Integral Sliding Mode Convex Optimization in Uncertain Lagrangian Systems Driven by PMDC Motors: Averaged Subgradient Approach

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***Abstract*—An uncertain Lagrangian dynamic controlled plant with permanent magnet dc-actuator, governed by a system of ordinary differential equations, is treated. The state variables (generalized coordinates and their velocities) are assumed to be measurable. The controller design is based on sliding mode concept, aimed to minimize a given convex (not obligatory strongly convex) function of the current state. The subgradient of this cost function is supposed to be measurable online. An optimization type algorithm is developed and analyzed using ideas of the averaged subgradient technique. The main results consist in proving the reachability of the "practical desired regime" (nonstationary analogue of sliding surface) from the beginning of the process and obtaining an explicit upper bound for the cost function decrement, that is, a functional convergence is proven and the rate of convergence is estimated. Numerical example, dealing with a robot manipulator of three freedom degrees illustrates the effectiveness of the suggested approach.**

***Index Terms*—Actuators, sliding mode control, torque control.**

I. Introduction

A. Brief Survey

Online optimization by an uncertain dynamic plant is a form of robust control, where the steady-state input–output characteristic of an extremal function is optimized, without requiring any explicit knowledge about a dynamic controllable model—only the existence of its output and exact measurability are required. Two principle types of optimization problems via dynamic plants may be mentioned:

Problems where the *optimized function is unknown*, but admits to be measurable, providing the possibility to estimate online its gradient. Such problems are referred to as extremum seeking. The review of the published papers dealing with this topic can be found in [17]. Originally it was developed as a method of adaptive control for hard-to-model systems, solving some of the same problems as today's neural network techniques (see [2], [9]). This paradigm was applied in [14] for two levels plant's economic optimization. In [1], an extremum-seeking control problem is posed for a class of nonlinear systems with unknown dynamic parameters, whose states are subject to convex, point-wise inequality constraints. The article [16] describes a new algorithm for extremum seeking using stochastic online gradient estimation.

Problems where the *optimized function* or *its gradient are available online.* So, the article [6] deals with the problem of constrained optimization in dynamic linear time-invariant systems characterized by a control vector dimension less than that of the system state vector. The finite-time convergence to a vicinity of order ε of the optimal equilibrium point is proved. In [5], a variable structure convex programming based control for a class of linear uncertain systems with accessible state is presented. A convex programming problem is solved, online, by reformulating the problem in terms of a piecewise smooth penalty function. The passivity property has been effectively used for designing of a robust set-valued control of a certain class of uncertain Lagrangian systems in [3]. In [10], the robust trajectory tracking of fully actuated Lagrange systems was studied. A family of set-valued passivity-based controllers was proposed, but actuator effects were not considered.

In this article, we consider the second type of problems and present the convex optimization concept applied for the Lagrangian dynamic model, given by the second-order ordinary differential equation with unknown right-hand side, but with accessible states and their rates. The joint model, including both Lagrange system and actuators [permanent magnet dc (PMDC) motors], is analyzed. Such problem formulation has a wide spectrum of real engineering applications. Here, we suggest to use a feedback, which designing is very close to the widely used integral sliding mode (ISM) approach [7] together with the, socalled, averaged subgradient technique, which in turn is close to the primal-dual inertial gradient descent method [4], [15] (supplied by an "inertial" term [11]), supposing that the current subgradient $a(x_t)$ of the convex function $F(x_t)$, to be optimized, is available online. With the specially selected parameters of the designed controller we prove the functional convergence of the convex (not obligatory strongly convex) loss function $F(x_t)$ to its minimal value F^*.

B. Main Contributions

1) A new form of ISM variable is introduced and analyzed.
2) Completely new functional convergence analysis in ideal and practical dynamic regimes is presented.
3) The rate of functional convergence and its dependence on initial conditions and some parameters of optimization algorithm are estimated.
4) High level of uncertainties of the considered model is admitted: tensor of inertia, nonpotential forces (friction, hysteresis, Coriolis, damping, centripetal effects, and others) may be unknown *a* priory.
5) New form of Lyapunov function with dead zone is used to prove the realization of the desired dynamics from the beginning of the process.
6) Effectiveness of the suggested approach is demonstrated in a real practical example, dealing with a two links robot manipulator of three degrees of freedom.

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II. Dynamic Plant and Loss Function

A. System Description

The dynamic model of a Lagrangian mechanical system with n-degrees of freedom in the standard form, driven by n-independent PMDC motors [12], is given by the following set of differential equations:

$$\left.\begin{array}{c} D(q_t)\ddot{q}_t + C(q_t,\dot{q}_t)\dot{q}_t + G(q_t) = \tau_t + \vartheta_t \\ \tau_t = W K_a I_{at} \\ L_a \dot{I}_{at} + R_a I_{at} + K_e W^\intercal \dot{q}_t = v_{at} \end{array}\right\} \tag{1}$$

where $q_t, \dot{q}_t \in R^n$ is the state vector (generalized coordinates and their velocities), $\tau_t \in R^n$ is a vector of external torques acting to the mechanical system, $I_{at} \in R^n$—the armature current vector, $W \in R^{n\times n}$ is the electromotive force constant matrix (possibly taking into account the gear ratios of the motors), $K_a \in R^{n\times n}$—the direct—electromotive forces constants matrix, $D(q_t) = M(q_t) + WJW^\intercal$ is the inertia matrix, which is positive definite ($D(q_t) = D^\intercal(q_t) \geq d_- I_{n\times n}$) and therefore invertible for all q_t, $J = \mathrm{diag}\{J_1, J_2, \ldots, J_n\}$—the rotors inertia matrix, $M(q_t)$ is the matrix of Lagrangian system corresponding to the armature inertia matrix in original coordinates, $C(q_t,\dot{q}_t) \in R^{n\times n}$ is the matrix corresponding the generalized nonpotential forces $C(q_t,\dot{q}_t)\dot{q}_t$, which may describe friction, hysteresis, Coriolis, damping, centripetal effects, and etc., $G(q_t) \in R^n$—the vector corresponding the generalized potential forces, $K_e = \mathrm{diag}\{K_{e1}, K_{e2}, \ldots, K_{en}\}$—the back-electromotive forces constants matrix, $L_a = \mathrm{diag}\{L_{a1}, L_{a2}, \ldots, L_{an}\}$ and $R_a = \mathrm{diag}\{R_{a1}, R_{a2}, \ldots, R_{an}\}$ is the armature inductances and resistants *positive* matrices, respectively, $\vartheta_t \in R^n$ is the disturbance (or uncertainty) vector, $v_{at} \in R^n$—the armature voltage vector, which below is considered as *a control* to be designed to obtain a desired behavior. In fact, the second equation in (1) describes the dynamics of the *actuator*, realizing the applied control action v_{at}. The model (1) assumes fully allocated control.

