CHAPTER 21

THE WEIGHTED DISTRIBUTION PROBLEM

21-1. THE NEAR-TRIANGULARITY OF THE BASIS

The standard transportation model uses a particularly simple constraint matrix in which each variable has at most two non-zero coefficients whose values are $+1$ and -1. We propose to examine a model which is similar in form, for it allows at most two non-zero coefficients for each variable.

Problem 1. The Row-Column Weighted Distribution Problem.

Find nonnegative y_{ij} and Min z satisfying

(1) The Row Equations: $(i = 1, 2, \ldots, m)$ $\quad \sum_{j=1}^{n} a_{ij}y_{ij} = a_i$

(2) The Column Equations: $(j = 1, 2, \ldots, n)$ $\quad \sum_{i=1}^{m} b_{ij}y_{ij} = b_j$

(3) The Objective Function: $\quad \sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}y_{ij} = z$

where a_i and b_j are nonnegative, and a_{ij} is positive. Before proceeding, we will transform the variables so that all row coefficients become unity. Thus, when we replace the y_{ij} by new variables, defined as $x_{ij} = a_{ij}y_{ij}$, the row equations take on unit coefficients in place of the a_{ij}, while the coefficients of the column equations become $p_{ij} = b_{ij}/a_{ij}$ and the cost equation becomes $c_{ij} = c'_{ij}/a_{ij}$. In this way we arrive at

The Standard Form of the Weighted Distribution Problem.

(4) $$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{(Row equations, } i = 1, 2, \ldots, m)$$

(5) $$\sum_{i=1}^{m} p_{ij}x_{ij} = b_j \qquad \text{(Column equations, } j = 1, 2, \ldots, n)$$

(6) $$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} = z$$

where $x_{ij} \geq 0$, and z is to be minimized.

In a typical application, the a_i represent availabilities which must not be exceeded (as in the machine-task example to be discussed). In this case (4) is replaced by (7).

$$\sum_{j=1}^{m} x_{ij} \leq a_i \qquad \text{for } i = 1, 2, \ldots, m \tag{7}$$

The theory and technique of solution are virtually the same for the system, {(5), (6), and (7)} as for {(4), (5), and (6)}.

The Dual of the Standard Weighted Distribution Problem is: Find u_i, v_j, and the Maximum q, such that

$$u_i + p_{ij}v_j \leq c_{ij} \qquad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \tag{8}$$

where

$$\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j = q \text{ (Max)} \tag{9}$$

Criterion of Optimality: A set of x_{ij} satisfying the primal problem is an optimal solution if there are u_i and v_j satisfying the dual, such that

$$\begin{aligned} x_{ij} > 0 &\Rightarrow u_i + p_{ij}v_j = c_{ij} \\ u_i + p_{ij}v_j < c_{ij} &\Rightarrow x_{ij} = 0 \end{aligned} \tag{10}$$

Illustrative Applications.

1. A number of different tasks can be accomplished on one of several types of available machines, some more efficiently than others. The tasks are to be assigned to machines in such a way that all tasks are completed within the machine-time available and at a minimum over-all cost.

To set up the mathematical model, let

a_i = number of hours available on i^{th} type machine,
b_j = number of units of jth type task to be performed,
c_{ij} = cost to do one unit of the jth type task on the ith type machine,
p_{ij} = number of units of the jth type task that can be processed per hour on the ith type machine,
x_{ij} = number of hours machine i is to work on task j.

2. A fleet, consisting of various types of aircraft, is to be assigned to airline routes in order to satisfy the passenger demand at the least operating costs. In this case, for some period, let

a_i = number of aircraft of type i in the fleet,
b_j = number of passengers requiring passage on the jth route,
c_{ij} = operating costs per aircraft of type i assigned to route j in the period,

p_{ij} = the total number of passengers that can be accommodated by one aircraft of type i if assigned to route j during the period,

x_{ij} = number of aircraft of type i assigned to the jth route.

Tableau and Implicit Prices for the Weighted Distribution Problem.

The tableau for a 2×3 problem takes the form (11)

(11)

x_{11} p_{11} c_{11}	x_{12} p_{12} c_{12}	x_{13} p_{13} c_{13}	a_1 u_1
x_{21} p_{21} c_{21}	x_{22} p_{22} c_{22}	x_{23} p_{23} c_{23}	a_2 u_2
b_1 v_1	b_2 v_2	b_3 v_3	

One practical observation is in order: since the entries, x_{ij}, u_i, and v_j, are numerical and must be changed from one iteration to the next, it is important for hand computation that the chart be arranged in some convenient manner, as above, to facilitate the manual labor.

Finding a Starting Basic Solution.

The weighted distribution problem differs from the standard transportation problem in that no simple rule for directly obtaining an initial basic feasible solution has been found as yet. However, for the case in which all the p_{ij} are *nonnegative*, we shall describe and illustrate a method analogous to the rule of solution given in § 15-3, using the simple numerical problem which appears below.

(12)

x_{11} $p_{11} = 1$ $c_{11} = 4$	x_{12} $p_{12} = 2$ $c_{12} = -8$	x_{13} $p_{13} = 1$ $c_{13} = 3$	$a_1 = 4$
x_{21} $p_{21} = 2$ $c_{21} = 6$	x_{22} $p_{22} = 1$ $c_{22} = 4$	x_{23} $p_{23} = 1$ $c_{23} = 7$	$a_2 = 5$
$b_1 = 5$	$b_2 = 2$	$b_3 = 3$	

Step 1. Select a basic variable by choosing a square arbitrarily (a good choice would be one having a smallest c_{ij}), and increasing the corresponding x_{ij} to the largest value consistent with its row and column totals. Delete the row or the column that becomes *saturated*, i.e., the one whose total has

just been attained (should a row and a column be saturated simultaneously, delete one or the other, but not both). Now repeat this cycle as necessary. For the example (12), squares were chosen as follows:

Square	Value of the Variable	Saturated Row or Column
(1, 3)	$x_{13} = 3$	col. 3
(1, 1)	$x_{11} = 1$	row 1
(2, 2)	$x_{22} = 2$	col. 2
(2, 1)	$x_{21} = 2$	col. 1

Step 2. One column or row total, in general, will remain unsaturated. We then introduce an artificial variable by adding a "deficit" square, (r, s), either $(0, s)$ or $(r, 0)$, to the unsaturated row or column. Next, we minimize the infeasibility form, $\Sigma\Sigma d_{ij}x_{ij}$, where $d_{ij} = 0$ for all i and j, with one exception, for the supplementary square $d_{rs} = 1$. If the deficit occurs in a column, we arbitrarily set $p_{rs} = 1$, while for a row, we need not define p_{rs} at all (since there is no column equation corresponding to the deficit).

EXERCISE: Modify this rule to cover the case where some p_{ij} may be negative.

The Phase I multipliers, u_i and v_j, must be such that $u_i + p_{ij}v_j = 0$ for x_{ij} a basic variable, except that $u_r = 1$, if the rth row is left unsaturated, and $v_j = 1$ if the jth column is left unsaturated. In our example, row 2 is not saturated; the initial tableau for Phase I, therefore, takes the form (13).

(13)

	$(j = 1)$	$(j = 2)$	$(j = 3)$	a_i	u_i
Deficit	1 / 1 / 0	2 / 0	3 / 1 / 0	4 / $\frac{1}{2}$	
1	● / 2 / 0	2 / 1 / 0	1 / 0	5 / 1	
b_j	5	2	3		
v_j	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	← Implicit prices	↑

The dot in square (2, 0) is to indicate that p_{ij} is undefined (i.e., there is no equation for column 0). Because of the deficit the price of $u_2 = 1$; the remainder of the prices, u_i and v_j, are computed using equation (10) with c_{ij} replaced by d_{ij}. Since $d_{23} - (u_2 + p_{23}v_3) = 0 - [1 + 1 \cdot (-1/2)] = -(1/2)$, the infeasibility can be diminished by increasing x_{23} and adjusting the basic variables to compensate for this increase; see (14).

(14)

Deficit	$1+\theta$ 1 0	$3-\theta$ 2 1 0 0	4
$1-(1/2)\theta$ ● 1	$2-(1/2)\theta$ 2 0	2 θ^* 1 1 0 0	5
	5	2 3	

It is clear that Max $\theta = 2$, and that the infeasibility vanishes at this value. Thus, x_{23} becomes a new basic variable replacing the deficit variable, x_{20}, and Phase I is already complete; we drop the deficit box. The starting tableau for Phase II, showing the u_i and v_j as well as θ entries, both of which we shall explain in a moment, is (15).

(15)

$3+2\theta$ 1 4	θ^* 2 -8	$1-3\theta$ 1 3	4 6	
$1-\theta$ 2 6	$2-2\theta$ 1 4	$2+3\theta$ 1 7	5 10	
5 -2	2 -6	3 -3	$\leftarrow v_j$	$\uparrow u_i$

Computing Implicit Prices (Simplex Multipliers).

The values of u_i and v_j from (10), must satisfy a system of five equations in the five unknowns, u_1, u_2, v_1, v_2, and v_3:

(16)
$$\begin{aligned} u_1 + v_1 &= 4 \\ u_1 + v_3 &= 3 \\ u_2 + 2v_1 &= 6 \\ u_2 + v_2 &= 4 \\ u_2 + v_3 &= 7 \end{aligned}$$

In contradistinction to the standard transportation model, there need not be a redundancy in the system of equations {(1), (2)}. *In general, therefore, it is not possible to choose one of the prices arbitrarily.* Similarly, the bases of system {(1), (2)} need not be triangular. Nevertheless, the system is nearly triangular in the following sense: Choose any variable, say u_2, and treat it as a parameter in terms of which the other variables are to be evaluated.

This leads to equations in a single variable, which can be immediately evaluated in terms of the parameter. From (16), we get

$$(17) \qquad v_1 = 3 - \tfrac{1}{2}u_2, \qquad v_2 = 4 - u_2, \qquad v_3 = 7 - u_2$$

$$\begin{aligned} u_1 &= 4 - (3 - \tfrac{1}{2}u_2) \\ &= 1 + \tfrac{1}{2}u_2 \end{aligned} \qquad\qquad \begin{aligned} u_1 &= 3 - (7 - u_2) \\ &= -4 + u_2 \end{aligned}$$

We have arrived at two expressions for u_1 in terms of the parameter u_2. Equating them leads to a numerical evaluation of u_2 and, hence, of all the variables it defines. Thus, since $1 + \frac{1}{2}u_2 = -4 + u_2$, we have $u_2 = 10$ whence $v_1 = -2$, $v_2 = -6$, $v_3 = -3$, and $u_1 = 6$.

A complete set of prices for our example has now been brought to hand. However, in certain other cases, some implicit prices might still remain unevaluated after such a procedure. In that event, any unevaluated price may, as before, be treated as a parameter and additional variables evaluated in terms of it by successive substitution until two equations in two variables appear which allow its evaluation. We now generalize these remarks.

THEOREM 1: *Assuming a basis of rank $n + m$, implicit prices can be evaluated by treating any one of them as a parameter, and solving a sequence of equations in one unknown, repeating the procedure as necessary for any residual set of unevaluated prices.*

PROOF: Each equation of (10) contains two prices. If one of them is selected as a parameter, several others can be expressed in terms of it. By successive substitutions this leads to a set of variables evaluable in terms of this parameter, which have no variable in common with any equation still containing an unevaluated variable (for this in turn could be immediately evaluated in terms of the parameter and added to the set). We shall show in this set that there is just one variable that is linearly expressed in exactly two different ways. By equating the two expressions, we can determine the value of the parameter. If more than one price in the set were doubly expressed (i.e., if one variable could be evaluated in more than two ways), this would mean that the equations contain either a redundancy or an inconsistency. On the other hand, if no variable were doubly evaluable, then the value of the parameter could be chosen arbitrarily. If the basis is of rank $n + m$, however, neither one of these situations is possible, for the set of equations associated with the transpose of a basis must always lead to a unique solution.

Improving the Basic Solution.

The implicit prices, computed as above, are used to determine whether the current solution is minimal, according to condition (10), and to compute

an improved solution if it is not. The solution we have computed for our example is not yet optimal, since

$$c_{12} - u_1 - p_{12}v_2 = -8 - 6 - 2(-6) = -2$$

Accordingly, set $x_{12} = \theta$, and adjust the basic variables. In our example, we compute the values of the basic variables, which result by changing the constant terms to

$$a_1' = 4 - \theta,\ a_2' = 5,\ b_1' = 5,\ b_2' = 2 - 2\theta, \text{ and } b_3' = 3$$

The new values of the basic variables are expressed linearly, in the form $\alpha + \beta\theta$, where α is the old value and $\beta\theta$ is the compensatory change necessitated by the increase of θ in the value of the non-basic variable coming into the basic set. The old values, α, are known, and therefore only the amount by which basic variables change, Δx_{ij}, need be computed. They must satisfy five equations in five unknowns:

$$(18)\qquad \begin{cases} \Delta x_{11} + \Delta x_{13} = -\theta \\ \Delta x_{21} + \Delta x_{22} + \Delta x_{23} = 0 \\ \Delta x_{11} + 2\Delta x_{21} = 0 \\ \Delta x_{22} = -2\theta \\ \Delta x_{13} + \Delta x_{23} = 0 \end{cases}$$

This, of course, is the transpose of the system used earlier for evaluating the implicit prices. After we evaluate Δx_{22} from the fourth equation and substitute its value in the others, each of the remaining equations has precisely two variables that are still unknown. Moreover, it is clear that if one unknown is introduced parametrically, the others may be evaluated in terms of it. Thus, if Δx_{11} is chosen as a parameter, it can be used in turn to express Δx_{13}, Δx_{23}, Δx_{21} and then back to Δx_{11}. This gives an equation in Δx_{11} alone, and the latter can therefore be numerically evaluated. The cycle of dashes in (18) indicates this order of expression. Explicitly,

$$(19)\qquad \begin{aligned} \Delta x_{22} &= -2\theta \\ \Delta x_{13} &= -\theta - \Delta x_{11} \\ \Delta x_{23} &= -\Delta x_{13} = \theta + \Delta x_{11} \\ \Delta x_{21} &= -\Delta x_{22} - \Delta x_{23} = +\theta - \Delta x_{11} \\ \Delta x_{11} &= -2\Delta x_{21} = -2\theta + 2\Delta x_{11} \text{ or } \Delta x_{11} = 2\theta. \end{aligned}$$

Substituting the value $\Delta x_{11} = 2\theta$ in the remaining equations yields $\Delta x_{13} = -3\theta$, $\Delta x_{23} = 3\theta$, and $\Delta x_{21} = -\theta$.

This method of evaluating changes in the basic variables is perfectly general. Indeed, the procedure applied here to nearly triangular bases is an analogue of the process by which one exploits the basic triangularity which occurs in transportation problems.

THEOREM 2: *Given any basis of rank $n + m$ for a weighted distribution*

problem, either (a) there exists a row or column with just one basic variable, or (b) each row and column has precisely two basic variables and $m = n$. If (a) is true, then the sub-basis, resulting by deletion of the row or column which contains only this basic variable, has the same properties.

PROOF: Each row and column has at least one basic variable. If none of them have precisely one, then every row and column has two or more. In the latter case, the number of basic variables, $n + m$, cannot be less than twice the number of rows or twice the number of columns; hence,

$$n + m \geq 2m \quad \text{and} \quad n + m \geq 2n \tag{20}$$

or, by adding,

$$2(n + m) \geq 2m + 2n$$

where equality holds only if equality holds in both expressions (20), i.e., if each row and each column has precisely two basic variables, and $m = n$. But the equality *must* hold. This argument can be repeated for the sub-basis if there is one equation having a single variable and if the row and column in which it occurs have been deleted.

21-2. LINEAR GRAPH STRUCTURE OF THE BASIS

When the weighted distribution problem is interpreted in the context of a linear network, the linear graph whose arcs correspond to the variables of a basic set possesses a characteristic structure.

The discussion will be facilitated if we consider the model simply as a set of equations having certain well-defined properties, and for this purpose it will be convenient to make use of single-subscripted letters to represent the variables involved. It will be recalled that the classical transportation model, § 14-2, had a system of constraints composed of two subsystems, one of which (with row equations) referred to *exports from* each source to the various destinations, while the second subsystem (with column equations) dealt with *imports to* each destination from the various sources. In the transshipment model, § 16-1, each variable had at most two non-zero coefficients ± 1. In the weighted model, we remove the restriction ± 1 and consider a class of problems in which at most two coefficients of each variable x_j are non-zero, one in equation $i = g(j)$ and the other in $i = h(j)$. When x_j has only one non-zero coefficient $g(j) = h(j)$.

Problem 2. The General Weighted Distribution Problem.

Choose a set of nonnegative numbers, x_j, and the Minimum z, such that

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, m) \tag{1}$$

$$\sum_{j=1}^{n} c_j x_j = z \text{ (Min)}$$

where $a_{ij} = 0$ unless $i = g(j)$ or $i = h(j)$. We will refer to any x_j having either a_{gj} or a_{hj} zero (but not both) as a *slack variable*.

Each equation of (1) corresponds to a node in the network form of the model, while each non-slack variable, x_i, corresponds to an undirected arc joining node $g(j)$ to node $h(j)$. Slack arcs may be considered (as pointed out by F. Harary) as arcs which connect two nodes that are identical.

In drawing a linear graph, the nodes may be placed in any convenient position. Thus Fig. 21-2-I and Fig. 21-2-II are pictures of the same network, but the nodes in Fig. 21-2-II have been repositioned, so as to illustrate more clearly its essential structure. If slack variables are to be admissible, then the network must also include *slack arcs*, associated with these variables, each having only a single node.

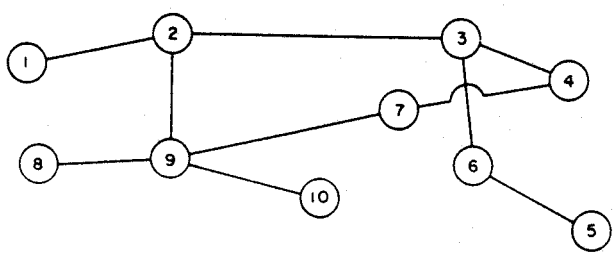


Figure 21-2 I. Graph of a weighted distribution problem.

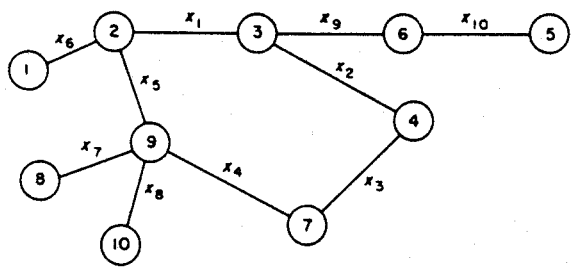


Figure 21-2-II. Rearranged graph of a weighted distribution problem.

In Fig. 21-2-III is depicted a linear graph whose arcs correspond to the variables of a basis. This graph is composed of four isolated, connected subgraphs.

THEOREM 1: *Each connected subgraph of a basic graph for system* (1) *has precisely one loop.*

PROOF: It is clear that each connected part consists of an equal number of nodes and arcs, for the variables associated with the arcs appear only in the equations associated with the nodes of the subgraph, and these equations could not, in general, be satisfied by fewer variables than equations. On the other hand, the total number of basic variables equals the total number

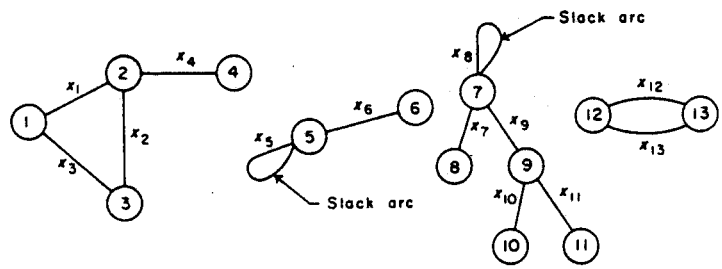


Figure 21-2-III. The basis graph has one or more connected parts each consisting of a tree and one added arc.

of equations, so equality must hold for each isolated subgraph in the network.

Now a connected subgraph having no slack arc(s) must contain at least one loop, since it would otherwise have fewer arcs than nodes. If we were to delete an arc belonging to a loop in this part, we would be left with a connected subgraph in which the number of arcs would be one less than the number of nodes, which therefore [Theorem 1, § 17-1] constitutes a tree. But a tree contains no loops; hence, the subgraph must initially have had exactly one loop.

On the other hand, if a connected subgraph has one or more slack arcs, then the graph formed by deleting them has a smaller number of arcs than nodes, the difference being equal to the number of arcs deleted. But since only the slack arcs (i.e., arcs having one node) were deleted, the resulting subgraph is still connected, and this is possible only if the number of slack arcs was one and the remaining subgraph, a tree. The proof is completed by defining an arc as a one-arc loop.

Computing an Associated Solution.

To evaluate the variables of a basis, one may begin with nodes having exactly one arc. Thus, for Fig. 21-2-II, we have

$$\begin{aligned} a_{1,6}x_6 &= b_1 \\ a_{8,7}x_7 &= b_8 \\ a_{10,8}x_8 &= b_{10} \\ a_{5,10}x_{10} &= b_5 \end{aligned} \tag{2}$$

If a node has several arcs, all but one of whose variables have been evaluated, then the excepted variable can be evaluated immediately. Thus, after x_{10} is determined as above, x_9 can be evaluated at node 6 from the equation,

$$a_{6,9}x_9 + a_{6,10}x_{10} = b_6 \tag{3}$$

By this means, all variables, except those corresponding to arcs of the

loop, can successively be evaluated. The evaluation of such loop variables may proceed by the method we shall now illustrate.

$$
\begin{array}{llllll}
(4) & a_{31}x_1 + a_{32}x_2 & & & & = \bar{b}_3 \\
 & & a_{42}x_2 + a_{43}x_3 & & & = \bar{b}_4 \\
 & & & a_{73}x_3 + a_{74}x_4 & & = \bar{b}_7 \\
 & & & & a_{94}x_4 + a_{95}x_5 & = \bar{b}_9 \\
 & a_{21}x_1 & & & + a_{25}x_5 & = \bar{b}_2
\end{array}
$$

The number $\bar{b}_i$ is the value of b_i adjusted by subtraction of such terms in the original array as are missing from (4). These missing terms belong to arcs, whose numerical values have already been determined by the foregoing procedure. Because of the nearly triangular structure of system (4), we may treat one variable of the loop as the parameter and then evaluate all the others in terms of it as we proceed around the loop. Upon completion of this circuit, a second expression for the parameter will result, and by equating the two expressions we may evaluate it numerically.

Thus, by proceeding clockwise about the loop in network Fig. 21-2-II, one arrives at the following explicit formula for x_1:

$$
(5) \qquad x_1 = \frac{\dfrac{1}{a_{21}}\left[\bar{b}_2 - \dfrac{a_{25}}{a_{95}}\left(\bar{b}_9 - \dfrac{a_{94}}{a_{74}}\left[\bar{b}_7 - \dfrac{a_{73}}{a_{43}}\left(\bar{b}_4 - \dfrac{a_{42}}{a_{32}}\bar{b}_3\right)\right]\right)\right]}{1 - \dfrac{-a_{25}}{a_{95}} \cdot \dfrac{-a_{94}}{a_{74}} \cdot \dfrac{-a_{73}}{a_{43}} \cdot \dfrac{-a_{42}}{a_{32}} \cdot \dfrac{-a_{31}}{a_{21}}}
$$

EXERCISE: Derive an equivalent expression by proceeding counterclockwise about the loop, and then show algebraically that the two formulas are identical.

Evaluation of the Implicit Prices Associated with a Basis.

If the basis includes slack variables, then the implicit price for any equation associated with the single node of a slack arc can, of course, be immediately evaluated, and, from this, all the prices for the entire subgraph.

If, on the other hand, an isolated subgraph contains a loop, then the prices may first be determined at nodes sequenced around the loop, and the remainder evaluated by successively proceeding to nodes which have an arc in common with nodes whose prices have already been determined. If x_j is a basic variable and π_i is the price associated with the ith equation, then

$$
(6) \qquad a_{gj}\pi_g + a_{hj}\pi_h = c_j
$$

where $g = g(j)$ and $h = h(j)$ are the node designations of the equations corresponding to the non-zero coefficients of x_j.

For Fig. 21-2-II the arcs around the loop give rise to the system:

$$
(7) \qquad \begin{aligned}
a_{31}\pi_3 \qquad\qquad\qquad\qquad\qquad\qquad + a_{21}\pi_2 &= c_1 \\
a_{32}\pi_3 + a_{42}\pi_4 \qquad\qquad\qquad\qquad\qquad &= c_2 \\
a_{43}\pi_4 + a_{73}\pi_7 \qquad\qquad\qquad\qquad &= c_3 \\
a_{74}\pi_7 + a_{94}\pi_9 \qquad\qquad &= c_4 \\
a_{95}\pi_9 + a_{25}\pi_2 &= c_5
\end{aligned}
$$

Moreover, since the coefficient matrix in (7) is the transpose of the one in (4), the systems have analogous structures, and the same technique of evaluation may be used for the implicit pricing as for the basic solution itself.

21-3. A SUBCLASS WITH TRIANGULAR OPTIMUM BASES

The technique we have described for exploiting near-triangularity of the basis structure is a little more complex than the methods we have applied to the standard transportation problem. This is partly because the equations of the weighted distribution can have non-triangular bases.

However, even though non-optimal bases need not be triangular, H. Markowitz [1954-2] found that, for an important class of these problems, any basis corresponding to an optimal solution *is* triangular, regardless of the values of the constant terms [Theorem 1, below]. Unfortunately, if the usual simplex process is employed, no computational advantage results from the fact that the final basis is triangular. Markowitz's idea was to vary the procedure, so that each basis occurring in the course of the algorithm would be made triangular. He noted that, for certain values of the constant terms, an optimal basic solution is immediately available, and, by *parametrizing* these terms, one could subsequently adjust them to any desired values. Since the bases are, in this way, kept both feasible and optimal throughout the process, they must remain triangular. This is the idea we will develop in the present section.

Two forms of the problem were considered by Markowitz; the first appears below and the second will be discussed later under (5).

Determine nonnegative numbers and the minimum z satisfying

$$
(1) \qquad \begin{cases}
\displaystyle\sum_{j=1}^{n} a_{ij}x_{ij} + x_i = a_i & \text{for } i = 1, 2, \ldots, m \\
\displaystyle\sum_{i=1}^{m} b_{ij}x_{ij} = b_j & \text{for } j = m+1, m+2, \ldots, m+n
\end{cases}
$$

$$
(2) \qquad -\sum_{i=1}^{m} c_i x_i = z \qquad (c_i > 0)
$$

where a_{ij} and b_{ij} are nonnegative, while a_i and b_j are positive. The n column equations have been numbered $j = m + 1, m + 2, \ldots, m + n$, to bring out the one-to-one correspondence between the $m + n$ nodes and the $m + n$ equations. Problems of this kind first came to light in attempts to assign machines to a fixed set of tasks in such a way as to minimize the use of machine time by maximizing the total value of the machine time left unused after the tasks have been completed; that is, by building up certain of the slack variables, x_i. (See illustrative applications in § 21-1.)

THEOREM 1: *Every optimal basis for* {(1), (2)} *is triangular.*

PROOF: If, on the contrary, a basis is not triangular, there is a subset of the equations whose linear subgraph is connected, contains one loop, and has no slack variable arcs, by Theorem 1 of § 21-2. The prices corresponding to nodes about the loop may be evaluated by means of equations, such as § 21-2-(7). However, since there are no slack variables, the coefficients in the objective form are all zero for the subset of the basic variables contained in these equations. Hence, the implicit prices must all vanish for this subset of equations. In particular, if x_{rs} is one of the basic variables in the subset, then $u_r = 0$. However, the conditions for an optimal basis are that

$$(3) \qquad \begin{aligned} &a_{ij}u_i + b_{ij}v_j \leq 0 \\ &-u_i \geq c_i > 0 \qquad \text{for } i = 1, 2, \ldots, m \\ &\qquad \text{and } j = m + 1, m + 2, \ldots, m + n \end{aligned}$$

which contradicts the assertion that $u_r = 0$. Hence, loops are not possible (except slack-arc loops) and the basis must be triangular.

Finding an Initial Feasible Dual Solution.

If the a_i are replaced by sufficiently large values a_i^*, a starting solution with a triangular basis is immediately available. As basic variables, choose the slacks and one variable, x_{rj}, from each column j, such that

$$(4) \qquad \operatorname*{Min}_i \frac{a_{ij}c_i}{-b_{ij}} = \frac{a_{rj}c_r}{-b_{rj}} = v_j \qquad (b_{ij}, b_{rj} \neq 0)$$

Then (assuming that all b_{ij} are nonnegative), it is easy to see that the prices $u_i = c_i$ and v_j above satisfy (3), so that the basic solution is both feasible and optimal.

EXERCISE: Determine explicitly how large to make a_i^* in order to guarantee feasibility.

The next step is to replace the constants a_i by $a_i + \lambda(a_i^* - a_i)$ and determine the values of the basic variables as a function of λ. The solution is feasible and optimal when $\lambda = 1$. Finally, the parametric linear programming algorithm (§ 11-3) can be applied to reduce λ to zero. At the end of this section we will describe an adaptation of the algorithm to this problem.

The second form of the problem, considered by Markowitz, is as follows: Determine numbers $x_j \geq 0$ and maximum λ, such that

$$\sum_j a_{ij}x_{ij} + x_i = a_i \qquad (i = 1, 2, \ldots, m) \tag{5}$$

$$\sum_i b_{ij}x_{ij} = \lambda b_j \qquad (j = m+1, m+2, \ldots, m+n)$$

In the previous formulation we sought to maximize the unused machine time after performing a fixed set of tasks, but here we seek that allocation which will turn out the most work when the *proportion* b_j of each type task is fixed. This type of problem arises naturally if the different type tasks are combined in fixed proportions to form completed assemblies (e.g., parts of a calculator to be used for completed machines).

THEOREM 2: *For any fixed λ, a basis can be determined that is triangular.*

PROOF: Let λ be any fixed value in (5). Maximize $\Sigma c_i x_i$. By Theorem 1, the final basis is triangular, completing the proof. To solve (5), one can proceed as follows: for $\lambda = 0$, use the same basic set as found by (4). The corresponding basic solution, $x_{ij} = 0$, $x_i = a_i$ for all i, j is feasible and optimal for (5). Apply the parametric linear programming algorithm to increase λ to a maximum. The successive bases will be triangular.

Iterative Procedure. To simplify the calculations, we assume $a_{ij} > 0$; it is convenient to consider system {(1), (2)} in the form:

Find $x' \geq 0$, $x'_i \geq 0$, Max z' satisfying

$$\begin{cases} \sum_{j=1}^{n} x'_{ij} + x'_i = a'_i, \\ -\sum_{i=1}^{m} \beta_{ij}x'_{ij} = -b_j, & (\beta_{ij} \geq 0) \\ \sum_{i=1}^{m} x'_i = z' \text{ (Max)} \end{cases} \tag{6}$$

where we have chosen to maximize instead of minimize. This can be done by a simple transformation of variables $x'_{ij} = c_i a_{ij} x_{ij}$ ($x_{ij} = x'_{ij}$, if $a_{ij} = 0$) and $c_i x_i = x'_i$, and by setting $\beta_{ij} = b_{ij}/c_i a_{ij}$ ($\beta_{ij} = b_{ij}$, if $a_{ij} = 0$) and $c_i a_i = a'_i$. We have assumed $a_{ij} \neq 0$ in (6).

EXERCISE: Explain why in practice $a_{ij} = 0$ usually implies $b_{ij} = 0$.

The *dual* of (6) is: Find u_i, v_j, Min $\bar{z}'$ satisfying:

$$\begin{aligned} u_i - \beta_{ij}v_j &\geq 0 \\ u_i &\geq 1 \\ a'_i u_i - b_j v_j &= \bar{z}' \text{ (Min)} \end{aligned} \tag{7}$$

where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Replacing a_i' by $\theta a_i'$: Let us suppose that the $a_i' > 0$ have been replaced by $\theta a_i'$ for $i = 1, 2, \ldots, m$ and that for θ sufficiently large, we have at hand an optimal basic solution to (6). Then the basis is triangular and consists of a number of trees each connected to a slack arc, as shown in Fig. 21-3-I, which we will refer to as an sl+tree where *sl* is short for *slack arc*.

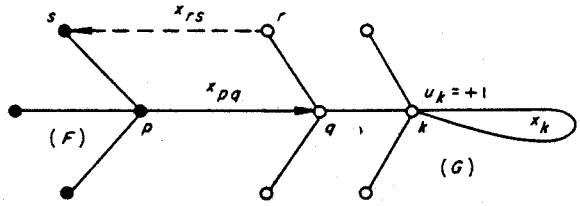


Figure 21-3-I. Improving a basic solution.

However, if θ is reduced, the value of some basic variable associated with an arc of some sl+tree may become negative. By the rules of the parametric programming algorithm (§ 11-3), the first basic variable to change sign below a critical value $\theta = \theta^*$ will be dropped from the basic set in the next iteration. In case the variable to be dropped is not a slack, say x_{pq} as in Fig. 21-3-I, then the removal of the arc (p, q) from the sl+tree separates it into parts (F) and (G). We assume for convenience that node $p \in F$ and that nodes $k, q \in G$. Now (F), containing no slack variables, must join in the graph of the new basis with either (G) or some other sl+tree by an arc (r, s) associated with the new basic variable. In case x_k, the slack variable, is dropped, then, of course, (G) is vacuous and (F) must join up with some other sl+tree.

We shall now determine which basic variable to introduce into the basic set. We note that the prices on all nodes on all other sl+trees do not change with the change in basis, nor is there any change on the nodes of (G). Hence, the only changes are the prices on the nodes of (F). However, the prices on the nodes of (F) were determined before a change of basis by a set of relations:

$$u_i - \beta_{ij} v_j = 0 \qquad (i, j \in F) \tag{8}$$

where (i, j) is an arc of (F) and the price u_p on the node p where u_p was determined via similar relations on (G) and $u_k = 1$. Relations (8) still hold after a change of basis, but the price on p can change from u_p^o to, say, μu_p^o. Since relations (8) are *homogeneous*, it follows that the prices on all nodes i of (F) will all change proportionally from u_i^o to μu_i^o. We have assumed $p \in F$ and $q, k \in G$; similar remarks hold if $q \in F$ and $p, k \in G$.

EXERCISE: Show that all implicit prices are nonnegative.

From these observations it is easy to put together the following rules for

deciding the factor, μ, of proportionality and the variable, x_{rs} or x_r, to introduce into the basic set:

Case I: If x_{pq} is dropped, where p is in F and q is not in F, or if a slack variable x_k is dropped, then $\mu > 1$. If it exists, choose x_{rs} to enter where r is not in F and s is in F, such that

$$\mu = \frac{u_r}{\beta_{rs} v_s} = \text{Min} \frac{u_i}{\beta_{ij} v_j} > 1 \qquad (j \text{ in } F,\ i \text{ not in } F). \tag{9}$$

If no such (r, s) exists, terminate.

Case II: If x_{pq} is dropped, where p is not in F and q is, then $\mu < 1$. If it exists, choose x_{rs} to enter where r is in F and s is not in F, such that

$$\mu' = \frac{\beta_{rs} v_s}{u_r} = \text{Max} \frac{\beta_{ij} v_j}{u_i} < 1 \qquad (i \text{ in } F, j \text{ not in } F) \tag{10}$$

or choose slack variable x_r to enter, where r is in F

$$\mu'' = \frac{1}{u_r} = \text{Max} \frac{1}{u_i} < 1 \qquad (i \text{ in } F) \tag{11}$$

depending on which ratio, $\mu = \mu'$ or $\mu = \mu''$, is the larger. If no such (r, s) exists, terminate.

The following can easily be shown:

EXERCISE: If the x_{rs} or x_r chosen above is introduced into the basic set in the place of x_{pq} or x_p, show that the new solution (if non-degenerate) will be feasible in some range of values $\theta < \theta^*$.

EXERCISE: If it is not possible to find an i not in F, j in F for Case I or an i in F, j not in F for Case II, show that there can be no feasible solution for $\theta < \theta^*$.

Example: Consider the array:

(12)

	$(j = 3)$	$(j = 4)$	$(j = 5)$	(slack)	a_i
$(i = 1)$	x_{13} $\beta_{13} = 1$	x_{14} $\beta_{14} = 2$	x_{15} $\beta_{15} = 3$	x_1	8
$(i = 2)$	x_{23} $\beta_{23} = 4$	x_{24} $\beta_{24} = 3$	x_{25} $\beta_{25} = 6$	x_2	2
b_j	8	6	12		

which represents the equations

(13) Row Equations: $x_{13} + x_{14} + x_{15} + x_1 = 8$ ($= 8\theta$ when right-hand side is parametrized)

$x_{23} + x_{24} + x_{25} + x_2 = 2$ ($= 2\theta$ when right-hand side is parametrized)

(14) Column Equations:

$$\begin{aligned} x_{13} + 4x_{23} &= 8 \\ 2x_{14} + 3x_{24} &= 6 \\ 3x_{15} + 6x_{25} &= 12 \\ x_1 + x_2 &= z \text{ (Max)} \end{aligned}$$

To construct a starting solution choose for basic variables the variable in each column j corresponding to Max β_{ij} and the slack variables. For the example the basic variables are x_{23}, x_{24}, x_{25}, x_1, x_2. The graph of the basis, the prices, and the values of the basic variables are shown in Fig. 21-3-II.

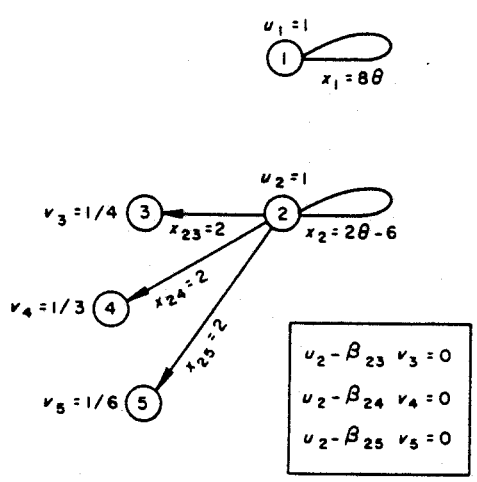
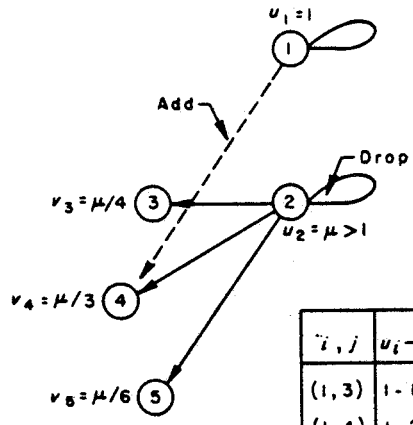


Figure 21-3-II. Graph of basis for cycle 0.

It is easy to verify $u_1 - \beta_{1j}v_j \geq 0$ for $j = 3, 4, 5$, so that the solution is optimal and feasible for $\theta \geq 3$. Reducing θ below the critical value $\theta^* = 3$, x_2 changes sign; hence x_2 is to be dropped as basic variable. To do this, prices are modified as shown in Fig. 21-3-III. By the calculation shown at the right of the figure, x_{14} becomes the new basic variable to replace x_2. The graph of the new basis, the prices, and the values of basic variables are given in Fig. 21-3-IV.

When θ is reduced below the critical value $\theta^* = 2$, x_{24} changes sign and accordingly will be dropped on the next iteration. Hence, prices are modified as shown in Fig. 21-3-V, and by the calculation shown at the right, x_{15} will be the new basic variable.



i, j	$u_i - \beta_{ij} v_j \geq 0$	
(1, 3)	$1 - 1(\mu/4) \geq 0$	$\mu \leq 4$
(1, 4)	$1 - 2(\mu/3) \geq 0$	$\mu \leq 3/2$: (Min)
(1, 5)	$1 - 3(\mu/6) \geq 0$	$\mu \leq 2$

Figure 21-3-III. Graph of basis for cycle 1.

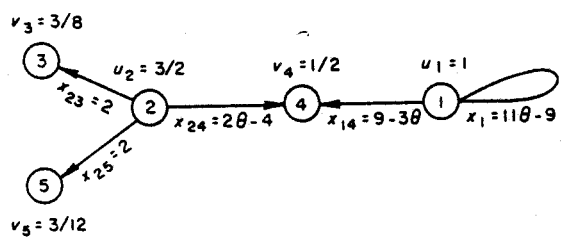
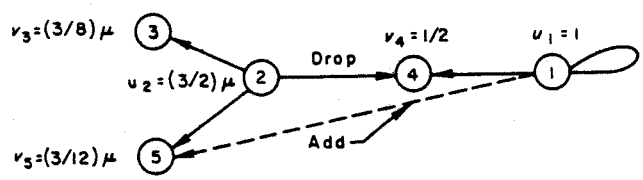


Figure 21-3-IV. Graph of basis for cycle 2 (start).



i, j	$u_i - \beta_{ij} v_j \geq 0$	
(1, 3)	$1 - 1(3/8)\mu \geq 0$	$\mu \leq 8/3$
(1, 5)	$1 - 3(3/12)\mu \geq 0$	$\mu \leq 4/3$: Min

Figure 21-3-V. Graph of basis for cycle 2 (end).

The graph of the new basis is given by Fig. 21-3-VI.

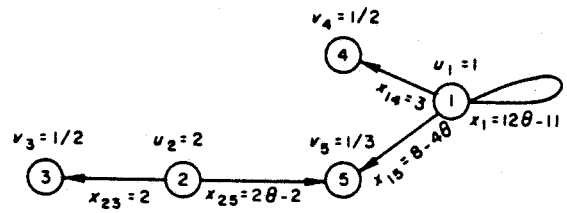


Figure 21-3-VI. Graph of basis for cycle 3 (optimum).

The value of θ can be reduced to 1 without affecting feasibility. At $\theta = 1$ the parametrized right-hand side of (1) attains the desired a_i values, hence the optimal solution is given by (15) for $\theta = 1$. This solution is recapped below (boxes indicate position of basic variables).

$$
(15) \qquad \begin{array}{llll}
x_{13} = 0 & x_{14} = \boxed{3} & x_{15} = \boxed{4} & x_1 = \boxed{1} \\
x_{23} = \boxed{2} & x_{24} = 0 & x_{25} = \boxed{0} & x_2 = 0
\end{array}
$$

21-4. PROBLEMS

1. (Review.) What is the change of variables referred to in the second paragraph of § 21-1? Why does it fail if a coefficient is negative in a row equation? How would you work the simplex method if some variables were allowed to be negative? How is p_{ij} related to the previous coefficients?
2. What is the dual for the system of equations (4), (5), and (6) of § 21-1?
3. Change the constants a_i and b_j in such a way that more than one iteration is needed in Phase I of the example in § 21-1.
4. (Review.) In § 21-3, show how to choose a_i^* sufficiently large so that the basic solution will also be feasible. Interpret the meaning of the optimal solution in this case.
5. How is the selection made if some β_{ij} is negative?
6. Apply the procedure and solve a 3×4 example. Use linear graphs to guide computation of prices and adjustments in the values of basic variables.
7. Solve the 3×4 example as in Example 1, § 21-1, where b_i are replaced by λb_i. Choose λ initially sufficiently small so that feasibility is attained, and then parametrically increase λ to 1.
8. Set up a machine-task model in the standard form of the weighted distribution problem, § 21-1-(4), (5), (6). Assume the cost per hour, c_{ij}, for the ith machine is the same, regardless of the task, and c_i is the revenue per hour derived from other uses of the left-over machine time. If the

objective is to minimize net costs, show for such a model that system (6) of § 21-3 can be obtained from system {(1), (2)} of § 21-3.

9. If u_i and v_j are the prices associated with the ith row equation and the jth column equation of § 21-3-(6) respectively, show that the prices of an optimal solution satisfy $u_i \geq 0$, $v_j > 0$.
10. Starting from (15) of § 21-3, continue to reduce θ and find the value of θ below which there is no feasible solution. Show that, in this case, it is not possible to determine which variable is to enter the basic set.
11. Review problems and illustrative examples in Chapters 3, 4, 5 and determine which ones are weighted distribution problems. Solve those which are numerical.
12. (Unsolved.) Does there exist a transformation of variables and constants that will convert the first type of system considered by Markowitz § 21-3-(1), (2) to his second type § 21-3-(5)?

REFERENCES

Eisemann and Lourie, 1959-1
Ferguson and Dantzig, 1954-1, 1956-1
Hadley, 1962-2
Jewell, 1960-1
Kantorovich, 1939-1
Markowitz, 1954-2

CHAPTER 22

PROGRAMS WITH VARIABLE COEFFICIENTS

22-1. WOLFE'S GENERALIZED PROGRAM[1]

In this chapter, we will consider problems in which there is some freedom in the choice of coefficients of an activity. Such problems arise when a system is being designed or when the input and output characteristics of a process depend on one or more parameters, such as temperature, which can be regulated.

First, a matter of notation; it will be convenient to consider the linear programming problem in vector form: Find $x_j \geq 0$ for $j \neq 0$ and Max x_0 satisfying

$$P_0x_0 + P_1x_1 + P_2x_2 + \ldots + P_nx_n = Q \tag{1}$$

For example, if we had to find $x_j \geq 0$ and Min z satisfying

$$\begin{aligned} x_1 + x_2 - 4x_3 + 2x_4 &= 5 \\ -x_1 + x_2 - x_3 + 3x_4 &= 1 \\ 6x_1 + 4x_2 + x_3 - 2x_4 &= z \end{aligned} \tag{2}$$

we would set $z = -x_0$, obtaining

$$\begin{bmatrix}0\\0\\1\end{bmatrix} x_0 + \begin{bmatrix}1\\-1\\6\end{bmatrix} x_1 + \begin{bmatrix}1\\1\\4\end{bmatrix} x_2 + \begin{bmatrix}-4\\-1\\+1\end{bmatrix} x_3 + \begin{bmatrix}2\\3\\-2\end{bmatrix} x_4 = \begin{bmatrix}5\\1\\0\end{bmatrix} \tag{3}$$

It will also be convenient to redefine the simplex multipliers $\pi = (\pi_1, \pi_2, \ldots, \pi_m, \pi_{m+1})$, so that

$$\begin{aligned} \pi P_j &= 0 \qquad \text{if } x_j \neq x_0 \text{ is a basic variable} \\ \pi P_0 &= 1 \end{aligned} \tag{4}$$

The last of these multipliers, π_{m+1}, is always 1 because $P_0 = U_{m+1}$ is a unit

[1] The approach to this class of problems was first developed in the joint work of Philip Wolfe and the author on a *decomposition principle* for large-scale programs (discussed in Chapter 23); because it was Wolfe who suggested that the procedure developed there could be formalized as a special case of a "Generalized Linear Program," we associate his name with it here [Dantzig and Wolfe, 1960-1; Gomory and Hu, 1960-1].

vector and x_0 is unrestricted in sign. Hence these multipliers would generate a modified cost equation $\Sigma \bar{c}_j x_j$, where (referring to a standard linear program)

$$\bar{c}_j = \pi P_j = \sum_{i=1}^{m} \pi_i a_{ij} + 1 \cdot c_j$$

It is now evident, since we are adding the sum term to c_j instead of subtracting, that the multipliers $\pi_1, \pi_2, \ldots, \pi_m$ will have the opposite sign to those defined in § 9-2. As usual, in order for a basic feasible solution to be optimal $\bar{c}_j = \pi P_j \geq 0$ for all j.

DEFINITION: A *Generalized Linear Program* is a linear programming problem with variable coefficients as follows: *Find* $x_j \geq 0$ *for* $j \neq 0$ *and* Max x_0 *satisfying*

$$P_0 x_0 + P_1 x_1 + \ldots + P_n x_n = Q \tag{5}$$

where each P_j *for* $j \neq 0$ *may be freely chosen to be any* $P_j \in C_j$, *where each* C_j *is a convex set.* By simple extension the fixed vectors in (5), P_0 and Q, may be replaced by any vectors P_0, Q drawn from convex sets C_0 and C_q. This is further discussed in the subsection on "Equivalent Formulations" at the end of this section. The convex sets we shall consider here are defined by systems of linear inequalities. However, the methods of Chapter 24 can be used to extend the results to general convex sets.

For example, (3) becomes a program with variable coefficients if we generalize it to the following form: Find $x_j \geq 0$ for $j \neq 0$, Max x_0 *satisfying*

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_0 + \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} x_4 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

where y_i may be chosen as any values satisfying

$$y_1 + 2y_2 + 3y_3 = 2 \qquad (y_i \geq 0) \tag{7}$$

One may wonder whether a system such as (6), (7) is formally a linear programming system. However, if we set $y_1 x_4 = u_1$, $y_2 x_4 = u_2$, $y_3 x_4 = u_3$ and multiply (7) by x_4, it is easily seen that the system may be re-expressed linearly in x_j and u_i (see Problem 2). In general we have:

COMMENT: *A system* (5) *in which some column* $P_t = (y_1, y_2, \ldots, y_m)$ *must satisfy a system* S_t *of linear inequalities in variables* y_i *for* $i = 1, 2, \ldots, m$, *and auxiliary variables* $y_{m+1}, \ldots, y_{m+k}$ *independent of the rest of the system, can be replaced by a linear inequality system by multiplying the relations of* S_t *by* $x_t \geq 0$ *and substituting new variables* $u_{it} = y_i x_t$.

While the above remark permits us formally to expand the system to make it linear, this expansion does not necessarily lead to a linear program which is equivalent to the original system because it is possible for the

linear program to have a solution such that $x_t = 0$ and $u_{it} \neq 0$ at the same time. For further discussion of this point, see Problem 2. Great advantage accrues by *not* using this approach. Using instead the variable coefficient concept, the general program is solved by a series of adjustments of the values of y_i obtained by solving an auxiliary program or *subprograms* in the y_i above. In effect, a large linear program is decomposed into smaller linear programs.

The Method Illustrated.

The working out of example {(6), (7)} will illustrate clearly the general procedure. Suppose we initiate the computation with the basic set of variables x_0, x_1, x_2. We wish to ascertain whether or not the basic feasible solution $x_0 = -24$, $x_1 = 2$, $x_2 = 3$, $x_3 = x_4 = 0$ is optimal for (6), which we will refer to as *a master program*. The simplex multipliers

$$\pi = [-5, 1, 1] \tag{8}$$

are defined so that $\pi P_0 = 1$, $\pi P_1 = 0$, $\pi P_2 = 0$. Multiplying equations (6) on the left by π, we obtain

$$x_0 + (\pi P_3)x_3 + (\pi P_4)x_4 = (\pi Q) \tag{9}$$

where

$$\pi P_3 = 20, \qquad \pi Q = -24 \tag{10}$$

$$\pi P_4 = -5y_1 + y_2 + y_3 \tag{11}$$

It is clear that the test for maximum x_0 is $\pi P_j \geq 0$ for all j. On the other hand, if $\pi P_4 < 0$, it is possible to find a better solution by increasing the value of x_4. Therefore, let us set $z_4 = \pi P_4$ and consider the auxiliary linear programming problem of finding $y_i \geq 0$, and Min z_4 satisfying

$$y_1 + 2y_2 + 3y_3 = 2 \qquad (y_i \geq 0) \tag{12}$$

$$-5y_1 + \ \ y_2 + \ \ y_3 = z_4 \text{ (Min)} \qquad (z_4 = \pi P_4) \tag{13}$$

What we are doing is in keeping with the usual simplex procedure of bringing into the basis that column which prices out the least. The solution yielding minimal πP_4 is immediate, namely, $y_1 = 2$, $y_2 = 0$, $y_3 = 0$; $\pi P_4 = -10$.

We can now obtain an improved solution by introducing the column ($y_1' = 2$, $y_2' = 0$, $y_3' = 0$) into the basis of the master program. However, we must allow for the possibility of revising later the values of y_i. We can do both by rewriting the problem in the form

$$\begin{array}{ccccccccc} \begin{bmatrix}0\\0\\1\end{bmatrix} x_0 & + & \begin{bmatrix}1\\-1\\6\end{bmatrix} x_1 & + & \begin{bmatrix}1\\1\\4\end{bmatrix} x_2 & + & \begin{bmatrix}-4\\-1\\1\end{bmatrix} x_3 & + & \begin{bmatrix}2\\0\\0\end{bmatrix} x_4' & + & \begin{bmatrix}y_1''\\y_2''\\y_3''\end{bmatrix} x_4'' & = & \begin{bmatrix}5\\1\\0\end{bmatrix} \\ \bullet & & \text{(drop)}\ \bullet & & \bullet & & & & \star \end{array} \tag{14}$$

where the y_i'' satisfy the same relations as the y_i:

$$y_1'' + 2y_2'' + 3y_3'' = 2 \qquad (y_i'' \geq 0) \tag{15}$$

We shall refer to the column $\langle y_1'', y_2'', y_3'' \rangle$ as the *generic column*. It appears we have changed our original problem; they are equivalent, however. To see this we rewrite any solution to (14) back in the form (6) by setting $x_4 = x_4' + x_4''$ and letting the coefficients of x_4 be given by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \frac{x_4'}{x_4' + x_4''} + \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} \frac{x_4''}{x_4' + x_4''} \qquad (x_4' + x_4'' > 0) \tag{16}$$

The right-hand side is clearly a convex combination of two points lying in a convex set defined by (12); as a consequence the point (y_1, y_2, y_3) must lie in the convex set also. Conversely, to any solution of (12) we can associate a solution of (16), for example, set $x_4' = 0$; in this case $x_4 = x_4''$, $y_i = y_i''$.

When x_4' is introduced into the basic set of the master program, x_1 will be found to drop out. The new basic solution is $x_0 = -4$, $x_2 = 1$, $x_4' = 2$; $x_1 = x_3 = x_4'' = 0$. The corresponding simplex multipliers are

$$\pi = [0, -4, 1] \tag{17}$$

where π is defined by (4), so that $\pi P_0 = 1$, $\pi P_2 = 0$, $\pi P_4' = 0$. To test for maximum x_0 we form

$$\begin{aligned} \pi P_1 &= 10 \\ \pi P_3 &= 5 \\ \pi P_4'' &= -4y_2'' + y_3'' \end{aligned} \tag{18}$$

The only possibility for increasing values of x_0 is to find values of y_i'' so that $\pi P_4'' < 0$. Accordingly, we consider the new subprogram

$$\begin{aligned} y_1'' + 2y_2'' + 3y_3'' &= 2 \qquad (y_i'' \geq 0) \\ -4y_2'' + y_3'' &= \pi P_4'' \text{ (Min)} \end{aligned} \tag{19}$$

This yields $y_1'' = 0$, $y_2'' = 1$, $y_3'' = 0$ and $\pi P_4 = -4$.

Our augmented master problem now becomes

(20)

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_0 + \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} x_4' + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_4'' + \begin{bmatrix} y_1''' \\ y_2''' \\ y_3''' \end{bmatrix} x_4''' = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccccc} \bullet & \text{drop} \bullet & \bullet & \star \end{array}$$

where again we have allowed for the possibility that we may again revise the values of y_i by the introduction of the "generic" column y_i''' and variable x_4'''.

Introducing x_4'' into the basic set, x_2 drops out and the new basic solution is $x_0 = 0$, $x_4' = \frac{5}{2}$, $x_4'' = 1$; $x_1 = x_2 = x_3 = 0$; the new simplex multipliers

$$\pi = [0, 0, 1] \tag{21}$$

are defined by $\pi P_0 = 1$, $\pi P_4' = 0$, $\pi P_4'' = 0$. We now have

$$\begin{aligned} \pi P_1 &= 6 \\ \pi P_2 &= 4 \\ \pi P_3 &= 1 \\ \pi P_4''' &= y_3''' \end{aligned} \tag{22}$$

The auxiliary linear program is: Find y_i''' and Min $\pi P_4'''$ satisfying

$$\begin{aligned} y_1''' + 2y_2''' + 3y_3''' &= 2 \qquad (y_i''' \geq 0) \\ y_3''' &= \pi P_4''' \text{ (Min)} \end{aligned} \tag{23}$$

But this yields as one optimal solution $y_1''' = 2$, $y_2''' = 0$, and $y_3''' = \text{Min}\, \pi P_4''' = 0$. Thus, at this stage, no improved solution to (20) can be found modifying the values of y_i. Since all other $\pi P_j \geq 0$ the basic solution is optimal.

EXERCISE: From the optimal solution to (20), (23) derive the optimal solution to (6), (7).

