expanding truncation incorporating with analysis along convergent subsequences (Trajectory-Subsequence Method) plays an important role.

3) Although the conditions used for guaranteeing optimality of the adaptive regulator are not complicated, the open-loop stability is a restrictive condition.

For further research there are many problems of interest, e.g., the multidimensional systems, the robustness issue, the problem of adaptive tracking and other performance indices. It is also of interest to consider $f(\cdot)$ without any growth rate restriction and to speed up the convergence rate of algorithms.

ACKNOWLEDGMENT

The author would like to thank Mr. W. Zhao for his numerical simulation.

REFERENCES

[1] K. J. Åström and B. Wittenmark, "On self-tuning regulator," *Automatica*, vol. 9, pp. 195–199, 1973.

[2] E. W. Bai, "Frequency domain identification of Hammerstein models," *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 530–542, Apr. 2003.

[3] E. W. Bai and D. Li., "Convergence of the iterative Hammerstein system identification algorithm," *IEEE Trans. Autom. Control*, vol. 49, no. 11, pp. 1929–1940, Nov. 2004.

[4] P. Celka, N. J. Bershad, and J. M. Vesin, "Stochastic gradient identification of polynomial Wiener systems: Analysis and application," *IEEE Trans. Signal Process.*, vol. 49, no. 2, pp. 301–313, Feb. 2001.

[5] H. F. Chen, *Stochastic Approximation and Its Applications*. Dordrecht, The Netherlands: Kluwer, 2002.

[6] ——, "Pathwise convergence of recursive identification algorithms for Hammerstein systems," *IEEE Trans. Autom. Control*, vol. 49, no. 10, pp. 1641–1649, Oct. 2004.

[7] H. F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*. Boston, MA: Birkhäuser, 1991.

[8] Y. S. Chow and H. Teicher, *Probability Theory*. New York: Springer-Verlag, 1978.

[9] R. C. Emerson, M. J. Korenberg, and M. C. Citron, "Identification of complex-cell intensive nonlinearities in a cascade model of cat visual cortex," *Bio. Cybern.*, vol. 66, pp. 291–300, 1992.

[10] E. Eskinat and S. H. Johwon, "Use of Hammerstein models in identification of nonlinear systems," *Amer. Inst. Chem. Eng.*, vol. 37, no. 2, pp. 255–268, 1991.

[11] W. Greblicki, "Stochastic approximation in nonparametric, identification of Hammerstein systems," *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1800–1810, Nov. 2002.

[12] L. Guo and H. F. Chen, "Åström-Wittenmark self-tuning regulator revisited and ELS-based adaptive trackers," *IEEE Trans. Autom. Control*, vol. 36, no. 7, pp. 802–812, Jul. 1991.

[13] P. R. Kumar and P. Varaiya, *Stochastic Systems: Estimation, Identification and Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall, 1986.

[14] M. Pawlak, "On the series expansion approach to the identification of Hammerstein system," *IEEE Trans. Autom. Control*, vol. 36, no. 6, pp. 763–767, Jun. 1991.

[15] T. Wigren, "Convergence analysis of recursive algorithms based on the nonlinear Wiener model," *IEEE Trans. Autom. Control*, vol. 39, no. 11, pp. 2191–2206, Nov. 1994.

[16] J. Vörös, "Recursive identification of Wiener systems with discontinuous nonlinearities," *Syst. Control Lett.*, vol. 44, pp. 363–372, 2001.

[17] W. Wang and R. Henrikson, "Generalized predictive control of nonlinear systems of the Hammerstein form," *Model. Ident. Control*, vol. 15, no. 4, pp. 253–262, 1994.

[18] D. T. Westwick and R. E. Kearney, "A new algorithm for the identification of multiple input Wiener systems," *Bio. Cybern.*, vol. 68, pp. 75–85, 1992.

On the Discrete-Time Integral Sliding-Mode Control

Khalid Abidi, Jian-Xin Xu, and Yu Xinghuo

***Abstract*—A new discrete-time integral sliding-mode control (DISMC) scheme is proposed for sampled-data systems. The new control scheme is characterized by a discrete-time integral sliding manifold which inherits the desired properties of the continuous-time integral sliding manifold, such as full order sliding manifold with pole assignment, and elimination of the reaching phase. In particular, comparing with existing discrete-time sliding-mode control, the new scheme is able to achieve more precise tracking performance. It will be shown in this work that, the new control scheme achieves $O(T^2)$ steady-state error for state regulation with the widely adopted delay-based disturbance estimation. Another desirable feature is, the proposed DISMC prevents the generation of overlarge control actions due to deadbeat response, which is usually inevitable due to the existence of poles at the origin for a reduced order sliding manifold designed for sampled-data systems. Both the theoretical analysis and illustrative example demonstrate the validity of the proposed scheme.**

***Index Terms*—Discrete-time controller, integral sliding-mode control, sampled-data system.**

I. INTRODUCTION

Research in discrete-time control has been intensified in recent years. A primary reason is that most control strategies nowadays are implemented in discrete-time. This also necessitated a rework in the sliding-mode control (SMC) strategy for sampled-data systems [1]–[4]. In such systems, the switching frequency in control variables is limited by T^{-1}; where T is the sampling period. This has led researchers to approach discrete-time sliding-mode control from two directions. The first is the emulation that focuses on how to map continuous-time SMC to discrete-time, and the switching term can be preserved [3], [4]. The second is based on the equivalent control design and disturbance observer [1], [2]. In the former, although high-frequency switching is theoretically desirable from the robustness point of view, it is usually hard to achieve in practice because of physical constraints, such as processor computational speed, A/D and D/A conversion delays, actuator bandwidth, etc. The use of a discontinuous control law in a sampled-data system will bring about chattering phenomenon in the vicinity of the sliding manifold, hence lead to a boundary layer with thickness $O(T)$, [1].