Suppose that q_t, $\dot{q}_t$, and I_{at} are available online. From (1) it follows that:

$$I_{at} - I_{at_0} = -L_a^{-1} R_a \int_{\tau=t_0}^{t} I_{a\tau} d\tau - L_a^{-1} K_e W^\intercal (q_t - q_{t_0}) + L_a^{-1} \int_{\tau=t_0}^{t} v_{a\tau} d\tau \tag{2}$$

($t_0 \geq 0$ is any fixed time) and selecting (neglecting the Joule effect, related to the dependence of the winding motor resistance)

$$v_{at} = v_{at}^{(1)} + v_{at}^{(2)},\, v_{at}^{(1)} = R_a I_{at} + K_e W^\intercal \dot{q}_t \tag{3}$$

the relation (2) becomes

$$I_{at} = I_{at_0} + L_a^{-1} \int_{\tau=t_0}^{t} v_{a\tau}^{(2)} d\tau. \tag{4}$$

Substituting (4) into (1) gives

$$D(q_t)\ddot{q}_t + C(q_t,\dot{q}_t)\dot{q}_t + G(q_t) = u_t + \tilde{\vartheta}_t \tag{5}$$

$$u_t := W K_a L_a^{-1} \int_{\tau=t_0}^{t} v_{a\tau}^{(2)} d\tau,\, \tilde{\vartheta}_t := W K_a I_{at_0} + \vartheta_t. \tag{6}$$

In standard matrix format with *new state vectors* $x_1 = q \in R^n$ and $x_2 = \dot{q} \in R^n$ the considered Lagrange dynamics (5) has the following form:

$$\dot{x}_t = \begin{pmatrix} \dot{x}_{1t} \\ \dot{x}_{2t} \end{pmatrix} = H(x_{1t}, x_{2t}) \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} + Bu_t + \xi_t$$

$$H(x_1, x_2) = \begin{pmatrix} 0 & I_{n\times n} \\ 0 & -D^{-1}(x_1) C(x_1, x_2) \end{pmatrix}$$

$$B = \begin{pmatrix} 0_{n\times n} \\ D^{-1}(x_1) \end{pmatrix},\, \xi_t = \begin{pmatrix} 0_{n\times n} \\ D^{-1}(x_{1t}) \left[\tilde{\vartheta}_t - G(x_1)\right] \end{pmatrix}$$

So, in this representation the size of control actions u is n and the extended state dimension $x = (x_1^\intercal, x_2^\intercal)^\intercal$ is $2n$.

B. Loss Function

Consider also the convex (not obligatory strongly) loss function $F : R^n \to R^1$, which defines the quality of control actions v_a, that is, $F = F(x_1)$. For example, the following two functions belong to the considered class of the convex loss functions to be optimized:

$$F(x_1) = \sum_{i=1}^{n} \left|x_{1,i} - x_{1,i}^*\right|,\, F(x_1) = \sum_{i=1}^{n} \left|x_{1,i} - x_{1,i}^*\right|_\varepsilon^+$$

$$|z|_\varepsilon^+ := \begin{cases} z - \varepsilon & \text{if } z \geq \varepsilon \\ -z - \varepsilon & \text{if } z \leq -\varepsilon \\ 0 & \text{if } |z| < \varepsilon \end{cases}$$

III. Main Assumptions

Suppose that the following assumptions hold throughout of this article.

1) The extended vector $x = (q_t^\intercal, \dot{q}_t^\intercal)^\intercal$ and I_{at} are available (physically measurable) at any time $t \geq 0$.
2) The motors parameters such as positive matrix L_a, K_a, R_a, and matrices K_e and W $(\det W \neq 0)$ are supposed to be *a priori* known;
3) The inertial matrix, which is usually unknown, is positive definite and bounded, i.e., $0 < d_- I_{n\times n} \leq D(x_1) \leq d_+ I_{n\times n}$, and differentiable such that $\|\frac{\partial}{\partial x_1}\mathrm{col}\dot{D}(x_1)\| \leq D'_+$.
4) The unmeasurable noise term ϑ_t is bounded, i.e., $\|\vartheta_t\| \leq \vartheta_+ < \infty$.
5) The matrix functions $C(q_t, \dot{q}_t)$ and $G(q_t)$ are unknown, but bounded, namely
$$\|C(q_t,\dot{q}_t)\| \leq c_0, \|G(q_t)\| \leq g_0 \tag{7}$$
for all $q_t, \dot{q}_t \in R^n$. The constants $c_0 > 0$ and $g_0 > 0$ are assumed to be known.
6) The loss function $F : R^n \to R^1$ is assumed to be unknown, convex (not obligatory, strongly convex), differentiable for almost all $x_1 \in R^n$ (see Radamacher theorem in [13]) and its *subgradient* $a(x_1)$ is supposed to be *measurable*[1] and bounded at any point x_1, that is, the reaction $a(x_1)$ $(\|a(x_{1t})\| \leq d_g < \infty)$ is available for any argument $x_1 \in R^n$.
7) The minimum of the loss function $F(x_1)$ exists, namely, $F^* = \min_{x_1\in R^n} F(x_1) > -\infty$.