It would appear that this process, iterated many times, could expand the problem by an indefinite number of columns. Such is not the case, however, since no more of these added columns need be retained than are currently used in the basis. We shall show that any column which drops out of the basis may be "dropped" because it is included in the convex sets defining the generic columns associated with the master program and these convex sets are each defined by a system of linear inequalities. We now formalize some of the terms and concepts used so far.

DEFINITION: *A restricted master program* (at the kth stage of the algorithm) consists of variables $x_j^{(k)}$ with specified columns of coefficients $P_j^{(k)}$ drawn from the convex sets C_j. Its optimal solution determines values for the simplex multipliers, $\pi = \pi^0$, for use in the subprograms.

DEFINITION: The jth *subprogram* is: Find $P_j \in C_j$ which minimizes the linear form, $\pi^0 P_j$ (in the unknown components of P_j), where $\pi = \pi^0$ is known. Its optimal solution $P_j = P_j^*$ generates an additional specified column of coefficients P_j^* for the next restricted master program.

THEOREM 1: *If terms $\sum_1^n P_j x_j$ are added to a restricted master program, where the "generic columns," P_j, are general elements of convex sets C_j, the new problem is equivalent to the original generalized linear program.*

The General Theory for Polyhedral C_j.

We assume here that each convex set C_j is polyhedral, i.e., defined by a system of linear inequalities. In this case a general P_j can be represented

by a convex linear combination of a *finite set of extreme points* of C_j plus a nonnegative linear combination of a *finite set of homogeneous solutions* (in case C_j is unbounded).

DEFINITION: By a *homogeneous solution* $\bar{P}_j$ is meant one with the property that if $P_j \in C_j$, then $P_j + k\bar{P}_j \in C_j$ for all $k \geq 0$.

THEOREM 2: *A solution* (x_j^*, P_j^*) *for* $j = 0, 1, 2, \ldots, n$, *is optimal if there exists a* π, *such that* $\pi P_0 = 1$, $\pi P_j \geq 0$ *for all* $P_j \in C_j$ *and* $\pi P_j^* = 0$ *for all* j *for which* $x_j^* > 0$, $j \neq 0$.

THEOREM 3: *Only a finite number of iterations of the simplex algorithm is required if each basic feasible solution is improved by introducing into the basis either an extreme point* $P_j^* \in C_j$ *chosen so that*

$$\pi P_j^* = \operatorname*{Min}_{P_j \in C_j} \pi P_j < 0 \qquad (j = 1, 2, \ldots, n) \tag{24}$$

where π *are the simplex multipliers of the basis or by introducing into the basis any homogeneous solution* $\bar{P}_j^*$ *from a finite set such that* $\pi \bar{P}_j^* < 0$.

PROOF: From our earlier remarks, Theorem 2 is obvious. With regard to Theorem 3, finiteness of the algorithm is also obvious if we can show that columns of any basis must be drawn from a finite class. Since C_j is a convex set defined by a set of inequalities, each P_j^* is obtained by solving a linear program which minimizes the linear form πP_j where π is fixed and the components of P_j are unknown.

If C_j is bounded, P_j^* will be *one of a finite number of basic solutions*. If C_j is unbounded, then it may happen that πP_j has no lower bound. In this case, on some iteration of the simplex method a homogeneous solution will be obtained (see Chapter 6, Problem 19). In the notation of the standard simplex method as applied to the canonical form for the subproblem for C_j, on some iteration a column will be found such that all coefficients, say $\bar{a}_{is}$, will be nonpositive and the relative cost factor $\bar{c}_s < 0$. In this case, letting y_{j_i} represent the ith basic variable of the subproblem, the set of values $y_{j_i} = -\bar{a}_{is} \geq 0$, $y_s = 1$, and all other $y_i = 0$ forms a homogeneous solution $\bar{P}_j$ for C_j with the requisite property; namely $\bar{c}_s = \pi \bar{P}_j \leq 0$, where π is the set of simplex multipliers for the basis of the master program. Since the number of canonical forms for the various subprograms (omitting the objective equation) is finite, the number of columns with the property $\bar{a}_{is} \leq 0$ for all i is finite. Hence the set of $\bar{P}_j$ is also finite.

Equivalent Formulations.

It was assumed in (5) that P_0 and Q were fixed vectors. However, if it is desired to have Q freely chosen from a convex set C_q, it is possible to do so by considering the system

$$\begin{aligned} P_0x_0 + P_1x_1 + \ldots + P_nx_n - Qx_{n+1} &= 0 \\ x_{n+1} &= 1 \end{aligned} \tag{25}$$

EXERCISE: Extend the results so that P_0 can be drawn from a convex set C_0.

EXERCISE: Reduce the following problem to a generalized linear program: Find vectors $P_j \in C_j$, $Q \in C_q$ such that

$$P_1 + P_2 + \ldots + P_n = Q \tag{26}$$

EXERCISE: Suppose in place of condition (7), we have $y_1^2 + y_2^2 + y_3^2 \leq 1$. Show that this is a generalized linear program. Apply methods of this section and contrast with the polyhedral case.

Convex Programs. (See Chapter 24.)

Kuhn and Tucker [1950-2] considered a broad class of problems of the form: Find $x = (x_1, x_2, \ldots, x_n)$ and Min z satisfying

$$\begin{aligned} G_i(x) &\leq 0 \qquad (i = 1, 2, \ldots, m) \\ G_0(x) &= z \text{ (Min)} \end{aligned} \tag{27}$$

where $G_i(x)$ are convex functions and x is restricted to a convex set R.

We replace this by the equivalent problem: Find λ, λ_i, y_i satisfying

$$\begin{array}{lllll} \lambda & & & = 1 & (\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_m \geq 0) \\ y_1\lambda + \lambda_1 & & & = 0 & \\ y_2\lambda & + \lambda_2 & & = 0 & \\ \vdots & & \ddots & \vdots & \\ y_m\lambda & & + \lambda_m & = 0 & \\ y_0\lambda & & & = z \text{ (Min)} & \end{array} \tag{28}$$

where $y_i \geq G_i(x)$ for some $x \in R$. To show that the set of possible $y = (y_0, y_1, \ldots, y_m)$ forms a convex set, C_y, suppose $y_i' \geq G_i(x')$, $y_i'' \geq G_i(x'')$, and let $\lambda + \mu = 1$ and $\lambda \geq 0$, $\mu \geq 0$. Then

$$y_i = \lambda y_i' + \mu y_i'' \geq \lambda G_i(x') + \mu G_i(x'') \geq G_i(\lambda x' + \mu x'') \tag{29}$$

where $(\lambda x' + \mu x'') \in R$ because R is convex. The methods of Chapter 24 can be used to extend the results to solve in general this essentially *one variable* linear program with coefficients generated by a set of general convex functions of a point x in n dimensions.

EXERCISE: Prove that if $y_i^o \geq G_i(x^o)$ solves the linear program with $\lambda = \lambda^o$, $\lambda_i = \lambda_i^o$, then $y_i^* = G_i(x^o)$ also solves the program and $G_0(x^o) \leq y_0^o$.

EXERCISE: Suppose a feasible solution to (28) exists with $y = y^o$, where y^o is a convex combination of several y^k such that for each k, $y_i^k \geq G_i(x^k)$, and suppose that feasible solutions may or may not exist for these y^k, prove there exists an x^o such that $y_i^o \geq G_i(x^o)$ where $y^o = (1, y_1^o, \ldots, y_m^o, y_0^o)$.

EXERCISE: Prove that the simplex multipliers π_i^o of the optimal solution of each master program for (28) are nonnegative. Show that the subprogram can be stated: Find $x \in R$ such that $F(x)$ is minimum where

$$F(x) = \sum \pi_i^o G_i(x)$$

Prove that $F(x)$ is convex. Relate this to the method of Lagrange multipliers, § 6-5.

22-2. NOTES ON SPECIAL CASES

The Case of One Control Parameter T.

Many industrial processes have a continuous spectrum of alternative activities depending on the settings of certain controls such as temperature (T), pressure (P), recycle ratio (R), etc. This is particularly true in refinery applications where, to keep things simple, only one standard way to operate the equipment is often assumed in any one problem; or sometimes a few typical sets of values are selected that "span" the range of possibilities.

Suppose there is one *control* T, whose range of settings is $T_1 \leq T \leq T_2$; let $j = 1$ be a single *activity* whose technological coefficients depend on T.

Its coefficients for $T = T_1$ and $T = T_2$ and general T are

(1)

T_1	T_2	T
$a_{11}^{(1)}$	$a_{11}^{(2)}$	y_1
$a_{21}^{(1)}$	$a_{21}^{(2)}$	y_2
.	.	.
.	.	.
.	.	.
$a_{m1}^{(1)}$	$a_{m1}^{(2)}$	y_m
$c_1^{(1)}$	$c_1^{(2)}$	y_{m+1}

where $y_i = y_i(T)$ for $i = 1, 2, \ldots, m, m + 1$ are functions of T.

In many applications a linear interpolation between the first two columns of coefficients is a satisfactory approximation for a general T. If this is the case, the equivalent linear programming problem becomes:

(2)
$$\begin{aligned}
a_{11}^{(1)}x_{11} + a_{11}^{(2)}x_{21} + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}^{(1)}x_{11} + a_{21}^{(2)}x_{21} + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{m1}^{(1)}x_{11} + a_{m1}^{(2)}x_{21} + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \\
c_1^{(1)}x_{11} + c_1^{(2)}x_{21} + c_2x_2 + \ldots + c_nx_n &= z
\end{aligned}$$

where $x_1 = x_{11} + x_{21}$ is the level of the first activity. It is clear we are assuming that the variable coefficients y_i are given by the linear interpolation (3), where $\lambda_1 = x_{11}/x_1$ corresponds to some temperature setting T.

$$
(3) \qquad \begin{aligned} \lambda_1 \quad + \lambda_2 \quad &= 1, \\ \lambda_1 a_{i1}^{(1)} + \lambda_2 a_{i1}^{(2)} &= y_i \qquad (\lambda_i \geq 0;\ i = 1, 2, \ldots, m) \\ \lambda_1 c_1^{(1)} + \lambda_2 c_1^{(2)} &= y_{m+1} \end{aligned}
$$

In particular, if the temperature setting T has also the same linear interpolation, then

$$
(4) \qquad \frac{x_{11}T_1 + x_{21}T_2}{x_{11} + x_{21}} = T
$$

The set of relations (3) defines a convex set of possible values for $(y_1, y_2, \ldots, y_m)$. The method of § 22-1 is therefore applicable. If π is the set of multipliers associated with a basis and $\bar{c}_1^{(1)} = c_1^{(1)} - \Sigma\pi_i a_{i1}^{(1)}$ and $\bar{c}_1^{(2)} = c_1^{(2)} - \Sigma\pi_i a_{i1}^{(2)}$, then the subprogram reduces to

$$
\begin{aligned} \lambda_1 + \lambda_2 &= 1 \qquad (\lambda_i \geq 0) \\ \bar{c}_1^{(1)}\lambda_1 + \bar{c}_1^{(2)}\lambda_2 &= \pi P_1 \text{ (Min)} \end{aligned}
$$

The extreme point solutions for (λ_1, λ_2) are either (1, 0) or (0, 1). Thus either the first or the second column of (2) would be introduced into the basis. Because of this small number of possible extreme points of the subprogram (two in this case), it does not pay to use the method of § 22-1.

In order to interpolate between several possible values of T, say $T = T_1$, $T_2, \ldots, T_k$, let us consider the linear program

$$
(5) \qquad \begin{aligned} a_{11}^{(1)}x_{11} + a_{11}^{(2)}x_{21} + \ldots + a_{11}^{(k)}x_{k1} + \sum_2^n a_{1j}x_j &= b_1 \\ a_{21}^{(1)}x_{11} + a_{21}^{(2)}x_{21} + \ldots + a_{21}^{(k)}x_{k1} + \sum_2^n a_{2j}x_j &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}^{(1)}x_{11} + a_{m1}^{(2)}x_{21} + \ldots + a_{m1}^{(k)}x_{k1} + \sum_2^n a_{mj}x_j &= b_m \\ c_1^{(1)}x_{11} + c_1^{(2)}x_{21} + \ldots + c_1^{(k)}x_{k1} + \sum_2^n c_j x_j &= z \end{aligned}
$$

where $a_{i1}^{(r)}$ is the value of $y_i = y_i(T)$ for $T = T_r$, $r = 1, 2, \ldots, k$. In general, letting

$$
(6) \qquad x_1 = x_{11} + x_{21} + \ldots + x_{k1}
$$

it is clear that we are *assuming* the linear interpolations

$$
(7) \qquad \frac{x_{11}}{x_1} a_{i1}^{(1)} + \frac{x_{21}}{x_1} a_{i1}^{(2)} + \ldots + \frac{x_{k1}}{x_1} a_{i1}^{(k)} = y_i
$$

and a similar expression for c_1.

However, in some applications *a linear interpolation between two columns of coefficients is satisfactory, providing the range of their T values is sufficiently small. If not, then it is necessary to break up the range into several parts* $T_1 \leq T_2 \ldots \leq T_r \leq \ldots \leq T_k$, so that a linear interpolation between the vectors $[y_1(T), y_2(T), \ldots, y_{m+1}(T)]$ for adjacent pairs $T = T_i$ and $T = T_{i+1}$ is an acceptable approximation. For this situation, an optimal solution to system (5) is acceptable only if the positive x_{ri} occur in adjacent pairs as underlined in (8):

$$
\begin{array}{llll}
(8) & \text{either} & (a) & \underline{x_{11} \geq 0,\ x_{21} \geq 0},\ x_{31} = 0,\ x_{41} = 0, \ldots \\
 & \text{or} & (b) & x_{11} = 0,\ \underline{x_{21} \geq 0,\ x_{31} \geq 0},\ x_{41} = 0, \ldots \\
 & \text{or} & (c) & x_{11} = 0,\ x_{21} = 0,\ \underline{x_{31} \geq 0,\ x_{41} \geq 0}, \ldots \\
 & \multicolumn{3}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}
\end{array}
$$

The following results can be established: (a) If the coefficients y_i except y_{m+1} are linear in T and y_{m+1} is a strictly convex function of T, then (8) will hold. (b) Let T' be some strictly monotonic function of the control parameter T; if the coefficients y_i, $i \neq m + 1$, are linear functions of T' and y_{m+1} is a strictly convex function of T', then (8) must hold. (c) If y_{m+1} is just a convex (but not necessarily strictly convex) function of T', then (8) need not hold, but if not, then the *average* values a_{i1} and c_{i1} obtained at the minimum are the coefficients y_i corresponding to some fixed T' and the optimum solution to (5) is exact—not an approximation to the original nonlinear problem. (d) Suppose the system has the property that the optimal prices $\pi_i \leq 0$ for some subset of the items (where π_i are defined so that $\Sigma a_{ij}\pi_i = c_i$ for basic x_i); suppose that y_i depend on T' only for these items and these are convex functions of T', then either property (8) holds for an optimum solution or the average solution is exact for some fixed T'.

Interpreting a Mixed Solution.

An interesting question arises if (8) does not hold and moreover any adjustment of the solution so that it does hold only increases the value of z. In this situation we have no physical interpretation of the solution in the sense there are values x_1 given by (6) and a_{i1}, c_1 given by (7) which correspond to some value of T. This may or may not be acceptable. If the activity is such that T is not an adjustable control, but rather a design characteristic that once settled for some value cannot be changed, then the answer is, of course, not acceptable. (We shall discuss what to do in this case in a moment.)

If T is easily adjustable, there may be an "out." Let us suppose the optimal solution yields $x_{11} > 0$, $x_{12} = x_{13} = 0$, $x_{14} > 0$. We may interpret this to mean that it pays to use more than one setting for T. Thus, if the activity is actually performed with several pieces of equipment, some can be

set for the value $T = T_1$ and the others for $T = T_4$. Or, if only one piece of equipment is used, it may be set part of the time at T_1 and the rest of the time at T_4.

If Only One Value of T Is Acceptable.

This case can be treated by limiting the number of alternative T values. For example, suppose a basic solution is at hand using a fixed value $T = T_r$. There are no alternative columns for the moment corresponding to other values of T. It is now desired to see if the solution can be improved by changing the value of T. Accordingly, new columns are introduced corresponding to adjacent values of T, say T_{r-1} and T_{r+1}, these being selected sufficiently close that linear interpolation of their a_{i1} values is acceptable. These columns are then priced out; and, if it pays to do so, one of the alternative T columns is introduced. If, as a result, T_r is replaced in the basis by, say T_{r+1}, then, T_{r+2} is added as a possible alternative column and T_{r-1} is dropped from consideration in the next iteration. The new solution is then priced out on the added column and on all P_j, $j = 2, \ldots, n$, since the shift in T may make it profitable to make other choices for the basic variables. On the other hand if the introduction of the alternative column T_{r+1} causes not the T_r column, but some other column to drop, then a T_{r+2} column is added as a possible alternative and the T_{r-1} column is retained and both are priced out along with the other columns; if T_{r+2} becomes a candidate by the pricing out procedure, it is allowed to enter the basis only if it replaces T_r, if not it is dropped as a candidate in the next iteration; similarly T_{r-1} is allowed to be a candidate only if it replaces T_{r+1}.

This procedure will eventually arrive at a value of $T = T_r^*$, such that the alternatives T_{r-1}^* or T_{r+1}^* result in no admissible improvement, or it will eventually arrive at an interpolation between two T values T_r, T_{r+1}. This does not mean that this T is best; all that has been found is a T that is *locally* best. If there is any suspicion that there may be other local optima that are superior, it is necessary to revise the procedure so as to drive T in turn through all values $T = T_1, T_2, \ldots, T_k$. If there are other local optima superior to $T = T_r$, their z values must lie between the z value for $T = T_r$ and the z value obtained by allowing free choice among all alternative T_r columns. From a practical point of view, it might be best to allow unrestricted choice initially. If the solution satisfies (8), then of course the solution admits physical interpretation and is optimal. If not, it is often feasible to use the average T value

$$T = \sum_r x_{r1} T_r / x_1 \tag{9}$$

as a starting value for the above procedure. For example, in gasoline blending problems there is a nonlinear "octane change" as a function of the amount

T of tetraethyl lead added to a mixture of blending stocks whose proportions are also to be determined. While the structure of such a problem is more complex than that described here, a linear programming solution can arise for which there is no physical interpretation. This is due to a lack of convexity of a certain octane "response surface." However, it is so slight that the approximation (9) gives excellent results and no further iterations are used in practice [Kawaratani, Ullman, and Dantzig, 1960-1].

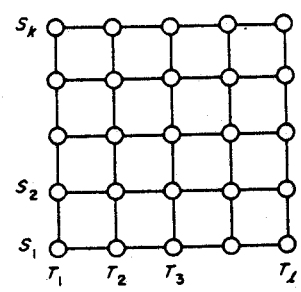
The Case of Several Independent Control Parameters, Each Affecting a Different Activity.

While the discussion so far has centered about one control T and one activity $j = 1$, it should be noted that, if there were several independent control parameters, each affecting a different activity, the values of all the parameters could be determined *simultaneously* by splitting up each such activity column into parts corresponding to the different control settings.

The Case of Several Control Parameters Applying to the Same Activity.

For the case of only two control parameters whose range is $T_1 \leq T \leq T_2$ and $S_1 \leq S \leq S_2$, one can introduce the four extreme cases (T_1, S_1), (T_1, S_2), (T_2, S_1) and (T_2, S_2) as four alternative columns. Thus in (5) the coefficients of x_{11} could be interpreted as those obtained by setting $(T, S) = (T_1, S_1)$, etc. Again, if the resulting linear interpolations given by (6) and (7) are acceptable, the problem may be solved in this manner. Nothing prevents the introduction of any number of alternative columns corresponding to a grid of (T_i, S_j) values in order to obtain a more accurate approximation.

(10)



Again the only difficulty that can arise is one of physical interpretation if the optimal solution chooses non-adjacent grid points with positive weights.

When there are more than two control parameters associated with an activity, it is recommended that variables be used that measure the *change in the value of the parameters*. To illustrate, let the coordinates of P_1, the first activity, be some function of the three control parameters, R, S, T (for each component of the vector P_1) which we denote by $P_1 = P_1(R, S, T)$. Let us

suppose that for $(R, S, T) = (R_0, S_0, T_0)$ a solution to the linear programming problem is known, and that the linear approximation,

$$(11)\quad P_1(R_0 + \Delta R, S_0 + \Delta S, T_0 + \Delta T) \doteq P_1^o + \Delta R \cdot E^o + \Delta S \cdot F^o + \Delta T \cdot G^o$$

where P_1^o, E^o, F^o, G^o are fixed vectors, is acceptable within the ranges

$$(12)\qquad -\alpha_1' \leq \Delta R \leq \alpha_1,\ -\alpha_2' \leq \Delta S \leq \alpha_2,\ -\alpha_3' \leq \Delta T \leq \alpha_3$$

This formulation permits immediate and easy solution via the methods of § 22-1 and the upper bounding methods of § 18-1.

Let π be the simplex multipliers associated with the known basis of the master program. By Theorem 3, § 22-1, we set up the subprogram: Minimize $\pi P_1 = \pi P_1^o + \Delta R \pi E^o + \Delta S \pi F^o + \Delta T \pi G^o$ subject to the constraints (12). This subprogram divides into three independent linear programs:

$$\begin{aligned} -\alpha_1' \leq \Delta R \leq \alpha_1 \qquad & (\pi E^o)\Delta R = z_R(\text{Min}) \\ -\alpha_2' \leq \Delta S \leq \alpha_2 \qquad & (\pi F^o)\Delta S = z_S\ (\text{Min}) \\ -\alpha_3' \leq \Delta T \leq \alpha_3 \qquad & (\pi G^o)\Delta T = z_T\ (\text{Min}) \end{aligned}$$

These are readily solved; for example, setting $\bar{e} = \pi E^0$, the solution for ΔR is $\Delta R = -\alpha_1'$, 0, or α_1 according as $\bar{e} > 0$, $\bar{e} = 0$, or $\bar{e} < 0$. Substituting these values into (11) yields a new column, P_1', to be introduced into the basis in the next iteration. After that iteration, expression (11) may be retained for further iterations, or it may be replaced by a new linear approximation about the new values of R, S, and T determined by the iteration. If, however, expression (11) is changed while P_1^o vector is still in the basis, we may encounter difficulties in physical interpretation of the results.

As an alternative to the subprogram of § 22-1, an equivalent linear program can be obtained by multiplying all expressions in (11) and (12) through by x_1 and introducing new variables $x_1\Delta R = x_{11}$, $x_1\Delta S = x_{21}$, $x_{11}\Delta S = x_{31}$. This yields, setting $P_1 = \langle a_{11}, a_{12}, \ldots, a_{1m}, c_1\rangle$,

$$(13)\qquad \begin{gathered} a_{i1}x_1 = a_{i1}^o x_1 + e_{i1}x_{11} + f_{i1}x_{21} + g_{i1}x_{31} \\ -\alpha_1' x_1 \leq x_{11} \leq \alpha_1 x_1,\ -\alpha_2' x_1 \leq x_{21} \leq \alpha_2 x_1,\ -\alpha_3' x_1 \leq x_{31} \leq \alpha_3 x_1 \end{gathered}$$

Thus the final linear programming problem requiring solution becomes

$$(14)\qquad \begin{aligned} a_{11}^o x_1 + e_{11}x_{11} + f_{11}x_{21} + g_{11}x_{31} + (a_{12}x_2 + \ldots + a_{1n}x_n) &= b_1 \\ a_{21}^o x_1 + e_{21}x_{11} + f_{21}x_{21} + g_{21}x_{31} + (a_{22}x_2 + \ldots + a_{2n}x_n) &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}^o x_1 + e_{m1}x_{11} + f_{m1}x_{21} + g_{m1}x_{31} + (a_{m2}x_2 + \ldots + a_{mn}x_n) &= b_m \\ c_1^o x_1 + \quad ex_{11} + \quad fx_{21} + \quad gx_{31} + \quad (c_2x_2 + \ldots + \quad c_nx_n) &= z\ (\text{Min}) \end{aligned}$$

where $x_j \geq 0$, and x_{r1} and x_1 are subject to (13).

22-3. PROBLEMS

1. *Review:* Extend the theory of § 22-1 to cover the case where C_0 and C_q are general convex sets instead of convex sets each consisting of one fixed vector.
2. (a) In system § 22-1-(6) and (7), replace each $y_i x_4$ by u_i and reduce to a linear inequality system and solve numerically. Can this solution be used to solve the original system?
   (b) Review § 22-1, Comment, in general. Suppose a solution is obtained for the new system with $x_t = 0$; show that $y_i x_t \neq 0$ may be possible. Construct an example where this is the case.
   (c) Show that the linear program is equivalent to the original generalized program for the system (6) and (7).
   (d) Show in general that the linear program is equivalent to the original generalized program if the linear program implies a relation $\Sigma a_i u_i \leq a_0 x_t$ with positive a_i, where $u_i = y_i x_t$.
   (e) Consider the general problem of the existence of $a_i > 0$ such that $\Sigma a_i u_i \leq a_0 x_t$ for all $x_t \geq 0$ and $u_i \geq 0$ satisfying a linear inequality system $(x, u)D \geq d$. Set up a linear program for finding such a_i if they exist.

   Hint: Because of homogeneity, let $a_i = v_i + 1$ where $v_i \geq 0$. Then the problem is equivalent to finding u_i, x_j, and v_i such that

$$\begin{array}{r|l} x \geq 0, u \geq 0 & v_i \geq 0 \\ (x, u)D \geq d & v_t = 1 \\ \Sigma u_i - x_t = \bar{x}_t & \end{array}$$

$$\theta = \min_{v} \max_{x|v} \left[\sum v_i u_i - v_0 x_t + v_t \bar{x}_t \right]$$

   If $\theta \leq 0$, then $a_i > 0$ exists. Use Problem 2 of Chapter 13 to complete this discussion.

3. Complete the exercises given in § 22-1.
4. Simplify (13) of § 22-2 by substituting $\bar{x}$:

$$\bar{x}_{r1} = x_{r1} + \alpha_r' x_1 \qquad (r = 1, 2, 3)$$

5. (Unsolved.) Develop a theory for the case of one control parameter affecting simultaneously several activities.
6. (Unsolved.) In connection with (d) following § 22-2-(8), devise a procedure for converting k given functions, $F_i(T)$, to convex functions of a parameter T' where T' is a monotonic function of T, providing such a conversion is possible. (See Problem 7.)
7. Formulate Problem 6 as a linear program if the k given functions are defined for discrete T_i and it is desired to find increasing T_i' corresponding

to increasing T_i such that the broken line fit through the discrete points $[T'_i, F(T_i)]$ is a convex function of T'.

8. Establish the assertions (a), (b), (c), (d) following § 22-2-(8).

REFERENCES

Dantzig, 1957-3
Dantzig and Wolfe, 1960-1
Ford and Fulkerson, 1958-1
Gomory and Hu, 1960-1, 2
Kawaratani, Ullman and Dantzig, 1960-1
Kuhn and Tucker, 1950-2

CHAPTER 23

A DECOMPOSITION PRINCIPLE FOR LINEAR PROGRAMS

23-1. THE GENERAL PRINCIPLE

To introduce a typical situation that suggests the application of the decomposition principle, consider the problem facing a manager of a plant with two almost independent shops. Within each shop, there are many constraints which are unaffected by the activities of the other shop, but there are a few constraints and a common objective that tie the two shops together. The manager's problem may be formulated in linear programming terms as follows: Find $X \geq 0$, $Y \geq 0$, and Max x_0 satisfying

$$
\begin{aligned}
A_1X \qquad\qquad &= b_1 \\
A_2Y &= b_2 \\
P_0x_0 + \bar{A}_1X + \bar{A}_2Y &= \bar{b}
\end{aligned}
\tag{1}
$$

We are using an extension of the notation set up at the beginning of Chapter 22. X is the vector of activity levels in the first shop; Y, that in the second. The first line of (1) expresses the constraints which involve directly only the first shop; the second line does the same for the second shop; the last line expresses the objective function and those constraints which bind together the shops.

On looking at (1), the manager feels that the size of the problem has gotten out of hand. Both A_1 and A_2 are moderately large, and together they make a problem that exceeds the capacity of available computers. "But what I really have," reflects the manager, "is not this *one* big problem but *two* moderate sized ones, one for each shop. All I need is a way to break the problem into two parts and still take account of their interconnections."

In this chapter we will follow through on the manager's hunch by developing a technique which decomposes linear programs similar to (1) into

(a) *subprograms corresponding to its almost independent parts, and*
(b) *a master program which ties together the subprograms.*

The price paid for this decomposition is that the master program and the subprogram may have to be solved several times. First the master program is solved, and from its solution, objective functions are generated for each of the subprograms. Then these are solved, and from their solution new columns are generated to be added to the master program. The process is

then repeated until, after a finite number of cycles, an optimality test is passed.[1] In the next section we will show how this technique can be applied to certain problems arising in dynamic systems, and in the last section it will be used to show how central planning can be accomplished without complete information at the center.

For our discussion it will be convenient to think of problem (1) in the following form: Solve the linear program

$$P_0x_0 + \bar{A}_1X + \bar{A}_2Y = \bar{b} \tag{2}$$

subject to the additional constraints

$$\begin{aligned} \mathscr{L}_1: \quad & A_1X = b_1 && (X \geq 0) \\ \mathscr{L}_2: \quad & A_2Y = b_2 && (Y \geq 0) \end{aligned} \tag{3}$$

The A_i, $\bar{A}_i$ are, of course, matrices; P_0, $\bar{b}$, and b_i are vectors.

It will simplify the discussion to assume the feasible sets for $\mathscr{L}_1$ and $\mathscr{L}_2$ to be bounded convexes and to indicate later the minor modifications in the formulas needed to take care of the unbounded case. *Under this assumption*, any $X \geq 0$ solving $A_1X = b_1$ can be represented by a convex combination of the *extreme points* of the set of feasible solutions of $\mathscr{L}_1$. Since the set of different basic feasible solutions $X = X_1$, $X = X_2$, . . ., $X = X_K$ defines the *finite* set of extreme points, we can represent any solution X by

$$X = \sum_1^K \lambda_i X_i \qquad \left(\sum_1^K \lambda_i = 1;\ \lambda_i \geq 0\right) \tag{4}$$

Conversely, any X represented by (4) is feasible for $\mathscr{L}_1$. Similarly, any $Y \geq 0$ solving $A_2Y = b_2$ can be represented by

$$Y = \sum_1^L \mu_j Y_j \qquad \left(\sum_1^L \mu_j = 1;\ \mu_j \geq 0\right) \tag{5}$$

where Y_1, Y_2, . . ., Y_L are the finite set of basic feasible solutions of $\mathscr{L}_2$. Hence, any solution X and Y solving (1) can be re-expressed in terms of λ_i, μ_i; thus

$$\left\{\begin{aligned} P_0x_0 + \sum_1^K \lambda_i(\bar{A}_1X_i) + \sum_1^L \mu_j(\bar{A}_2Y_j) &= \bar{b} \quad (\lambda_i \geq 0;\ \mu_j \geq 0) \\ \sum_1^K \lambda_i &= 1 \\ \sum_1^L \mu_j &= 1 \end{aligned}\right. \tag{6}$$

[1] Historically, it was this special case that first gave rise to the more general concept of a generalized linear program developed in § 22-1 [Dantzig and Wolfe, 1960-1]. The decomposition principle approach was inspired by the proposals of Ford and Fulkerson [1958-1] for solving multistage commodity network problems. W. S. Jewell [1958-1] should also be credited with using a similar approach for the latter.

and, conversely, any λ and μ satisfying (6), determine an X and Y by (4) and (5), and give a feasible solution to (1). Denoting, in general, the linear transforms of X and Y,

$$S = \bar{A}_1 X, \quad T = \bar{A}_2 Y \tag{7}$$

and, in particular,

$$S_i = \bar{A}_1 X_i, \quad T_j = \bar{A}_2 Y_j \quad (i = 1, 2, \ldots, K; j = 1, 2, \ldots, L) \tag{8}$$

the original linear program is *equivalent* to the problem:

Find $\lambda_1 \geq 0, \ldots, \lambda_K \geq 0; \mu_1 \geq 0, \ldots, \mu_L \geq 0$ and Max x_0 satisfying

$$\begin{aligned} P_0 x_0 + \sum_1^K S_i \lambda_i + \sum_1^L T_j \mu_j &= \bar{b} \qquad (\lambda_i \geq 0;\ \mu_i \geq 0) \\ \sum_1^K \lambda_i \qquad\qquad &= 1 \\ \sum_1^L \mu_j \quad &= 1 \end{aligned} \tag{9}$$

DEFINITION: The linear program (9) generated from the extreme point solutions of $\mathscr{L}_i$ by (8) is called the equivalent *extremal problem*, or the *full master program*.

The basic solutions of $\mathscr{L}_1$ and $\mathscr{L}_2$ are probably far too numerous for us ever to express this extremal problem explicitly; rather, we propose to solve it by generating only those columns S_i and T_j which the simplex method brings into the successive bases.

Let us suppose that we have at hand an *initial, basic feasible solution* $\lambda_i = \lambda_i^o$ and $\mu_j = \mu_j^o$ to the extremal problem. If $\bar{b}$ has m components, then there will be $m + 2$ columns, say $S_1, S_2, \ldots, S_k$; $T_1, T_2, \ldots, T_l$ in the corresponding basis, where $k + l = m + 2$. Corresponding to this basic feasible solution to the extremal problem, is the solution $X = X^o$, $Y = Y^o$, $x_0 = x_0^o$ to (1) given by

$$\begin{aligned} X^o &= \lambda_1^o X_1 + \lambda_2^o X_2 + \ldots + \lambda_k^o X_k \quad (\Sigma \lambda_i^o = 1;\ \lambda_i^o \geq 0) \\ Y^o &= \mu_1^o Y_1 + \mu_2^o Y_2 + \ldots + \mu_l^o Y_l \quad (\Sigma \mu_i^o = 1;\ \mu_i^o \geq 0) \end{aligned} \tag{10}$$

At the end of this section we will give a Phase I procedure by which such a starting solution and representation may be found.

In order to test optimality of the basic solution or to generate a better solution to the master program, let the row vector $(\pi^o;\ -s^o,\ -t^o)$ denote the simplex multipliers associated with starting basis

$$B = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} P_0 & S_1 & S_2 \ldots & S_k & T_1 & T_2 \ldots & T_l \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{Multipliers} \\ : \ (\pi^o) \\ : \ (-s^o) \\ : \ (-t^o) \end{array} \tag{11}$$

where s^o, t^o correspond to the last two rows of B. As discussed in § 22-1, $(\pi^o; -s^o, -t^o)$ is determined such that

$$\pi^o P_0 = 1, \quad \pi^o S_i = s^o, \quad \pi^o T_j = t^o \tag{12}$$

for $i = 1, 2, \ldots, k; j = 1, 2, \ldots, l$.

The standard simplex method now requires us to bring into the basis that column of (9) which has the lowest relative cost, $\pi^o S_i - s^o < 0$ or $\pi^o T_j - t^o < 0$, if any. That is, we must determine S_* and T_* such that

$$\begin{aligned} \pi^o S_* &= \operatorname*{Min}_i \pi^o S_i \\ \pi^o T_* &= \operatorname*{Min}_j \pi^o T_j \end{aligned} \tag{13}$$

But

$$\operatorname{Min} \pi^o S_i = \operatorname{Min} (\pi^o \bar{A}_1) X_i = \operatorname*{Min}_{\substack{X \geq 0 \\ A_1 X = b_1}} \gamma_i^o X \tag{14}$$

where $\gamma_1^o = \pi^o \bar{A}_1$. That is, we determine S_* by finding the solution, X_*, to

The Subprogram $\mathscr{L}_1$:

$$\begin{aligned} A_1 X &= b_1 && (X \geq 0) \\ \gamma_1^o X &= z_1 \text{ (Min)} && (\gamma_1^o = \pi^o \bar{A}_1) \end{aligned} \tag{15}$$

If the simplex method is used on the subprogram, X_* will always be one of the $X_1, \ldots, X_K$. We can then form $S_* = \bar{A}_1 X_*$ and bring it into the basis. If

$$\operatorname{Min} z_1 = \gamma_1^o X_* < s^o \tag{16}$$

then the relative cost factor for S_*, $\pi^o S_* - s^o = \gamma_1^o X_* - s^o$, is negative and the introduction of S_* into the basis will (assuming nondegeneracy) bring about a finite increase in x_0.

Likewise, we can determine T_* from the Y_* which solves

The Subprogram $\mathscr{L}_2$:

$$\begin{aligned} A_2 Y &= b_2 && (Y \geq 0) \\ \gamma_2^o Y &= z_2 \text{ (Min)} && (\gamma_2^o = \pi^o \bar{A}_2) \end{aligned} \tag{17}$$

If $\operatorname{Min} z_2 < t^o$, then introducing $T_* = \bar{A}_2 Y_*$ into the basis will (assuming nondegeneracy) increase x_0 by a finite amount. We introduce S_* or T_*, whichever has the lower relative cost factor.

On the other hand, if $\operatorname{Min} z_1 = s^o$ and $\operatorname{Min} z_2 = t^o$, then all the relative cost factors for (9) are nonnegative. Consequently, we are at the optimal solution of the equivalent extremal problem, and therefore, X^o and Y^o given by (10) constitute an optimal solution to (1).

DEFINITION: The program obtained from the full master program (9) by dropping all the columns except those in the basis and the new S_* (or T_*)

about to be introduced, is known as *The Restricted Master Program*: Find $\lambda_i \geq 0$, $\mu_j \geq 0$, and Max x_0 such that

$$(18)\quad \begin{aligned} P_0x_0 + S_1\lambda_1 + \ldots + S_k\lambda_k + S_*\lambda_* + T_1\mu_1 + \ldots + T_l\mu_l &= \bar{b} \\ \lambda_1 + \ldots + \lambda_k + \lambda_* &= 1 \\ \mu_1 + \ldots + \mu_l &= 1 \end{aligned}$$

Iterative Process and Final Solution.

We now proceed to solve the restricted master program. The resulting optimal solution yields a new (π, s, t); these, in turn, determine new γ_1 and γ_2, which constitute new objective forms for $\mathscr{L}_1$ and $\mathscr{L}_2$. The subprograms are then solved with the new objective forms, and the optimality test above is applied. If the test fails, a new column is added to the restricted master program, corresponding to the subprogram failing the test. The whole process is then repeated, and the cycle continued until the optimality test is passed. The optimal solution is then given by $X = \sum_i \lambda_i X_i$ and $Y = \sum_j \mu_j Y_j$, where the λ_i and μ_j are the solution to the final restricted master program and the X_i and Y_j are the solutions of $\mathscr{L}_1$ and $\mathscr{L}_2$ corresponding to columns in the final basis of the last restricted master program. If the restricted master programs are nondegenerate (or ε-perturbed), the introduction of each S_* or T_* will, as we remarked, increase x_0 by a finite amount. Hence, none of the finite number of bases of (9) can reappear, so the iterative procedure is finite. We have established

THEOREM 1: *The solution $X = X^o$, $Y = Y^o$ corresponding to a basic solution of the master program is optimal if there exist no solutions to the subprograms with z_1, z_2 values less than those of the solutions which were used to generate the basis of the master program, i.e., it is optimal if*

$$(19)\qquad \text{Min } z_1 = s^o, \quad \text{Min } z_2 = t^o$$

If the restricted master programs are nondegenerate (or ε-perturbed) such an optimum will be reached in a finite number of iterations.

THEOREM 2: *An upper bound for the values x_0 is*

$$(20)\qquad \begin{aligned} \text{Max } x_0 &\leq x_0^o + (s^o - \text{Min } z_1) + (t^o - \text{Min } z_2) \\ &\leq x_0^o + (s^o - \pi^o S_*) \quad + (t^o - \pi^o T_*) \end{aligned}$$

PROOF: Multiply (9) by the multipliers, $(\pi^o;\ -s^o,\ -t^o)$, and sum. The constant term is x_0^o, which is the value of x_0 for the basic solution. We have, therefore,

$$(21)\qquad x_0 + \sum_1^K (\pi^o S_i - s^o)\lambda_i + \sum_1^L (\pi^o T_j - t^o)\mu_j = x_0^o$$

$$(22)\qquad x_0 + \sum_1^K (\text{Min } z_1 - s^o)\lambda_i + \sum_1^L (\text{Min } z_2 - t^o)\mu_j \leq x_0^o$$

and the result follows.

Computational Remarks.

The change of basis in the restricted master program is performed, of course, by the simplex method using multipliers (Chapter 9). In this format, the *lexicographic ordering rule* may be used to resolve degeneracy and assure that no basis is repeated (Chapter 10). As various columns of the master program are generated during the iterative process, one of the following three variants is customarily used:

(1) The restricted master program is augmented by each new column, but each column that drops out of the basis is dropped from the current restricted master.

(2) The restricted master program is augmented by more and more columns and those dropping out of the basis are retained as supplementary columns.

(3) The restricted master program is augmented by more and more columns, and those dropping out of the basis are retained up to the available memory capacity within the electronic computer; at this point columns that price out most positive are dropped.

Observe also that on each cycle after the first we can start the solution of a subprogram or the master program from the basic solution which was optimal on the preceding cycle. Thus, no Phase I procedures are necessary except before the first cycle, and there is a good chance that only a few pivot operations will be necessary for the re-solution of the subprograms and master program.

Modifications Necessary for Unbounded $\mathscr{L}_i$.

Even though the set of feasible solutions to the entire problem (1) is bounded, it is quite possible that the sets of solutions for some $\mathscr{L}_i$ are unbounded. In this case, there may be no lower bound for z_i, and the minimization of z_i via the simplex procedure will result in a *homogeneous* solution X_*, Y_*, satisfying

$$\begin{aligned} A_1X_* = 0 \text{ or } A_2Y_* = 0 \\ \gamma_1X_* < 0 \text{ or } \gamma_2Y_* < 0 \end{aligned} \tag{23}$$

These X_*, Y_* as pointed out in a similar discussion at the end of § 22-1, belong to a finite class (extreme rays, Chapter 7). In this case *we can generate the full class of solutions to $\mathscr{L}_i$ by considering solutions that are convex combinations of the nonhomogeneous solutions and nonnegative combinations of the homogeneous solutions*. Thus, in place of (4) and (5)

$$\begin{aligned} X' &= X_1\bar{\lambda}_1 + X_2\bar{\lambda}_2 + \ldots + X_K\bar{\lambda}_K \\ Y' &= Y_1\bar{\mu}_1 + Y_2\bar{\mu}_2 + \ldots + Y_L\bar{\mu}_L \end{aligned} \tag{24}$$

where $\bar{\lambda}_i \geq 0$, $\bar{\mu}_i \geq 0$, satisfy a condition that the λ_i and μ_j must each sum

to unity when restricted to those i and j corresponding to nonhomogeneous solutions, i.e.,

$$\begin{aligned} \delta_1\bar{\lambda}_1 + \delta_2\bar{\lambda}_2 + \ldots + \delta_K\bar{\lambda}_K &= 1 \\ \delta_1'\bar{\mu}_1 + \delta_2'\bar{\mu}_2 + \ldots + \delta_L'\bar{\mu}_L &= 1 \end{aligned} \tag{25}$$

where

$$\begin{cases} \delta_i = 1 \text{ if } X_i \text{ is a basic feasible solution,} \\ \delta_i = 0 \text{ if } X_i \text{ is a homogeneous solution,} \end{cases} \tag{26}$$

$$\begin{cases} \delta_i' = 1 \text{ if } Y_i \text{ is a basic feasible solution,} \\ \delta_i' = 0 \text{ if } Y_i \text{ is a homogeneous solution} \end{cases}$$

This means that the basis B is redefined to be

$$B = \begin{bmatrix} P_0 & S_1 \ldots S_k & T_1 \ldots T_l \\ 0 & \delta_1 \ldots \delta_k & 0 \ldots 0 \\ 0 & 0 \ldots 0 & \delta_1' \ldots \delta_l' \end{bmatrix} \tag{27}$$

In the event $b_1 = 0$ or $b_2 = 0$, only homogeneous solutions are of interest. In this case the corresponding restrictions (25) can be dropped from the master program.

Since there is only a finite number of homogeneous solutions generated by the simplex process, the proof of the finiteness of the iterative procedure remains valid.

Initiating the Algorithm.

We assumed in (10) an initial solution X^o, Y^o represented in terms of basic feasible solutions X_i, Y_j and—we may now add—homogeneous solutions of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Such an initial solution can be obtained by a Phase I procedure.

Let X_1 and Y_1 be any arbitrary solution to $\mathscr{L}_1$ and $\mathscr{L}_2$, and S_1, T_1 their transforms under (7). Let the starting basis of the master program of Phase I be

$$\begin{aligned} P_0x_0 + \lambda_1S_1 + \mu_1T_1 \pm U_1\varepsilon_1 \pm U_2\,\varepsilon_2 \pm \ldots \pm U_m\varepsilon_m &= \bar{b} \\ \lambda_1 &= 1 \\ \mu_1 &= 1 \\ \varepsilon_1 + \quad \varepsilon_2 + \ldots + \quad \varepsilon_m - w &= 0 \end{aligned} \tag{28}$$

where U_i is an $m + 1$ component unit vector with 1 in row i, where $m + 1$ is the number of rows in $[\bar{A}_1, \bar{A}_2]$. The variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$; x_0, λ_1, μ_1, and w form a basic set. The signs of the U_i are chosen so that the artificial variables ε_i are nonnegative in the basic solution. During Phase I, the objective is to minimize w. Accordingly, the simplex multipliers are defined so that all columns price out to zero except the $(-w)$ column, which prices out to unity. From the Phase I-Phase II process, we deduce:

THEOREM 3: *An optimal basic feasible solution to the original system can be represented as the sum of at most $m + 2$ basic feasible solutions of the two*

parts where m is the number of equations in the interconnecting part (2) *exclusive of the objective form.*

23-2. DECOMPOSITION PRINCIPLE, ANIMATED

PROLOGUE

The coordinator, "Staff," of the Central Agency must procure tankers to assist his distributor, "Sub," in the shipping of their product from two plants to four terminals. Staff hates details and has asked Sub to furnish him with only two numbers, the cost, c, of the proposed shipping plan, and the number of tankers, t, required to support it.

Distributor, Sub, has two arrays, a unit cost matrix $[c_{ij}]$, and a unit tanker requirement matrix $[t_{ij}]$ (tankers required per unit shipped):

$$[c_{ij}] = \begin{bmatrix} 3 & 6 & 6 & 5 \\ 8 & 1 & 3 & 6 \end{bmatrix}; \; [t_{ij}] = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

(The zeros in the $[t_{ij}]$ array indicate shipments by pipeline instead of by boat.) Since the general objective is to minimize costs, Sub sets out to solve the transportation problem, below, where a_i are the known availabilities at the plants and b_j are the known requirements at the terminals.

				Available
x_{11} / 3	x_{12} / 6	x_{13} / 6	x_{14} / 5	$9 = a_1$
x_{21} / 8	x_{22} / 1	x_{23} / 3	x_{24} / 6	$8 = a_2$

Required: $2 = b_1 \quad 7 = b_2 \quad 3 = b_3 \quad 5 = b_4$

Having taken a ten-day operations research course where he learned how to solve small transportation problems, Sub soon discovers the *minimum cost* solution to this problem to be

$$[x_{ij}]_1 = \begin{bmatrix} 2 & 0 & 2 & 5 \\ 0 & 7 & 1 & 0 \end{bmatrix}$$

In this case $c = \Sigma\Sigma c_{ij}x_{ij} = 53$, $t = \Sigma\Sigma t_{ij}x_{ij} = 18$. Hence, the two numbers that Sub furnishes Staff, in his proposed plan, are

$$P_1 = \begin{bmatrix} c_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} 53 \\ 18 \end{bmatrix}$$

where P_1 represents the vector made up of the components of his first plan.

Staff discovers, however, that a sudden shortage of tankers has developed, and the most he can muster for Sub is 9 tankers. Noting that two out of Sub's eight possible shipping activities have to use tankers, Staff requests Sub to find a solution that "goes easy on the use of tankers." Sub, being literal, forgets costs and immediately comes up with the following solution that minimizes the use of tankers

$$[x_{ij}]_2 = \begin{bmatrix} 0 & 7 & 0 & 2 \\ 2 & 0 & 3 & 3 \end{bmatrix}$$

In this case $\Sigma\Sigma c_{ij}x_{ij} = 95 = c_2$, $\Sigma\Sigma t_{ij}x_{ij} = 0 = t_2$. Hence, Sub's new proposed plan is

$$P_2 = \begin{bmatrix} c_2 \\ t_2 \end{bmatrix} = \begin{bmatrix} 95 \\ 0 \end{bmatrix}$$

ACT I.

STAFF: Well, whatever else one might say about Sub's new plan, it certainly has gone easy on the use of tankers; in fact, none are used. But look what has happened to costs—they have nearly doubled! I just can't tell Sub that he can have only 9 tankers and let him find his own least-cost solution; I tried that the last time there was a tanker shortage, and costs soared. Somehow I wish I had sent Sub to that six-week operations research course, instead of the ten-day one. It would probably have been a lot cheaper in the long run. Note: *At this point, Staff has decided to call in his economist friend, F. M. Dalks,*[2] *as a consultant.*

DALKS: According to good economic theory, what you should do is to tell Sub that there may be an extra premium charge for the use of tankers. This will teach Sub to keep the costs down and at the same time not use tankers excessively, because they are now part of the charges in the total bill.

STAFF (*Enthusiastic and not above a bit of subterfuge if it gets results*): Let's do just that. (F.M. *is crestfallen; theory is theory, but putting it into practice is another matter.*)

DALKS: Let's go a little more slowly. It is not always easy to calculate what the prices should be on scarce commodities. It depends on many factors. But let me assure you it has been rigorously established, beyond any doubt, that such prices always exist. (*Seeing distrust creep into Staff's eyes,* F.M. *begins to toy with the idea of coming down out of the "ivory tower."*) In this case, however, we have only one scarce commodity, tankers, and we could try various premium charges for tankers and see what happens to Sub's use of them. This way we could eventually get Sub to come up with a plan that offers both feasibility and least-cost.

[2] F. M. Dalks, like the letters of his name, is a composite of leading mathematical economists of his time, who have applied linear programming to economic theory.

STAFF: Sounds like a good approach.

DALKS: According to theory, if tankers are in surplus, the price on tankers should be zero. Since Sub's latest plan, P_2, doesn't use any tankers at all, this is certainly the case. As a start, we can tell Sub the price, $\pi = 0$, on tankers. How's that?

STAFF (*Sarcastically*): Terrific! The only trouble with it is that we have already tried placing no value on tankers, and Sub came up with some plan or other which we couldn't use. I threw it away, but maybe Marge saved a copy. . . . Ah! See how valuable a secretary can be; here is a copy. See Sub's old plan: $P_1 = \langle 53, 18 \rangle$. It isn't feasible, so what good is it?

DALKS: I have another idea. I was reading an O.R. journal the other day. Of course, I don't usually read that kind of journal, but when I do, I find it is not so peripheral to econometrics as some people think. Anyway, a friend wanted my opinion about an article on a "decomposition principle." He thought it might throw some light on how to handle price problems when there are breakpoints due to discontinuity in the derivatives of the underlying production functions. But this is beside the point. The article suggests that we should take a weighted average of the two plans (*he moves to the blackboard*) like this:

$$P_1\lambda_1 + P_2\lambda_2 = \begin{bmatrix} 53 \\ 18 \end{bmatrix} \lambda_1 + \begin{bmatrix} 95 \\ 0 \end{bmatrix} \lambda_2$$

where $\lambda_1 + \lambda_2 = 1$, and, of course, $0 \leq \lambda_1 \leq 1$. For example, we could try $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$. Say, this gives

$$\begin{aligned} \tfrac{1}{2}(53 + 95) &= 74 \\ \tfrac{1}{2}(18 + \ \ 0) &= \ \ 9 \end{aligned}$$

which is a lot cheaper than 95 and happens to use up the available tankers. How's that?

STAFF: That's splendid, simply splendid! But how do we know that we can average Sub's proposals in this way? How will Sub know what to do? I don't want to get into the details of Sub's shipping schedule, you know.

DALKS: Oh! Don't worry about that. Just tell Sub to average the detailed schedules he used for generating plans P_1 and P_2, and he has the answer.

STAFF: That is a neat trick—averaging the old and new plans—and to think I was smart enough not to throw away that old plan. The boss is going to be very pleased when I tell him how we figured this one out. Before doing so, let's study this average plan of yours a little to make sure it's okay. Let's see, 9 tankers and an over-all transportation cost of 74. Isn't 74 rather high, considering that the

least-cost solution might have been 53, if it were not for the tanker shortage?

DALKS: Well, according to the article, this does not necessarily give the best solution. What we are supposed to do is set up a little linear program that tells us how to "blend" old and new plans in the best way. Here it is.

$$\begin{aligned} 53\lambda_1 + 95\lambda_2 + (-z) &= 0 \qquad (\lambda_i \geq 0) \\ \lambda_0 + 18\lambda_1 + 0\lambda_2 \qquad &= 9 \\ \lambda_1 + \lambda_2 \qquad &= 1 \\ \bullet \quad \bullet \quad \bullet \end{aligned}$$

where λ_0 is the slack (if any) in the use of tankers. It's fairly obvious, in this case, that $\lambda_0 = 0$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, with a cost $z^1 = 74$ so that we don't learn anything we don't already know. The important thing is that it gives us an idea of what price we should tell Sub to set on tankers. We do this by solving the dual linear program.

STAFF: I'm afraid you are in over my head. What's the solution?

DALKS: Well, the dual solution is very interesting. It works out this way. You see, the basic variables associated with the optimal primal solution are indicated by the three dots ● ● ● under the λ_1, λ_2, and $-z$ columns. It means we should choose multipliers $(1, \pi^1, -s^1)$ for the three equations in such a way that when we multiply and sum, the λ_1 and λ_2 terms vanish. It is easy to see that the following conditions must hold:

$$\begin{aligned} 53 + 18\pi^1 - s^1 &= 0 \quad \text{(coefficient of } \lambda_1) \\ 95 + 0\pi^1 - s^1 &= 0 \quad \text{(coefficient of } \lambda_2) \end{aligned}$$

or $s^1 = +95$ and $\pi^1 = +\frac{42}{18} = 2\frac{1}{3}$.

STAFF: I don't understand why $\pi^1 = 2\frac{1}{3}$ is positive. As I understand linear programming price-conventions, this means Sub is paid to use tankers, instead of having to pay a premium for their use (§ 12-3).

DALKS: Yes, but not in this case. For once, for this kind of problem, the linear programmers have set up their sign conventions sensibly, just as we economists do. So, in this case, π means what it says. In other words, we tell Sub that the premium charge on tankers might be $2\frac{1}{3}$ units per tanker. Let's try this and see what happens.

STAFF: Quite an interesting game we are playing with Sub. What is this $s^1 = +95$?

DALKS: Oh, that is to see whether or not we should try to blend Sub's proposal number three (when it comes in) with P_1 and P_2. But we can talk more about this later (§ 23-1, Theorem 1).

STAFF: But I don't understand. Does that mean that Sub is going to send

us another "infeasible" proposal, and we will have to do some more combining of proposals?

DALKS: Yes, I am afraid it does, unless we are awfully lucky!

Note: *At this point Sub is instructed to set up a new unit cost matrix,*

$$[c_{ij}] + 2\tfrac{1}{3}[t_{ij}] = \begin{bmatrix} 3 & 6 & 6 + 4\tfrac{2}{3} & 5 \\ 8 & 1 + 4\tfrac{2}{3} & 3 & 6 \end{bmatrix}$$

In due course, Sub arrives at the following optimal solution to his transportation problem:

$$[x_{ij}]_3 = \begin{bmatrix} 2 & 2 & 0 & 5 \\ 0 & 5 & 3 & 0 \end{bmatrix}$$

In this case, $\Sigma\Sigma c_{ij}x_{ij} = 57$, *and* $\Sigma\Sigma t_{ij}x_{ij} = 10$. *Therefore, the third plan he submits, is*

$$P_3 = \begin{bmatrix} c_3 \\ t_3 \end{bmatrix} = \begin{bmatrix} 57 \\ 10 \end{bmatrix}$$

ACT II.

STAFF: I begin to understand it now. What we do is set up a new little linear program. Is there some special name by which we can refer to it?

