Assignment 2 - 1930026143

1. We know that $\varepsilon = 0.2$ and $L_0 = x_s - x_t = 5 - (-3) = 8$.
   For Exhaustive Search Method:
   $$N > \frac{2 * L_0}{\varepsilon} = \frac{16}{0.2} - 1 = 79$$
   Thus, there are 80 function values at least should be calculated in this case.

   For Golden Section Method:
   $$N \geq \log_\tau\left(\frac{\varepsilon}{L_0}\right) \; where\ \tau \approx 0.618\ (golden\ ratio)$$
   After computing, we can get $N \geq 7.6649$. Since N must be an integer, there are 8 function values at least should be calculated in this case.

2. $f(x) = x^2 + 2x + 1$. We can calculate and determine the case step by step.
   Step1: $\bar{x} = x_s + \tau L_0 = 1.944$
   $\hat{x} = x_s + (1 - \tau)L_0 = 0.056$
   $f(\bar{x}) = 1.944^2 + 2 * 1.944 + 1 = 8.6671$
   $f(\hat{x}) = 0.056^2 + 2 * 0.056 + 1 = 1.1151$
   Since $f(\hat{x}) > f(\hat{x})$, the interval $[x_s, x_t] \to [x_s, \bar{x}]$, which means that $x_t^1 = \bar{x}$.
   According to the question (1), there are have 8 steps in the case that the length of the final interval of uncertainty needs to be less than 0.2. So we can solve this problem by coding:

In [1]:
```python
def fx(x):
    return x**2 + 2*x + 1

def Golden_Section_Method(tol):
    x_s = -3 # x_s
    x_t = 5  # x_t
    cnt = 0  # Record the number of iterations
    while (1):
        cnt += 1
        x_bar = x_s + 0.618 * (x_t-x_s)
        x_hat = x_s + (1-0.618) * (x_t-x_s)
        # detemine the update condition
        if fx(x_bar) > fx(x_hat):
            x_t = x_bar
        else:
            x_s = x_hat
        L = abs(x_s - x_t)
        if L <= tol:
            print(x_s, x_t)
            return x_s, x_t, cnt
            break
x_s, x_t, cnt = Golden_Section_Method(0.2)
# Show the number of iterations and the result
cnt, fx((x_s+x_t)/2)
```

```
-1.1115903533668563 -0.941375369605261
```

Out[1]:
```
(8, 0.0007013419524897202)
```

As we can see that the final interval is $(-1.1116, -0.9414)$ and there are 8 function values at least should be calculated. And the minimum point of function is

0.0070134.

3. Let $g(x) = x^4 - 1$, we should minimize it by using the Newton's Method, which means that we should transform the "find the minimum problem" to "Zero problem" by the first derivative. $g^{'}(x) = 4x^3$ and $g^{''}(x) = 12x^2$.
   According to Newton's Method, value update formula as follow:

$$x_{k+1} = x_k - \frac{f(x_k)}{f^{'}(x_k)}$$

   In this case, $f(x) = g'(x)$, $x_0 = 4$

   Step1: $x_1 = x_0 - \frac{4x_0^3}{12x_0^2} = 4 - \frac{156}{192} = 2.67$

   $g(2.67) = 2.67^4 - 1 \approx 49.8212$, $f(2.67) = 2.67^3 * 4 \approx 76.1367$

   Step2: $x_2 = x_1 - \frac{4x_1^3}{12x_1^2} = 2.67 - \frac{76.1367}{185.5468} = 1.78$

   $g(1.78) = 1.78^4 - 1 \approx 9.0388$, $f(1.78) = 1.78^3 * 4 \approx 22.5590$

   Step3: $x_3 = x_2 - \frac{4x_2^3}{12x_2^2} = 1.78 - \frac{22.5590}{38.0208} = 0.9750$

   $g(0.975) = 0.975^4 - 1 \approx - 0.0963$.

4. Let $g(x) = x^4 - 1$, we should minimize it by using the Scant Method, which means that we should transform the "find the minimum problem" to "Zero problem" by the first derivative. $g^{'}(x) = 4x^3$ and $g^{''}(x) = 12x^2$.
   According to Newton's Method, value update formula as follow:

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f^{'}(x_k) - f(x_{k-1})} * f'(x_k)$$

   In this case, $f(x) = g'(x)$, $x_0 = 4, x_{-1} = 6$.

   1: $x_1 = x_0 - \frac{x_0 - x_{-1}}{f^{'}(x_0) - f(x_{-1})} f^{'}(x_0) = 4 - \frac{4-6}{256-864} * 256 \approx 3.1579$

   $f(3.1579) = 3.1579^3 * 4 \approx 125.9785$

   2: $x_2 = x_1 - \frac{x_1 - x_0}{f^{'}(x_1) - f(x_0)} f^{'}(x_1) = 3.1579 - \frac{3.1579-4}{125.9785-256} 125.9785 \approx 2.3422$

   $f(2.3422) = 2.3422^3 * 4 \approx 51.3963$

   3: $x_3 = x_2 - \frac{x_2 - x_1}{f^{'}(x_2) - f(x_1)} f^{'}(x_2) = 2.3422 - \frac{2.3422-3.1579}{51.3963-125.9785} 51.3963 \approx 1.7801$

   $g(1.7801) \approx 9.0410$

5. We know that the general term $x_k = 1 + 5 * 10^{-2k}$, so $x_{k+1} = 1 + 5 * 10^{-2k-2}$.

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} 1 + 5 * 10^{-2k} = 1 = (x^*)$$

Thus, when $p = 1$ we have:

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = \lim_{k\to\infty} \left| \frac{5 * 10^{-2k-2}}{5 * 10^{-2k}} \right| = \lim_{k\to\infty} 10^{-2} = 0.02$$

Therefore, the general term is linear convergence and the order of convergence is 1.

(b) We know that the general term $x_{k+1} = \frac{x_k}{2} + \frac{2}{x_k}$ with the $x_0 = 4$.

In this case, it is difficult to find the limit of x_k. But we know that as k goes to infinity, x_k tends to the smallest. Since we can

$$\left(\frac{x_k}{2} + \frac{2}{x_k}\right)^2 = (\frac{4}{{x_k}^2} + x_k^2) + 2$$

which has the minimum point when $x_k = 2$. Thus $x^* = 2$ and let $p = 2$, we have

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = \lim_{k\to\infty} \frac{\left|\frac{x_k}{2} + \frac{2}{x_k} - 2\right|}{|x_k - 2|^2} = \lim_{k\to\infty} \left| \frac{\frac{x_k^2 + 4 - 4x_k}{2x_k}}{(x_k - 2)^2} \right| = \lim_{k\to\infty} \left(\frac{1}{2x_k}\right) = \frac{1}{4}$$

Therefore, the convergence is quadratic and the order of convergence is 2.

6. We know that the sequence $y_k = cx_k$ where x_k converges to x^* with order p and $c \neq 0$. For sequence x_k, it satisfies:

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = \beta$$

Thus, we have:

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} cx_k = c \lim_{k\to\infty} x_k = cx^* = y^*$$

$$\lim_{k\to\infty} \frac{|y_{k+1} - y^*|}{|y_k - y^*|^p} = \lim_{k\to\infty} \left| \frac{cx_{k+1} - cx^*}{(cx_k - cx^*)^p} \right| = c \lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = c\beta$$

which means that for the sequence y_k, the order of convergence of it is also equal to p then it can converge.