DS4003 Optimization Method
Assignment 1 — 2022 Spring

1. Express $(2,2)^T$ as a convex combination of $(0,0)^T, (1,4)^T$, and $(3,1)^T$.

2. Let f be a convex function on a convex set $S \subseteq R^n$. Let k be a non-zero scalar; and define $g(x) = kf(x)$. Prove that if $k > 0$ then g is a convex function on S; and if $k < 0$ then g is a concave function on S.

3. Let g be a concave function; and let f be a convex function. Let both g and f be defined on R^n; and let μ be a positive-valued constant. Prove that the function $\beta(x) = f(x) - \mu \log(g(x))$ is convex on the set $S = \{x : g(x) > 0\}$.

4. Let $\mathbf{x} \in R^n$, and $f(x) = ||\mathbf{x}||_2^2 = \mathbf{x}^T\mathbf{x}$. Calculate the gradient of f.

5. Consider the function $f(x) = x_1^2 + x_2^2 + 2x_3^2 - x_1x_2 - x_2x_3 - x_1x_3, x \in R^3$
   (a) Write the function into the form $f(x) = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3)^T$ and A is a 3×3 matrix.
   (b) Find the $\nabla f(x)$.
   (c) Find the $\nabla^2 f(x)$.
   (d) Test the convexity of function $f(x)$.

6. Let $A \in R^{m\times n}, \mathbf{x}, \mathbf{b} \in R, Q(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|_2^2$.
   (a) Find the gradient of $Q(\mathbf{x})$.
   (b) When there is a unique stationary point for $Q(\mathbf{x})$. (Hint: stationary point is where gradient equals to zero)

7. Take the symmetric matric of order 2,

$$A = \begin{pmatrix} 4 & \alpha \\ \alpha & 2 \end{pmatrix}$$

with α a real parameter. Determine when the matrix A is positive definite, positive semi-definite and indefinite.