The effort to eliminate the chattering has been paid over 30 years. In continuous-time SMC, smoothing schemes such as boundary layer (saturator) are widely used, which in fact results in a continuous nonlinear feedback instead of switching control. Nevertheless, it is widely accepted by the community that this class of controllers can still be regarded as SMC. Similarly, in discrete-time SMC, by introducing a continuous control law, chattering can be eliminated. In such circumstance, the central issue is to guarantee the precision bound or the smallness of the error.

Manuscript received May 10, 2005; revised November 4, 2005, March 20, 2006, August 22, 2006, and November 20, 2006. Recommended by Associate Editor I. Kolmanovsky

K. Abidi and J.-X. Xu are with the National University of Singapore, Singapore 119260, Singapore (e-mail: kabidi@nus.edu.sg; elexujx@nus.edu.sg).

Y. Xinghuo is with the School of Electrical and Computer Engineering, Royal Melbourne Institute of Technology, Melbourne 3000, Australia (e-mail: x.yu@rmit.edu.au).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2007.894537

In [2] a discrete-time equivalent control was proposed. This approach results in the motion in $O(T^2)$ vicinity of the sliding manifold. The main difficulty in the implementation of this control law is that we need to know the disturbances for calculating the equivalent control. Lack of such information leads to an $O(T)$ error boundary.

The control proposed in [1] drives the sliding-mode to $O(T^2)$ in one-step owing to the deadbeat nature of the closed-loop system (due to poles at the origin). State regulation was not considered in [1]. In fact, as far as the state regulation is concerned, the same SMC design will produce an accuracy in $O(T)$ instead of $O(T^2)$ boundary. Moreover, the SMC with deadbeat response requires large control efforts that might be undesirable in practice. Introducing saturation in the control input endangers the global stability or accuracy of the closed-loop system. In [5] a second-order SMC designed in continuous-time is applied to a sampled-data system resulting in a worst case error of order $O(T)$ which is at the same level of that of first-order SMC [1], [2].

In this work, aiming at improving control performance for sampled-data systems, a discrete-time integral sliding manifold (ISM) is proposed. With the full control of the system closed-loop poles and the elimination of the reaching phase, like the continuous-time integral sliding-mode control [6]–[8], the closed-loop system can achieve the desired control performance while avoiding the generation of overly large control inputs. It is worth highlighting that the discrete-time ISM control does not only drive the sliding-mode into the $O(T^2)$ boundary, but also achieve the $O(T^2)$ boundary for state regulation.

In this work, pole assignment of the full-order sliding-mode, as well as the closed-loop dynamics in the sliding motion, will be discussed.

The note is organized as follows: The problem formulation and a revisit of the existing SMC properties in sampled-data systems are presented in Section II. Appropriate discrete-time integral sliding manifold and SMC design for state regulation will be presented in Section III. An illustrative example demonstrating the validity of the proposed scheme is shown in Section IV. Section V gives the conclusions.

II. Problem Formulation

A. Sampled-Data System

Consider the following continuous-time system with a nominal linear time invariant model and matched disturbance

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B(\mathbf{u}(t) + \mathbf{f}(t)) \tag{1}$$

where the state $\mathbf{x} \in \Re^n$, the control $\mathbf{u} \in \Re^m$, and the disturbance $\mathbf{f} \in \Re^m$ is assumed smooth and bounded. The discretized counterpart of (1) can be given by

$$\mathbf{x}_{k+1} = \Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{2}$$

where

$$\Phi = e^{AT}, \quad \Gamma = \int_0^T e^{A\tau} d\tau B$$

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau) d\tau$$

and T is the sampling period. Here, the disturbance $\mathbf{d}_k$ represents the influence accumulated from kT to $(k+1)T$, in the sequel it shall directly link to $\mathbf{x}_{k+1} = \mathbf{x}((k+1)T)$. From the definition of Γ, it can be shown that

$$\begin{aligned}\Gamma &= BT + \frac{1}{2!}ABT^2 + \cdots = BT + MT^2 + O(T^3)\\ &\Rightarrow BT = \Gamma - MT^2 + O(T^3)\end{aligned} \tag{3}$$

where M is a constant matrix because T is fixed. From (3), it can be concluded that the magnitude of Γ is $O(T)$.

The control objective is to design a discrete-time integral sliding manifold and a discrete-time SMC law that will stabilize the sampled-data system (2) and achieve as precisely as possible state regulation. Meanwhile the closed-loop dynamics of the sampled-data system has all its closed-loop poles assigned to desired locations.

Remark 1: The smoothness assumption made on the disturbance is to ensure that the disturbance bandwidth is sufficiently lower than the controller bandwidth, or the ignorance of high frequency components does not significantly affect the control performance. Indeed, if a disturbance has frequencies nearby or higher than the Nyquist frequency, for instance a non-smooth disturbance, a discrete-time SMC will not be able to handle it.

Note that, as a consequence of sampling, the disturbance originally matched in continuous-time will contain mismatched components in the sampled-data system. This is summarized in the following lemma.

Lemma 1: If the disturbance $\mathbf{f}(t)$ in (1) is bounded and smooth, then

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau) d\tau = \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_k T + O(T^3) \tag{4}$$

where $\mathbf{v}_k = \mathbf{v}(kT)$, $\mathbf{v}(t) = (d/dt)\mathbf{f}(t)$, $\mathbf{d}_k - \mathbf{d}_{k-1} = O(T^2)$, and $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = O(T^3)$.

Proof: See the Appendix .

Note that the magnitude of the mismatched part in the disturbance $\mathbf{d}_k$ is of the order $O(T^3)$

B. Discrete-Time Sliding-Mode Control Revisited

Consider the well established discrete-time sliding manifold [1], [2] shown below

$$\boldsymbol{\sigma}_k = D\mathbf{x}_k \tag{5}$$

where $\boldsymbol{\sigma} \in \Re^m$ and D is a constant matrix of rank m. D is chosen such that $D\Gamma$ is invertible, and the matrix $(\Phi - \Gamma(D\Gamma)^{-1}D\Phi)$ has m zero poles and $n-m$ poles inside the unit disk in the complex z-*plane*. The objective is to steer the states towards and force them to stay on the sliding manifold $\boldsymbol{\sigma}_k = 0$ at every sampling instant. The control accuracy of this class of sampled-data SMC is given by the following lemma.

