |q|vB \\
p &= |q|BR
\end{aligned}$$

This is very useful for particle physicists, since they can measure the strength of the electric field and the radius of the circle. Cyclotrons, a type of particle accelerator, take advantage of this behavior. Our equation for momentum naturally leads to some useful expressions for cyclotrons.

The Lorentz factor γ is here to ensure our equation for momentum is still valid for relativistic speeds. If the speeds are much lower than the speed of light , $\gamma \approx 1$.

$$\begin{aligned}
\omega &= \frac{|q|B}{m} \\
f &= \frac{|q|B}{2\pi m} \\
T &= 2\pi\frac{m}{|q|B}
\end{aligned}$$

We can inject different particles into a cyclotron, observe their frequency, and draw conclusions about their mass. This allows us to do things such as determine isotopes, identify fundamental particles, and impress the Nobel prize committee. Chemists actually often need to seperate ions by mass and measure the mass of each type of ion, so we've made a specific instrument for them called a *mass spectrometer* (fig. 22). We can do a little bit of work and find that

$$\frac{m}{q} = \frac{B^2R^2}{2|\Delta V|}.$$

Recall the Biot-Savart law, which states that for a current-carrying wire

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\vec{l}\times\hat{r}}{|r|^2}.$$

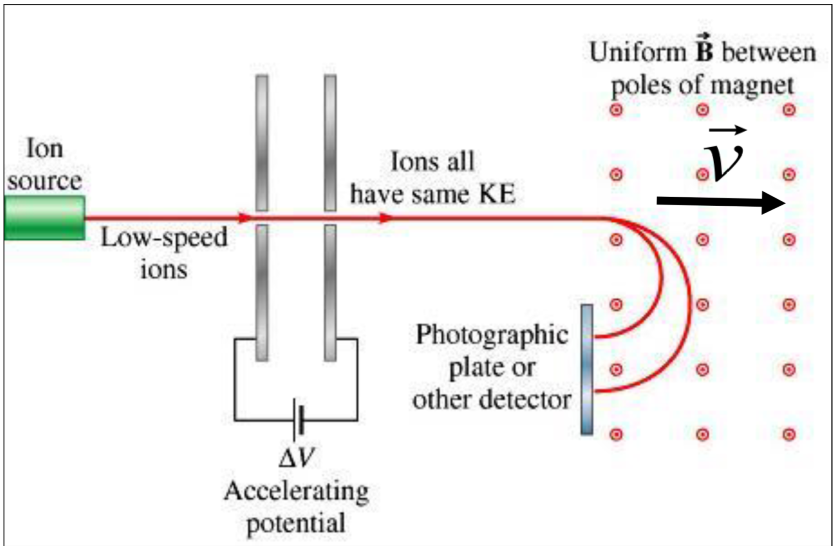


Figure 22: Mass spectrometer

Since this wire creates a magnetic field, it will exert a force on other wires. If these wires have current I_1, I_2 and are seperated by a distance d, this force is given by

$$F = \frac{\mu_0}{4\pi}\frac{2I_1I_2}{d}\Delta l.$$

If you have a wire twisted into a non-circular shape and run current through it, the reflexive magnetic force will expand the wire into a circle.

We can use the fact that a magnetic field exerts a force on a wire with current to create machines that turn electrical energy into physical energy. Consider fig. 23. In this example,

$$\begin{aligned}
\vec{F} &= I\Delta\vec{l} \times \vec{B} \\
|F| &= IwB \\
|F_\perp| &= IwB\sin(\theta) \\
\mu &= IA \\
&= Iwh \\
\tau &= IwhB\sin(\theta) \\
&= \mu B\sin(\theta) \\
&= \vec{\mu} \times \vec{B}
\end{aligned}$$

This equation, $\vec{\mu} \times \vec{B}$, is valid for a loop of any shape.

