Adaptive AEM Controller For A Wide Class of Nonlinear Discrete-Time Systems Using On-Line State Estimation

David Velázquez[1] and Alexander Poznyak[2]

***Abstract*— This paper describes the application of the Attractive Ellipsoid Method (AEM), which uses the state estimates obtained by a sliding mode observer (SMO), for a wide class of quasi-Lipschitz nonlinear stochastic discrete time systems. For the extended vector, containing state estimation and tracking errors as its components, we prove the mean square convergence to an attractive ellipsoid, which "size" is done as small as possible by the corresponding optimal selection of the gain matrices in both the sliding mode observer and in the linear feedback, using obtained current state estimates. It is shown that the procedure of the gain matrices optimization consists in the numerical solution of a corresponding matrix optimization problem subject a set of bilinear matrix inequalities (BMIs), which by a special transformation procedure can be converted to a set of linear matrix inequalities (LMIs). Two illustrative examples for 2 and 4 dimensional systems illustrate the effectiveness of the suggested approach.**

I. INTRODUCTION

The *Attractive Ellipsoid Method* (AEM) provides researchers a special tool for designing of linear feedbacks for a wide class of nonlinear systems containing both uncertainties in the description of the model and possible external bounded perturbations [1]. Usually the application of this method requires the exact knowledge (availability) of all current states and control actions in the use. When the required variables or their part are not available on-line, a possible approaches consists in the realization of a state estimation process with the direct usage of them in the applied control actions. In some sense such construction may be treated as an adaptive controller, which in our case is referred as to the adaptive AEM. As an example of such approach, in the deterministic case, we can mention the recent paper [2], presents a method to identify an unknown discrete-time nonlinear system, using high-order neural networks and high-order sliding mode algorithms, which are subject to internal and external disturbances. A SMO have also been applied in systems with deterministic bounded perturbations [3], [4], [5]. In [6], an adaptive sliding mode control strategy, based on the extended equivalent control, is developed. The adaptation rule combines the qualities of monotonically increasing gains and the equivalent control. Here we consider the wide class of quasi-Lipschitz nonlinear *stochastic discrete time systems* where the state estimates are obtained by the special version of SMO, providing an acceptable mean square level of state space estimation accuracy. Specific feature of stochastic systems consists in the consideration of unbounded random external perturbation that makes impossible the direct application of AEM and SMO approaches: some special constractions and extentions are required. So, in [7] the network-based sliding mode observer is investigated for a class of discrete nonlinear time-delay systems with stochastic communication protocol. The stochastic communication protocol is governed by a Markov chain, which converts the protocol-constrained system into a Markovian jump system. The purpose is to design a sliding mode observer such that, with the stochastic communication protocol, the trajectories of the estimation error system are driven into a band of the sliding surface and, in subsequent time, the sliding motion is mean-square asymptotically stable. By solving a minimization problem, the sufficient conditions for the desired sliding mode observer are established. In [8] an adaptive sliding mode observer is designed to reconstruct the states of non-linear stochastic continuous time systems with uncertainties from the measurable system output and the reconstructed states are employed to construct a sliding mode controller for the stabilization control of complex non-linear systems. It takes the advantages of the sliding mode schemes to design both the observer and the controller. The convergence of the observer and the globally asymptotic stability of the controller are analyzed in terms of stochastic Lyapunov stability, and the effectiveness of the control strategy is verified with numerical simulation studies. The number of works, in which the methodology of sliding modes is applied to observe or control the Discrete Time Stochastic Systems, is in fact very limited [9], [10], [11], [12], [13], [14] and basically deals with linear models (see, for example, [15]). The stability analysis of nonlinear discrete time stochastic systems can be found in [19]. The recent and most advanced studies, concerning the sliding mode observers design for discrete time systems, can be found in [16], [17] and [18]. The merit of this paper is to propose the exact mechanism for designing an adaptive version of AEM and SMO, which, working simultaneously, provide a good behavior in some probabilistic sense for a wide class of uncertain nonlinear stochastic systems.

