Linear Complementarity Problems

A short Introduction to Definitions and Numerical Methods

Kenny Erleben
kenny@diku.dk

Department of Computer Science
University of Copenhagen

2010

The Definition

The Complementarity Problem

If given $x, y \in \mathbb{R}$ where

$$x \geq 0$$
$$y \geq 0$$

and

$$x > 0 \Rightarrow y = 0$$
$$y > 0 \Rightarrow x = 0$$

Then we have a complementarity problem. If for some $a, b \in \mathbb{R}$

$$y = ax + b$$

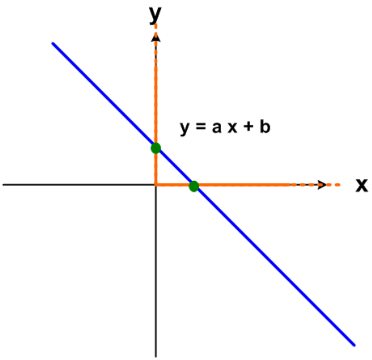
We have a Linear Complementarity Problem (LCP).

The Linear Complementarity Problem

More compact notation

$$
\begin{aligned}
x &\geq 0 \\
ax + b &\geq 0 \\
x(ax + b) &= 0
\end{aligned}
$$

The Geometry



How many #Solutions of LCP?

Hint: Try to examine signs of a and b

	$b < 0$	$b = 0$	$b > 0$
$a < 0$			
$a = 0$			
$a > 0$			

Answer of # Solutions

Verify this using geometry

	$b < 0$	$b = 0$	$b > 0$
$a < 0$	0	1	2
$a = 0$	0	∞	1
$a > 0$	1	1	1

Going to Higher Dimensions

Let $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ so $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$,

$$\begin{aligned} \mathbf{x}_i &\geq 0 \quad \forall i \in [1..n] \\ (\mathbf{A}\mathbf{x} + \mathbf{b})_i &\geq 0 \quad \forall i \in [1..n] \\ \mathbf{x}_i(\mathbf{A}\mathbf{x} + \mathbf{b})_i &= 0 \quad \forall i \in [1..n] \end{aligned}$$

In Matrix-Vector Notation

$$\begin{aligned} \mathbf{x} &\geq 0 \\ (\mathbf{A}\mathbf{x} + \mathbf{b}) &\geq 0 \\ \mathbf{x}^T(\mathbf{A}\mathbf{x} + \mathbf{b}) &= 0 \end{aligned}$$

How can we solve this?

What kind of problem is a LCP formulation?

What do you think?

- A constrained minimization problem?
- A nonlinear equation (root search problem) ?
- A fixed-point problem?
- A combinatorial problem?
- Something else?

A NAÏVE Active Set/Pivoting Method

Guessing A Solution

Given the index set $\mathcal{I} = \{1, \dots, n\}$ define

$$\mathcal{F} = \{i \quad | \quad i \in \mathcal{I} \quad \wedge \quad \mathbf{y}_i > 0\}$$
$$\mathcal{A} = \{i \quad | \quad i \notin \mathcal{F} \quad \wedge \quad \mathbf{y}_i = 0\}$$

Make "lucky" guess of $\mathcal{F}$ and $\mathcal{A}$,

$$\begin{bmatrix} \mathbf{y}_\mathcal{A} \\ \mathbf{y}_\mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}} & \mathbf{A}_{\mathcal{A}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathcal{A} \\ \mathbf{x}_\mathcal{F} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_\mathcal{A} \\ \mathbf{b}_\mathcal{F} \end{bmatrix}$$

By assumption $\mathbf{y}_\mathcal{F} > 0 \Rightarrow \mathbf{x}_\mathcal{F} = 0$

$$\begin{bmatrix} 0 \\ \mathbf{y}_\mathcal{F} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}} & \mathbf{A}_{\mathcal{A}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathcal{A} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_\mathcal{A} \\ \mathbf{b}_\mathcal{F} \end{bmatrix}$$

Verify if Guess was a Solution

So

$$\begin{bmatrix} 0 \\ \mathbf{y}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{A}\mathcal{A}}\mathbf{x}_{\mathcal{A}} + \mathbf{b}_{\mathcal{A}} \\ \mathbf{A}_{\mathcal{F}\mathcal{A}}\mathbf{x}_{\mathcal{A}} + \mathbf{b}_{\mathcal{F}} \end{bmatrix}$$

