Notes for ECE 30100 - Signals and Systems

Shubham Saluja Kumar Agarwal

January 31, 2025

These are lecture notes for Fall 2025 ECE 30100 at Purdue. Modify, use, and distribute as you please.

Contents

Introduction

A signal can be continuous time (CT) signal, which has an independent continuous variable indexed $t \in \mathbb{R}$, or discrete time (DT) which has a discrete independent variable indexed $n \in \mathbb{N}(\mathbb{Z})$.
$(.) \rightarrow CT$
$[.] \rightarrow DT$

A system, on the other hand, is something that transforms inputs into outputs.

$$input \rightarrow [\textbf{SYSTEM}] \rightarrow output$$

Another way this could be represented is:

$$system(input, t) = output$$

These can also be divided into CT and DT.
A CT system is of the form:

$$x(t) \rightarrow [CT] \rightarrow y(t)$$

On the other hand, a DT is of the form:

$$x[n] \rightarrow [DT] \rightarrow y[n]$$

> *Note*: For most of the course, continuous and discrete will be analyzed separately. That is, only a couple topics will have CT inputs with DT outputs, or DT inputs with CT outputs.

> *Note*: Most of the analyzed systems we will be linear and time invariant.

Linearity

A system is linear if superposition holds. That is, if it can be analyzed by analyzing the individual components of the system and combining it.
A more formal definition would be "given an input, which can be represented as the weighted sum of several inputs, the output can be represented as the sum of several weighted outputs".
The necessary and sufficient conditions for linearity are:

- CT: $\alpha_1 x_1(t) + \alpha_2 x_2(t) \cdots \xrightarrow{S} \alpha_1 y_1(t) + \alpha_2 y_2(t) \cdots$
- DT: $\alpha_1 x_1[n] + \alpha_2 x_2[n] \cdots \xrightarrow{S} \alpha_1 y_1[n] + \alpha_2 y_2[n] \cdots$

for all values of α.
Linearity gives us an alternative way to represent and analyze a system. That is, if we know the responses of all the subcomponents of the input, we can calculate the response of the input without having to calculate it for the input directly.

Signal Classification

1. DT vs. CT:

   - DT: $x[n]$ is a sequence of either real or complex valued numbers. $x[n]$ can be written as $x_{Re}[n] + jx_{Im}[n]$ or as $A[n]e^{j\phi[n]}$.
     We can transform between the two notations using Euler's formula:
     $$x[n] = A[n]e^{j\phi[n]} = A[n]\cos(\phi[n]) + jA[n]\sin(\phi[n])$$
   - CT: $x(t)$ behaves similarly, with the only difference being that it is in terms of t. Euler's formula still applies.

2. Energy and Power:

   - Energy is the area under the squared magnitude of the signal, and it represents how costly it is to store and/or transmit the signal.
     For CT signals, energy over times $t \in (t_1, t_2)$ is
     $$E = \int_{t_1}^{t_2} |x(t)|^2 dt = \int_{t_1}^{t_2} (x_{Re}^2(t) + jx_{Im}^2(t))dt$$
     For DT signals, energy over $n \in [n_1, n_2]$ is:
     $$E = \sum_{n=n_1}^{n_2} |x[n]|^2 = \sum_{n=n_1}^{n_2} (x_{Re}^2[n] + jx_{Im}^2[n])$$
     The total energy can be written as:
     $$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
   - Power:
     For CT signals, average power over (t_1, t_2) is
     $$P = \frac{1}{t_2 - t_1} E(t_1, t_2)$$
     For DT signals:
     $$P = \frac{1}{n_2 - n_1 + 1} E[n_1, n_2]$$

And the overall average power will be:

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

And finally, instantaneous power is $|x(t)|^2$ or $|x[n]|^2$

There are three realistic types of signals:

(a) E_∞ finite: Must have $P_\infty = 0$

(b) P_∞ finite: Must have $e_\infty = \infty$, since we are integrating over time.

(c) Neither E_∞ nor P_∞ are finite: not practical, but mathematically possible.

> *Note*: We cannot have finite energy, ∞ power signals.

Transformations in Time

1. Time Shift:

$$x(t) \to [TS] \to x(t-t_0)$$
$$x[n] \to [TS] \to x[n-n_0]$$

- $t_0, n_0 > 0$: signal is shifted to the right, or delayed by t_0
- $t_0, n_0 < 0$: signal is shifted to the left, or advanced by t_0

2. Time Reversal:

$$x(t) \to [TR] \to x(-t)$$
$$x[n] \to [TR] \to x[-n]$$

3. Time Scaling:

$$x(t) \to [TSc] \to x(\alpha t)$$
$$x[n] \to [TSc] \to x[cn]$$

Assume $\alpha, c > 0$. If they corresponded to something negative, it can be considered as a scaling and a reversal together.

> *Note*: c must be a value such that cn is an integer. We are also assured of the fact that we will be losing information for any value of $c \neq 1$.