Lemma 2: With $\boldsymbol{\sigma}_k = D\mathbf{x}_k$ and equivalent control based on a disturbance estimate

$$\hat{\mathbf{d}}_k = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1}$$

then the steady state response of the closed-loop system is specified by an ultimate bound $O(T)$.

Proof: Discrete-time equivalent control is defined by solving $\boldsymbol{\sigma}_{k+1} = 0$, [1]. This leads to

$$\mathbf{u}_k^{eq} = -(D\Gamma)^{-1}D(\Phi\mathbf{x}_k + \mathbf{d}_k) \tag{6}$$

with D selected such that $D\Gamma$ is invertible. Under practical considerations, the control cannot be implemented in the same form as in (6) because of the lack of prior knowledge regarding the discretized disturbance $\mathbf{d}_k$. However, with some continuity assumptions on the disturbance, $\mathbf{d}_k$ can be estimated by its previous value $\mathbf{d}_{k-1}$, [1]. The substitution of $\mathbf{d}_k$ by $\mathbf{d}_{k-1}$ will at most result in an error of $O(T^2)$. With reasonably small sampling interval as in motion control or mechatronics, such a substitution will be effective. Let

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1} \tag{7}$$

where $\hat{\mathbf{d}}_k$ is the estimate of $\mathbf{d}_k$. Thus, analogous to the equivalent control law (6), the practical control law is

$$\mathbf{u}_k = -(D\Gamma)^{-1}D(\Phi\mathbf{x}_k + \mathbf{d}_{k-1}). \tag{8}$$

Substituting the sampled-data dynamics (2), applying the above control law, and using the conclusions in *Lemma* 1, yield

$$\boldsymbol{\sigma}_{k+1} = D(\Phi \mathbf{x}_k + \Gamma \mathbf{u}_k + \mathbf{d}_k) = D(\mathbf{d}_k - \mathbf{d}_{k-1}) = O(T^2) \quad (9)$$

which is the result shown in [1]. The closed-loop dynamics is

$$\begin{aligned}\mathbf{x}_{k+1} &= \left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)\mathbf{x}_k \\ &\quad + \left(I - \Gamma(D\Gamma)^{-1}D\right)\mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1} \quad (10)\end{aligned}$$

where the matrix $(\Phi - \Gamma(D\Gamma)^{-1}D\Phi)$ has m zero poles and $n-m$ poles to be assigned inside the unit disk in the complex z-*plane*. It is possible to simplify (10) further to

$$\mathbf{x}_{k+1} = \left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)\mathbf{x}_k + \boldsymbol{\delta}_k \quad (11)$$

where $\boldsymbol{\delta}_k = \left(I - \Gamma(D\Gamma)^{-1}D\right)\mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1}$. From Lemma 1

$$\begin{aligned}\boldsymbol{\delta}_k &= \mathbf{d}_k - \mathbf{d}_{k-1} + (I - \Gamma(D\Gamma)^{-1}D) \\ &\quad \times \left(\Gamma \mathbf{f}_{k-1} + \frac{1}{2}\Gamma \mathbf{v}_{k-1}T + O(T^3)\right) \\ &= O(T^2) + (I - \Gamma(D\Gamma)^{-1}D)O(T^3) = O(T^2). \quad (12)\end{aligned}$$

In the previous derivation, we use the relations $(I - \Gamma(D\Gamma)^{-1}D)\Gamma = 0$, $\|I - \Gamma(D\Gamma)^{-1}D\| \le 1$ and $O(1)\cdot O(T^3) = O(T^3)$. Note that since m poles of $\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)$ are at the origin, it can be written as

$$\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right) = PJP^{-1} \quad (13)$$

where P is a transformation matrix and J is the Jordan matrix of the poles of $\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)$. The matrix J can be written as

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \quad (14)$$

where $J_1 \in \Re^{m\times m}$ and $J_2 \in \Re^{(n-m)\times(n-m)}$ and are given by

$$J_1 = \begin{bmatrix} \mathbf{0} & I_{m-1} \\ 0 & \mathbf{0} \end{bmatrix} \quad J_2 = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where λ_j are the poles of $\left(\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right)$. For simplicity it is assumed that the non-zero poles are designed to be distinct and that their continuous time counterparts are of order $O(1)$. Then, the solution of (11) is

$$\mathbf{x}_k = PJ^kP^{-1}\mathbf{x}_0 + P\left(\sum_{i=0}^{k-1} J^iP^{-1}\boldsymbol{\delta}_{k-i-1}\right). \quad (15)$$

Rewritting (15) as

$$\begin{aligned}\mathbf{x}_k &= PJ^kP^{-1}\mathbf{x}_0 + P\left(\sum_{i=0}^{k-1}\begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}P^{-1}\boldsymbol{\delta}_{k-i-1}\right) \\ &\quad + P\left(\sum_{i=0}^{k-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i\end{bmatrix}P^{-1}\boldsymbol{\delta}_{k-i-1}\right) \quad (16)\end{aligned}$$

it is easy to verify that $J_1^i = \mathbf{0}$ for $i \ge m$. Thus, (16) becomes (for $k \ge m$)

$$\begin{aligned}\mathbf{x}_k &= PJ^kP^{-1}\mathbf{x}_0 + P\left(\sum_{i=0}^{m}\begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}P^{-1}\boldsymbol{\delta}_{k-i-1}\right) \\ &\quad + P\left(\sum_{i=0}^{k-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i\end{bmatrix}P^{-1}\boldsymbol{\delta}_{k-i-1}\right). \quad (17)\end{aligned}$$