[1] By the definition (see [13]) a vector $a \in R^n$, satisfying the inequality

$$F(x+y) \geq F(x) + a^\intercal(x) y$$

for all $y \in R^n$, is called the subgradient of the function $F(x)$ at the point $x \in R^n$ and is denoted by $a(x) \in \partial F(x)$—the set of all subgradients of $F(x)$ at the point x. If $F(x)$ is differentiable at a point x, then $a(x) = \nabla F(x)$. In the minimal point x^* we have $0 \in \partial F(x^*)$.

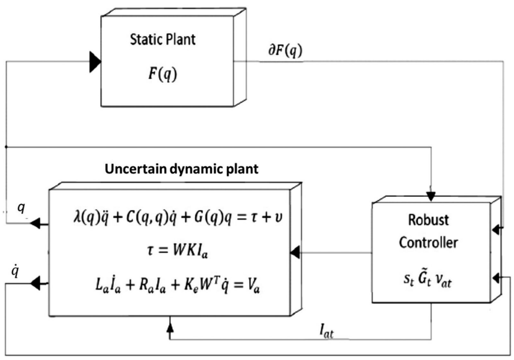

Fig. 1. Dynamical optimization system.

IV. Problem Setting

Problem 1: Under the Assumptions 1–7 we need to find the control action $v_{at}(t \geq 0)$ [and hence, the action u_t connected with v_{at} by the integral relation (6)], providing the functional convergence of the loss function $F(x_{1t}(v_{at}))$ to its minimal value F^* in the presence of uncertain factors H and ξ in (Section II-A), i.e.,

$$F(x_{1t}(v_{at})) \underset{t\to\infty}{\to} F^*. \tag{8}$$

The block-scheme of the considered system is shown in Fig. 1

V. On Properties of the Conventional Gradient Descent Method

Consider here the simple optimization problem of a convex function $F: R^n \to R^1$, which gradient satisfies Assumption 6

$$F(x) \to \min_{\bar{x}\in R^n}. \tag{9}$$

The corresponding *gradient descent* (GD) algorithm is (x_0 is given)

$$\frac{d}{dt}x_t = -\gamma a(x_t), 0 < \gamma \in R^1, t \geq 0. \tag{10}$$

A. Strongly Convex Functions

Under Assumptions 6 for the class of *strongly convex functions* the following properties hold for all $x \in R^n$ [13]:

$$\left.\begin{array}{c}(x-x^*)^\intercal \nabla F(x) \geq l\|x-x^*\|^2, \\ \|\nabla F(x)\|^2 \geq l(F(x)-F^*), l>0,\end{array}\right\} \tag{11}$$

where $x^* = \arg\min_{x\in R^n} F(x)$ is unique extremal point. The property (11) provides both argument and functional convergence and permits to estimate their rate of convergence for GD-algorithm (10).

1) Argument Convergence: For the Lyapunov function $V(x) = \frac{1}{2}\|x-x^*\|^2$ along the trajectories of the GD algorithm (10), in view of the property (11), we have

$$\left.\begin{array}{c}\dot{V}(x_t) = (x_t - x^*)^\intercal \frac{d}{dt}x_t = -\gamma (x_t - x^*)^\intercal a(x_t) \\ \leq -\gamma l \|x_t - x^*\|^2 = -2\gamma l V(x_t)\end{array}\right\} \tag{12}$$

which implies the argument exponential convergence $V(x_t) \leq V(x_0)e^{-2\gamma l t} \underset{t\to\infty}{\to} 0$ with the rate $e^{-2\gamma l t}$.

2) Functional Convergence: For the Lyapunov function $V(x) = F(x) - F^*$ along the trajectories of the GD-algorithm (10), in view of the property (11), we have

$$\left.\begin{array}{c}\dot{V}(x_t) = a^\intercal(x_t)\frac{d}{dt}x_t = -\gamma\|a(x_t)\|^2 \\ \leq -\gamma l(F(x_t)-F^*) = -\gamma l V(x_t),\end{array}\right\} \tag{13}$$

which also implies the functional exponential convergence $F(x_t) - F^* \leq (F(x_0) - F^*)e^{-\gamma l t} \underset{t\to\infty}{\to} 0$ with the rate $e^{-\gamma l t}$.

B. Convex Functions

For the more wide class of convex functions the constant l in (11) is equal to 0. That's why both main inequalities (12) and (13) become to be trivial providing the property $\dot{V}(x_t) \leq 0$, that guarantees only boundedness of the argument and function deviation from the optimal values. But for the general convex functions we can use the inequality (see [13] for $y = x^*$ and $f = F$)

$$(x-x^*)^\intercal a(x) \geq F(x) - F^* \tag{14}$$

valid for any $x \in R^n$. The details of the corresponding consideration is given below.

Remark 2: Let us consider the additional averaged variable

$$\widehat{x}_t = \frac{1}{t}\int_0^t x_\tau d\tau, t>0, \widehat{x}_0 = x_0 \tag{15}$$

One gets the following upper bound for $F(\widehat{x}_t)$ due to the *Jensen's inequality*, (14), taking $V(x) = F(x) - F^*$:

$$\left.\begin{array}{c}F(\widehat{x}_t) - F^* \leq \frac{1}{t}\int_0^t (F(x_\tau)-F^*)\,d\tau \\ \leq \frac{1}{t}\int_0^t (x_\tau - x^*)^\intercal a(x_\tau)\,d\tau = \\ -\frac{1}{\gamma t}\int_0^t \dot{V}(x_\tau)d\tau \leq \frac{\gamma^{-1}}{t}[F(x_0)-F^*]\end{array}\right\} \tag{16}$$

where the last equality is due to algorithm (10). This *functional convergence* results in algorithm (10) with *additional averaging* (15) looks nice for the convex optimization problem (without constraints), but it is not relevant for the control problem considered in this article, because the averaged variable $\widehat{x}_t$ is not, in this case, the output of the considered dynamic plant: the input to the "static plant", given by the function $F = F(x_t)$, may be only the vector x_t, but not the vector $\widehat{x}_t$.

