Numerical Computation - Assignment 6

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Q1.

The number of flops to create L and U is equal to Gaussian Elimination is:

$$2\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 = 2\left(\frac{1}{6}n(n+1)(2n+1)\right) - \frac{1}{2}n(n+1) - n$$

$$= \left(\frac{2}{3}n - \frac{1}{6}\right)n(n+1) - n = \frac{2}{3}n^3 + \frac{2}{3}n^2 - \frac{1}{6}n^2 - \frac{1}{6}n - n$$

$$= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n \Rightarrow \frac{2}{3}n^3$$

The number of flops for forward substitution is:

$$0 + 2 + 4 + \dots + 2*(n-1) = 2*(1 + 2 + \dots + n - 1)$$

$$= 2 * \frac{1}{2}(n-1)(n-2) = n^2 - 3n + 2 \Rightarrow n^2$$

The number of flops for backward substitution is:

$$1 + 3 + 5 + \dots + 2n - 1 = \frac{(1+2n-1)n}{2} = n^2 \Rightarrow n^2$$

Solve the first problem: $\frac{2}{3}n^3$

100 problems by LU method: $100 * (n^2 + n^2)$

Build equation:

$$\frac{2}{3}n^3 = 100 * (n^2 + n^2)$$

```
>> syms x
>> X=solve((2/3)*x^3==100*(x^2+x^2),x)

X =

   0
   0
 300
```

$n \neq 0$， so $n = 300$

Q2.

$$||Ax||_\infty = \max_{1 \le i \le n}\left[\sum_{j=1}^{n} \left|a_{ij} * x_j\right|\right] \le \max_{1 \le i \le n}\left[\sum_{j=1}^{n} \left|a_{ij}\right| * |x_j|\right]$$

Expand this expression:

$$\max_{1 \le i \le n}\left[\sum_{j=1}^{n} \left|a_{ij}\right| * |x_j|\right] = \begin{bmatrix} |a_{11}||x_1| + & \cdots & \cdots & + |a_{1n}||x_1| \\ |a_{21}||x_1| + & \cdots & \cdots & + |a_{2n}||x_1| \\ \cdots\cdots & & & \cdots\cdots \\ |a_{n1}||x_1| + & \cdots & \cdots & + |a_{nn}||x_n| \end{bmatrix}$$

Then use $\left||x|\right|_\infty$ to replace the all $|x_j|$, and since $\left||x|\right|_\infty = \max_{1 \le j \le n}|j|$

$$\max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}| * |x_j|\right] \le \begin{bmatrix} (|a_{11}| + |a_{12}| + & \cdots & \cdots & + |a_{1n}|) * ||x||_{\infty} \\ (|a_{21}| + |a_{22}| + & \cdots & \cdots & + |a_{2n}|) * ||x||_{\infty} \\ \ldots\ldots & & & \ldots\ldots \\ (|a_{n1}| + |a_{n2}| + & \cdots & \cdots & + |a_{nn}|) * ||x||_{\infty} \end{bmatrix}$$

So $\max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}| * |x_j|\right] \le \max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}| * ||x||_{\infty}\right]$.

Take the $||x||_{\infty} = 1$:

$$\max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}| * ||x||_{\infty}\right] = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}|\right] = ||A||_{\infty}.$$

Q3.

A is a diagonal matrix = $\begin{bmatrix} 1 & 0 & 0 & & \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ & & & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n} \end{bmatrix}$

L^1 norm: $||A||_1 = \max_{1\le j\le n}\left[\sum_{i=1}^{n}|a_{ij}|\right] = 1$.

L^2 norm: A is a diagonal matrix, then A^TA is also a diagonal matrix. The eigenvalues of A^TA are equal to the element values in this diagonal, so the maximum eigenvalue of A^TA is 1.

$||A||_2 = \sqrt{\lambda_{max}(A^TA)} = 1$.

L^{∞} norm: $||A||_{\infty} = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}|\right] = 1$.

Q4.

(a) Initially, transform the linear system to matrix $A * x = b$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Then use the Gaussian elimination to get the solution:

$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \end{bmatrix}$ by $r_2 - 3r_1$.