Notice $\|J_1\| = 1$ and $\|J_2\| = \lambda_{\max} = \max\{\lambda_{m+1}, \ldots, \lambda_n\}$ ($\|\cdot\|$ indicates $\|\cdot\|_2$). Hence, from (17)

$$\begin{aligned}\|\mathbf{x}_k\| &\le \|P\|\left(\sum_{i=0}^{m}\left\|\begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right\|^i \|P^{-1}\|\|\boldsymbol{\delta}_{k-i-1}\|\right. \\ &\quad \left. + \sum_{i=0}^{k-1}\left\|\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2\end{bmatrix}\right\|^i \|P^{-1}\|\|\boldsymbol{\delta}_{k-i-1}\|\right). \quad (18)\end{aligned}$$

Since $\lambda_{\max} < 1$ for a stable system,

$$\sum_{i=0}^{\infty}\|J_2\|^i = \frac{1}{1-\lambda_{\max}}, \quad \sum_{i=0}^{m}\|J_1\|^i = m.$$

Using Tustin's approximation

$$\lambda_{\max} = \frac{2+Tp}{2-Tp} \Rightarrow \frac{1}{1-\lambda_{\max}} = \frac{1}{1-\frac{2+Tp}{2-Tp}} = \frac{2-Tp}{-2Tp} = O(T^{-1}) \quad (19)$$

where $p = O(1)$ is the corresponding pole in continuous-time. Assuming $m = O(1)$, and using the fact $\|P^{-1}\| = \|P\|^{-1}$, it can be derived from (18) that, at the steady state when k is sufficiently large, the ultimate bound of $\mathbf{x}_k$ is

$$O(1)\cdot O(T^2) + O(T^{-1})\cdot O(T^2) = O(T). \quad (20)$$

■

Remark 2: Under practical considerations, it is generally advisable to select the pole p large enough such that the system has a fast enough response. With the selection of a small sampling time T, a pole of order $O(T)$ would lead to an undesirably slow repsonse. Thus, it makes sense to select a pole of order $O(1)$ or larger.

Remark 3: The SMC in [1] guarantees that the sliding variable σ is of order $O(T^2)$, but cannot guarantee the same order of magnitude of steady-state errors for the system state variables. In the next section, we show that an integral sliding-mode design can achieve a more precise state regulation.

Remark 4: Note that the design of matrix D such that the SMC closed-loop system (11) has desired response is not so straightforward, [9]. It will be shown that in addition to improving the accuracy of the closed-loop system, the control based on integral sliding manifold is straightforward to design.

III. State Regulation With ISMC

Consider the new discrete-time integral sliding manifold defined as follows:

$$\begin{aligned}\boldsymbol{\sigma}_k &= D\mathbf{x}_k - D\mathbf{x}_0 + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + E\mathbf{x}_{k-1} \quad (21)\end{aligned}$$

where $\boldsymbol{\sigma} \in \Re^m$, $\boldsymbol{\varepsilon} \in \Re^m$, and matrices D and E are constant and of rank m. The term $D\mathbf{x}_0$ is used to eliminate the reaching phase. Equation (21) is the discrete-time counterpart of the following sliding manifold [7]:

$$\boldsymbol{\sigma}(t) = D\mathbf{x}(t) - D\mathbf{x}(0) + \int_0^t E\mathbf{x}(\tau)d\tau = 0. \quad (22)$$

Theorem 1: Assume the pair (Φ, Γ) in (2) is controllable. There exists a matrix K such that the poles of $\Phi - \Gamma K$ are distinct and within the unit circle. Choose the control law

$$\mathbf{u}_k = (D\Gamma)^{-1}D\mathbf{x}_0 - (D\Gamma)^{-1}\left((D\Phi + E)\mathbf{x}_k + D\hat{\mathbf{d}}_k + \boldsymbol{\varepsilon}_k\right) \quad (23)$$

where $D\Gamma$ is invertible,

$$E = -D(\Phi - I - \Gamma K) \quad (24)$$

and $\hat{\mathbf{d}}_k$ is the disturbance compensation (7). Then, the closed-loop dynamics is

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K)\mathbf{x}_k + \boldsymbol{\zeta}_k \quad (25)$$

with $\boldsymbol{\zeta}_k \in \Re^n$ is $O(T^3)$, and the ultimate bound of $\mathbf{x}_k$ is set by $O(T^2)$.

Proof: Consider a forward expression of (21)

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= D\mathbf{x}_{k+1} - D\mathbf{x}_0 + \boldsymbol{\varepsilon}_{k+1} \\ \boldsymbol{\varepsilon}_{k+1} &= \boldsymbol{\varepsilon}_k + E\mathbf{x}_k. \end{aligned} \quad (26)$$

Substituting $\boldsymbol{\varepsilon}_{k+1}$ and (2) into the expression of the sliding manifold in (26) leads to

$$\boldsymbol{\sigma}_{k+1} = (D\Phi + E)\mathbf{x}_k + D(\Gamma\mathbf{u}_k + \mathbf{d}_k) + \boldsymbol{\varepsilon}_k - D\mathbf{x}_0. \quad (27)$$

The equivalent control is found by solving for $\boldsymbol{\sigma}_{k+1} = 0$

$$\mathbf{u}_k^{eq} = (D\Gamma)^{-1}D\mathbf{x}_0 - (D\Gamma)^{-1}((D\Phi + E)\mathbf{x}_k + D\mathbf{d}_k + \boldsymbol{\varepsilon}_k). \quad (28)$$

Similar to the classical case with control given by (6), implementation of (28) would require *a priori* knowledge of the disturbance $\mathbf{d}_k$. By replacing the disturbance in (28) with its estimate $\hat{\mathbf{d}}_k$, which is defined in (7), the practical control law is

$$\mathbf{u}_k = (D\Gamma)^{-1}D\mathbf{x}_0 - (D\Gamma)^{-1}\left((D\Phi + E)\mathbf{x}_k + D\hat{\mathbf{d}}_k + \boldsymbol{\varepsilon}_k\right). \quad (29)$$