II. STOCHASTIC DISCRETE - TIME NONLINEAR PLANT

A. Model of the process

Consider the stochastic discrete-time system

$$\left.\begin{array}{r} x(k+1)=f(k,x(k))+Bu(k)+\xi(k+1)\in\mathbb{R}^n \\ y(k)=Cx(k)+\zeta(k)\in\mathbb{R}^m \\ u(k)\in\mathbb{R}^l,\quad k=0,1,2... \end{array}\right\} \quad (1)$$

[1] Automatic Control Department, CINVESTAV-Zacatenco, Av. IPN 2508, CDMX, México dvelazquez@ctrl.cinvestav.mx

[2] Automatic Control Department, CINVESTAV-Zacatenco, Av. IPN 2508, CDMX, México apoznyak@ctrl.cinvestav.mx

Random sequence $\{y(k)\}_{k\geq 0}$ is available during the process, but $\{x(k)\}_{k\geq 0}$ not. Mesurable input is $\{u(k)\}_{k\geq 0}$. $\xi(k+1)$ and $\zeta(k)$ are the input and output stochastic noises, respectively. This sequences are defined on the probability space $\left(\Omega, \{\mathscr{F}_k\}_{k\geq 0}, P\right)$, where $\{\mathscr{F}_k\}_{k\geq 0}$ is a flow of the σ -algebras $\mathscr{F}_k$, which for each $k=0,1,...$ is a minimal sigma-algebra, generated by the prehistory of the process, i.e.,

$$\mathscr{F}_k = \sigma\left\{x(0), u(0), \xi_y(0); ...; x(k), u(k), \xi_x(k), \xi_y(k)\right\}. \tag{2}$$

B. Main assumptions

Suppose that

A1) Random variables $\xi_x(k+1)$ and $\xi_y(k)$ are independent martingal-differences, namely,

$$\left.\begin{array}{c} \mathrm{E}\left\{\xi_x(k+1) \mid \mathscr{F}_k\right\} \overset{a.s.}{=} 0,\ \mathrm{E}\left\{\xi_y(k) \mid \mathscr{F}_k\right\} \overset{a.s.}{=} 0, \\ \mathrm{E}\left\{\xi_x(k+1)\xi_y^{\mathsf{T}}(k) \mid \mathscr{F}_k\right\} \overset{a.s.}{=} 0 \end{array}\right\} \tag{3}$$

with bounded conditional covariation matrices

$$\left.\begin{array}{c} \mathrm{E}\left\{\xi_x(k+1)\xi_x^{\mathsf{T}}(k+1) \mid \mathscr{F}_k\right\} \overset{a.s.}{\leq} \Xi_x, \\ \mathrm{E}\left\{\xi_y(k)\xi_y^{\mathsf{T}}(k) \mid \mathscr{F}_k\right\} \overset{a.s.}{\leq} \Xi_y; \end{array}\right\} \tag{4}$$

A2) the nonliner mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ is supposed to be *a* priory *unknown* but belonging to the class $C(A, f_0, f_1)$ of *quasi-Lipchitz functions* (see [1]), which means

$$\|f(x(k), k) - Ax(k)\|^2 \leq f_0 + f_1\|x(k)\|^2 \tag{5}$$

globaly on $\mathbb{R}^n$;

A3) The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$ and $C \in \mathbb{R}^{m\times n}$ are assumed to be known such that the pair (C, A) is observableand, and (A, B) is controllable.

Here $\mathrm{E}\{\cdot/\mathscr{F}_k\}$ and $\mathrm{E}\{\cdot\}$ represent the operators of conditional and complete mathematical expectation.

III. Problem statement

Before the formulation problem we need to describe the class of observer and controller which will be considered.

A. Sliding mode observer

The on-line state estimates $\{\hat{x}(k)\}_{k\geq 0}$ of $\{x(k)\}_{k\geq 0}$ is generated by the SMO:

$$\left.\begin{array}{c} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L\sigma(k) + L_a \mathrm{Sign}(\sigma(k)) \in \mathbb{R}^n, \\ \sigma(k) = y(k) - C\hat{x}(k) \in \mathbb{R}^m. \end{array}\right\} \tag{6}$$

B. Robust controller

The control actions will be deigned as a linear feedback

$$\left.\begin{array}{c} u(k) := K\hat{x}(k) + v(k), \\ v(k) := -Kx_k^* - B^+[Ax_k^* - \varphi(k+1, x_k^*)], \\ BB^+B = B,\ B^+BB^+ = B^+ \end{array}\right\} \tag{7}$$

depending on the desired dynamics given by $x^*(k) = \varphi(k, x^*(k-1)) \in \mathbb{R}^n$.

C. Problem formulation

The problem which we are inteded to resolve in this paper can be formulated as follows.

