Linear Complementarity Problems and their Sources

The *Linear Complementarity Problem* (LCP) (q, M) is defined as follows:

Given a real $n \times n$ matrix M and an n-vector q, find $z \in R^n$ such that

$$z \geq 0, \qquad q + Mz \geq 0, \qquad z^{\mathrm{T}}(q + Mz) = 0.$$

Define the mapping $F(z) := q + Mz$.

Then F is an affine transformation from R^n into itself.

Some notation

Given the LCP (q, M) we write

$$\mathrm{FEA}(q, M) = \{z : q + Mz \geq 0, \quad z \geq 0\}$$

and

$$\mathrm{SOL}(q, M) = \{z : q + Mz \geq 0, \quad z \geq 0, \quad z^{\mathrm{T}}(q + Mz) = 0.\}$$

These are the *feasible set (region)* and *solution set* of the LCP (q, M), respectively.

Note that if $\mathrm{FEA}(q, M) \neq \emptyset$, it is a closed polyhedral set.

How do such problems arise?

Optimality criterion for Linear Programming (LP)

Consider the LP

$$
\text{(P)} \qquad \begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array}
$$

According to the theory of linear programming, a vector $\bar{x}$ is optimal for (P) if and only if it is feasible and there exists a vector $\bar{y}$ such that

$$
\bar{y}^{\mathrm{T}}A \leq c^{\mathrm{T}}, \quad \bar{y} \geq 0, \quad \bar{y}^{\mathrm{T}}(A\bar{x} - b) = 0, \quad (\bar{y}^{\mathrm{T}}A - c^{\mathrm{T}})\bar{x} = 0.
$$

Now arrange these conditions as follows:

$$
\begin{array}{rcrcrcrcl}
u & = & c & + & & - & A^{\mathrm{T}}y & \geq & 0 \\
v & = & -b & + & Ax & & & \geq & 0
\end{array}
$$

$$x \geq 0, \qquad y \geq 0$$

$$x^{\mathrm{T}}u = 0, \qquad y^{\mathrm{T}}v = 0.$$

Next define

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The optimality conditions of the LP then become the LCP (q, M).

Optimality conditions for Quadratic Programming (QP)

Consider the QP

$$
\text{(P)} \qquad \begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array}
$$

According to the Karush-Kuhn-Tucker (KKT) Theorem, if the vector $\bar{x}$ is a local minimizer for (P), there exists a vector $\bar{y}$ such that

$$
c+Q\bar{x}-A^{\mathrm{T}}\bar{y} \geq 0, \quad \bar{y} \geq 0, \quad \bar{y}^{\mathrm{T}}(A\bar{x}-b) = 0, \quad \bar{x}^{\mathrm{T}}(c+Q\bar{x}-A^{\mathrm{T}}\bar{y}) = 0.
$$

If we assemble these conditions along with the feasibility of the vector x, we obtain the LCP (q, M) where

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad M = \begin{bmatrix} Q & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Remark. The above necessary conditions of optimality for QP are also sufficient for (global) optimality when Q is positive semi-definite.

Note that if Q is positive semi-definite, then so is

$$M = \begin{bmatrix} Q & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}.$$

Bimatrix Games as LCPs

The initial set up

Let A and B denote two $m \times n$ matrices.

These are "payoff matrices" for Players I and II, respectively.

Let $\sigma_m = \{x \in R^m_+ : e^{\mathrm{T}}x = 1\}$ and $\sigma_n = \{y \in R^n_+ : e^{\mathrm{T}}y = 1\}$.

If $x \in \sigma_m$ and $y \in \sigma_n$, the *expected losses* of Players I and II are, respectively:

$$x^{\mathrm{T}}Ay \quad \text{and} \quad x^{\mathrm{T}}By.$$

Let $\Gamma(A, B)$ denote the corresponding two person game.

Nash Equilibrium Point of $\boldsymbol{\Gamma(A, B)}$

The pair $(x^*, y^*) \in \sigma_m \times \sigma_n$ is a *Nash Equilibrium Point* (NEP) for $\Gamma(A, B)$ if

$$\begin{array}{rcll} (x^*)^{\mathrm{T}}Ay^* & \leq & x^{\mathrm{T}}Ay^* & \text{for all } x \in \sigma_m \\ (x^*)^{\mathrm{T}}By^* & \leq & (x^*)^{\mathrm{T}}By & \text{for all } y \in \sigma_n \end{array}$$

It is crucial to note that given $(x^*, y^*) \in \sigma_m \times \sigma_n$, each of the vectors x^*, y^* is optimal in a simple linear program defined in terms of the other. The LP's are:

$$\text{minimize } (Ay^*)^{\mathrm{T}}x \quad \text{subject to} \quad e^{\mathrm{T}}x = 1,\ x \geq 0$$

and

$$\text{minimize } (B^{\mathrm{T}}x^*)^{\mathrm{T}}y \quad \text{subject to} \quad e^{\mathrm{T}}y = 1,\ y \geq 0$$

Let E be the $m \times n$ matrix whose entries are all 1. For a suitable scalar $\theta > 0$, all the entries of the matrices $A + \theta E$ and $B + \theta E$ are positive.