Compute

$$\mathbf{x}_{\mathcal{A}} = -\mathbf{A}_{\mathcal{A}\mathcal{A}}^{-1}\mathbf{b}_{\mathcal{A}}$$

Verify

$$\mathbf{x}_{\mathcal{A}} \geq 0$$

Compute

$$\mathbf{y}_{\mathcal{F}} = \mathbf{A}_{\mathcal{F}\mathcal{A}}\mathbf{x}_{\mathcal{A}} + \mathbf{b}_{\mathcal{F}}$$

Verify

$$\mathbf{y}_{\mathcal{F}} > 0$$

How Many Guesses?

We only need

$$\mathbf{A}_{\mathcal{A}\mathcal{A}}^{-1}$$

Hopefully

$$\| \mathcal{A} \| \ll n$$

Cool this will be fast!

How many guesses do we need?

Answer: Non-Polynomial Complexity

Worst case time complexity of guessing

$$\mathcal{O}(n^3 2^n)$$

Not computational very efficient!

Connection to Positive Cones (Linear Programming)

First some algebra on $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$,

$$\mathbf{I}\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \mathbf{b}$$

Make guesses $\mathcal{F} = \{i | \mathbf{y}_i \geq 0\}$ and $\mathcal{A} = \{i | \mathbf{x}_i \geq 0\}$ so $\mathcal{F} \cap \mathcal{A} = \emptyset$ and $\mathcal{F} \cup \mathcal{A} = \{1, \ldots, n\}$

$$\underbrace{\begin{bmatrix} \mathbf{I}_{\cdot\mathcal{F}} & -\mathbf{A}_{\cdot\mathcal{A}} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \mathbf{y}_{\mathcal{F}} \\ \mathbf{x}_{\mathcal{A}} \end{bmatrix}}_{\mathbf{z}} = \mathbf{b}$$

Verify if LP

$$\mathbf{M}\mathbf{z} = \mathbf{b} \quad \text{subject to} \quad z \geq 0$$

Has a solution (same as $\mathbf{b}$ in positive cone of $\mathbf{M}$).

The Projected Gauss–Seidel Method

Use a Splitting Method

Use the splitting

$$\mathbf{A} = \mathbf{M} - \mathbf{N}$$

then

$$\begin{aligned} \mathbf{Mx} - \mathbf{Nx} + \mathbf{b} &\geq 0 \\ \mathbf{x} &\geq 0 \\ (\mathbf{x})^T(\mathbf{Mx} - \mathbf{Nx} + \mathbf{b}) &= 0 \end{aligned}$$

Use Discretization $\Rightarrow$ Fixed Point Formulation

Create a sequence of sub problems

$$\begin{aligned} \mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k &\geq 0 \\ \mathbf{x}^{k+1} &\geq 0 \\ (\mathbf{x}^{k+1})^T(\mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k) &= 0 \end{aligned}$$

where

$$\mathbf{c}^k = \mathbf{b} - \mathbf{N}\mathbf{x}^k$$

Use Minimum Map Reformulation

Given sub problem

$$\begin{aligned} \mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k &\geq 0 \\ \mathbf{x}^{k+1} &\geq 0 \\ (\mathbf{x}^{k+1})^T(\mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k) &= 0 \end{aligned}$$

Same as (Why?)

$$\underbrace{\min(\mathbf{x}^{k+1}, \mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k)}_{\mathbf{H}(\mathbf{x}^{k+1})} = 0$$

A root search problem: $\mathbf{H}(\mathbf{x}^{k+1}) = 0$.

The Minimum Map Formulation

Say $a, b \in \mathbb{R}$ are complementary

$$a > 0 \Rightarrow b = 0$$
$$b > 0 \Rightarrow a = 0$$

Let us look at the minimum map

$\min(a, b)$	$a > 0$	$a = 0$	$a < 0$
$b > 0$	$+$	0	$-$
$b = 0$	0	0	$-$
$b < 0$	$-$	$-$	$-$

Same solutions as complementarity problem.