C. Sliding Variable

Introduce the vector function $s_t \in R^n$, which is referred to below as "*sliding variable*"

$$\left.\begin{array}{c}s_t = x_{2t} + \frac{x_{1t}+\eta}{t+\theta} + \tilde{G}_t, \eta = \text{const} \\ \tilde{G}_t := \frac{1}{t+\theta}\int_{\tau=t_0}^t a(x_{1\tau})\,d\tau, \theta > 0.\end{array}\right\} \tag{17}$$

Remark 3: Notice that the *sliding variable* s_t is available (physically measurable) since both vectors x_{1t} and x_{2t} are measured online and contains an integral term, resemble those as in ISM approach [7].

D. Ideal and Practical Desired Dynamics

1) Ideal Desired Dynamics: Let us associate the sliding variable s_t with the "*idealdesired dynamics*" given by

$$\left.\begin{array}{c}\dot{\zeta}_t = -a(x_{1t}), \zeta_{t_0} = 0, x_{1t_0} \text{ is given} \\ (t+\theta)x_{2t} + x_{1t} + \eta = \zeta_t, \;\; t \geq t_0 \geq 0 \\ t_0 \text{ is a moment when the desired regime may begin.}\end{array}\right\} \tag{18}$$

As one can see the ideal desired dynamics (18) exactly corresponds to the situation when the sliding variable s_t is equal to zero for all $t \geq t_0$

$$s_t = \dot{s}_t = 0. \tag{19}$$

Why the dynamics (18) is *desired*? To answer this question we need to show that if (19) holds, then $F(x_{1t}) \underset{t\to\infty}{\to} F^*$. The following lemma proves this fact.

Lemma 4 (ideal functional convergence): For the ideal desired variable (17), with any $\theta > 0$ and any η, for any $t \geq t_0 \geq 0$ we may guarantee that

$$F(x_{1t}) - F^* \leq \frac{\Phi_{t_0}}{t+\theta} \to 0 \text{ when } t \to \infty \tag{20}$$

where

$$\begin{aligned}\Phi_{t_0} = \Phi(x_1^*, x_{1t_0}, \zeta_{t_0}, \theta, \eta) &:= (t_0+\theta)[F(x_{1t_0})\text{-}F^*] \\ &+ \frac{1}{2}\|\zeta_{t_0}\|^2 - \zeta_{t_0}^\intercal x_1^* + \frac{1}{2}\|x_1^* - \eta\|^2.\end{aligned} \tag{21}$$

Proof: Following [11] and defining $\mu_t := t + \theta$, we have

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|\zeta_t\|^2 - \zeta_t^\intercal x_1^*\right] = \dot{\zeta}_t^\intercal (\zeta_t - x_1^*) \\ &= -a^\intercal(x_{1t})[\mu_t x_{2t} + x_{1t} + \eta - x_1^*] \\ &= -a^\intercal(x_{1t})(x_{1t} - x_1^*) - a^\intercal(x_{1t})(\mu_t x_{2t} + \eta).\end{aligned}$$

Applying the inequality (14) to the first term in the right-hand side and using the identity

$$a^\intercal(x_{1t}) x_{2t} = a^\intercal(x_{1t})\dot{x}_{1t} = \frac{d}{dt}[F(x_{1t}) - F^*]$$

we have

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\|\zeta_t\|^2 - \zeta_t^\intercal x_1^*\right] &\leq -[F(x_{1t}) - F^*] \\ -\mu_t \frac{d}{dt}[F(x_{1t}) - F^*] &- a^\intercal(x_{1t})\eta.\end{aligned}$$

Then, integrating this inequality on interval $[t_0, t]$, we get

$$\begin{aligned}&\int_{\tau=t_0}^t [F(x_{1\tau}) - F^*]d\tau \leq \frac{1}{2}\left(\|\zeta_{t_0}\|^2 - \|\zeta_t\|^2\right) \\ &+ (\zeta_t - \zeta_{t_0})^\intercal x_1^* - (\mu_t[F(x_{1t}) - F^*])_{t_0}^t \\ &+ \int_{\tau=t_0}^t [F(x_{1\tau}) - F^*]\dot{\mu}_\tau d\tau - \left[\int_{\tau=t_0}^t a^\intercal(x_{1\tau})d\tau\right]\eta.\end{aligned}$$

Since $\dot{\mu}_\tau = 1$, the last inequality leads (using of the integration by parts) to the following relation:

$$\begin{aligned}&\mu_t[F(x_{1t}) - F^*] \leq \mu_{t_0}[F(x_{1t_0}) - F^*] \\ &\frac{1}{2}\left(\|\zeta_{t_0}\|^2 - \|\zeta_t\|^2\right) + (\zeta_t - \zeta_{t_0})^\intercal x_1^* + \zeta_t^\intercal \eta \\ &= (t_0+\theta)[F(x_{1t_0}) - F^*] + \left(\frac{1}{2}\|\zeta_{t_0}\|^2 - \zeta_{t_0}^\intercal x_1^*\right) \\ &+\frac{1}{2}\|x_1^*\text{-}\eta\|^2 \text{-} \frac{1}{2}\underbrace{\left[\|\zeta_t\|^2 \text{-}2\zeta_t^\intercal(x_1^*\text{-}\eta) + \|x_1^*\text{-}\eta\|^2\right]}_{\|\zeta_t - (x_1^* - \eta)\|^2} \\ &\leq (t_0+\theta)[F(x_{1t_0})\text{-}F^*] - \frac{1}{2}\|\zeta_t\text{-}(x_1^*\text{-}\eta)\|^2 \\ &\left(\frac{1}{2}\|\zeta_{t_0}\|^2 - \zeta_{t_0}^\intercal x_1^*\right) + \frac{1}{2}\|x_1^* - \eta\|^2 = \Phi_{t_0}\end{aligned}$$

that gives (21). ■

Remark 5: The parameter η will be selected below [see (32)] in such a way that the desired optimization regime will start from the beginning of the process.