DALKS: It is called a "restricted master." But first, we want to test to see if P_3 is worth considering. Remember $s^1 = 95$? What we do is "price out" the new proposal and compare it with 95, the "break-even value." Since the new plan has a transportation cost of 57 and there will be a charge of $2\frac{1}{3}$ units for each of the 10 tankers used, we have $57 + 23\frac{1}{3} = 80\frac{1}{3}$. This is *less* than $s^1 = 95$, so we can now go ahead with the restricted master. Here it is:

$$\begin{aligned} 53\lambda_1 + 95\lambda_2 + 57\lambda_3 + (-z) &= 0 \\ \lambda_0 + 18 \cdot \lambda_1 + 0\lambda_2 + 10\lambda_3 \qquad &= 9 \\ \lambda_1 + \lambda_2 + \lambda_3 \qquad &= 1 \end{aligned}$$

STAFF: Does it take very long to solve? I'm curious to see what the answer is.

DALKS: Well, in this case, no! According to linear programming theory, only two of the three plans that Sub submitted are going to be used. Since no average of plans No. 1 and 3 will have less than 10 tankers, this leaves only one new possibility, a combination of P_2 and P_3. The new solution is easy. It is obviously $\lambda_2 = \frac{1}{10}$, $\lambda_3 = \frac{9}{10}$. This gives a cost of $z^2 = 95(\frac{1}{10}) + 57(\frac{9}{10}) = 60.8$.

STAFF: That's a big improvement! We have cut the cost from $z^o = 95$ to $z^1 = 74$ and now to $z^2 = 60.8$. How much more do you think we can reduce it if we keep this up?

DALKS: Well, not too much. It can't go any lower than 53, because we know that the value of z for the cheapest cost solution is 53 when there is no restriction on tankers. But perhaps we can do a little better. According to decomposition theory, we can always get an estimate of a lower bound by going back to the previous basic solution. Here is the formula (§ 23-1, Theorem 2):

$$\text{Min } z \geq z^1 - s^1 + c_3 + \pi^1 t_3 = 74 - 95 + 57 + 23\tfrac{1}{3} = 59\tfrac{1}{3}$$

STAFF: Fantastic! If you are right, very little saving below $z^2 = 60.8$ is possible. I would like to see what the true minimum is and see how good your guess really is.

DALKS: Okay, but it is not really a guess. Let us determine a new price on tankers. Since λ_2 and λ_3 are new basic variables, we have

$$\begin{aligned} 95 + \ \ 0\pi^2 - s^2 &= 0 \\ 57 + 10\pi^2 - s^2 &= 0 \end{aligned}$$

or $s^2 = 95$ and $\pi^2 = \frac{38}{10} = 3.8$. It shows we should raise the ante on tankers from $\pi^1 = 2\frac{1}{3}$ to $\pi^2 = 3.8$.

STAFF: I hope Sub won't become too unhappy with our changing our minds so often. He might start complaining to the big boss before this noble experiment is finished.

Note: *In due course, Sub sets up his new array*

$$[c_{ij}] + 3.8[t_{ij}] = \begin{bmatrix} 3 & 6 & 6 + 7.6 & 5 \\ 8 & 1 + 7.6 & 3 & 6 \end{bmatrix}$$

and derives a new optimal solution

$$[x_{ij}]_4 = \begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

Here $\Sigma\Sigma c_{ij}x_{ij} = 87$ *and* $\Sigma\Sigma t_{ij}x_{ij} = 0$. *So this, Sub's fourth plan, becomes*

$$P_4 = \begin{bmatrix} c_4 \\ t_4 \end{bmatrix} = \begin{bmatrix} 87 \\ 0 \end{bmatrix}$$

ACT III.

STAFF: This gets better and better. Obviously we can substitute $\frac{1}{10}$ of P_4 for the $\frac{1}{10}P_2$ that we used earlier and get a better solution. So, let's go ahead with the restricted master. Here it is.

$$\begin{aligned} 53\lambda_1 + 95\lambda_2 + 57\lambda_3 + 87\lambda_4 + (-z) &= 0 \\ \lambda_0 + 18\lambda_1 + 0 \cdot \lambda_2 + 10\lambda_3 + 0 \cdot \lambda_4 \qquad\qquad &= 9 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \qquad\qquad &= 1 \end{aligned}$$

Let us see what the new value of z is. Not much improvement, $z^3 = 57(\frac{9}{10}) + 87(\frac{1}{10}) = 60$. Maybe your lower bound of $59\frac{1}{3}$ is not so bad after all. Let's try your formula again for the lower bound.

For this purpose, we go back to the previous basic solution. Is that not so? That's funny! This time don't we get

$$\text{Min } z \geq z^2 - s^2 + c_4 + \pi^2 t_4 = 60.8 - 95 + 87.0 = 52.8?$$

DALKS: It appears that our previous estimate of $59\frac{1}{3}$ was just a very lucky one. I guess we should make the lower bound estimate each cycle, because some are better than others.

STAFF: I suppose now we will have to get a new price

$$57 + 10\pi^3 - s^3 = 0$$
$$87 + \ \ 0\pi^3 - s^3 = 0$$

or $s^3 = 87$ and $\pi^3 = 3$. This time we tell Sub the price on tankers has dropped to 3.

Note: *Sub's new problem becomes*

$$[c_{ij} + 3t_{ij}] = \begin{bmatrix} 3 & 6 & 6+6 & 5 \\ 8 & 1+6 & 3 & 6 \end{bmatrix}$$

which he quickly solves because his last solution, $[x_{ij}]_4$, *is still optimal. Also he finds that his previous one,* $[x_{ij}]_3$, *is also optimal.*

ACT IV.

STAFF: What does this mean? Sub has come up with the same solution as last time.

DALKS: It means that there is no improvement. Let us try our lower bound estimate and see what it says. If we use $c_4 = 87$, $t_4 = 0$ in the formula, then

$$\text{Min } z \geq z^3 - s^3 + c_4 + \pi^3 t_4 = 60 - 87 + 87 + 0 = 60$$

If we use $c_3 = 57$, $t_3 = 10$, instead of c_4, t_4, which is Sub's other optimum solution, we also get

$$z \geq 60 - 87 + 57 + 3(10) = 60$$

Since our lower bound 60 is the same as our value of z, this proves we are done.

STAFF: I must tell the boss about our new decentralized decision process and send out orders to Sub to form the weighted average of his plans, namely, $.9P_3 + .1P_4$.

Note: *When Sub carries out his orders he finds his optimal plan to be*

$$.9[x_{ij}]_3 + .1[x_{ij}]_4 = \begin{bmatrix} 2 & 2\frac{1}{2} & 0 & 4\frac{1}{2} \\ 0 & 4\frac{1}{2} & 3 & \frac{1}{2} \end{bmatrix}$$

DALKS: Technically, this is not exactly what we economists mean when we say "decentralized planning." A better term would be "central planning without full information at the center." A very interesting experiment! I may write a paper on how for the first time prices were used to control, in a precisely defined way, a real life situation.

The End

23-3. CENTRAL PLANNING WITHOUT COMPLETE INFORMATION AT THE CENTER[3]

The theory developed in § 23-1 makes it possible to plan the over-all operation of an organization without the central staff having full knowledge of the technology of each part. We shall deal with the case in which the technology matrix is constant over time and there is no capital expansion except for current needs.

Consider, then, an economy or industrial complex with T plants and m items which are traded between plants. Plant p has a technological matrix, A_p, of m_p rows expressing constraints on intermediate products, plant capacity, and local demands. In addition to A_p, each plant has a trade matrix, $\bar{A}_p$, which has m rows that correspond to m commodities traded between plants or supplied to the consumption sector of the economy. Plant p also has a vector of constraint constants, b_p. The constraints for plant p are then

$$A_p X_p = b_p, \qquad X_p \geq 0 \qquad (p = 1, 2, \ldots, T) \tag{1}$$

where X_p is the vector of its activity levels.

The Central Trade Agency has constraints specifying that the amounts of item j procured from various plants minus the amounts supplied to them be greater than or equal to the amount it turns over to final *minimum basic demand* (consumption, defense, exports, and the like). Using I to denote the identity matrix, these constraints may be written

$$\bar{A}_1 X_1 + \bar{A}_2 X_2 + \ldots + \bar{A}_T X_T + IY = \bar{b} \qquad (Y \geq 0) \tag{2}$$

Y being the vector of final incremental demands in excess of the minimum basic demand $\bar{b}$. (With our sign conventions the components of $\bar{b}$ are all negative.)

The planners attach a certain set of values, specified by the row vector $-c$, to final incremental demands so that $-cY$ represents the value of the plan to them. The total problem facing the economy may then be written

$$\begin{array}{llllll} A_1 X_1 & & & & & = b_1 \\ & A_2 X_2 & & & & = b_2 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ & & & & A_T X_T & = b_T \\ \bar{A}_1 X_1 + & \bar{A}_2 X_2 + & \ldots + & \bar{A}_T X_T & + IY & = \bar{b} \\ & & & & cY & = z \text{ (Min)} \end{array} \tag{3}$$

[3] This section was also contributed by C. Almon to bring out some of the potentialities of the decomposition principle for decentralized planning.

This is an angular system which the theory of § 23-1 will decompose into T subprograms and one master program. If the A_p matrices were all available they could be sent to the Central Planning Agency where a large computer, programmed to use our decomposition principle, would soon produce the optimal solution.

In reality, however, the A_p matrices are hard to compose. It may require substantial engineering time to specify each new activity even though no real technological change, or basic innovation is involved. There may be many things which no one doubts could be done, but which it would require much valuable time to specify in detail. For instance, no engineer would, just for the sake of adding an activity to the A_p matrix, spend a week to carefully specify the characteristics of a transformer using copper wire instead of aluminum if a few rough calculations showed that the resulting product would cost about twice what the present one does and offer no compensating advantages.

For such reasons, the complete A_p are probably not explicitly known; rather we may regard them as potentially known to the plants, with some effort being required to make them explicit. Since potential knowledge can hardly be sent to the Central Planning Agency, this Agency lacks complete information. It does not, however, completely lack information, for it has available the records of Central Trade which show how much each plant used and supplied of each common item for several preceding years, say *two* for simplicity. Thus it knows the values of $S_p = \bar{A}_p X_p$ determined by two feasible solutions to the entire program and hence to each subprogram:

$$(4) \qquad \begin{aligned} A_p X_p^0 &= b_p; \qquad S_p^0 = \bar{A}_p X_p^0 \\ A_p X_p^1 &= b_p; \qquad S_p^1 = \bar{A}_p X_p^1 \end{aligned} \qquad (p = 1, \ldots, T)$$

(If there was no capital expansion, the b_p are the same in the two years. If there was capital expansion, the first year's plan could be converted into a feasible plan for the second year and used in (4).)

The Central Planning Agency uses this data to make up a *master program* with $m + T$ constraints as follows:

$$(5) \quad \begin{aligned} S_1^0\lambda_1^0 + S_1^1\lambda_1^1 + S_2^0\lambda_2^0 + S_2^1\lambda_2^1 + \ldots + S_T^0\lambda_T^0 + S_T^1\lambda_T^1 + IY &= b \\ \lambda_1^0 + \lambda_1^1 &= 1 \\ \lambda_2^0 + \lambda_2^1 &= 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad

speaking, π is, of course, a vector of prices. For each plant we can then make an objective form

$$\text{Minimize } z_p = \pi S_p = (\pi \bar{A}_p) X_p$$

In words, this objective form states: Minimize the excess of the cost of purchases over the value of sales, i.e., minimize losses or, what comes to the same thing, *maximize profits*.

The planners then announce these prices, π, and instruct all plant managers to propose plans for the operation of their plant in the next year. Specifically, the plan is to state how much of each item they will buy and sell to Central Trade. In making these proposals, they are to assume that they can buy, at the announced prices, all they need of any of the m items and likewise sell all they produce. They are further instructed to propose the plan which will maximize their profits.

On receipt of these instructions, the managers call in their engineers, give them the prices, and tell them to leave no stone unturned in looking for ways to cut costs and increase profits *at these prices*. The engineers now have the guidance they need to set about making explicit a portion of their potential knowledge. In a month or so they produce what they believe to be their optimal plans, and the managers give the plans to Central Planning.

It would be extraordinary good luck if these plans will satisfy (2) and constitute a feasible program for the whole economy; usually Central Planning will have to coordinate them. For each plant p for which the proposed profits exceed s_p, the Agency adds an activity, S_p^2, to the master program (5). Then it re-solves this program, obtaining a new $(\pi; -s)$, say $(\pi'; -s')$ and also a λ solution. The λ are used to combine the new proposals with the previous solutions to get what we may call the optimal feasible plan, given the information possessed by the Central Planning Agency.

In principle, the planners could then announce π', get new proposals, and repeat the process until the optimal is found. In practice, planning takes time, production must go on, and the planners reason that consistent plans that get better and better are to be preferred over no plans. Hence, they announce *in quantitative terms* their feasible plan. They tell each plant manager how much of each traded commodity he must produce and how much he is allowed to purchase; this information is summarized for plant p in the column vector $S_p = \sum_i \lambda_p^i S_p^i$ where the values of λ_p^i are those of the optimum solution of the restricted master. They also announce the prices $(\pi'; -s')$ and direct that trade be conducted at these prices. They may also instruct the managers that, subject to their meeting the quantitative goals S_p, they should also maximize profits. Such a rule is intended as a guide to avoid possible waste in the event that S is not precisely achieved for one reason or another. It is important to note that they cannot tell the manager simply to maximize the profits (omitting production goals, S_p) for

if they did, Central Trade would almost certainly have difficulty with its constraints (2).

Toward the end of the period for which production was planned, the order again goes out to propose plans for the next period, this time using π' as a guide. The managers, knowing that this order was coming, have been at work all year looking for ways to make profits, and they soon have their proposals. From these, Central Planning makes up new activities to add to the master program and matters proceed as before, the values of the plans always increasing. The essential point to bear in mind is that the master program remembers all previous proposals, except those which it is no longer using at all.

We leave to the reader's judgment the problem of the economic significance of the finiteness proposition of § 23-1. He should also consider the related, though not identical, question of whether it will ever be possible for the planners to abandon the quota and allotment system and simply direct the managers to maximize their profits.

Concluding Remark.

Our discussion has been intended to elucidate the workings of the decomposition principle as a planning tool, rather than to explain the methods actually used by a present-day industrial complex or by a socialist economy. We have shown that there exists a special method of allocation by a central authority and a specially devised system of prices that can induce the separate plants to submit *summarized proposals*, which can be combined into better and better over-all plans. Whether or not the system of allocation, prices, and proposals used by a particular economy or complex approximate those envisaged by our method, we cannot say. Our iterative process can be diagrammed as follows:

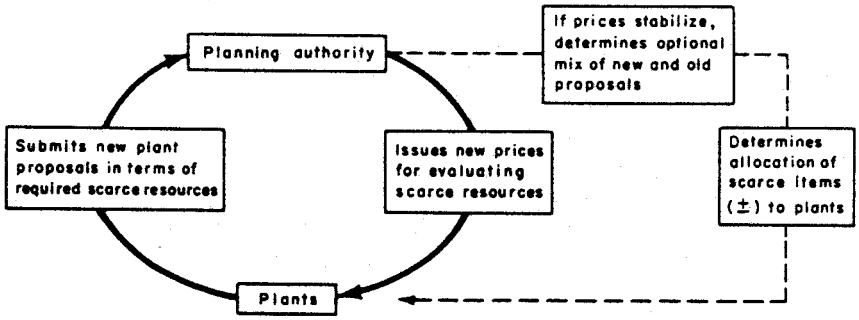


Figure 23-3-I. Decentralized planning using the decomposition principle.

Many people familiar with planning in large organizations are conscious of a flow of information similar to the above. *The difference is that the*

decomposition principle replaces the not-too-well-understood procedure by one that is rigorous and well defined.

23-4. DECOMPOSING MULTI-STAGE PROGRAMS

The methods developed in § 23-1 extend immediately to problems where there are more than two subprograms, the so-called "angular" systems of the form:

$$
(1) \qquad \begin{array}{lll}
A_1X_1 & & = b_1 \\
\quad \ddots & & \vdots \\
\qquad A_tX_t & & = b_t \\
\qquad\quad \ddots & & \vdots \\
\qquad\qquad A_TX_T & & = b_T \\
\bar{A}_1X_1 + \ldots + \bar{A}_tX_t + \ldots + \bar{A}_TX_T & & = \bar{b} \\
C_1X_1 + \ldots + C_tX_t + \ldots + C_TX_T & & = z \text{ (Min)}
\end{array} \qquad (X_t \geq 0)
$$

where the A_i and $\bar{A}_i$ are matrices, and X_i and C_i are vectors.

Let us now consider another important class of structures, those of the form: Find X_t and Max x_0 satisfying

$$
(2) \qquad \begin{array}{ll}
A_1X_1 & = e_1 \\
\bar{A}_1X_1 + A_2X_2 & = e_2 \\
\qquad \bar{A}_2X_2 + A_3X_3 & = e_3 \\
\qquad\qquad \bar{A}_3X_3 + A_4X_4 + P_0x_0 & = e_4
\end{array} \qquad (X_t \geq 0)
$$

where X_t are vectors, A_t matrices, e_t and P_0 vectors. These are called *multi-stage* systems (so-called "staircase" systems) and often arise in the study of processes through time in which the activities of one period are directly connected with those of the preceding and following periods but with no others. In such cases, the several A_t may all be the same, as may be the $\bar{A}_t$. Although our results would permit computational advantage to be taken of such constancy, we shall not assume it.

Following the general lines of what we did in § 23-1, we begin by making a subprogram out of *every other* stage by rewriting (2) in the following form: Solve the program: Find $X_t \geq 0$ and Max x_0 satisfying

$$
(3) \qquad \begin{array}{ll}
\bar{A}_1X_1 + A_2X_2 & = e_2 \\
\qquad\qquad \bar{A}_3X_3 + A_4X_4 + P_0x_0 & = e_4
\end{array}
$$

subject to the constraints

$$(4) \qquad \mathscr{L}_1\colon A_1X_1 = e_1 \qquad \mathscr{L}_2\colon \bar{A}_2X_2 + A_3X_3 = e_3$$

Thus, equations (4) have been selected to play the role of the independent parts, while equations (3) that of the binding constraints. The reader, of course, recognizes that the roles of (3) and (4) could be interchanged and that there are several other ways we could choose the subprograms.

We denote the extreme points of $\mathscr{L}_1$ by $X_1 = X_{1i}$ for $i = 1, 2, \ldots, K$ and those of $\mathscr{L}_2$ by $[X_{2j}, X_{3j}]$ for $j = 1, 2, \ldots, L$ and denote their transforms by

$$(4.1) \qquad S_i = \bar{A}_1X_{1i},\ [T_j = A_2X_{2j},\ \bar{T}_j = \bar{A}_3X_{3j}]$$

In this case the master program takes the form: Find Max x_0, $\lambda_i \geq 0$, $\mu_j \geq 0$ satisfying

$$(5) \qquad \boxed{\begin{aligned} &\sum_{i=1}^{K} S_i\lambda_i + \sum_{j=1}^{L} T_j\mu_j && = e_2 \\ &\sum_{i=1}^{K} \lambda_i && = 1 \end{aligned}}$$

$$\boxed{\begin{aligned} &\sum_{j=1}^{L} \bar{T}_j\mu_j + A_4X_4 + P_0x_0 && = e_4 \\ &\sum_{j=1}^{L} \mu_j && = 1 \end{aligned}}$$

Starting with some basic set of variables λ_i, μ_j and components X_{4p} of X_4 of the master program, we can solve for simplex multipliers, derive objective forms for $\mathscr{L}_1$ and $\mathscr{L}_2$, solve, use transforms of their solutions to add columns to the restricted master, re-solve, and repeat the process until the optimality test is passed.

When we set about to solve (5), however, the first thing we observe is that the master program itself is of the staircase form (2), but with half the number of stages. Hence, the logical thing to do is to *decompose this master program*. Accordingly, let us refer to (5) as the *first level master program* and proceed to its decomposition. We rewrite (5) in the form:

Find Max x_0 such that

$$(6) \qquad \sum_{j=1}^{L} \bar{T}_j\mu_j + A_4X_4 + P_0x_0 = e_4 \qquad (\mu_j \geq 0;\ X_4 \geq 0)$$

subject to

$$(7) \qquad \mathscr{L}_3: \quad \sum_{i=1}^{K} S_i\lambda_i + \sum_{j=1}^{L} T_j\mu_j = e_2 \qquad (\lambda_i \geq 0)$$

$$\sum_{i=1}^{K} \lambda_i = 1$$

$$\sum_{j=1}^{L} \mu_j = 1$$

We denote the extreme points of $\mathscr{L}_3$ by $\lambda_i = \lambda_i^q$, $\mu_j = \mu_j^q$ for $q = 1, 2, \ldots, Q$, and denote for the latter its transform $\sum_{j=1}^{L} \bar{T}_j\mu^q = R_q$. Substituting in (6) we get the *second level master program*:

$$(8) \qquad \sum_{q=1}^{Q} R_q v_q + A_4X_4 + P_0x_0 = e_4 \qquad (v_q \geq 0\,;\ X_4 \geq 0)$$

$$\sum_{q=1}^{Q} v_q = 1, \quad \text{where } R_q = \sum_{j=1}^{L} \bar{T}_j\mu_j^q$$

We have now decomposed the four-stage program into four programs, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ and (8), whose total number of equations is three more than the corresponding number of equations of the four original stages.

Let us now outline the iterative procedure by which these programs are solved. First, we solve *a restricted second level master program* (8) and determine, in addition to v_q, the simplex multipliers $(\pi_4^o;\ -u_4^o)$ such that $\pi_4^oP_0 = 1$, the basis columns R_q price out to u_4^o, and those of A_4 price out nonnegative. (Compare with § 23-1-(12).) Next, we determine an objective form for $\mathscr{L}_3$, namely

$$(9) \qquad \text{Minimize } z_4 = \sum_{j=1}^{L} (\pi_4\bar{T}_j)\mu_j$$

We assume we have at hand some feasible basis for $\mathscr{L}_3$. The values of simplex multipliers $(\pi_3^o;\ u_3,\ v_3^o)$ for this basis can now be determined using the values of $(\pi_4\bar{T}_j)$ pertaining to basis columns. We now wish to test $\mathscr{L}_3$ for optimality by pricing out (using these multipliers) all columns $i = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, L$. However, since K and L can be large, we wish to discover the columns $i = i_0$ and $j = j_0$ that price out the least *without actually generating all the columns* in advance; we note by (4.1)

$$(10) \quad \pi_4^o\bar{T}_{j_0} - \pi_3^oT_{j_0} = \operatorname*{Min}_j\,[\pi_4^o\bar{T}_j - \pi_3^oT_j]$$
$$= \text{Min}\,[(-\pi_3^oA_2)X_2 + (\pi_4^o\bar{A}_3)X_3] \leq v_3^o, \quad ((X_2, X_3) \in \mathscr{L}_2)$$

and

$$(11) \qquad -\pi_3 S_{i_0} = \operatorname*{Min}_i [-\pi_3^o S_i] = \operatorname{Min} [-(\pi_3^o \bar{A}_1) X_1] \leq u_3^o \qquad (X_1 \in \mathscr{L}_1)$$

Hence, the problem reduces to minimizing the two linear objective forms; the first, (10), in variables (X_2, X_3), where $(X_2, X_3) \in \mathscr{L}_2$, and the second, (11), linear in X_1, where $X_1 \in \mathscr{L}_1$. If the solutions to these two linear subprograms yield equality, then subprogram $\mathscr{L}_3$ was optimum and the problem terminates. If not, then the transforms of the extreme point solution $X_1 = X_{1*}$ or $(X_2, X_3) = (X_{2*}, X_{3*})$ associated with the most negative one is adjoined to the restricted $\mathscr{L}_3$ problem and used, in turn, to generate a new extreme solution for the second-order master problem.

As in § 23-1, we note that it is not necessary to carry over from one cycle to the next any columns of (7) or (8) not in the basis. Non-basic columns in these programs may be dropped and forgotten; they will be generated again if needed.

Although we have been working with a four-stage program, our procedure has been perfectly general. Letting $[p]$ mean the largest integer not exceeding p, we can see the following:

THEOREM 1: *A K-stage problem can be decomposed into K-subprograms; one corresponding to each stage, the total number of equations in the set of subprograms is $K - 1$ more than the total number of equations of the original problem; the subprograms form a hierarchy with $[K/2]$ or $[K/2] + 1$ in the lowest rung, $[K/4]$ or $[K/4] + 1$ in the next, etc.*

It will be noted that the procedure yields the *optimal multipliers*, but it was necessary to do some side calculations to get the optimal values of the primal variables. It may be useful to consider the *dual* of a multi-stage program, for it has the same structure. The calculations carried out on the dual permit direct evaluation of the primal variables and indirect evaluation of the multipliers.

23-5. PROBLEMS

1. Consider the method used of decomposing the multi-stage system § 23-4-(2); develop the analogous method for decentralized planning in such a structure. Generalize.
2. With reference to Problem 1, develop alternative methods corresponding to other ways to decompose such a system.
3. With reference to the discussion following § 23-4, Theorem 1, dualize § 23-4-(2) and develop a method based on the decomposition principle that will evaluate directly the primal variables.
4. Establish in general the statement in § 23-4, Theorem 1, that there are $K - 1$ more equations in the set of subprograms than in the original system.

5. Analogous to § 23-1-(20), develop a lower bound for z for the simplex method for the case where one of the equations is of the form $\sum_1^n x_j = 1$.
6. Let α_j be an upper bound for x_j and let $x_0 + \sum_1^n \bar{c}_j x_j = x_0^o$; show, in general, that another upper bound for x_0 is $x_0^o - \sum \bar{c}_j \alpha_j$, where the summation is restricted to those j such that $\bar{c}_j < 0$.

REFERENCES

Dantzig, 1957-3
Dantzig and Wolfe, 1960-1
Ford and Fulkerson, 1958-1
Gomory and Hu, 1960-2
Jewell, 1958-1
Kawaratani, Ullman, and Dantzig, 1960-1
Kuhn and Tucker, 1950-2

CHAPTER 24

CONVEX PROGRAMMING

24-1. GENERAL THEORY

Kuhn and Tucker in their paper on "Nonlinear Programming" [1950-2] [Tucker, 1957-1] considered the problem of minimizing a convex function with variables $x_1, x_2, \ldots, x_n$ subject to the condition that the values assumed by a system of concave functions in these variables be nonnegative. They showed that if the concave functions were *differentiable*, the method of Lagrange multipliers could be appropriately extended to inequality restrictions on concave functions (see discussion in § 6-5). Instead of concave functions, their negative, convex functions (for definition, see § 7-1 and Fig. 7-1-VIII) will be used whose values must be nonpositive. Following Slater [1950-1] and Uzawa [1958-1], our object will be to show that their results hold even if the functions are not differentiable, provided: (a) the domain of variation is restricted to a closed bounded convex set R; (b) there exists at least one point where the convex functions are all negative; and, (c) the convex functions are continuous[1] in R. We shall also give a constructive procedure for solving such systems using the generalized programming approach. Philip Wolfe first discussed this idea at the RAND Symposium on Mathematical Programming, 1959, for convex objectives; at the same meeting, H. O. Hartley discussed the case of variable coefficients in a column of a special form. The first proof of convergence can be found in Dantzig [1960-3]; see also A. C. Williams [1960-1].

Problem A: Find a point $x = (x_1, x_2, \ldots, x_n)$ in a closed bounded convex set R and the minimum value of z satisfying

$$\begin{aligned} &\phi_i(x) \leq 0 \qquad (i = 1, 2, \ldots, m) \\ &\phi_0(x) = z \text{ (Min)} \end{aligned} \tag{1}$$

where the $\phi_i(x)$ are continuous convex functions.

THEOREM 1: *If there exists $x^o \in R$ such that $\phi_i(x^o) < 0$ for $i = 1, 2, \ldots, m$, then there exist multipliers $\hat{\pi}_1 \geq 0, \hat{\pi}_2 \geq 0, \ldots, \hat{\pi}_m \geq 0$ and an $\hat{x} \in R$ which solves* (1) *with Min z and satisfies*

$$F(\hat{x}) = \text{Min } F(x) = \underset{x \in R}{\text{Min}} \left[\phi_0(x) + \sum_1^m \hat{\pi}_i \phi_i(x) \right] \tag{2a}$$

[1] Since a general convex function is always continuous in the interior of the domain of definition, R, we are essentially assuming that this continuity extends to the boundary.

or, if the $\phi_i(x)$ are differentiable, satisfies the Kuhn-Tucker condition,

$$\text{(2b)} \qquad \sum_{j=1}^{n} (x_j - \hat{x}_j) \frac{\partial F}{\partial x_j}\bigg|_{x = \hat{x}} \geq 0 \qquad \textit{for all } x \in R,$$

where $F(x)$ is defined by the bracket term in (2a).

EXERCISE: Show that a convex differentiable function $F(x)$ defined over a closed bounded convex set R attains a minimum at $x = \hat{x}$ if and only if (2b) is satisfied.

EXERCISE: Show that the usual primal-dual complementary slackness for optimality is a special case of (2b).

We shall consider Problem A in a slightly more general form.

Problem B: Find a point $x = (x_1, x_2, \ldots, x_n)$ in a closed bounded convex set R and Min z satisfying

$$\text{(3)} \qquad \begin{aligned} L_i(x) &= 0 && (i = 1, 2, \ldots, r) \\ \phi_i(x) &\leq 0 && (i = r + 1, \ldots, m) \\ \phi_0(x) &= z \text{ (Min)} \end{aligned}$$

where $\phi_i(x) = L_i(x)$, for $i = 1, 2, \ldots, r$, are *linear*, and $\phi_i(x)$, for $i = 0$ and $r + 1, \ldots, m$, *are continuous convex functions.*

As noted in § 22-1, we may rewrite (3) in the form

$$\text{(4)} \qquad \begin{aligned} {}_0\lambda &= 1 && (\mu_i \geq 0) \\ y_1\lambda_0 &= 0 \\ &\vdots \\ y_r\lambda_0 &= 0 \\ y_{r+1}\lambda_0 + \mu_{r+1} &= 0 \\ &\vdots \\ y_m\lambda_0 + \mu_m &= 0 \\ y_0\lambda_0 &= z \text{ (Min)} \end{aligned}$$

where y_i are variable coefficients that may be freely chosen subject to the conditions that $y_i = L_i(x)$ for $1 \leq i \leq r$ and $y_i \geq \phi_i(x)$ for $r + 1 \leq i \leq m$ for some $x \in R$.

EXERCISE: Show that the set of possible $P = \langle 1, y_1, y_2, \ldots, y_m, y_0 \rangle$ is a *convex set* and that (4) is a generalized program; see § 22-1-(27), (28).

Nondegeneracy Assumption. There exists at least one nondegenerate basic feasible solution to a restricted master problem, (5), generated by some p choices of admissible P:

$$\text{(5)} \quad \lambda_1 P_1 + \ldots + \lambda_p P_p + \mu_{r+1} U_{r+1} + \ldots + \mu_m U_m + (-z) U_{m+1} = U_0 \qquad (\lambda_j \geq 0,\ \mu_i \geq 0)$$

where U_i denotes the unit vector with its unit component in equation i of (4), where i may range from $i = 0$ to $i = m + 1$. This restricted master, which we assume for convenience to be the initial restricted master, whose columns form a *basis*, is shown explicitly in detached coefficient form and in greater detail in (6). Because the basic solution is nondegenerate, the values of λ_i and μ_{r+l} are positive.

(6)

Initial Restricted Master Program for Problem B, Cycle k (for Initial Basis set $k = 0, p = r + 1$)										
$\lambda_1 \geq 0$	. . .	$\lambda_p \geq 0$	. . .	$\lambda_{p+k} \geq 0$	$\mu_{r+1} \geq 0$	. . .	$\mu_m \geq 0$	$(-z)$	Constants	Multipliers
1	. . .	1	. . .	1	0			0	1	(π_0)
y_1^1	. . .	y_1^p	. . .	y_1^{p+k}	0			0	0	(π_1)
.		.		.	.			.	.	.
.		.		.	.			.	.	.
.		.		.	.			.	.	.
y_r^1	. . .	y_r^p	. . .	y_r^{p+k}	0			0	0	.
y_{r+1}^1	. . .	y_{r+1}^p	. . .	y_{r+1}^{p+k}	1			0	0	.
.		.		.		.		.	.	.
.		.		.			.	.	.	.
.		.		.			.	.	.	.
y_m^1	. . .	y_m^p	. . .	y_m^{p+k}			1 .	0	0	(π_m)
y_0^1	. . .	y_0^p	. . .	y_0^{p+k}	0	. . .	0	1	0	(1)

EXERCISE: Show that the nondegeneracy assumption implies the existence of a point x^o such that $L_i(x^o) = 0$ for $i = 1, 2, \ldots, r$ and $\phi_i(x^o) < 0$ for $i = r + 1, \ldots, m$.

EXERCISE: Show that if there exists a point x^o satisfying the above conditions and a nondegenerate basis for the system of r linear equations $L_i(x) = 0$ above, then there exists a nondegenerate basis for the system as a whole.

Since we have shown that (4) is a generalized program, we are in a position to apply the methods of § 22-1 to affect a solution. This is an iterative procedure that was shown to converge in a finite number of steps when the variable coefficients associated with any column are drawn from a convex polyhedron. However, in both Problems A and B we are dealing with *general convex sets* and our purpose is to show that the process, if infinite, converges to a solution.

EXERCISE: Assuming that the iterative procedure converges, formulate a "Phase I" type problem that will yield, in a *finite* number of iterations, a nondegenerate basic feasible solution satisfying (5), if one exists. *Hint:* Make use of the two previous exercises.

Finding an Initial Nondegenerate Basic Feasible Solution to a Restricted Master Program.

Under the hypothesis of Theorem 1, where there are no linear equations $L_i(x)$ to be satisfied, there exists an $x = x^o$ such that $\phi_i(x^o) < 0$ for $i = 1, 2, \ldots, m$. The nondegeneracy condition is then satisfied by the basic solution formed by using m slack variables and the variable $\lambda_1 = 1$, where the coefficients of λ_1 are $y_i^1 = \phi_i(x^o)$ [see (7)]. Hence for Problem A, there *always exists at least one nondegenerate basic feasible solution.*

(7)

Initial Restricted Master (and Basis) for Problem A							
λ_1	μ_1	.	.	.	μ_m	$(-z)$	Constants
1							1
$\phi_1(x^o)$	1						0
.		.					.
.			.				.
.				.			.
$\phi_m(x^o)$					1		0
$\phi_0(x^o)$						1	0

For Problem B, we assume that we have at hand a nondegenerate basic feasible solution to some restricted master with which to initiate the algorithm. See the first and second exercises above.

Iterative Procedure.

We now review the iterative procedure given in Chapter 22 for a generalized program. The restricted master (5) for some cycle k is optimized, yielding a new basic feasible solution λ^k, and a new set of simplex multipliers π^k. These multipliers are used to generate a new column P_{p+k+1} for the restricted master for cycle $k + 1$ *where all columns of the master program for cycle k are retained and used in cycle $k + 1$.* (Note that retention of the columns was optional in Chapter 22, but here it is required for the proof of convergence.)

The components of λ^k, the kth basic feasible solution, will be denoted by

$$\lambda_1^k, \lambda_2^k, \ldots, \lambda_{p+k}^k, \mu_{r+1}^k, \ldots, \mu_m^k \geq 0$$

and

$$z^k = \sum_{j=1}^{p+k} \lambda_j^k y_0^j$$

In order to express conveniently the kth approximation, $\hat{x}^k$, to a minimizing solution of Problem B, we will assume that each column P_j of the master program is defined by choosing some value $x = x^j$ and setting

$$P_j = \langle 1, \phi_1(x^j), \phi_2(x^j), \ldots, \phi_m(x^j); \phi_0(x^j) \rangle \qquad (j = 1, 2, \ldots, p + k)$$

In this case the kth approximation to the minimizing solution of (3) is

$$\hat{x}^k = \sum_{j=1}^{p+k} \lambda_j^k x^j; \quad \hat{z}^k = \phi_0(\hat{x}^k) \tag{8}$$

The simplex multipliers for cycle k satisfy the following conditions:

$$\begin{aligned} &\pi^k U_{m+1} = 1 \\ &\left.\begin{aligned} \pi^k P_j &= 0 \\ \pi^k U_i &= 0 \end{aligned}\right\} \text{ if } \lambda_j \text{ or } \mu_i \text{ is a basic variable} \\ &\left.\begin{aligned} \pi^k P_j &\geq 0 \\ \pi^k U_i &\geq 0 \end{aligned}\right\} \text{ if } \lambda_j \text{ or } \mu_i \text{ is a nonbasic variable} \end{aligned} \tag{9}$$

It follows that the components $i = r + 1, \ldots, m$ of $\pi^k = [\pi_0^k, \pi_1^k, \ldots, \pi_m^k; 1]$ are *nonnegative* because $\pi_i^k = \pi^k U_i \geq 0$ for these i.

To test whether or not the kth approximate solution (8) is optimal, the function

$$\Delta(x|\pi^k) = \phi_0(x) + \sum_1^m \pi_i^k \phi_i(x) + \pi_0^k \tag{10}$$

is *minimized* over all $x \in R$.

THEOREM 2: *If Min* $\Delta(x|\pi^k) \geq 0$ *for* $x \in R$, *then* $\hat{x}^k$ *is optimal.*

PROOF: Let $x \in R$ satisfy

$$\begin{aligned} L_i(x) &= 0 && (i = 1, 2, \ldots, r) \\ \phi_i(x) &\leq 0 && (i = r + 1, \ldots, m) \end{aligned} \tag{11}$$

Multiplying the ith relation by π_i^k, where $\pi_i^k \geq 0$ for $i = r + 1, \ldots, m$, and adding gives

$$\sum \pi_i^k \phi_i(x) \leq 0 \tag{12}$$

Adding $\phi_0(x) + \pi_0^k$ to both sides of this inequality yields

$$\Delta(x|\pi^k) \leq \phi_0(x) + \pi_0^k \tag{13}$$

Setting $\lambda = \lambda^k$ in the master program (6), multiplying its rows by the corresponding components of π^k, and summing yield, by the complementary slackness conditions (9) of the optimal solutions to the primal and dual systems,

$$-z^k = \pi_0^k \tag{14}$$

So that

$$\Delta(x|\pi_0^k) + z^k \leq \phi_0(x) \tag{15}$$

On the other hand, for the approximate solution $\hat{x}^k = \Sigma \lambda_j^k x^j$, it follows from the convexity of $\phi_0(x)$ that

$$\phi_0(\hat{x}^k) \leq \sum_{j=1}^{p+k} \lambda_j^k \phi_0(x^j) = \sum_{j=1}^{p+k} \lambda_j^k y_0^j = z^k \tag{16}$$

By first selecting that x which minimizes $\phi_0(x)$ in (15) and then that x which minimizes $\Delta(x|\pi_0^k)$, it is easy to see that

THEOREM 3: *Lower and upper bounds for* Min $\phi_0(x)$ *are*

$$\text{(17)} \qquad \operatorname*{Min}_{x \in R} \Delta(x|\pi^k) + z^k \leq \operatorname{Min} \phi_0(x) \leq \phi_0(\hat{x}^k) \leq z^k$$

Continuing with the proof of Theorem 2, setting $x = \hat{x}$ in (13), noting (16),

$$\text{(18)} \qquad \operatorname{Min} \Delta(x|\pi^k) \leq \Delta(\hat{x}|\pi^k) \leq \phi_0(\hat{x}^k) + \pi_0^k = \phi_0(\hat{x}^k) - z^k \leq 0$$

If now we employ the hypothesis Min $\Delta(x|\pi^k) \geq 0$, the above implies

$$\text{(19)} \qquad \operatorname{Min} \Delta(x|\pi_0^k) = 0$$

and, by Theorem 3,

$$\text{(20)} \qquad -\pi_0^k = z^k = \phi_0(\hat{x}^k) = \operatorname{Min} \phi_0(x)$$

which establishes Theorem 2.

EXERCISES:

(a) Prove Theorem 3 in detail.

(b) Show that a nonnegative weighted sum of convex functions is a convex function.

(c) Show that $\Delta(x|\pi^k)$ is a continuous convex function on R.

(d) Show that there exists an $\hat{x} \in R$, such that $\Delta(\hat{x}|\pi^k) = \operatorname{Min} \Delta(x|\pi^k)$.

Generating a New Column for the Master Program.

If Min $\Delta(x|\pi^k) < 0$, define $x^{k+1} \in R$ and P_{p+k+1} by

$$\text{(21)} \qquad \Delta(x^{k+1}|\pi^k) = \operatorname*{Min}_{x \in R} \Delta(x|\pi^k) < 0$$

$$\text{(22)} \qquad P_{p+k+1} = \langle 1, \phi_1(x^{k+1}), \ldots, \phi_m(x^{k+1}) ; \phi_0(x^{k+1}) \rangle$$

It follows from the definitions (10), (21), and (22), that

$$\text{(23)} \qquad \pi^k P_{p+k+1} = \Delta(x^{k+1}|\pi^k) < 0$$

If we could now show that Min $\Delta(x|\pi^k)$ tends to zero as $k \to \infty$, it would be easy to show convergence of $\phi_0(x^k)$ from Theorem 3, since z^k forms a monotonically decreasing sequence bounded from below [see Exercise (d)]. However, note that convergence can also be established if Min Δ tends to zero on some subset of the k's; indeed, the latter is all we shall be able to prove. For this purpose we first show

THEOREM 4: *If a nondegenerate basic feasible solution exists for some master program, there is an infinite subset K of values of k, such that* $\operatorname*{Lim}_{k \to \infty} \pi^k$ *exists for $k \in K$.*

PROOF: Let B be the basis of some master program associated with the

nondegenerate basic feasible solution. For convenience, let this be the solution for (6), the initial master program $k = 0$, and let

$$B = [P_1, P_2, \ldots, P_{m+1}, U_{m+1}] \tag{24}$$

where some of the P_j may be the U_i associated with the slack variables. Thus $\lambda^o = \langle \lambda_1^o, \lambda_2^o, \ldots, \lambda_{m+1}^o, -z^o \rangle$ satisfies

$$B\lambda^o = U_0 \tag{25}$$

and, by nondegeneracy, the components λ_i^o are positive. Thus

$$\lambda^o = B^{-1}U_0 = [\lambda_1^o, \lambda_2^o, \ldots, \lambda_{m+1}^o, -z^o]^T \qquad \text{where } \lambda_i^o > 0 \tag{26}$$

On the other hand, any π^k solving the dual master program for cycle k satisfies $\pi^k P_j \geq 0$, $\pi^k U_{m+1} = 1$ by (9), and therefore the components of a vector $\gamma^k = (\gamma_1^k, \gamma_2^k, \ldots, \gamma_m^k, 1)$ defined by (27) are nonnegative

$$\pi^k B = \gamma^k \geq 0 \qquad (\gamma_i^k \geq 0) \tag{27}$$

Observe that $\pi^k = \gamma^k B^{-1}$, and in particular, that

$$\begin{aligned} \pi_0^k = \pi^k U_0 &= \gamma^k(B^{-1}U_0) = \gamma^k \lambda^o \\ &= \gamma_1^k \lambda_1^o + \gamma_2^k \lambda_2^o + \cdots + \gamma_{m+1}^k \lambda_{m+1}^o - z^o \end{aligned}$$

Hence, noting that $\pi_0^k = -z^k \leq -\phi(\hat{x}^k) \leq -\operatorname{Min} \phi_0(x)$, we have

$$z^o - \operatorname{Min} \phi_0(x) \geq (\gamma_1^k \lambda_1^o + \gamma_2^k \lambda_2^o + \cdots + \gamma_{m+1}^k \lambda_{m+1}^o) \qquad (\lambda_i^o > 0) \tag{28}$$

It is now easy to see, because $\gamma_i^k \geq 0$, and $\lambda_i^o > 0$, that by reducing the right-hand side of (28) by dropping all products except the ith,

$$0 \leq \gamma_i^k \leq [z^o - \operatorname{Min} \phi_0(x)]/\lambda_i^o \tag{29}$$

Therefore the components of γ^k have finite upper and lower bounds and Theorem 4 follows from the exercises below.

EXERCISE: If the components of γ^k have finite upper and lower bounds, then the same is true for any linear transform, in particular, the components of π^k, where $\pi^k = \gamma^k B^{-1}$.

EXERCISE: If the components of π^k have finite upper and lower bounds, there exists an infinite subset, K, of values k such that $\operatorname{Lim} \pi^k$ exists for $k \in K$ as $k \to \infty$.

THEOREM 5: *For x satisfying the conditions of Problem B,*

$$\lim_{k \to \infty} \phi_0(\hat{x}^k) = \operatorname{Min} \phi_0(x)$$

PROOF: Either the process is finite and an optimum solution is obtained or for all k we have

$$\operatorname{Min} \Delta(x|\pi^k) = \pi^k P_{k+1} < 0 \tag{30}$$

and, since the master program for any cycle $l > k$ contains the column P_{k+1}, we have also for all k and all $l > k$,

$$\pi^l P_{k+1} \geq 0 \qquad (l > k) \tag{31}$$

For $k \in K$ and $l \in K$ the difference, $(\pi^k - \pi^l) \to 0$ as $k \to \infty$ and $l > k$, by Theorem 4. Since R is bounded and $\phi_i(x)$ is continuous in R, $\phi_i(x)$ is bounded in R and so is the ith component, $\phi_i(x_j)$, of P_j over all j. Because the components of P_{k+1} are bounded from above and below, the difference $(\pi^k - \pi^l)P_{k+1}$ must also tend to zero. But, if the difference between the left sides of (30) and (31) goes to zero and $\pi^k P_{k+1}$ is negative and $\pi^l P_{k+1}$ is nonnegative, then $\pi^k P_{k+1}$ must come arbitrarily close to zero; that is, for any $\varepsilon > 0$, there exists a k_ε such that for any $k \in K$ and greater than k_ε,

$$-\varepsilon < \operatorname*{Min}_{x \in R} \Delta(x|\pi^k) < 0 \qquad (k \geq k_\varepsilon,\ k \in K) \tag{32}$$

From Theorem 3 and (32), the convergence of z_k to Min $\phi_0(x)$ now follows on the set $k \in K$. But the z_k are monotonically decreasing for all k. We conclude z_k tends to Min $\phi_0(x)$ for any set $k \to \infty$ (not just for $k \in K$). Since z_k is an upper bound for $\phi_0(\hat{x}^k)$, the latter also must converge to this limit. This establishes Theorem 5.

THEOREM 6: *Let $\hat{\pi} = Lim_{k \to \infty} \pi^k$ for $k \in K$ and let $\hat{x}$ be any optimum solution to Problem B, then*

$$F(\hat{x}) = \text{Min } F(x) = \operatorname*{Min}_{x \in R} \left[\phi_0(x) + \sum_{i=1}^{m} \hat{\pi}_i \phi_i(x) \right] \tag{33}$$

where $\hat{\pi}$ has nonnegative components for $i = r + 1, \ldots, m$.

PROOF: From (32), for $k > k_\varepsilon$ and $k \in K$

$$-\varepsilon \leq \Delta(x|\pi^k) \leq \phi_0(x) + \pi_0^k \tag{34}$$

where the inequality on the right is from (13). In the limit, letting $k \to +\infty$, $\varepsilon \to 0$, Lim $\pi^k = \hat{\pi}$ and

$$0 \leq \Delta(x|\hat{\pi}) = F(x) + \hat{\pi}_0 \leq \phi_0(x) + \hat{\pi}_0 \tag{35}$$

where $\Delta(x|\hat{\pi})$ is defined by (10).

On the other hand, by (20)

$$\hat{\pi}_0 = \text{Lim } \pi_0^k = \text{Lim } (-z_k) = -\operatorname*{Min}_{x \in R} \phi_0(x) = -\phi_0(\hat{x}) \tag{36}$$

Hence, substituting $x = \hat{x}$ into (35) and noting the above, we have

$$0 = F(\hat{x}) + \hat{\pi}_0 \leq F(x) + \hat{\pi}_0 \tag{37}$$

thus establishing Theorem 6 and, as a special case, Theorem 1.

24-2. HOMOGENEOUS OBJECTIVES AND THE CHEMICAL EQUILIBRIUM PROBLEM

Homogeneous Objective Functions of the First Degree.

In some applications, such as the chemical problem to be discussed later, we have to minimize a convex function

$$G(x) = \bar{x}G\left(\frac{x_1}{\bar{x}}, \frac{x_2}{\bar{x}}, \ldots, \frac{x_n}{\bar{x}}\right) \tag{1}$$

where

$$\bar{x} = x_1 + x_2 + \ldots + x_n \tag{2}$$

subject to a system of linear equations in nonnegative variables,

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad x_j \geq 0 \quad (i = 1, 2, \ldots, m) \tag{3}$$

Note that if each component of x is multiplied by t, the value of $G(x)$ is multiplied by t:

$$G(tx) = tG(x) \tag{4}$$

That is to say, G is a *homogeneous function of first degree.*

We will assume that $G(u) > 0$ is a continuous convex function on the set of possible u satisfying

$$u_1 + u_2 + \ldots + u_n = 1 \qquad (u_j \geq 0) \tag{5}$$

During the iterative process, we shall generate solutions $\hat{x}^1, \hat{x}^2, \ldots$ satisfying (3), such that $G(\hat{x}^k)$ is monotonically decreasing. Letting z_0 be an upper bound for $G(x)$, for $x = \hat{x}^k$ we have for such x

$$\begin{gathered} G(x) = \bar{x}G(u) < z_0 \qquad (x = \hat{x}^k) \\ 0 \leq \bar{x} < z_0/\text{Min } G(u) \end{gathered} \tag{6}$$

It follows that the components of such x are bounded from above and below.

Consider the equivalent *single-variable* generalized program

$$\begin{cases} \bar{x}\left[\sum_{j=1}^{n} a_{1j}u_j\right] = b_1 \qquad \left(\bar{x} \geq 0,\ u_j \geq 0,\ \sum u_j = 1\right) \\ \bar{x}\left[\sum_{j=1}^{n} a_{2j}u_j\right] = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \bar{x}\left[\sum_{j=1}^{n} a_{mj}u_j\right] = b_m \\ \bar{x}G(u) = z \text{ (Min)} \end{cases} \tag{7}$$

where $\bar{x} \geq 0$ is the variable whose coefficients are linear in u_j subject to (5). This gives rise to a master program that forms a general solution out of a nonnegative linear combination of columns of coefficients generated by various solutions u^k to the subprograms. This is in contradistinction to the usual case where convex combinations are required. This greater flexibility is possible, because the set of coefficients are homogeneous functions in u. To be explicit, we start with some set of m vectors $u = u^p$ satisfying (5) and, setting $[a_{ij}] = A$, generate

$$S_p = Au^p \qquad (p = 1, 2, \ldots, m) \tag{8}$$
$$c_p = G(u^p)$$

Assuming that the m columns form a feasible basis the first restricted master becomes

$$\sum_1^m S_p \lambda_p^1 = b \qquad (\lambda_p^1 \geq 0) \tag{9}$$
$$\sum_1^m c_p \lambda_p^1 = z^1 \text{ (Min)}$$

This determines the simplex multipliers π^1 which satisfy

$$\pi^1 S_p + c_p = 0 \qquad (p = 1, 2, \ldots, m) \tag{10}$$

To test optimality, the expression

$$\Delta(u|\pi^1) = G(u) + (\pi^1 A)u \tag{11}$$

is minimized over all u satisfying (5). If Min $\Delta \geq 0$, the solution

$$\hat{x} = \sum_1^m \lambda_p^1 u^p \tag{12}$$

is optimal. If not, choose u^{m+1} such that

$$G(u^{m+1}) + (\pi^1 A)u^{m+1} = \text{Min } \Delta\ (u|\pi^1) \tag{13}$$

Augment the master system (9) by $S_{m+1} = Au^{m+1}$ and $c_{m+1} = G(u^{m+1})$, optimize the new restricted master with $\lambda = \lambda^2$, and repeat the cycle with new multipliers $\pi = \pi^2$, etc.

A slight variation in our earlier proof given in § 24-1 (to take account of the fact that we no longer have the $\Sigma\lambda_i = 1$ constraint or its multiplier s) yields the following

THEOREM 1: *If there exists at least one nondegenerate basic feasible solution to a master program, then the modified algorithm converges to an optimal solution.*

EXERCISE: Show that, if $G(x)$ is a linear objective form, then it is also

convex and homogeneous of the first degree, and that the procedure just developed reduces to the standard simplex method.

EXERCISE: Prove Theorem 1. What role does (6) play in the proof?

The Chemical Equilibrium Problem.

For an application of the preceding theory, let us consider the problem of determining the molecular composition of the equilibrium state of a gaseous mixture containing m different types of atoms [White, Johnson, and Dantzig, 1958-1]. While in theory these will combine into all chemically possible molecular species, in practice only the standard types which occur in measurable amounts are considered.

Let

b_i = the number of atomic weights of atom type i present in the mixture,

x_j = the number of moles of molecular species j present in the mixture, where

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n) \tag{14}$$

$\bar{x}$ = the total number of moles of gas in the mixture, i.e.,

$$\bar{x} = \sum x_j \tag{15}$$

a_{ij} = the number of atoms of type i in a molecule of species j.

Then the mass-balance equations are

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \text{for} \quad i = 1, 2, \ldots, m \tag{16}$$

The determination of the equilibrium composition of a gaseous mixture is equivalent to the determination of the values of the mole numbers x_j that obey constraints (16) and minimize the total free energy of the mixture given by

$$\begin{aligned} G(x_1, \ldots, x_n) &= \sum_1^n c_j x_j + \sum_1^n x_j \log (x_j/\bar{x}) \\ &= \bar{x}\left[\sum_1^n c_j(x_j/\bar{x}) + \sum_1^n (x_j/\bar{x}) \log (x_j/\bar{x})\right] \\ &= \bar{x} \sum_1^n (c_j u_j + u_j \log u_j) \qquad (u_j = x_j/\bar{x} \geq 0) \end{aligned} \tag{17}$$

which can be shown to be a convex function. The c_j are the values of *Gibbs free energy functions* F^o/RT of the atomic species at a given temperature plus the natural logarithm of the pressure in atmospheres.

Our problem is to minimize (17), a first degree homogeneous form, subject

to the linear equality and inequality constraints (14), (15), and (16). The function

$$G(u) = \sum (c_j u_j + u_j \log u_j) \qquad (u_j \geq 0) \tag{18}$$

is clearly convex.

For this application, then,

$$\begin{aligned} \Delta &= G(u) + \sum \pi_i^k \sum a_{ij} u_j \\ &= \sum (u_j \log u_j + \bar{c}_j^k u_j) \end{aligned} \tag{19}$$

where π_i^k are the simplex multipliers of some iteration defined by relations analogous to (10). Let

$$\bar{c}_j^k = c_j + \sum_{i=1}^{m} \pi_i^k a_{ij} \tag{20}$$

To find the Min Δ subject to

$$\sum u_j = 1 \qquad (u_j \geq 0) \tag{21}$$

we ignore the relations $u_j \geq 0$ and find the unconditional minimum of the function

$$\bar{\Delta} = \sum \left(u_j \log u_j + \bar{c}_j^k u_j\right) - \theta \left(\sum u_j - 1\right) \tag{22}$$

where θ is a Lagrange multiplier (see § 6-5). We set the partial derivatives of $\bar{\Delta}$ with respect to u_j to zero; thus

$$\frac{\partial \bar{\Delta}}{\partial u_j} = 1 + \log u_j + \bar{c}_j^k - \theta = 0 \tag{23}$$

whence u_j may be written in the form

$$u_j = A e^{-\bar{c}_j^k} \tag{24}$$

where $A = e^{\theta - 1} > 0$. Substituting into (21) determines A and

$$u_j = e^{-\bar{c}_j^k} \Big/ \sum_{i=1}^{n} e^{-\bar{c}_j^k} \tag{25}$$

so that the conditions $u_j \geq 0$ hold at the minimum. These, then, are the values $u_j = u_j^{k+1}$ with which to initiate the next iteration.