More Clever Manipulation

So

$$\min(\mathbf{x}^{k+1}, \mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k) = 0$$

Subtract $\mathbf{x}^{k+1}$

$$\min(0, \mathbf{M}\mathbf{x}^{k+1} + \mathbf{c}^k - \mathbf{x}^{k+1}) = -\mathbf{x}^{k+1}$$

Multiply by minus one

$$\underbrace{\max(0, -\mathbf{M}\mathbf{x}^{k+1} - \mathbf{c}^k + \mathbf{x}^{k+1})}_{\mathbf{F}(\mathbf{x}^{k+1})} = \mathbf{x}^{k+1}$$

A fixed point formulation: $\mathbf{F}(\mathbf{x}^{k+1}) = \mathbf{x}^{k+1}$.

Do A Case-by-Case Analysis

So

$$\max(0, -\mathbf{M}\mathbf{x}^{k+1} - \mathbf{c}^k + \mathbf{x}^{k+1}) = \mathbf{x}^{k+1}$$

If

$$(-\mathbf{M}\mathbf{x}^{k+1} - \mathbf{c}^k + \mathbf{x}^{k+1})_i \leq 0$$

Then

$$\mathbf{x}_i^{k+1} = 0$$

Else

$$(-\mathbf{M}\mathbf{x}^{k+1} - \mathbf{c}^k + \mathbf{x}^{k+1})_i > 0$$

and

$$(\mathbf{x}^{k+1} - \mathbf{M}\mathbf{x}^{k+1} - \mathbf{c}^k)_i = \mathbf{x}_i^{k+1}$$

That is

$$(\mathbf{M}\mathbf{x}^{k+1})_i = -c_i^k$$

Putting it Together

For suitable choice of $\mathbf{M}$

$$(\mathbf{M}\mathbf{x}^{k+1})_i = -c_i^k \Rightarrow \mathbf{x}_i^{k+1} = (-\mathbf{M}^{-1}\mathbf{c}^k)_i$$

Back-substitution of $\mathbf{c}^k = \mathbf{b} - \mathbf{N}\mathbf{x}^k$ we have

$$\left(\mathbf{M}^{-1}\left(\mathbf{N}\mathbf{x}^k - \mathbf{b}\right)\right)_i = \mathbf{x}_i^{k+1}$$

Insert in fixed point formulation

$$\underbrace{\max\left(0, \left(\mathbf{M}^{-1}\left(\mathbf{N}\mathbf{x}^k - \mathbf{b}\right)\right)\right)}_{\mathbf{G}(\mathbf{x}^k)} = \mathbf{x}^{k+1}$$

Closed form solution for sub problem: $\mathbf{x}^{k+1} \leftarrow \mathbf{G}(\mathbf{x}^k)$.

Final Iterative Scheme – The Projected Gauss–Seidel (PGS) method

Given $\mathbf{x}^1$ set $k = 1$

Step 1 Compute

$$\mathbf{z}^k = \left(\mathbf{M}^{-1}\left(\mathbf{N}\mathbf{x}^k - \mathbf{b}\right)\right)$$

Step 2 Compute

$$\mathbf{x}^{k+1} = \max(0, \mathbf{z}^k)$$

Step 3 If convergene then return $\mathbf{x}^{k+1}$ otherwise goto Step 1

Study Group Work

Derive an algebraic equation of the PGS method for the i^{th} component only. That is rewrite

$$\mathbf{x}_i^{k+1} = \max(0, \mathbf{z}_i^k)$$

using $\mathbf{r} = \mathbf{A}\mathbf{x}^k + \mathbf{b}$ and letting

$$\mathbf{M} = \mathbf{diag}(\mathbf{diag}(\mathbf{A})) + \mathbf{tril}(\mathbf{A}, -1)$$
$$\mathbf{N} = -\mathbf{triu}(\mathbf{A}, 1)$$

Here we used Matlab like notation. The final result should only include the terms $\mathbf{r}_i$, $\mathbf{A}_{ii}$ and $\mathbf{x}_i$.

The Projected Gauss–Seidel Method...

AGAIN!!!

Connection to Quadratic Programming (QP) Problems

Consider the minimization problem

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\geq 0} \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x}$$

where $\mathbf{A}$ is symmetric. First order optimality (KKT) conditions

$$\begin{aligned}\mathbf{A}\mathbf{x} + \mathbf{b} - \mathbf{I}\mathbf{y} &= 0\\ \mathbf{x} &\geq 0\\ \mathbf{y} &\geq 0\\ \mathbf{y}^T\mathbf{x} &= 0\end{aligned}$$

Same as LCP problem.