2) Practical Desired Dynamics With a Zone Convergence: Suppose now that we are dealing with the, socalled, practical desired regime, namely when instead of (19) we have $\|s_t\| \leq \varepsilon_{wt}$.

Lemma 6 (practically functional convergence): For the practical desired variable, with any $\theta > 0$ and any η, for any $t \geq t_0 \geq 0$ we may guarantee that

$$F(x_{1t}) - F^* \leq \frac{\Phi_{t_0}}{t+\theta} + \frac{d_g}{t+\theta}\int_{\tau=t_0}^t (\tau+\theta)\varepsilon_{w\tau}d\tau. \tag{22}$$

Proof: Defining $\tilde{x}_{2t} := x_{2t} - w_t$, we have

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|\zeta_t\|^2 - \zeta_t^\intercal x_1^*\right] = \dot{\zeta}_t^\intercal(\zeta_t - x_1^*) \\ &= -a^\intercal(x_{1t})[\mu_t\tilde{x}_{2t} + x_{1t} + \eta - x_1^*] \\ &= -a^\intercal(x_{1t})(x_{1t} - x_1^*) + \dot{\zeta}_t(\mu_t\tilde{x}_{2t} + \eta).\end{aligned}$$

Applying the inequality (14) to the first term in the right-hand side and using the identity

$$\begin{aligned}a^\intercal(x_{1t})\tilde{x}_{2t} &= a^\intercal(x_{1t})[x_{1t} - w_t] \\ &= \frac{d}{dt}[F(x_{1t}) - F^*] - \partial^\intercal F(x_{1t})w_t\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|\zeta_t\|^2 - \zeta_t^\intercal x_1^*\right] \leq -[F(x_{1t}) - F^*] \\ &- \mu_t\frac{d}{dt}[F(x_{1t}) - F^*] + \dot{\zeta}_t^\intercal(\eta - \mu_t w_t).\end{aligned}$$

Then, integrating this inequality on interval $[t_0, t]$, we get

$$\begin{aligned}&\int_{\tau=t_0}^t [F(x_{1\tau}) - F^*]d\tau \leq \frac{1}{2}\left(\|\zeta_{t_0}\|^2 - \|\zeta_t\|^2\right) \\ &+ (\zeta_t - \zeta_{t_0})^\intercal x_1^* - (\mu_t[F(x_{1t}) - F^*])_{t_0}^t \\ &+ \int_{\tau=t_0}^t [F(x_{1\tau}) - F^*]\dot{\mu}_\tau d\tau + (\zeta_t - \zeta_{t_0})^\intercal \eta \\ &+ \int_{\tau=t_0}^t \mu_\tau\|a^\intercal(x_{1\tau})\|\|w_\tau\|d\tau.\end{aligned}$$

Since $\dot{\mu}_\tau = 1$ and $\zeta_{t_0} = 0$, the last inequality leads (using of the integration by parts) to the following relation:

$$\begin{aligned}&\mu_t[F(x_{1t}) - F^*] \leq \mu_{t_0}[F(x_{1t_0}) - F^*] \\ &+ \frac{1}{2}\left(\|\zeta_{t_0}\|^2 - \|\zeta_t\|^2\right) + (\zeta_t - \zeta_{t_0})^\intercal x_1^* + \zeta_t^\intercal\eta \\ &+ \int_{\tau=t_0}^t \mu_\tau d_g\varepsilon_{w\tau}d\tau = (t_0+\theta)[F(x_{1t_0}) - F^*] \\ &+ \left(\frac{1}{2}\|\zeta_{t_0}\|^2 - \zeta_{t_0}^\intercal x_1^*\right) \\ &- \frac{1}{2}\left[\|\zeta_t\|^2 - 2\zeta_t^\intercal(x_1^* - \eta) + \|x_1^* - \eta\|^2\right] \\ &+ \frac{1}{2}\|x_1^* - \eta\|^2 + d_g\int_{\tau=t_0}^t \mu_\tau\varepsilon_{w\tau}d\tau \\ &= (t_0+\theta)[F(x_{1t_0}) - F^*] - \frac{1}{2}\|\zeta_t - (x_1^* + \eta)\|^2\end{aligned}$$

$$+\left(\frac{1}{2}\|\zeta_{t_0}\|^2-\zeta_{t_0}^{\intercal}x_1^*\right)+\frac{1}{2}\|x_1^*-\eta\|^2$$
$$+d_g\int_{\tau=t_0}^{t}\mu_\tau\varepsilon_{w\tau}d\tau\le\Phi_{t_0}+d_g\int_{\tau=t_0}^{t}\mu_\tau\varepsilon_{w\tau}d\tau.$$

that gives (22). ■

Corollary 7: If

$$\varepsilon_{wt}=\frac{\varepsilon_{w0}}{(t+\theta)^{1+\alpha}},\quad \alpha,\varepsilon_{w0}>0 \tag{23}$$

then

$$\frac{d_g}{t+\theta}\int_{\tau=t_0}^{t}(\tau+\theta)\varepsilon_{w\tau}d\tau=\frac{d_g\varepsilon_{w0}}{(t+\theta)^{\alpha}}$$
$$-\frac{d_g\varepsilon_{w0}(t_0+\theta)^{1-\alpha}}{t+\theta}\le\frac{d_g\varepsilon_{w0}}{(t+\theta)^{\alpha}}$$

and, finally

$$F(x_{1t})-F^*\le\frac{\Phi_{t_0}}{t+\theta}+\frac{d_g\varepsilon_{w0}}{(t+\theta)^{\alpha}}\to 0. \tag{24}$$