We may also go in the opposite direction. By rotating a loop in a magnetic field, we may convert mechanical energy to electrical energy. Working through the math, we will find that

$$emf(t) = A\omega B\sin(\omega t,$$

where ω is the rotational speed, A is the area, and B is the strength of the magnetic field. At any time during the rotation, the loop has

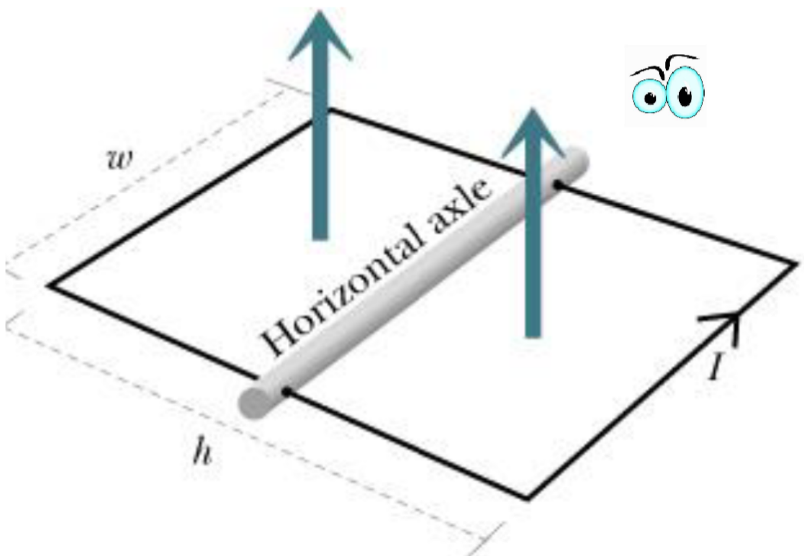


Figure 23: Torque

potential given by

$$U_m == -\vec{\mu} \cdot \vec{B},$$

which is true for a general magnetic dipole. The force on a magnetic dipole is

$$\begin{aligned} F_x &= \mu \frac{dB_x}{dx} \\ &= -\frac{\mu_0}{4\pi} \frac{6\mu_1\mu_2}{x^4} \text{ (in the case of two dipoles)} \end{aligned}$$

Consider a moving bar in an applied magnetic field. The work done on the bar is $W = ILB\Delta x$. Moving the bar through the field exerts a force on the electrons within the bar, causing a current to flow throughout the bar. If the bar has a wire hooked up, the current can flow through the wire and be used to deliver power to a load. If there is no wire, charge will just pile up at each end.

Hall Effect

Hall effect: The Hall effect is the production of a potential difference (the Hall voltage) across an electrical conductor that is transverse to an electric current in the conductor and to an applied magnetic field perpendicular to the current. Since moving charges in a magnetic field experience a force perpendicular to their direction of motion, positive

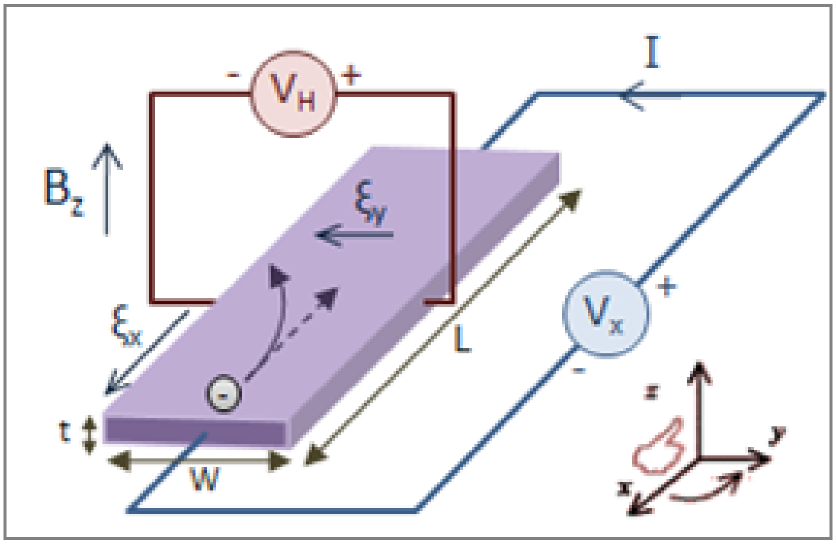


Figure 24: Hall effect

charges will be forced to one side of the material and negative charges to the other. This creates a potential difference across the material, called the Hall voltage. In steady-state, the force of these accumulated charges will exactly cancel the magnetic force. By measuring current, voltage across the material in the direction of current, and voltage across the transverse, you can figure out if the charge carriers in a material are positive or negative. The steps to do so are easy:

1. Apply a magentic field.
2. Apply a current across the material.
3. Measure the Hall voltage.

Special Relativity

Special relativity is based on two postulates:

- Laws of physics are invariant in all inertial frames of reference (with constant v).
- The speed of light in the vacuum is the same for all observers.

Special relativity predicts how magnetic and electric fields transform when you change from one inertial frame of reference to another.

Imagine you have a current I flowing in a wire, and an electron travelling next to it. From the viewpoint of a stationary observer, we would observe that the electron moves down due to the force from the wire. However, imagine you are travelling with the electron. Then its velocity relative to you is zero, and so the force it experiences is zero. How can we resolve this conundrum? The answer is with relativity. In classical physics, we would desribe this system as

The apostrophe "′" indicates that we are transforming from one frame of reference to another.

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

In relativity, a corrective factor arises and we have that

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t' &= \gamma \left(t - \frac{vx}{c^2} \right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

What we find is that the electron still experiences a force, but now, there is an electric field from the wire, and this is what repels the electron. We can find expressions for the electric and magnetic fields incorporating relativity as well.

Space and time warp to keep the speed of light constant for all observers! Namely,

$$L_{obs} = \frac{L}{\gamma} \tag{1}$$

$$t_{obs} = \gamma \times t \tag{2}$$

It is possible to deduce the Boit-Savart law from special relativity alone! Consider a frame where a charged particle is still. The electric field is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r},$$

while the magentic field B is zero. Say you have another frame, the *primed* frame, what is moving to the right with speed v. Then we observe that

$$B_z' = \gamma(B_z - \frac{vE_y}{c^2}).$$

Substituting in and assuming $v << c$, we find that

$$B_z' = -\frac{\mu_0}{4\pi} \frac{qv}{r^2}.$$

So we find that the magnetic force is just the electric force under a relativistic transformation! That hints at a relation between the mangetnic and electric fields, which we shall soon see are intertwined.

Gauss's Law

To get into Gauss's law, it will first be useful to understand *flux*. **Flux**: a measure of the number of electric or magnetic field lines passing through a surface. It may be useful to imagine flux as the flow of water through an area. Just as with water flowing, there must be a source. In the case of electric field the source is a charge. Suppose you have water flowing through a loop of area A. The flux is the volume of water per time, so we have that

$$\begin{aligned} \phi &= \frac{V}{\Delta t} \\ &= \frac{v \Delta t A}{\Delta t} \\ &= vA \end{aligned}$$

However, the amount of water flowing through depends on the angle between the velocity and the loop. If the loop just cuts the water in half (if it is perpendicular) then there is no water flow through the loop. The effective area in that case becomes $A\cos(\theta)$, and our equation becomes

$$\phi = vA\cos(\theta).$$

In the case of the electric field, the flux is defined as

$$\begin{aligned} \phi &= \sum (\vec{E} \cdot \hat{n}) dA \\ &= \sum E \cos(\theta) dA \\ &= \int \vec{E} \cdot d\vec{A} \\ &= \oint \vec{E} \cdot d\vec{A} \end{aligned}$$

Now, here is Gauss's law: If you have a charge Q enclosed in a surface S, then

Don't let the symbol $\oint$ scare you. It simply means that this is an integral on a closed surface, such as a sphere, a cube, etc. We will usually not even need to compute the integral if we can exploit the symmetry of a problem.

$$\begin{aligned} \phi &= \oint_S \vec{E} \cdot d\vec{A} \\ &= \frac{Q}{\epsilon_0} \end{aligned}$$