- $\alpha, c > 1$: shorter timescale, or faster
- $\alpha, c < 1$: longer timescale, or slower

4. Composite Transformation: a transformation that involves multiple transformations from 1., 2., 3.
For example:

$$x(t) \to [CT] \to x(-\alpha t + \beta)$$

Usually, to decompose these transformations, we shift, then scale, then reverse, as and if necessary.

More Signal Classifications

3. By the period (symmetry under time shift)
CT signal is periodic $\iff \exists T \neq 0 | x(t) = x(t+T) \forall T$. That signal is periodic with period t.
DT signal is periodic $\iff \exists N \neq 0 | x[n] = x[n+N] \forall N$. That signal is periodic with period n.

Fundamental Period: Smallest T such that $x(t) = x(t+T) \forall t$, and similar for discrete. These are delineated by T_0, N_0.

Note: If a signal is periodic, it means that both its real and imaginary parts are periodic.

If there are two periodic signals, their sum $x_1(t) + x_2(t)$ is periodic if $T = LCM(x_1, x_2)$ exists.

Note: T might not exist, if the ration between the signals is irrational.

4. Even and Odd Signals:

Even Signal: $x(t) = x(-t)$, or, the reversal of the signal is identical.

Odd Signal: $x(t) = -x(-t)$, or, it is symmetric across the $x = y$. This has, as a requirement, $x(0) = 0$.

If we were to assign (+1) to EVEN, and (-1) to ODD:

- EVEN*EVEN=EVEN
- EVEN*ODD=ODD
- ODD*ODD = EVEN

Any signal can be decomposed as $EVEN + ODD$:

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

5. Complex Exponential Signal: signals of the form

$$x(t) = Ce^{\alpha t} \text{ where C and } \alpha \text{ tend to be complex}$$

This is often rewritten as

$$\begin{aligned} C &= |C|e^{j\phi} = |C|\underline{/\phi} \\ \alpha &= \sigma + j\omega \\ \implies x(t) &= |C|e^{\sigma t}e^{j(\omega t + \phi)} \end{aligned}$$

The resulting equation can then be divided into three terms:

(a) $|C|$ scales the signal

(b) $e^{\phi t}$ is a real exponential signal:

- $\sigma < 0$: exponential decay
- $\sigma > 0$: exponential growth
- $\sigma = 0$: constant 1

(c) $e^{j(\omega t + \phi)}$ is a periodic complex exponential:

$$e^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

This means we have a period of $T = \frac{2\pi}{\omega}$.
The real part will be the imaginary part shifted by $\pi/2$.
ω is known as the fundamental frequency, and ϕ is known as the phase.
So, the resulting function is a scaled sinusoidal of frequency ω which either grows or decays exponentially, and is phase shifted by ϕ.
The instantaneous power of such a function is $|C|^2 e^{2\sigma t}$.
If

Harmonically Related Complex Exponential (HRCE) Signals

Defined by

$$x_k(t) = e^{jk\omega_0 t}$$

Where $k \in \mathbb{Z}$, ω_0 is the fundamental frequency, and $\omega_k = |k|\omega_0$.
The fundamental period of k can be defined as:

$$T_k = \frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

In general, we can consider

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Discrete Time Complex Exponentials

$x[n] = Ce^{\alpha n}$ where α is not necessarily e. But, when it is, we can rewrite the previous equation as:

$$x[n] = Ce^{\beta n}, \beta \in \mathbb{C}$$

We can then decompose these variables into their components:

$$C = |C|\underline{/\phi}$$
$$B = \sigma + j\omega$$

as we mentioned before.
Thus:

$$x[n] = x_{Re}[n] + jx_{Im}[n] = |C|e^{\sigma n}\cos(\omega n + \phi) + j|C|e^{\sigma n}\sin(\omega n + \phi)$$

This signal can be surrounded by an "envelope" defined by the equation(s) $\pm|C|e^{\sigma t}$.

Periodicity in Discrete Time

The periodicity in discrete time is not entirely analogous to that of continuous time.

> *Note*: Let us remember that in CT, if ω increases, the oscillation rate increases.

The difference can be seen in non-integer periods for CT.
Instead of the fundamental period being $T_0 = \frac{2\pi}{\omega}$, it will be

$$N_0 = LCM(T_0 = \frac{2\pi}{\omega}, 1) \text{ if } x[n] = e^{j\omega_0 n}$$

If instead we had multiple added frequencies, with different fundamental frequencies, added, the fundamental period would be:

$$N_0 = LCM(LCM(T_1 = \frac{2\pi}{\omega_1}, 1), LCM(T_2 = \frac{2\pi}{\omega_2}, 1), \cdots)$$

If this LCM does not exist, that is, if the period is irrational, $x[n]$ will be aperiodic.

Rate of Oscillation

> *Note*: Just increasing ω does not continuously increase the rate of oscillation of $x[n]$.
> For example, increasing it by 2π would produce an identical signal.

Note: Due to the previous note, we can conclude that $x[n]$ is not necessarily unique as a function of ω. Due to this, ω is usually restricted to intervals of length 2π.

When $\omega = k\pi$ with $k = 2n + 1 | n \in \mathbb{Z}^+$ will always be $(-1)^n$.