VI. ISM Controller Design

A. Robust Controller Structure

Following (3) select the control v_a as

$$\left.\begin{aligned} v_{at}&=v_{at}^{(1)}+v_{at}^{(2)}\\ v_{at}^{(1)}&=R_aI_{at}+K_eW^{\intercal}\dot q_t\\ v_{at}^{(2)}&=-L_aK_a^{-1}W^{-1}\frac{d}{dt}\left(k_t\frac{s_t}{\|s_t\|+\varepsilon_t}\right).\end{aligned}\right\} \tag{25}$$

The smooth positive functions k_t and ε_t will be defined below.[2]

B. Lyapunov Function Analysis

Let us show that this controller provides a practical desired dynamics with special selection of the function ε_t. Introduce the Lyapunov function

$$V_t=\frac{1}{2}\left(\left[\sqrt{s_t^{\intercal}D(x_{1t})s_t}-\varkappa_t\right]_+\right)^2,\varkappa_t>0 \tag{26}$$

where $[z]_+=\begin{cases} z & \text{if } z\ge 0,\\ 0 & \text{if } z<0.\end{cases}$ Notice that the function $([z]_+)^2$ is differentiable. Defining

$$\begin{aligned}\chi_t&:=\frac{\left[\sqrt{s_t^{\intercal}D(x_{1t})s_t}-\varkappa_t\right]_+}{\sqrt{s_t^{\intercal}[D(x_{1t})]s_t}}\\ &\begin{cases}\in(0;1) & \text{if } \sqrt{s_t^{\intercal}D(x_{1t})s_t}>\varkappa_t\\ =0 & \text{if } \sqrt{s_t^{\intercal}D(x_{1t})s_t}\le\varkappa_t\end{cases}\\ z_t^{(0)}&=(t+\theta)^{-1}\left[x_{2t}+a(x_{1t})-\tilde G_t\right]-\frac{(x_{1t}+\eta)}{(t+\theta)^2},\\ z_t^{(1)}&:=g_0+c_0\|x_{2t}\|+\frac{D'_+}{2}\|x_{2t}\|\|s_t\|\\ &+\tilde\vartheta^++d_+\left(\left\|z_t^{(0)}\right\|+|\dot\varkappa_t|\right).\end{aligned} \tag{27}$$

[2]Since functions k_t, ε_t, and s_t in (25) are known and smooth the control action $v_{at}^{(2)}$ can be calculated analytically. But in practice it is much easily to use, for example, the super-twist differentiator (see [8]).

in view of (7) and (27) for $t\ge t_0$ we have

$$\left.\begin{aligned}&\frac{d}{dt}V_t(s_t)=\left[\sqrt{s_t^{\intercal}D(x_{1t})s_t}-\varkappa_t\right]_+\cdot\\ &\left(\frac{s_t^{\intercal}\dot D(x_{1t})s_t+2s_t^{\intercal}[D(x_{1t})]\dot s_t}{2\sqrt{s_t^{\intercal}D(x_{1t})s_t}}-\dot\varkappa_t\right)\\ &=\chi_t\left[\frac{1}{2}s_t^{\intercal}\dot D(x_{1t})s_t+\right.\\ &\left.s_t^{\intercal}D(x_{1t})\left(\dot x_{2t}+z_t^{(0)}\right)-\dot\varkappa_t\sqrt{s_t^{\intercal}D(x_{1t})s_t}\right]\\ &=\chi_t\tfrac{1}{2}s_t^{\intercal}\left[\dot D(x_{1t})\right]s_t+\chi_ts_t^{\intercal}[D(x_{1t})]\cdot\\ &[-D^{-1}(x_{1t})G(x_{1t})\ -D^{-1}(x_{1t})C(x_{1t},x_{2t})x_{2t}\\ &x_{2t}+D^{-1}(x_{1t})u_t+D^{-1}(x_{1t})\tilde\vartheta_t+z_t^{(0)}\Big]-\\ &\chi_t\dot\varkappa_t\sqrt{s_t^{\intercal}D(x_{1t})s_t}.\end{aligned}\right\} \tag{28}$$

Since

$$\left\|\dot D(x_{1t})\right\|\le D'_+\|\dot x_{1t}\|=D'_+\|x_{2t}\|$$
$$\|C(x_{1t},x_{2t})\|\le c_0$$
$$\|G(x_{1t})\|\le g_0,\left\|\tilde\vartheta_t\right\|\le\tilde\vartheta^+$$

the last relation (28) becomes

$$\left.\begin{aligned}&\frac{d}{dt}V_t(s_t)\le\chi_t\frac{D'_+}{2}\|x_{2t}\|\|s_t\|^2+\chi_ts_t^{\intercal}u_t\\ &\chi_t\|s_t\|\left(g_0+c_0\|x_{2t}\|+\tilde\vartheta^+\right.\\ &\left.+d_+\left(\left\|z_t^{(0)}\right\|+|\dot\varkappa_t|\right)\right)=\chi_t\|s_t\|z_t^{(1)}+\chi_ts_t^{\intercal}u_t\end{aligned}\right\} \tag{29}$$

Taking $v_{a\tau}^{(2)}$ as in (25), namely

$$v_{a\tau}^{(2)}=-L_aK_a^{-1}W^{-1}\frac{d}{dt}\left(k_t\frac{s_t}{\|s_t\|+\varepsilon_t}\right) \tag{30}$$

we get

$$\begin{aligned}&\frac{d}{dt}V_t(s_t)\le\chi_t\|s_t\|z_t^{(1)}+\chi_ts_t^{\intercal}u_t=\chi_t\|s_t\|z_t^{(1)}\\ &-\chi_ts_t^{\intercal}\left[k_t\frac{s_t}{\|s_t\|+\varepsilon_t}-k_t\frac{s_{t_0}}{\|s_{t_0}\|+\varepsilon_t}\right].\end{aligned} \tag{31}$$