Not so bad! To get a little intuition, consider the case where you have a closed surface with no charge within, but the surface is still in an electric field. Then any electric field lines that flow in must pass out again, so the net flux is zero, just as $\frac{0}{\epsilon_0}$ is zero! Note that the law works for any closed surface, so the flux through a sphere, cube, cylinder, etc. will all be the same if the same charge in enclosed. The size doesn't matter, either! Gauss's law often allows us to solve complicated system simply, if we exploit symmetry. Here, let's work through an example together. Say we have a uniformly charged sphere with charge Q. Gauss's law tells us that

$$\begin{aligned}\oint \vec{E} \cdot d\vec{A} &= \frac{Q}{\epsilon_0} \\ &= \oint E\hat{r} \cdot \hat{r} dA \\ &= \oint EdA\end{aligned}$$

Here is an example of where we may exploit symmetry. Since the surface in question is spherical and uniform, the electric field will be radially uniform. Therefore we may take it out of the integral.

$$\begin{aligned}&\oint EdA \\ &= E \oint dA \\ &= E(4\pi r^2) \\ &= \frac{Q}{\epsilon_0}\end{aligned}$$

Rearranging to solve for E, we find that

$$E = \frac{Q}{4\pi\epsilon_0 r^2},$$

which we already know as Coulomb's law.

Gauss's law works for magnetism as well, stating that

$$\oint \vec{B} \cdot \hat{n} dA = \frac{1}{\epsilon_0} \sum q_i.$$

However, since no experiment has ever found a magnetic charge (a.k.a. magnetic monopole), this simplifies to

$$\oint \vec{B} \cdot \hat{n} dA = 0.$$

Ampere's Law

There exists a comparable law for the magnetic field, which states the path integral of the magnetic field over a closed curve C is proportional to the current passing through any surface bounded by C. Mathematically, this is expressed as

$$\int_C \vec{B} \cdot d\vec{L} = \mu_0 I_{enc}.$$

This is known as *Ampere's law* and can make complex problems very easy. Let's see if we can use it to find the magnetic field at a distance r

from a current-carrying wire. If we draw a circle around the wire, we have that

$$\begin{aligned}\int_C \vec{B} \cdot d\vec{l} &= \mu_0 I_{enc} \\ &= B \int_0^{2\pi r} dl \\ &= \mu_0 I \\ &= 2\pi r B.\end{aligned}$$

Which implies that

$$B = \frac{\mu_0 I}{2\pi r}.$$

We already know this is the formula for the magnetic field due to a wire! Let's try to find something new: the magnetic field *inside* a wire. Once more, draw a circle around the center of the wire and you have that

$$\begin{aligned}\int_C \vec{B} \cdot d\vec{l} &= \mu_0 I_{enc} \\ &= \mu_0 \frac{\pi r^2}{\pi R^2} I \\ &= \mu_0 \frac{r^2}{R^2} I \\ &= B \int_0^{2\pi r} dl \\ &= B 2\pi r\end{aligned}$$

Taking all of this together, we find that

$$B = \frac{\mu_0 I r}{2\pi R^2}.$$

Maxwell's Equations

Taking Coulomb's law, Gauss's law for electric/magnetic charge, and Ampere's laws together, we get four equations, known as *Maxwell's equations*:

$$\begin{aligned}\oint_S \vec{E} \cdot \hat{n} dA &= \frac{1}{\epsilon_0} q_{enc} \\ \oint_S \vec{B} \cdot \hat{n} dA &= 0 \\ \int_C \vec{B} \cdot d\vec{L} &= \mu_0 I_{enc} \\ \oint_C \vec{E} \cdot d\vec{l} &= 0\end{aligned}$$

However, this is not the full form of even the equations we have. Here are Maxwell's equations in all their glory:

$$\oint_S \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_0} q_{enc}$$
$$\oint_S \vec{B} \cdot \hat{n} dA = 0$$
$$\int_C \vec{B} \cdot d\vec{L} = \mu_0 \left(I_{enc} + \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} dA \right)$$
$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dA$$

We have not yet seen where all these extra terms come from, but fear not, for time uncovers all secrets.