Substitution of $\mathbf{u}_k$ defined by (29) into (2) leads to the closed-loop equation in the sliding-mode

$$\begin{aligned}\mathbf{x}_{k+1} = {} & [\Phi - \Gamma(D\Gamma)^{-1}(D\Phi + E)]\mathbf{x}_k - \Gamma(D\Gamma)^{-1}\boldsymbol{\varepsilon}_k \\ & + \Gamma(D\Gamma)^{-1}D\mathbf{x}_0 + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\hat{\mathbf{d}}_k. \end{aligned} \quad (30)$$

Let us derive the sliding dynamics. Rewriting (26)

$$\boldsymbol{\sigma}_{k+1} = D\mathbf{x}_{k+1} + E\mathbf{x}_k - D\mathbf{x}_0 + \boldsymbol{\varepsilon}_k. \quad (31)$$

Substitution of (30) into (31) leads to

$$\boldsymbol{\sigma}_{k+1} = D\mathbf{d}_k - D\hat{\mathbf{d}}_k = D\mathbf{d}_k - D\mathbf{d}_{k-1} = O(T^2) \quad (32)$$

that is, the introduction of ISMC leads to the same sliding dynamics as in [1].

Next, solving $\boldsymbol{\varepsilon}_k$ in (21) in terms of $\mathbf{x}_k$ and $\boldsymbol{\sigma}_k$

$$\boldsymbol{\varepsilon}_k = \boldsymbol{\sigma}_k - D\mathbf{x}_k + D\mathbf{x}_0 \quad (33)$$

and substituting it into (30), the closed-loop dynamics becomes

$$\begin{aligned}\mathbf{x}_{k+1} = {} & \left[\Phi - \Gamma(D\Gamma)^{-1}(D(\Phi - I) + E)\right]\mathbf{x}_k \\ & - \Gamma(D\Gamma)^{-1}\boldsymbol{\sigma}_k + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\hat{\mathbf{d}}_k. \end{aligned} \quad (34)$$

In (34), $\boldsymbol{\sigma}_k$ can be substituted by $\boldsymbol{\sigma}_k = D\mathbf{d}_{k-1} - D\mathbf{d}_{k-2}$ as can be inferred from (32). Also, under (24), $D(\Phi - I) + E = D\Gamma K$. Therefore, $\Phi - \Gamma(D\Gamma)^{-1}(D(\Phi - I) + E) = \Phi - \Gamma K$. Since the pair (Φ, Γ) is controllable, there exists a matrix K such that poles of $\Phi - \Gamma K$ can be placed anywhere inside the unit disk. Note that, the selection of matrix D is arbitrary as long as it guarantees the invertibility of $D\Gamma$ while matrix E, computed using (24), guarantees the desired closed-loop performance. Thus, we have

$$\begin{aligned}\mathbf{x}_{k+1} = {} & (\Phi - \Gamma K)\mathbf{x}_k + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-1} \\ & - \Gamma(D\Gamma)^{-1}D(\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \end{aligned} \quad (35)$$

Note that in (35), the disturbance estimate $\hat{\mathbf{d}}_k$ has been replaced by $\mathbf{d}_{k-1}$. Further simplification of (35) leads to

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K)\mathbf{x}_k + \boldsymbol{\zeta}_k \quad (36)$$

where

$$\boldsymbol{\zeta}_k = \mathbf{d}_k - 2\Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-1} + \Gamma(D\Gamma)^{-1}D\mathbf{d}_{k-2}. \quad (37)$$

The magnitude of $\boldsymbol{\zeta}_k$ can be evaluated as below. Adding and substracting $2\mathbf{d}_{k-1}$ and $\mathbf{d}_{k-2}$ from the right-hand side of (37) yield

$$\boldsymbol{\zeta}_k = (\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}) + (I - \Gamma(D\Gamma)^{-1}D)(2\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \quad (38)$$

In Lemma 1, it has been shown that $(\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}) = O(T^3)$. On the other hand, from (4) we have

$$\begin{aligned}(I - \Gamma(D\Gamma)^{-1}D)(2\mathbf{d}_{k-1} - \mathbf{d}_{k-2}) = {} & (I - \Gamma(D\Gamma)^{-1}D) \\ & \times \left(\Gamma(2\mathbf{f}_{k-1} - \mathbf{f}_{k-2}) + \frac{T}{2}\Gamma(2\mathbf{v}_{k-1} - \mathbf{v}_{k-2}) + O(T^3)\right).\end{aligned}$$

Note that $(I - \Gamma(D\Gamma)^{-1}D)\Gamma = 0$, thus

$$(I - \Gamma(D\Gamma)^{-1}D)\left(\Gamma(2\mathbf{f}_{k-1} - \mathbf{f}_{k-2}) + \frac{T}{2}\Gamma(2\mathbf{v}_{k-1} - \mathbf{v}_{k-2})\right) = 0.$$

Furthermore, $\|(I - \Gamma(D\Gamma)^{-1}D)\| \leq 1$, thus $(I - \Gamma(D\Gamma)^{-1}D)O(T^3)$ remains $O(T^3)$. This concludes that

$$\boldsymbol{\zeta}_k = O(T^3).$$

Comparing (36) with (11), the difference is that $\boldsymbol{\delta}_k = O(T^2)$ whereas $\boldsymbol{\zeta}_k = O(T^3)$. Further, by doing a similarity decomposition for dynamics of (36), only the J_2 matrix of dimension n exists. Thus, the derivation procedure shown in (11)–(20) holds for (36), and the solution is