It is easy to see that $\Gamma(A, B)$ and $\Gamma(A+\theta E, B+\theta E)$ have the same equilibrium points (if any).

Thus, it is not restrictive to assume that A and B are (elementwise) positive matrices.

Now consider the LCP

$$\begin{aligned} u &= -e_m + Ay \geq 0, \quad x \geq 0, \quad x^{\mathrm{T}}u = 0 \\ v &= -e_n + B^{\mathrm{T}}x \geq 0, \quad y \geq 0, \quad y^{\mathrm{T}}v = 0 \end{aligned}$$

In this case, we have

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad q = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix}, \quad M = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$e_m = (1, \ldots, 1) \in R^m \qquad e_n = (1, \ldots, 1) \in R^n.$$

We wish to show that

> *to every solution of this LCP, there corresponds a Nash equilibrium point of* $\Gamma(A, B)$*—and vice versa.*

The correspondences are as follows:

- If (x^*, y^*) is a Nash equilibrium of $\Gamma(A, B)$, then

  $$(x', y') = (x^*/(x^*)^{\mathrm{T}} B y^*,\ y^*/(x^*)^{\mathrm{T}} A y^*)$$

  solves the LCP (q, M) given above.

- If (x', y') solves the LCP (above), then

  $$(x^*, y^*) = (x'/e_m^{\mathrm{T}} x',\ y'/e_n^{\mathrm{T}} y')$$

  is a Nash equilibrium point for $\Gamma(A, B)$.

A Market Equilibrium Problem

Here we seek to determine prices at which there is a balance between supplies and demands.

The supply side

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax \geq b \\ & Bx \geq r^* \\ & x \geq 0 \end{array}$$

The demand side

$$r* = Q(p^*) = Dp^* + d$$

Equilibration

$$p^* = \pi^*$$

Formulation as an Equilibrium Problem

$$
\begin{aligned}
y^* &= c - A^{\mathrm{T}}v^* - B^{\mathrm{T}}\pi^* \geq 0, & x^{\mathrm{T}} &\geq 0, & (x^*)^{\mathrm{T}}y^* &= 0 \\
u^* &= -b + Ax^* \geq 0, & v^* &\geq 0, & (v^*)^{\mathrm{T}}u^* &= 0 \\
\delta^* &= -r^* + Bx^* \geq 0, & \pi^* &\geq 0, & (\pi^*)^{\mathrm{T}}\delta^* &= 0
\end{aligned}
$$

Substitute $Dp^* + d$ for r^* and π^* for p^*.

Then we get the LCP (q, M) with

$$
q = \begin{bmatrix} c \\ -b \\ -d \end{bmatrix} \qquad M = \begin{bmatrix} 0 & -A^{\mathrm{T}} & -B^{\mathrm{T}} \\ A & 0 & 0 \\ B & 0 & -D \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} x^* \\ v^* \\ \pi^* \end{bmatrix}
$$

What sort of matrix is D? Having it be negative (semi)definite and symmetric would be nice.

Convex Hulls in the Plane

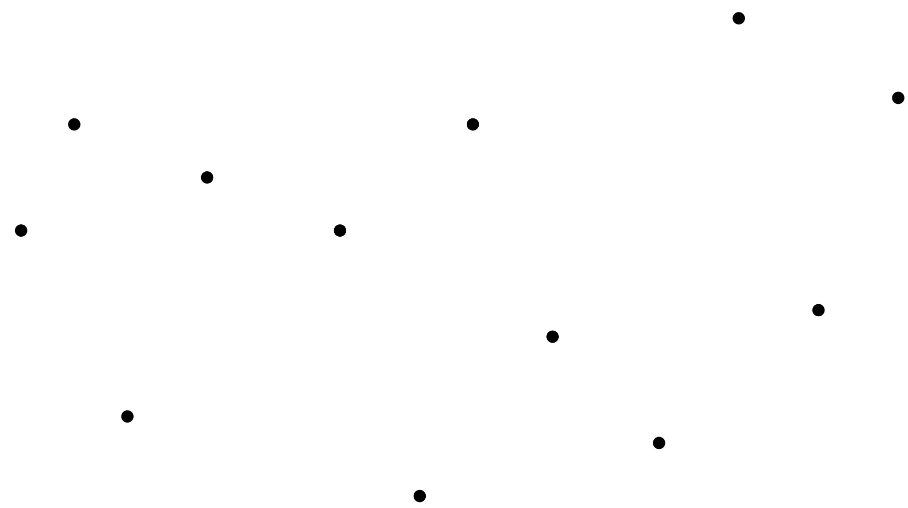
Given $\{(x_i, y_i)\}_{i=0}^{n+1} \subset R^2$ find the *extreme points* and the *facets* of their convex hull and the order in which they appear.