Faraday's Law

Recall that magnetic fields must always be "curly", since there may be no magnetic monopoles. However, electric fields can be either "curly"

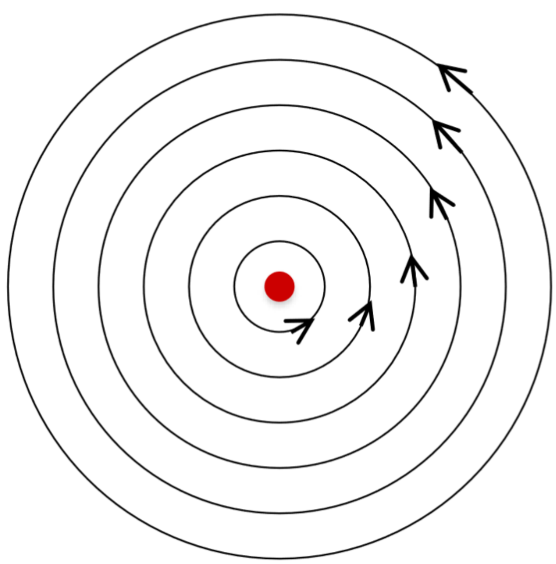


Figure 25: Curly force field

or radial, as is the case with a point charge. Here is an interesting empirical observation: just as moving charges induce a magnetic field, a changing magnetic field creates an electric field! The electric field produced by this phenomenon is curly. Say we have a region with a changing magnetic field and we place a conductive ring within it. The curly electric field will drive current through the ring. This electric field is fundementally different from the standard, Coulomb

The current within the ring in a changing magnetic field is unconventional and has no surface charge gradient, since there is no battery.

electric field produced by charges, which is radially outward. While the latter is governed by Coulomb's law, the former curly electric field is given by

$$emf = -\frac{d\phi_{mag}}{dt}$$

where

$$\phi_{mag} = \int \vec{B} \cdot \hat{n} dA.$$

This is *Faraday's law*. It tells us that a changing magnetic flux induces an electric field and thus a current in a wire. Faraday's law is typically written as

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA.$$

With this curly electric field comes a new right hand rule: if you place your thumb in the direction of $-\frac{dB}{dt}$, your fingers will curl in the direction of E. When dealing with current, it can be easier to remember Lenz's Law: an induced current has a direction such that the magnetic field due to the induced current opposes the change in the magnetic flux that induces the current. We can calculate the current induced in a ring by a changing magnetic flux. We know that

$$\begin{aligned} emf &= -\frac{d\phi_{mag}}{dt} \\ &= -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA \\ &= -\frac{d}{dt}(B \int dA). \end{aligned}$$

For a circular ring,

$$-\frac{d}{dt}(B \int dA) = -\frac{dB}{dt} \pi r^2$$

Just to give us something to work with, let's say that the magnetic field is due to a solenoid and therefore that $B = \frac{\mu_0 N I}{L}$.

$$\begin{aligned} -\frac{dB}{dt} \pi r^2 &= -\frac{d}{dt} \left(\frac{\mu_0 N I}{L} \right) \pi r^2 \\ &= -\frac{dI}{dt} \left(\frac{\mu_0 N}{L} \right) \pi r^2 \qquad = emf \end{aligned}$$

Now, let's use Ohm's law to finally obtain

$$\begin{aligned} I &= \frac{emf}{R} \\ &= -\frac{dI}{dt} \left(\frac{\mu_0 N}{LR} \right) \pi r^2 \end{aligned}$$

If we recall a previous problem of moving a bar along two conducting rods through a magnetic field, the area was changing there. Can you use Faraday's law to find the current induced?

The astute reader will notice that it is not required that we change the magnitude of the magnetic field, merely the magnetic flux, which is a function of both area and the magnetic field. Therefore, changing the area of the surface in consideration will induce an electric field.