24-3. SEPARABLE CONVEX OBJECTIVES

DEFINITION: If the objective function to be minimized, instead of being linear, is of the form

$$\sum_{j=1}^{n} \phi_j(x_j) = z \text{ (Min)} \qquad (0 \leq x_j \leq h_j) \tag{1}$$

where $\phi_j(x_j)$ is a convex function, it is called *convex-separable*, a term used by Charnes to describe this class of objective forms [Charnes and Lemke, 1954-1; Dantzig, 1956-2]. We shall assume that h_j is a given finite upper bound for x_j and the x_j are subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, m) \tag{2}$$

To illustrate, if the first 100 units of an activity can be performed at \$1 per unit, the next 50 units at \$1.25 per unit and the next 50 units at \$1.50 per unit, then the total cost $\phi(x)$ is convex in the range $0 \leq x \leq 200$. In general, if the first α_1 units cost s_1 per unit, the second α_2 units cost s_2 per unit, . . ., the last α_k units cost s_k per unit, then the total cost $\phi(x)$ is convex if $s_1 \leq s_2 \leq \ldots \leq s_k$, but is not convex if, for any r, $s_r > s_{r+1}$.

EXERCISE: The definition of a convex function is given in § 7-1. Show that it is equivalent to the definition: a function $\phi(x)$ is convex if

$$\phi\left(\sum \lambda_r x_r\right) \leq \sum \lambda_r \phi(x_r) \tag{3}$$

for any $\lambda_r \geq 0$, such that $\Sigma\lambda_r = 1$.

EXERCISE: Show that, if the cost of the jth activity is proportional to the square of the jth activity, $\phi_j(x_j) = c_j x_j^2$, then $\phi(x_j)$ is convex for $c_j \geq 0$ but is not convex for $c_j < 0$.

As a third example, if the total cost $\phi(x)$ consists of a fixed charge f and the remaining cost is proportional to the activity level, then $\phi(x)$ is convex. On the other hand, if there is a fixed charge f only if $x > 0$, then we can write

$$\phi(x) = \begin{cases} f + hx, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{4}$$

In this case $\phi(x)$ is not convex, if $f > 0$.

EXERCISE: Show that if $f \leq 0$, $\phi(x)$, above, is convex.

To replace equations (1) and (2) by a standard linear programming problem, we assume that $\phi_j(x)$ is a broken-line function. Later we shall remove this restriction. (In the event that $\phi_j(x)$ has a continuous derivative, one can select $k + 1$ points on the curve such that the broken line fit through these points is a sufficiently close approximation.)

The Equivalent Linear Program.

For $r = 0, 1, 2, \ldots, k$, let (f_{rj}, g_{rj}) be the coordinates of the break points $(x_j, \phi_j(x_j))$ of the function $\phi_j(x)$ (see Fig. 24-3-I). Any x_j in the range $f_{0j} \leq x_j \leq f_{kj}$ may be represented by

$$x_j = \lambda_{0j}f_{0j} + \lambda_{1j}f_{1j} + \ldots + \lambda_{kj}f_{kj} \qquad (\lambda_{rj} \geq 0) \tag{5}$$

where

(6) $$1 = \lambda_{0j} + \lambda_{1j} + \ . \ . \ . \ + \lambda_{kj}$$

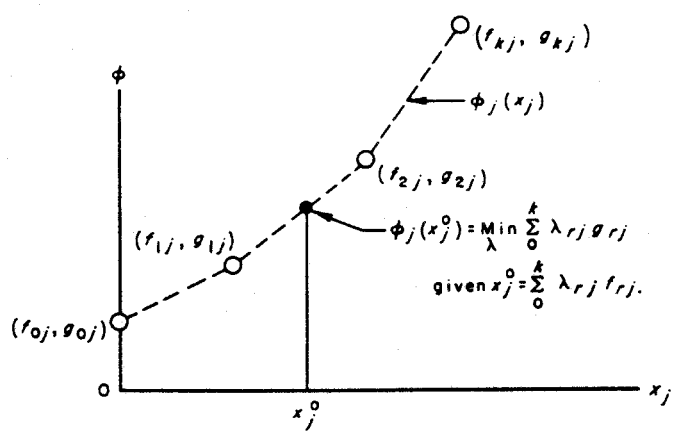


Figure 24-3-I. Converting a convex function into a linear programming format.

Since ϕ_j is a convex function, we then have, by (3),

(7) $$\phi_j(x_j) \leq \lambda_{0j}g_{0j} + \lambda_{1j}g_{1j} + \ . \ . \ . \ + \lambda_{kj}g_{kj}$$

To solve (1) and (2), determine $\lambda_{rj} \geq 0$, and Min z' satisfying

(8) $$\sum_{j=1}^{n} a_{ij}[\lambda_{0j}f_{0j} + \lambda_{1j}f_{1j} + \ . \ . \ . \ + \lambda_{kj}f_{kj}] = b_i \qquad (i = 1, 2, \ . \ . \ ., m)$$

(9) $$\lambda_{0j} + \lambda_{1j} + \ . \ . \ . \ + \lambda_{kj} = 1 \qquad (j = 1, 2, \ . \ . \ ., n)$$

(10) $$\sum_{j=1}^{n} [\lambda_{0j}g_{0j} + \lambda_{1j}g_{1j} + \ . \ . \ . \ + \lambda_{kj}g_{kj}] = z' \text{ (Min)}$$

and substitute the resulting λ into (5) to determine x_j. To prove that solving (8)–(10) is equivalent to solving (1) and (2), we have only to show that Min $z \geq$ Min z'. From (7), Min $z \leq$ Min z'. To show that Min $z \geq$ Min z', take the x_j yielding the minimum to (1) and (2) and represent them as combinations of the abscissas of the two breakpoints immediately to the left and right. The resulting λ satisfy (8) and (9) and satisfy (7) with *equality*, so that the z' resulting in (10) equals Min z. Hence, Min $z \geq$ Min z', as we were to show.

Bounded-Variable Method.

It will be noted that the above procedure increases the number of variables and increases the number of equations by one for each $\phi_j(x_j)$. By use of the upper bounding method, it is possible to maintain the original number of equations. The original variables will be replaced by sets of new bounded variables which (except in the cost row) will all have their several columns of coefficients identical (so that a number of short cuts are possible;

see Chapter 18). In Fig. 24-3-II the slopes of the broken line are denoted by s_i and the width of the intervals by α_i.

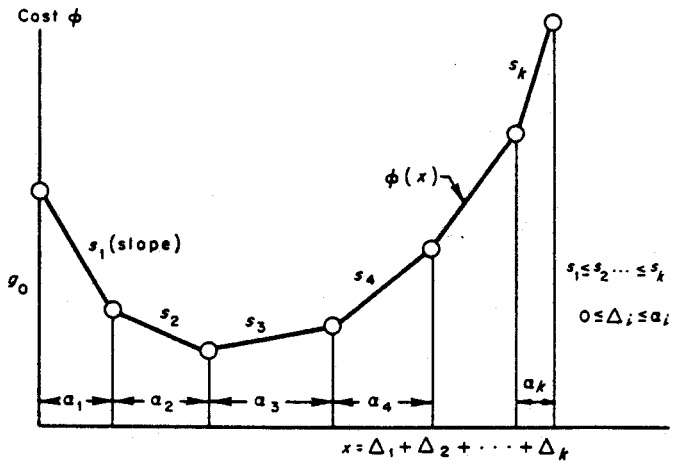


Figure 24-3-II. Converting a convex function into a linear programming format using bounded variables.

From the convexity of $\phi(x)$ follow the relations

(11) $$s_1 \le s_2 \le \ldots \le s_k$$

We now assert that

(12) $$\phi(x) = g_0 + \text{Min}\,[s_1\Delta_1 + s_2\Delta_2 + \ldots + s_k\Delta_k]$$

where

(13) $$x = \Delta_1 + \Delta_2 + \ldots + \Delta_k$$

and

(14) $$0 \le \Delta_i \le \alpha_i$$

Indeed, since $s_1 \le s_2 \le \ldots \le s_k$, it is obvious that the way to find the minimum is to choose Δ_1 as large as possible until it hits its upper bound α_1, then take Δ_2 in turn, etc., until a Δ_r is reached, such that setting $\Delta_r = \alpha_r$ the value of x is exceeded, in which case the value of Δ_r is reduced so that (13) holds. It is clear that this process is simply generating the curve $\phi(x)$ from 0 up to the value x.

Equivalent Linear Program.

To solve (1) and (2), determine $\Delta_{rj} \ge 0$ and Min z' satisfying

(15) $$\sum_{j=1}^{n} a_{ij}[\Delta_{1j} + \Delta_{2j} + \ldots + \Delta_{kj}] = b_i \qquad (i = 1, 2, \ldots, m)$$

$$\sum_{j=1}^{n} [s_{1j}\Delta_{1j} + s_{2j}\Delta_{2j} + \ldots + s_{kj}\Delta_{kj}] = z' \text{ (Min)}$$

where

$$0 \leq \Delta_{rj} \leq \alpha_{rj} \tag{16}$$

and determine x_j by

$$x_j = \Delta_{1j} + \Delta_{2j} + \ldots + \Delta_{kj} \tag{17}$$

The Variable-Coefficient Approach.

This method has the advantage that it uses the original convex functions instead of a broken line fit. While the procedure appears to be the continuous analogue of our earlier procedure, we do require the convergence proof of § 24-1 to justify it. Referring to equations (5), (6), and (7), we may write in a purely formal way

$$\begin{aligned} x_j &= \lambda_j f_j \\ 1 &= \lambda_j \\ \phi_j(x_j) &\leq \lambda_j g_j \end{aligned} \tag{18}$$

where (f_j, g_j) is a pair of variable coefficients. The set, C_j, of possible values for these coefficients consists of those points (f_j, g_j) for which $g_j \geq \phi_j(f_j)$. Since the function ϕ_j is convex, this set is convex by definition; it is therefore appropriate to use the variable coefficient method given in Chapter 22. In that chapter, however, we assumed that the C_j were defined by a finite number of linear equations. In that case the iterative process is finite. The process is an infinite one if the derivative of ϕ_j is continuous and non-constant. (See § 24-1 for convergence proof.)

Formally, the full system with variable coefficients takes the form: Find $x_j \geq 0$, f_j, $g_j \geq \phi_j(f_j)$, and Min z satisfying

(19)

										Multipliers
$a_{11}x_1$		$+ a_{12}x_2$	$+ \ldots$	$+ a_{1n}x_n$					$= b_1$	π_1
$a_{21}x_1$		$+ a_{22}x_2$	$+ \ldots$	$+ a_{2n}x_n$					$= b_2$	π_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$a_{m1}x_1$		$+ a_{m2}x_2$	$+ \ldots$	$+ a_{mn}x_n$					$= b_m$	π_m
x_1	$- \lambda_1 f_1$								$= 0$	$-s_1$
	λ_1								$= 1$	t_1
		x_2	$- \lambda_2 f_2$						$= 0$	$-s_2$
			λ_2						$= 1$	t_2
				$\cdot$					$\cdot$	$\cdot$
					$\cdot$				$\cdot$	$\cdot$
						$\cdot$			$\cdot$	$\cdot$
							x_n	$- \lambda_n f_n$	$= 0$	$-s_n$
								λ_n	$= 1$	t_n
	$\lambda_1 g_1$		$+ \lambda_2 g_2$	$+ \ldots$		$+ \lambda_n g_n$			$= z$ (Min)	

where the multipliers associated with the system are shown on the right. In all, there are $2n + m$ constraints.

EXERCISE: Show that the set of possible coefficients (f_1, g_1) in (19) is a convex set.

According to the variable coefficient theory (Chapter 22), a basis is formed using particular values for f_j and g_j where one is allowed to form columns using more than one set of (f_j, g_j) values for a given j. Certain of the j will have associated with them just one column of values (f_{j1}, g_{j1}) in the basis, and others will have two associated values (f_{j1}, g_{j1}) and (f_{j2}, g_{j2}).

EXERCISE: Show that it is not possible to have zero or more than two different (f_{ji}, g_{ji}), $i = 1, 2, 3, \ldots$ in a basis.

We will use λ_{ji} for the variable associated with (f_{ji}, g_{ji}). It can be arranged that all the x_j are in the basic set. For example, we could initiate Phase I with x_j, λ_j, and a set of m artificial variables as a basic set. Since x_j can be assumed unrestricted in sign, they will remain in all subsequent basic sets. Assuming no artificial variables remain at the end of Phase I, it follows, because there are $2n + m$ basic variables, that n are x_j; $n - m$ are singles, λ_{j1}; and m are pairs, λ_{j1} and λ_{j2}.

Interpreted another way, m of the x_j, say $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$, satisfy a relation

$$f_{j1} \leq x_j \leq f_{j2} \qquad (j = j_i) \tag{20}$$

while the remaining x_j satisfy

$$x_j = f_{j1} \qquad (j \neq j_i) \tag{21}$$

EXERCISE: Discuss the case where there may be artificial variables in Phase II.

THEOREM 1: *The simplex multipliers are determined by*

$$s_j = (g_{j2} - g_{j1})/(f_{j2} - f_{j1}) \qquad (j = j_1, j_2, \ldots, j_m) \tag{22}$$

$$\sum_{i=1}^{m} a_{ij}\pi_i = s_j \qquad (j = j_1, j_2, \ldots, j_m) \tag{23}$$

$$s_j = \sum_{i=1}^{m} a_{ij}\pi_i \qquad \text{for } j \neq j_i \tag{24}$$

$$t_j = g_{j1} - s_j f_{j1} \qquad \text{all } j \tag{25}$$

PROOF: From (19) we have that

$$\begin{aligned} s_j f_{j1} + t_j &= g_{j1} \\ s_j f_{j2} + t_j &= g_{j2} \end{aligned} \qquad (j = j_1, j_2, \ldots, j_m) \tag{26}$$

Subtracting the first from the second and solving for s_j give (22). Equations

(23)–(25) follow directly from (19). Note that in (23) the s_j are known and the π_i are to be determined, while in (24) the π_i are known and the s_j are determined. In (25), only t_j is unknown.

For optimality of a solution $x_j = x_j^o$, we require that

$$g_j - (s_j f_j + t_j) \geq 0 \qquad (j = 1, 2, \ldots, n) \tag{27}$$

for all possible choices of (f_j, g_j).

EXERCISE: Why is this sufficient?

Since s_j and t_j are fixed, we solve the subproblem of minimizing $z_j = \phi_j(f_j) - s_j f_j$. If ϕ_j has a continuous slope, $\phi_j'(f_j)$, we seek $f_j \geq 0$ such that

$$\phi_j'(f_j) = s_j \tag{28}$$

In Fig. 24-3-III, we graph the function $g_j = \phi_j(f_j)$, and for each point (f_j^o, g_j^o) on the curve we consider the line passing through it, $g_j - s_j f_j = g_j^o - s_j^o f_j^o$. We then seek that line with minimum g_j intercept, denoted by z_j. By (28) this occurs at the point of tangency if such a point exists.

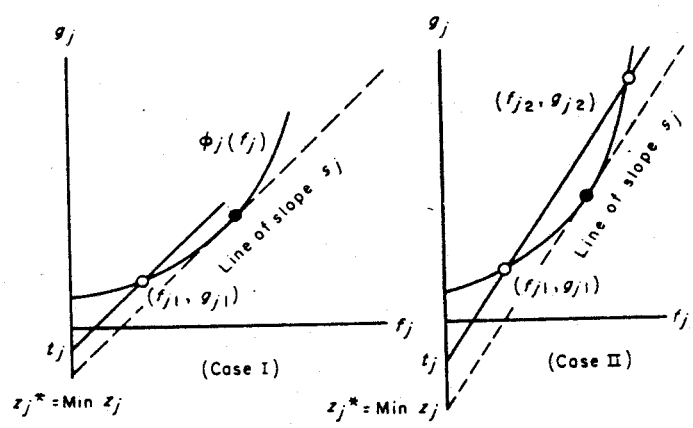


Figure 24-3-III. Solving a separable objective problem using a generalized programming approach.

It may not be possible to find such a tangency point (f_j, g_j), for it may be that the slope at $f_j = 0$ is greater than s_j; then the best choice is $f_j = 0$, $g_j = \phi_j(0)$, since convexity of $\phi_j(x)$ implies a monotone increasing slope. If $\phi_j'(f_j) \leq s_j$ for all $f_j > 0$, then the best choice is $f_j = h_j$. If j corresponds to a variable having two associated λ's in the basis, then the slope of the line joining (f_{j1}, g_{j1}) to (f_{j2}, g_{j2}) is s_j, by (22); in this case there is always somewhere between these two points, a point satisfying (28), provided the slope is a continuous function of f_j. We conclude

THEOREM 2: *A solution* $(x_1^o, x_2^o, \ldots, x_n^o)$ *is optimal if for each* j *the simplex multipliers* s_j *determined by Theorem* 1 *satisfy*

$$\begin{aligned} s_j &= \phi_j'(x_j^o) \quad \text{if } 0 < x_j^o < h_j \\ s_j &\leq \phi_j'(0) \quad \text{if } 0 = x_j^o \\ s_j &\geq \phi_j'(h_j) \quad \text{if } x_j^o = h_j \end{aligned}$$

or, more generally, $\phi_j^L(x_j^o) \leq s_j \leq \phi_j^R(x_j^o)$ *where* ϕ^L *and* ϕ^R, *the left or right derivatives, are omitted for* $\phi_j^L(0)$ *or* $\phi_j^R(h_j)$.

Algorithm.

The foregoing analysis suggests a variant of the simplex method in which m variables x_j are considered basic, with bounds $f_{j1} \leq x_j \leq f_{j2}$ and $n - m$ variables are non-basic at fixed values f_{j1}. It is assumed a basic feasible solution $x_j = x_j^o$ of this type is at hand to initiate the algorithm whose steps are as follows:

Step 1. Compute slopes $s_j = \phi_j'(x_j^o)$; for basic variables x_j, either decrease upper bound, f_{j2}, or increase lower bound, f_{j1}, so that the adjusted bounds are closer together and satisfy

$$s_j = \phi_j'(x_j^o) = \frac{g_{j2} - g_{j1}}{f_{j2} - f_{j1}} \qquad (x_j \text{ basic}) \tag{29}$$

(In case $\phi_j'(x_j)$ is discontinuous, replace $\phi_j'(x_j)$ by the left derivative $\phi_j^L(x_j)$ in the above discussion.)

Step 2. Compute the simplex multipliers π_i by (23).

Step 3. Compute the s_j and t_j for non-basic variables by (24) and (25).

Step 4. For non-basic j, determine f_j^* such that

(a) $f_j^* = 0 \quad \text{if } s_j < \phi_j^R(0)$

(b) $f_j^* = h_j \quad \text{if } s_j \geq \phi_j^L(h_j)$

(c) $\phi_j^L(f_j^*) \leq s_j \leq \phi_j^R(f_j^*)$ otherwise

Step 5. Compute $g_j^* = \phi_j(f_j^*)$ and $z_j^* = g_j^* - s_j f_j^*$.

Step 6. Determine p, non-basic, such that

$$(z_p^* - t_p) = \underset{j \text{ non-basic}}{\text{Min}} (z_j^* - t_j) \leq 0$$

If Min $(z_j - t_j) = 0$, *terminate*. The basic solution is optimal.

Step 7. Keeping all other non-basic variables at their values f_{j1}, adjust the value of x_j and the basic variables as follows:

Decrease or increase x_p from its fixed value f_{p1} according to whether f_p^* is less than or greater than f_{p1}. Drop the basic variable which hits its upper or lower bound first or stop if x_p hits the value f_p^* first. In the latter case, f_p^* is the new fixed value of x_p which is still non-basic. In the

former case, x_p becomes a new basic variable with f_p^* and f_{p1} as its new bounds; the basic variable which hits its bound is then dropped from the basis and fixed at the value of this bound.

Convergence: It is not known whether or not the algorithm as given above converges to Min z. In order that the proof of convergence given in § 24-1 apply, some provision must also be made in the algorithm for first re-introducing into the basis any previously generated column such that $g_j - s_j f_j - t_j < 0$.

EXERCISE: Expand the algorithm to allow the re-introduction of previously generated columns. Review the proof of convergence and show why it breaks down if the latter is not done.

EXERCISE: Develop a simpler form of the algorithm by replacing the upper and lower bounds f_{j1} and f_{j2}, as defined above, by $f_{j1} = 0$ and $f_{j2} = h_j$ and setting $s_j = \phi_j'(x_j^0)$ for basic x_j.

24-4. QUADRATIC PROGRAMMING

Although a convex quadratic objective can be treated by the methods of § 24-1 and can be reduced to the convex separable case discussed in § 24-3, the linear nature of its partial derivatives has given rise to an elegant theory important in its own right. It is doubtful at this writing that the full potentiality of this theory has been realized.

Quadratic programs can arise in several ways; four listed by Wolfe in his [1959-1] paper are as follows:

Regression: To find the best least-square fit to given data, where certain parameters are known a priori to satisfy linear inequalities constraints.

Efficient Production: Maximization of profit, assuming linear production functions and linearly varying marginal costs [Dorfman, 1951-1].

Minimum Variance: To find the solution of a linear program with variable cost coefficients which will have given expected costs and minimum variance [Markowitz, 1956-1, 1959-1].

Convex Programming: To find the minimum of a general convex function under linear constraints and quadratic approximation [White, Johnson, and Dantzig, 1958-1].

Historically, it was Barankin and Dorfman [1958-1] who first pointed out that, if the linear Lagrangian conditions of optimality were combined with those of the original system, the optimum solution was a basic solution in the enlarged system with the property that only one of certain pairs of variables were in the basic set. Markowitz [1956-1, 1959-1], on the other hand, showed that it was possible to modify the enlarged system and then parametrically generate a class of basic solutions with the above special property which converged to the optimum in a finite number of iterations. Finally, Wolfe [1959-1] proved that an easy way to do this is by slightly modifying the simplex algorithm so as not to allow a variable to enter the

basic set if its "complementary" variable is already in the basic set. Thus, by modifying a few instructions in a simplex code for linear programs, it was possible to solve a convex quadratic program! We shall present here a variant of Wolfe's elegant procedure. The chief difference is that ours is more nearly a strict analogue of the simplex method; it has a tighter selection rule and a monotonically decreasing objective.

Preliminaries.

Before stating the problem, let us note that every quadratic form can be conveniently expressed in terms of a *symmetric matrix* associated with its coefficients. For example, for $n = 3$ variables,

$$(1) \qquad Q(x) = c_{11}x_1^2 + c_{22}x_2^2 + c_{33}^2x_3^2 + 2c_{12}x_1x_2 + 2c_{23}x_2x_3 + 2c_{13}x_1x_3$$

$$= [x_1, x_2, x_3] \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^TCx$$

where T stands for *transpose*.

DEFINITION: A quadratic form is called *positive definite* if $x^TCx > 0$ for all $x \neq 0$; it is called *positive semi-definite* if $x^TCx \geq 0$ for all x.

Problem: Find $x = (x_1, x_2, \ldots, x_n) \geq 0$ and Min $Q(x)$ satisfying

$$(2) \qquad \begin{array}{lll} Ax = b & A = [a_{ij}] & (i = 1, 2, \ldots, m) \\ x^TCx = Q(x) & C = [c_{ij}] & \end{array}$$

where $Q(x)$ is a convex quadratic function.

LEMMA 1: *x^TCx is convex if and only if it is positive semi-definite.*

PROOF: Assume x^TCx is a convex function. To prove $x^TCx \geq 0$, suppose on the contrary, $(x^o)^TCx^o < 0$ for some $x = x^o$. Then, for $x' = -x^o$, it is also true $(x')^TCx' < 0$ and for any convex combination, x^*, of x^o and x' we also have $(x^*)^TCx^* < 0$ because x^TCx is convex. In particular, for $x^* = \frac{1}{2}x^o + \frac{1}{2}x'$ we have $(\frac{1}{2}x^o + \frac{1}{2}x')^TC(\frac{1}{2}x^o + \frac{1}{2}x') < 0$ or $(0)^TC(0) < 0$, a contradiction, since $\frac{1}{2}x^o + \frac{1}{2}x' = 0$.

The convexity of a positive semi-definite form follows from Lemma 2 below because a linear transform, $\bar{x}$, of the variables x reduces $Q(x)$ to a sum of convex functions in $\bar{x}$.

LEMMA 2: *If x^TCx is positive semi-definite, there exists a non-singular matrix E such that a change of variables $x = E\bar{x}$ yields*

$$(3) \qquad x^TCx = \sum_1^n \lambda_j\bar{x}_j^2, \qquad (\lambda_j \geq 0)$$

where $\lambda_j \geq 0$ is the jth diagonal element of a diagonal matrix E^TCE.

PROOF: Select any variable, say x_1, with $c_{11} > 0$. (See first exercise below.) Express $Q(x)$ as a quadratic polynomial in x_1 and "complete the square"; thus

$$(4) \qquad x^TCx = c_{11}x_1^2 + 2x_1 \sum_{j=2}^{n} c_{1j}x_j + \sum_{i=2}^{n}\sum_{j=2}^{n} c_{ij}x_ix_j, \qquad (c_{11} > 0)$$

$$= \frac{1}{c_{11}}\left[\sum_{j=1}^{n} c_{1j}x_j\right]^2 - \frac{1}{c_{11}}\left[\sum_{j=2}^{n} c_{1j}x_j\right]^2 + \sum_{i=2}^{n}\sum_{j=2}^{n} c_{ij}x_ix_j$$

$$= \frac{1}{c_{11}}\bar{x}_1^2 + \sum_{i=2}^{n}\sum_{j=2}^{n} c'_{ij}x_ix_j$$

where $\bar{x}_1 = \sum_{j=1}^{n} c_{1j}x_j$ and $c'_{ij} = (c_{11}c_{ij} - c_{1i}c_{1j})/c_{11}$. The process may now be repeated using the quadratic expression in $(x_2, \ldots, x_n)$ on the right. The process terminates in $k \leq n$ steps. Set $\bar{x}_j = x_j$ and $\lambda_j = 0$ for $j = k + 1, \ldots, n$.

EXERCISE: Show that either $c_{11} > 0$ or all $c_{11} = c_{12} \ldots = c_{1n} = 0$ in a positive semi-definite quadratic form.

EXERCISE: Show that the determinants of all the principal minors of C are positive if $Q(x)$ is positive definite, in particular $c_{11} > 0$.

EXERCISE: Show that if $k = n$, $Q(x)$ is positive definite; and that if $k < n$, it is semi-definite.

EXERCISE: Apply Lemma 2 to show that, if x^TCx is positive semi-definite and if $(x^o)^TCx^o = 0$, for $x = x^o$, then $Cx^o = 0$.

EXERCISE: Complete the proof of Lemma 1.

Optimality Conditions.

Let A_j, C_j denote the jth columns of A and C, respectively, and let

$$(5) \qquad y_j = C_j^Tx - \pi A_j \qquad (\pi = \pi_1, \pi_2, \ldots, \pi_m)$$

THEOREM 1: *A solution $x = x^o$ is minimal if there exist $\pi = \pi^o$ and $y = y^o$ such that*

$$(6) \qquad Ax^o = b, \qquad x^o \geq 0$$

$$(7) \qquad y_j^o = C_j^Tx^o - \pi^oA_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

$$(8) \qquad y_j^o = 0 \text{ if } x_j^o > 0$$

PROOF: Rewrite $Q(x)$ in the form

$$(9) \quad Q(x) - Q(x^o) = 2\sum_{j=1}^{n}\left[\sum_{i=1}^{n} c_{ij}x_i^o\right](x_j - x_j^o) + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}(x_i - x_i^o)(x_j - x_j^o)$$

$$= 2\sum_{j=1}^{n}(C_j^Tx^o)(x_j - x_j^o) + (x - x^o)^TC(x - x^o)$$

Let $x \geq 0$ be any solution satisfying $Ax = b$, then

$$A(x - x^o) = \sum_{j=1}^{n} A_j(x_j - x_j^o) = 0 \tag{10}$$

Multiplying on the left by $2\pi^o$ and subtracting from (9) yield

$$\begin{aligned} Q(x) - Q(x^o) &= 2\sum_{j=1}^{n} (C_j^T x^o - \pi^o A_j)(x_j - x_j^o) + (x - x^o)^T C(x - x^o) \\ &= 2\sum_{j=1}^{n} y_j^o(x_j - x_j^o) + (x - x^o)^T C(x - x^o) \end{aligned} \tag{11}$$

For the class of solutions with the property $y_j^o = 0$ for $x_j^o > 0$, (11) simplifies to

$$Q(x) - Q(x^o) = 2\sum_{y_j^o \neq 0} y_j^o x_j + (x - x^o)^T C(x - x^o) \tag{12}$$

Note that (12) holds by (8) and $y_j^o \geq 0$ holds by (7), thus all terms in (12) are nonnegative, therefore $Q(x) \geq Q(x^o)$.

Improving a Non-optimal Solution.

Consider the system

$$\begin{aligned} Ax \qquad\qquad &= b \qquad\qquad (x \geq 0) \\ Cx - A^T\pi^T - I_n y &= 0 \end{aligned} \tag{13}$$

where $x^T Cx$ is assumed to be positive semi-definite. Let x^o, π^o, y^o be a basic feasible solution associated with a basic set *with the complementarity property*, namely, for each j either x_j or y_j, but not both, is in the basic set. We shall assume further that the right-hand side has been perturbed to insure that all basic solutions are nondegenerate. Note that neither π nor y is sign restricted; only $x \geq 0$ is required for a feasible solution to (13); an optimal solution will have been obtained if $y_j \geq 0$ and $x_j y_j = 0$ for all j.

THEOREM 2: *If a basis is complementary and $y_s^o < 0$, then any increase of the non-basic variable x_s, with adjustment of only the basic variables, generates a class of solutions x', π', y', such that $x^T Qx$ decreases as long as $y_s' < 0$.*

PROOF: Let x be any solution in the class above, i.e., generated by increasing x_s; in particular, let x' be generated by $x_s = x_s'$. Analogous to (11), $Q(x) - Q(x') = 2y_s'(x_s - x_s') + (x - x')^T C(x - x')$ since for all $j \neq s$ either $x_j = x_j' = 0$ if x_j is non-basic or if x_j is basic $y_j = y_j' = 0$ by the complementarity assumption. The adjusted values of the basic variables are linear functions of x_s, hence it follows that $(x - x') = (x_s - x_s')v$ where v is a *constant vector*. Hence, $Q(x) - Q(x') = (x_s - x_s')[2y_s' + (x_s - x_s')(v^T Cv)]$ and it is clear that, if $y_s' < 0$, the right-hand side is negative for sufficiently small $(x_s - x_s') > 0$.

Moreover for $Q(x)$ to decrease with an increase of $x_s \geq 0$ from, say, x_s' to x_s'', it must be accompanied by $y_s' < y_s''$ because

$$Q(x'') - Q(x') = 2(x_s'' - x_s')y_s' + (x_s'' - x_s')^2 v^T C v$$

and, by interchanging the roles of x' and x'',

$$Q(x') - Q(x'') = 2(x_s' - x_s'')y_s'' + (x_s'' - x_s')^2 v^T C v$$

whence $2(y_s'' - y_s') = 2(x_s'' - x_s')v^T C v \geq 0$. But $v^T C v \neq 0$ because $v^T C v = 0$ implies $Cv = 0$ for positive semi-definite forms (see last exercise following proof of Lemma 2) and if $Cv = 0$, then from (9), $Q(x'') - Q(x') = 2(x_s'' - x_s')(x')^T C v + (x_s'' - x_s')^2 v^T C v = 0$ which contradicts $Q(x'') - Q(x') < 0$; we conclude $y_s'' > y_s'$.

As in the simplex method, we require that all solutions generated by increasing x_s and adjusting the basic variables remain feasible, i.e., $x_j \geq 0$ for all j. In this process, either y_s attains the value zero first, and thus can be dropped from the basic set, or the value of some basic x_r attains the zero value first and is dropped.

THEOREM 3: *If x_r drops as basic variable, introduction of y_r either causes $x^T C x$ to decrease (and some x_{r_1} or y_s to be dropped) or causes $x^T C x$ to stay fixed and y_s to be dropped; if x_{r_1} is dropped, this theorem may be reapplied; if y_s is dropped either initially or upon increase of y_r Theorem 2 may be reapplied.*

PROOF: Our proof is completely general; however, for convenience we will illustrate it on system (14)

(14)

	x_1	x_2	x_3	x_4	x_5	$-\pi_1$	$-\pi_2$	y_1	y_2	y_3	y_4	y_5	Constants
	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}								b_1
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}								b_2
	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	a_{11}	a_{21}	-1					0
	c_{12}	c_{22}	c_{23}	c_{24}	c_{25}	a_{12}	a_{22}		-1				0
	c_{13}	c_{23}	c_{33}	c_{34}	c_{35}	a_{13}	a_{23}			-1			0
	c_{14}	c_{24}	c_{34}	c_{44}	c_{45}	a_{14}	a_{24}				-1		0
	c_{15}	c_{25}	c_{35}	c_{45}	c_{55}	a_{15}	a_{25}					-1	0
B:	●	●	●	●	★	●	●					●	
B':	●	●	●		●	●	●				★	●	

Let system (14) in vector form be

$$(15) \quad P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + P_5x_5 + (P_6\pi_1 + P_7\pi_2) + \bar{P}_1y_1 + \bar{P}_2y_2 + \bar{P}_3y_3 + \bar{P}_4y_4 + \bar{P}_5y_5 = P_0$$

We suppose that we have on some cycle a basis B and a basic feasible complementary solution (x^o, π^o, y^o) with basic variables $x_1, x_2, x_3, x_4, \pi_1, \pi_2, y_5$

and the value of $y_5 = y_5^o < 0$. In this case, x_5 will be the new basic variable and we assume that x_4 will drop. This yields a new basis B'. In (14), the heavy dots (●) indicate that the column is in the basis B, and the star indicates that the column P_5 associated with x_5 is replacing a vector P_4 of the basis B to form basis B'; see second row of dots. The dropping of x_4 automatically requires that y_4 become the new basic variable for the basis following B'; see ★ in the B' row of (14).

Let the representation of both P_5 and $\bar{P}_4$ in terms of the basis B be:

$$P_5 = P_1\alpha_1 + P_2\alpha_2 + P_3\alpha_3 + P_4\alpha_4 + (P_6\alpha_6 + P_7\alpha_7) + \bar{P}_5\bar{\alpha}_5 \tag{16}$$

$$\bar{P}_4 = P_1\lambda_1 + P_2\lambda_2 + P_3\lambda_3 + P_4\lambda_4 + (P_6\lambda_6 + P_7\lambda_7) + \bar{P}_5\bar{\lambda}_5 \tag{17}$$

We first show that $\lambda_4 \leq 0$ in (17). Setting $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, the first six rows of representation (17) yield (18) and (19):

$$\begin{aligned} [a_{11} \quad a_{12} \quad a_{13} \quad a_{14}]\,\lambda^T &= 0 \\ [a_{21} \quad a_{22} \quad a_{23} \quad a_{24}]\,\lambda^T &= 0 \end{aligned} \tag{18}$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{bmatrix} \lambda^T + \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} \lambda_6 + \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \end{bmatrix} \lambda_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{19}$$

Multiplying (19) by λ on the left and denoting the square matrix by C_4 yield, by (18), $\lambda C_4 \lambda^T = -\lambda_4$. Since $\lambda C_4 \lambda^T$ is positive semi-definite (C_4 is a principal minor of C), $\lambda C \lambda^T \geq 0$ and $\lambda_4 \leq 0$.

Case $\lambda_4 < 0$: Our objective is to show that, if x_4 drops out of the basic set upon introduction of x_5 into the basic set and if the non-basic complementary variable to x_4, namely y_4, is subsequently increased (with adjustment of the values of the new basic variables), then x_5 and y_5 will continue to increase and x^TCx to decrease as long as y_5 remains negative. This assumes $\lambda_4 < 0$. (Later, for the case $\lambda_4 = 0$, we shall show that x_5 and x^TCx will remain unchanged but y_5 will decrease to zero when y_4 is increased in value.) Let the representation of P_4 in terms of the basis B' be

$$\bar{P}_4 = P_1\lambda_1' + P_2\lambda_2' + P_3\lambda_3' + P_5\lambda_5' + (P_6\lambda_6' + P_7\lambda_7') + \bar{P}_5\bar{\lambda}_5' \tag{20}$$

and let the basic solution associated with B' be

$$P_0 = P_1x_1' + P_2x_2' + P_3x_3' + P_5x_5' + (P_6\pi_1' + P_7\pi_2') + \bar{P}_5y_5' \tag{21}$$

We observe that in the representation (16) of P_5 in terms of B, the weight α_4 on P_4 is positive (since x_4 decreased when x_5 increased). Since (20) is obtained by eliminating P_4 from (16) and (17) and since $\lambda_4 < 0$ and $\alpha_4 > 0$, it follows that $\lambda_5' < 0$. If $y_4 = \theta > 0$ units of $\bar{P}_4$ are introduced into

the solution and the values of the basic variables are adjusted, we obtain from (20) and (21),

$$(22)\quad P_0 = P_1(x_1' - \theta\lambda_1') + \ldots + P_3(x_3' - \theta\lambda_3') + P_5(x_5' - \theta\lambda_5') + \ldots + \bar{P}_4\theta + \bar{P}_5(y_5' - \theta\bar{\lambda}_5')$$

Thus $x_5 = x_5' - \theta\lambda_5'$ will increase when $y_4 = \theta > 0$ is increased since $\lambda_5' < 0$. Moreover, we may adopt the point of view for the purpose of the proof, that it is the increase in x_5 that is "causing" the increase in y_4 (instead of the other way around), so that we are, in fact, repeating the situation just considered of increasing x_5 and adjusting the other "basic" variables, except here y_4 is in the basic set instead of x_4. It follows, therefore, as before, that an increase in x_5 decreases x^TCx as long as y_5 remains negative in value in the adjustment of the basic solution by the increase of x_5.

Case $\lambda_4 = 0$: On the other hand, if $\lambda_4 = 0$ in (17), then we must set $\lambda_i = \lambda_i'$ in (20) because the representation of $\bar{P}_4$ is the same, whether in terms of B or B'; hence, $\lambda_5' = 0$. In this case $\lambda C_4\lambda^T = -\lambda_4 = 0$ and therefore, because C is positive semi-definite, $C_4\lambda = 0$. In addition, $\lambda = 0$ must hold because $\lambda \neq 0$ implies a dependence of the first four columns of (18) and (19) which is *impossible* because then the square array of coefficients of (18) and (19), and in turn B, would be singular. Setting $\lambda_1', \ldots, \lambda_5' = 0$ in (20) and (21), we observe (and this holds in general) that $\bar{P}_4$ is dependent only on the columns of π_i and of y_j, and therefore the values of $x_j' - \theta\lambda_j'$ remain unchanged in (21) with increasing values of $y_4 = \theta$.

Because the y_j are not sign restricted, y_4 can be increased until y_5 is dropped out of the basic set at value zero (since all x_j values are unaffected). Hence, in this shift of basis there is no change in the value of x^TCx; however, the introduction of y_4 into the basic set and the dropping of y_5 give rise to a new basic set that satisfies the complementarity property. We may thus reapply Theorem 2 to reduce x^TCx.

The Quadratic Algorithm.

Step I. Initiate: Let x^o be a basic feasible solution for $Ax = b, x \geq 0$, with basic variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$; choose for the initial set of basic variables x_j for the enlarged problem these x_{j_i}, the complements y_j of the non-basic x_j, and the set of π_i.

Step II. For the values of y_j^o of the basic solution, determine Min $y_j^o = y_s^o$. If $y_s^o \geq 0$, *terminate; the solution is optimal.* If $y_s^o < 0$ introduce into the basic set x_s; if y_s drops from the basic set, repeat Step II. Go to Step III if x_r drops.

Step III. Introduce y_r into basic set. If y_s drops, return to Step II; otherwise, if some x_{r_2} drops, repeat Step III with r_2 playing the role of r.

THEOREM 4. *The iterative process is finite.*

EXERCISE: Prove Theorem 4.

EXERCISE: Extend the results of this section to cover the case of a convex objective form consisting of mixed quadratic and linear terms.

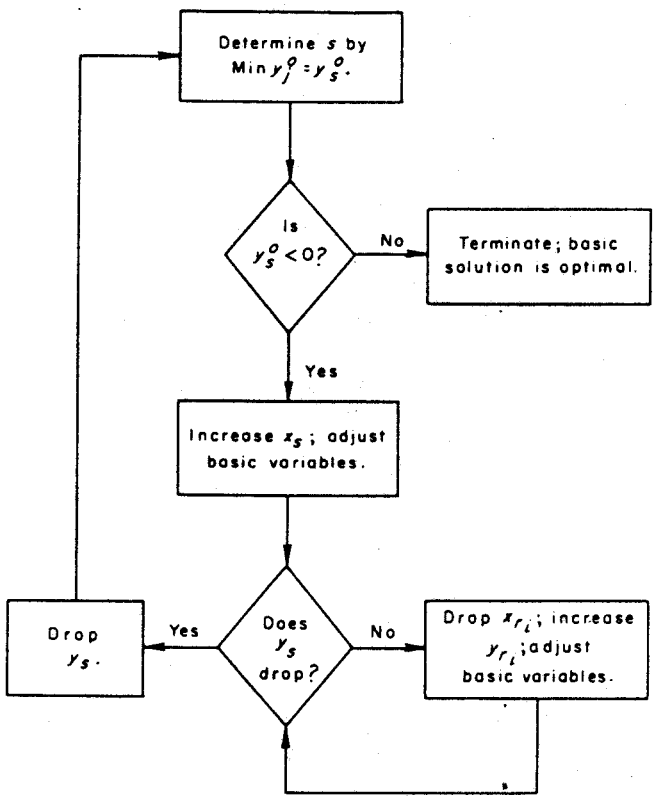


Figure 24-4-I. Flow chart of the quadratic programming algorithm.

24-5. PROBLEMS

1. Show that the bounded variable problem of § 24-3-(12), (13), (14) is equivalent to the original convex-separable problem.
2. Show for § 24-3-(15), (16), (17) that, if Δ_{pq} is in the basis, Δ_{tq} is not, where $t \neq p$. Show under the simplex criterion that Δ_{tq} is not a candidate to enter the basic set if Δ_{pq} is in the basic set.
3. Solve the distribution problem below by both methods of § 24-3. Which is easier?

					Total
	x_{11}	x_{12}	x_{13}	x_{14}	9
	x_{21}	x_{22}	x_{23}	x_{24}	9
	x_{31}	x_{32}	x_{33}	x_{34}	9
Total	5	10	4	8	

where the cost per unit shipped from (i) to (j) has incrementally increasing costs per total amount shipped as follows:

	Cost per Unit
1st unit	1
2nd unit	2
3rd unit	3
4th unit	4
.....	
9th unit	9

4. Show that the total cost $\phi(x)$ of shipping $x = x_{ij}$ units from i to j in Problem 3 is $\phi(x) = x(x + 1)/2$ if x is an integer. Show that, if we set $\phi(x) = x(x + 1)/2$ for fractional values of x, the optimal solution has fractional values for x_{ij}.
5. Show how to get a solution to a distribution problem with integer-valued variables when the objective form is convex-separable. To be precise, find integers $x_{ij} \geq 0$, Min z satisfying

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1, 2, \ldots, n)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \phi_{ij}(x_{ij}) = z \text{ (Min)}$$

where a_i and b_j are integers, and $\phi_{ij}(x_{ij})$ are convex functions (for example, $\phi_{ij}(x_{ij}) = x_{ij}^2$ or $x_{ij}(x_{ij} + 1)/2$).

REFERENCES

Arrow, Hurwicz, and Uzawa, 1958-1
Barankin and Dorfman, 1958-1
Beale, 1955-2, 1959-2
Beckenbach, 1948-1
Charnes and Lemke, 1954-1
Dantzig, 1955-1, 1956-2, 1960-3
Dorfman, 1951-1
Dorn, 1961-1
Duffin, 1956-1
Forsythe, 1955-1
Hartley and Loftsgard, 1958-1
Karlin, 1959-1
Kelley, 1960-1
Kuhn and Tucker, 1950-2
Markowitz, 1952-1, 1956-1, 1959-1
Miller, 1960-1
Rosen, 1960-1
Shettz, 1959-1
Simon, 1956-1
Slater, 1950-1
Thrall, 1955-1
Tucker, 1957-1
Uzawa, 1958-1
Vazsonyi, 1955-1
White, Johnson, and Dantzig, 1958-1
Williams, A. C., 1960-1
Wolfe, 1957-1, 1959-1, 1960-1
Zoutendijk, 1959-1, 1960-1

CHAPTER 25

UNCERTAINTY

In the final analysis, most applied programming problems involve uncertainty in either the technology matrix or the constant terms. The techniques discussed so far, however, do not take into account the uncertain nature of the coefficients of the program. In the period 1955–60, various individuals have tried to extend linear programming methods to deal with the problem of optimizing in some sense an objective function, subject to constraints whose constants are subject to random variation [Dantzig, 1955-1; Ferguson and Dantzig, 1956-1]. One of the basic difficulties is that the problem is capable of many formulations, with only fragmentary results for each of the formulations [Madansky, 1959-1]. In this chapter we shall examine some of the solved problems in this area, cautioning the reader that the treatment is incomplete and that much research remains to be done.

For the concepts of probability and statistical theory used in this chapter, the reader is referred to [Feller, 1957-1].

25-1. SCHEDULING TO MEET VARIABLE COST

By way of background let us recall that there are in common use two essentially different types of scheduling applications—one designed for the short run and one for the long run. For the latter the effect of probabilistic or chance events is reduced to a minimum by the usual technique of providing plenty of *fat* in the system. For example, consumption rates, attrition rates, and wear-out rates are all planned on the high side; times to ship, times to travel, and times to produce are always made well above actual needs. Indeed, the entire system is put together with plenty of fat with the hope that it will be a *shock absorber*, which will permit the general objective and timing of the plan to be executed in spite of unforeseen events. More precisely, the fat is introduced into the system so that, whatever be the random unforeseen event, the activities chosen will still be feasible. Activities which satisfy this proviso are called *permanently feasible*.

The effect of chance events is also reduced to a minimum by the technique of providing plenty of *slack* in the system. By this we mean that scarce exogenous inputs to the system are estimated on the low side, so that it is highly unlikely for the set of chosen activities to be infeasible because of shortages.

In the general course of things, long-range plans are revised frequently

because the stochastic elements of the problem have a nasty way of intruding. For this reason the chief contribution, if any, of the long-range plan, is to effect an immediate decision—such as the appropriation of funds or the initiation of an important development contract.

For short-run scheduling, many of the *slack* and *fat* techniques of its long-range brother are employed. The principal differences are attention to detail and the short time-horizon. As long as *capabilities* are well above *requirements* (or demands) or if the *demands can be shifted in time*, this approach presents no problems, since it is feasible to implement the schedule in detail. However, where there are shortages, the projected plan based on such techniques may lead to actions far from optimal, whereas these new methods, where applicable, may result in considerable savings.

Minimum Expected Costs.

A nutrition expert wishes to advise his followers on a minimum-cost diet without prior knowledge of the prices [Stigler, 1945-1]. Because prices of food (except for general inflationary trends) are likely to show variability due to weather conditions, supply, etc., he wants to assume a distribution of possible prices rather than a fixed price for each food, and determine a diet that meets specified nutritional requirements and minimizes the expected total cost. Let x_j be the quantity in pounds of jth food purchased, p_j its price; let a_{ij} be the quantity of the ith nutrient (e.g., Vitamin A) contained in a unit quantity of the jth food; and let b_i be the minimum quantity required by an individual for good health. Then the x_j must be chosen so that

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \qquad x_j \geq 0 \qquad (i = 1, 2, \ldots, m) \tag{1}$$

The cost of the diet will be

$$C = \sum_{j=1}^{n} p_j x_j \tag{2}$$

However, the x_j are chosen *before* the prices are known, so that for the *fixed* selected values of x_j the total cost C becomes a random variable that is a weighted sum of random variables p_j. We shall denote the *expected value* of any variable, say u, by $\mathscr{E}u$ or $\mathscr{E}(u)$. Accordingly, the expected cost E of such a diet is clearly

$$E = \mathscr{E}C = \sum_{j} \bar{p}_j x_j \tag{3}$$

where $\bar{p}_j$ is the expected price of food j. Since the $\bar{p}_j$ are assumed known in advance, the best choices of x_j are those which satisfy (1), and minimize E. We have, therefore, in general

THEOREM 1: *If the unit costs p_j in (2) are randomly distributed independently of the x_j, then the minimum expected total cost solution is obtained by finding $x_j, \geq 0$ satisfying (1) and minimizing C with p_j replaced by $\bar{p}_j = \mathscr{E}(p_j)$.*

On the other hand if each p_j depends on x_j, we write $p_j x_j = \phi_j(x_j, p_j)$; remembering the expectation of a sum is equal to the sum of the expectations:

$$E = \mathscr{E}C = \sum_{j=1}^{n} \mathscr{E}\phi_j(x_j, p_j) = \sum_{j=1}^{n} \phi_j(x_j) \tag{4}$$

where $\phi(x_j)$ is some (not necessarily linear) function of x_j. Special methods for minimizing E for the case where $\phi(x_j, p_j)$ is convex in the x_j were given in § 24-3.

The following example illustrates a case where the expected cost is not linear in the x_j. Let x_j be the quantity of the jth good manufactured and let the constraints be manufacturing capacity restraints. Assume that costs are threefold: there are a non-random manufacturing cost c_{0j}, a non-random stockage cost c_{1j} for those items not bought by a random demand, and a non-random shortage cost c_{2j}, if the random demand exceeds the supply; then

$$E = \sum_j c_{0j}x_j + \sum_j c_{1j}\mathscr{E}(x_j - d_j | x_j > d_j) + \sum_j c_{2j}\mathscr{E}(d_j - x_j | x_j < d_j)$$

where the symbol $\mathscr{E}(A|B)$ is read *the expected value of A given that B is true.* This problem, though set up as a one-stage problem with the uncertainty appearing only in the objective function, could be set up as a two-stage problem with the uncertainty in the constraints instead. As such, it becomes a problem of the type described in § 25-2 and treated more fully in § 25-3 and in Elmaghraby's paper [1960-1].

Minimum Variance for Fixed Expected Costs.

Referring again to our nutrition problem, it may be desirable to control the variance V of the expected costs. Thus, a solution to (1) and (3) that results in a low expected cost, but one that has great variability, may not be as desirable as one which shows greater cost stability. This certainly was the case considered by H. Markowitz in his analysis of "portfolio" selections [Markowitz, 1952-1]. Stockbrokers often advise their customers to buy a variety of stocks, some of which they regard as very safe, low-yield stocks, while others (like oil exploration stocks) may have a high average yield but show great variation (depending, say, on whether or not oil is discovered). The objective of such an analysis is to produce for each of a variety of expected profit levels $-E$, portfolio selections that minimize variance (Fig. 25-1-I).

It is then left to the customer to decide what combination of "yield level" and "risk level" (Min V) he wants.

Let us now assume that the price is *independent* of x_j and that we know the *variance* (or standard error squared) σ_j^2 of p_j, the price of an individual item, and the *covariance* σ_{jk} between two prices p_j and p_k. Let $\sigma_{jk} = \sigma_j\sigma_k\rho_{jk}$ where the *correlation coefficient* between the two prices ρ_{jk} satisfies $-1 \leq \rho_{jk} \leq +1$. Since all x_j units are purchased at the *same* cost p_j, the variance of

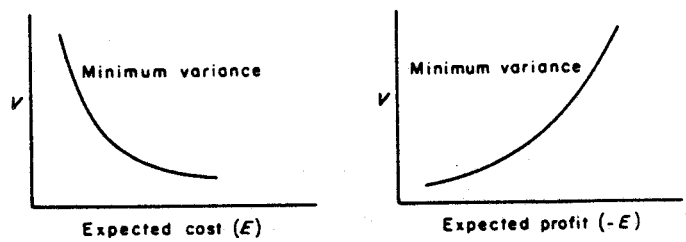


Figure 25-1-I. Minimum variance is a convex function of expected cost or profit.

x_jp_j is given by $x_j^2\sigma_j^2$ and the covariance between x_jp_j and x_kp_k is given by $x_jx_k\sigma_{jk}$. From this it follows that the variance V of E is given by the *quadratic expression*

$$V = \mathscr{E}[C - E]^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} x_jx_k\sigma_{jk} \qquad (\sigma_{jj} = \sigma_j^2) \tag{5}$$

If, in particular, food prices are highly correlated so that for all practical purposes $\rho_{jk} \doteq 1$ and $\sigma_{jk} \doteq \sigma_j\sigma_k$, then in this case it would be advisable to replace $V^{\frac{1}{2}}$ by the *linear expression* bounding it given in (6),

$$V^{\frac{1}{2}} \leq x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \tag{6}$$

where equality holds if all correlation coefficients $\rho_{jk} = 1$. On the other hand, if the prices are independent so that $\rho_{jk} = 0$, then

$$V = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + \cdots + x_n^2\sigma_n^2 \qquad (\rho_{jk} = 0) \tag{7}$$

We now address ourselves to the general problem of determining the solution to (1), (3), and (5) that minimizes V for fixed E.

Case I: $V^{\frac{1}{2}}$ is Linear. If (6) holds, then we solve the problem of determining Min $V^{\frac{1}{2}}$ subject to $E \leq E^*$, where E^* is an upper bound which we wish to impose on expected costs. Since the restriction is linear, we may study the effect on V of varying E^* as a parameter. In this case we have a standard parametric programming problem (see § 11-3).

Case II: V is a Sum of Squares. If (7) holds, then V is convex-separable and the convex functions x_j^2 may be approximated by broken-line functions. This reduces the problem again to a standard parametric programming problem (combined with an upper-bounding technique, if desired). Here

we are varying one of the constant terms. See § 11-3 and § 24-3. However, the following alternative is recommended. Replace V by

$$\bar{V} = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + \ldots + x_n^2\sigma_n^2 + k \cdot E^* \qquad (E \leq E^*) \tag{8}$$

Again using a broken-line fit for x_j^2, the problem is first solved using $k = 0$; it is easy to see that this corresponds to setting $E^* = +\infty$. If now k is allowed to increase gradually, a critical value $k = k_1$ would be determined for which the solution would no longer be optimal. This will result in one or more basis changes until the solution is again optimal. After this, k can again be increased until a new critical value $k = k_2$ is obtained, etc.

EXERCISE: Prove the latter procedure is the first parametric linear programming method in disguise. Develop other alternatives based on § 24-3. Why is E^* instead of E used above?

It is probably worth while, however, instead of choosing to increase k gradually, to choose a number of discrete values in advance. The solution will generate a number of different pairs of E^* and $\bar{V}$ values that can be used to spot points on the curve. A plot of E^* against k can be used to determine what new k values to use if a better spacing of E^* values is desired.

Case III: V Is General. Since V is positive (semi-) definite, it can, by a suitable linear transformation, be reduced to Case II; hence this procedure can be used in general to effect a solution to the problem. Alternatively, a general quadratic programming procedure such as that developed by Markowitz [1956-1] or by Wolfe [1959-1] may be used, combined with a parametric programming method for the right-hand side (see § 11-3). However, an analogue of (8) using an E^* term weighted by a parameter k is recommended.

EXERCISE: Combine methods for solving quadratic programs and parametric programs as given in § 24-4 and § 11-3 to solve the minimum-variance problem.

25-2. SCHEDULING TO MEET AN UNCERTAIN DEMAND

Let us consider a simple case: A factory has 100 items on hand which may be shipped to an outlet at the cost of \$1 apiece to meet an uncertain demand d. In the event that the demand should exceed the supply, it is necessary to meet the unsatisfied demand by purchases on the local market at \$2 apiece. The equations that the system must satisfy are

$$\begin{aligned} 100 &= x + y \\ d &= x \quad + v - s \\ C &= x \quad + 2v \end{aligned} \qquad (x, y, v, s \geq 0) \tag{1}$$

where $\begin{cases} x = \text{number shipped from the factory,} \\ y = \text{number stored at factory;} \end{cases}$

$\begin{cases} v = \text{number purchased on open market,} \\ s = \text{excess of supply over demand;} \end{cases}$

d = unknown demand uniformly distributed between 70 and 80;

C = total costs.

We view the shipping and purchasing as part of a two-stage process. In the first stage a decision is made consistent with the initial inventory of amounts to ship. In the second stage, the unknown demands occur.