Problem 1: For the extended vector

$$z(k) = \left(\begin{array}{cc} \delta^{\mathsf{T}}(k) & e^{\mathsf{T}}(k) \end{array}\right)^{\mathsf{T}} \in \mathbb{R}^{2n} \tag{8}$$

with the components, defined as

$$\delta(k) := x(k) - x^*(k),\ e(k) := x(k) - \hat{x}(k), \tag{9}$$

where $\delta(k)$ is the tracking error and $e(k)$ the state estimation error, to find the gain matrices $K \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{n\times m}$ and $L_a \in \mathbb{R}^{n\times m}$ such that the joint mean square weighted error $\mathrm{E}\{z^{\mathsf{T}}(k)P_z z(k)\}$ belongs asymptotically to the **stochastic attractive ellipsoid**, fulfilling the inequality

$$\limsup_{k\to\infty} \mathrm{E}\{z^{\mathsf{T}}(k)P_z z(k)\} \leq 1 \tag{10}$$

for any admissible nonlinearity $f \in C(A, f_0, f_1)$.

IV. Zone-convergence analysis

A. Tracking error

The tracking error dynamics in (9) with the control (7) results in:

$$\left.\begin{array}{c} \delta(k+1) = (A+BK)\delta(k) - BKe(k) + \vartheta(k) \\ \vartheta(k) := \hat{\xi}(k+1) - \hat{\delta}(k), \\ \hat{\delta}(k) := \varphi(k+1, x^*(k)) - Ax^*(k). \end{array}\right\} \tag{11}$$

B. Observation error

For the observation error $e(k)$ in (9) it follows

$$\left.\begin{array}{c} e(k+1) = (A-LC)e(k) - L_a \mathrm{Sign}(\sigma(k)) + \omega(k), \\ \omega(k) := \hat{\xi}(k+1) - L\zeta(k), \\ \mathrm{Sign}(\sigma) := (\mathrm{sign}(\sigma_1), ..., \mathrm{sign}(\sigma_n))^{\mathsf{T}}, \\ \mathrm{sign}(\sigma_i) := \left\{\begin{array}{ccc} 1 & \text{if} & \sigma_i > 0 \\ -1 & \text{if} & \sigma_i < 0 \\ [-1,1] & \text{if} & \sigma_i = 0 \end{array}\right. . \end{array}\right\} \tag{12}$$

C. Storage function analysis

Theorem 1: If matrices P, K, L, L_a and scalars α, β, γ are selected in such a way that

$$\left.\begin{array}{c} \tilde{W}(P, K, L, L_a \mid \alpha, \lambda, \gamma) = \\ \left(\begin{array}{ccc} 3\tilde{A}^{\mathsf{T}}P\tilde{A} + \tilde{\Lambda}_\delta - \alpha P & 0 & 0 \\ 0 & 2Q^{\mathsf{T}}PQ - \lambda I & Q^{\mathsf{T}}P \\ 0 & PQ & 2P - \gamma I \end{array}\right) \leq 0, \end{array}\right\} \tag{13}$$

with

$$\tilde{\Lambda}_\delta = \left[\begin{array}{cc} 6\gamma f_1 I_{n\times n} & 0 \\ 0 & 0 \end{array}\right], \tag{14}$$

then for the storage (Lyapunov-like) function

$$V(k) = z^{\mathsf{T}}(k)Pz(k),\ 0 < P = \left(\begin{array}{cc} P_1 & 0 \\ 0 & P_2 \end{array}\right), \tag{15}$$

we may guarantee that

$$\mathrm{E}\{V_{k+1}\} \leq \alpha \mathrm{E}\{V(k)\} + \tilde{\beta}_k(L). \tag{16}$$

where

$$\left.\begin{array}{c}\tilde{\beta}_k(L) := m\lambda + \gamma\left(2\mathrm{tr}\left\{\Sigma_x\right\} + 3f_0 + 6f_1\left\|x_k^*\right\|^2 + 2\|\tilde{\delta}_k\|^2\right) \\ +\gamma\mathrm{tr}\left\{L\Sigma_y L^\intercal\right\},\ \tilde{\delta}_k = (I - BB^+)\,\hat{\delta}(k).\end{array}\right\} \tag{17}$$

Proofs of this and the following next statements can be found in Appendix.