More on QP relation

From optimization theory we know

- If **A** is positive definite then we have a strict convex QP with a unique solution
- If **A** is positive semi-definite then we have a convex QP where a solution exist but it is no longer unique

This is cool if we have this prior knowledge[1] of **A**.

[1] The LCP solution needs not be a global solution of the QP or even a minimizer. It is sufficient that the LCP solution fulfills first-order optimality only.

The QP problem – Again

The LCP problem can be restated as

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \geq \mathbf{0}} f(\mathbf{x})$$

where

$$f(\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x} + \mathbf{x}^T \mathbf{b}$$

The Idea – Sweep over Coordinates

The i^{th} unit axis vector

$$\mathbf{e}^i_j = \begin{cases} 1 & i = i \\ 0 & i \neq j \end{cases}$$

The i^{th} relaxation step solves the one dimensional problem

$$\tau^* = \arg \min_{\mathbf{x}+\tau\mathbf{e}^i \geq \mathbf{0}} f(\mathbf{x} + \tau\mathbf{e}^i)$$

and computing $\mathbf{x} \leftarrow \mathbf{x} + \tau\mathbf{e}^i$ same as

$$x_i \leftarrow x_i + \tau$$

One relaxation cycle consist of one sequential sweep over all i's.

A Closed Form Solution for the i^{th} Coordinate

The object function of the one-dimensional problem

$$\begin{aligned} f(\mathbf{x} + \tau\mathbf{e}^i) &= \frac{1}{2}(\mathbf{x} + \tau\mathbf{e}^i)^T\mathbf{A}(\mathbf{x} + \tau\mathbf{e}^i) + (\mathbf{x} + \tau\mathbf{e}^i)^T\mathbf{b}, \\ &= \underbrace{\frac{1}{2}\tau^2(\mathbf{A})_{ii} + \tau(\underbrace{\mathbf{A}\mathbf{x} + \mathbf{b}}_{\mathbf{r}})_i}_{g(\tau)} + \underbrace{\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{b}}_{f(\mathbf{x})\equiv\text{const}}. \end{aligned}$$

So

$$f(\mathbf{x} + \tau\mathbf{e}^i) = g(\tau) + f(\mathbf{x})$$

We just need to minimize $g(\tau)$

Found PGS Again

The unconstrained minimizer of $g(\tau)$

$$\tau^u = -\frac{\mathbf{r}_i}{(\mathbf{A})_{ii}}.$$

Considering the constraint $\mathbf{x}_i + \tau \geq 0$ we have

$$\tau^c = \max\left(-\frac{\mathbf{r}_i}{(\mathbf{A})_{ii}}, -\mathbf{x}_i\right)$$

The final update rule

$$\mathbf{x}_i \leftarrow \max\left(0, \mathbf{x}_i - \frac{\mathbf{r}_i}{\mathbf{A}_{ii}}\right)$$

Algebraic equivalent to the i^{th} step in PGS shown by splitting.

Lessons Learned

- PGS can be derived from a QP reformulation or by a splitting method
- Each derivation assumes different matrix properties of the **A**-matrix

Study Group Work

- Show that both derivations end up in algebraic equivalent update formulas for $\mathbf{x}_i$.
- What properties should the $\mathbf{A}$ have for splitting derivation to work?
- What properties should the $\mathbf{A}$ have for QP derivation to work?
- What properties should the $\mathbf{A}$ have for PGS to converge?
- Speculate what to do if $\mathbf{A}_{ii} \ngtr 0$ (Hint: look at how to minimize $g(\tau)$)?

The Projected Succesive Over Relaxation Method

Relaxing the Steps

Given the polynomial

$$g(\tau) \equiv \frac{1}{2}\tau^2 \mathbf{A}_{ii} + \tau \mathbf{r}_i$$

where $\mathbf{A}_{ii} > 0$ then

- One trival root at $\tau^1 = 0$
- One global minima at $\tau^u = -\frac{\mathbf{r}_i}{\mathbf{A}_{ii}}$ where $g(\tau^0) < 0$
- Second root at $\tau^2 = -2\frac{\mathbf{r}_i}{\mathbf{A}_{ii}}$.