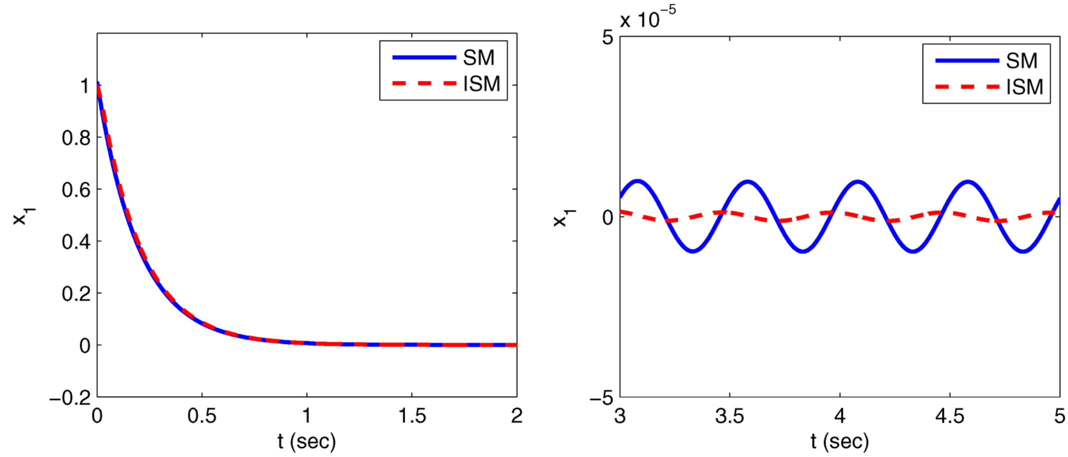
$$\mathbf{x}_k = (\Phi - \Gamma K)^k\mathbf{x}_0 + \sum_{i=0}^{k-1}(\Phi - \Gamma K)^i\boldsymbol{\zeta}_{k-i-1}. \quad (39)$$

Assuming distinct poles of $\Phi - \Gamma K$ and following the procedure that resulted in (20), it can be shown that

$$\lim_{k\to\infty}\left\|\sum_{i=0}^{k-1}(\Phi - \Gamma K)^i\boldsymbol{\zeta}_{k-i-1}\right\| = O(T^2). \quad (40)$$

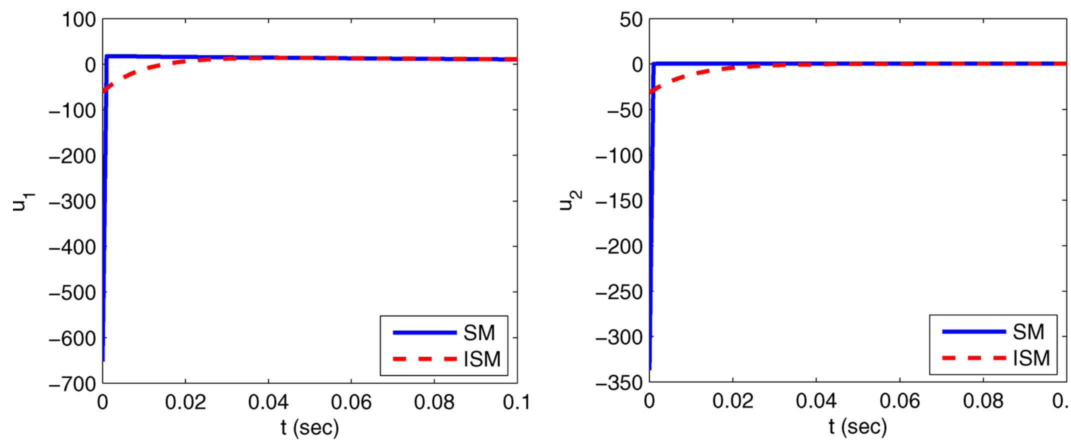
Thus, it is concluded that for sufficiently large k the ultimate bound of $\mathbf{x}_k$ is set by $O(T^2)$. ■

Remark 5: From foregoing derivations, it can be seen that the state errors are always one order higher than the disturbance term $\boldsymbol{\zeta}$ in the worst case due to convolution as shown by (40). After incorporating the integral sliding manifold, the off-set from the disturbance can be better compensated, in the sequel leading to a smaller steady state error boundary.



(a) (b)

Fig. 1. System state x_1.



(a) (b)

Fig. 2. System control inputs u_1 and u_2.

Remark 6: It is evident from the above analysis that, for the class of systems considered in this work and in [1], [2], the equivalent control based SMC with disturbance observer guarantees the motion of the states within an $O(T^2)$ bound, which is smaller than $O(T)$ for T sufficiently small, and is lower than what can be achieved by SMC using switching control [3], [4]. In such circumstances, without the loss of precision we can relax the necessity of incorporating a switching term, in the sequel avoid exciting chattering.

IV. Illustrative Example

Consider (1) with the following parameters:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$$
$$\mathbf{f}(t) = \begin{bmatrix} 0.3\sin(4\pi t) \\ 0.3\cos(4\pi t) \end{bmatrix}.$$

The initial states are $\mathbf{x}_0 = [1 \quad 1 \quad -1]^T$. The system will be simulated for a sample interval $T = 1$ ms. For the classical SMC, the D matrix is chosen such that the non-zero pole of the sliding dynamics is $p = -5$ in continuous-time, or $z = 0.9950$ in discrete-time. Hence, the poles of the system with SM are $[0 \quad 0.9950 \quad 0]^T$, respectively. Accordingly the D matrix is

$$D = \begin{bmatrix} 0.2621 & -0.3108 & -0.0385 \\ 3.4268 & 2.4432 & 1.1787 \end{bmatrix}.$$

Using the same D matrix given above, the system with ISMC is designed such that the dominant (non-zero) pole remains the same, but, the remaining poles are not at the origin. The poles are selected as $z = [0.9048 \quad 0.9950 \quad 0.8958]^T$, respectively.