D. Analytical representation of attractive ellipsoid

Taking θ as upper bound of $\mathrm{tr}\left\{L^\intercal\Sigma_y L\right\}$ in (17), we can write:

$$\mathrm{tr}\left\{\frac{\theta}{n}I_{n\times n} - L\Sigma_y L^\intercal\right\} \geq 0, \tag{18}$$

then, $\limsup\limits_{k\to\infty}$ of (16), is:

$$\begin{array}{c}\limsup\limits_{k\to\infty}\mathrm{E}\left\{V_k\right\} \leq \frac{\tilde{\beta}_k(L)}{1-\alpha} \leq \frac{\psi(\beta,\gamma,\theta)}{1-\alpha}, \\ \psi(\beta,\gamma,\theta) := m\beta + \gamma\left(2\mathrm{tr}\left\{\Sigma_x\right\} + 3f_0 + 6f_1X_+^* + 2\Delta_+^* + \theta\right), \\ \left\|x_k^*\right\|^2 \leq X_+^*, \ \text{ and } \ \left\|\tilde{\delta}_k\right\|^2 \leq \Delta_+^*,\end{array}$$

This may be represented as

$$\limsup\limits_{k\to\infty}\mathrm{E}\left\{z_k^\intercal\ P_z\ z_k\right\} \leq 1,\ P_z := \frac{1-\alpha}{\psi(\beta,\gamma,\theta)}\begin{pmatrix} P_1 & 0 \\ 0 & P_2\end{pmatrix},$$

which, according to the difinition (10), defines the stochastic attractive ellipsoid with matrix P_z.

E. Gain matrix optimization

To minimize the errors e_k and δ_{k+1}, we need to maximize P_z with respect to the matrices L_a, L, K and the scalar positive parameters α,β,γ. The optimal matrix gains L_a^*, L^*, K^* are suggested to be found as the solution of the following optimization problem

$$\mathrm{tr}\left\{\frac{1-\alpha}{\psi(\beta,\gamma,\theta)}\begin{pmatrix} P_1 & 0 \\ 0 & P_2\end{pmatrix}\right\} \to \sup_{P_1>0,P_2>0,L_a,L,K;\alpha>0,\beta>0,\gamma>0}$$

under constraints (13) and (18). These constraints are bilinear ones, to apply Sedumi Matlab Package, we need to transform them into linear ones. The next theorem show this.

Theorem 2: Using change of variable

$$X_1^* = P_1^*,\ X_2^* = P_2^*,\ K^* = G^*, L^* = (X_2^*)^{-1}Y_1^*,\ L_a^* = (X_2^*)^{-1}Y_2^*,$$

inequalities (13) and (18) are fulfilled if the following LMIs hold:

$$\bar{W}^+ \leq 0,\ W_{Q_1} \geq 0,\ W_{Q_2} \geq 0,\ Q_1 \geq 0,\ Q_2 \geq 0, W_\theta \geq 0 \tag{19}$$

here

$$\bar{W}^+ := \begin{pmatrix} -Q_1 & 0 & 0 \\ 0 & -Q_2 & W_{Y_2} \\ 0 & W_{Y_2} & 2X - \gamma I\end{pmatrix}, W_{Y_2} := \begin{bmatrix} 0 & 0 \\ 0 & Y_2\end{bmatrix}$$

$$W_{Q_1} := \begin{pmatrix} \frac{1}{3}\left(\alpha X - \tilde{\Lambda}_\delta - Q_1\right) & W_{Q_1}(1,2) \\ W_{Q_1}(2,1) & \begin{bmatrix}\frac{\gamma}{2}I & 0 \\ 0 & X_2\end{bmatrix}\end{pmatrix},$$

$$W_{Q_1}(1,2) = W_{Q_1}^\intercal(2,1) := \begin{pmatrix} (A+BG)^\intercal & 0 \\ -G^\intercal B^\intercal & A^\intercal X_2 - C^\intercal Y_1\end{pmatrix}$$

$$W_{Q_2} := \begin{pmatrix} \frac{1}{2}\left(\lambda I - Q_2\right) & W_{Y_2} \\ W_{Y_2} & X\end{pmatrix}, W_\theta := \begin{bmatrix} \frac{\theta}{n}I_{n\times n} & \frac{\gamma}{2}Y_1 \\ \frac{\gamma}{2}Y_1^\intercal & \Sigma_y^{-1}\end{bmatrix}.$$

Notice that, with Theorem 2, for the parameters α, λ, γ and θ after some transformations the matrix inequalities (13) and (18) become LMIs. They can be solved using the LMItoolbox, SeDuMi and Yalmip. Our optimization problem can be also solved following the next two-steps:

1 we fix the scalar parameters α, λ, γ and θ, and solve the LMIs with respect to the matrix variables.
2 for the found matrix variables X_1, Y_1, X_2, Y_2 and G, solve our optimization problem only with respect to scalar parameters α, λ, γ and θ.

