STAT2013 Regression Analysis
Assignment 2

1. An experiment was performed on a certain metal to determine if the strength is a function of heating time. Results based on 10 metal sheets are given below. Use the simple linear regression model.

$$\sum \mathbf{x} = 30, \quad \sum \mathbf{x^2} = 104, \quad \sum \mathbf{y} = 40$$

$$\sum \mathbf{y^2} = 178, \quad \sum \mathbf{xy} = 134$$

a) Find the estimated intercept and slope and write the equation of the least squares regression line.

b) Provide the interpretation of $\hat{\beta}_0$ and $\hat{\beta}_1$, the least square point estimations of the intercept and the slope.

c) Estimate σ, the standard deviation of the random error term (standard error of the estimate).

2. Let X and Y be two random variables with $E(X) = 1$, $E(Y) = 2$, $\text{Var}(X) = 3, \text{Var}(Y) = 4$ and $\text{Cov}(X, Y) = 2$. Let $Z = 2X + Y$ and $W = X - 2Y$. Answer the following questions:

a) Find mean and variance of Z and W, respectively.

b) Find $\text{Cov}(X, Z), \text{Cov}(W, Y)$, and $\text{Cov}(Z, W)$.

c) Find correlation coefficient between of X and Y.

d) Find correlation coefficient between of Z and W.

3. Assume that the random vector $\boldsymbol{x} = (X_1, X_2, X_3, X_4)'$ has mean vector and covariance matrix as follows

$$E(\boldsymbol{x}) = \begin{bmatrix} 1.1 \\ 2.3 \\ 3.2 \\ 1.7 \end{bmatrix}, \quad \boldsymbol{S} = \mathrm{Cov}(\boldsymbol{x}) = \begin{bmatrix} 1.0 & 0.5 & 0.4 & 0.3 \\ 0.5 & 2.0 & 0.5 & 0.4 \\ 0.4 & 0.5 & 3.0 & 0.6 \\ 0.3 & 0.4 & 0.6 & 1.5 \end{bmatrix}.$$

a) Find $E(\boldsymbol{x}_1)$ and $\mathrm{Cov}(\boldsymbol{x}_1)$, where $\boldsymbol{x}_1 = (X_1, X_3)'$.

b) Find $\mathrm{Cov}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ where $\boldsymbol{x}_1$ is given in a) and $\boldsymbol{x}_2 = (X_2, X_4)'$.

c) Let $Z_1 = X_1 + 2X_2 + 3X_3 + 4X_4$, $Z_2 = 4X_1 - 3X_2 - 2X_3 + X_4$, and $\boldsymbol{z} = (Z_1, Z_2)'$. Find $E(\boldsymbol{z})$ and $\mathrm{Cov}(\boldsymbol{z})$.

d) Find $\mathrm{Cov}(X_1 - 2X_2, X_2 - 3X_3 + 1.5X_4)$.

4. The matrix

$$H = X(X'X)^{-1}X'$$

is usually called the hat matrix because it maps Y to $\hat{Y}$ the vector of fitted values.

a) Show that in the linear regression model

$$Var(\hat{Y}) = \sigma^2 H$$

b) Prove that the matrices H and $I - H$ are idempotent, that is

$$H^2 = H$$

$$(I - H)^2 = (I - H)$$