By back substitution, $x_2 =- 1, x_1 = 1$. So actual solution $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The approximation solution $x_{\alpha} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, so $error\ \Delta x = x_{\alpha} - x = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

$||x||_{\infty} = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|x_{ij}|\right] = 1$, $||\Delta x||_{\infty} = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|\Delta x_{ij}|\right] = 2$.

And then calculate $b_{\alpha} = A * x_{\alpha} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} * \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $residual\ \Delta b = b_{\alpha} - b = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$.

$||b||_{\infty} = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|b_{ij}|\right] = 7$, $||\Delta b||_{\infty} = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|\Delta b_{ij}|\right] = 6$.

The relative forward error: $\frac{||\Delta x||_\infty}{||x||_\infty} = \frac{2}{1} = 2$

The relative backward error: $\frac{||\Delta b||_\infty}{||b||_\infty} = \frac{6}{7} \approx 0.8571$

The error magnification factor: $\frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\frac{||\Delta x||_\infty}{||x||_\infty}}{\frac{||\Delta b||_\infty}{||b||_\infty}} = \frac{2}{\frac{6}{7}} \approx 2.3333$

(b) Let $M = max\frac{||Ax||_\infty}{||x||_\infty} = \left||A|\right|_\infty = 7,\ m = min\frac{||Ax||_\infty}{||x||_\infty}.$

If $Ax = y$, then $x = A^{-1}y$ and $m = min\frac{||Ax||_\infty}{||x||_\infty} = min\frac{||y||_\infty}{\left||A^{-1}y|\right|_\infty} = \frac{1}{max\frac{\left||A^{-1}y|\right|_\infty}{||y||_\infty}} = \frac{1}{\left||A^{-1}|\right|_\infty}$

$A^{-1} = \begin{bmatrix} -2 & 1 \\ -1.5 & 0.5 \end{bmatrix},\ \left||A^{-1}|\right|_\infty = 3$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$, condition number of $A = k(A) = \frac{M}{m} = \left||A|\right|_\infty * \left||A^{-1}|\right|_\infty = 21$

Q5.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4.001 \end{bmatrix},\ A^{-1} = \begin{bmatrix} 4001 & -2000 \\ 2000 & 1000 \end{bmatrix}$

$\left||A|\right|_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|a_{ij}|\right] = 6.001,\ \left||A^{-1}|\right|_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|A_{ij}^{-1}|\right]$ =6001

Condition number of $A = k(A) = \left||A|\right|_\infty * \left||A^{-1}|\right|_\infty = 6001 * 6.001 = 36012.001.$

(b) $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x_\alpha = \begin{bmatrix} -6000 \\ 3001 \end{bmatrix}$. So $error\ \Delta x = x_\alpha - x = \begin{bmatrix} -6001 \\ 3000 \end{bmatrix}$

$\left||x|\right|_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|x_{ij}|\right] = 1,\ ||\Delta x||_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|\Delta x_{ij}|\right] = 6001$

The relative forward error: $\frac{||\Delta x||_\infty}{||x||_\infty} = \frac{6001}{1} = 6001$

$b = \begin{bmatrix} 3 \\ 6.001 \end{bmatrix}, b_\alpha = A * x_\alpha = \begin{bmatrix} 1 & 2 \\ 2 & 4.001 \end{bmatrix} * \begin{bmatrix} -6000 \\ 3001 \end{bmatrix} = \begin{bmatrix} 2 \\ 7.001 \end{bmatrix}$

So residual $\Delta b = b_\alpha - b = \begin{bmatrix} 2 \\ 7.001 \end{bmatrix} - \begin{bmatrix} 3 \\ 6.001 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\left||b|\right|_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|b_{ij}|\right] = 6.001,\ ||\Delta b||_\infty = \max_{1\le i\le n}\left[\sum_{j=1}^{n}|\Delta b_{ij}|\right] = 1$

The relative backward error: $\frac{||\Delta b||_\infty}{||b||_\infty} = \frac{1}{6.001}$

The error magnification factor: $\frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\frac{||\Delta x||_\infty}{||\mathrm{x}||_\infty}}{\frac{||\Delta b||_\infty}{||\mathrm{b}||_\infty}} = \frac{6001}{\frac{1}{6.001}} = 36012.001$

(c) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4.001 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_\alpha = \begin{bmatrix} 1 - 6001\delta \\ 1 + 3000\delta \end{bmatrix}$, So $error\ \Delta x = x_\alpha - x = \begin{bmatrix} 1 - 6001\delta \\ 1 + 3000\delta \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6001\delta \\ 3000\delta \end{bmatrix}$

$||x||_\infty = \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |x_{ij}|\right] = 1,\ ||\Delta x||_\infty = \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\Delta x_{ij}|\right] = 6001\delta$ since $\delta > 0$.