Recall that by (17) $s_{t_0}=x_{2t_0}+\theta^{-1}(x_{1t_0}+\eta)$. Therefore, taking

$$\eta=-x_{1t_0}-\theta x_{2t_0} \tag{32}$$

we have that $s_{t_0}=0$, and as the result from (31) we have

$$\left.\begin{aligned}&\frac{d}{dt}V_t(s_t)\le\chi_t\|s_t\|z_t^{(1)}-\chi_tk_ts_t^{\intercal}\frac{s_t}{\|s_t\|+\varepsilon_t}\\ &=\chi_t\|s_t\|\left(z_t^{(1)}-k_t\frac{\|s_t\|}{\|s_t\|+\varepsilon_t}\right).\end{aligned}\right\} \tag{33}$$

Since the equality (28) is nontrivial only when $\chi_t\ge\varepsilon_t>0$, or equivalently, when $\sqrt{s_t^{\intercal}D(x_{1t})s_t}>\varkappa_t+\varepsilon_t$. But it is fulfilled if $\|s_t\|$ satisfies

$$\sqrt{s_t^{\intercal}D(x_{1t})s_t}\ge\sqrt{s_t^{\intercal}d_-s_t}=\sqrt{d_-}\|s_t\|>\varkappa_t+\varepsilon_t \tag{34}$$

namely, when

$$\|s_t\|>(\varkappa_t+\varepsilon_t)/\sqrt{d_-}. \tag{35}$$

Therefore, within this region we have

$$\frac{\|s_t\|}{\|s_t\|+\varepsilon_t}\ge\frac{(\varkappa_t+\varepsilon_t)/\sqrt{d_-}}{(\varkappa_t+\varepsilon_t)/\sqrt{d_-}+\varepsilon_t}=\frac{\varkappa_t+\varepsilon_t}{\varkappa_t+\varepsilon_t\left(1+\sqrt{d_-}\right)}.$$

So, (33) becomes

$$\frac{d}{dt}V_t(s_t)\le\chi_t\|s_t\|\left(z_t^{(1)}-k_t\frac{\varkappa_t}{\varkappa_t+\left(1+\sqrt{d_-}\right)\varepsilon_t}\right) \tag{36}$$

Selecting $\varkappa_t = \epsilon\varepsilon_t$ $(\epsilon > 0)$ and k_t in such a way that the following inequality $\frac{k_t}{1+(1+\sqrt{d_-})\epsilon^{-1}} - z_t^{(1)} = \frac{\rho}{\sqrt{d_+}} > 0$ is fulfilled, we get that for

$$k_t = \left(z_t^{(1)} + \frac{\rho}{\sqrt{d_+}}\right)\left[1 + \left(1 + \sqrt{d_-}\right)\epsilon^{-1}\right] \tag{37}$$

(36) becomes

$$\begin{aligned}\frac{d}{dt}V(s_t) &\le -\frac{\rho}{\sqrt{d_+}}\chi_t \|s_t\| \\ &= -\frac{\rho}{d_+}\chi_t\left(\sqrt{s_t^\intercal D^{1/2}(x_{1t})\left[D^{-1}(x_{1t})\right]D^{1/2}(x_{1t})\, s_t} - \varkappa_t\right) \\ &\quad - \chi_t\frac{\rho}{d_+}\varkappa_t \overset{\varkappa_t = \epsilon\varepsilon_t}{\ge} -\rho\chi_t\left(\sqrt{V(s_t)} - \varepsilon_t\frac{\epsilon}{d_+}\right).\end{aligned} \tag{38}$$

From (38) it follows a finite-time convergence of $V(s_t)$ to zero when the condition (35) holds. Indeed, for $\epsilon < \frac{d_+}{\sqrt{2}}$ by (34) we have

$$\sqrt{V(s_t)} - \frac{\varkappa_t}{d_+} \ge \varepsilon_t\left(\frac{1}{\sqrt{2}} - \frac{\epsilon}{d_+}\right) > 0$$

$$\chi_t = \frac{\left[\sqrt{s_t^\intercal D(x_{1t})\, s_t} - \varkappa_t\right]_+}{\left(\sqrt{s_t^\intercal\left[D(x_{1t})\right]s_t} - \varkappa_t\right) + \varkappa_t} \ge \frac{1}{1 + \sqrt{2}\frac{\varkappa_t}{\varepsilon_t}} = \frac{1}{1+\sqrt{2}\epsilon}$$

$$\frac{d}{dt}V(s_t) \le -\rho\chi_t\left(\sqrt{V(s_t)} - \frac{\varkappa_t}{d_+}\right) \le -\tilde{\rho}\varepsilon_t$$

$$\tilde{\rho} := \frac{\rho}{1+\sqrt{2}\epsilon}\left(\frac{d_+ - \sqrt{2}\epsilon}{\sqrt{2}d_+}\right), 0 \le V(s_t) \le V(s_{t_0})$$

$$-\tilde{\rho}\int_{\tau=t_0}^{t}\varepsilon_\tau d\tau \le \sqrt{\frac{d_+}{2}}\|s_{t_0}\| - \tilde{\rho}\int_{\tau=t_0}^{t}\varepsilon_\tau d\tau \tag{39}$$

implying $V(s_t) = 0$ for any $t \ge t_{\text{reach}} = \min_t\{t : \tilde{\rho}\int_{\tau=t_0}^{t}\varepsilon_\tau d\tau = \sqrt{\frac{d_+}{2}}\|s_{t_0}\|\} = 0$ since $\|s_{t_0}\| = 0$.

Now, we are ready to formulate the main results of the article.