For any τ between τ^1 and τ^2

$$\tau^\lambda = -\lambda \frac{\mathbf{r}_i}{\mathbf{A}_{ii}} \quad \Rightarrow \quad g(\tau^\lambda) < 0, \quad \forall \lambda \in [0..2].$$

The Projected SOR Method

From this it follows that

$$f(\mathbf{x} + \tau^{\lambda}\mathbf{e}^{i}) = g(\tau^{\lambda}) + f(\mathbf{x}) \leq f(\mathbf{x}), \quad \forall \lambda \in [0..2]$$

With equality for $\tau^{\lambda} = 0$. This results in the over-relaxed version

$$\mathbf{x}_i \leftarrow \max\left(0, \mathbf{x}_i - \lambda \frac{\mathbf{r}_i}{\mathbf{A}_{ii}}\right).$$

Algebraic equivalent to the i^{th} step in the projected SOR[2].

[2] PGS is special case when $\lambda = 1$

The Fischer–Newton Method

The Fischer Function

The Fischer function

$$\phi(a, b) = \sqrt{a^2 + b^2} - (a + b) \quad \text{for some } a, b \in \mathbb{R}.$$

If one has the complementarity problem $a \geq 0 \perp b \geq 0$ then a solution (a^*, b^*) is only a solution if and only if $\phi(a^*, b^*) = 0$

The Fischer Reformulation

Given the LCP

$$\mathbf{x} \geq 0 \quad \perp \quad \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \geq 0$$

We reformulate

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} \phi(\mathbf{x}_1, \mathbf{y}_1) \\ \vdots \\ \phi(\mathbf{x}_n, \mathbf{y}_n) \end{bmatrix} = \mathbf{0}$$

This is a nonsmooth root search problem

The Fischer–Newton Method

Solved using a generalized Newton–Method. In an iterative fashion solves the generalized Newton subsystem

$$\mathbf{J}\Delta\mathbf{x}^k = -F(\mathbf{x}^k)$$

for the Newton direction $\Delta\mathbf{x}^k$. Here $\mathbf{J} \in \partial F(\mathbf{x}^k)$ is any member from the generalized Jacobian $\partial F(\mathbf{x}^k)$. Then the Newton update yields

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \tau^k \Delta\mathbf{x}^k$$

where τ^k is the step length of the k^{th} iteration.

The Generalized Jacobian

Given $\mathbf{F}$

- let $\mathcal{D} \subset \mathbb{R}^n$ be the set of all $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{F}$ is continuously differentiable
- Assume $\mathbf{F}$ is Lipschits continuous at $\mathbf{x}$

The B–subdifferential of $\mathbf{F}$ at $\mathbf{x}$ is defined as

$$\partial_B \mathbf{F}(\mathbf{x}) \equiv \left\{ \mathbf{H} \in \mathbb{R}^{n \times n} \quad \middle| \quad \exists (\mathbf{x}_k) \subset \mathcal{D} \quad \text{and} \quad \lim_{\mathbf{x}_k \to \mathbf{x}} \frac{\partial \mathbf{F}(\mathbf{x}_k)}{\partial \mathbf{x}} = \mathbf{H} \right\}$$

Clarke's generalized Jacobian of $\mathbf{F}$ at $\mathbf{x}$ is defined as the convex hull of the B–subdifferential,

$$\partial \mathbf{F}(\mathbf{x}) \equiv \mathbf{co}\left(\partial_B \mathbf{F}(\mathbf{x})\right)$$

Example – The Euclidean Norm

Consider the Euclidean norm $e : \mathbb{R}^2 \mapsto \mathbb{R}$ then for $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we have

$$\partial e(\mathbf{x}) = \partial_B e(\mathbf{x}) = \frac{\partial e(\mathbf{x})}{\partial \mathbf{x}} = \frac{\mathbf{x}^T}{\| \mathbf{x} \|} \quad \forall \mathbf{x} \neq \mathbf{0}$$

For $\mathbf{x} = \mathbf{0}$ we have

$$\begin{aligned} \partial_B e(\mathbf{0}) &= \{\mathbf{v}^T \quad | \quad \mathbf{v} \in \mathbb{R}^2 \quad \text{and} \quad \| \mathbf{v} \| = 1\} \\ \partial e(\mathbf{0}) &= \{\mathbf{v}^T \quad | \quad \mathbf{v} \in \mathbb{R}^2 \quad \text{and} \quad \| \mathbf{v} \| \leq 1\} \end{aligned}$$