Finally, iterating this process we find the optimal solution $K^* = G^*$, $L^* = (X_2^*)^{-1}Y_1^*$, $L_a^* = (X_2^*)^{-1}Y_2^*$.

V. ILLUSTRATIVE NUMERICAL ACADEMIC EXAMPLES

Consider the system

$$\begin{array}{c} x(k+1) = f(x(k)) + Bu + \xi(k+1), \\ y(k) = x_1(k) + \zeta(k),\end{array}$$

which in the quasi-linear format is presented as

$$\begin{array}{c} x(k+1) = Ax(k) + Bu(k) + \hat{\xi}(k+1), \\ y(k) = Cx(k) + \zeta(k),\end{array}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k)\end{bmatrix}^\intercal$ and

$$\begin{array}{c}\hat{\xi}(k+1) = \xi(k+1) + f(x(k)) - Ax(k), \\ f(x(k)) = \begin{bmatrix} x_2(k)\sin(x_1(k)) \\ -0.1(x_1(k) + x_2(k))\end{bmatrix} \\ A = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.1\end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},\ C = \begin{bmatrix} 1 & 0\end{bmatrix}.\end{array}$$

For (5), $f_0 = 0$ and $f_1 = 1$. Notice that (A,B) is controllable and (C,A) is observable. Here, $\xi(k+1) \leq \Xi_x = 0.01I_{2\times 2}$ and $\zeta(k) \leq \Xi_y = 0.01$. The desired dynamics is given by

$$x^*(k) = \begin{bmatrix} C_1^* \\ C_2^*\sin(k)\end{bmatrix},\ C_1^* = 1,\ C_2^* = 0.5.$$

The gain optimization procedure (IV-E) leads to

$$\begin{array}{c}\alpha^* = 0.1,\ \beta = 0.1,\ \gamma = 0.9, \\ P^* = \begin{bmatrix} 0.0503 & 0 & 0 & 0 \\ 0 & 0.0503 & 0 & 0 \\ 0 & 0 & 0.4499 & -0.0012 \\ 0 & 0 & -0.0012 & 0.0021\end{bmatrix}, \\ K^* = \begin{bmatrix} 0 & -0.2596 \\ -0.2596 & 0.0499\end{bmatrix}, \\ L^* = \begin{bmatrix} -0.3 \\ -0.6\end{bmatrix},\ L_a^* = \begin{bmatrix} 0.015 \\ 0.01\end{bmatrix}.\end{array}$$

Trajectories of the real, observed and desired of states are showed on Fig.1 and Fig.2, respectively. Fig.3 and Fig.4 shows the convergence errors to the attractive ellipsoid.

VI. CONCLUSIONS

In this paper we prove the *mean square convergence* of state estimation and tracking errors by the application of AEM for the robust control design of a sufficiently large class of quasi-Lipschitz nonlinear stochastic discrete-time systems using the on-line state estimates, obtained by the corresponding sliding mode observer; the size of the convergence

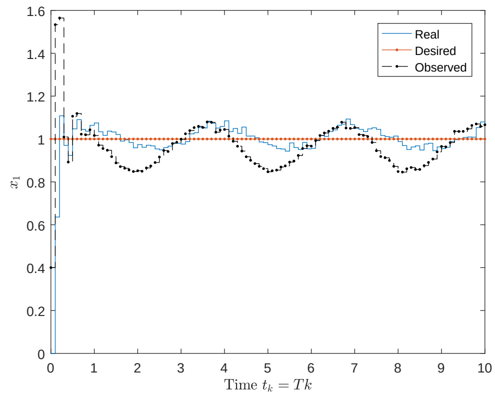

Fig. 1. Real, desired and observed trajectories of the state $x_1(k)$.

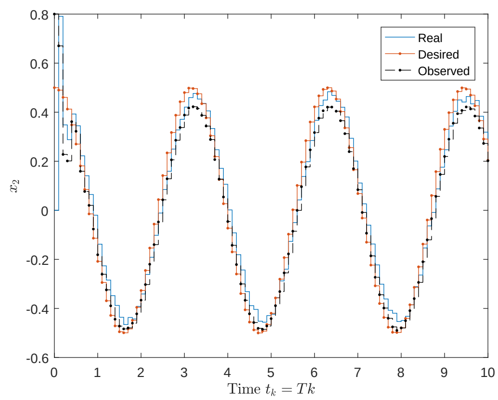

Fig. 2. Real, desired and observed trajectories of state $x_2(k)$.