Using the pole placement command of Matlab, the gain matrices can be obtained

$$K = \begin{bmatrix} 66.6705 & 9.4041 & 15.8872 \\ 18.2422 & 21.3569 & 8.5793 \end{bmatrix}.$$

According to (24)

$$E = \begin{bmatrix} 0.0297 & -0.0313 & -0.0034 \\ 0.3147 & 0.2366 & 0.1115 \end{bmatrix}.$$

The delayed disturbance compensation is used. Fig. 1(a) shows that the system state $x_1(t)$ is asymptotically stable for both discrete-time SMC and ISMC, which show almost the same behavior globally. On the other hand, the difference in the steady-state response between discrete-time SMC and ISMC can be seen from Fig. 1(b). The control inputs are shown in Fig. 2. Note that the control inputs of the SMC has much larger values at the initial phase in comparison with ISMC, due to the existence of poles at the origin. Another reason for the lower value of the control inputs in the ISMC is the elimination of the reaching phase by compensating for the nonzero intial condition in (29).

To demonstrate the quality of both designs, the open-loop transfer function matrices $G_{i,j}^{OL}$, for the systems with SM and ISM are com-

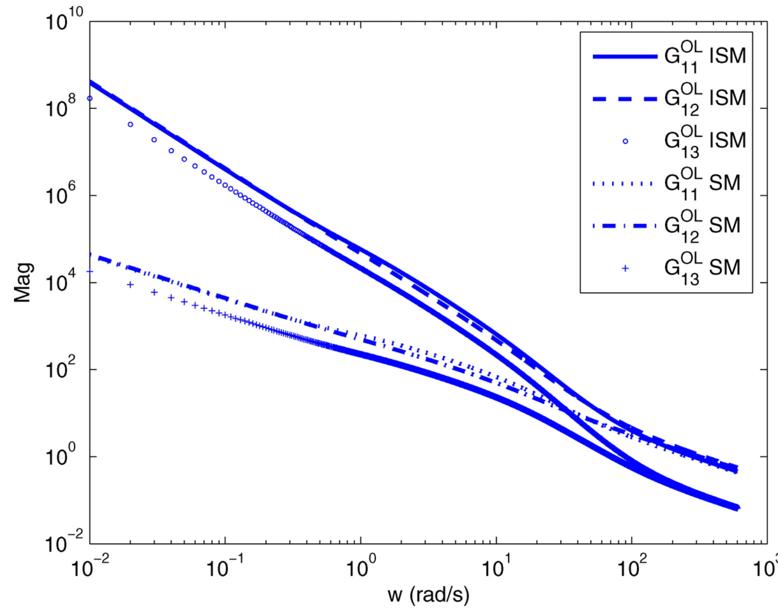


Fig. 3. Bode plot of some of the elements of the open-loop transfer matrix.

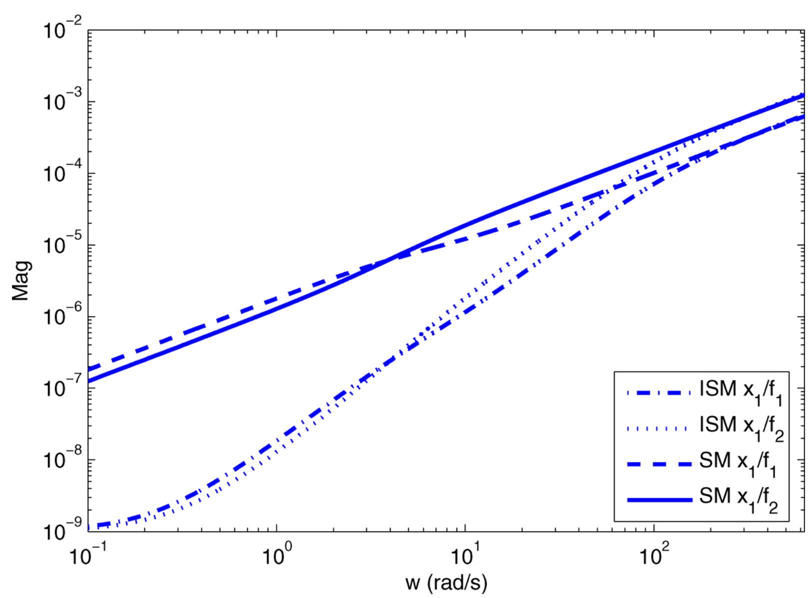


Fig. 4. Sensitivity function of x_1 with respect to f_1 and f_2.

puted and Bode plots of some elements are shown in Fig. 3. In addition, the sensitivity function of state x_1 with respect to the disturbance components $f_1(t)$ and $f_2(t)$ is plotted in Fig. 4. It can be seen from Figs. 3 and 4 that ISMC greatly reduces the effect of the disturbance as compared to SMC. Moreover ISMC presents a larger open-loop gain at the lower frequency band due to the effects of integral action in the sliding manifold, which ensures a more accurate closed-loop response to possible reference inputs.

V. Conclusion

This work presents a new discrete-time integral sliding control design for sampled-date systems under state regulation. Using the new discrete-time integral type sliding manifold, the SMC design retains the deadbeat structure of the discrete-time sliding-mode in the sense that the sliding variable reaches the origin in one step, and at the same time allocates the closed-loop poles for the full-order multiple-input system. The discrete-time ISMC achieves accurate control performance for both the sliding-mode and state regulation, meanwhile eliminates the reaching phase and avoids overlarge control efforts. The theoretical results were confirmed through both theoretical analysis and a numerical example.