The simple example above belongs to a general class of two-stage problems that have the following structure. In the first stage, $x_j \geq 0$, $u_k \geq 0$ are determined such that

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, m) \tag{2}$$

$$\sum_{j=1}^{n} \bar{a}_{kj}x_j = u_k \qquad (k = 1, 2, \ldots, p)$$

where the initial inventories b_i for $i = 1, 2, \ldots, m$ are known; the $x_j \geq 0$ represent decisions in the first stage resulting in specified quantities u_k being made available for the second stage. For the second stage, the quantities v_k and s_k are determined such that

$$d_k = u_k + v_k - s_k \qquad (k = 1, 2, \ldots, p) \tag{3}$$

where d_k is the *unknown demand* whose probability distribution is known, v_k is the shortage of supply, and s_k is the excess of supply over demand. Assuming for convenience no purchases on the open market in case of shortage, the total cost is of the form

$$C = \sum_{j=1}^{n} c_j x_j - \sum_{k=1}^{p} f_k(d_k - v_k) \text{ where } v_k = 0 \text{ if } u_k \geq d_k \tag{4}$$

where c_j is the cost of performing the jth activity and $f_k \geq 0$ is the revenue from satisfying one unit of demand. Thus, it always pays to sell as much of the amount, u_k, supplied as possible so that $(d_k - v_k) = \text{Min } (u_k, d_k)$ and

$$C = \sum_{j=1}^{n} c_j x_j - \sum_{k=1}^{p} f_k \text{ Min } (u_k, d_k) \tag{5}$$

It is clear that, for any particular choice of x_j and u_k consistent with the first-stage equations, the value of C depends on this choice and on the

unknown demand. Hence, for fixed choice of x_j and u_j the expected value of C is given by

$$\mathscr{E}C = \sum_{j=1}^{n} c_j x_j - \sum_{k=1}^{p} f_k \mathscr{E} \operatorname{Min}(u_k, d_k) \tag{6}$$

Since the expected value of $\operatorname{Min}(u_k, d_k)$ depends on u_k, let us denote this function (of u_k) by

$$\phi_k(u_k) = \mathscr{E} \operatorname{Min}(u_k, d_k) \tag{7}$$

Given the demand distribution of d_k, the function $\phi_k(u_k)$ is easily calculated. Suppose $d_k = 1, 2$ with probability $\frac{1}{2}, \frac{1}{2}$, respectively, then $\phi_k(u_k)$ is easily determined.

(8)

Amount Supplied (u_k)	Amount Sold $d_k - v_k = \operatorname{Min}(u_k, d_k)$		Expected Value[1] (revenue) $\phi_k(u_k)$ (3)
	where $d_k = 1$ (1)	where $d_k = 2$ (2)	
$u_k = 0$	0	0	0
$0 < u_k < 1$	u_k	u_k	u_k
$u_k = 1$	1	1	1
$1 < u_k < 2$	1	u_k	$1/2 + u_k/2$
$u_k = 2$	1	2	3/2
$2 < u_k$	1	2	3/2

[1] Entries in column (3) are formed by multiplying the entries in column (1) by $p_1 = \frac{1}{2}$, those in column (2) by $p_2 = \frac{1}{2}$, and summing.

In general, to compute $\phi(u) = \mathscr{E} \operatorname{Min}(u, d)$ where d can take on successive values $e_1 \leq e_2, \ldots, \leq e_t$ with probability $p_1, p_2, \ldots, p_t$ and $\sum_1^t p_i = 1$, we note

$$\begin{aligned} \operatorname{Min}(u, d) &= u \qquad \text{if } d > u \\ \operatorname{Min}(u, d) &= d \qquad \text{if } d \leq u \end{aligned} \tag{9}$$

Suppose $e_{r-1} \leq u < e_r$, then

$$\phi(u) = u \operatorname{Prob}(d > u) + \operatorname{Prob}(d \leq u)\mathscr{E}(d|d \leq u)$$

whence

$$(10)\quad \phi(u) = u(p_r + p_{r+1} + \cdots + p_t) + (e_1p_1 + e_2p_2 + \cdots + e_{r-1}p_{r-1})$$
$$= u(1 - p_1 - p_2 - \cdots - p_{r-1}) + (e_1p_1 + e_2p_2 + \cdots + e_{r-1}p_{r-1})$$

where we define, for $r = 1$ in (10), $e_0 = 0$ and $p_0 = 0$. Thus, $\phi(u)$ is a broken-line function starting at the origin with initial slope $s_1 = 1$; at $u = e_1$, the *slope* decreases to $s_2 = 1 - p_1$, etc. (see Fig. 25-2-I).

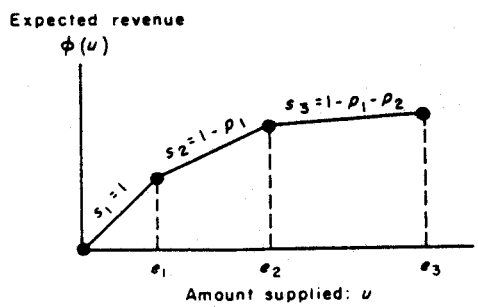


Figure 25-2-I. Maximum expected revenue is a concave function of the amount supplied.

Referring to our section on convex separable functions, § 24-3, it should be noted that $-\phi(u)$ is *convex* because of the decreasing slopes in Fig. 25-2-I. Indeed, $s_i - s_{i+1} = p_i > 0$. We have therefore shown a result of H. Scarf (informal demonstration to the author).

THEOREM 1: *The total expected costs under uncertain demand is a convex separable function*

$$(11)\qquad \mathscr{E}C = \sum_{j=1}^{n} c_j x_j' - \sum_{k=1}^{p} f_k \phi_k(u_k)$$

where $\phi_k(u_k)$ is a broken-line function whose slope between two successive demands $d_k = e_{r,k}$, and $e_{r+1,k}$ is equal to the probability of exceeding the demand $e_{r,k}$.

THEOREM 2: *To minimize expected costs, determine $x_j \geq 0$, $u_k \geq 0$, satisfying* (2), *such that the convex separable function* (11) *is minimized.*

Continuous Demand Distribution.

To illustrate an example involving a continuous distribution, consider the small two-stage case described earlier in (1). The costs are given by

$$(12)\qquad C = x_1 + 2\,\mathrm{Max}\,(0, d - x_1)$$

where d is uniformly distributed between 70 and 80. The determination of

expected costs requires the evaluation of $\mathscr{E}$ Max $[0, d - x_1]$ as a function of x_1:

$$\mathscr{E}\,\mathrm{Max}\,(0, d - x_1) = \begin{cases} 75 - x_1 & \text{if } 0 \leq x_1 \leq 70 \\ \frac{1}{10}\int_{x_1}^{80} (t - x_1)dt & \text{if } 70 \leq x_1 \leq 80 \\ 0 & \text{if } 80 \leq x_1 \end{cases} \tag{13}$$

whence from (12)

$$\mathscr{E}C = \begin{cases} 150 - x_1 & \text{if } 0 \leq x_1 \leq 70 \\ x_1 + (80 - x_1)^2/10 & \text{if } 70 \leq x_1 \leq 80 \\ x_1 & \text{if } 80 \leq x_1 \end{cases} \tag{14}$$

It then follows that $\mathrm{Min}_{x_1} \mathscr{E}C = 77.5$ occurs at $x_1 = 75$, i.e., it is best to ship the expected demand in this case.

EXERCISE: Modify the above problem to show that it is not always best to ship the expected demand.

When is it best to ship the expected demand? The following result, due to Madansky [1960-1] (see also Reiter, [1957-1]), is a generalization of earlier results [Theil, 1957-1; Simon, 1956-1], and gives a sufficient condition for the solution of the linear program when the demand is replaced by its expected value also to solve the problem of scheduling to meet uncertain demand. If C can be expressed as $C_1(d, x) + C_2(d)$ where $C_1(d, x)$ is a linear function of d for each x and $C_2(d)$ involves only d and not x, then the x which solves the linear program with d replaced by $\mathscr{E}(d)$ also solves the uncertainty problem.

EXERCISE: Prove this result. Does the example above satisfy the sufficiency condition?

25-3. ON MULTI-STAGE PROBLEMS[2]

The Two-stage Problem with General Linear Structure.

We shall prove a general theorem on convexity for the two-stage problem that forms the inductive step for the multi-stage problem. We shall say a few words about the significance of this convexity later on. The assumed structure of the general two-stage model (of which § 25-2-(2) is a special case) is

$$\begin{aligned} b_1 &= A_{11}X_1 \qquad (X_1, X_2 \geq 0) \\ b_2 &= A_{21}X_1 + A_{22}X_2 \\ C &= \phi(X_1, X_2|E_2) \end{aligned} \tag{1}$$

where A_{ij} are known matrices; b_1 a known vector of initial inventories; b_2 an unknown vector whose components depend on a set of parameters E_2; and C, the cost, depends on X_1, X_2, and E_2.

[2] The material for this section is based on [Dantzig, 1955-1].

We assume the following sequence of events:

(1) $X_1 \geq 0$ is chosen to satisfy $b_1 = A_{11}X_1$. We denote by Ω_1 the set of possible vectors X_1. It is assumed that Ω_1 is nonempty.

(2) E_2 is drawn randomly as a point from a multidimensional sample space S_2 with known probability distribution. E_2 determines b_2.

(3) $X_2 \geq 0$ is chosen to satisfy $b_2 = A_{21}X_1 + A_{22}X_2$. We denote by $\Omega_2 = \Omega_2(X_1|E_2)$ the set of possible vectors X_2. It is assumed that Ω_2 is nonempty, that is to say, there exists at least one such vector whatever be the values of X_1 and E_2 chosen above.[3]

The problem is to select X_1 and later X_2 so that the expected value of C is minimum.

Observing that the nonnegative weighted sum of convex functions is convex and that an expected value is such a sum (or more generally a Stieltjes integral), the following useful lemma results:

LEMMA 1: *If $\phi(X|E)$ is convex in $X \in \Omega$ whatever E be chosen from a sample space S with known probability distribution, then the function $\phi(X) = \underset{E}{\mathscr{E}}\phi(X|E)$ is convex.*

THEOREM 1: *If $\phi(X_1, X_2|E_2)$ is a convex function in $X_1 \in \Omega_1$, $X_2 \in \Omega_2(X_1|E_2)$, then the function*

$$\phi_0(X_1) = \underset{E_2}{\mathscr{E}}\,[\underset{X_2}{\mathrm{Inf}}\,\phi(X_1, X_2|E_2)] \tag{2}$$

is convex and has the property that $X_1 = X_1^*$ solves the uncertainty problem if

$$\phi_0(X_1^*) = \mathrm{Min}\,\phi_0(X_1)$$

The expectation ($\mathscr{E}$) is taken with respect to the distribution of E_2 and the greatest lower bound (Inf)[4] is taken with respect to all $X_2 \in \Omega_2(X_1|E_2)$.

PROOF:[5] In order to minimize $\mathscr{E}\phi(X_1, X_2|E_2)$, it is clear that, once X_1 has been selected and E_2 determined by chance, X_2 must be selected so that $\phi(X_1, X_2|E_2)$ is minimized for fixed X_1 and E_2. Thus, the cost for *given* X_1 and E_2 is given by

$$\phi_1(X_1|E_2) = \underset{X_2 \in \Omega_2}{\mathrm{Inf}}\ \phi(X_1, X_2|E_2) \tag{3}$$

The expected cost for a *given* X_1 is then simply the expected value of $\phi_1(X_1|E_2)$ and this we denote by $\phi_0(X_1)$. The optimal choice of X_1 to minimize expected cost is thus reduced to choosing X_1 so as to minimize $\phi_0(X_1)$.

[3] This assumption can be interpreted as assuring either that there is enough *fat* in the system or that there are enough *slack* variables in the system so that the set of permanently feasible (X_1, X_2) is not null.

[4] The greatest lower bound instead of minimum is used to avoid the possibility that the minimum value is not attained for any admissible point $X_2 \in \Omega_2$ or $X_1 \in \Omega_1$. In the case where the latter occurs, it should be understood that while there exists no X_i where the minimum is attained, there exists X_i for which values as close to minimum as desired are attained.

[5] This proof is along lines suggested by I. Glicksberg.

There remains only to establish the convexity property. We shall show first that $\phi_1(X_1|E_2)$ for bounded ϕ_1 is convex for X_1 in Ω_1. If true, then applying the lemma, the result that $\phi_0(X_1)$ is convex readily follows. Let us suppose that $\phi_1(X_1|E_2)$ is not convex; then there exist three points X_1', X_1'', X_1''' in Ω_1, $X_1''' = \lambda X_1' + \mu X_1''$, $(\lambda + \mu = 1, 0 \leq \lambda \leq 1)$ that violate the condition for convexity, i.e.,

$$\lambda\phi_1(X_1'|E_2) + \mu\phi_1(X_1''|E_2) < \phi_1(X_1'''|E_2) \tag{4}$$

or

$$\lambda\phi_1(X_1'|E_2) + \mu\phi_1(X_1''|E_2) = \phi_1(X_1'''|E_2) - \varepsilon_0 \qquad (\varepsilon_0 > 0) \tag{5}$$

For any $\varepsilon_0 > 0$, however, there exist X_2' and X_2'' such that

$$\begin{aligned} \phi_1(X_1'|E_2) &= \phi(X_1', X_2'|E_2) - \varepsilon_1 \qquad (0 \leq \varepsilon_1 < \varepsilon_0) \\ \phi_1(X_1''|E_2) &= \phi(X_1'', X_2''|E_2) - \varepsilon_2 \qquad (0 \leq \varepsilon_2 < \varepsilon_0) \end{aligned} \tag{6}$$

Setting $X_2''' = \lambda X_2' + \mu X_2''$ we note because of the assumed linearity of the model (1) that $(\lambda X_2' + \mu X_2'') \in \Omega_2(X_1'''|E_2)$ and hence by convexity of ϕ

$$\lambda\phi(X_1', X_2'|E_2) + \mu\phi(X_1'', X_2''|E_2) \geq \phi(X_1''', X_2'''|E_2) \tag{7}$$

whence by (6)

$$\lambda\phi_1(X_1'|E_2) + \mu\phi_1(X_1''|E_2) \geq \phi(X_1''', X_2'''|E_2) - \lambda\varepsilon_1 - \mu\varepsilon_2 \tag{8}$$

and by (5)

$$\phi_1(X_1'''|E_2) > \phi(X_1''', X_2'''|E_2) - \lambda\varepsilon_1 - \mu\varepsilon_2 + \varepsilon_0 \tag{9}$$

where $0 < \lambda\varepsilon_1 + \mu\varepsilon_2 < \varepsilon_0$, which contradicts the assumption that

$$\phi_1(X_1'''|E_2) = \operatorname*{Inf}_{X_2 \in \Omega_2} \phi(X_1''', X_2|E_2)$$

The proof for unbounded ϕ_1 is omitted; see [Dantzig, 1955-1]. For an illustration of the use of this theorem in solving a linear program under uncertainty, see Chapter 28.

The Multi-stage Problem with General Linear Structure.

The structure assumed is

$$\begin{aligned} b_1 &= A_{11} X_1 \\ b_2 &= A_{21} X_1 + A_{22} X_2 \\ b_3 &= A_{31} X_1 + A_{32} X_2 + A_{33} X_3 \\ b_4 &= A_{41} X_1 + A_{42} X_2 + A_{43} X_3 + A_{44} X_4 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_m &= A_{m1}X_1 + A_{m2}X_2 + A_{m3}X_3 + \cdots \quad \cdots + A_{mm}X_m \\ C &= \phi(X_1, X_2, \ldots, X_m|E_2, E_3, \ldots, E_m) \end{aligned} \tag{10}$$

where b_1 is a known vector; b_i is a chance vector ($i = 2, \ldots, m$) whose components are functions of a point E_i drawn from a known multidimensional distribution; A_{ij} are known matrices. The sequence of decisions is as follows: X_1, the vector of nonnegative activity levels in the first stage, is chosen so as to satisfy the first-stage restrictions $b_1 = A_{11}X_1$; the values of components of b_2 are chosen by chance by determining E_2; X_2 is chosen to satisfy the second-stage restrictions $b_2 = A_{21}X_1 + A_{22}X_2$, etc., iteratively for the third and higher stages. It is further assumed that:

(a) The components of X_j are nonnegative.
(b) There exists at least one X_j satisfying the jth-stage restraints, whatever be the choice of $X_1, X_2, \ldots, X_{j-1}$ satisfying the earlier restraints or the outcomes $b_1, b_2, \ldots, b_m$.
(c) The total cost C is a convex function in $X_1, \ldots, X_m$ which depends on the values of the sample points $E_2, E_3, \ldots, E_m$.

THEOREM 2: *An equivalent $(m - 1)$-stage programming problem with a convex payoff function can be obtained by dropping the mth-stage restrictions and replacing the convex cost function ϕ by*

$$(11) \quad \phi_{m-1}(X_1, X_2, \ldots, X_{m-1}|E_2, \ldots, E_{m-1}) = \mathscr{E}_{E_m} \operatorname*{Inf}_{X_m \in \Omega_m} \phi(X_1, X_2, \ldots, X_m|E_2, \ldots, E_m)$$

where Ω_m is the set of possible X_m that satisfy the mth-stage restrictions.

Since the proof of the above theorem is identical to the two-stage case, no details will be given. The fact that a cost function for the $(m - 1)^{\text{st}}$ stage can be obtained from the mth stage is simply a consequence of the fact that optimal behavior for the mth stage is well defined, that is, given any state, $(X_1, X_2, \ldots, X_{m-1})$ at the beginning of this stage, the best possible actions can be determined and the minimum expected cost evaluated. This is a standard technique in "dynamic programming." The reader interested in methods built around this approach is referred to R. Bellman's book on dynamic programming [1957-1].

While the *existence* of convex functions has been demonstrated that permit reduction of an m-stage problem to equivalent $m - 1, m - 2, \ldots,$ one-stage problems, it appears unlikely that such functions can be computed except in very simple cases. The convexity theorem was demonstrated not as a solution to an m-stage problem but only in the hope that it will spur the development of an efficient computational theory for such models. It should be remembered that any procedure that yields a local optimum will be a true (global) optimum if the function is convex. This is important because multidimensional problems in which non-convex functions are defined over non-convex domains lead, as a rule, to local optima and an

almost hopeless task, in general, of exploring other parts of the domain for the other extremes [Dantzig and Madansky, 1960-1]. See § 26-3.

The General Two-stage Case.

When the set of possibilities for the chance vector b_2 is $b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(k)}$ with probabilities $p_1, p_2, \ldots, p_k$, $(\Sigma p_i = 1)$, it is not difficult to obtain a direct linear programming solution for small k, say $k = 3$. Since this type of structure is very special, it appears likely that techniques can be developed to handle large k, which could be used to approximate the solution when b_2 has a general distribution. For $k = 3$, the problem is equivalent to determining vectors X_1 and vectors $X_2^{(1)}, X_2^{(2)}, X_2^{(3)}$ such that

$$
\begin{aligned}
(12) \qquad b_1 &= A_{11}X_1 \\
b_2^{(1)} &= A_{21}X_1 + A_{22}X_2^{(1)} \\
b_2^{(2)} &= A_{21}X_1 \qquad\qquad + A_{22}X_2^{(2)} \\
b_2^{(3)} &= A_{21}X_1 \qquad\qquad\qquad\qquad + A_{22}X_2^{(3)} \\
\mathscr{E}C &= \gamma_1 X_1 + p_1\gamma_2 X_2^{(1)} + p_2\gamma_2 X_2^{(2)} + p_3\gamma_2 X_2^{(3)} = z \text{ (Min)}
\end{aligned}
$$

where for simplicity we have assumed a linear objective function. Thus a general two-stage linear program with an uncertain constant vector for the second stage reduces to a linear program of a special structure like (12).

EXERCISE: Develop an algorithm for solving systems like (12) for large k, by dualizing and then using the Decomposition Principle (Chapter 23). Take advantage of the repetitive appearance of A_{22} and γ_2.

25-4. PROBLEMS

1. Prove that the quadratic expression given in § 25-1-(5) is positive semi-definite, i.e., $V \geq 0$.
2. Prove that any quadratic expression Q can by linear transformation be reduced to sum and difference of squares of the new variables. Show that Q cannot be positive definite if it involves differences of squares. (Review.)
3. Show that if Q is positive (semi-) definite, Q is a convex function. (Review.)
4. Show for V and x satisfying § 25-1-(1, 3, 5) and $E \leq E^*$, where E^* is a parameter, that Min V is a monotonically decreasing function of E^* if we allow E^* to take on increasing values.
5. Show above that if Q is any convex function of x, then Min Q is a convex function of E (where E and x satisfy § 25-1-(1, 3).
6. Solve the same problem as § 25-2-(1) using the discrete distribution $d = 70, 71, 72, \ldots, 80$ with probability $1/11$ each.

7. (a) Solve the transportation problem

				Available:
x_{11} 2	x_{12} 3	x_{13} 4	x_{14} 1	3
x_{21} 7	x_{22} 2	x_{23} 5	x_{24} 1	2
x_{31} 4	x_{32} 3	x_{33} 2	x_{34} 2	5

Required: d_1 d_2 d_3 d_4

where the demands are

$d_1 = 3$ with probability 1
$d_2 = 3$,, ,, ,
$d_3 = 2$,, ,, ,,
$d_4 = 2$,, ,, ,,

(b) Solve, if

$d_1 = 2, 3, 4$ with probability $\frac{1}{3}$ each
$d_2 = 2, 3, 4$,, ,, ,, ,,
$d_3 = 1, 2, 3$,, ,, ,, ,,
$d_4 = 1, 2, 3$,, ,, ,, ,,

Compare this solution with that of 7(a) which uses expected demands in place of the variable demands 7(b).

8. Consider a linear program where all coefficients are subject to uncertainty. Suppose (for $i = 1, 2, \ldots, m$) that

$$\varepsilon_i(x) = \sum_{j=1}^{n} a_{ij}x_j + a_0 \leq 0 \qquad (x_j \geq 0)$$

$$\varepsilon_0(x) = \sum_{j=1}^{n} a_{0j}x_j = z \text{ (Min)}$$

is desired but, unfortunately, all x_j must be selected *prior* to a random choice of the coefficients a_{ij} whose distributions are, however, known. Denote by $\sigma_i(x)$ the standard error of $\varepsilon_i(x)$. Show that

$$\sigma_i(x) = \left[\sum_{j=1}^{n} \sum_{k=1}^{n} x_j x_k \mathscr{E}(a_{ij} - \bar{a}_{ij})(a_{ik} - \bar{a}_{ik})\right]^{\frac{1}{2}}$$

Suppose we solve the program (for $i = 1, 2, \ldots, m$)

$$\bar{\varepsilon}_i(x) + t_i\sigma_i(x) \leq 0 \qquad (x \geq 0)$$

$$\bar{\varepsilon}_0(x) + t_0\sigma_0(x) = z \text{ (Min)}$$

where $t_i = 3$, say, means that we have built in a safety factor so that $\bar{\varepsilon}_i(x)$, the expected value of $\varepsilon_i(x)$, is three standard errors below zero. Prove that this is a *convex program*. Apply § 24-1 to solve such a problem. Show by Tchebycheff's inequality that

$$\text{Prob}\left[\varepsilon_i(x) > 0\right] < \frac{1}{t^2}$$

What is the above probability if $\varepsilon_i(x)$ is approximately normally distributed? Show that if the a_{ij} are independent and normally distributed, then $\varepsilon_i(x)$ is normally distributed.

REFERENCES

Babbar, 1955-1
Bellman, 1957-1
Charnes and Cooper, 1959-1
Charnes and Lemke, 1954-1
Cramér, 1955-1
Dantzig, 1954-1, 1954-3, 1955-1, 1956-2
Dantzig and Madansky, 1960-1
El-Maghraby, 1959-1, 1960-1
Feller, 1957-1
Ferguson and Dantzig, 1956-1
Freund, 1956-1
Kuhn and Tucker, 1950-2
Madansky, 1959-1, 1960-1, 2
Markowitz, 1952-1, 1956-1
Neyman, 1950-1
Radner, 1955-1
Reiter, 1957-1
Simon, 1956-1
Stigler, 1945-1
Talacko, 1959-1
Theil, 1957-1
Tintner, 1955-1
Vajda, 1958-2
Vazsonyi, 1955-1
Wagner, 1958-2
Wald, 1950-1
Wolfe, 1959-1

CHAPTER 26

DISCRETE-VARIABLE EXTREMUM PROBLEMS

Our purpose now is to solve programs involving variables that have integer values. This first section is confined to a general survey; the second section describes Gomory's Method of Integer Forms, which has now replaced the earlier incomplete work in the field; and the third section gives an appreciation of these results by describing a large class of difficult mathematical problems which are reducible to integer programs. In § 26-3-(14) a simple device is given for transforming a discrete-valued variable to an integer-valued variable.

26-1. SURVEY OF METHODS

A number of important scheduling problems, such as the assignment of flights for an airline or the arrangement of stations on an assembly line, require the study of an astronomical number of arrangements to determine which one is "best." The mathematical problem is to find some short-cut way of getting this best assignment without going through all the combinations. By allowing the unknown assignments to vary *continuously* over some range, one can obtain pseudo-solutions in which one or more assignments turn out to be fractions instead of whole numbers. It is common practice to adjust such values to whole numbers. Because mathematical models are often imperfect mirrors of reality, this approach is recommended for most practical problems. But since such procedures can occasionally give far from the best answer, mathematicians have been working on improved techniques.

The purpose of the present section is to review some recent successes using linear programming methods in this difficult area. We shall also say a few words about the functional-equation approach of dynamic programming; one example is presented in which this method provides an efficient algorithm.

To be more explicit, certain classes of problems are combinatorial in nature and easy to formulate, but mathematicians have had only partial success in solving them. These arise often in the form of discrete-variable programming problems, such as:

1. *The empty-containers problem.* A transport company has a large

number of objects for shipment that it places in empty containers of fixed size. What is the least number of containers required?

2. *The multi-stage machine-scheduling problem.* A machine shop has a large number of different types of tasks to be performed. Each task must be processed first on machine A, then B, then C, . . .; the time required depends on the task. In what order should the processing be done to complete all the tasks in the least time? [Johnson, 1958-1]

3. *The flight-scheduling problem.* Given a number of sources that must ship specified quantities to a number of destinations, arrange an efficient flight schedule [Markowitz and Manne, 1957-1].

4. *The trim problem.* Newsprint comes in rolls of varying widths that are cut from rolls many times these widths. How are these to be cut to minimize trim? [Paull and Walter, 1955-1; Eisemann, 1957-1; Land and Doig, 1957-1; Doig and Belz, 1956-1.]

5. *The fixed-charge problem.* See § 26-3.

6. *The traveling-salesman problem.* See § 26-3.

Examples of problems that have yielded to analysis, as we have seen in earlier chapters, are the following:

7. *The assignment problem.* See Chapter 15.

8. *The problem of the shortest route in a network.* See Chapter 17.

The mind seems to have a remarkable facility for scanning many combinations and arriving at what appears to be either a best one or a very good one. The number of possible combinations can be extremely large, however, making it difficult to verify that the choice is, indeed, a good one. Any ideas, therefore, that help verify that a conjectured solution is optimal are of interest. We shall consider the following class of problems:

Find x_j satisfying

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, m) \tag{1}$$

and

$$x_j = 0 \text{ or } 1 \tag{2}$$

that minimize the linear form

$$\sum_{j=1}^{n} c_j x_j = z \tag{3}$$

For a programming problem to be discrete, it is not necessary that the variables be 0 or 1. In flight-scheduling problems, for example, the variables that represent the number of flights are required to be nonnegative integers. There exists, however, a very simple device by which such problems can be reduced to the "0 or 1" form if the variables have known upper bounds. Indeed, let x be a variable that can take on only nonnegative integral values

and let the integer k be an upper bound for x, so that $x \leq k$; then x can be replaced by the sum

$$x = y_1 + y_2 + \cdots + y_k \qquad (y_j = 0 \text{ or } 1)$$

For this reason the representation (1), (2), and (3) of a discrete programming problem is often referred to as the *standard discrete form.*

An important property of any set of points whose coordinates satisfy equation (2) is that the points are vertices of a convex polyhedral set in n-dimensional space. This is perhaps intuitively obvious since a point such as $(0, 1, 0, \ldots, 1)$ is one of the vertices of the unit n-cube (which of course is convex). As we know in a linear programming problem, if an optimal solution exists, there is one that is an extreme point of feasible solutions. This suggests that, in seeking a solution to the standard discrete problem, we first weaken the hypothesis as follows:

$$\text{(4)} \qquad \text{Replace} \begin{Bmatrix} x_j = 0 \text{ or } x_j = 1 \\ \text{a discontinuous range} \end{Bmatrix} \text{by} \begin{Bmatrix} 0 \leq x_j \leq 1 \\ \text{a continuous range} \end{Bmatrix}$$

Because the replacement given in (4) is less restrictive than the condition (2), it follows that the set of feasible solutions to the linear programming problem (1) and (4) forms a convex polyhedral set C that contains the convex polyhedral set C^*, which is the convex hull of the solution points of (1) and (2). It is easy to see, however, that every extreme point (vertex) of C^* is an extreme point of C (see open dots in Fig. 26-1-I); but there may be extreme points of C that are outside of C^* (see closed dots in Fig. 26-1-I). The

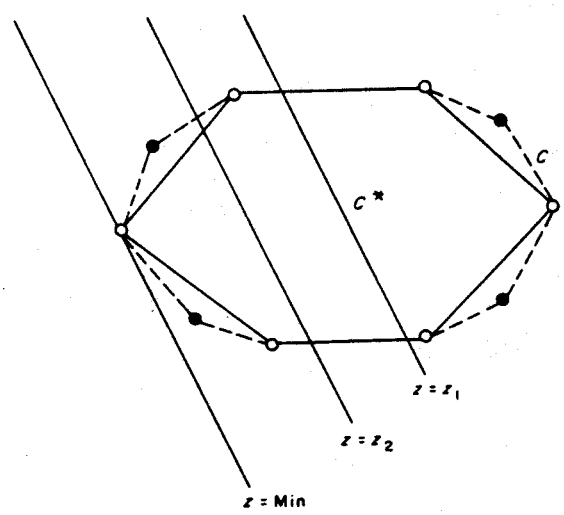


Figure 26-1-I. Schematic representation (two-dimensional case) of the polyhedral set C^* whose vertices are the solutions for a discrete program and of the convex polyhedral set C of solutions for the corresponding continuous problem.

parallel lines in Fig. 26-1-I represent different positions of the hyperplane

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = z = \text{constant}$$

and it is clear that, depending on the values of the c_j, an extreme point corresponding to minimum z may belong to C^* (as in Fig. 26-1-I) or may belong to C and not to C^* [Hoffman and Kruskal, 1956-1].

A remarkable property of the "assignment" problem, and the same holds true for the "shortest-route" problem, is that

$$C^* = C \tag{5}$$

Indeed, this result holds true for a general class of "transportation" problems of which these are special cases. Thus, in the marriage problem (Chapter 15), when we replace the condition $x_{ij} = 0$ or 1 by $0 \leq x_{ij} \leq 1$, we are, in effect, allowing the class of solutions to be extended from the monogamous to the polygamous situation in which sharing mates is possible. The fact that $C^* = C$ states that monogamy will turn out to be the best after all!

The Knapsack Problem.

In certain types of problems, we can get extreme-point solutions for which not all the values of the x_j are either zero or one. When any of the x_j have fractional values, the corresponding extreme points are referred to as *fractional extreme points*. An example of this occurs in the knapsack problem: A person is planning a hike and has decided not to carry more than 70 pounds of various items, such as bed roll, geiger counters (these days), cans of food, etc.

We try to formulate this in mathematical terms. Let a_j be the weight of the jth object and let b_j be its relative value determined by the hiker in comparison with the values of the other objects he would like to have on his trip. Let $x_j = 1$ mean that the jth item is selected, and $x_j = 0$ mean that it is not selected. We express the weight limitation by

$$\sum_{j=1}^{n} a_jx_j \leq 70 \tag{6}$$

$$x_j = 0 \text{ or } 1 \tag{7}$$

and wish to choose the x_j, so that the total value

$$\sum_{j=1}^{n} b_jx_j = z \tag{8}$$

is a maximum.

Now we can show this pictorially in the plane (Fig. 26-1-II) if one coordinate axis measures weight, a, and the other measures value, b. Each object then is represented by a point having coordinates (a_j, b_j). The problem,

graphically, is to select a subset of these points that represents the set of items that he carries with him on his hike; the others he rejects. Let us see what type of graphical solution is obtained if the condition $x_j = 0$ or 1 is replaced by the condition that the variables can lie anywhere in the interval from 0 to 1. The latter problem can be solved by regular linear programming methods; indeed, because of its very simple form, it admits an immediate

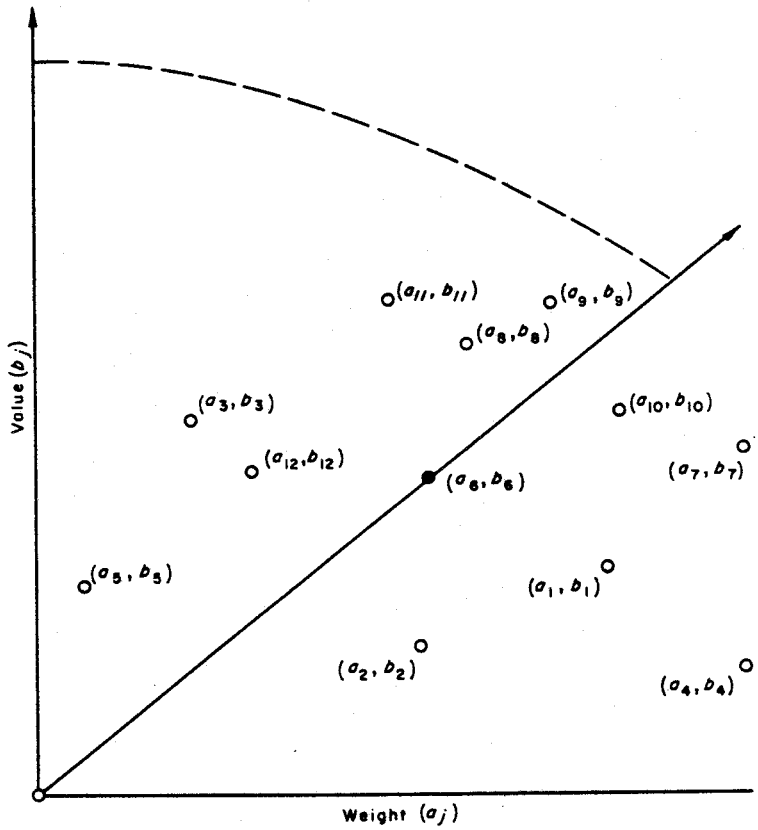


Figure 26-1-II. Graphical solution of the knapsack problem (continuous case).

solution: Rotate clockwise a ray with the origin as pivot point and b axis as starting position. Items corresponding to points swept out by the ray are selected in turn until the sum of their weights exceeds the weight limitation. If upon selection of item j the weight limitation is exceeded, the value x_j is chosen as that fractional part of its weight a_j that would make the sum come exactly to 70 lbs. With the exception of this one item, all the items swept out by the ray have the value $x_j = 1$, while those not swept out have the value $x_j = 0$. It will be noted that this is very close to the kind of

solution desired; all the x_j's are either 0 or 1 with the exception of the one that has a fractional value.

Now at this point the natural question is, "What happens if the solution is *rounded*?" The effect of rounding up or down is of course to change the total weight carried to a number different from 70 lbs. If the model is imperfect (in other words, if the hiker really has a limitation of *roughly* 70 lbs) this may be a satisfactory way of getting rid of the fractional solution; this is particularly true if the weight of individual items is small relative to 70 lbs. *For most practical problems, this is probably all that is needed.* Our object here, however, is to explore ways of getting an exact mathematical solution.

The functional-equation approach of dynamic programming (see § 1-4) is the best technique so far devised for the case where there are only a few items and only one kind of limitation. Extensions to two or more limitations—say one on total weight and another on total volume—can be done, but there would be a considerable increase in the amount of computational work. The method consists in ordering the items in any arbitrary way and determining what items would be carried if (a) the selections were restricted to only the first k items, and (b) the weight limitations were $w = 1, 2, \ldots,$ or 70. For example, if $k = 1$ and $w < a_1$ (where a_1 is the weight of the first item), then the item would not be selected; but if $w \geq a_1$, it would be. From this it is easy to decide what the selections would be for the first two items ($k = 2$) for every total weight $w = 1, 2, \ldots, 70$, and then, inductively, for $k = 3, 4, \ldots, n$. To see this, suppose we wish to determine whether we select the $(k + 1)^{\text{st}}$ item if our weight limitation is w when we know how to make the selections for the first k items for any weight $w = 1, 2, \ldots, 70$.

Let $F_k(w)$ be the highest total value that can be obtained with the first k objects under weight limitation w. Then, under the same weight limitation w, the highest total value that can be obtained with first $k + 1$ objects is $F_k(w)$ if the $(k + 1)^{\text{st}}$ object is not selected, but is $b_{k+1} + F_k(w - a_{k+1})$ if the $(k + 1)^{\text{st}}$ object is selected. Hence, the $(k + 1)^{\text{st}}$ object is or is not selected depending on which of these is higher. Thus, not only is the selection for the first $k + 1$ objects determined, but it also is clear that $F_{k+1}(w)$ is given by

$$(9) \qquad F_{k+1}(w) = \begin{cases} \text{Max}\,[F_k(w), \quad b_{k+1} + F_k(w - a_{k+1})] & \text{for} \quad w \geq a_{k+1} \\ F_k(w) \quad \text{for} \quad w < a_{k+1} \end{cases}$$

The procedure is iterated for each w and repeated for $k = 1, 2, \ldots, n$. Although for the functional-equation approach the terms were ordered in an arbitrary way, it is recommended that in a practical application they be ordered initially in sequence corresponding to decreasing b_j/a_j (see Fig. 26-1-II).

The linear programming approach consists in adding new linear-inequality constraints to the system, so that the fractional extreme points of C in the neighborhood of Max z will be excluded, but the set of extreme points of the convex hull C^* of admissible solutions will be in the new C. The procedure would be complete except that the rules for systematically generating additional constraints require the use of methods discussed in the next section.

DEFINITION: A *cutting plane* is the hyperplane boundary of an added linear inequality constraint. It is thought of as "cutting off" part of the convex of feasible solutions to form a new convex.

Let us suppose, as in Fig. 26-1-II, that the ray swept out items 5, 3, 12, 11, 8, and 9 before the weight limitation was exceeded, but included item 6 just after it was exceeded; then

$$a_5 + a_3 + a_{12} + a_{11} + a_8 + a_9 \qquad < 70$$
$$a_5 + a_3 + a_{12} + a_{11} + a_8 + a_9 + a_6 > 70$$

We wish to exclude the fractional extreme-point solution

$$x_5 = x_3 = x_{12} = x_{11} = x_8 = x_9 = 1, \quad x_6 = f \qquad (0 < f < 1) \tag{10}$$

and $x_j = 0$ for all other j.

It is clear that for an *admissible solution* not all $x_j = 1$ for the seven points $j = 5, 3, 12, 11, 8, 9, 6$, because the weight limitation would be violated. This means that the sum of these variables cannot exceed 6, or

$$x_5 + x_3 + x_{12} + x_{11} + x_8 + x_9 + x_6 \leq 6 \tag{11}$$

Since the fractional extreme-point solution (10) does not satisfy this constraint, it is clear that the effect of adding the particular inequality (11) is to exclude this fractional extreme point. Form (8) is maximized under conditions (6) and $0 \leq x_j \leq 1$, but with the constraint (11) added. Again a new fractional extreme point may turn up for the new convex C, and it will be necessary again to seek a condition that will exclude it. For the most part the conditions added will be other partial sums of the x_j similar to (11). However, at times more subtle relations will be required until at last an extreme point is obtained that is admissible. The method discussed in the next section provides a straightforward way to determine these constraints.

Many experiments by the author and others indicate that very often a practical problem can be solved using only such obvious supplementary conditions as (11). In an experiment with a number of randomly chosen traveling-salesman problems involving nine cities, simple upper bounds and so-called simple "loop conditions" on the variables were often sufficient to yield the desired discrete solution [Dantzig, Fulkerson, and Johnson, 1954-1, 1959-1]. To appreciate the power of this procedure it should be noted that, for each nine-city case solved, the tour that minimized the total distance covered was one from among 362,880 ways of touring nine cities and was selected in about two hours of hand-computation time.

26-2. GOMORY'S METHOD OF INTEGER FORMS

In this section we shall present a method due to R. Gomory [1958-1, 2] of automatically generating cutting planes or *integer forms* which gives promise of providing an efficient solution to linear programs in integers in a finite number of steps. This approach has been generalized to the case where some variables are continuous and some are constrained to be integers. This was first done by E. M. L. Beale [1958-1], and by Gomory [1958-3] in a different way, which we follow here.

We begin by giving the method in a precise form and then discuss various ways that it may be relaxed in practice. Our concern is with linear-programming-type problems where, however, some of the variables must have integer values and the others may have fractional values. The former we shall call *integer variables* and the latter *fractional variables*.

PROBLEM: Determine Min v and $y_j \geq 0$ for $j = 1, 2, \ldots, n$ such that v and y_j are integers for a subset J of the indices j and such that

$$
\begin{aligned}
(1) \qquad & a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n = b_1 \qquad (y_j \geq 0)\\
& a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n = b_2\\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
& a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n = b_m\\
& c_1y_1 + c_2y_2 + \cdots + c_ny_n = v \text{ (Min)}
\end{aligned}
$$

Note that v as well as y_j for $j \in J$ are required to be integers while the remaining variables are allowed to take on fractional values.

The Initial Primal and Dual Systems.

To initiate the algorithm, the problem is first solved ignoring the integer restrictions. Suppose the canonical form obtained on the final iteration is

$$
\begin{array}{llllllll}
(2) & & & & & & & \\
\bar{a}_{11}y_1 + \bar{a}_{12}y_2 + \cdots + \bar{a}_{1,n-m}y_{n-m} + y_{n-m+1} & & & & & & & = \bar{b}_1\\
\bar{a}_{21}y_1 + \bar{a}_{22}y_2 + \cdots + \bar{a}_{2,n-m}y_{n-m} & + y_{n-m+2} & & & & & & = \bar{b}_2\\
\cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & & \cdot & & & \\
\cdot & \cdot & \cdot & & & \cdot & & \\
\bar{a}_{m1}y_1 + \bar{a}_{m2}y_2 + \cdots + \bar{a}_{m,n-m}y_{n-m} & & & & & & + y_n & = \bar{b}_m\\
\bar{c}_1y_1 + \bar{c}_2y_2 + \cdots + \bar{c}_{n-m}y_{n-m} & & & & & & & = v - \bar{v}_0
\end{array}
$$

where we have assumed, for convenience, $y_{n-m+1}, \ldots, y_n$ as basic variables. Because our solution is optimal, we have for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$(3) \qquad \bar{b}_i \geq 0, \ \bar{c}_j \geq 0$$

If we now set

$$y_i = \pi_i \qquad (i = 1, 2, \ldots, n - m) \tag{4}$$

for all the non-basic variables and solve for basic variables in terms of π_i

$$y_{n-m+k} = \bar{b}_k - \sum_{i=1}^{n-m} \bar{a}_{ki}\pi_i \qquad (k = 1, \ldots, m) \tag{5}$$

$$v = \bar{v}_0 + \sum_{i=1}^{n-m} \bar{c}_i\pi_k$$

then all the variables $y_1, y_2, \ldots, y_n$ are expressed *parametrically* in terms of π_i.

In the expression for y_j let us now denote by $+\alpha_{ij}$ the coefficient of π_i, by α_{0j} the constant terms, and by α_{i0} the *coefficients* of v. Setting

$$\bar{m} = n - m \tag{6}$$

then the original problem can be cast in the following form:

Find Min v, $y_j \geq 0$, π_i where v and y_j are integers for $j \in J$, such that

$$\begin{aligned} y_1 &= \alpha_{01} + \alpha_{11}\pi_1 + \ldots + \alpha_{\bar{m}1}\pi_{\bar{m}} \\ y_2 &= \alpha_{02} + \alpha_{12}\pi_1 + \ldots + \alpha_{\bar{m}2}\pi_{\bar{m}} \\ &\vdots \\ y_n &= \alpha_{0n} + \alpha_{1n}\pi_1 + \ldots + \alpha_{\bar{m}n}\pi_{\bar{m}} \\ v &= \alpha_{00} + \alpha_{10}\pi_1 + \ldots + \alpha_{\bar{m}0}\pi_{\bar{m}} \end{aligned} \tag{7}$$

If we set aside the integer restrictions, we can replace (7) by

$$\alpha_{0j} + \sum \alpha_{ij}\pi_i \geq 0 \qquad (j = 1, 2, \ldots n)$$

$$\alpha_{00} + \sum \alpha_{i0}\pi_i = v \text{ (Min)}$$

and it is clear that the dual problem for (7) becomes

Dual: Find Max x_0 and $x_j \geq 0$ for $j \neq 0$ satisfying

$$\begin{aligned} x_0 + \alpha_{01}x_1 + \alpha_{02}x_2 + \ldots + \alpha_{0n}x_n &= \alpha_{00} \\ \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n &= \alpha_{10} \\ &\vdots \\ \alpha_{\bar{m}1}x_1 + \alpha_{\bar{m}2}x_2 + \ldots + \alpha_{\bar{m}n}x_n &= \alpha_{\bar{m}0} \end{aligned} \tag{8}$$

Because system (7) is by definition the same as (4), (5), system (8) is the same as (9) below

$$
\begin{array}{lllllll}
(9) & x_0 & & + \bar{b}_1 x_{\bar{m}+1} + \cdots + & \bar{b}_m x_n = \bar{v}_0 \\
 & & x_1 & - \bar{a}_{11} x_{\bar{m}+1} - \cdots - & \bar{a}_{m1} x_n = \bar{c}_1 \\
 & & \ddots & \vdots & \vdots \\
 & & & x_{\bar{m}} - \bar{a}_{1\bar{m}} x_{\bar{m}+1} - \cdots - & \bar{a}_{m\bar{m}} x_n = \bar{c}_{\bar{m}}
\end{array}
$$

which is in canonical form and the basic solution is feasible and optimal ($\bar{b}_i \geq 0$, $\bar{c}_j \geq 0$). The actual order of the variables x_j can be the same as in the primal, and the order of the equations can correspond to the order of the non-basic variables in the primal. We shall assume that this is the case. Moreover, we assume that *the variables of the primal have been previously arranged so that all the integer variables are ahead of the others*. This assumption is not made merely for convenience of exposition; indeed, the finiteness of Gomory's algorithm depends on its being satisfied.

To illustrate, consider the problem of finding Min v, $y_j \geq 0$, where v, y_1, y_2 are integers satisfying

$$
(10) \qquad \begin{aligned}
2y_1 + y_2 + \tfrac{4}{3}y_3 \qquad &= \tfrac{13}{3} \\
\tfrac{1}{2}y_1 \qquad + \tfrac{3}{4}y_3 + y_4 &= \tfrac{9}{4} \\
\tfrac{1}{4}y_1 \qquad + \tfrac{4}{3}y_3 \qquad &= v + \tfrac{7}{12}
\end{aligned}
$$

If we set $y_1 = \pi_1$, $y_3 = \pi_3$, then

$$
(11) \qquad \begin{aligned}
y_1 &= \pi_1 \\
y_2 &= \tfrac{13}{3} - 2\pi_1 - \tfrac{4}{3}\pi_3 \\
y_3 &= \pi_3 \\
y_4 &= \tfrac{9}{4} - \tfrac{1}{2}\pi_1 - \tfrac{3}{4}\pi_3 \\
v &= -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{4}{3}\pi_3
\end{aligned}
$$

where the parameter π_1 is an integer. Ignoring the integer restriction, the dual of (11), is

$$
(12) \qquad \begin{aligned}
x_0 \quad + \tfrac{13}{3}x_2 \qquad + \tfrac{9}{4}x_4 &= -\tfrac{7}{12} \\
x_1 - \quad 2x_2 \qquad - \tfrac{1}{2}x_4 &= \tfrac{1}{4} \\
- \quad \tfrac{4}{3}x_2 + x_3 - \tfrac{3}{4}x_4 &= \tfrac{4}{3}
\end{aligned}
$$

Generating New Restrictions.

If the minimizing solution to the primal problem happens to satisfy the integer conditions, this solves the original problem. Thus, if the values of $v = \alpha_{00}$ and $y_j = \alpha_{0j}$ are integers for the integer variables, the basic solution is integral and optimal. If *not*, then the constant term is fractional for at

least one expression for an integer variable or for the expression for v as given in (7). In our example, we have two such:

$$(13) \qquad \begin{aligned} v &= -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{4}{3}\pi_3 \\ y_2 &= \tfrac{13}{3} - 2\pi_1 - \tfrac{4}{3}\pi_3 \end{aligned}$$

Any such expression can be used to generate a linear inequality (indeed a class of inequalities) which is not satisfied by the current basic solution formed by setting $\pi_i = 0$.

It will be convenient to introduce[1] the symbol $[\alpha_j]^*$ which represents here the greatest *integer* $\leq \alpha_j$; we define

$$(14) \qquad f_j = \alpha_j - [\alpha_j]^* \geq 0$$

to be the *positive proper fractional part of* α_j. For example, if $\alpha_j = \frac{4}{3}$, then $[\alpha_j]^* = 1$ and $f_j = \frac{1}{3}$; if $\alpha_j = -\frac{4}{3}$, then $[\alpha_j]^* = -2$ and $f_j = \frac{2}{3}$. We also define the *complement*, $\bar{f}_j$, of the positive fractional part of α_j, as the positive proper fractional part of $-\alpha_j$. It is easy to see that

$$(15) \qquad \bar{f}_j = \begin{cases} 1 - f_j & \text{if } f_j > 0 \\ 0 & \text{if } f_j = 0 \end{cases}$$

THEOREM 1: *If y is an integer variable and $\pi_i \geq 0$ are parameters related by*

$$(16) \qquad y = \alpha_0 + \alpha_1\pi_1 + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}} \qquad ([\alpha_0]^* < \alpha_0)$$

where $\alpha_i \geq 0$, for $i \neq 0$, then the linear inequality

$$(17) \qquad 1 \leq f_0 + \alpha_1\pi_1 + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}}$$

holds for all π_i generating integral y but is not satisfied by the basic solution generated by $\pi_i = 0$ for $i = 1, \ldots, m$.

PROOF: Since $\alpha_i \geq 0$, and $\pi_i \geq 0$, Min $y \geq \alpha_0$; but the possible values of y are integers, so

$$(18) \qquad y \geq [\alpha_0]^* + 1$$

Subtracting from (16) yields (17). Moreover, writing $y - y^* = [\alpha_0]^* + 1$ where $y^* \geq 0$ is an integer and subtracting from (16) yield

$$(19) \qquad y^* = -\bar{f}_0 + \alpha_1\pi_1 + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}} \qquad (y^* \geq 0)$$

This is a stronger form of (17) because the new slack variable, y^*, is required to be a *nonnegative integer*.

In our illustrative example

$$(20) \qquad v = -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{4}{3}\pi_3$$

[1] The customary symbol for the largest integer part of a number is a bracket without a star; however, with a star there is less possibility of confusion with ordinary brackets.

where v is an integer variable unrestricted in sign and $\pi_1 \geq 0$, $\pi_3 \geq 0$. (Actually $\pi_1 \geq 0$ is required to be an integer also.) Applying our theorem we have, setting $y^* = y_5$, say

$$y_5 = -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{4}{3}\pi_3 \qquad (y_5 \geq 0) \tag{21}$$

where $y_5 \geq 0$ is a new integer variable.

System (11) may now be *augmented* by the expression for this new (basic) variable in terms of π_1, π_3. It will be noted that the basic solution generated by setting $\pi_1 = \pi_3 = 0$ is no longer feasible, because $y_5 = -\tfrac{7}{12}$.

It will be also noted that the dual of the augmented system is formed by adding a new variable, say x_5, with coefficients $(-\tfrac{7}{12}, \tfrac{1}{4}, \tfrac{4}{3})$; see (57).[2] The dual is still in canonical form, and its basic solution is feasible but no longer optimal. It will be shown later that the sequence of basic changes required to make it optimal, correspond to a change of the parameters π_i used to represent the y_i.

We may also generate other inequalities by multiplying expression (20) by any integer k, since, if v is an integer, so is kv; thus

$$kv = -\frac{7}{12}k + \frac{k}{4}\pi_1 + \frac{4k}{3}\pi_3 \qquad (k \geq 0) \tag{22}$$

$$y^* = \left[+\frac{7}{12}k\right]^* - \frac{7}{12}k + \frac{k}{4}\pi_1 + \frac{4k}{3}\pi_3 \qquad (y^* \geq 0)$$

where y^* is an integer.

The knowledge that one or more of the π_i are integers can be used, however, to generate in general new, even stronger inequalities.

THEOREM 2: *If y is an integer variable and the $\pi_i \geq 0$ are integer-valued variables related by*

$$y = \alpha_0 + \alpha_1\pi_1 + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}} \qquad ([\alpha_0]^* < \alpha_0) \tag{23}$$

Then the linear inequality,

$$1 \leq f_0 + f_1\pi_1 + \ldots + f_{\bar{m}}\pi_{\bar{m}} \tag{24}$$

holds for all π_i generating integral y but is not satisfied by the basic solution generated by setting all $\pi_i = 0$.

PROOF: Note that in this theorem the α_i may have either sign, and the π_i are nonnegative integers. Substituting $\alpha_j = f_j + [\alpha_j]^*$ in (23) yields

$$\left\{y - [\alpha_0]^* - \sum [\alpha_i]^*\pi_i\right\} = f_0 + f_1\pi_1 + \ldots + f_{\bar{m}}\pi_{\bar{m}} \tag{25}$$

The left member is an integer and $f_j \geq 0$; hence, Theorem 1 may be applied to yield (24) or the stronger form

$$y^{**} = -f_0 + f_1\pi_1 + f_2\pi_2 + \ldots + f_{\bar{m}}\pi_{\bar{m}} \qquad (y^{**} \geq 0) \tag{26}$$

where y^{**} is an integer-valued variable.

[2] In (57) we have used a stronger condition than (21) by applying Theorem 2.

EXERCISE: Interpret (24) if $f_1 = f_2 = \ldots = f_{\bar{m}} = 0$.

For example, suppose $y \geq 0$ and the $\pi_i \geq 0$ are integer variables related by

$$y = \tfrac{5}{3} + (7/4)\pi_1 - (8/3)\pi_2 + (1/2)\pi_3$$

then a new condition is

$$y^* = -\tfrac{1}{3} + (3/4)\pi_1 + (1/3)\pi_2 + (1/2)\pi_3 \quad (y^* \geq 0 \text{ integer})$$

Another new condition can be generated using $2y$, thus

$$2y = \tfrac{10}{3} + (14/4)\pi_1 - (16/3)\pi_2 + (2/2)\pi_3$$

implies

$$y^{**} = -\tfrac{2}{3} + (1/2)\pi_1 + (2/3)\pi_2 \qquad (y^{**} \geq 0 \text{ integer})$$

A simple extension of these two theorems occurs in the case where some α_i refer to integer parameters π_i, for $i = 1, 2, \ldots, k$, and the remaining parameters may take a fractional value *with the property* all $\alpha_i \geq 0$ (*or* all $\alpha_i \leq 0$) for $i > k$; then the mixed expression

$$y^* = -\bar{f}_0 + f_1\pi_1 + \ldots + f_k\pi_k + (\alpha_{k+1}\pi_{k+1} + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}}) \tag{27}$$

holds for the case $\alpha_i \geq 0$, for all $i > k$, and

$$y^* = -f_0 + \bar{f}_1\pi_1 + \ldots + \bar{f}_k\pi_k - (\alpha_{k+1}\pi_{k+1} + \ldots + \alpha_{\bar{m}}\pi_{\bar{m}})$$

holds for the case $\alpha_i \leq 0$ for all $i > k$, for any π_i generating integral y where $y^* \geq 0$, an integer. (This y^* is not the same y^* used earlier.)

In our illustrative example

$$y_2 = \tfrac{13}{3} - 2\pi_1 - \tfrac{4}{3}\pi_3 \tag{28}$$

where $y_2 \geq 0$, $\pi_1 \geq 0$ are integers and $\pi_3 \geq 0$. Applying the second case, setting $y^* = y_6$, then

$$y_6 = -\tfrac{1}{3} + \tfrac{4}{3}\pi_3 \tag{29}$$

where $y_6 \geq 0$ is a new integer variable whose expression in terms of the π_i may also be used to augment the system (11) and (21).