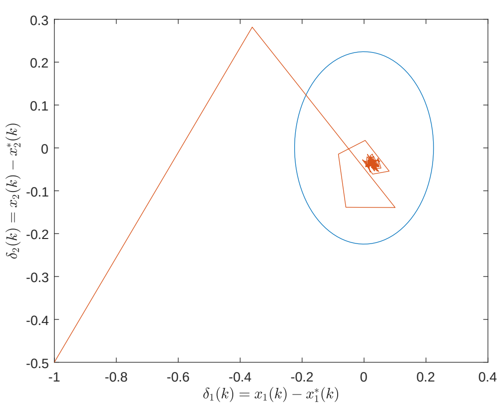

Fig. 3. Traking error $\delta(k)$ convergence.

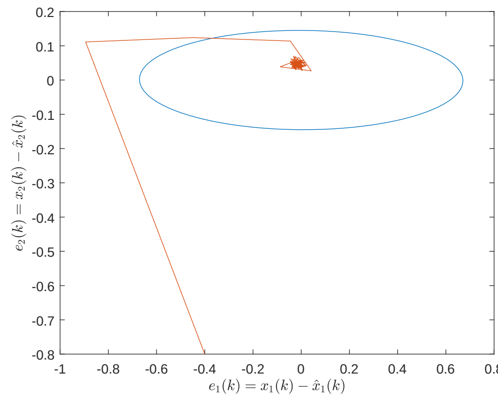

Fig. 4. The observation error $e(k)$ convergence.

zone (in this case, the trace of the corresponding ellipsoidal matrix) is minimized by the corresponding optimal selection of the gain matrices in the sliding mode observer and in the linear feedback, containing the current state estimates, the optimal selection is achieved transformation of BMIs into LMIs and using standard MATLAB packages. A numerical example illustrate the effectiveness of the suggested technique.

APPENDIX

A. Theorem 1

a) For the vector $z(k) \in \mathbb{R}^{2n}$ (8) with the control (7), we have

$$z(k+1) = \tilde{A}(K,L)z(k) + Q(L_a)s(k) + \eta(k,k+1) \tag{20}$$

where

$$\left.\begin{array}{c} \tilde{A}(K,L) = \begin{pmatrix} (A+BK) & -BK \\ 0_{n\times n} & (A-LC) \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \\ Q(L_a) = \begin{pmatrix} 0_{n\times m} \\ -L_a \end{pmatrix} \in \mathbb{R}^{2n\times m}, \\ s(k) = \mathrm{Sign}(\sigma(k)) \in \mathbb{R}^m,\ \eta(k) = \begin{pmatrix} \vartheta(k) \\ \omega(k) \end{pmatrix} \in \mathbb{R}^{2n}. \end{array}\right\} \tag{21}$$

Now, for the storage function $V(k+1)$ it follows

$$\left.\begin{array}{c} V(k+1) = z^{\intercal}(k+1)Pz(k+1) \\ = \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} \begin{pmatrix} \tilde{A}^{\intercal}P\tilde{A} & \tilde{A}^{\intercal}PQ & \tilde{A}^{\intercal}P \\ Q^{\intercal}P\tilde{A} & Q^{\intercal}PQ & Q^{\intercal}P \\ P\tilde{A} & PQ & P \end{pmatrix} \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} \end{array}\right\} \tag{22}$$

By applying Λ-matrix inequality $W^{\intercal}Z + Z^{\intercal}H \leq H^{\intercal}\Lambda H + Z^{\intercal}\Lambda^{-1}Z$, valid for $H, Z \in \mathbb{R}^{K\times M}$ and $\Lambda = \Lambda^{\intercal} > 0$, to the terms $2z^{\intercal}(k)\tilde{A}^{\intercal}PQs(k)$ and $2z^{\intercal}(k)\tilde{A}^{\intercal}P\eta(k,k+1)$, we get

$$\left.\begin{array}{c} V(k+1) \leq \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} + \\ \alpha z^{\intercal}(k)Pz(k) + \lambda\|s(k)\|^2 + \gamma\|\eta(k)\|^2,\ |\alpha| < 1, \end{array}\right\} \tag{23}$$

where

$$W := \begin{pmatrix} 3\tilde{A}^{\intercal} P\tilde{A} - \alpha P & 0 & 0 \\ 0 & 2Q^{\intercal} PQ - \lambda I & Q^{\intercal} P \\ 0 & PQ & P - \gamma I \end{pmatrix} \quad (24)$$