Illustration of Generalized Jacobian in 1D

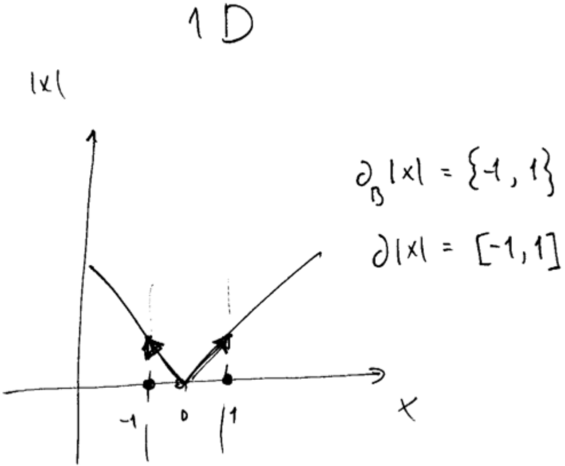
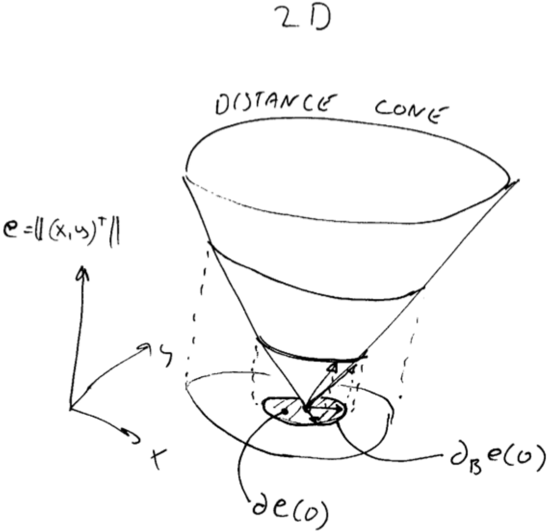


Illustration of Generalized Jacobian in 2D



Example – The Fischer Function

For $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2$ we may write the Fischer function as

$$\phi(\mathbf{x}) = e(\mathbf{x}) - f(\mathbf{x})$$

where $f(\mathbf{x}) = \left(\begin{bmatrix} 1 & 1 \end{bmatrix}^T \mathbf{x}\right)$. From this we find

$$\partial_B \phi(\mathbf{x}) = \partial_B e(\mathbf{x}) - \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}$$

$$\partial \phi(\mathbf{x}) = \partial e(\mathbf{x}) - \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}$$

Hence for $\mathbf{x} \neq 0$,

$$\partial \phi(\mathbf{x}) = \partial_B \phi(\mathbf{x}) = \left\{ \frac{\mathbf{x}^T}{\| \mathbf{x} \|} - \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}$$

and

$$\partial_B \phi(\mathbf{0}) = \{ \mathbf{v}^T - \begin{bmatrix} 1 & 1 \end{bmatrix}^T \quad | \quad \mathbf{v} \in \mathbb{R}^2 \quad \text{and} \quad \| \mathbf{v} \| = 1 \}$$

$$\partial \phi(\mathbf{0}) = \{ \mathbf{v}^T - \begin{bmatrix} 1 & 1 \end{bmatrix}^T \quad | \quad \mathbf{v} \in \mathbb{R}^2 \quad \text{and} \quad \| \mathbf{v} \| \leq 1 \}$$

Generalized Jacobian of The Fischer Reformulation

Written as

$$\partial F(\mathbf{x}) \equiv \mathbf{D}_p(\mathbf{x}) + \mathbf{D}_q(\mathbf{x})\mathbf{A}$$

where $\mathbf{D}_p(\mathbf{x}) = \mathbf{diag}\left(p_1(\mathbf{x}), \ldots, p_n(\mathbf{x})\right)$ and $\mathbf{D}_q(\mathbf{x}) = \mathbf{diag}\left(q_1(\mathbf{x}), \ldots, q_n(\mathbf{x})\right)$ are diagonal matrices. If $\mathbf{y}_i \neq 0$ or $\mathbf{x}_i \neq 0$ then

$$p_i(\mathbf{x}) = \frac{\mathbf{x}_i}{\sqrt{\mathbf{x}_i^2 + \mathbf{y}_i^2}} - 1,$$

$$q_i(\mathbf{x}) = \frac{\mathbf{y}_i}{\sqrt{\mathbf{x}_i^2 + \mathbf{y}_i^2}} - 1,$$

else if $\mathbf{y}_i = \mathbf{x}_i = 0$ then

$$p_i(\mathbf{x}) = \alpha_i - 1,$$

$$q_i(\mathbf{x}) = \beta_i - 1$$

for any $\alpha_i, \beta_i \in \mathbb{R}$ such that $\| \begin{bmatrix} \alpha_i & \beta_i \end{bmatrix}^T \| \leq 1$