If all parameters π_i are integers, then it is interesting to observe that a particularly simple relation exists consisting of a *partial sum* of π_i. This condition is not as strong as earlier ones, since it can be found by combining two known inequalities. Thus for $k = \bar{m}$, (27) implies both

$$\begin{aligned} y^* &= -\bar{f}_0 + f_1\pi_1 + \ldots + f_{\bar{m}}\pi_{\bar{m}} && (y^* \geq 0) \\ y^{**} &= -f_0 + \bar{f}_1\pi_1 + \ldots + \bar{f}_{\bar{m}}\pi_{\bar{m}} && (y^{**} \geq 0) \end{aligned} \tag{30}$$

where y^* and y^{**} are integers, and we have by adding and setting $y' = y^* + y^{**}$

$$y' = -1 + \delta_1\pi_1 + \ldots + \delta_{\bar{m}}\pi_{\bar{m}} \qquad (y' \geq 0) \tag{31}$$

where $\delta_j = 1$, if $f_j > 0$, and $\delta_j = 0$, if $f_j = 0$. For example, if π_1 and π_3 were both integer variables in (28), then a new relation would be

$$y' = -1 + \pi_3 \tag{32}$$

It is also interesting to observe that a strong condition can sometimes be generated from an integer variable y, which has an integer value in a solution when all $\pi_i = 0$. For example, suppose

$$y = 2 + \tfrac{4}{3}\pi_1 - \tfrac{1}{2}\pi_2 \tag{33}$$

where π_1 and π_2 constitute *all* the parameters. Writing

$$y - 2 - \pi_1 + \pi_2 = \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_2 \tag{34}$$

we note that the left member is integral. Because the right member is nonnegative, we conclude

$$y^* = \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_2 \tag{35}$$

where $y^* \geq 0$ is an integer. However $y^* = 0$ is *not* possible, because then $\pi_1 = \pi_2 = 0$ and the other integral y_j would have to take on the fractional values of the current basic solution. Hence $y^* = y^{**} + 1 \geq 1$ and

$$y^{**} = -1 + \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_2 \tag{36}$$

becomes the new restriction. In general, if for some j, $\alpha_{0j} = [\alpha_{0j}]^*$ and $f_i = \alpha_{ij} - [\alpha_{ij}]^* > 0$ for *all* $i \neq 0$, then a new restriction is

$$y^{**} = -1 + \sum f_i \pi_i \qquad (f_i \neq 0) \tag{37}$$

Generating New Conditions for the General Mixed Integer Case.

For problems involving mixed integer and "fractional" variables (variables which may assume fractional values), we must however, in general, be able to generate new inequalities when the coefficients of the fractional variables have either sign. Let

$$y = \alpha_0 + \sum_{i=1}^{\bar{m}} \alpha_i \pi_i = \alpha_0 + P - N \qquad ([\alpha_0]^* < \alpha_0) \tag{38}$$

where P and $-N$ are the partial sums of the positive and negative terms, α_0 is not an integer and y is an integer variable.

If for certain values of π_i, $P - N \geq 0$, then, since $y = \alpha_0 + P - N$ must be an integer so must $f_0 + P - N$; but the latter is strictly positive (for we are assuming that α_0 is non-integral). Hence we must have

$$\begin{aligned} 1 &\leq f_0 + P - N \\ f_0 &\leq P - N \\ 1 &\leq \frac{1}{f_0} P - \frac{1}{f_0} N \leq \frac{1}{f_0} P + \frac{1}{f_0} N \end{aligned} \tag{39}$$

If for other values of π_i, $N - P \geq 0$, then $-y = -\alpha_0 + N - P$ is an integer; therefore

$$(40) \qquad \begin{aligned} 1 &\leq \bar{f}_0 + N - P \\ f_0 &\leq N - P \\ 1 &\leq \frac{1}{f_0} N - \frac{1}{f_0} P \leq \frac{1}{f_0} N + \frac{1}{\bar{f}_0} P \end{aligned}$$

Hence for any set of π_i values,

$$(41) \qquad 1 \leq \frac{1}{\bar{f}_0} P + \frac{1}{f_0} N$$

THEOREM 3: *If y is an integer variable, and $\pi_i \geq 0$ satisfy*

$$(42) \qquad y = \alpha_0 + \sum_1^{\bar{m}} \alpha_i \pi_i \qquad \text{where } \alpha_0 > [\alpha_0]^*$$

then

$$(43) \qquad 1 \leq \frac{1}{\bar{f}_0} \left(\sum_{I_1} f_i \pi_i + \sum_{I_2} \alpha_i \pi_i \right) + \frac{1}{f_0} \left(\sum_{I_3} \bar{f}_i \pi_i - \sum_{I_4} \alpha_i \pi_i \right)$$

holds for all π_i generating integral y but is not satisfied by the basic solution generated by $\pi_i = 0$. Here

$$(44) \qquad \begin{aligned} &i \in I_1 \text{ if } f_i \leq \bar{f}_0 \text{ and } \pi_i \text{ integral} \\ &i \in I_2 \text{ if } \alpha_i > 0 \text{ and } \pi_i \text{ fractional} \\ &i \in I_3 \text{ if } \bar{f}_i < f_0 \text{ and } \pi_i \text{ integral} \\ &i \in I_4 \text{ if } \alpha_i < 0 \text{ and } \pi_i \text{ fractional} \end{aligned}$$

PROOF: If π_i is an integer, set in (42) $\alpha_i = [\alpha_i]^* + f_i$ or $\alpha_i = [\alpha_i]^* + 1 - \bar{f}_i$ according to whether $f_i < \bar{f}_0$ or $\bar{f}_i < f_0$. Moving all integer terms to the left and calling the left member y', we have

$$(45) \qquad y' = \alpha_0 + \left\{ \sum_{I_1} f_i \pi_i + \sum_{I_2} \alpha_i \pi_i \right\} - \left\{ \sum_{I_3} \bar{f}_i \pi_i - \sum_{I_4} \alpha_i \pi_i \right\}$$

where y' is an integer variable. Identifying (45) with (38), relation (43) corresponds to (41) and the theorem follows.

If we let $y^* \geq 0$ represent the slack in the inequality (43), then the new relation may be written

$$(46) \qquad y^* = -1 + \frac{1}{\bar{f}_0} \left\{ \sum_{I_1} f_i \pi_i + \sum_{I_2} \alpha_i \pi_i \right\} + \frac{1}{f_0} \left\{ \sum_{I_3} \bar{f}_i \pi_i - \sum_{I_4} \alpha_i \pi_i \right\}$$

However, it no longer follows that the added new variable is an integer as was the case with all previous relations that have been developed. (This is too bad because, as we have seen, when y^* is an integer this fact can be

used to advantage to develop, at a later stage, new stronger inequalities.) To illustrate, suppose

$$y = \tfrac{5}{3} + \tfrac{3}{5}\pi_1 + \tfrac{4}{2}\pi_2 - \tfrac{9}{7}\pi_3 - \tfrac{5}{2}\pi_4 \qquad (\pi_j \geq 0) \tag{47}$$

where y, π_1, and π_3 are integers. We may rewrite this

$$(y - 1 + \pi_3) = \tfrac{2}{3} + \tfrac{3}{5}\pi_1 + \tfrac{4}{2}\pi_2 - \tfrac{2}{7}\pi_3 - \tfrac{5}{2}\pi_4 \tag{48}$$

which, if we simply group positive and negative terms, leads to the inequality

$$1 \leq \frac{1}{\frac{1}{3}}\left(\tfrac{3}{5}\pi_1 + \tfrac{4}{2}\pi_2\right) + \frac{1}{\frac{2}{3}}\left(\tfrac{2}{7}\pi_3 + \tfrac{5}{2}\pi_4\right) \tag{49}$$

We may alternatively in (48) set $\frac{3}{5}\pi_1 = \pi_1 - \frac{2}{5}\pi_1$. Then grouping integer terms

$$y - 1 + \pi_3 - \pi_1 = \tfrac{2}{3} + \left(\tfrac{4}{2}\pi_2\right) - \left(\tfrac{2}{5}\pi_1 + \tfrac{2}{7}\pi_3 + \tfrac{5}{2}\pi_4\right) \tag{50}$$

leads to the inequality

$$1 \leq \frac{1}{\frac{1}{3}}\left(\tfrac{4}{2}\pi_2\right) + \frac{1}{\frac{2}{3}}\left(\tfrac{2}{5}\pi_1 + \tfrac{2}{7}\pi_3 + \tfrac{5}{2}\pi_4\right) \tag{51}$$

This is a stronger inequality than (49), because it has a smaller coefficient of π_1.

Indeed, we may arrange matters, in general, so that the coefficients of all the integer variables never exceed unity when the constant terms on the new constraints are unity. To see this, suppose $\pi_i \geq 0$ is an integer; replace α_i by either $[\alpha_i]^* + f_i$ or $[\alpha_i]^* + 1 - \bar{f}_i$, according to whether $f_i \leq \bar{f}_0$ or $\bar{f}_i < f_0$. In the first case the coefficient of π_i becomes $f_i/\bar{f}_0 < 1$ and in the second case $\bar{f}_i/f_0 \leq 1$.

It is also possible to develop a second type of inequality for the general case that introduces a *sharper inequality* but increases the size of the system. Let $i = 1, \ldots, k$ refer to integer variables and $i > k$ to fractional variables. Set

$$\bar{y} = \sum_{i>k} \alpha_i \pi_i - \pi^* \qquad (0 \leq \pi^* \leq 1) \tag{52}$$

where $\bar{y}$ is a new integer variable unrestricted in sign and π^* is the *positive proper fractional part* of the sum of fractional variable terms; then the new restriction becomes

$$y^* = -\bar{f}_0 + \sum_{1}^{k} f_i \pi_i + \pi^* \qquad (0 \leq \pi^* \leq 1) \tag{53}$$

where y^* is an integer. Since the new restrictions now involve a *bounded* variable π^*, it is probably better to stay in the primal system rather than pass to the dual system.

Iterative Procedure.

Any new restriction

$$y_t = \alpha_{0t} + \alpha_{1t}\pi_1 + \cdots + \alpha_{\bar{m}t}\pi_{\bar{m}} \tag{54}$$

used to augment the initial system will result in a new variable x_t for the dual system with coefficients α_{it}. The *dual* system is still of course in canonical form but the basic solution is no longer optimal (since $\alpha_{0t} < 0$). Hence, x_t will be introduced into the basis by pivoting on some element α_{rt} in the dual system. This transforms the dual matrix $[\alpha_{ij}]$ into $[\alpha'_{ij}]$.

On the other hand, if we go back to the primal system and introduce a new parameter y_t in place of π_r (by solving equation (54) for π_r in terms of the other π_i and y_t), and substitute in the expressions for y_j and v, then it is easy to see that the matrix $[\alpha_{ij}]$ will also be transformed into $[\alpha'_{ij}]$.

Thus the new dual corresponds to the primal being represented by a set of parameters such that the slack variable of the new restriction becomes a new parameter in place of one of the old parameters. In general, each subsequent dual cycle corresponds to using one of the variables y_j as a new parameter in place of a previous one.

If, after a pivot operation, the dual is still not optimal because some $\alpha_{0t} < 0$, the simplex algorithm is applied until it is. Formally, this corresponds to successively using as "new" restrictions for the primal any existing relation (54) with $\alpha_{0t} < 0$.

EXAMPLE: Find integers Min v, $y_j \geq 0$ satisfying

$$\begin{aligned} 2y_1 + y_2 + \tfrac{4}{3}y_3 \qquad &= \tfrac{13}{3} \\ \tfrac{1}{2}y_1 \qquad + \tfrac{3}{4}y_3 + y_4 &= \tfrac{9}{4} \\ \tfrac{1}{4}y_1 \qquad + \tfrac{4}{3}y_3 \qquad &= v + \tfrac{7}{12} \end{aligned} \tag{55}$$

Fortunately, the system is in optimal canonical form if the integer constraints are set aside. Next, set non-basic variables y_j equal to π_j and represent all variables in terms of π_j. (In the steps below, each y_j used as a parameter is, for convenience, identified by setting $\pi_j = y_j$.)

Primal Cycle 0:

$$\begin{aligned} y_1 &= \pi_1 \\ y_2 &= \tfrac{13}{3} - 2\pi_1 - \tfrac{4}{3}\pi_3 \\ y_3 &= \pi_3 \\ y_4 &= \tfrac{9}{4} - \tfrac{1}{2}\pi_1 - \tfrac{3}{4}\pi_3 \\ v &= -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{4}{3}\pi_3 \\ \boxed{y_5 = -\tfrac{7}{12} + \tfrac{1}{4}\pi_1 + \tfrac{1}{3}\pi_3} \end{aligned} \tag{56}$$

where the new restriction y_5 is derived from v. We now dualize and optimize:

Dual Cycle 0:

(57)
$$\begin{array}{rrrrrrrl}
x_0 & & +\tfrac{13}{3}x_2 & & +\tfrac{9}{4}x_4 & -\tfrac{7}{12}x_5 & = & -\tfrac{7}{12} \\
 & x_1 & -\ 2x_2 & & -\tfrac{1}{2}x_4 & +\tfrac{1}{4}x_5 & = & \tfrac{1}{4} \\
 & & -\tfrac{4}{3}x_2 & +x_3 & -\tfrac{3}{4}x_4 & +\tfrac{1}{3}x_5 & = & \tfrac{4}{3} \\
\bullet & \circ & & \bullet & & \star & &
\end{array}$$

Dual Cycle 1:

(58)
$$\begin{array}{rrrrrl}
x_0 & +\tfrac{7}{3}x_1 & -\tfrac{1}{3}x_2 & & +\tfrac{13}{12}x_4 & = 0 \\
 & 4x_1 & -\ 8x_2 & & -\ 2x_4 & +\ x_5 = 1 \\
 & -\tfrac{4}{3}x_1 & +\tfrac{4}{3}x_2 & +x_3 & -\tfrac{1}{12}x_4 & = 1 \\
\bullet & & \star & \circ & & \bullet
\end{array}$$

This corresponds to eliminating π_1 from the primal, using

$$\pi_5 = y_5 = -\frac{7}{12} + \frac{\pi_1}{4} + \frac{\pi_3}{3}$$

where we have let $y_5 = \pi_5$ be a new parameter; operationally, this is accomplished by *pivoting* on the $\frac{1}{4}\pi_1$ term in (56) after first moving the π_5 term to the right-hand side, next to the π_1 term. After pivoting, the pivot equation is dropped. The π_1 terms after elimination are all zero and do not appear. Thus

Primal Cycle 1:

(59)
$$\begin{aligned}
y_1 &= \tfrac{7}{3} + 4\pi_5 - \tfrac{4}{3}\pi_3 \\
y_2 &= -\tfrac{1}{3} - 8\pi_5 + \tfrac{4}{3}\pi_3 \\
y_3 &= \pi_3 \\
y_4 &= \tfrac{13}{12} - 2\pi_5 - \tfrac{1}{12}\pi_3 \\
y_5 &= \pi_5 \\
v &= \pi_5 + \pi_3
\end{aligned}$$

Returning to (58), since the basic solution is not optimal, we perform the indicated pivot step. This yields (except for the boxed column):

Dual Cycle 2:

(60)
$$\begin{array}{rrrrrrrl}
x_0 & +2x_1 & & +\tfrac{1}{4}x_3 & +\tfrac{17}{16}x_4 & & -\tfrac{3}{4}x_6 & = \tfrac{1}{4} \\
 & -4x_1 & & +6x_3 & -\ \tfrac{5}{2}x_4 & +x_5 & & = 7 \\
 & -\ x_1 & +x_2 & +\tfrac{3}{4}x_3 & -\tfrac{1}{16}x_4 & & +\tfrac{3}{4}x_6 & = \tfrac{3}{4} \\
\bullet & & \circ & & & \bullet & \star &
\end{array}$$

This corresponds to eliminating π_3 from the primal by formally adjoining one of the existing relations, namely,

$$\pi_2 = y_2 = -\tfrac{1}{3} - 8\pi_5 + \tfrac{4}{3}\pi_3$$

and letting $y_2 = \pi_2$ be a new parameter; we can accomplish this by pivoting

on the $\frac{4}{3}\pi_3$ term above with respect to (59) (after first moving $\pi_2 = y_2$ to the right). This will eliminate π_3 from the other equations, yielding

Primal Cycle 2:

(61)
$$\begin{aligned}
y_1 &= 2 - 4\pi_5 - \pi_2 \\
y_2 &= \pi_2 \\
y_3 &= \tfrac{1}{4} + 6\pi_5 + \tfrac{3}{4}\pi_2 \\
y_4 &= \tfrac{17}{16} - \tfrac{5}{2}\pi_5 - \tfrac{1}{16}\pi_2 \\
y_5 &= \pi_5 \\
v &= \tfrac{1}{4} + 7\pi_5 + \tfrac{3}{4}\pi_2 \\
\boxed{y_6 = -\tfrac{3}{4} + \tfrac{3}{4}\pi_2}
\end{aligned}$$

The new column for (60) is derived from the coefficients of x_3 which correspond to $y_3 = \frac{1}{4} + 6\pi_5 + \frac{3}{4}\pi_2$; namely, the new column corresponds to the new restriction $y_6 = -\frac{3}{4} + \frac{3}{4}\pi_2 \geq 0$ shown in (61). Iterating

Dual Cycle 3: (Optimal Integral Solution)

(62)
$$\begin{aligned}
x_0 + x_1 + x_2 + x_3 + x_4 &= 1 \\
-4x_1 + 6x_3 - \tfrac{5}{2}x_4 + x_5 &= 7 \\
-\tfrac{4}{3}x_1 + \tfrac{4}{3}x_2 + x_3 - \tfrac{1}{12}x_4 + x_6 &= 1
\end{aligned}$$

Primal Cycle 3:

(63)
$$\begin{aligned}
y_1 &= 1 - 4\pi_5 - \tfrac{4}{3}\pi_6 \\
y_2 &= 1 + \tfrac{4}{3}\pi_6 \\
y_3 &= 1 + 6\pi_5 + \pi_6 \\
y_4 &= 1 - \tfrac{5}{2}\pi_5 - \tfrac{1}{12}\pi_6 \\
v &= 1 + 7\pi_5 + \pi_6
\end{aligned}$$

where the optimal solution is found by setting $\pi_5 = \pi_6 = 0$.

Proof of Finiteness of Algorithm. [Gomory, 1958-2, 3.]

For this purpose we regard the initial dual's right-hand side (8) as perturbed by the columns appearing on the left for $j \neq 0$. (See Chapter 10.)

(64)
$$x_0 + \sum_1^n \alpha_{0j}x_j = \sum_0^n \alpha_{0j}\varepsilon^j = \alpha_0(\varepsilon)$$
$$\sum_1^n \alpha_{ij}x_j = \sum_0^n \alpha_{ij}\varepsilon^j = \alpha_i(\varepsilon) \qquad (i = 1, \ldots, \bar{m})$$

Thus the values of the basic variables become *polynomials* in ε. *It will be assumed that the leading term (i.e., the lowest-power non-zero term) of each* $\alpha_i(\varepsilon)$ *is positive except possibly* $i = 0$.

If the basic variables can be arranged ahead of the non-basic variables as in (9), this assumption is initially satisfied. However, as noted in the paragraph following (9), we require for the convergence proof that x_j corresponding to integer variables y_j *precede* the fractional variables.

EXERCISE: Use artificial variables and the simplex algorithm with perturbation to obtain in the latter case an initial canonical system with the requisite properties, provided one exists. What happens if none exists?

After each augmentation with a supplementary variable (corresponding to a new restriction) we require that the simplex method with perturbation be applied to (64) until it is rendered optimal. We shall refer to this as an *optimal stage*; at such a stage, the coefficients of $\alpha_0(\varepsilon)$ are just the values of v and y_j of a *feasible* solution to the primal, i.e., $\alpha_{0j} \geq 0$ for $j = 1, 2, \ldots, n$. Because we assume that the convex set of feasible solutions to the primal is bounded, the values of the α_{0j} will have, at each such optimal stage, a finite upper bound. If the $k + 1$ leading coefficients $\alpha_{00}, \alpha_{01}, \ldots, \alpha_{0k}$ of $\alpha_0(\varepsilon)$ are all *integers* then the nonnegative coefficients solve the primal and the process terminates. However, if any among the first $k + 1$ are fractional, let $j = j_0$ be the first such. Let $j = n + 1$ be the index of the new column generated from $j = j_0$; then since $\alpha_{0,n+1} < 0$, the new pivot (r, s) is chosen in column $s = n + 1$ and row $r = i$ where $\alpha_{i,n+1} > 0$ and

$$0 < \frac{\alpha_r(\varepsilon)}{\alpha_{r,n+1}} = \operatorname*{Min}_{\varepsilon \to 0} \frac{\alpha_i(\varepsilon)}{\alpha_{i,n+1}} \tag{65}$$

Because (64) is in canonical form it is not difficult to show that the choice of pivot is *unique* (see Chapter 10). As a result of the pivot the *new value* of $\alpha_0(\varepsilon)$ is

$$\alpha_0^*(\varepsilon) = \alpha_0(\varepsilon) - \alpha_r(\varepsilon)(\alpha_{0,n+1}/\alpha_{r,n+1}) > \alpha_0(\varepsilon) \tag{66}$$

For clarity we give again the definition of $j = j_0$ and also define $j = j^*$:

(a) Let $j = j_0$ be the subscript of the first term of $\alpha_0(\varepsilon)$ with non-integral coefficient.

(b) Let $j = j^*$ be the subscript of the first term of $\alpha_0^*(\varepsilon) - \alpha_0(\varepsilon)$ with non-zero coefficient.

THEOREM 4: *Either* $j^* < j_0$ *or* $j^* = j_0$ *and* $\alpha_{0j_0}^* \geq [\alpha_{0j_0}]^* + 1$.

PROOF: The j_0 coefficient of $\alpha_0^*(\varepsilon)$ is

$$\alpha_{0j_0}^* = \alpha_{0j_0} - \alpha_{rj_0}(\alpha_{0,n+1}/\alpha_{r,n+1}) \tag{67}$$

The first non-zero term of $\alpha_0^*(\varepsilon) - \alpha_0(\varepsilon)$ is positive by (66). If it does not occur before $j^* = j_0$, then $\alpha_{rj_0} \geq 0$ (since $\alpha_{0,n+1} < 0$ and the pivot $\alpha_{r,n+1} > 0$,

it follows that $\alpha_{rj_0} < 0$ would contradict the inequality in (66)). The remainder of the theorem follows from the relation between

$$y_{j_0} = \alpha_{0j_0} + \sum_i \alpha_{ij_0}\pi_i \tag{68}$$

and the new restriction on y^*. *If all π_i are integer parameters*, then

$$y^* = -\bar{f}_0 + \sum f_i\pi_i \tag{69}$$

where $\alpha_{ij_0} = [\alpha_{ij_0}]^* + f_i$, so that

$$\alpha^*_{0j_0} = \alpha_{0j_0} - \{[\alpha_{rj_0}]^* + f_r\}(-\bar{f}_0/f_r) \geq (\alpha_{0j_0} + \bar{f}_0) = [\alpha_{0j_0}]^* + 1 \tag{70}$$

which establishes the theorem for this case. *If some π_i are fractional parameters*, we have (46) as the new relation for y^*. In this case $\alpha_{0,n+1} = -1$. If π_r is integral, then

$$\alpha_{r,n+1} = \text{Min}\left[\frac{f_r}{\bar{f}_0}, \frac{\bar{f}_r}{\bar{f}_0}\right] < 1 \tag{71}$$

where the latter relation was established in the discussion following (51). The remainder of the proof parallels (70) if $\alpha_{r,n+1} = f_r/\bar{f}_0$. On the other hand, if $0 < \alpha_{r,n+1} = \bar{f}_r/f_0 < 1$, then

$$\begin{aligned}\alpha^*_{0j_0} &= \alpha_{0j_0} - \{[\alpha_{rj_0}] + 1 - \bar{f}_r\}(-f_0/\bar{f}_r) \\ &\geq \alpha_{0j_0} + (f_0/\bar{f}_r) - f_0 > \alpha_{0j_0} + 1 - f_0 = [\alpha_{0j_0}]^* + 1\end{aligned} \tag{72}$$

Finally, if π_r is a fractional parameter, then (since $\alpha_{r,j_0} \geq 0$)

$$\alpha_{r,n+1} = \alpha_{r,j_0}/\bar{f}_0 > 0 \tag{73}$$

so that (67) reduces simply to

$$\alpha^*_{0j_0} = \alpha_{0j} + \bar{f}_0 = [\alpha_{0j}]^* + 1 \tag{74}$$

THEOREM 5: *If the convex of fractional solutions is bounded, then the algorithm is finite.*

PROOF: Suppose, on the contrary, the algorithm is infinite; then $\alpha_0(\varepsilon)$ forms a monotonically increasing sequence. However, the first term must assume some finite integer value after a finite number of iterations and remain unchanged thereafter. Otherwise (because of the assumption of a bounded convex) for some infinite sequence of optimal stages it would take on a set of values increasing toward or attaining some upper bound, and hence, an infinite sequence of non-decreasing fractional values, whose difference tends to zero. By the preceding theorem, *at each optimum stage* a solution is obtained such that the first term of $\alpha_0(\varepsilon)$ which takes on a fractional value, must on the next cycle be at least equal to the next higher integer value. Here $j_0 = j^* = 0$. Hence, the fractional value of one optimal stage and that of the one after the one which follows are separated by at least unity—a contradiction.

Having established that the first term has a constant integer value after p_1 iterations, the second must also after $p_1 + p_2$ iterations. The argument is the same: Since $\alpha_0(\varepsilon)$ is monotonically increasing for sufficiently small ε, it means that after p_1 iterations the second term must be non-decreasing. Because of boundedness of the convex, the values at the end of successive optimum stages must approach or attain a finite upper bound. It cannot take on an infinite set of fractional values because by the preceding theorem it would (because the first component is fixed) on each subsequent stage assume values at least equal to the next higher integer values, etc. The argument may thus be repeated until the first k components which correspond to integer variables are all integers.

Variations.

In practice the selection of the new restriction is not always made by the lowest-index rule; i.e., generated by the integer variable y_j with the lowest index whose value is fractional in a basic solution. Instead, j is often selected so that $-f_{0j}$ is minimal. This has sometimes cut down the number of iterations in a number of test runs. If this rule is periodically mixed with the lowest index rule, convergence is still guaranteed. Another device is to impose not one but many new constraints simultaneously. If this is done for our example, the simultaneous imposing of all constraints generated by the integer variables and their multiples yields the required integral solution. If it did not, new simultaneous sets of conditions would have to be imposed and the process repeated. Using perturbation and noting that $\alpha_0^*(\varepsilon)$ is at least as large as before, this variation will converge to a solution in a finite number of steps.

Another weakness of the lowest-index rule is that it requires that v be an integer corresponding to the lowest index. It would be much more satisfactory if v could also be a fractional variable. Gomory has devised special rules to guarantee convergence where v is a fractional variable and all other variables are integers. The proof, however, breaks down in the mixed case if conditions (46) are used. It is not known what happens if (53) are used instead. In practice, of course, the above variants are often used even when v is not an integer.

EXERCISE: If v is a fractional variable and all other variables are integer variables, show for rational coefficients that another form can replace the v form which can be maximized instead and whose value is an integer.

26-3. ON THE SIGNIFICANCE OF SOLVING LINEAR PROGRAMMING PROBLEMS WITH SOME INTEGER VARIABLES

Our purpose is systematically to review and classify problems that can be reduced to linear programs, some or all of whose variables are integer

valued [Dantzig, 1960-2]. We shall show that a host of difficult, indeed seemingly impossible, problems of a nonlinear, nonconvex, and combinatorial character are now open for direct attack. The outline for this section is as follows:

I. General Principles.
   (a) Discussion.
   (b) Dichotomies.
   (c) k-fold Alternatives.
   (d) Selection from Many Pairs of Regions.
   (e) Discrete-variable Problems.
   (f) Nonlinear-objective Problems.
   (g) Conditional Constraints.
   (h) Finding a Global Minimum of a Concave Function.
II. The Fixed-charge Problem.
III. The Traveling-salesman Problem.
IV. The Orthogonal Latin-Square Problem.
V. Four-Coloring a Map (if possible).

I. General Principles

(a) *Discussion.* Let us now turn to the main subject, types of problems that are reducible to linear programs some or all of whose variables are integer-valued.

Quite often in the literature, papers appear which formulate a problem in linear programming format *except* for certain side conditions such as $x_1 \cdot x_2 = 0$, or the sum of products of this type, such as $x_1 \cdot x_2 + x_3 \cdot x_4 = 0$, which imply for nonnegative variables that at least one variable of each pair must be zero. Superficially this seems to place the problem in the area of *quadratic programming*. However, the presence of such conditions can change entirely the character of the problem (as we shall see in a moment) and should serve a warning to those who would apply willy-nilly a general nonlinear programming method. If we graph the conditions $x_1 \cdot x_2 = 0$, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 1$, the double lines depict the domain of feasible solutions (see Fig. 26-3-I). It will be noted that this domain has two disconnected parts. If there are many such *dichotomies* in a larger problem, the result can be a domain of feasible solutions with many disconnected parts or connected non-convex regions. For example, k pairs of variables whose products are zero might lead to 2^k disconnected parts. Usual mathematical approaches can guarantee at best a local optimum solution to such problems, i.e., a solution which is optimum only over some connected convex part.

It is well known that in many cases, local optimum solutions could be avoided by the introduction of integer-valued variables, but this fact has been of only passing interest until the recent developments rendered this approach practical. Our purpose here will be to systematize this knowledge.

(b) *Dichotomies.* Let us begin with the important class of problems that have "either-or" conditions. For such a problem to be computationally

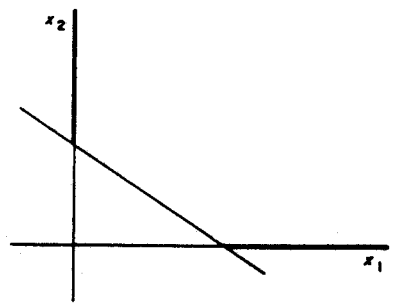


Figure 26-3-I. A disconnected region arising from linear inequalities and conditions like $x_1x_2 = 0$.

difficult, there must be many sets of such conditions. Let us focus our attention on one of them; say

(1) $$\textit{either:}\ G(x_1, x_2, \ldots, x_n) \geq 0$$

(2) $$\textit{or:}\ H(x_1, x_2, \ldots, x_n) \geq 0$$

must hold for vectors $(x_1, x_2, \ldots, x_n)$ chosen from some set S. We do not exclude the case of both holding if this is possible. For example, a contractor in a bid might stipulate either $x_1 \geq \$10{,}000$ or $x_1 = 0$. If all bids are nonnegative so that $x_1 \geq 0$, then we can write

(3) $$\begin{aligned} \textit{either:}\quad & x_1 - 10{,}000 \geq 0 \\ \textit{or:}\quad & -x_1 \qquad\qquad \geq 0 \end{aligned}$$

From other considerations it may be known that no bid can exceed \$1,000,000, so that the set S of interest is $0 \leq x_1 \leq 1{,}000{,}000$.

We now assume that lower bounds for the functions G and H are known for all vectors $(x_1, x_2, \ldots, x_n)$ in S. If L_G is a lower bound for G and L_H for H, then for $\delta = 1$ the condition

$$G(x_1, x_2, \ldots, x_n) - \delta L_G \geq 0$$

holds for all $(x_1, x_2, \ldots, x_n)$ in S. Similarly for $\delta = 0$ the condition

$$H(x_1, x_2, \ldots, x_n) - (1 - \delta)L_H \geq 0$$

holds for all $(x_1, x_2, \ldots, x_n)$ in S. For our example we would have

(4) $$\begin{aligned} x_1 - 10{,}000 - \delta(-10{,}000) &\geq 0 \\ -x_1 - (1 - \delta)(-1{,}000{,}000) &\geq 0 \end{aligned}$$

The either-or condition (1), (2) can now be replaced by

$$G(x_1, x_2, \ldots, x_n) - \delta L_G \geq 0 \qquad (\delta = 0, 1) \tag{5}$$

$$H(x_1, x_2, \ldots, x_n) - (1 - \delta) L_H \geq 0 \tag{6}$$

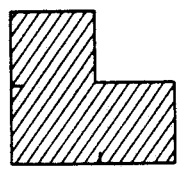
$$0 \leq \delta \leq 1 \tag{7}$$

where δ is an integer variable. The effect of $\delta = 1$ is to relax the G condition when H holds and that of $\delta = 0$ is to relax H when G holds. If G and H are linear functions, we have reduced the either-or condition to three simultaneous linear inequalities in which the variable δ must be 0 or 1.

A dichotomy can be used to describe an L-shaped region (non-convex): for example, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 2$, $x_2 \leq 2$, and either $x_1 \leq 1$ or $x_2 \leq 1$. We replace this by

(8)

$$\begin{aligned} 0 &\leq x_1 \leq 1 + \delta \\ 0 &\leq x_2 \leq 2 - \delta \\ 0 &\leq \delta \leq 1 \end{aligned} \qquad (\delta = 0, 1)$$



If now a problem contains not one but several such pairs of dichotomies (1) and (2), each one would be replaced by a simultaneous set (5), (6), (7) in integer variables δ_i.

(c) *k-fold Alternatives.* More generally suppose that we have a set of conditions

$$\begin{aligned} G_1(x_1, x_2, \ldots, x_n) &\geq 0 \\ G_2(x_1, x_2, \ldots, x_n) &\geq 0 \\ &\vdots \\ G_p(x_1, x_2, \ldots, x_n) &\geq 0 \end{aligned} \tag{9}$$

Suppose a solution is required in which at least k of the conditions must hold simultaneously. We replace (9) by

$$\begin{aligned} G_1(x) - \delta_1 L_1 &\geq 0 \\ G_2(x) - \delta_2 L_2 &\geq 0 \\ &\vdots \\ G_p(x) - \delta_p L_p &\geq 0 \end{aligned} \tag{10}$$

where L_i is the lower bound for $G_i(x)$ for $x = (x_1, x_2, \ldots, x_n)$ in S, and δ_i are integer-valued variables satisfying

$$\delta_1 + \delta_2 + \ldots + \delta_p \leq p - k \tag{11}$$

and

$$0 \leq \delta_i \leq 1 \qquad (\delta_i = 0 \text{ or } 1) \tag{12}$$

An example of this type of problem might occur if one wishes to find the minimum over the shaded regions described by $G_1 \geq 0$, $G_2 \geq 0$, $G_3 \geq 0$, and at least two of the conditions $G_4 \geq 0$, $G_5 \geq 0$, $G_6 \geq 0$ as in Fig. 26-3-II.

(d) *Selection from Many Pairs of Regions.* The six-pointed "Star of David" region shown in Fig. 26-3-II (lower part) can best be described as

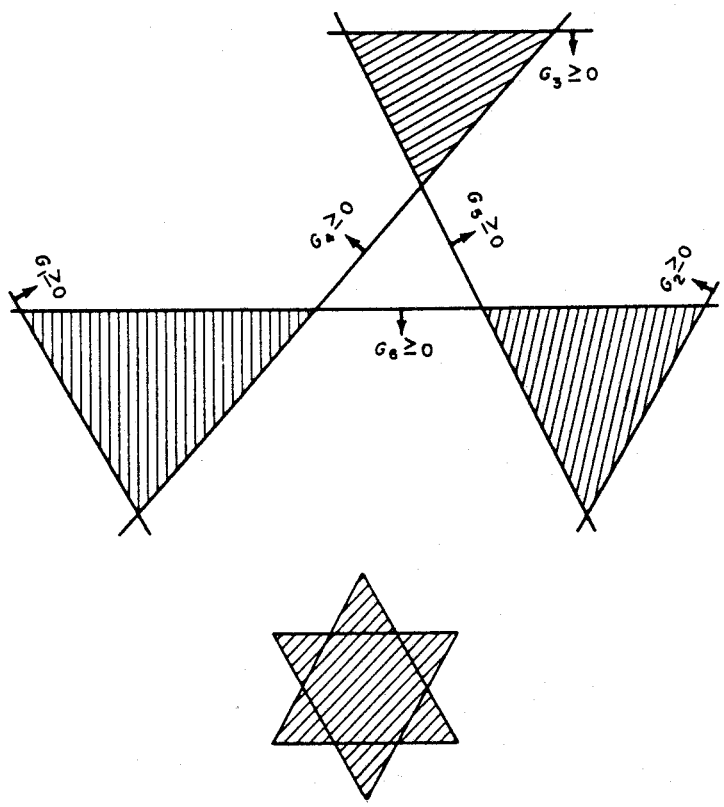


Figure 26-3-II. Examples of non-convex regions representable by a mixed-integer program.

a dichotomy in which a point must be taken from one of two triangles. It is only when there are many such pairs to be chosen at the same time that the problem becomes significant. In general we might have several pairs of regions (R_1, R_1'), (R_2, R_2'), . . ., (R_n, R_n'), and the solution point x must lie in either R_i or R_i' for each i. For each pair R and R' we proceed as follows. Let region R be described by a set of inequalities $G_1(x) \geq 0$, $G_2(x) \geq 0$,

. . ., $G_m(x) \geq 0$, and R' by $H_1(x) \geq 0$, $H_2(x) \geq 0$, . . ., $H_n(x) \geq 0$. The condition that the point must be selected from either the first or the second region can be written

$$
\begin{aligned}
(13) \qquad & G_1(x) - \delta L_1 \geq 0, && H_1(x) - (1-\delta)L_1' \geq 0, \\
& G_2(x) - \delta L_2 \geq 0, && H_2(x) - (1-\delta)L_2' \geq 0, \\
& \vdots && \vdots \\
& G_m(x) - \delta L_m \geq 0, && H_n(x) - (1-\delta)L_n' \geq 0, \\
& & 0 \leq \delta \leq 1 && (\delta = 0 \text{ or } 1)
\end{aligned}
$$

where L_i, L_i' are lower bounds for G_i and H_i. The more general case of selection from several regions can be done by introducing several δ_i as in (11) and (12).

(e) *Discrete-variable Problems.* Suppose that a variable is constrained to take one of several values: $x_1 = a_1$ or $x_1 = a_2$, . . ., or $x_1 = a_k$ and at the same time several other variables are also constrained the same way. It would be a formidable task to test all the combinations. Instead we replace each k-fold dichotomy by

$$x_1 = a_1\delta_1 + a_2\delta_2 + \ldots + a_k\delta_k \tag{14}$$

$$1 = \delta_1 + \delta_2 + \ldots + \delta_k \qquad (\delta_i = 0 \text{ or } 1) \tag{15}$$

Similarly let $x = (x_1, x_2, \ldots, x_n)$ represent a vector which may only take on specified vector values $x = a^1$ or $x = a^2$ or $x = a^3$ . . . . This may be replaced by

$$x = a^1\delta_1 + a^2\delta_2 + \ldots + a^k\delta_k \tag{16}$$

$$1 = \delta_1 + \delta_2 + \ldots + \delta_k \qquad (\delta_i = 0 \text{ or } 1) \tag{17}$$

This device permits the replacement of a nonlinear function $F_{ij} = F_{ij}(x_j)$, in a system $\sum_{j=1}^{n} F_{ij}(x_j) = 0$ for $i = 1, 2, \ldots, m$, by the values of the function corresponding to a sprinkling of representative values of x_j, say $x_j = x_j^r$ where $r = 1, 2, \ldots, k$. In this case the vector takes on the set of values $F_{1j}, F_{2j}, \ldots, F_{mj}$ for each value $x_j = x_j^r$.

(f) *Nonlinear-objective Problems.* Referring to Fig. 26-3-III, suppose the objective form can be written

$$\sum_{j=1}^{n} \phi_j(x_j) = z \text{ (Min)} \tag{18}$$

where ϕ_j is nonlinear and non-convex. Let each $\phi(x)$ be approximated by a broken line function. These define a set of intervals $i = 1, 2, \ldots, k$ of

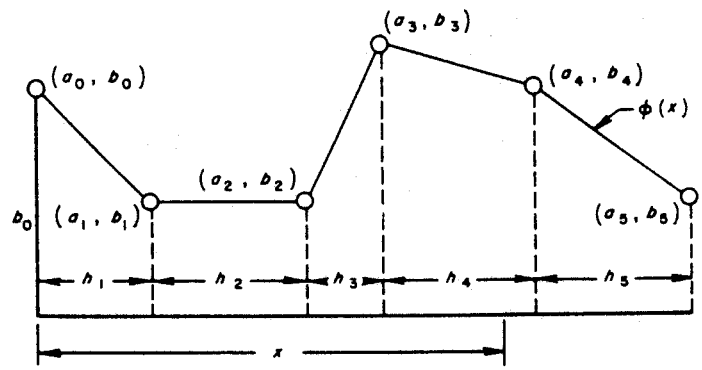


Figure 26-3-III. Converting a non-convex function into a mixed-integer-programming format.

width h_i and slopes s_i for the approximating chords. We now define y_i as the amount of overlap of the interval from 0 to x with interval i. Then

(19) $$x = y_1 + y_2 + \cdot \cdot \cdot + y_k$$

and $\phi(x)$ is given approximately by

(20) $$\phi(x) \doteq b_0 + s_1 y_1 + s_2 y_2 + \cdot \cdot \cdot + s_k y_k$$

where

(21) $$0 \leq y_i \leq h_i \qquad (i = 1, 2, \ldots, k)$$

In the case of *convex* ϕ, the procedure is to replace x and $\phi(x)$ by (19) and (20) and conditions (21). Here the slopes are monotonically increasing so that

(22) $$s_1 \leq s_2 \leq \cdot \cdot \cdot \leq s_k$$

For a fixed x, $\phi(x)$ would be minimum if y_1 is chosen maximum and if for y_1 maximum, y_2 is chosen maximum, etc. In other words, for the minimizing solution the y_i's are the overlap of the ith interval with the interval 0 to x, and all is well. See § 24-3.

However, if $\phi(x)$ is *not convex* as in Fig. 26-3-III, then simple replacement of x and $\phi(x)$ would result, for fixed x, in those y_i with smaller slopes being maximized first. In this case the segments that comprise x would be disconnected and our approximation for $\phi(x)$ would no longer be valid. In order to avoid this, we impose the condition that

(23) $$\text{either } h_i - y_i = 0 \qquad \text{or } y_{i+1} = 0$$

which implies that unless y_i is maximum, $y_{i+1} = 0$, and if y_i is maximum, then $y_{i+1} > 0$ is possible. We rewrite this condition

(24) $$\text{either } y_i - h_i \geq 0 \qquad \text{or } -y_{i+1} \geq 0$$

and then replace it formally by

$$(25)\qquad \begin{cases} -y_1 + h_1 \geq 0 & \\ y_i - h_i - (-h_i)\delta_i \geq 0 & (i = 1, 2, \ldots, k-1) \\ -y_{i+1} - (-h_{i+1})(1 - \delta_i) \geq 0 & \\ 0 \leq \delta_i \leq 1 & (\delta_i = 0 \text{ or } 1) \end{cases}$$

Upon substitution of $\delta_i' = 1 - \delta_i$, (25) simplifies to

$$(26)\qquad \begin{cases} y_1 \leq h_1 & \\ y_i \geq h_i\delta_i' & (i = 1, 2, \ldots, k-1) \\ y_{i+1} \leq h_{i+1}\delta_i' & \\ 0 \leq \delta_i' \leq 1 & (\delta_i' = 0 \text{ or } 1) \end{cases}$$

which together with (19) and (20) formulates the problem (note that (21) is not required). The above procedure for the non-convex case was discussed in the paper of Markowitz and Manne [1957-1]. The convex case will be found in [Dantzig, 1956-2] and [Charnes and Lemke, 1954-1].

A second method based on (16) is worth noting. Any point on the curve $\phi(x)$ can be represented as a weighted average of two *successive* breakpoints. Hence we may replace x and $\phi(x)$ by

$$(27)\qquad \begin{cases} x = \lambda_0 a_0 + \lambda_1 a_1 + \ldots + \lambda_k a_k & (0 \leq \lambda_i) \\ \phi(x) = \lambda_0 b_0 + \lambda_1 b_1 + \ldots + \lambda_k b_k & \end{cases}$$

$$(28)\qquad 1 = \lambda_0 + \lambda_1 + \ldots + \lambda_k$$

and then impose the conditions that all $\lambda_i = 0$ except for one pair λ_i and λ_{i+1}. For $k = 4$ this may be expressed by

$$(29)\qquad \begin{aligned} \lambda_0 &\leq \delta_0 \\ \lambda_1 &\leq \delta_0 + \delta_1 \\ \lambda_2 &\leq \delta_1 + \delta_2 \\ \lambda_3 &\leq \delta_2 + \delta_3 \\ \lambda_4 &\leq \delta_3 + \delta_4 \\ \lambda_5 &\leq \delta_4 \end{aligned}$$

where the δ_i are integer-valued variables satisfying

$$(30)\qquad \delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \qquad (\delta_i = 0, 1)$$

Indeed it will be noted that, when $\delta_{i_0} = 1$ for some $i = i_0$, the inequalities involving λ_{i_0} and λ_{i_0+1} are relaxed but the remainder satisfy $\lambda_i \leq 0$ since their $\delta_i = 0$ by (30).

(g) *Conditional Constraints.* Suppose x and y are functions of several variables. We assume that an upper bound U_x is known for x, and lower

bounds L_x and L_y are known for x and y. We wish to impose conditions such as

$$x > 0 \Rightarrow y \geq 0 \tag{31}$$

We can write (31) as

$$\text{either } x > 0, y \geq 0 \text{ or } x \leq 0 \tag{32}$$

which we rewrite as

$$\begin{cases} x \geq \delta L_x \\ y \geq \delta L_y \\ x \leq (1 - \delta)U_x \end{cases} \qquad (\delta = 0, 1) \tag{33}$$

where the first inequality is written ($\geq$) instead of ($>$) because the condition $y \geq 0$ is automatically relaxed for $x = 0$ by selecting $\delta = 1$.

We can now elaborate this method to impose conditions such as

$$\begin{aligned} x > 0 &\Rightarrow u \geq 0 \\ x < 0 &\Rightarrow w \geq 0 \end{aligned} \tag{34}$$

which may be written as

$$\begin{aligned} x &\geq (1 - \delta_3)L_x \\ x &\leq (1 - \delta_3)U_x \\ x &\geq \delta_1 L_x \\ u &\geq \delta_1 L_u \\ x &\leq \delta_2 U_x \\ w &\geq \delta_2 L_w \\ \delta_1 + \delta_2 &= 1 + \delta_3 \qquad (\delta_i = 0, 1) \end{aligned} \tag{35}$$

For example, suppose in a T-period program we wish to complete a specified work load by the *earliest* period possible. Let x_t be the *cumulative sum* of activity levels from the tth period through the last period T, then we wish to arrange matters so that $x_t = 0$ for the smallest t. Note in this case $x_k > 0$ implies $x_s > 0$ for $s < k$. In this case we can define for $t = 1, 2, \ldots, T$,

$$\delta_t = 0 \Rightarrow x_t = 0 \tag{36}$$

We may rewrite (36)

$$0 \leq x_t \leq \delta_t U_t \qquad (\delta_t = 0, 1) \tag{37}$$

where U_t is an upper bound for x_t, and then determine Min z where

$$z = \delta_1 + \delta_2 + \ldots + \delta_T \tag{38}$$

(h) *Finding a Global Minimum of a Concave Function.*[3] Suppose the

[3] This application was developed jointly with P. Wolfe.

concave function $Z = Z(x_1, x_2, \ldots, x_n)$ is to be minimized over a region R. If R is convex, this is intrinsically a difficult problem because the concave function could have local minima at many, indeed at *all*, the extreme points of R. We shall in fact assume R convex, for we note that the devices discussed earlier allow us to use a convex domain coupled with integer-valued variables to solve a wide class of problems expressible by either-or conditions. We suppose R to be given, after suitable change in variables, in standard linear programming form

$$Ex = e, \qquad x \geq 0 \tag{39}$$

where E is a given $m \times n$ matrix and e a given m-component vector.

The concave function Z may be given explicitly or be given implicitly. As an example of the latter, suppose vector y and quantity Z for *fixed* x is given by

$$\begin{aligned} Fy &= f + \bar{E}x, \qquad y \geq 0 \\ \operatorname*{Max}_{y|x} \beta y &= Z \text{ (Min)} \end{aligned} \tag{40}$$

where $\bar{E}$ and F are given matrices and f and β are given vectors.

EXERCISE: Prove $\text{Max}_{y|x}\ \beta y$ is a concave function of x. Prove that $\phi(x) = \text{Min}\{\phi_1(x), \phi_2(x) \ldots\}$ is a concave function if $\phi_i(x)$ is concave for all i.

An illustration having a striking parallel in the real world can be given: Consider a two-move game in which the first player, A, by choosing his activity levels x consistent with (39) modifies the inventory f of his opponent by an amount $\bar{E}x$. The second player, B, by choosing his activity levels y consistent with (40) obtains a payoff βy. A's problem is to choose x so as to minimize the maximum payoff to B.

We shall suppose that Z can reasonably be approximated at all points x in R by the minimum Z of a *finite set of k tangent hyperplanes*,

$$Z = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - b_i \quad (i = 1, 2, \ldots, k) \tag{41}$$

to the surface $Z = Z(x)$. The problem reduces to choosing x, Min Z satisfying (39) such that (x, Z) satisfies *at least one* of the conditions

$$\begin{aligned} &Z - [a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - b_1] \geq 0 \\ &Z - [a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - b_2] \geq 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &Z - [a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n - b_k] \geq 0 \end{aligned} \tag{42}$$

We may rewrite (42) as

$$\begin{aligned} Z - [a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n] &\geq M(1 - \delta_i) \quad (i = 1, 2, \ldots, k) \\ \delta_1 + \delta_2 + \ldots + \delta_k &= 1 \qquad\qquad (\delta_i = 0, 1) \end{aligned} \tag{43}$$

where M is some assumed lower bound for the differences. This solution

depends on the approximation of the function $Z = Z(x)$ by k hyperplanes. The solution, given in [Dantzig, 1958-2], for the case where Z is given implicitly by (40), requires finding x, y, Min z, and auxiliary variables $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ and $\eta \geq 0$, for $j = 1, 2, \ldots, n'$, satisfying

$$(44) \qquad Ex = e, \qquad Fy = f + \bar{E}x, \qquad z = \beta y$$
$$\pi F + \eta - \beta = 0$$
$$\text{either } \eta_j \geq 0 \text{ or } y_j \geq 0$$

where $\pi = (\pi_1, \pi_2, \ldots, \pi_{m'})$ is a row vector, y_j is the jth component of y, and η_j the jth component of η.

EXERCISE: Verify (44).

II. The Fixed-charge Problem.

Earlier we described a problem where a bidder required that the size of the bid satisfies either $x = 0$ or $x \geq a$. In this and many other problems there is an underlying notion of a fixed charge that is independent of the size of the order. In this case $x = a$ represents the break-even point to the bidder. In general, the cost C is characterized by

$$(45) \qquad C = \begin{cases} kx + b & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where b is the fixed charge. We may write this in the form

$$(46) \qquad C = kx + \delta b \qquad (\delta = 0, 1)$$

where $x = 0$ if $\delta = 0$, which we impose by

$$(47) \qquad x \leq \delta U$$

and

$$(48) \qquad 0 \leq \delta \leq 1 \qquad (\delta = 0, 1)$$

where U is some upper bound for x. A discussion of the fixed-charge problem including this device will be found in the paper by W. Hirsch and the author [1954-1].

III. The Traveling-salesman Problem.

The Problem. In what order should a traveling salesman visit n cities to minimize the total distance covered in a complete circuit? We shall give three formulations of this well-known problem. Let $x_{ijt} = 1$ or 0 according to whether the tth directed arc on the route is from node i to node j or not. Letting $x_{ijn+1} \equiv x_{ij1}$, the conditions

$$(49) \qquad \sum_i x_{ijt} = \sum_k x_{j,k,t+1} \qquad (j, t = 1, \ldots, n)$$

$$\sum_{j,t} x_{ijt} = 1 \qquad (i = 1, \ldots, n) \tag{50}$$

$$\sum_{i,j,t} d_{ij}x_{ijt} = z \text{ (Min)} \tag{51}$$

express (a) that if one arrives at city j on step t, one leaves city j on step $t + 1$, (b) that there is only one directed arc leaving node i, and (c) the length of the tour is minimum. It is not difficult to see that an integer solution to this system is a tour [Flood, 1956-1].

In two papers by Dantzig, Fulkerson, and Johnson [1954-1, 1959-1] the case of a *symmetric distance* $d_{ij} = d_{ji}$ was formulated with only two indices. Here $x_{ji} \equiv x_{ij} = 1$ or 0 according to whether the route from i to j or from j to i was traversed at some time on a route or not. In this case

$$\sum_{i} x_{ij} = 2 \qquad (j = 1, 2, \ldots, n) \tag{52}$$

and

$$\sum_{i,j} d_{ij}x_{ij} = z \text{ (Min)} \tag{53}$$

express the condition that the sum of the number of entries and departures for each node is two. Note in this case that no distinction is made between the two possible directions that one could traverse an arc between two cities. These conditions are not enough to characterize a tour even though the x_{ij} are restricted to be integers in the interval

$$0 \leq x_{ij} \leq 1 \tag{54}$$

since sub-tours like those in Fig. 26-3-IV also satisfy the conditions. However,

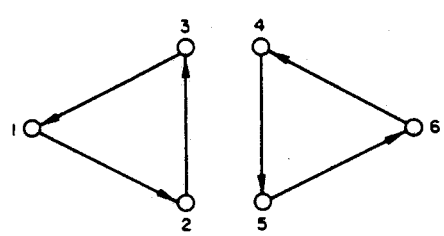


Figure 26-3-IV. Loop conditions are added to rule out sub-tours in the traveling-salesman problem.

if so-called *loop conditions* like

$$x_{12} + x_{23} + x_{31} \leq 2 \tag{55}$$

are imposed as added constraints as required, these will rule out integer solutions which are not admissible.

EXERCISE: Construct a fractional solution to (49), (50), (51) that is extremal. Show that any integer solution without sub-tours is a full tour.

A third way to reduce a traveling-salesman problem to an integer program is due to A. W. Tucker [1960-1]. It has less constraints and variables than those above. Let $x_{ij} = 1$ or 0, depending on whether the salesman travels from city i to j or not, where $i = 0, 1, 2, \ldots, n$. Then an optimal solution can be found by finding integers x_{ij}, arbitrary real numbers u_i, and Min z satisfying

$$\sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \ldots, n) \tag{56}$$

$$\sum_{j=0}^{n} x_{ij} = 1 \qquad (i = 1, 2, \ldots, n)$$

$$u_i - u_j + nx_{ij} \leq n - 1 \qquad (1 \leq i \neq j \leq n)$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} d_{ij}x_{ij} = z \text{ (Min)}$$

The third group of conditions is violated whenever we have an integer solution to the first two groups that is *not* a tour, for in this case it contains two or more loops with $k < n$ arcs. In fact, if we add all inequalities corresponding to $x_{ij} = 1$ around such a loop *not* passing through city 0, we will cancel the differences $u_i - u_j$ and obtain $nk \leq (n - 1)k$, a *contradiction*. We have only to show for *any tour* starting from $i = 0$ that we can find u_i that satisfies the third group of conditions. Choose $u_i = t$ if city i is visited on the tth step where $t = 1, 2, \ldots, n$. It is clear that the difference $u_i - u_j \leq n - 1$ for all (i, j). Hence the conditions are satisfied for all $x_{ij} = 0$; for $x_{ij} = 1$ we obtain $u_i - u_j + nx_{ij} = (t) - (t + 1) + n = n - 1$.

IV. The Orthogonal Latin-Square Problem.

A latin square consists of n sets of n objects (1), (2), . . ., (n) assigned to an $n \times n$ square array so that no object is repeated in any row or column. Two latin squares are *orthogonal* if the n^2 ordered pairs of corresponding entries are all different; for example

(57)

(1)	(2)	(3)	(2)	(3)	(1)
(2)	(3)	(1)	(1)	(2)	(3)
(3)	(1)	(2)	(3)	(1)	(2)

It was conjectured by Euler that there are no orthogonal latin squares for certain n. After a great deal of research, the case for $n = 10$, for example,

was settled[4] in 1959. It has been suggested informally by David Gale that integer programming be tried in this area.