Taking $\mathrm{E}\{\cdot|\mathscr{F}_k\}$, in both sides of (23), we obtain

$$\begin{aligned} \mathrm{E}\{V(k+1)|\mathscr{F}_k\} &\overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathscr{F}_k \right\} \\ &+ \alpha V(k) + \lambda \|s(k)\|^2 + \gamma \mathrm{E}\left\{ \|\eta(k)\|^2 |\mathscr{F}_k \right\}. \end{aligned} \quad (25)$$

Expanding $\mathrm{E}\left\{ \|\eta(k)\|^2 |\mathscr{F}_k \right\}$ and taking into acount the relations (3), (4) and (5), we derive

$$\left. \begin{aligned} \mathrm{E}\{V(k+1)|\mathscr{F}_k\} &\overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} W \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathscr{F}_k \right\} \\ &+ \alpha \mathrm{E}\{V(k)|\mathscr{F}_k\} + \beta_k(L), \\ \beta_k(L) &:= m\lambda + \gamma\left(2\mathrm{tr}\{\Sigma_x\} + 3\left(f_0 + f_1 \|x(k)\|^2\right)\right) + \\ &\gamma\left(2\left\|\tilde{\delta}(k)\right\|^2 + \mathrm{tr}\left\{L\Sigma_y L^{\intercal}\right\}\right) \end{aligned} \right\} \quad (26)$$

b) Since $x(k) = \delta(k) + x^*(k)$, it follows $\|x(k)\|^2 \leq 2\|\delta(k)\|^2 + 2\|x^*(k)\|^2$. The term $2\|\delta(k)\|^2$ can be included in quadratic form. We also may include the term $2\left\|x_k^*\right\|^2$ into $\beta_k(L)$ that leads to

$$\left. \begin{aligned} \mathrm{E}\{V(k+1)|\mathscr{F}_k\} &\overset{a.s.}{\leq} \mathrm{E}\left\{ \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix}^{\intercal} \tilde{W} \begin{pmatrix} z(k) \\ s(k) \\ \eta(k) \end{pmatrix} |\mathscr{F}_k \right\} \\ &+ \alpha \mathrm{E}\{V(k)|\mathscr{F}_k\} + \tilde{\beta}_k(L). \end{aligned} \right\} \quad (27)$$

If the matrices P, K, L, L_a and scalars α, β, γ are selected in such a way that $\tilde{W} \leq 0$, from (23) we get

$$\mathrm{E}\{V(k+1)|\mathscr{F}_k\} \leq \alpha \mathrm{E}\{V(k)|\mathscr{F}_k\} + \tilde{\beta}_k(L). \quad (28)$$

Taking the complete mathematical expectation of (28) we finally obtain (16).

B. Theorem 2

Let us introduce the change of variable $X_1 = P_1$, $X_2 = P_2$, $K = G$, $Y_1 = P_2 L$, $Y_2 = P_2 L_a$. Entering the matrices $Q_1 > 0$ and $Q_2 > 0$ such that:

$$3\tilde{A}^{\intercal}(K,L) P\tilde{A}(K,L) + \tilde{\Lambda}_{\delta} - \alpha P \leq -Q_1 \leq 0, \quad (29)$$

$$2Q^{\intercal}(L_a) PQ(L_a) - \lambda I \leq -Q_2 \leq 0. \quad (30)$$

Substituting this in (13) we get $\tilde{W} \leq \tilde{W}^+$. Representing (30) as $0 \leq \frac{1}{3}\left(\alpha P - Q_1 - \tilde{\Lambda}_{\delta}\right) - \tilde{A}^{\intercal}(K,L) PP^{-1} P\tilde{A}(K,L)$, and applying the Schur's complement we get

$$\begin{bmatrix} \frac{1}{3}\left(\alpha P - \tilde{\Lambda}_{\delta} - Q_1\right) & \tilde{A}^{\intercal} \begin{bmatrix} I & 0 \\ 0 & P_2 \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & P_2 \end{bmatrix} \tilde{A} & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} \geq 0. \quad (31)$$

Now, from $\bar{W}^+ \leq 0$, (31) and taking into acount the change of varibale introduced above, it's easy to see that $X \leq \frac{\gamma}{2} I \leftrightarrow X^{-1} \geq \frac{2}{\gamma} I$, which implies $W_{Q_1} \geq 0$. In turn, by the same reasoning, applying the Schurs complement to inequality (30) and substitution $Y_2 = P_2 L_a$, results $W_{Q_2} \geq 0$. Aditionally, in view of the relation $Y_1 = LX_2$ and applying, again, the Schur's complement into (18), we have $W_{\theta} \geq 0$

References

[1] A. Poznyak, A. Polyakov, V. Azhmyakov, Attractive ellipsoids in robust control, Springer International Publishing, MA: Birkhauser, 2014.