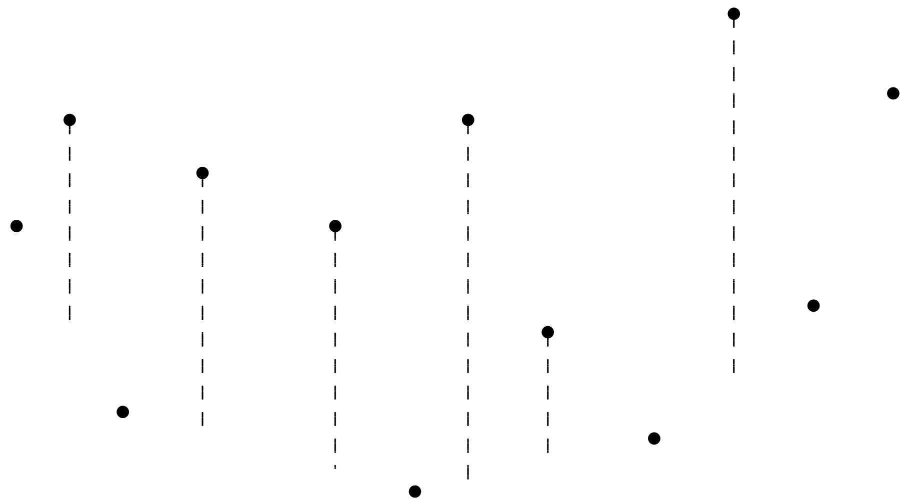
First find the *lower envelope* of the convex hull.

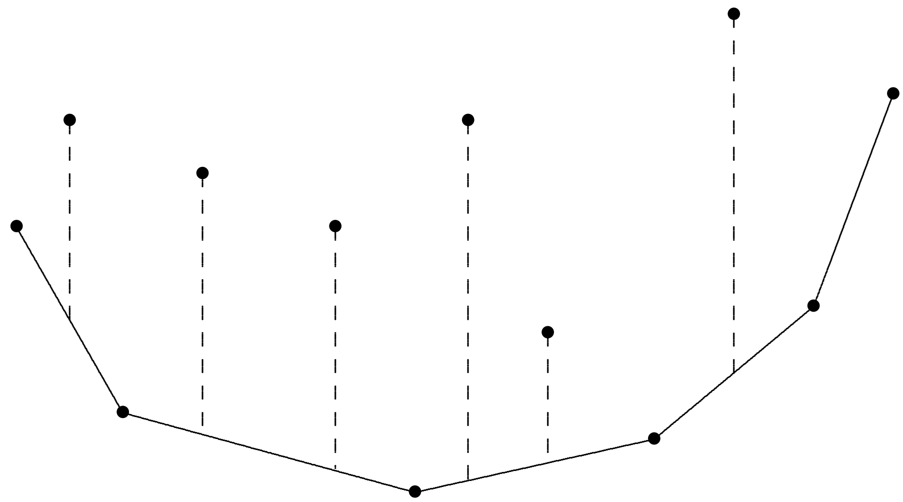
If $x_i = x_j$ and $y_i \leq y_j$, we can ignore (x_j, y_j) without changing the lower envelope.

Thus, assume $x_0 < x_1 < \cdots < x_n < x_{n+1}$. In practice, this would require *sorting*.

The lower envelope is a piecewise linear convex function $f(x)$, the pointwise maximum of all convex functions $g(x)$ such that $g(x_i) \leq y_i$ for $i = 0, 1, \ldots, n+1$.







Define $t_i = f(x_i)$ and let $z_i = y_i - t_i$, for $i = 0, 1, \ldots, n+1$.

Note that $z_0 = z_{n+1} = 0$.

If (x_i, y_i) is a breakpoint, then $t_i = y_i$ and $z_i = 0$.

The segment of the lower envelope between (x_{i-1}, t_{i-1}) and (x_i, t_i) has a different slope than the segment between (x_i, t_i) and (x_{i+1}, t_{i+1}).

Since $f(x)$ is convex, the former (left-hand) segment must have a smaller slope than the latter (right-hand) segment.

Hence strict inequality holds in

$$\frac{t_i - t_{i-1}}{x_i - x_{i-1}} \leq \frac{t_{i+1} - t_i}{x_{i+1} - x_i}.$$

If $z_i > 0$, then (x_i, y_i) cannot be a breakpoint of $f(x)$.

In that case, equality holds in the inequality above.

The vector $z = \{z_i\}_{i=1}^n$ must solve the LCP (q, M) where $q \in R^n$ and $M \in R^{n\times n}$ are defined by

$$q_i \;=\; \beta_i - \beta_{i-1} \qquad \text{and} \qquad m_{ij} \;=\; \begin{cases} \alpha_{i-1} + \alpha_i & \text{if } j = i, \\ -\alpha_i & \text{if } j = i+1, \\ -\alpha_j & \text{if } j = i-1, \\ 0 & \text{otherwise,} \end{cases}$$

and where

$$\alpha_i = 1/(x_{i+1} - x_i) \text{ and } \beta_i = \alpha_i(y_{i+1} - y_i) \text{ for } i = 0, \ldots, n.$$

This LCP has a unique solution.

The matrix M associated with this LCP has several nice properties which can be exploited to produce very efficient solution procedures.