With Faraday's law now in our toolbox, we can finally assemble all of Maxwell's equations:

Technically we have yet to cover the final equation, the Ampere-Maxwell law, but I am impatient.

$$\text{Gauss's law: } \oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_0} \sum Q_{enc}$$

$$\text{Gauss's law (magnetism): } \oint \vec{E}\vec{B} \cdot \hat{n} dA = 0$$

$$\text{Faraday's law: } \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$$

$$\text{Ampere-Maxwell law: } \oint \vec{B} \cdot d\vec{l} = \mu_0 \left[\sum I_{enc} + \epsilon_0 \frac{d}{dt} \int \vec{E} \hat{n} dA \right]$$

Conduction and induction

We love metals! They're shiny, conduct well, and malleable. We make wires out of them, and as the non-ideal components they are, they have some resistance and therefore dissipate some power. As metal heats up, the resistance increases (Ohm's law only holds up to a point!). There are some materials where the converse is taken to an extreme. If they are cooled below a certain point, the resistance drops to zero! Not just an infinitesimally small amount, but actually zero. There are the so-called *superconductors*, and have some interesting properties. For starters, the electric field inside one of these must be zero, or else there would be an infinite current. What's more, the magnetic flux through a superconducting ring cannot change! Superconductors carry current perfectly and lose no energy. Another interesting thing is that superconductors expel magnetic fields via the Meissner effect. These expelled fields will exactly cancel out any applied magnetic field. The first material discovered to be superconductive was mercury, with a critical temperature below 5 K.

Now, on to inductors. Thinking about a solenoid, $B = \mu_0 n I$. Now, if the current changes the emf in each loop will change as well. It can be shown that the inductance of a solenoid is

$$L = \frac{\mu_0 N^2}{d} \pi r^2.$$

A solenoid will tend to oppose a change in current. Inductors are instrumental in transformers, which can change the voltage across two coils. Notably, if there are two coils with N_1 and N_2 loops,

$$\frac{emf_2}{emf_1} = \frac{N_2}{N_1}.$$

Let's now consider an RL circuit, a circuit with a resistor and inductor in series. The voltage across the inductor is given by $V_L = L\frac{dI}{dt}$.

The current is given by

$$I = \frac{V}{R}\left[1 - e^{-Rt/L}\right].$$

The voltage is given by

$$V = L\frac{dI}{dt}.$$

The energy stored in an inductor is given by

$$U_L = \frac{1}{2}LI^2.$$

Energy density is given by

$$\frac{U_L}{volume} = \frac{1}{2\mu_0}B^2.$$

For a general electromagnetic field, energy density is given by

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}.$$

LC circuits

Thus far we have mainly been concerned with RL and RC circuits. Let us now turn our attention to LC circuits such as the one shown in fig. 26. Since a capacitor and inductor are the only elements in this circuit,

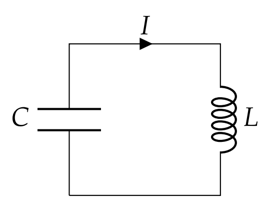


Figure 26: LC circuit

the circuit's total energy U will be equal to the sum of the energy in the capacitor and the energy in the inductor. Therefore,

$$\begin{aligned}
U_L &= \frac{1}{2}LI^2 \\
U_C &= \frac{1}{2}CV^2 \\
&= \frac{Q^2}{2C} \\
U &= \frac{Q^2}{2C} + \frac{1}{2}LI^2
\end{aligned}$$

Since no resistor is present, an ideal LC circuit will have no power dissapation. The energy will simply trade back and forth from the magnetic field in the inductor to the electric field in the capacitor. If we apply Kirchhoff's loop rule to the LC circuit, we find that

$$\begin{aligned} V_C + V_L &= 0 \\ \frac{Q}{C} + L\frac{dI}{dt} &= 0 \\ \frac{Q}{C} + L\frac{d^2Q}{dt^2} &= 0 \end{aligned}$$

This is a second-order differential equation, and solving it will reveal that the LC circuit is actually a harmonic oscillator with period $2\pi\sqrt{LC}$ and frequency $\frac{1}{2\pi\sqrt{LC}}$.