C. Convergence to a Decreasing Zone Around the Desired Regime

Theorem 8: Under the Assumptions 1–7 and using ISM controller (25), (37) we may guarantee that the practical desired optimization regime (23) starts from the beginning of the process, so that for any $t \ge 0$ the asymptotic functional convergence (24) with [see (35)] $\varepsilon_t = \varepsilon_{wt}$ and

$$\varkappa_t = \epsilon\varepsilon_t = \frac{\epsilon\varepsilon_0}{(t+\theta)^{1+\alpha}}\,(\epsilon, \alpha, \varepsilon_0 > 0)$$

takes place.

Proof: From (18) and (32) it follows that for $t_0 = 0$ we have $\zeta_0 = 0$, and the relation (21) becomes as in (24). ∎

VII. Illustrative Example

To illustrate the suggested approach let us consider the two links robot manipulator with three revolute joints (with coordinates $q = (\phi_1, \phi_2, \phi_3) \in R^3$), which are powered by individual PMDC motors (see Fig. 2). The low base allows to realize a 360°- rotation over the XY plane. It has the radius of 10 cm and the thickness of 3 cm. This base supports a link, composed by a cylinder with a longitude (l_1) and 1 cm radius, coupled to a second link conformed by another cylinder of the same thickness and the longitude (l_2).

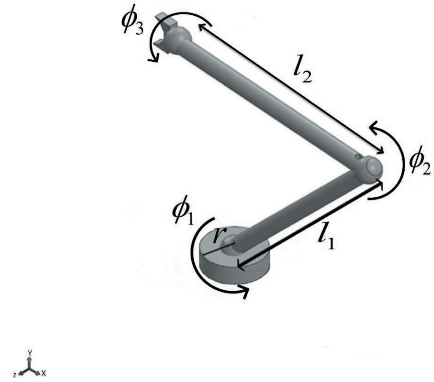

Fig. 2. Manipulator robot of three degrees of freedom.

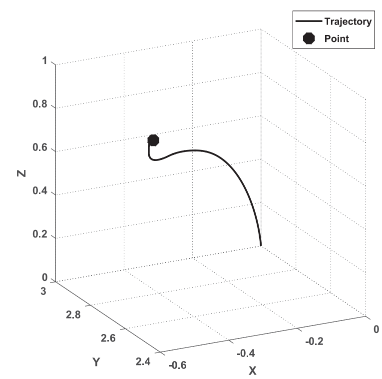

Fig. 3. Trajectory in 3-D arriving to the desired point.

The problem consists in moving the robot to a desired point and stabilize it there. During the simulation the following form of nonpotential forces (friction, hysteresis, Coriolis, damping, centripetal effects, and others) have been modeled as

$$\begin{gathered}Q_{\text{nonpot}} = C(q_t, \dot{q}_t)\,\dot{q}_t = -k_{\text{res}}||\dot{q}_t||^2\text{Sign}(\dot{q}_t) \\ k_{\text{res}} > 0.\end{gathered}$$

The corresponding cost function F to be minimized is as follows:

$$\begin{gathered}F = |q_1 - q_1^*| + |q_2 - q_2^*| + |q_3 - q_3^*|, \\ \partial F = \left(\text{sign}(q_1\text{-}q_1^*), \text{sign}(q_2\text{-}q_2^*), \text{sign}(q_3\text{-}q_3^*)\right)\end{gathered} \tag{40}$$

with $q^* = (-0.59, 2.48, 0.9)$. The parameters, given below, have been used for the computer simulations

Parameter	Numerical Value	Description
k_{res}	$1.1e^{-6}$	environmental (air) resistance
g	9.81	Gravitational acceleration
m_1, m_2	1	Mass
l_1, l_2	$0.35m, 0.67m$	Length

$$W = \begin{bmatrix} 0.05 & 0.04 & 0.06 \\ 0.04 & 0.044 & 0.022 \\ 0.06 & 0.022 & 0.057 \end{bmatrix}, L_a = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} 0.021 & 0.0006 & 0 \\ 0.0006 & 0.021 & 0 \\ 0 & 0 & 0.0407 \end{bmatrix}, K_a = \begin{bmatrix} 0.02 & 0.04 & 0.06 \\ 0.04 & 0.08 & 0.1 \\ 0.06 & 0.1 & 0.04 \end{bmatrix}$$

and $K_e = \text{diag}\{11, 13, 21\}$, $\theta = 10^{-3}$, $\eta = [-0.65\ \ -0.065\ \ 0.65]$. Fig. 3 shows the obtained trajectory in 3-D and the corresponding

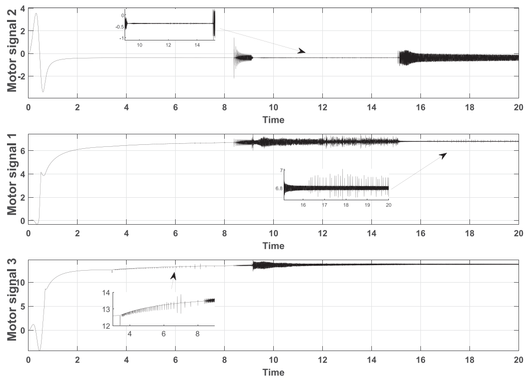

Fig. 4. Components of the control action v_{at}.

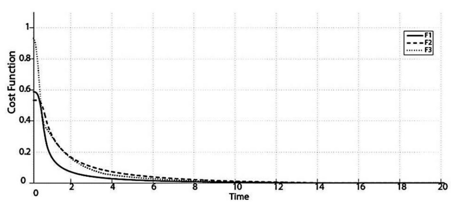

Fig. 5. Component-wise cost functions $F_i = |x_i - x_i^*|$.

components. Fig. 4 displays the realized control signal v_{at} and the component-wise cost functions are shown in Fig. 5.

VIII. Conclusion

Here, a new algorithm for convex optimization, realized by a Lagrangian uncertain dynamic plant, is proposed and analyzed. It is based on the ISM approach application, using a continuous-time version of the, socalled, averaged subgradients technique for online optimization of a unknown loss function (when only its subgradient is available online). The numerical example, dealing with a two links robot manipulator, demonstrates a good workability of the suggested method.

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