[2] M. Hernandez, M.V. Basin, E. A. Hernandez, Discrete-time high-order neural network identifier trained with high-order sliding mode observer and unscented Kalman filter, Neurocomputing, 2019.

[3] F. J. Bejarano, L. Fridman, A. Poznyak, Exact state estimation for linear systems with unknown inputs based on hierarchical super-twisting algorithm, Wiley Online Library, International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal, vol. 17, no. 18, pp. 1734–1753, 2007.

[4] Moreno, A. Jaime, M. Osorio, A Lyapunov approach to second-order sliding mode controllers and observers, 2008 47th IEEE conference on decision and control, pp. 2856–2861.

[5] J. Davila, L. Fridman, A. Poznyak, Observation and identification of mechanical systems via second order sliding modes, International Journal of Control, vol. 79, no. 10, pp. 1251–1262, 2006.

[6] T.R. Oliveira, J. P. Cunha, L. Hsu, Adaptive sliding mode control based on the extended equivalent control concept for disturbances with unknown bounds, Advances in Variable Structure Systems and Sliding Mode Control Theory and Applications, Springer, pp. 149–163, 2018.

[7] S. Chen, J. Guo, L. Ma, Sliding Mode Observer Design for Discrete Nonlinear Time-delay Systems with Stochastic Communication Protocol, International Journal of Control, Automation and Systems, vol. 17, no. 7, pp. 1666-1676, 2019.

[8] F. Qiao, Q. M. Zhu, J. Liu, F. Zhang, Adaptive observer-based nonlinear stochastic system control with sliding mode schemes, Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering, vol. 222, no. 7, pp. 681-690, 2008.

[9] L. Wu, P. Shi, H. Gao, State estimation and sliding-mode control of Markovian jump singular systems, IEEE Transactions on Automatic Control, vol. 55, no. 5, pp. 1213-1219, 2010.

[10] K. Abidi, J. X. Xu, Y. Xinghuo, On the discrete-time integral sliding-mode control, IEEE Transactions on Automatic Control, vol. 52, no. 4, pp. 709-715, 2007.

[11] L. Wu, W. Zheng, H. Gao, Dissipativity-based sliding mode control of switched stochastic systems, IEEE Transactions on Automatic Control, vol. 1, no. 1, pp. 1, 2013.

[12] T. Kailath, Linear systems, Prentice-Hall Englewood Cliffs, vol. 156, 1980.

[13] M. A. Alcorta, M. Basin, Y. Sánchez, Risk-sensitive approach to optimal filtering and control for linear stochastic systems, International Journal of Innovative Computing, Information and Control, vol. 5, no. 6, pp. 1599-1614, 2009.

[14] M. Basin, P. C. Rodríguez, Sliding mode controller design for stochastic polynomial systems with unmeasured states, IEEE Transactions on Industrial Electronics, vol. 61, no. 1, pp. 387–396, 2013.

[15] S. Janardhanan, S. Singh, Sliding mode control-based linear functional observers for discrete-time stochastic systems, International Journal of Systems Science, vol. 48, no. 15, pp. 3246-3253, 2017.

[16] S. Janardhanan, B. Bandyopadhyay, Multirate output feedback based robust quasi-sliding mode control of discrete-time systems, IEEE Transactions on Automatic Control, vol. 52, no. 3, pp. 499-503, 2007.

[17] S. Janardhanan, B. Bandyopadhyay, Discrete-time sliding mode control: a multirate output feedback approach, Springer Science & Business Media, vol. 323, 2005.

[18] H. Alazki, A. Poznyak, Constraint robust stochastic discrete-time tracking: Attractive ellipsoids technique, IEEE, 2010 7th International Conference on Electrical Engineering Computing Science and Automatic Control, pp. 99–104.

[19] M. Bensoubaya, A. Ferfera, A. Iggidr, Stabilization of nonlinear stochastic discrete systems, Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No. 99CH36304), vol. 4, pp. 3180-3181, 1999.