Maxwell's equations

We have constructed nearly all of Maxwell's equations already. As of now, we have

$$\begin{aligned} &\text{Gauss's law: } \oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_0} \sum Q_{enc} \\ &\text{Gauss's law (magnetism): } \oint \vec{E}B \cdot \hat{n} dA = 0 \\ &\text{Faraday's law: } \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA \\ &\text{Ampere's law: } \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \end{aligned}$$

However, the reader may find it strange that these laws are called "Maxwell's equations" when, so far, they are all named after someone else. Well, there is actually a problem with Ampere's law that Maxwell solved. We know that a time-varying magnetic field creates an electric field. Maxwell discovered that a time-varying electric field creates a magnetic field as well. He added a term to Ampere's law to account for this phenomenon, the displacement current I_d.

$$\begin{aligned} I_d &= \epsilon_0 \frac{d\phi_E}{dt} \\ &= \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A} \end{aligned}$$

Adding displacement current to Ampere's law gives us

$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \mu_0 (I_{enc} + I_d) \\ \oint \vec{B} \cdot d\vec{l} &= \mu_0 (I_{enc} + \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A}) \end{aligned}$$

This corrected form is the Ampere-Maxwell law, and finally we have our complete Maxwell's equations.

Gauss's law: $\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_0} \sum Q_{enc}$

Gauss's law (magnetism): $\oint \vec{EB} \cdot \hat{n} dA = 0$

Faraday's law: $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$

Ampere-Maxwell law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 \left[\sum I_{enc} + \epsilon_0 \frac{d}{dt} \int \vec{E} \hat{n} dA \right]$

These are Maxwell's equations as stated in the integral form. Integrals are ugly and bad, so we prefer to write them in their differential form. If we let the nabla operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, then

Gauss's law: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Gauss's law (magnetism): $\vec{\nabla} \cdot \vec{B} = 0$

Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere-Maxwell law: $\vec{\nabla} \times B = \mu_0 J + \mu_0 + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

This form lets us do a lot more than the integral form. One thing that arises out of it is that the electric field may be expressed as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}.$$

The reader may notice that this satisfies the condition for a wave equation, namely that

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2}.$$

In fact, in free space both $\vec{E}$ and $\vec{B}$ satisfy wave equations.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Let's see if we can find the velocity of an eletromagnetic wave then, using the wave equation.

$$\begin{aligned} \mu_0 \epsilon_0 &= \frac{1}{v^2} \\ v &= \frac{1}{\sqrt{\mu_0 \epsilon_0}} \\ &= \frac{1}{\sqrt{(1.257 \times 10^{-6})(8.85 \times 10^{-12})}} \\ &= c \end{aligned}$$

This is how we first discovered that light is an eletromagnetic wave. We now know much more about the nature of these waves:

- $\vec{E}$ and $\vec{B}$ are perpendicular to one another.
- $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propogation, making them transverse waves.
- $\vec{E}$ and $\vec{B}$ are in phase.
- $|B| = \frac{|E|}{c}$.

Electromagnetic waves have energy. We can use the expressions for the energy stored in the electric and magnetic fields and the fact that $|B| = \frac{|E|}{c}$ to derive

$$\begin{aligned}
u &= u_E + u_B \\
&= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} \\
&= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{E^2}{\mu_0 c^2} \\
&= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{E^2}{\mu_0 \frac{1}{\mu_0 \epsilon_0}} \\
&= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2 \\
&= \epsilon_0 E^2 \\
&= \frac{B^2}{\mu_0}
\end{aligned}$$

Now, we already know that the direction of propogation is orthogonal to the eletric and magnetic waves. Mathematically,

$$\hat{s} = \hat{E} \times \hat{B}.$$

We define the Poynting vector $\vec{S}$ as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}.$$

Not only does this vector tell us the direction of travel of the electromagnetic wave, but its magnitude is directly related to the amount of energy being transported by the wave. The energy transported, or *intensity*, is given by

$$I = \frac{E^2}{\mu_0 c}.$$

If you need a break from studying, take a momement to fill out our quiz! We'll bake the cookie with the most votes and bring enough for all the students to the final exam. Here's the link: github.com/ezekielulrich/Notes/discussions/3