4.

1. $f(x) = 1/x \Rightarrow f'(x) = -\frac{1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$, $\quad f'''(x) = -\frac{6}{x^4}$

$f^{(n-1)}(x) = \frac{(n-1)!}{x^n}(-1)^{n-1}$ $\quad f^{(n)}(x) = \frac{n!}{x^{n+1}}(-1)^n$

So the Taylor Series generated by $f(x)$ at $a=2$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \frac{f^{(n)}(\xi)(x-a)^n}{(n)!} \quad (a \le \xi \le x)$$

$$= \frac{1}{2} + (-\tfrac{1}{4})\times(x-2) + \frac{\frac{2!}{8}\times(x-2)^2}{2!} + \cdots + \frac{\frac{(n-1)!}{2^n}\times(x-2)^{n-1}\times(-1)^{n-1}}{(n-1)!} + \frac{\frac{n!}{\xi^{n+1}}\times(x-2)^n\times(-1)^n}{(n)!}$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{2} + \frac{1}{8}(x-2)^2 + \cdots + \frac{(-1)^{n-1}\times(x-2)^{n-1}}{2^n} + \frac{(-1)^n\times(x-2)^n}{\xi^{n+1}}$$

$$= 1 - \frac{1}{4}x + \frac{1}{8}(x-2)^2 + \cdots + \frac{(-1)^{n-1}(x-2)^{n-1}}{2^n} + \frac{(-1)^n(\xi-2)^n}{2^{n+1}}$$

$(2 \le \xi \le x)$

2. $f(x) = \cos(x)$

since $f'(x) = -\sin(x)$ $\quad f''(x) = -\cos(x)$ $\quad f'''(x) = \sin(x)$

$f^{(4)}(x) = \cos(x)$ $\quad f^{(5)}(x) = -\sin(x)$

So the Taylor polynomial of $f(x)$ about $x=0$ at order 5:

$$p(x) = \cos(0) - \sin(0)\cdot(x-0) - \frac{\cos(0)(x-0)^2}{2!} + \frac{\sin(0)(x-0)^3}{3!} + \frac{\cos(0)(x-0)^4}{4!} + \frac{-\sin(0)\cdot(x-0)^5}{5!}$$

$$= 1 - 0 - \frac{x^2}{2} + \frac{x^4}{60} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

3. $f(x) = x\cdot e^x \Rightarrow f'(x) = e^x + x\cdot e^x$. $f^{(2)}(x) = e^x + e^x + xe^x$.

$f^{(3)}(x) = (3+x)e^x$ $\quad f^{(n-1)}(x) = (n-1+x)e^x$

So the Maclaurin series $(a=0)$ for $f(x) = x\cdot e^x$:

$$f(x) = 0e^0 + (1+0)\cdot e^0 \times (x-0) + \frac{(2+0)e^0 \times (x-0)^2}{2!} + \cdots + \frac{(n-1+0)\cdot e^0\cdot(x-0)^{n-1}}{(n-1)!}$$

$$+ \frac{(n+0)\cdot e^0\cdot(\xi-0)^n}{(n)!} = x + x^2 + \cdots + \frac{x^{n-1}}{(n-2)!} + \frac{\xi^n}{(n-1)!}$$

4. $f(x) = \ln x$

since $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $a = 1$

① at order 0 : $P(x) = f(a) = \ln a = 0$

② at order 1 : $P(x) = f(a) + f'(a)(x-a) = \ln a + \frac{1}{a}(x-a) = x-1$

③ at order 2 : $P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} = \ln a + \frac{1}{a}(x-a) + \frac{-\frac{1}{a^2}(x-a)^2}{2!}$

$= (x-1) - \frac{(x-1)^2}{2}$

④ at order 3 : $P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

$= \ln a + \frac{1}{a}(x-a) + \frac{-\frac{1}{a^2}(x-a)^2}{2!} + \frac{\frac{2}{a^3}(x-a)^3}{3!}$

$= (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

So when $x = 1.5$, ① $= 0$ ② 0.5 ③ $= 0.375$ ④ $= 0.41667$

and the real value ≈ 0.4055.

5. Taylor Series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

$$f^{(n)}(x) = f^{(n)}(a) + f^{(n+1)}(a)(x-a) + \frac{f^{(n+2)}(a)(x-a)^2}{2!} + \cdots + \frac{f^{(2n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \frac{f^{(2n)}(a)(\xi-a)^n}{n!}$$

when $x=a$, $f^{(n)}(a) + 0 + \cdots 0 = f^{(n)}(a)$

$$p(x) = \sum_{k=0}^{n} \frac{f^{k}(a)(x-a)^k}{k!}$$, first n derivatives at $x=a$.

$$= f^{(n)}(a).$$

So the Taylor polynomial of order n and its first n derivatives have the same value.

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