Q1.

(1)

$l(\beta_0, \beta_1) = y \ln p(x) + (1-y)\ln(1-p(x))$ where $p(x) = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}}$

Thus,

$$l(\beta_0, \beta_1) =- y\ln\left(1+e^{-(\beta_0+\beta_1 x)}\right) + (1-y)\ln\left(\frac{e^{-(\beta_0+\beta_1 x)}}{1+e^{-(\beta_0+\beta_1 x)}}\right)$$

$$=- y\ln\left(1+e^{-(\beta_0+\beta_1 x)}\right) + (1-y)[\ln(e^{-(\beta_0+\beta_1 x)}) - \ln(1+e^{-(\beta_0+\beta_1 x)})]$$

$$=- y\ln\left(1+e^{-(\beta_0+\beta_1 x)}\right) + (y-1)\ln\left(1+e^{-(\beta_0+\beta_1 x)}\right) - (1-y)\ln(e^{-(\beta_0+\beta_1 x)})$$

$$=- \ln\left(1+e^{-(\beta_0+\beta_1 x)}\right) + (y-1)(\beta_0+\beta_1 x)$$

Take the partial derivative of β_0:

$$\frac{\partial l}{\partial \beta_0} = (y-1) - \frac{-e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} = y - \frac{1+e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} + \frac{e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)}$$

$$= y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} = y - p(x)$$

Take the partial derivative of β_1:

$$\frac{\partial l}{\partial \beta_1} = (y-1)x - \frac{-xe^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} = x(y - \frac{1+e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} + \frac{e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)})$$

$$= \left(y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)}\right)x = [y-p(x)]x$$

To conclude, $\nabla l = \begin{bmatrix} y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)} \\ \left(y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)}\right)x \end{bmatrix} = \begin{bmatrix} y-p(x) \\ [y-p(x)]x \end{bmatrix}$

(2)

From the question (1), we know the partial derivative of β_0 and β_1, where:

$$\frac{\partial l}{\partial \beta_0} = y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)}$$

$$\frac{\partial l}{\partial \beta_1} = \left(y - \frac{1}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)}\right)x$$

And $\nabla^2 l = \begin{bmatrix} \frac{\partial^2 l}{\partial^2 \beta_0} & \frac{\partial^2 l}{\partial \beta_0 \beta_1} \\ \frac{\partial^2 l}{\partial \beta_0 \beta_1} & \frac{\partial^2 l}{\partial^2 \beta_1} \end{bmatrix}$, thus we can get the $\nabla^2 l$:

$$\frac{\partial^2 l}{\partial^2 \beta_0} =- \frac{-e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)^2} =- p(x)(1-p(x))$$

$$\frac{\partial^2 l}{\partial^2 \beta_1} =- \frac{-x^2 e^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)^2} =- x^2 p(x)(1-p(x))$$

$$\frac{\partial^2 l}{\partial \beta_0 \beta_1} = -\frac{-xe^{-(\beta_0+\beta_1 x)}}{\left(1+e^{-(\beta_0+\beta_1 x)}\right)^2} = -x\,p(x)(1-p(x))$$

To conclude, $\nabla^2 l = \begin{bmatrix} \frac{\partial^2 l}{\partial^2 \beta_0} & \frac{\partial^2 l}{\partial \beta_0 \beta_1} \\ \frac{\partial^2 l}{\partial \beta_0 \beta_1} & \frac{\partial^2 l}{\partial^2 \beta_1} \end{bmatrix} = \begin{bmatrix} -p(x)(1-p(x)) & -x\,p(x)(1-p(x)) \\ -x\,p(x)(1-p(x)) & -x^2 p(x)(1-p(x)) \end{bmatrix}$