Appendix
Proof of Lemma 1

Consider the Taylor's series expansion of $\mathbf{f}((k+1)T-\tau)$

$$\begin{aligned}\mathbf{f}(kT+T-\tau) &= \mathbf{f}_k + \mathbf{v}_k(T-\tau) + \frac{1}{2!}\mathbf{w}_k(T-\tau)^2 + \cdots \\ &= \mathbf{f}_k + \mathbf{v}_k(T-\tau) + \boldsymbol{\xi}(T-\tau)^2 \qquad (41)\end{aligned}$$

where $\mathbf{v}(t) = (d/dt)\mathbf{f}(t), \mathbf{w}(t) = (d^2/dt^2)\mathbf{f}(t)$ and $\boldsymbol{\xi} = (1/2!)\mathbf{w}(\mu)$ and μ is a time value between kT and $(k+1)T$, [10]. Substituting (41) into the expression of $\mathbf{d}_k$

$$\mathbf{d}_k = \int_0^T e^{A\tau}\mathbf{B}\mathbf{f}_k d\tau + \int_0^T e^{A\tau}B\mathbf{v}_k(T-\tau)d\tau + \int_0^T e^{A\tau}B\boldsymbol{\xi}(T-\tau)^2 d\tau. \quad (42)$$

For clarity, each integral will be analyzed separately. Since $\mathbf{f}_k$ is independent of τ it can be taken out of the first integral

$$\int_0^T e^{A\tau}B\mathbf{f}_k d\tau = \int_0^T e^{A\tau}B d\tau \mathbf{f}_k = \Gamma\mathbf{f}_k. \quad (43)$$

In order to solve the second integral term, it is necessary to expand $e^{A\tau}$ into series form. Thus

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T-\tau)d\tau = \int_0^T \left[e^{A\tau}B - \left(B + AB\tau + \frac{1}{2!}A^2B\tau^2 + \cdots\right)\tau\right] d\tau\mathbf{v}_k. \quad (44)$$

Solving the integral leads to

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T-\tau)d\tau = \left[\Gamma - \left(\frac{1}{2!}BT + \frac{1}{3!}ABT^2 + \cdots\right)\right]T\mathbf{v}_k. \quad (45)$$

Simplifying the result with the aid of (3)

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T-\tau)d\tau = \left[\Gamma - \frac{1}{2}\Gamma + \frac{1}{2}MT^2 - \left(\frac{1}{3!}ABT^2 + \frac{1}{4!}A^2BT^2 + \cdots\right)\right]T\mathbf{v}_k. \quad (46)$$

Simplifying the aforementioned expression further

$$\int_0^T e^{A\tau}B\mathbf{v}_k(T-\tau)d\tau = \frac{1}{2}\Gamma\mathbf{v}_kT + \hat{M}\mathbf{v}_kT^3 \quad (47)$$

where $\hat{M}$ is a constant matrix. Finally, note that in (42) the third integral is $O(T^3)$, since, the term inside the integral is already $O(T^2)$, therefore

$$\int_0^T e^{A\tau}B\boldsymbol{\xi}(T-\tau)^2 d\tau = O(T^3). \quad (48)$$

Thus, combining (42), (43), (47), and (48) leads to

$$\begin{aligned}\mathbf{d}_k &= \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_kT + \hat{M}T^3\mathbf{v}_k + O(T^3)\\ &= \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_kT + O(T^3). \quad (49)\end{aligned}$$

Now evaluate

$$\mathbf{d}_k - \mathbf{d}_{k-1} = \Gamma(\mathbf{f}_k - \mathbf{f}_{k-1}) + \frac{1}{2}\Gamma(\mathbf{v}_k - \mathbf{v}_{k-1})T + O(T^3). \quad (50)$$

From (41) and letting $\tau = 0$, $\mathbf{f}_k - \mathbf{f}_{k-1} = O(T)$. From (3), $\Gamma = O(T)$. In the sequel $\mathbf{d}_k - \mathbf{d}_{k-1} = O(T^2)$, if the assumptions on the boundedness and smoothness of $\mathbf{f}(t)$ hold. Finally, we notice that (50) is the difference of the first order approximation, whereas

$$\begin{aligned}&\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}\\ &\quad = \Gamma(\mathbf{f}_k - 2\mathbf{f}_{k-1} + \mathbf{f}_{k-2}) + \frac{1}{2}\Gamma(\mathbf{v}_k - 2\mathbf{v}_{k-1} + \mathbf{v}_{k-2}) + O(T^3)\end{aligned}$$

is the difference of the second order approximation. Accordingly the magnitude of $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}$ is $O(T^3)$, [10].

REFERENCES

[1] W. C. Su, S. Drakunov, and U. Ozguner, "An $O(T^2)$ boundary layer in sliding mode for sampled-data systems," *IEEE Trans. Autom. Control*, vol. 45, no. 3, pp. 482–485, Mar. 2000.

[2] V. Utkin, "Sliding mode control in discrete-time and difference systems," in *Variable Structure and Lyapunov Control*, A. S. I. Zinober, Ed. New York: Springer-Verlag, 1994, vol. 193, ch. 5, pp. 87–107.

[3] W. Gao, Y. Wang, and A. Homaifa, "Discrete-time variable structure control systems," *IEEE Trans. Ind. Electron.*, vol. 42, no. 2, pp. 117–122, Apr. 1995.

[4] G. Golo and C. Milosavljevic, "Robust discrete-time chattering free sliding mode control," *Syst. Control Lett.*, vol. 41, no. 1, pp. 19–28, 2000.

[5] J. P. Barbot and T. Boukhobza, "Discretization issues," in *Sliding Mode Control in Engineering*, W. Perruquetti and J. P. Barbot, Eds. New York: Marcel Dekker, 2002, ch. 8, pp. 217–232.

[6] V. Utkin and J. Shi, "Integral sliding mode in systems operating under uncertainty conditions," in *Proc. Conf. Decision Control*, Kobe, Japan, 1996, pp. 4591–4596.

[7] W. J. Cao and J. X. Xu, "Eigenvalue assignment in full-order sliding mode using integral type sliding surface," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1355–1360, Aug. 2004.

[8] L. Fridman, F. Castanos, N. M'Sirdi, and N. Khraef, "Decomposition and robustness properties of integral sliding mode controllers," in *Proc. IEEE VSS Workshop*, Vilanova i la Geltrú, Spain, 2004.

[9] J. Ackermann and V. Utkin, "Sliding mode control design based on Ackermann's formula," *IEEE Trans. Autom. Control*, vol. 43, no. 2, pp. 234–237, Feb. 1998.

[10] S. C. Chapra and R. P. Canale, *Numerical Methods for Engineers*. Singapore: McGraw-Hill, 1998.