The formulation is straightforward and well known. Let $x_{ijkl} = 0$ or 1 according to whether or not the pair (i, j) is assigned to row k, column l. The condition that each pair (i, j) is assigned to only one location is given by

$$\sum_{k,l} x_{ijkl} = 1 \qquad (i, j = 1, 2, \ldots, n) \tag{58}$$

The condition that one pair (i, j) is assigned to each location k, l is:

$$\sum_{i,j} x_{ijkl} = 1 \qquad (k, l = 1, 2, \ldots, n) \tag{59}$$

The condition that i appears only once in the first latin square in column l is given by (60). The condition that j appears only once in the second latin square in column k is given by (61).

$$\sum_{j,k} x_{ijkl} = 1 \qquad (i, l = 1, 2, \ldots, n) \tag{60}$$

$$\sum_{i,k} x_{ijkl} = 1 \qquad (j, l = 1, 2, \ldots, n) \tag{61}$$

Similarly the condition that i and j appear only once in the first and second latin square respectively in row k is given by

$$\sum_{j,l} x_{ijkl} = 1 \qquad (i, k = 1, 2 \ldots, n) \tag{62}$$

$$\sum_{i,l} x_{ijkl} = 1 \qquad (j, k = 1, 2 \ldots, n)$$

It is interesting to note that all pairs of subscripts possible out of four are summed to form the six sets of n^2 equations. For $n = 10$ there are 600 equations which are too many for any integer programming code devised up to 1962 to handle. However, with some short-cuts introduced, it might become tractable.

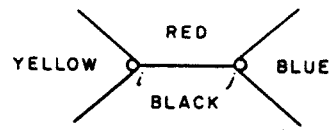
V. Four-Coloring a Map (If Possible).

A famous unsolved problem is to prove or disprove that any map *in the plane* can be colored using at most four colors; regions with a boundary in common (other than a point) must have different colors. We shall give two ways to color constructively a particular map, if possible. While this does not

[4] R. C. Bose and S. S. Shukhande proved that Euler's famous conjecture about the non-existence of orthogonal latin squares of certain even orders was false. E. T. Parker has constructed a pair of orthogonal latin squares of order 10. For further information the reader is referred to Abstract 558-27 of the August 1959 Notices of the American Mathematical Society. A non-technical report of their results has appeared in the *Scientific American*, Vol. 201, No. 5, November 1959, pp. 181–188.

contribute to a proof of the truth or falsity of this conjecture, nevertheless an efficient way for solving particular examples on an electronic computer may serve as an aid in finding a counter example.

Without difficulty it can be arranged (as below) so that three regions



have at most one point in common which will be called a *node*. There will be, accordingly, three arcs leading from any node i to other nodes j. It is well known that if it is possible to four-color a map, then (and this will be true conversely) it is possible (treating the nodes as cities and the arcs as routes between cities) either to tour all the nodes or to make a group of mutually exclusive sub-tours of the cities in several *even* (sub-cycle) loops as in Fig. 26-3-V.

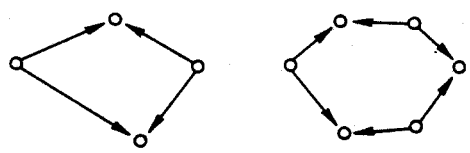


Figure 26-3-V. The four-color problem is equivalent to finding a set of even-order sub-tours.

EXERCISE: Show that the number of arcs is a multiple of 3 and the number of nodes is even.

We may associate with each such even-cycle sub-tour, directed arcs that *reverse their direction* as we pass from node to node. This means the nodes i can be divided into two classes: those which have two arcs pointing away from them and those which have two arcs pointing toward them. Let us set $x_{ij} = 1$, if the directed arc $(i \to j)$ is part of such a sub-tour; otherwise $x_{ij} = 0$. Hence

$$0 \leq x_{ij} \leq 1 \tag{63}$$

It is understood that only arcs (i, j) and variables x_{ij} are considered corresponding to regions that have a boundary in common. All arcs (i, j) that do not correspond to boundaries are omitted in the constraints.

The conditions

$$\sum_j x_{ij} = 2\delta_i \qquad (\delta_i = 0, 1) \tag{64}$$

express the fact there must be two arcs on some sub-tour leading away from node i if $\delta_i = 1$; otherwise there are none. The conditions

$$\sum_i x_{ij} = 2 - 2\delta_i \tag{65}$$

state there must be two arcs on some sub-tour leading into node i if $\delta_i = 0$, and otherwise none. The three sets of conditions (63), (64), and (65) are those of a *bounded transportation problem* and will be integers (at an extreme point) if the δ_i are integers. This would seem to imply that it is only necessary to assume that the δ_i are integers and the x_{ij} will come out automatically integral in an extremizing solution without further assumptions. However, the wrong choice of integral δ_i could lead to an empty solution set. As an alternative to using an integer programming method to solve this problem, it might be practical to allow the δ_i to vary $0 \leq \delta_i \leq 1$ but to randomly choose various objective forms (since these are open to choice) until an optimal extreme-point solution with integral x_{ij} is obtained.

A second formulation suggested informally by Gomory is straightforward. Let the regions be $r = 1, 2, \ldots, R$, and let t_r be an integer-valued variable such that

$$0 \leq t_r \leq 3$$

where the four values $t_r = 0, 1, 2, 3$ correspond to the four colors. If regions r and s have a boundary in common, their colors must be different. Hence for each such pair

$$t_r - t_s \neq 0 \tag{66}$$

This may be written in either-or form:

$$\text{either } t_r - t_s \geq 1 \text{ or } t_s - t_r \geq 1 \tag{67}$$

which we may rewrite as

$$\begin{aligned} t_r - t_s &\geq 1 - 4\delta_{rs} \\ t_s - t_r &\geq -3 + 4\delta_{rs} \end{aligned} \qquad (\delta_{rs} = 0, 1) \tag{68}$$

REFERENCES

Beale, 1958-1
Charnes and Lemke, 1954-1
Dantzig, 1956-2, 1957-2, 1958-1, 2, 1960-2
Dantzig, Fulkerson, and Johnson, 1954-1, 1959-1
Doig and Belz, 1956-1
Eisemann, 1957-1
Flood, 1956-1
Ford and Fulkerson, 1960-1
Fulkerson, 1956-1, 1959-1, 1960-1
Gale, 1956-1
Gomory, 1958-1, 2, 3
Heller, 1955-1, 1957-1
Hirsch and Dantzig, 1954-1
Hoffman, 1956-1, 1960-1
Hoffman and Kruskal, 1956-1
Johnson, 1954-1, 1958-1
Kruskal, 1956-1
Lambert, 1960-1
Land and Doig, 1957-1
LaVallee, 1955-1
Markowitz and Manne, 1957-1
Miller, Tucker, and Zemlin, 1960-1
Norman, 1955-1
Paull and Walter, 1955-1
Ryser, 1961-1
Saaty, 1955-1
Tucker, 1960-1
Tutte, 1946-1

CHAPTER 27

STIGLER'S NUTRITION MODEL: AN EXAMPLE OF FORMULATION AND SOLUTION

One of the first applications of the simplex algorithm was to the determination of an adequate diet that was of least cost.[1] In the fall of 1947, J. Laderman of the Mathematical Tables Project of the National Bureau of Standards undertook, as a test of the newly proposed simplex method, the first large-scale computation in this field. It was a system with nine equations in seventy-seven unknowns. Using hand-operated desk calculators, approximately 120 man-days were required to obtain a solution.

The particular problem solved was one which had been studied earlier by G. J. Stigler [1945-1], who had proposed a solution based on the substitution of certain foods by others which gave more nutrition per dollar. He then examined a "handful" of the possible 510 ways to combine the selected foods. He did not claim the solution to be the cheapest but gave good reasons for believing that the annual cost could not be reduced by more than a few dollars. Indeed, we shall see that Stigler's solution, when converted from a cost-per-day to a cost-per-year, was only 24 cents higher than the true minimum for the year, which was $39.69 (10.9 cents per day).

27-1. PROBLEMS IN FORMULATING A MODEL

Before launching into the mathematical characteristics of the nutrition problem, it is worthwhile to see just how to develop a "mathematical model." It will be seen to be far from a precise operation, and it is only natural to question the validity of refined techniques for solving what is clearly an approximate model. This situation is typical almost everywhere programming techniques are applied. One should remember, however, that one reason why only the approximate models exist today has been the historic inability of the investigator to solve any large-scale complex model. As the tools for handling these systems increase, so does the desire of the investigator increase to refine his models to take advantage of these new techniques. The next few years will probably see the end of this vicious

[1] J. Cornfield of the U.S. Government formulated such a mathematical problem as early as 1940.

circle of the past, in which poor model building justified rapid rough "solutions" and, conversely, the non-existence of methods of accurate solution justified poor model building. It is likely that both model building and solution techniques will begin to reinforce each other in a positive manner.

Stigler's paper [1945-1] provides a very frank discussion of the background of the nutrition model; the greater part of what follows is based upon this source and follows it very closely. In the years since his paper was published, many improvements in the model have been made. However, it is interesting to study some of the typical situations that confront a model builder in the early stages. Stigler began with some findings of nutrition studies:

(1) After certain minimum values of the nutrients are secured, additional quantities yield decreasing (and, in some cases, eventually negative) returns to health.
(2) The optimum quantity of any nutrient depends on the quantities of the other nutrients available.

Diminishing returns are illustrated by the fact that the amount of calcium in the body increases much more slowly than the input of calcium, and that increases of longevity are not proportional to calcium inputs. It appears possible in some cases to substitute one type of nutrition for another type. Stigler cites an example in which it was recommended that 30 micrograms of thiamine be substituted for 100 calories derived from sources other than fats. Another example cited is that a loss of riboflavin accompanies a deficiency of thiamine.

Stigler then turned to another question: How much of various nutrients are required? How do the requirements differ from one individual to another? In resolving this difficulty, he noted that the optimum quantity of calories is known fairly accurately, but that the requirements of other nutrients are known only roughly or not at all. Many minima (to which 50 per cent is usually added as a safety factor) are found by determining the lowest level of input compatible with a stable rate of loss of the nutrient through excreta. The interrelationships among various nutrients are even more obscure, and they are virtually ignored in dietary recommendations. Even the statement of what substances are necessary for health is very complex. Thus, in addition to calcium, the body requires about 13 minerals (some in minute quantities), many kinds of vitamins, a dozen or so types of amino acids, and perhaps many more nutrients yet to be discovered.

The diets developed by Stigler were considerably lower in cost than those developed by others. One of the reasons given is that the other diets included a greater variety of foods as a kind of "insurance" against omitting any of the unknown dietary elements. Another reason is that diet experts do give some weight to social and institutional pressures, particularly where they are not on firm grounds to support alternatives. On the other hand,

Stigler justifies his diet in this regard by citing the National Research Council's belief that these other minerals and vitamins are supplied in adequate amounts automatically when a certain group of common nutrients are obtained from natural foods. Based on considerations of this kind, the first step in setting up a mathematical model was to accept the Council's statement of daily nutritional requirements (given in Table 27-1-I). Note

TABLE 27-1-I

DAILY ALLOWANCES OF NUTRIENTS FOR A MODERATELY ACTIVE MAN (Weighing 154 pounds)[1]

Nutrient	Daily Allowance
Calories	3,000 $= b_1$ (calories)
Protein	70 $= b_2$ (grams)
Calcium	.8 $= b_3$ (grams)
Iron	12 $= b_4$ (milligrams)
Vitamin A	5,000 $= b_5$ (International Units)
Thiamine (B_1)	1.8 $= b_6$ (milligrams)
Riboflavin (B_2 or G)	2.7 $= b_7$ (milligrams)
Niacin (nicotinic acid)	18 $= b_8$ (milligrams)
Ascorbic acid (C)	75 $= b_9$ (milligrams)

[1] National Research Council, "Recommended Dietary Allowances," Reprint and Circular Series No. 115, January, 1943.

that only nine of the more common nutrients were used, and the others were assumed to be automatically satisfied. It should also be noted that the requirements (discussed earlier) are rough, and, possibly with the exception of calories, almost any number in a very broad range probably could equally well be justified.

In considering the nutritive values of foods, again we see a similar situation, for the nutritive values of common foods are known only approximately, and that is all that can be known about them. A large margin of uncertainty arises on several scores. For example, the milligrams of ascorbic acid per 100 grams of apples vary with the type of apple:

Jonathan	4.4
McIntosh	2.0
Northern Spy	11.0
Ontario	20.8
Winesap	5.8
Winter Banana	6.6.

The ascorbic acid in milk varies with season. Conditions of storage, such as temperature and length of time in storage, are important factors. The more corn matures, the greater is the amount of vitamin A, but the ascorbic acid content decreases. Long cooking decreases the nutritive value; well-done roasts of beef have roughly 70 per cent of the thiamine, riboflavin, and niacin of raw cuts. Not only is there a considerable variability in foods,

which conceivably could be taken into account in programming, by introducing probabilistic considerations (as is done in Chapter 25 on uncertainty), but there is also the fact, according to Stigler, that the nutritive values that had been established in 1944 for many foods had been determined by obsolete and inaccurate techniques, or may be just plain wrong for other reasons.

Ignoring these difficulties, a model was set up in which some kind of average nutritive per unit quantity for each food as purchased was developed. If x_j units of the jth food were purchased and each food contained, per unit quantity, a_{ij} units of the ith nutrient, then it was assumed that the individual would receive

$$\sum_{j=1}^{n} a_{ij}x_j$$

units of the ith nutrient, assuming there are no losses due to preparation of the foods. There is also a tacit assumption that there is no interaction between various foods; i.e., the total number of units of a nutrient available in a food is unaffected by the presence of some other food in the diet.

Finally, a list of potential foods was selected for which retail prices were reported by the Bureau of Labor Statistics. The list was not complete since it excluded almost all fresh fruits, many cheap vegetables rich in nutrients, and fresh fish. If these could have been included, it would seem that the minimum cost diet could be reduced by a substantial amount. However, other investigators have found that the optimal choice is quite insensitive to these particular prices due to the presence of certain staples in the optimum diet.

In Table 27-1-II, the coefficients a_{ij} per dollar expenditure are given for an abbreviated list of some 77 types of foods considered by Stigler. He recommended, however, where prices are subject to change because of local and seasonal conditions, and it is desired to compute not one but several such problems, that the units for measuring the quantity of foods be physical units such as weight, or possibly volume in case of liquids. If this is done, the price data and the nutritive data can be developed independently.

Mathematical Formulation.

The nutrition model may now be set up in linear programming terms. Let the set of possible activities (j) and activity levels x_j be:

Activities	Activity Level (unit = one dollar expended per day)
1. Buying the 1st type food (wheat flour)	x_1
2. Buying the 2nd type food (macaroni)	x_2
.....	...
n. Buying the $n = 77^{\text{th}}$ type food (strawberry preserves)	x_{77}

TABLE 27-1-II

NUTRITIVE VALUES OF COMMON FOODS PER DOLLAR OF EXPENDITURE

August 15, 1939

(Abbreviated List of 20 of the 77 Foods)

Nutritive Items (i)

	Commodity[1] (j)	Calories (1,000)	Protein (grams)	Calcium (grams)	Iron (mg.)	Vitamin A (1,000 I.U.)	Thiamine (mg.)	Riboflavin (mg.)	Niacin (mg.)	Ascorbic Acid (mg.)
**	1. Wheat flour (enriched)	44.7	1,411	2.0	365	—	55.4	33.3	441	—
	5. Corn meal	36.0	897	1.7	99	30.9	17.4	7.9	106	—
**	15. Evaporated milk (can)	8.4	422	15.1	9	26.0	3.0	23.5	11	60
*	17. Oleomargarine	20.6	17	.6	6	55.8	.2	—	—	—
**	19. Cheese (Cheddar)	7.4	448	16.4	19	28.1	.8	10.3	4	—
	21. Peanut butter	15.7	661	1.0	48	—	9.6	8.1	471	—
	24. Lard	41.7	—	—	—	.2	—	.5	5	—
**	30. Liver (beef)	2.2	333	.2	139	169.2	6.4	50.8	316	525
	34. Pork loin roast	4.4	249	.3	37	—	18.2	3.6	79	—
	40. Salmon, pink (can)	5.8	705	6.8	45	3.5	1.0	4.9	209	—
*	45. Green beans	2.4	138	3.7	80	69.0	4.3	5.8	37	862
**	46. Cabbage	2.6	125	4.0	36	7.2	9.0	4.5	26	5,369
*	50. Onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1,184
*	51. Potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2,522
**	52. Spinach	1.1	106	—	138	918.4	5.7	13.8	33	2,755
**	53. Sweet potatoes	9.6	138	2.7	54	290.7	8.4	5.4	83	1,912
*	64. Peaches, dried	8.5	87	1.7	173	86.8	1.2	4.3	55	57
*	65. Prunes, dried	12.8	99	2.5	154	85.7	3.9	4.3	65	257
**	68. Lima beans, dried	17.4	1,055	3.7	459	5.1	26.9	38.2	93	—
**	69. Navy beans, dried	26.9	1,691	11.4	792	—	38.4	24.6	217	—

[1] Commodity numbers refer to Stigler's original list. Starred lines refer to a reduced list explained in § 27-2.

Let the set of items (i) be, in this case, the nine different types of nutrients given in Table 27-1-I. Then the only question remaining with regard to formulation of the mathematical model is whether we want to specify that the requirements are met exactly or can be exceeded. In the first case we are considering a problem of the type

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (x_j \geq 0) \tag{1}$$

and in the second case, we are considering a problem of the type

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad (x_j \geq 0) \tag{2}$$

where we wish to choose x_j to minimize the total cost

$$z = \sum_{j=1}^{n} x_j \tag{3}$$

It might seem that asking for exact requirements is better than allowing the possibility of exceeding requirements, since getting more than one really needs should be more costly. This reasoning is fallacious, however, since any set of x_j which satisfies (1) automatically satisfies (2) and hence the minimal value of z attained in (2) is certainly no greater than that in (1). *In other words, it is always cheaper (or at least no more costly) to permit an excess over requirements than to insist that requirements be met exactly.*

Often people criticize a plan because they observe what they believe are "wastes." They note that not everybody is busy all the time, or an installation or machine is idle part of the time. They believe a good plan would find some way to put all these people and machines to work constructively. However, one of the first things one learns in programming is that it is not always efficient to try to remove these "defects." For example, putting idle people to work may require that they have tools and materials at their disposal that are badly needed elsewhere. Referring to the footnote of Table 27-2-I, it is seen for the nutrition problem that insistence on no surplus in nutritives would increase the minimum cost per day from 10.9 cents to 13.8 cents, an increase of over 25 per cent.

We shall formulate the nutrition problem allowing surpluses of all nutrients. There is one risk we take in such a procedure; namely, the solution may be a diet which contains an excess of calories or some other nutrient known to be harmful to health.

First, augment the system by introducing slack variables $x_{n+1}, \ldots, x_{n+m}$, yielding

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \qquad (i = 1, 2, \ldots, m) \tag{4}$$

$$\sum_{1}^{n} x_j = z \text{ (Min)} \tag{5}$$

We now further augment the system by introducing artificial variables $x_{n+m+1}, \ldots, x_{n+2m}$, so that now for $i = 1, 2, \ldots, m$

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} + x_{n+m+i} = b_i \tag{6}$$

27-2. NUMERICAL SOLUTION OF THE NUTRITION PROBLEM

Solution by Electronic Computer.

The nutrition problem that took 120 man-days to compute in 1948 (using desk computers) can be run in a few minutes on a modern electronic computer. The exact time depends on just how much information one wants to print regarding the problem. In 1953, using an IBM 701 machine, the RAND Simplex Code, and printing out each iteration, the total time was 12 minutes; without printing, the time would have been cut to about one-fourth. If run on a post-1960 computer, the time would be a fraction of a minute.

Since it was desired to compare the minimum cost solutions when exact requirements must be satisfied, § 27-1-(1), and when excesses of requirements are permitted, § 27-1-(2), the instruction code did not allow any slack variable to enter the basic set initially. In 12 iterations Phase I was completed and a basic feasible solution was obtained. On the 16th iteration, Phase II was completed and an optimal feasible solution was obtained for § 27-1-(1). At this point, excess variables were allowed to enter the basic set and an additional 8 iterations were required to obtain an optimal solution for § 27-1-(2). In order to see whether the number of iterations for the latter could be reduced, a second problem was run in which the excess variables were allowed to enter the basic set *at any time*. By some odd coincidence the same set of 24 iterations took place. See Problem 15 for a possible explanation. The results of these computations are summarized in Table 27-2-I and Fig. 27-2-I.

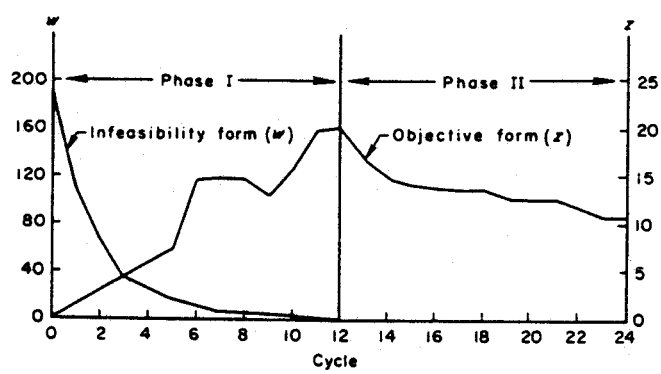


Figure 27-2-I. Decrease in the value of the feasibility and objective forms as a function of iteration in the nutrition problem.

TABLE 27-2-I

CHANGES IN BASIS ACTIVITIES: VALUES FOR w AND z

	Iteration	Basic Activities									Infeasibility Form (w)	Objective Form z (cents)
Phase I	0	Calories short	Protein short	Calcium short	Iron short	Vitamin A short	Thiamine short	Riboflavin short	Niacin short	Ascorbic Acid short	191.3	0
	1	,,	,,	,,	,,	,,	,,	,,	,,	Cabbage	110.3	1.4
	2	,,	,,	,,	Navy beans	,,	,,	,,	,,	,,	69.6	2.8
	3	,,	,,	,,	,,	,,	Flour	,,	,,	,,	38.2	4.4
	4	,,	,,	,,	Salmon	,,	,,	,,	,,	,,	27.8	5.6
	5	,,	,,	,,	,,	,,	,,	,,	Pork roast	,,	19.5	7.0
	6	,,	,,	,,	,,	,,	Beans	,,	,,	,,	11.0	14.3
	7	,,	,,	,,	,,	Spinach	,,	,,	,,	,,	7.3	14.6
	8	,,	Corn meal	,,	,,	,,	,,	,,	,,	,,	4.2	14.5
	9	,,	,,	,,	,,	Beef liver	,,	,,	,,	,,	2.6	13.0
	10	Lard	,,	,,	,,	,,	,,	,,	,,	,,	1.5	15.6
	11	,,	,,	Evaporated milk	,,	,,	,,	,,	,,	,,	.2	19.9
Phase II	12	,,	,,	,,	,,	,,	,,	Lima beans	,,	,,	—	20.5
	13	,,	,,	,,	,,	,,	,,	Flour	,,	,,	—	16.9
	14	,,	,,	,,	,,	,,	Spinach	,,	,,	,,	—	14.8
	15	,,	,,	,,	Peanut butter	,,	,,	,,	,,	,,	—	14.0
	16	,,	,,	,,	,,	,,	,,	,,	Potatoes	,,	Min z_1 =	13.8[1]
	17	,,	,,	,,	,,	,,	,,	,,	Iron excess	,,	—	13.5
	18	,,	Navy beans	,,	,,	,,	,,	,,	,,	,,	—	13.4
	19	,,	,,	,,	Thiamine excess	,,	,,	,,	,,	,,	—	12.8
	20	,,	Lima beans	,,	,,	,,	,,	,,	,,	,,	—	12.7
	21	,,	Niacin excess	,,	,,	,,	,,	,,	,,	,,	—	12.7
	22	,,	,,	,,	,,	Protein excess	,,	,,	,,	,,	—	11.8
	23	Navy beans	,,	,,	,,	,,	,,	,,	,,	,,	—	10.9
	24	,,	,,	Beef liver	,,	,,	,,	,,	,,	,,	Min z_2 =	10.9[1]

[redacted] Min [redacted]ched on iteration (16) is the least cost solution without excess nutrients [redacted]
Min[redacted]hed d[redacted]tion [redacted] the le[redacted]t solu[redacted]ith e[redacted]utrie[redacted]

Solution by Hand Techniques.

Stigler reduced the list of 77 foods to a list of 15 foods (starred lines in Table 27-1-II) by dropping from the list any food which had, per dollar, no more of each nutrient than did some other food. He effected a further reduction to a list of 9 foods (double starred lines) by dropping any food which had, per dollar, no more of each nutrient than did a mixture of other foods costing a dollar. However, this method can only be used if the nutritional requirements may be exceeded. Where nutritional requirements must be met exactly, this short cut is no longer valid. In Problems 1–5, proofs of these statements are suggested as exercises.

In the final solution for the case where excess nutrients are allowed, nutrients protein, iron, thiamine, and niacin exceeded requirements. With this foreknowledge, it is possible to present here a complete short cut numerical solution of the nutrition problem. To do this, we shall follow Stigler and use the reduced set of 9 foods in place of the 77, and use only 5 of the 9 equations of § 27-1-(2). Equations $i = 2, 4, 6, 8$ (corresponding to the excess slack variables for niacin, thiamine, protein, and iron) have been ignored. The complete computations are given in Table 27-2-II.

It is natural to question any procedure which ignores certain of the restrictions in the problem and solves a smaller problem with fewer than the total number of restraints. In certain cases, however, this approach can be made the basis of a very efficient algorithm. Let us suppose that k equations, each having a slack variable, are ignored in solving a linear programming problem. After solution, the values of the variables of the reduced system may be substituted into these k equations and the values of the omitted slack variables determined. If, by good luck, the values of these slack variables are positive or zero, then it is easy to see that the solution for the restricted system, together with the solution for the slack variables, constitutes an optimal solution for the entire system.

The nutrition problem requires a slight modification in the setup if the simplex multipliers π_i are to be interpreted as the implicit costs of a calorie, a gram of calcium, etc. The c_j represent costs in the model but the $a_{ij} > 0$ are *outputs* of the ith type of nutrient in a unit quantity of the jth food. It is inconvenient in this instance to follow our sign convention that inputs should be positive and outputs negative. In this case, the prices π_i described in Chapters 9 and 12 will turn out to have the wrong sign.

TAB[redacted]

SIMPLEX SOLUTION TO AN ABBREVIATED

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_{111}	3.0	44.7	8.4	7.4	2.2	2.6	1[redacted]
Calcium	x_{113}	.8	2.0	15.1	16.4	.2	4.0	
Vitamin A	x_{115}	5.0		26.0	28.1	169.2	7.2	918.4
Riboflavin	x_{117}	2.7	33.3	23.5	10.3	50.8	4.5	13[redacted]
Ascorbic acid	x_{119}	75.0		60.0		525.0	$\boxed{5369.0}$	2755[redacted]
Objective form	$(-z)$		1.0	1.0	1.0	1.0	1.0	1.0
Infeasibility form	$(-w)$	−86.5	−80.0	−133.0	−62.2	−747.4	−5387.3	−3688[redacted]
							★	

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_{111}	2.9637	44.7	8.3709	7.4	1.9458		−.2[redacted]
Calcium	x_{113}	.7441	2.0	15.0553	16.4	−.1911		−2.0525
Vitamin A	x_{115}	4.8994		25.9195	28.1	168.4960		$\boxed{914.7055}$
Riboflavin	x_{117}	2.6371	33.3	23.4497	10.3	50.3600		11.4[redacted]
Ascorbic acid	x_{46}	.0140		.0112		.0978	1.0	.51[redacted]
Objective form	$(-z)$	−.0140	1.0	.9888	1.0	.9022		.4869
Infeasibility form	$(-w)$	−11.2444	−80.0	−72.7955	−62.2	−220.6106		−923.9[redacted]
							●	★

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_{111}	2.9649	$\boxed{44.7}$	8.3776	7.4072	1.9889		[redacted]
Calcium	x_{113}	.7551	2.0	15.1135	16.4631	.1870		
Vitamin A	x_{32}	.0054		.0283	.0307	.1842		1.0
Riboflavin	x_{117}	2.5756	33.3	23.1241	9.9470	48.2433		[redacted]
Ascorbic acid	x_{46}	.0112		−.0034	−.0158	.0033	1.0	
Objective form	$(-z)$	−.0166	1.0	.9750	.9850	.8125		
Infeasibility form	$(-w)$	−6.2956	−80.0	−46.6151	−33.817	−50.4191		[redacted]
			★				●	●

27-2-II

FORM OF STIGLER'S NUTRITION PROBLEM

0

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
9.6	17.4	26.9	−1.0					1.0				
2.7	3.7	11.4		−1.0					1.0			
290.7	5.1				−1.0					1.0		
5.4	38.2	24.6				−1.0					1.0	
1912.0							−1.0					1.0
1.0	1.0	1.0										
−2220.4	−64.4	−62.9	1.0	1.0	1.0	1.0	1.0					
								●	●	●	●	○

1

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
8.6741	17.4	26.9	−1.0				.0005	1.0				−.0005
1.2755	3.7	11.4		−1.0			.0007		1.0			−.0007
288.1359	5.1				−1.0		.0013			1.0		−.0013
3.7975	38.2	24.6				−1.0	.0008				1.0	.0008
.3561							−.0002					.0002
.6439	1.0	1.0					.0002					−.0002
−301.8830	−64.4	−62.9	1.0	1.0	1.0	1.0	−.0034					1.0034
								●	●	○	●	Drop

2

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
8.7478	17.4013	26.9	1.0		.0003		.0005	1.0		.0003		.0005
1.9221	3.7114	11.4		1.0	.0022		.0007		1.0	.0022		.0007
.3150	.0056				−.0011					.0011		
.1778	38.1359	24.6			.0126	−1.0	.0008			−.0126	1.0	−.0008
.1945	−.0029				.0006		−.0002			−.0006		.0002
.4905	.9973	1.0			.0005		.0002			−.0005		−.0002
−10.8477	−59.2487	−62.9	1.0	1.0	−.0101	1.0	−.0021			1.0101		1.0021
								○	●	Drop	●	Drop

TAB[redacted]

SIMPLEX SOLUTION TO AN ABBREVIA[redacted]

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinac[redacted]
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0663	1.0	.1874	.1657	.0445		[redacted]
Calcium	x_{113}	.6225		14.7386	16.1316	.0980		[redacted]
Vitamin A	x_{52}	.0054		.0283	.0307	.1842		1.0
Riboflavin	x_{117}	.3668		16.8831	4.4289	$\boxed{46.7616}$		[redacted]
Ascorbic acid	x_{46}	.0112		−.0034	−.0158	.0033	1.0	[redacted]
Objective form	$(-z)$	−.0829		.7876	.8193	.7660		
Infeasibility form	$(-w)$	−.9893		−31.6217	−20.5605	−46.8596		
			●			★	●	● [redacted]

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinac[redacted]
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0660	1.0	.1714	.1615			[redacted]
Calcium	x_{113}	.6217		14.7033	$\boxed{16.1224}$			[redacted]
Vitamin A	x_{52}	.0039		−.0382	.0133			1.0
Riboflavin	x_{30}	.0078		.3610	.0947	1.0		[redacted]
Ascorbic Acid	x_{46}	.0112		−.0045	−.0161		1.0	[redacted]
Objective form	$(-z)$	−.0889		.5103	.7466			
Infeasibility form	$(-w)$	−.6217		−14.7033	−16.1224			
			●		★	●	●	● [redacted]

Cycle

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spina[redacted]
	Variable	Value $\bar{b}_i$	($j = 1$)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0598	1.0	.0241				[redacted]
Calcium	x_{19}	.0386		.9120	1.0			[redacted]
Vitamin A	x_{52}	.0034		−.0503				1.0
Riboflavin	x_{30}	.0042		$\boxed{.2747}$		1.0		[redacted]
Ascorbic acid	x_{46}	.0118		.0101			1.0	[redacted]
Objective form	$(-z)$	−.1177		−.1706				
Infeasibility form	$(-w)$							
			●	★	●	○	●	● [redacted]

-2-II (contd.)

FORM OF STIGLER'S NUTRITION PROBLEM

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
.1957	.3893	.6018	.0224					−.0224				
1.5307	2.9329	10.1964	.0447	−1.0	−.0022		.0007	−.0447	1.0	.0022		−.0007
.3150	.0056				−.0011					.0011		
−6.3391	25.1725	4.5604	.7450		.0128	−1.0	.0005	−.7450		−.0128	1.0	−.0005
.1945	−.0029				.0006		−.0002			−.0006		.0002
.2948	.6080	.3982	.0224		.0005		.0002	−.0224		−.0005		−.0002
4.8084	−28.1054	−14.7568	−.7897	1.0	−.0105	1.0	−.0012	1.7897		1.0105		1.0012
								Drop	●	Drop	○	Drop

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
.2017	.3653	.5975	−.0231			.0010		.0231			−.0010	
1.5440	2.8801	10.1869	.0432	−1.0	−.0023	.0021	.0007	−.0432	1.0	.0023	−.0021	−.0007
.3400	−.0936	−.0180	−.0029		−.0011	.0039		.0029		.0011	−.0039	
−.1356	.5383	.0975	.0159		.0003	−.0214		−.0159		−.0003	.0214	
.1949	−.0046	−.0003	−.0001		.0006	.0001	−.0002	.0001		−.0006	−.0001	.0002
.3989	.1945	.3233	.0101		.0003	.0164	.0002	−.0101		−.0003	−.0164	−.0002
−1.5440	−2.8801	−10.1869	−.0432	1.0	.0023	−.0021	−.0007	1.0432		.9977	1.0021	1.0007
								Drop	○	Drop	Drop	Drop

(Feasible)

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
.1863	.3365	.4954	−.0235	.0100		.0009		.0235	−.0100		.0009	
.0958	.1786	.6318	.0027	−.0620	−.0001	.0001		−.0027	.0620	.0001	−.0001	
.3387	−.0960	−.0264	−.0030	.0008	−.0011	.0039		.0030	−.0008	.0011	−.0039	
−.1446	.5214	.0377	.0157	.0059	.0003	−.0214		−.0157	−.0059	−.0003	.0214	
.1965	−.0017	.0098		−.0010	.0006	.0001	−.0002		.0010	−.0006	−.0001	.0002
.3274	.0612	−.1484	.0081	.0463	.0004	.0163	.0001	−.0081	−.0463	−.0004	−.0163	−.0001
								1.0000	1.0000	1.0000	1.0000	1.0000
								Drop	Drop	Drop	Drop	

TA[redacted]LE

SIMPLEX SOLUTION TO AN ABBREVIATE[redacted]

C[redacted]cl[redacted]

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	(j = 1)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0594	1.0			−.0876		
Calcium	x_{19}	.0246	1		1.0	−3.3203		
Vitamin A	x_{52}	.0042				.1830		1.0
Riboflavin	x_{15}	.0153		1.0		3.6407		
Ascorbic acid	x_{46}	.0117				−.0368	1.0	
Objective form	$(-z)$	−.1151				.6210		
Infeasibility form	$(-w)$							
			●	●	○		●	●

C[redacted]cl[redacted]

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	(j = 1)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0355	1.0		−.9711	3.1368		
Calcium	x_{69}	.0486			1.9734	−6.5523		
Vitamin A	x_{52}	.0051			.0384	.0556		1.0
Riboflavin	x_{15}	.0086		1.0	−.2707	$\boxed{4.5396}$		
Ascorbic acid	x_{46}	.0112			−.0167	.0185	1.0	
Objective form	$(-z)$	−.1090			.2467	−.1982		
Infeasibility form	$(-w)$							
			●	○		★	●	●

[redacted]cl[redacted]

Item	Basic Solution		Wheat Flour (Enriched)	Evaporated Milk	Cheddar Cheese	Beef Liver	Cabbage	Spinach
	Variable	Value $\bar{b}_i$	(j = 1)	(15)	(19)	(30)	(46)	(52)
Calories	x_1	.0295	1.0	−.6910	−.7841			
Calcium	x_{69}	.0610		1.4434	1.5827			
Vitamin A	x_{52}	.0050		−.0122	.0417			1.0
Riboflavin	x_{30}	.0019		.2203	−.0596	1.0		
Ascorbic acid	x_{46}	.0112		−.0041	−.0156		1.0	
Objective form	$(-z)$	−.1087		.0437	.2349			
Infeasibility form	$(-w)$							
			●			●	●	●

7-2-II (contd.)

FORM OF STIGLER'S NUTRITION PROBLEM

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
.1989	.2908	.4921	−.0249	.0095		.0028		.0249	−.0095		.0028	
.5760	−1.5525	$\boxed{.5067}$	−.0494	−.0815	−.0011	.0712		.0494	.0815	.0011	−.0712	
.3122	−.0005	−.0195	−.0001	.0019	−.0011			.0001	−.0019	.0011		
−.5266	1.8983	.1372	.0571	.0214	.0010	−.0779		−.0571	−.0214	−.0010	.0779	
.2018	−.0209	.0084	−.0006	−.0012	.0005	.0009	−.0002	.0006	.0012	−.0005	−.0009	.0002
.2376	.3849	−.1250	.0179	.0500	.0006	.0030	.0001	−.0179	−.0500	−.0006	−.0030	−.0001
								1.0000	1.0000	1.0000	1.0000	1.0000
		★						Drop	Drop	Drop	Drop	

7

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
−.3604	1.7985		.0231	.0887	.0010	−.0663		−.0231	−.0887	−.0010	.0663	
1.1366	−3.0638	1.0	−.0974	−.1609	−.0022	.1405	.0001	.0974	.1609	.0022	−.1405	−.0001
.3343	−.0601		−.0020	−.0012	−.0011	.0028		.0020	.0012	.0011	−.0028	
−.6825	2.3186		.0704	.0435	.0013	−.0972		−.0704	−.0435	−.0013	.0972	
.1922	.0049		.0002	.0001	.0006	−.0003	−.0002	−.0002	−.0001	−.0006	.0003	.0002
.3797	.0019		.0057	.0298	.0003	.0206	.0001	−.0057	−.0298	−.0003	−.0206	−.0001
								1.0000	1.0000	1.0000	1.0000	1.0000
		●						Drop	Drop	Drop	Drop	

8 (Optimal)

Sweet Potatoes	Lima Beans (Dried)	Navy Beans (Dried)	Excess Variables					Artificial Variables				
(53)	(68)	(69)	(101)	(103)	(105)	(107)	(109)	(111)	(113)	(115)	(117)	(119)
.1112	.1964		−.0256	.0586	.0001	.0008		.0256	−.0586	−.0001	−.0008	
.1516	.2827	1.0	.0042	−.0982	−.0002	.0002	.0001	−.0042	.0982	.0002	−.0002	−.0001
.3427	−.0885		−.0029	−.0018	−.0011	.0039		.0029	.0018	.0011	−.0039	
−.1503	.5107		.0155	.0096	.0003	−.0214		−.0155	−.0096	−.0003	.0214	
.1950	−.0045		−.0001		.0006	.0001	−.0002	.0001		−.0006	−.0001	.0002
.3499	.1031		.0088	.0317	.0004	.0164	.0001	−.0088	−.0317	−.0004	−.0164	−.0001
								1.0000	1.0000	1.0000	1.0000	1.0000
		●						Drop	Drop	Drop	Drop	

27-3. PROBLEMS

1. Solve the nutrition problem using the revised simplex algorithm.
2. What are the implicit prices of the nutrients? Comment on the sign of the π_i.
3. What must the price of evaporated milk drop to before it will affect the solution? What food will drop out of the basis?
4. Assuming all other food costs stable, what must be the relationship between the cost of evaporated milk and the cost of the food being dropped out of the basic set for one or the other to be in the solution.
5. Prove that in a nutrition-type problem where all requirements may be exceeded, that the quantity of any food, s, in an optimal solution can be replaced by purchase of an equal number of dollars of another food j, provided $a_{ij} \geq a_{is}$ for all i.
6. Prove, more generally, that the purchase of any food, s, in an optimal solution can be replaced, per dollar of expenditure, by the purchase of λ_1 dollars of food 1, λ_2 dollars of food 2, . . ., λ_r dollars of food r, provided

$$\begin{aligned} &\lambda_1 a_{i1} + \lambda_2 a_{i2} + \ldots + \lambda_r a_{ir} \geq a_{is} \qquad (i = 1, 2, \ldots, m) \\ &\lambda_1 \quad + \lambda_2 \quad + \ldots + \lambda_r \quad = 1 \\ &\quad \lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_r \geq 0 \end{aligned}$$

7. Prove for a general linear programming problem:

$$\begin{aligned} &\sum a_{ij} x_j \geq b_i \qquad (x_j \geq 0;\ i = 1, 2, \ldots, m) \\ &\sum c_j x_j = z \text{ (Min)} \end{aligned}$$

that if, by relabeling, for $k = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, m$

$$\begin{aligned} &\lambda_1 a_{i1} + \lambda_2 a_{i2} + \ldots + \lambda_r a_{ir} \geq a_{is} \qquad (\lambda_k \geq 0) \\ &\lambda_1 c_1 \ + \lambda_2 c_2 \ + \ldots + \lambda_r c_r \ \leq c_s \end{aligned}$$

then an optimal feasible solution exists with $x_s = 0$.

8. Prove that in a nutrition problem where requirements may be exceeded, at least one requirement must be satisfied exactly in an optimal solution.
9. Prove, as a Corollary to Problem 8, that no $x_s > 0$ can occur in an optimal solution if there exists a j satisfying $a_{ij} > a_{is}$ for all i or, more generally, a set $j = 1, 2, \ldots, r$ (by relabeling) satisfying the relations of Problem 6 with strict inequality.
10. Prove for the general linear programming system § 27-1-(2) with a general linear objective that strict inequalities cannot hold for all i in an optimal feasible solution when Min $z \neq 0$. Show, by a counter-example, that this theorem need not be true if Min $z = 0$.
11. Construct an example to show that, if requirements are to be met exactly, then it is possible that there exists an optimal solution with

$x_s > 0$ in spite of the existence of a j satisfying conditions of Problems 5 or 6 above.

12. In solving the nutrition problem, suppose a guess is made as to which slack variables are positive. Suppose the corresponding equations are omitted and an optimal solution is obtained for the resulting system. Suppose finally that substitutions of this optimal solution into the omitted equations yield positive values for their slack variables. Show, by eliminating the basic variables of the restricted system from the omitted equations, that (a) the entire system is reduced to canonical form, (b) the proposed solution to the entire system is a basic solution, and (c) it is an optimal solution.
13. Suppose in Problem 12 that, upon substitution into the omitted equations, one of the corresponding slack variables is negative. Show that one way to eliminate infeasibility is to introduce an artificial variable whose coefficients are the negative of the slack variable, and then proceed to minimize this variable. Generalize this procedure.
14. Show that, by increasing sufficiently the quantity of evaporated milk ($j = 15$), an immediate feasible solution can be obtained for the abbreviated nutrition problems of § 27-2 (see Table 27-2-II).
15. Show by altering the units of a variable that the rule to choose Max $\bar{c}_j > 0$ will choose any j such that $\bar{c}_j > 0$. Illustrate in Table 27-2-II how a change in the units of some variable would have altered the course of calculation. Use this to explain why the introduction of slack variables from the start of the computations could give the same sequence of iterations as their introduction after a minimum was reached without allowing slacks.

REFERENCES

Fisher and Schruben, 1953-1
Newman, 1955-1
Smith, V. E., 1959-1
Stigler, 1945-1
Swanson, 1955-2
Waugh, 1951-1

CHAPTER 28

THE ALLOCATION OF AIRCRAFT TO ROUTES UNDER UNCERTAIN DEMAND

28-1. STATEMENT AND FORMULATION

The purpose of this chapter is to illustrate an application of linear programming to the problem of allocating aircraft to routes to maximize expected profits when there is uncertain customer demand. The computational procedure is similar to the fixed demand case with only slightly more computational effort required. After solution of a numerical case we shall compare the allocations with those obtained under the common practice of assuming a fixed demand equal to the expected value. The material for this chapter has been taken, with only minor changes, from the joint papers by Alan Ferguson and the author [Ferguson and Dantzig, 1954-1, 1956-1].

Many business, economic, and military problems have the following characteristics in common: a limited quantity of capital equipment or final product must be allocated among a number of final-use activities, where the level of demand for each of these activities, and hence the payoff, is uncertain; further, once the allocation is made, it is not economically feasible to reallocate because of geographical separation of the activities, or because of differences in form of the final products, or because of a minimum lead time between the decision and its implementation. Examples of such problems are (1) scheduling transport vehicles over a number of routes to meet a demand in some future period and (2) allocating quantities of a commodity at discrete time intervals among several storage or distribution points while the future demand for the commodity is unknown. It is assumed, however, that demand can be forecast or estimated as a distribution of values, each with a specified probability of being the actual value.

The general area where the techniques of this chapter apply may be schematized broadly as problems where

(a) alternative sets of activity levels can be chosen consistent with given resources;
(b) each set of chosen activity levels provides the facilities or stocks to meet an unknown demand whose distribution is assumed known;
(c) profits depend on the costs of the facilities, stocks, and on the revenues from the demand,

and where the general objective is to determine that set of activity levels that maximizes profits.

Chapter 25, which is based on the paper entitled "Linear Programming under Uncertainty" [Dantzig, 1955-1], forms the theoretical basis for the present chapter. Here we shall illustrate the procedural steps on an example [Ferguson and Dantzig, 1956-1] which, in fact, originally inspired the theoretical work in this area. Thus, little in the way of rigorous theory will be attempted, although each step will be justified intuitively.

The method is explained by the use of a model for routing aircraft. Several types of aircraft are allocated over a number of routes; the monthly demand for service over each route is assumed to be known only as a distribution of probable values. The aircraft are so allocated as to minimize the sum of the cost of performing the transportation, plus the expected value of the revenue lost through the failure to serve all the traffic that actually developed.

For purposes of month-to-month scheduling, an air-transport operator would probably be more willing to make an estimate of the range and general distribution of future travel (or shipment) over his routes than to commit himself to a single expected value. Indeed, he might feel that the optimal assignment should be insensitive to a wide range of demand distribution, and that an assignment based on expected values (as if these were known fixed demands) would be misleading. It is suggested that the reader make sensitivity tests by modifying the demand distributions given in the illustrative example to develop this point. Sensitivity analysis is discussed in § 12-4.

Passenger demand, of course, occurs on a day-by-day or, in fact, on a flight-by-flight basis. The assumed number of passengers per type aircraft per given type flight may be thought of as an ideal number which can be increased slightly by decreasing the amount of air freight and by "smoothing" the demand by encouraging the customers to take open reservations on alternative flights as opposed to less certain reservations on the desired flight. In spite of these possible adjustments, traveler preferences and the inevitable last-minute cancellations do cause loss of seat-carrying capacity. However, the best way to reflect these effects of the daily variations in demand is beyond the scope of this chapter. For our purpose here, either the aircraft passenger capability or the demand may be thought of as adjusted downward to reflect the loss due to daily variations of demand. Our concern will be in over-all monthly variability.

The method employed is simple, and the example used can be solved by hand in an hour or two. Larger problems can be solved with computing machines.

After we formulate the problem, we will (a) briefly indicate the nature of the solution based on *fixed demand* [Ferguson and Dantzig, 1954-1], (b) show the method of solving the problem using stochastic values for

demand, and (c) compare the two solutions. Note that the example has been so constructed that the fixed demand is the same as the expected value of the uncertain demand.

A Fixed-demand Example.

The fixed-demand example, used to illustrate the method, assumes a fleet of four types of aircraft, as shown in Table 28-1-I. These aircraft have

TABLE 28-1-I

ASSUMED AIRCRAFT FLEET

Type	Description	Number Available
A	Postwar 4-engine	10
B	Postwar 2-engine	19
C	Prewar 2-engine	25
D	Prewar 4-engine	15

differences in speeds, ranges, payload capacities, and cost characteristics. The assumed routes and expected traffic loads (the distribution of demand will be discussed later) are shown in Table 28-1-II.

TABLE 28-1-II

TRAFFIC LOAD BY ROUTE

Route	Route Miles[1]	Expected Number of Passengers[2]	Price One-way Ticket ($)
(1) N.Y. to L.A. (1-stop)	2,475	25,000	130
(2) N.Y. to L.A. (2-stop)	2,475	12,000	130
(3) N.Y. to Dallas (0-stop)	1,381	18,000	70
(4) N.Y. to Dallas (1-stop)	1,439	9,000	70
(5) N.Y. to Boston (0-stop)	185	60,000	10

[1] *Official Airline Guide*, July, 1954, p. 276. The New York-Los Angeles routes are via Chicago and via Chicago-Denver; the stop en route between New York and Dallas is at Memphis.

[2] This is the expected number of full one-way fares per month to be carried on each route. If a passenger gets off en route and is replaced by another passenger, it is counted as one full fare.

Since this chapter proposes to illustrate the applicability of a method of solving problems in which several realistic elements are considered, it is assumed that not all aircraft can carry their full loads on all routes and that the obtainable utilization varies from route to route. Specifically, Type B is assumed to be able to operate at only 75 per cent payload on Route 3, and Type D at 80 per cent on Route 1; Type C cannot fly either Route 1 or Route 3, and Type B cannot fly Route 1. *Utilization* is defined as the

average number of hours of useful work performed per month by each aircraft assigned to a particular route. Utilization of 300 hours per month is assumed on Routes 1 and 2; 285 on Routes 3 and 4; and 240 on Route 5.

The assumed dollar costs per 100 passenger-miles are shown in Table 28-1-III. These do not include any capital costs such as those of the aircraft

TABLE 28-1-III
DOLLAR COSTS

Type of Aircraft	Route				
	(1) N.Y. to L.A. 1-stop ($)	(2) N.Y. to L.A. 2-stop ($)	(3) N.Y. to Dallas 0-stop ($)	(4) N.Y. to Dallas 1-stop ($)	(5) N.Y. to Boston 0-stop ($)
	Dollar Costs Per 100 Passenger-miles				
1—A	0.45	0.57	0.45	0.47	0.64
2—B	—	0.64	0.83	0.63	0.88
3—C	—	0.92	—	0.93	1.13
4—D	0.74	0.61	0.59	0.62	0.81
	Dollar Costs Per Passenger Turned Away[1]				
5—E	130 (13)	130 (13)	70 (7)	70 (7)	10 (1)

[1] Figures shown in parentheses are thousands of dollars lost per 100 passengers turned away. (Throughout this discussion, passengers are measured in units of hundreds.)

and ground facilities; they represent variable costs such as the cost of gasoline, salaries of the crew, and costs of servicing the aircraft.

There is, however, a second kind of "cost." It is the *loss of revenue* when not enough aircraft are assigned to the route to meet the passenger demand. We shall assume that this loss of revenue is the same as the price of a one-way ticket shown in the E row of Table 28-1-III.

Based on the speeds, ranges, payload capacities, and turn-around times, passenger-carrying capabilities were determined. The resultant potential number p_{ij} (in hundreds) of passengers that can be flown per month per aircraft of type i on route j is shown in Table 28-1-IV, as the upper right figure in each box. By multiplying these numbers by the corresponding costs per 100 passenger-miles given in Table 28-1-III and by the number of miles given in Table 28-1-II, the monthly cost per aircraft can also be obtained. This is given in the lower left figure c_{ij} in each box; explicitly, c_{ij} is the cost in thousands of dollars per month per aircraft of type i assigned to the route j. The revenue losses c_{5j}, in thousands of dollars per 100 passengers *not carried*, are given in the E row of Table 28-1-IV; finally, we define $p_{5j} = 1$.[1] The staggered layout of the table was chosen so as to

[1] This will make it easier to state the passenger-balance or "column" equations (2).

TABLE 28-1-IV

PASSENGER-CARRYING CAPABILITIES AND COSTS

Type of Aircraft	Route				
	(1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop
	Per Aircraft Per Month				
1—A	$p_{11} = 16$ $c_{11} = 18$	$p_{12} = 15$ $c_{12} = 21$	$p_{13} = 28$ $c_{13} = 18$	$p_{14} = 23$ $c_{14} = 16$	$p_{15} = 81$ $c_{15} = 10$
2—B	*	$p_{22} = 10$ $c_{22} = 15$	$p_{23} = 14$ $c_{23} = 16$	$p_{24} = 15$ $c_{24} = 14$	$p_{25} = 57$ $c_{25} = 9$
3—C	*	$p_{32} = 5$ $c_{32} = 10$	*	$p_{34} = 7$ $c_{34} = 9$	$p_{35} = 29$ $c_{35} = 6$
4—D	$p_{41} = 9$ $c_{41} = 17$	$p_{42} = 11$ $c_{42} = 16$	$p_{43} = 22$ $c_{43} = 17$	$p_{44} = 17$ $c_{44} = 15$	$p_{45} = 55$ $c_{45} = 10$
	Per 100 Passengers Not Carried (Losses)				
5—E	$p_{51} = 1$ $c_{51} = 13$	$p_{52} = 1$ $c_{52} = 13$	$p_{53} = 1$ $c_{53} = 7$	$p_{54} = 1$ $c_{54} = 7$	$p_{55} = 1$ $c_{55} = 1$

identify the corresponding data found in Table 28-2-II; the latter is the *work sheet* upon which the entire problem is solved.

The basic problem is that of determining the number of aircraft of each type to assign to each route consistent with aircraft availabilities, and of determining how much revenue will be lost due to failure of allocated aircraft to meet passenger demand on various routes. Since many alternative allocations are possible, our specific objective will be to find that allocation that minimizes total costs, where costs are defined as operating costs plus lost revenues based on the cost factors given in Table 28-1-III.

This may be formulated mathematically as a linear programming problem. Let x_{ij} denote the unknown quantity of the ith type aircraft assigned to j^{th} route, where $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n - 1$. If x_{in} denotes the surplus or unallocated aircraft, then (1) states that the sum of allocated and unallocated aircraft of each type accounts for the total available aircraft a_i. If x_{mj} denotes the number of passengers in hundreds turned away, then equation (2) states that the sum of passenger carrying capability of each type aircraft allocated to the jth route, $p_{ij}x_{ij}$, plus the unsatisfied demand accounts for the total demand, d_j. Relation (3) states

that all unknown quantities x_{ij} must be either positive or zero. Finally, if z is total cost, it is the sum of all the individual operating costs of each allocation, $c_{ij}x_{ij}$, plus the revenues lost by unsatisfied demands $c_{mj}x_{mj}$ (see equation (4)).

Fixed-demand Model	
Find numbers x_{ij}, and the minimum value of z such that for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$:	
(1) Row Sums:	$x_{i1} + x_{i2} + \ldots + x_{in} = a_i \quad (i \neq m)$
(2) Col. Sums:	$p_{1j}x_{1j} + p_{2j}x_{2j} + \ldots + p_{mj}x_{mj} = d_j$
(3)	$x_{ij} \geq 0$
(4)	$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} = z$

The optimal assignment of aircraft to routes based on fixed demand is a *weighted distribution problem*. The numerical solution shown in Table 28-2-II was obtained using the methods of Chapter 21. The values assigned to the unknowns x_{ij} appear boldfaced in the upper left of each box unless $x_{ij} = 0$ in which case it is omitted; the entire layout takes the form:

x_{ij}	
	p_{ij}
c_{ij}	

The sums by *rows* of x_{ij} entries in Table 28-2-II equated to availabilities yield equations (1). The sums by *columns* of x_{ij} weighted by corresponding values of p_{ij} equated to demands yield equations (2); the x_{ij} weighted by corresponding c_{ij} and summed over the entire table yields (4). As noted earlier, Table 28-2-II is actually the work sheet upon which the entire problem is solved. Later we shall discuss a revision of this work sheet for solving problems with variable demand. All figures in the table, except for the upper left entries, x_{ij} and values of the so-called "implicit prices" u_i and v_j shown in the margins, are constants which do not change during the course of computation. The values of the variables x_{ij}, u_i, and v_j, however, will change during the course of successive iterations of the simplex method as adapted for this problem. For this reason it is customary to cover the work sheet with clear acetate and to enter the variable information with a grease pencil which can be easily erased; alternatively, a blackboard or semi-transparent tissue paper overlays can be used. The detailed rules for obtaining the optimal solution shown are given in Chapter 21 and will not

be repeated here. Instead a more general set of rules for the uncertain demand case will be given which, of course, could be used for the expected demand case.

In Table 28-1-V we have a convenient summary serving to identify and define the numerical data entered in Table 28-2-II and to give the test for optimality.