$b = \begin{bmatrix} 3 \\ 6.001 \end{bmatrix}, b_\alpha = A * x_\alpha = \begin{bmatrix} 1 & 2 \\ 2 & 4.001 \end{bmatrix} * \begin{bmatrix} -6001\delta \\ 3000\delta \end{bmatrix} = \begin{bmatrix} 3 - \delta \\ \delta + 6.001 \end{bmatrix}$

So residual $\Delta b = b_\alpha - b = \begin{bmatrix} 3 - \delta \\ \delta + 6.001 \end{bmatrix} - \begin{bmatrix} 3 \\ 6.001 \end{bmatrix} = \begin{bmatrix} -\delta \\ \delta \end{bmatrix}$.

$||b||_\infty = 6.001,\ ||\Delta b||_\infty = \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |\Delta b_{ij}|\right] = \delta$

The relative forward error: $\frac{||\Delta x||_\infty}{||\mathrm{x}||_\infty} = \frac{2}{1} = 2$

The relative backward error: $\frac{||\Delta b||_\infty}{||\mathrm{b}||_\infty} = \frac{6}{7} \approx 0.8571$

The error magnification factor: $\frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\frac{||\Delta x||_\infty}{||\mathrm{x}||_\infty}}{\frac{||\Delta b||_\infty}{||\mathrm{b}||_\infty}} = \frac{6001\delta}{\frac{\delta}{6.001}} = 36012.001$

According to the (a), the condition number of A is 36012.001 which is equal to the error magnification factor of approximation solution $x_\alpha = \begin{bmatrix} 1 - 6001\delta \\ 1 + 3000\delta \end{bmatrix}$ for any $\delta > 0$.

Q6.

generation.m

```
function x = generation(n)
x=ones(n);
x=-triu(x,1);
x=x+eye(n);


```

Test the function by $n = 10$:

```
>> x = generation(10)

x =

     1    -1    -1    -1    -1    -1    -1    -1    -1    -1
     0     1    -1    -1    -1    -1    -1    -1    -1    -1
     0     0     1    -1    -1    -1    -1    -1    -1    -1
     0     0     0     1    -1    -1    -1    -1    -1    -1
     0     0     0     0     1    -1    -1    -1    -1    -1
     0     0     0     0     0     1    -1    -1    -1    -1
     0     0     0     0     0     0     1    -1    -1    -1
     0     0     0     0     0     0     0     1    -1    -1
     0     0     0     0     0     0     0     0     1    -1
     0     0     0     0     0     0     0     0     0     1
```

$$A = \begin{bmatrix} 1 & -1 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \dots & \dots & \ddots & \dots \\ 0 & \dots & 0 & 1 \end{bmatrix}, \ a_{ij} = \begin{cases} -1 & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}$$

When $n = 1$: $A = [1], A^{-1} = [1]$

When $n = 2$: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A^{-1} = [1]$

When $n = 3$: $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

When $n = 4$: $A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

…

By induction, we can find that the inverse matrix of A= $A^{-1} = \begin{bmatrix} 1 & 2^0 & 2^1 & \dots & 2^{n-2} \\ 0 & 1 & 2^0 & \dots & 2^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 2^0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$||A||_1 = \max_{1 \le j \le n} \left[\sum_{i=1}^{n} |a_{ij}|\right] = n * 1 = n$

Since $2^{n-1} + 2^{n-2} + \dots + 2^1 + 2^0 = 2^n - 1$, $||A^{-1}||_1 = \max_{1 \le j \le n} \left[\sum_{i=1}^{n} |A_{ij}^{-1}|\right] = (2^{n-2} + 2^{n-1} + \dots + 2^0) + 1 = 2^{n-2+1} = 2^{n-1}$.

Condition number of $A = k(A) = ||A||_1 * ||A^{-1}||_1 = n * 2^{n-1}$.