Proof of $\partial F(\mathbf{x})$

Assume $\mathbf{y}_i \neq 0$ or $\mathbf{x}_i \neq 0$ then the differential is

$$d\mathbf{F}_i(\mathbf{x}, \mathbf{y}) = d\left(\left(\mathbf{x}_i^2 + \mathbf{y}_i^2\right)^{\frac{1}{2}}\right) - d\left(\mathbf{x}_i + \mathbf{y}_i\right)$$

By chain rule

$$\begin{aligned} d\mathbf{F}_i(\mathbf{x}, \mathbf{y}) &= \frac{1}{2}\left(\mathbf{x}_i^2 + \mathbf{y}_i^2\right)^{-\frac{1}{2}} d\left(\mathbf{x}_i^2 + \mathbf{y}_i^2\right) - d\mathbf{x}_i - d\mathbf{y}_i \\ &= \frac{\mathbf{x}_i d\mathbf{x}_i + \mathbf{y}_i d\mathbf{y}_i}{\sqrt{\mathbf{x}_i^2 + \mathbf{y}_i^2}} - d\mathbf{x}_i - d\mathbf{y}_i \\ &= \left[\underbrace{\left(\frac{\mathbf{x}_i}{\sqrt{\mathbf{x}_i^2 + \mathbf{y}_i^2}} - 1\right)}_{p_i(\mathbf{x})} \quad \underbrace{\left(\frac{\mathbf{y}_i}{\sqrt{\mathbf{x}_i^2 + \mathbf{y}_i^2}} - 1\right)}_{q_i(\mathbf{x})} \right] \begin{bmatrix} d\mathbf{x}_i \\ d\mathbf{y}_i \end{bmatrix} \end{aligned}$$

Proof of $\partial F(\mathbf{x})$ (Contd)

Finally $d\mathbf{y} = \mathbf{A}d\mathbf{x}$, so $d\mathbf{y}_i = \mathbf{A}_{i*}d\mathbf{x}$ by substitution

$$d\mathbf{F}_i(\mathbf{x}, \mathbf{y}) = \underbrace{\left(p_i(\mathbf{x})\mathbf{e}_i^T + q_i(\mathbf{x})\mathbf{A}_{i*}\right)}_{\partial F_i(\mathbf{x})} d\mathbf{x}$$

The case $\mathbf{x}_i = \mathbf{y}_i = 0$ follows from the previous examples.

How to solve Generalized Newton System

- whenever $\mathbf{x}_i = \mathbf{y}_i = 0$ one would use $\mathbf{x}'_i = \varepsilon$ in-place $\mathbf{x}_i$ when evaluating the generalized Jacobian where $0 < \varepsilon \ll 1$
- If Newton system is solved with iterative method (GMRES) then we only need to compute $\mathbf{J}\Delta\mathbf{x}$. By definition of directional derivative

$$\mathbf{J}\Delta\mathbf{x} = \lim_{h \to 0^+} \frac{\mathbf{F}(\mathbf{x} + h\Delta\mathbf{x}) - \mathbf{F}(\mathbf{x})}{h}$$

  So we can numerically approximate $\mathbf{J}\Delta\mathbf{x}$ using finite differences
- A projected Armijo back-tracking can be very usefull to globalize the Newton method and to ensure feasibility of all iterates

Projected Armijo Backtracking Line Search

Define natural merit function

$$\theta(\mathbf{x}) = \frac{1}{2}\mathbf{F}(\mathbf{x})^T\mathbf{F}(\mathbf{x})$$

Project Newton Search Direction

$$\Delta x \leftarrow \max\left(\mathbf{0}, \Delta\mathbf{x}\right)$$

Find smallest $k \in \mathbf{Z}_0$ such that

$$\theta(\mathbf{x} + \alpha^k \Delta\mathbf{x}) \leq \theta(\mathbf{x}) + \underbrace{\left(\beta \frac{\partial\theta(\mathbf{x})}{\partial\mathbf{x}}\Delta\mathbf{x}\right)}_{c\equiv\text{const}} \alpha^k$$

for some user defined constants $0 \leq \beta < \alpha < 1$. Now

$$\tau = \alpha^k$$

$$\mathbf{x} \leftarrow \mathbf{x} + \tau\Delta\mathbf{x}$$

Study Group Work

As a challenge if you have time. Derive a Nonsmooth Newton method for the minimum map reformulation of the LCP

$$\min(\mathbf{x}, \mathbf{y}) = \mathbf{0}$$

(Hint: Look at Erleben and Ortiz: A non-smooth newton method for multibody dynamics, In Proc. of ICNAAM 06')

Further Reading

- K. G. Murty: Linear complementarity, linear and nonlinear programming. Sigma Series in Applied Mathematics. 3. Berlin: Heldermann Verlag, 1988. (Chapter 1 and 9)
- R. W. Cottle, J.-S. Pang, R. E. Stone. The Linear Complementarity Problem. Academic Press. 1992. (Chapter 1 and 5)
- Stephen C. Billups and Katta G. Murty: Complementarity problems, Journal of Computational and Applied Mathematics Volume 124, Issues 1-2, Pages 1-373 (1 December 2000)
- D.E. Stewart and J.C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction. International Journal of Numerical Methods in Engineering, 39:2673-2691

More Reading

- D.E. Stewart and J.C. Trinkle. Dynamics, friction, and complementarity problems. In M.C. Ferris and J.S. Pang, editors, Complementarity and Variational Problems, pages 425-439. SIAM, 1997.
- D.E. Stewart and J.C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with coulomb friction. In Proceedings, IEEE International Conference on Robots and Automation, pages 162-169, 2000.
- M. Silcowitz, S. Niebe, and K. Erleben. Nonsmooth newton method for fischer function reformulation of contact force problems for interactive rigid body simulation. In Proceedings of VRIPHYS, 2009.
- M. Silcowitz, S. Niebe, and K. Erleben. A nonsmooth nonlinear conjugate gradient method for interactive contact force problems. The Visual Computer, 2010.

Study Group Work

- Examine the Stewart–Trinkle (ST) LCP formulation of the contact force problem.
- What matrix properties can you identify?
- What do you know about the right hand side vector?
- What kind of reformulations are applicable to the ST LCP formulation?
- What kind of methods can be used to solve a ST LCP formulation problem?

Basic Programming Exercise

- Obtain Lemke's method from CPNET[3]
- Create a routine that can generate random N-dimensional LCP problems.
- Generate a sequence of random LCP test problems with increasing number of variables, $N = 2^k, k = 2, 3, \ldots, 10$.
- Use Lemke to solve random sequences of LCP test problems (run 10 sequences at least).
- Make plots of computing time as a function of increasing number of variables and make a histogram showing the fraction of solved problems.
- Discuss your results – what do you think of Lemke's method?

[3] http://www.cs.wisc.edu/cpnet/

Intermediate Programming Exercise

- Try to implement a PGS solver, a Fischer–Newton solver,and use a QP reformulation solved by using the Matlab **quadprog** function.
- Create a seqeunce of random problems of increasing size and use the solvers to find solutions.
  - Compare the accuracy of each iterative solver with the true solution, what is the error
  - Plot how the error of each solver behaves as a function of the number of iterations (Hint: make a log plot and determine convergence rate)
  - Try to measure the computing cost per iteration of each solver as a function of increasing variables (Hint: compare plot with your complexity analysis)
- Based on your experiments evaluate if you implementations behaves as expected, speculate for what purposes you want to use different solvers for.

Advanced Programming Exercise

Create a simple 2D rigid body simulator using sperical objects of varying size and mass. Ignore friction and use simple first-order time stepping. In each simulation step solve a LCP for the normal penetration constraints.

- Determine the eigenvalue spectrum of the coefficient-matrix
- Determine if the coefficient matrix is symmetric or not
- Try experimenting with using different LCP solvers (take those from the intermediate programming exercise)
- Which solver do you think is best for this particular simulator and why?
- If you have more time try to add friction to the contact forces and rerun all your tests. Did this chance on your conclusion on which solver is the best?