TABLE 28-1-V

SUMMARY OF NOTATION AND RULES FOR FIXED-DEMAND CASE

(as displayed in Table 28-2-II)

Indices:	$i = 1, 2, \ldots, m - 1$ refers to type of aircraft to which passengers are assigned $i = m$ if passengers are unassigned $j = 1, 2, \ldots, n - 1$ refers to type of route to which an aircraft is assigned $j = n$ if aircraft is unassigned (surplus)
Constants:	a_i = number of available aircraft of type i d_j = expected passenger demand in 100's per month on route j p_{ij} = passenger-carrying capability in 100's per month per aircraft of type i assigned to route j ($p_{m,j} = 1$ by definition) c_{ij} = costs in 1000's of dollars per month per aircraft of type i assigned to route j (c_{mj} is cost per 100 passengers turned away)
x_{ij} Entries:	x_{ij} = number of aircraft of type i assigned to route j (x_{mj} is 100's of passengers turned away)
Omitted x_{ij} Entries:	$x_{ij} = 0$ if upper left entry in box is missing
Implicit Prices:	u_i and v_j are determined such that $u_i + p_{ij}v_j = c_{ij}$ for (i, j) boxes corresponding to $x_{ij} > 0$, i.e., non-omitted x_{ij} entries ($u_m = v_n = 0$ by definition)
Test for Optimality:	Solution is optimal if, for all (i, j), the relation $u_i + p_{ij}v_j \leq c_{ij}$ holds

Extension of the Example to Uncertain Demand.

To introduce the element of uncertain demand, we assume not a known fixed demand on each route but a known frequency distribution of demand. The assumed frequency distributions are shown in Table 28-1-VI. Thus on Route 2 (N.Y. to L.A., 2-stop) either 5,000 or 15,000 passengers will want transportation during the month, with probabilities 30 or 70 per cent respectively. The assumed traffic distributions are, of course, hypothetical to illustrate our method. The demand distributions on the five routes vary over wide ranges and have different characteristics; Route 1 is flat, Route 2 is U-shaped, Routes 3, 4, and 5 are unimodular but have differing degrees of concentration about the mode. Route 4 has a distribution with a very long tail that may reflect a realistic traffic situation.

To illustrate the essential character of the linear programming problem for the case of uncertain demand, let us focus our attention on a single route—say, Route 1—with probability distribution of demand as given in Table 28-1-VI. Let us suppose that aircraft assigned to Route 1 are capable

TABLE 28-1-VI

ASSUMED DISTRIBUTION OF PASSENGER DEMAND

(λ_{hj} = Probability of Demand d_{hj})

Route	Passenger Demand (in hundreds)	Approx. Mean (in hundreds)	Probability of Passenger Demand	Probability of Equaling or Exceeding Demand
(1)	$200 = d_{11}$ $220 = d_{21}$ $250 = d_{31}$ $270 = d_{41}$ $300 = d_{51}$	250	$0.2 = \lambda_{11}$ $0.05 = \lambda_{21}$ $0.35 = \lambda_{31}$ $0.2 = \lambda_{41}$ $0.2 = \lambda_{51}$	$1.0 = \gamma_{11}$ $0.8 = \gamma_{21}$ $0.75 = \gamma_{31}$ $0.4 = \gamma_{41}$ $0.2 = \gamma_{51}$
(2)	$50 = d_{12}$ $150 = d_{22}$	120	$0.3 = \lambda_{12}$ $0.7 = \lambda_{22}$	$1.0 = \gamma_{12}$ $0.7 = \gamma_{22}$
(3)	$140 = d_{13}$ $160 = d_{23}$ $180 = d_{33}$ $200 = d_{43}$ $220 = d_{53}$	180	$0.1 = \lambda_{13}$ $0.2 = \lambda_{23}$ $0.4 = \lambda_{33}$ $0.2 = \lambda_{43}$ $0.1 = \lambda_{53}$	$1.0 = \gamma_{13}$ $0.9 = \gamma_{23}$ $0.7 = \gamma_{33}$ $0.3 = \gamma_{43}$ $0.1 = \gamma_{53}$
(4)	$10 = d_{14}$ $50 = d_{24}$ $80 = d_{34}$ $100 = d_{44}$ $340 = d_{54}$	90	$0.2 = \lambda_{14}$ $0.2 = \lambda_{24}$ $0.3 = \lambda_{34}$ $0.2 = \lambda_{44}$ $0.1 = \lambda_{54}$	$1.0 = \gamma_{14}$ $0.8 = \gamma_{24}$ $0.6 = \gamma_{34}$ $0.3 = \gamma_{44}$ $0.1 = \gamma_{54}$
(5)	$580 = d_{15}$ $600 = d_{25}$ $620 = d_{35}$	600	$0.1 = \lambda_{15}$ $0.8 = \lambda_{25}$ $0.1 = \lambda_{35}$	$1.0 = \gamma_{15}$ $0.9 = \gamma_{25}$ $0.1 = \gamma_{35}$

of taking $100Y_i$ passengers where Y_i is to be determined. Up to 200 units (in hundreds of passengers) of this capability are certain to be used, and revenues from this source (negative costs) will be $13 = k_1$ units (in thousands of dollars) per unit of capability. If $100Y_1 \geq 200$, up to an additional 20 units of this capability will be used with probability $\gamma_{21} = 0.8$. Indeed, 80 per cent of the time the demand will be 220 units or greater, while 20 per cent of the time it will be 200 units; hence, the expected revenue per unit from this increment of capability is $0.8 \times 13 = 10.4$, or $10.4 = k_1\gamma_{21}$ units. On the third increment of 30 units (22,001 to 25,000 seats), the expected revenue is $0.75 \times 13 = 9.8$ or $k_1\gamma_{31}$ units per unit of capability since there is a 25 per cent chance that none of these units of capability will be used and 75 per cent that all will be used. For the fourth increment of 20 units (25,001 to 27,000 seats) of capability, the expected revenue is $0.4 \times 13 = 5.2$ or $k_1\gamma_{41}$ units per unit of capability, while for the fifth increment of 30

units (27,001 to 30,000 seats) it is $0.2 \times 13 = 2.6$ or $k_1\gamma_{51}$ units per unit. For the sixth increment, which is the number of units assigned above the 30,000 seat mark, the expected revenue is $0.0 \times 13 = 0$ per unit, since it is certain that none of these units of capability can be used. It is clear that no assignments above 30,000 seats are worthwhile, and hence the last increment can be omitted. The index $h = 1, 2, 3, 4, 5$ will be used to denote the 1st, 2nd, . . ., 5th increment of demand.

The number of assigned units in each increment, however, can be viewed as an unknown that depends on the *total* (passenger-carrying) capability assigned to route $j = 1$. Thus, if the total assigned is $Y_1 = 210$ units of capability, then the part of this total belonging to the first increment, denoted by y_{11}, is $y_{11} = 200$ and the part belonging to the second increment, denoted by y_{21}, is $y_{21} = 10$. The amounts in the higher increments are $y_{hi} = 0$ for $i = 3, 4, 5$. To review, the passenger-carrying capability Y_j is determined by the number of aircraft assigned to route j, so that

$$Y_j = p_{1j}x_{1j} + p_{2j}x_{2j} + p_{3j}x_{3j} + p_{4j}x_{4j} \tag{5}$$

On the other hand, Y_j itself breaks down into five increments

$$Y_j = y_{1j} + y_{2j} + y_{3j} + y_{4j} + y_{5j} \tag{6}$$

for routes $j = 1, 3, 4$, and correspondingly fewer for $j = 2, 5$. Regardless of the total Y_j, the amount y_{hj} belonging to each increment is bounded by the total size b_{hj} of that increment; the latter, however, is simply the change in demand level, so that

$$\begin{aligned} 0 \leq y_{1j} &\leq d_{1j} &&= b_{1j} \\ 0 \leq y_{2j} &\leq d_{2j} - d_{1j} &&= b_{2j} \\ 0 \leq y_{3j} &\leq d_{3j} - d_{2j} &&= b_{3j} \\ 0 \leq y_{4j} &\leq d_{4j} - d_{3j} &&= b_{4j} \\ 0 \leq y_{5j} &\leq d_{5j} - d_{4j} &&= b_{5j} \end{aligned} \tag{7}$$

Letting

$$\lambda_{hj} = \text{probability of demand } d_{hj}, \tag{8}$$

the total expected revenue from route j is, therefore,

$$k_j(\gamma_{1j}y_{1j} + \gamma_{2j}y_{2j} + \cdot \cdot \cdot + \gamma_{5j}y_{5j}) \tag{9}$$

where k_j is revenue (in thousands) per 100 passengers carried on route j and, as seen in Table 28-1-VI, the probability, γ_{hj}, of exceeding or equaling demand d_{hj} is related to λ_{hj}, the probability of demand d_{hj}, by

$$\begin{aligned} 1 = \gamma_{1j} &= \lambda_{1j} + \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\ \gamma_{2j} &= \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\ \gamma_{3j} &= \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\ \gamma_{4j} &= \lambda_{4j} + \lambda_{5j} \\ \gamma_{5j} &= \lambda_{5j} \end{aligned} \tag{10}$$

The numerical values of λ_{hj} and γ_{hj} are given in Table 28-1-VI. Applying (9), the total expected revenues for Route 1 are

$$13(1.0y_{11} + .8y_{21} + .75y_{31} + .4y_{41} + .2y_{51})$$

The most important fact to note about this linear form is the decrease in the successive values of the coefficients γ_{hj}. Moreover, this will always be the case whatever the distribution of demand, since the probability of equaling or exceeding a given demand level d_{hj} decreases with increasing values of demand.

Suppose now that $y_{11}, y_{21}, \ldots$, are treated as unknown variables in a linear programming problem subject only to (6) and (7) where the objective is to maximize revenues. Let us suppose further that Y_1 is fixed. It is clear, since the coefficient of y_{11} is largest in the maximizing form (9), y_{11} will be chosen first and made as large as possible consistent with (6) and (7); for the chosen value y_{11}, the next increment y_{21} will be chosen as large as possible consistent with (6) and (7), etc.; as a result, *when the maximum is reached*, the values of the variables $y_{11}, y_{21}, \ldots$, are precisely the *incremental values* associated with Y_1, which we discussed earlier, (6). Even if passenger capability Y_1 is not fixed, as in the case about to be considered, it should be noted that whatever the value of Y_1, the values of $y_{11}, y_{21}, \ldots$, which minimize an over-all cost form such as in (14) below, must maximize (9) for $j = 1$ and hence the incremental values of Y_1 will be generated by $y_{11}, y_{21}, \ldots$.

The linear programming problem in the case of uncertain demand is shown by (11), (12), (13), and (14).

Uncertain Demand Model		
Find numbers x_{ij} and y_{hj} and the minimum value of z such that for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $h = 1, 2, \ldots, r$:		
(11) Row Sums:	$x_{i1} + x_{i2} + \ldots + x_{in} = a_i$	$(i \neq m)$
(12) Column Sums:	$p_{1j}x_{1j} + p_{2j}x_{2j} + \ldots + p_{nj}x_{nj} = y_{1j} + y_{2j} + \ldots + y_{rj}$	$(j \neq n)$
(13)	$x_{ij} \geq 0, \quad 0 \leq y_{hj} \leq b_{hj}$	
(14) Expected Costs:	$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \left[R_0 - \sum_{j=1}^{n} k_j \sum_{h=1}^{r} \gamma_{hj}y_{hj}\right]$	

Thus expected costs are defined as the total outlays (first term) plus the expected loss of revenue due to shortage of seats (last two terms), where R_0, a constant, is the expected revenue if sufficient seats were supplied for all customers.

For the problem at hand, the bounds, b_{hj}, and the expected revenues, $k_j\gamma_{hj}$, per unit for the "incremental variables" y_{hj} can be computed from probability distributions Table 28-1-VI via (7) and (10).

The numerical values of the constants for the stochastic case are tabulated in Table 28-1-VII.

TABLE 28-1-VII

INCREMENTAL BOUNDS, b_{hj}, AND EXPECTED REVENUES, $k_j\gamma_{hj}$, PER UNIT OF PASSENGER CARRYING CAPACITY ASSIGNED

Increment h	Route 1		Route 2		Route 3		Route 4		Route 5	
	b_{h1}	$k_1\gamma_{h1}$	b_{h2}	$k_2\gamma_{h2}$	b_{h3}	$k_3\gamma_{h3}$	b_{h4}	$k_4\gamma_{h4}$	b_{h5}	$k_5\gamma_{h5}$
1	200	$k_1 = 13$	50	$k_2 = 13$	140	$k_3 = 7$	10	$k_4 = 7$	580	$k_5 = 1$
2	20	$.8k_1 = 10.4$	100	$.7k_2 = 9.1$	20	$.9k_3 = 6.3$	40	$.8k_4 = 5.6$	20	$.9k_5 = .9$
3	30	$.75k_1 = 9.8$		**	20	$.7k_3 = 4.9$	30	$.6k_4 = 4.2$	20	$.1k_5 = .1$
4	20	$.4k_1 = 5.2$		**	20	$.3k_3 = 2.1$	20	$.3k_4 = 2.1$		**
5	30	$.2k_1 = 2.6$		**	20	$.1k_3 = .7$	240	$.1k_4 = .7$		**

** Only two increments for Route 2 and three increments for Route 5 are needed to describe distribution of demand.

Rules for Computation.

The work sheet for determining the optimal assignment under uncertain demand is shown in Table 28-2-III.

The entries in the x_{ij} boxes and y_{hj} boxes take the form:

x_{ij}	y_{hj}
p_{ij}	-1
c_{ij}	$-k_j\gamma_{hj}$

To form the new row equations (11), the x_{ij} entries are summed to yield the a_i values given in the aircraft-available column. To form the column equations (12), the x_{ij} entries are multiplied by p_{ij}, the y_{hj} by -1, and summed down to yield zero.

Step 1. To initiate the computation any set of nonnegative values may be assigned to the unknown x_{ij} and y_{hj}, provided they satisfy the equations and thereby constitute a feasible solution.

Step 2. Put a box around any $(m + n - 2)$ of x_{ij} and y_{hj} entries where $m + n$ is the number of row plus column equations. These boxed entries can be arbitrarily selected except that they must have the property that if the fixed values assigned to the other non-boxed variables and the constant

terms were *arbitrarily changed to other values*, then the boxed variables would be determined uniquely in terms of the latter. Such a boxed set of variables constitutes, of course, *a basic set of variables*; the array of coefficients associated with this set in the equations (11) and (12) forms a basis (Chapter 8).

Note: One simple way of selecting a basic set is shown in Table 28-2-IV. One x_{ij} entry is arbitrarily selected and boxed in each row corresponding to a row equation (it is suggested that entries be boxed that appear to have a chance of having a positive value in an optimum solution). Next, each y_{hj}, in turn, $h = 1, 2, \ldots$, in a column is assigned its upper bound value b_{hj} until for some $h = h_0$ the column "net" goes negative, in which case a value $y_{hj} < b_{hj}$ for $h = h_0$ is assigned so that the net is zero; the (h_0, j) entry is then boxed.

Step 3. For (i, j) and (h, j) combinations corresponding to basic entries, compute implicit prices u_i and v_j associated with equations by determining values of u_i and v_j satisfying the equations

$$u_i + p_{ij}v_j = c_{ij} \qquad \text{(for } x_{ij} \text{ basic)} \tag{15}$$

$$0 + (-1)v_j = -k_j\gamma_{hj} \qquad \text{(for } y_{hj} \text{ basic)} \tag{16}$$

There are always $m + n - 2$ equations (15) and (16) in $m + n - 2$ unknowns u_i and v_j that can be shown easily to have a unique solution (see § 21-1). They can be solved by inspection, for it can be shown that the system is either completely triangular or, at worst, contains subsystems, some triangular and some triangular if one unknown is specified.[2]

Step 4. Compute for all (i, j) and (h, j)

$$\bar{c}_{ij} = +c_{ij} - (u_i + p_{ij}v_j) \tag{17}$$

$$\bar{c}'_{hj} = +(-k_j\gamma_{hj}) - (0 - v_j) \tag{18}$$

It has been shown in § 5-2 that, if the x_{ij} or y_{hj} value associated with some non-basic entry is changed to

$$x_{ij} \pm \theta \quad \text{or} \quad y_{hj} \pm \theta \qquad (\theta \geq 0) \tag{19}$$

the other non-basic variables remaining fixed, and the basic variables adjusted, then the expected costs z will change to z' where

$$z' = z \pm \theta\bar{c}_{ij} \quad \text{or} \quad z' = z \pm \theta\bar{c}'_{hj} \tag{20}$$

Assuming, for generality, that x_{ij} may also be a bounded variable, it pays to increase x_{ij} or y_{hj} if $\bar{c}_{ij}$ or $\bar{c}'_{hj} < 0$, if either is at its upper bound, in which case no increase is allowed; also it pays to decrease x_{ij} or y_{hj} if $\bar{c}_{ij}$ or $\bar{c}'_{hj} > 0$, unless $x_{ij} = 0$ or $y_{hj} = 0$, in which case no *decrease* is allowed.

[2] This is the analogue for the weighted distribution problem (11), (12), (13), (14) of the well-known theorem for the standard transportation problem that all bases are triangular. Its proof is similar. See Chapters 14 and 21.

Test for Optimality.

According to the theory of the simplex method with bounded variables (see Chapter 18), if the non-basic variables satisfy the following conditions:

(a) they are all at either their upper or lower bounds,

(b) their corresponding $\bar{c}_{ij}$ and $\bar{c}'_{hj} \geq 0$, if they are at their lower bound, and

(c) their corresponding $\bar{c}_{ij}$ and $\bar{c}'_{hj} \leq 0$ if they are at their upper bound,

then the solution is optimal and the algorithm terminates. Otherwise there are $\bar{c}_{ij}$ or $\bar{c}'_{hj}$ for which (b) or (c) does not hold. In which case an increase or decrease (depending on whether the sign is negative or positive) in the corresponding variable is allowed; we will call these (i, j) or (h, j) combinations *out-of-kilter*; let the largest $\bar{c}_{ij}$ or $\bar{c}'_{hj}$ among them in absolute value be denoted by $\bar{c}_{rs}$ or $\bar{c}'_{rs}$.

Step 5. Leaving all non-basic entries fixed except for the value of the variable corresponding to the (r, s) determined in Step 4, modify the value of x_{rs} (or y_{rs}), if not at its upper bound, to

$$x_{rs} + \theta \text{ (or } y_{rs} + \theta) \text{ if } \bar{c}_{rs} < 0 \text{ (or } \bar{c}'_{rs} < 0) \tag{21}$$

or, if not at its lower bound, to

$$x_{rs} - \theta \text{ (or } y_{rs} - \theta) \text{ if } \bar{c}_{rs} > 0 \text{ (or } \bar{c}'_{rs} > 0) \tag{22}$$

where $\theta \geq 0$ is unknown for the moment, and recompute the values of the basic variables as linear functions of θ. Choose the value of $\theta = \theta^*$ at the largest value possible consistent with keeping all basic variables (whose values now depend on θ) between their upper and lower bounds; in the next cycle correct the values of the basic variables on the assumption $\theta = \theta^*$.

Also, if at the value $\theta = \theta^*$ one (or more) of the basic variables attains its upper or lower bound, in the next cycle drop any one of these variables (never drop more than one) from the basic set and box the variable x_{rs} instead. Should it happen that x_{rs} or y_{rs} attains its upper or lower bound at $\theta = \theta^*$, the set of basic variables is the same as before; their values, however, are changed to allow x_{rs} or y_{rs} to be fixed at its new bound.

Start the next cycle of the iterative procedure by returning to Step 3.

28-2. NUMERICAL SOLUTION OF THE ROUTING PROBLEM

For our starting solution in Table 28-2-IV, cycle o, we used for values of x_{ij} the best solution assuming fixed demands equal to the expected values of the distribution[3] shown in Table 28-2-II. These x_{ij} will meet the expected

[3] In the humorous parody by Paul Gunther [1955-1] entitled "Use of Linear Programming in Capital Budgeting," *Journal of the Operations Research Society of America*, May, 1955, it will be recalled that Mrs. Efficiency wondered why Mr. O. R. did not start out with a good guess. In this chapter you will note that we followed Mrs. Efficiency's suggestion and have started by guessing at the final solution rather than going through the customary use of artificial variables and a Phase I of the simplex process.

demands so that $Y_j = d_j$, except for Route 5 where there is a deficit of 100 and $Y_5 = 500$ (see § 28-1-(5)). These Y_j are broken down into the incremental values shown below the double line in Table 28-2-IV, cycle 0.

Next, one of the variables in each row is boxed. The selected variables are $x_{11}, x_{22}, x_{35}, x_{43}$; each appears likely to be in an optimal solution; however, x_{43} has been boxed rather than x_{41}, which may be a better choice. Next, the last positive entry in each column is boxed, i.e., the variables y_{31}, y_{22}, y_{33}, y_{44}, y_{15}. In all there are $m + n - 2 = 9$ boxed variables. The implicit values, u_i and v_j, shown in the table are determined by solving the nine equations:

$$
\begin{aligned}
(1) \qquad u_1 + p_{11}v_1 &= c_{11} & (p_{11} = 16;\ c_{11} = 18)\\
u_2 + p_{22}v_2 &= c_{22} & (p_{22} = 10;\ c_{22} = 15)\\
u_3 + p_{35}v_5 &= c_{35} & (p_{35} = 29;\ c_{35} = 6)\\
u_4 + p_{43}v_3 &= c_{43} & (p_{43} = 22;\ c_{43} = 17)\\
0 + (-1)v_1 &= -k_1\gamma_{31} & (k_1\gamma_{31} = 9.8)\\
0 + (-1)v_2 &= -k_2\gamma_{22} & (k_2\gamma_{22} = 9.1)\\
0 + (-1)v_3 &= -k_3\gamma_{33} & (k_3\gamma_{33} = 4.9)\\
0 + (-1)v_4 &= -k_4\gamma_{44} & (k_4\gamma_{44} = 2.1)\\
0 + (-1)v_5 &= -k_5\gamma_{15} & (k_5\gamma_{15} = 1.0)
\end{aligned}
$$

This permits the computation of $\bar{c}_{ij}$ and $\bar{c}'_{hj}$ (see § 28-1-(17), (18)). As a check, $\bar{c}_{ij} = 0$ and $\bar{c}'_{hj} = 0$ for (i, j) and (h, j) corresponding to basic variables. The $\bar{c}_{ij}$ or $\bar{c}'_{hj}$ of largest absolute value for those (i, j) or (h, j) that are out-of-kilter is

$$\bar{c}_{24} = 14 - [-76 + 15(2.1)] = +58.5$$

Hence a decrease in the variable x_{24} with adjustments of the basic variables will result in a decrease in the expected costs by an amount of 58.5 units per unit decrease in x_{24}. If $x_{24} = 6$ is changed to $x_{24} = 6 - \theta$, then, in order to satisfy the column 4 equation, the basic variable $y_{44} = 10$ must be modified to $y_{44} = 10 - 15\theta$ (all other variables in column 4 are fixed). Also, to satisfy the row equation 2, $x_{22} = 8$ must be modified to $x_{22} = 8 + \theta$; this in turn causes $y_{22} = 70$ to be changed to $y_{22} = 70 + 10\theta$ in order to satisfy column equation 2. The largest value of θ is $\theta^* = \frac{10}{15}$ at which value $y_{44} = 0$.

The numerical values of the variables appearing in Table 28-2-IV, cycle 1, are obtained from those of Table 28-2-IV, cycle 0, by setting $\theta = \theta^* = \frac{10}{15}$. The variable x_{24} becomes a new basic variable in place of y_{44} which hits its lower bound, zero. Computing the new set of implicit prices the largest $\bar{c}_{ij}$ in absolute value which can increase or decrease according to sign of $\bar{c}_{ij}$ is $\bar{c}_{23} = -23.4$. Changing x_{23} to $5 - \theta$ requires that the variables x_{22}, y_{22}, y_{33} be modified as shown, Table 28-2-IV, cycle 1. The maximum value of θ is $\theta = \theta^* = \frac{20}{14}$ at which value $y_{33} = 0$. The new solution in which x_{23} replaces y_{33} as a basic variable is given in Table 28-2-IV, cycle 2. In Table 28-2-IV,

cycle 2, the decrease in non-boxed variable x_{41} causes changes in the variables x_{43}, x_{22}, x_{23}, y_{31}, y_{22}. The largest value of $\theta = \frac{9}{16}$, at which value y_{22} hits its *upper bound* $b_{22} = 100$.

In the passage from Table 28-2-IV, cycle 3, to Table 28-2-IV, cycle 4, we have become a little fancy and have taken a "double" step. The maximum increase is $\theta = \frac{80}{29}$ at which point y_{15} hits its upper bound $b_{15} = 580$. It is easy to see that if next the incremental variable y_{25} is increased, $\bar{c}_{32}$ associated with x_{32} should be changed to $\bar{c}_{32} - 29(\gamma_{15} - \gamma_{25})k_5 = +4.5 - 29(1.0 - .9) = +1.6$; therefore, it is economical to increase y_{25} as well as y_{15}. However, it can be shown that signs of $\bar{c}_{32}$ would become negative if the next increment, y_{35}, were considered. The maximum value of $\theta = \theta^* = \frac{100}{29}$.

It will be noted that in the passage from cycle 4 to cycle 5 of Table 28-2-IV, the variable y_{33} is again brought into solution after having been dropped earlier. The maximum value of θ is $\frac{22}{20}$ at which value y_{33} reaches its upper bound, so that the solution, Table 28-2-IV, cycle 5, has the same set of basic variables and hence the same implicit values as Table 28-2-IV, cycle 4. Moreover, the solution is *optimal* since all non-basic variables are either at their upper or lower bounds; those at upper bounds have corresponding $\bar{c}_{ij} \leq 0$ and those at lower bounds have $\bar{c}_{ij} \geq 0$.

In comparing this solution (Table 28-2-IV, cycle 5) with the optimal solution for the fixed-demand case (Table 28-2-II), it is interesting to note that the chief difference appears to be a general tendency to shift the total seats made available on a route to the *mode* of the distribution rather than to the *mean* of the distribution for those distributions with sharp peaks. The total seats made available to routes with flat distributions of demand appear to be at highest level attainable with the residual passenger-carrying potential.

TABLE 28-2-I

COMPARATIVE COSTS OF VARIOUS SOLUTIONS

(*Refer to Table 28-2-IV*)	Expected Revenues for Seats Supplied (1)	Expected Lost Revenues[1] (2)	Operating Costs (3)	Net Expected Cost (Thousands) (2) + (3)
Cycle 0	−6,534	766	900	1,666
Cycle 1	−6,574	726	901	1,627
Cycle 2	−6,607	693	901	1,594
Cycle 3	−6,638	662	899	1,561
Cycle 4	−6,641	659	883	1,542
Cycle 5	−6,659	641	883	1,524

[1] Data in column (2) are obtained by subtracting the expected revenues for seats supplied, column (1), from $R_0 = 7{,}300$, the expected revenues if an unlimited number of seats were supplied.

To compute the expected costs of the various solutions the first step (see § 28-1-(14)), is to determine what the expected revenues R_0 would be if sufficient seating capacity were furnished at all times to supply all passengers that show up. Referring to Table 28-1-II, it is easy to see that $R_0 = 13(250 + 13(120) + 13(120) + 7(180) + 7(90) + 1(600) = 7{,}300$ or \$7,300,000.

It is seen that the solution presented in Table 28-2-II (the same as the starting solution, Table 28-2-IV, cycle 0, which assumes demands to be exactly equal to the expected values of demand) has a net *expected cost* of \$1,666,000. (It is interesting to note that if the demands were not variable, but were fixed and equal to expected demands, the costs would only be \$1,000,000 (see Table 28-2-II). The 67 per cent increase in net cost for the variable demand case is due to 13,400 additional passengers (on the average) being turned away because of the distributions of demand assumed.[4]

TABLE 28-2-II

OPTIMAL ASSIGNMENT FOR FIXED DEMAND

Operating Costs and Lost Revenues = \$1,000,000

Type of Aircraft	Route (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	**10** 16 18	15 21	28 18	23 16	81 10	0 0	$10 = a_1$	−171
(2) B	**	8 10 15	5 14 16	6 15 14	57 9	0 0	$19 = a_2$	−51
(3) C	**	7.8 5 10	**	7 9	17.2 29 6	0 0	$25 = a_3$	−23
(4) D	**10** 9 17	11 16	5 22 17	17 15	55 10	0 0	$15 = a_4$	−89
(5) E Deficit	1 13	1 13	1 7	1 7	**100** 1 1	0 0	***	0
Demand d_j	250	120	180	90	600	***		
Implicit Prices v_j	11.8	6.6	4.8	4.33	1	0		

** Box not used; aircraft cannot fly range. *** Row or column has no equation.

[4] For concluding remarks, turn to page 591.

TABLE 28-2-III

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

Type of Aircraft	Route						Aircraft Available	Implicit Prices u_i
	(1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft		
(1) A	x_{11} $p_{11} = 16$ $c_{11} = 18$	x_{12} 15 21	x_{13} 28 18	x_{14} 23 16	x_{15} 81 10	x_{16} 0 0	10	u_1
(2) B	**	x_{22} 10 15	x_{23} 14 16	x_{24} 15 14	x_{25} 57 9	x_{26} 0 0	19	u_2
(3) C	**	x_{32} 5 10	**	x_{34} 7 9	x_{35} 29 6	x_{36} 0 0	25	u_3
(4) D	x_{41} 9 17	x_{42} 11 16	x_{43} 22 17	x_{44} 17 15	x_{45} 55 10	x_{46} 0 0	15	u_4
Increment (1)	$y_{11} \leq 200$ -1 -13	$y_{12} \leq 50$ -1 -13	$y_{13} \leq 140$ -1 -7	$y_{14} \leq 10$ -1 -7	$y_{15} \leq 580$ -1 -1	**	***	0
(2)	$y_{21} \leq 20$ -1 -10.4	$y_{22} \leq 100$ -1 -9.1	$y_{23} \leq 20$ -1 -6.3	$y_{24} \leq 40$ -1 -5.6	$y_{25} \leq 20$ -1 $-.9$	**	***	0
(3)	$y_{31} \leq 30$ -1 -9.8	**	$y_{33} \leq 20$ -1 -4.9	$y_{34} \leq 30$ -1 -4.2	$y_{35} \leq 20$ -1 $-.1$	**	***	0
(4)	$y_{41} \leq 20$ -1 -5.2	**	$y_{43} \leq 20$ -1 -2.1	$y_{44} \leq 20$ -1 -2.1	**	**	***	0
(5)	$y_{51} \leq 30$ -1 -2.6	**	$y_{53} \leq 20$ -1 $-.7$	$y_{54} \leq 240$ -1 $-.7$	**	**	***	0
Net Implicit Prices v_j	0 v_1	0 v_2	0 v_3	0 v_4	0 v_5	*** 0		

** Box not used because aircraft type cannot fly required range, or fewer increments are needed to describe the distribution of demand on the route.

*** Corresponding row or column has no equation.

TABLE 28-2-IV

Cycle 0

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\bar{c}_{24} = +58.4$, $\theta^* = \frac{10}{15}$, Expected Cost = \$1,666,000

Type of Aircraft	Route (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available / Implicit Prices u_i
(1) A	$\boxed{10}$ / 16 / 18	15 / 21	28 / 18	23 / 16	81 / 10	0 / 0	10 / −139
(2) B	**	$\boxed{8+\theta}$ / 10 / 15	5 / 14 / 16	$6-\theta^*$ / 15 / 14	57 / 9	0 / 0	19 / −76
(3) C	**	7.8 / 5 / 10	**	7 / 9	$\boxed{17.2}$ / 29 / 6	0 / 0	25 / −23
(4) D	10 / 9 / 17	11 / 16	$\boxed{5}$ / 22 / 17	17 / 15	55 / 10	0 / 0	15 / −91
Increment (1)	200 / −1 / −13	50 / −1 / −13	140 / −1 / −7	10 / −1 / −7	$\boxed{500}$ / −1 / −1	**	*** / 0
(2)	20 / −1 / −10.4	$\boxed{70+10\theta}$ / −1 / −9.1	20 / −1 / −6.3	40 / −1 / −5.6	−1 / −.9	**	*** / 0
(3)	$\boxed{30}$ / −1 / −9.8	**	$\boxed{20}$ / −1 / −4.9	30 / −1 / −4.2	−1 / −.1	**	*** / 0
(4)	−1 / −5.2	**	−1 / −2.1	$\boxed{10-15\theta}$ / −1 / −2.1	**	**	*** / 0
(5)	−1 / −2.6	**	−1 / −.7	−1 / −.7	**	**	*** / 0
Net Implicit Prices v_j	0 / 9.8	0 / 9.1	0 / 4.9	0 / 2.1	0 / 1	*** / 0	

TABLE 28-2-IV

Cycle 1

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\bar{c}_{23} = +23.4$, $\theta^* = \frac{30}{14}$, Expected Cost = $1,627,000

Type of Aircraft	Route: (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	$\boxed{10}$ 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	**	$\boxed{8.7 + \theta}$ 10 15	$5 - \theta^*$ 14 16	$\boxed{5.3}$ 15 14	57 9	0 0	19	−76
(3) C	**	5 10	**	7 9	$\boxed{17.2}$ 29 6	0 0	25	−23
(4) D	10 9 17	11 16	$\boxed{5}$ 22 17	17 15	55 10	0 0	15	−91
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	$\boxed{500}$ −1 −1	**	***	0
(2)	20 −1 −10.4	$\boxed{77 + 10\theta}$ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	**	***	0
(3)	$\boxed{30}$ −1 −9.8	**	$\boxed{20 - 14\theta}$ −1 −4.9	30 −1 −4.2	−1 −.1	**	***	0
(4)	−1 −5.2	**	−1 −2.1	−1 −2.1	**	**	***	0
(5)	−1 −2.6	**	−1 −.7	−1 −.7	**	**	***	0
Net Implicit Prices v_j	0 9.8	0 9.1	0 4.9	0 6	0 1	*** 0		

TABLE 28-2-IV

Cycle 2

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\bar{c}_{41} = +56.8$, $\theta^* = \frac{9}{16}$, Expected Cost = \$1,594,000

Type of Aircraft	Route (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	$\boxed{10}$ 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	**	$\boxed{10.1+1.6\theta}$ 10 15	$\boxed{3.6+1.6\theta}$ 14 16	$\boxed{5.3}$ 15 14	57 9	0 0	19	−76
(3) C	**	7.8 5 10	**	7 9	$\boxed{17.2}$ 29 6	0 0	25	−23
(4) D	$10-\theta^*$ 9 17	11 16	$\boxed{5+\theta}$ 22 17	17 15	55 10	0 0	15	−128
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	$\boxed{500}$ −1 −1	**	***	0
(2)	20 −1 −10.4	$\boxed{91+16\theta}$ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	**	***	0
(3)	$\boxed{30-9\theta}$ −1 −9.8	**	−1 −4.9	30 −1 −4.2	−1 −.1	**	***	0
(4)	−1 −5.2	**	−1 −2.1	−1 −2.1	**	**	***	0
(5)	−1 −2.6	**	−1 −.7	−1 −.7	**	**	***	0
Net Implicit Prices v_j	0 9.8	0 9.1	0 6.6	0 6	0 1	*** 0		

TABLE 28-2-IV

Cycle 3

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\delta_{32} = +5.5$, $\theta^* = \frac{100}{29}$, Expected Cost = $1,561,000

Type of Aircraft	Route (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	$\boxed{10}$ 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	**	$\boxed{11 + .5\theta}$ 10 15	$\boxed{2.7 - .5\theta}$ 14 16	$\boxed{5.3}$ 15 14	57 9	0 0	19	−40
(3) C	**	$7.8 - \theta^*$ 5 10	**	7 9	$\boxed{17.2 + \theta}$ 29 6	0 0	25	−23
(4) D	$\boxed{9.4 - .3\theta}$ 9 17	11 16	$\boxed{5.6 + .3\theta}$ 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −1	$\boxed{500 + 29\theta}$ −1 −1	**	***	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	**	***	0
(3)	$\boxed{25 - 2.7\theta}$ −1 −9.8	**	−1 −4.9	30 −1 −4.2	−1 −.1	**	***	0
(4)	−1 −5.2	**	−1 −2.1	−1 −2.1	**	**	***	0
(5)	−1 −2.6	**	−1 −.7	−1 −.7	**	**	***	0
Net Implicit Prices v_j	0 9.8	0 5.5	0 4	0 3.6	0 1	*** 0		

TABLE 28-2-IV

Cycle 4

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\bar{c}'_{33} = -.9$, $\theta^* = \frac{19}{22}$, Expected Cost = \$1,542,000

Type of Aircraft	Route (1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	$\boxed{10}$ 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	**	$\boxed{12.8}$ 10 15	$\boxed{.9}$ 14 16	$\boxed{5.3}$ 15 14	57 9	0 0	19	−40
(3) C	**	$\boxed{4.3}$ 5 10	**	7 9	$\boxed{20.7}$ 29 6	0 0	25	−18
(4) D	$\boxed{8.3 - \theta}$ 9 17	11 16	$\boxed{6.7 + \theta}$ 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	580 −1 −1	**	***	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	20 −1 −.9	**	***	0
(3)	$\boxed{15 - 9\theta}$ −1 −9.8	**	$+22\theta^*$ −1 −4.9	30 −1 −4.2	−1 −.1	**	***	0
(4)	−1 −5.2	**	−1 −2.1	−1 −2.1	**	**	***	0
(5)	−1 −2.6	**	−1 −.7	−1 −.7	**	**	***	0
Net Implicit Prices v_j	0 9.8	0 5.5	0 4	0 3.6	0 .8	*** 0		

TABLE 28-2-IV

Cycle 5 (Optimal)

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

Minimum Expected Cost $1,524,000

Type of Aircraft	Route						Aircraft Available	Implicit Prices u_i
	(1) N.Y. to L.A. 1-stop	(2) N.Y. to L.A. 2-stop	(3) N.Y. to Dallas 0-stop	(4) N.Y. to Dallas 1-stop	(5) N.Y. to Boston 0-stop	(6) Surplus Aircraft		
(1) A	$\boxed{10}$ 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	**	$\boxed{12.8}$ 10 15	$\boxed{.9}$ 14 16	$\boxed{5.3}$ 15 14	57 9	0 0	19	−40
(3) C	**	$\boxed{4.3}$ 5 10	**	7 9	$\boxed{20.7}$ 29 6	0 0	25	−18
(4) D	$\boxed{7.4}$ 9 17	11 16	$\boxed{7.6}$ 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	580 −1 −1	**	***	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	20 −1 −.9	**	***	0
(3)	$\boxed{7}$ −1 −9.8	**	20 −1 −4.9	30 −1 −4.2	−1 −.1	**	***	0
(4)	−1 −5.2	**	−1 −2.1	−1 −2.1	**	**	***	0
(5)	−1 −2.6	**	−1 −.7	−1 −.7	**	**	***	0
Net Implicit Prices v_j	0 9.8	0 5.5	0 4	0 3.6	0 .8	0		

The successive improvements in the solution, Table 28-2-IV, cycles 0–5, reduced the net expected costs from $1,666,000 to $1,524,000 for the optimal solution.

Thus, the best solution obtained by pretending that demands are fixed at their expected values has a 9 per cent higher expected cost than that for the best solution obtained by using the assumed distributions of demand. It is also seen that very little additional computational effort was required to take account of this uncertainty of demand.

REFERENCES

Dantzig, 1955-1
Ferguson and Dantzig, 1954-1, 1956-1
Gunther, 1955-1
Manne, 1956-2
Zimmern, 1957-1

BIBLIOGRAPHY

Ablow, C. M., and Georges Brigham, 1955-1. "An Analog Solution of Programming Problems," *Operations Research*, Vol. 3, No. 4, November, 1955, pp. 388–394.

Agmon, Shmuel, 1954-1. "The Relaxation Method for Linear Inequalities," *Canad. J. Math.*, Vol. 6, No. 3, June, 1954, pp. 382–392.

Allen, R. G. D., 1956-1. *Mathematical Economics*, Macmillan & Company, Ltd., London, 1956. 786 pp.

Allendoerfer, C. B., and C. O. Oakley, 1955-1. *Principles of Mathematics*, McGraw-Hill Book Company, Inc., New York, 1955. 448 pp.

Antosiewicz, H. A. (ed.), 1955-1. *Proceedings of the Second Symposium in Linear Programming*, Vols. 1 and 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955.

Arrow, K. J., 1951-1. "Alternative Proof of the Substitution Theorem for Leontief Models in the General Case," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 155–164.

Arrow, K. J., T. E. Harris, and J. Marschak, 1951-1, "Optimal Inventory Policy," *Econometrica*, Vol. 19, No. 3, July, 1951, pp. 250–272.

Arrow, K. J., L. Hurwicz, and H. Uzawa, 1958-1. *Studies in Linear and Non-Linear Programming*, Stanford University Press, Stanford, California, 1958. 229 pp.

Arrow, K. J., S. Karlin, and H. Scarf, 1958-1. *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, California, 1958. 340 pp.

Arrow, K. J., S. Karlin, and P. Suppes (eds.), 1960-1. "Mathematical Methods in the Social Sciences," *Proceedings of the First Stanford Symposium*, Stanford University Press, 1960. 365 pp.

Babbar, M. M., 1955-1. "Distribution of Solutions of a Set of Linear Equations (with Applications to Linear Programming)," *J. Amer. Statist. Assoc.*, Vol. 50, 1955, pp. 854–869.

Barankin, E. W., and R. Dorfman, 1958-1. *On Quadratic Programming*, University of California Publications in Statistics, Vol. 2, No. 13, University of California Press, Berkeley, California, 1958, pp. 285–318.

Batchelor, James H., 1959-1. *Operations Research, An Annotated Bibliography*, 2nd ed., St. Louis University Press, St. Louis, Missouri, 1959. 866 pp.

Baumol, William J., 1959-1. *Economic Dynamics—An Introduction*, 2nd ed., The Macmillan Company, New York, 1959. 396 pp.

———, 1961-1. *Economic Theory and Operations Analysis*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1961. 438 pp.

Beale, E. M. L., 1954-1. "Linear Programming by the Method of Leading Variables," *Report of the Conference on Linear Programming*, May, 1954, arranged by Ferranti, Ltd., London.

———, 1955-1. "Cycling in the Dual Simplex Algorithm," *Naval Res. Logist. Quart.*, Vol. 2, No. 4, December, 1955, pp. 269–276.

———, 1955-2. "On Minimizing a Convex Function Subject to Linear Inequalities," *J. Roy. Statist. Soc., Ser. B*, Vol. 17, No. 2, 1955, pp. 173–184.

———, 1958-1. "A Method of Solving Linear Programming Problems When Some but Not All of the Variables Must Take Integral Values," Statistical Techniques Research Group, Princeton University, Princeton, New Jersey, approximate date March, 1958.

———, 1959-1. "An Algorithm for Solving the Transportation Problem When the Shipping Cost over Each Route Is Convex," *Naval Res. Logist. Quart.*, Vol. 6, No. 1, March, 1959, pp. 43–56.

———, 1959-2. "On Quadratic Programming," *Naval Res. Logist. Quart.*, Vol. 6, No. 3, September, 1959, pp. 227–243.

Beckenbach, E. F., 1948-1. "Convex Functions," *Bull. Amer. Math. Soc.*, Vol. 54, 1948, pp. 439–460.

Beckmann, M. J., 1955-1. "Comparative Studies in Linear Programming and the Giffen Paradox," *Rev. Econ. Studies*, Vol. 23, No. 3, 1955–1956, pp. 232–235.

Beckmann, M. J., C. B. McGuire, and C. Winsten, 1956-1. *Studies in the Economics of Transportation*, The RAND Corporation, Research Memorandum RM-1488, May 12, 1955. 347 pp. Published by the Yale University Press, New Haven, Connecticut, 1956. 232 pp.

Bellman, Richard, 1954-1. *Dynamic Programming of Continuous Processes*, The RAND Corporation, Report R-271, July, 1954.

———, 1957-1. *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957. 342 pp.

———, 1958-1. "On a Routing Problem," *Quart. Appls Math.*, Vol. XVI, No. 1, April, 1958, pp. 87–90.

Bellman, Richard, and David Blackwell, 1951-1. "Red Dog, Blackjack, Poker," *Sci. American*, Vol. 184, No. 1, January, 1951, pp. 44–47.

Bellman, Richard, I. Glicksberg, and O. A. Gross, 1958-1. *Some Aspects of the Mathematical Theory of Control Processes*, The RAND Corporation, Report R-313, January, 1958.

Berge, Claude, 1958-1. *Théorie des graphes et ses applications*, Dunod, Paris, 1958. 275 pp.

Birkhoff, Garrett, 1946-1. "Three Observations on Linear Algebra," *Rev. univ. nac. Tucumán, Ser. A.*, Vol. 5, 1946, pp. 147–151.

Birkhoff, Garrett, and Saunders MacLane, 1953-1. *A Survey of Modern Algebra*, rev. ed., The Macmillan Company, New York, 1953. 276 pp.

Blackwell, David, and M. A. Girshick, 1954-1. *Theory of Games and Statistical Decisions*, John Wiley & Sons, Inc., New York, 1954. 355 pp.

Bodewig, E., 1959-1. *Matrix Calculus*, North-Holland Publishing Company, Amsterdam, 1959.

Boelter, L. M. K., 1955-1. "Some Reflections on Automation," *Symposium on Electronics and Automatic Production*, San Francisco, California, August, 1955.

Bohnenblust, H. F., S. Karlin, and L. S. Shapley, 1950-1. "Solutions of Discrete, Two-person Games," in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 51–72.

Boldyreff, A. W., 1955-1. "Determination of the Maximal Steady State Flow of Traffic through a Railroad Network," The RAND Corporation, Research Memorandum RM-1532, August 5, 1955, 36 pp. Published in *JORSA*, Vol. 3, No. 4, November, 1955, pp. 443-465.

Borel, Emile, 1921-1. "La théorie du jeu et les équations intégrales à noyau symétrique," *Compt. Rend. Acad. Sci.*, Vol. 173, December 19, 1921, pp. 1304–1308. Translated by Leonard J. Savage in *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 97–100.

———, 1924-1, "Sur les jeux où interviennent le hasard et l'habileté des joueurs," *Théorie des probabilités*, Librairie scientifique, Hermann, Paris, 1924, pp. 204–224. Translated by Leonard J. Savage in *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 101–115.

———, 1927-1. "Sur les systèmes de formes linéaires à déterminant symétrique gauche et la théorie générale du jeu," from "Algèbre et calcul des probabilités," *Compt. Rend. Acad. Sci.*, Vol. 184, 1927, pp. 52–53. Translated by Leonard J. Savage in *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 116–117.

———, 1953-1. Translation of three papers by Borel on the theory of games, with commentary, in "Emile Borel, Initiator of the Theory of Psychological Games and Its Application," by Maurice Fréchet, *Econometrica*, Vol. 21, No. 1, January, 1953, pp. 95–127.

Boulding, K. E., and W. A. Spivey, 1960-1. *Linear Programming and the Theory of the Firm*, The Macmillan Company, New York, 1960. 227 pp.

Bowman, E. H., and R. B. Fetter (eds.), 1959-1. *Analyses of Industrial Operations*, Richard D. Irwin, Inc., Homewood, Illinois, 1959. 485 pp.

Brown, G. W., 1951-1. "Iterative Solution of Games by Fictitious Play," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 374–376.

Brown, G. W., and T. C. Koopmans, 1951-1. "Computational Suggestions for Maximizing a Linear Function Subject to Linear Inequalities," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 377–380.

Brown, G. W., and J. von Neumann, 1950-1. "Solutions of Games by Differential Equations," The RAND Corporation, Paper P-142, April 19, 1950, 11 pp. Published in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 73–79.

Cahn, A. S., 1948-1. "The Warehouse Problem," *Bull. Amer. Math. Soc.*, Vol. 54, 1948, p. 1073 (abstract).

Charnes, A., 1952-1. "Optimality and Degeneracy in Linear Programming," *Econometrica*, Vol. 20, No. 2, April, 1952, pp. 160–170. Published in *Operational Res. Quart.*, Vol. 3, No. 3, September, 1952, p. 54 (abstract).

Charnes, A., and W. W. Cooper, 1955-1. "Generalizations of the Warehousing Model," *Operational Res. Quart.*, Vol. 6, 1955, pp. 131–172.

———, 1957-1. "Management Models and Industrial Applications in Linear Programming," *Management Sci.*, Vol. 4, No. 1, October, 1957, pp. 38–91.

———, 1959-1. "Chance-Constrained Programming," *Management Sci.*, Vol. 6, No. 1, October, 1959, pp. 73–79.

———, 1959-2. "Some Uses of Model Prototypes in an Operations Research Study," Graduate School of Industrial Administration, Reprint No. 35, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, 1959.

———, 1961-1. *Management Models and Industrial Applications of Linear Programming*, John Wiley & Sons, Inc., New York, 1961.

Charnes, A., W. W. Cooper, and A. Henderson, 1953-1. *An Introduction to Linear Programming*, John Wiley & Sons, Inc., New York, 1953. 74 pp.

Charnes, A., W. W. Cooper, and B. Mellon, 1952-1. "Blending Aviation Gasolines—A Study in Programming Interdependent Activities in an Integrated Oil Company," *Econometrica*, Vol. 20, No. 2, April, 1952.

Charnes, A., and H. J. Greenberg, 1951-1. "Plastic Collapse and Linear Programming—Preliminary Report," *Bull. Amer. Math. Soc.*, Vol. 57, No. 6, November, 1951, p. 480 (abstract).

Charnes, A., and C. E. Lemke, 1954-1. "Minimization of Non-Linear Separable Convex Functionals," *Naval Res. Logist. Quart.*, Vol. 1, No. 4, December, 1954.

Churchman, C. W., R. L. Ackoff, E. L. Arnoff, et al, 1957-1. *Introduction to Operations Research*, John Wiley & Sons, Inc., New York, 1957. 645 pp.

Clarke, G. B., and I. G. Simpson, 1959-1. "A Theoretical Approach to the Profit Maximization Problems in Farm Management," *J. of Agric. Econ.*, University of Leeds, England, Vol. 13, No. 3, June, 1959, pp. 150–164.

Cornfield, J., W. D. Evans, and M. Hoffenberg, 1947-1. "Full Employment Pattern, 1950," Parts I and II, *Monthly Labor Review*, Vol. 64, February and March, 1947, pp. 163–190, 420–432.

Courant, R., and D. Hilbert, 1953-1. *Methods of Mathematical Physics*, I, 1st English ed., translated and revised from the German original, Interscience Publishers, Inc., New York, 1953, pp. 164–167.

Cramér, H., 1955-1. *The Elements of Probability Theory and Some of Its Applications*, John Wiley & Sons, Inc., New York, 1955. 281 pp.

Dantzig, George B., 1948-1. "Programming in a Linear Structure," Comptroller, USAF, Washington, D.C., February, 1948.

———, 1949-1. "Programming of Interdependent Activities, II, Mathematical Model," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 330–335; also published in *Econometrica*, Vol. 17, Nos. 3 and 4, July–October, 1949, pp. 200–211.

———, 1951-1. "A Proof of the Equivalence of the Programming Problem and the Game Problem," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 330–335.

———, 1951-2. "Application of the Simplex Method to a Transportation Problem," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 359–373.

———, 1951-3. "Maximization of a Linear Function of Variables Subject to Linear Inequalities," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 339–347.

———, 1954-1. *Notes on Linear Programming: Part XI, Composite Simplex-Dual Simplex Algorithm—I*, The RAND Corporation, Research Memorandum RM-1274, April 26, 1954, 15 pp. Also The RAND Corporation, Paper P-516.

———, 1954-2. *Notes on Linear Programming: Part VII—The Dual Simplex Algorithm*, The RAND Corporation, Research Memorandum RM-1270, July 3, 1954. 10 pp.

———, 1954-3. *Notes on Linear Programming: Parts VIII, IX, X—Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming*, The RAND Corporation, Research Memorandum RM-1367, October 4, 1954. Published in *Econometrica*, Vol. 23, No. 2, April, 1955, pp. 174–183.

———, 1955-1. "Linear Programming under Uncertainty," *Management Sci.*, Vol. 1, 1955, pp. 197–206.

———, 1955-2. *Notes on Linear Programming: Part XIII—Optimal Solution of a Dynamic Leontief Model with Substitution*, The RAND Corporation, Research Memorandum RM-1281-1, April 6, 1955, 13 pp. Published in *Econometrica*, Vol. 23, No. 3, July, 1955, pp. 295–302.

———, 1956-1. "Constructive Proof of the Min-Max Theorem," *Pacific J. Math.*, Vol. 6, No. 1, 1956, pp. 25–33.

———, 1956-2. "Recent Advances in Linear Programming," *Management Sci.*, Vol. 2, No. 2, January, 1956, pp. 131–144.

———, 1957-1. "Thoughts on Linear Programming and Automation," The RAND Corporation, Paper P-824, March 2, 1956, 16 pp. Published in *Management Sci.*, Vol. 3, No. 2, January, 1957, pp. 131–139.

———, 1957-2. "Discrete Variable Extremum Problems," *Operations Research*, Vol. 5, No. 2, April, 1957, pp. 266–277.

———, 1957-3. "On the Status of Multistage Linear Programs," The RAND Corporation, Paper P-1028, February 20, 1957. Published in *Proc. Intern. Statist. Inst.*, Stockholm, Sweden, 1957; also in *Management Sci.*, October, 1959.

———, 1958-1. "On Integer and Partial Integer Linear Programming Problems," The RAND Corporation, Paper P-1410, June 20, 1958.

———, 1958-2. "Solving Two-Move Games with Perfect Information," The RAND Corporation, Paper P-1459, August 11, 1958.

———, 1959-1. "Inductive Proof of the Simplex Method," The RAND Corporation, Paper P-1851, December 28, 1959. Published in *IBM J. Res. Develop.*, Vol. 4, No. 5, November, 1960, pp. 505–506.

———, 1960-1. "On the Shortest Route through a Network," *Management Sci.*, Vol. 6, No. 2, January, 1960, pp. 187–190.

———, 1960-2. "On the Significance of Solving Linear Programming Problems with Some Integer Variables," *Econometrica*, Vol. 28, No. 1, January, 1960, pp. 30–44.

———, 1960-3. "General Convex Objective Forms," in K. J. Arrow, S. Karlin, and P. Suppes (eds.), *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, California, 1960, pp. 151–158.

Dantzig, George B., Lester R. Ford, Jr., and Delbert R. Fulkerson, 1956-1. "A Primal-Dual Algorithm for Linear Programs," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 171–181.

Dantzig, George B., and Delbert R. Fulkerson, 1954-1. "Minimizing the Number of Tankers to Meet a Fixed Schedule," *Naval Res. Logist. Quart.*, Vol. 1, No. 3, September, 1954, pp. 217–222.

———, 1956-1, "On the Max-Flow Min-Cut Theorem of Networks," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of

Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 215–221.

Dantzig, George B., Delbert R. Fulkerson, and Selmer M. Johnson, 1954-1. "Solution of a Large-Scale Traveling-Salesman Problem," *JORSA*, Vol. 2, No. 4, November, 1954, pp. 393–410.

———, 1959-1. "On a Linear Programming Combinatorial Approach to the Traveling-Salesman Problem," *Operations Research*, Vol. 7, No. 1, January-February, 1959, pp. 58–66.

Dantzig, George B., and Albert Madansky, 1960-1. "On the Solution of Two-Stage Linear Programs under Uncertainty," The RAND Corporation, Paper P-2039, July 28, 1960.

Dantzig, George B., and William Orchard-Hays, 1953-1. *Notes on Linear Programming: Part V—Alternate Algorithm for the Revised Simplex Method Using Product Form for the Inverse*, The RAND Corporation, Research Memorandum RM-1268, November 19, 1953.

Dantzig, George B., and Alex Orden, 1953-1. *Notes on Linear Programming: Part II—Duality Theorems*, The RAND Corporation, Research Memorandum RM-1265, October 30, 1953.

Dantzig, George B., Alex Orden, and Philip Wolfe, 1954-1. *Notes on Linear Programming: Part I—The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints*, The RAND Corporation, Research Memorandum RM-1264, April 5, 1954, 17 pp. Published in *Pacific J. Math.*, Vol. 5, No. 2, June, 1955, pp. 183–195.

Dantzig, George B., and A. Wald, 1951-1. "On the Fundamental Lemma of Neyman and Pearson," *Ann. Math. Statist.*, Vol. 22, 1951, pp. 87–93.

Dantzig, George B., and Philip Wolfe, 1960-1. "Decomposition Principle for Linear Programs," *Operations Research*, Vol. 8, No. 1, January-February, 1960, pp. 101–111.

Dennis, J. B., 1959-1. *Mathematical Programming and Electrical Networks*, John Wiley & Sons, Inc., New York, 1959. 185 pp.

Dilworth, R. P., 1950-1. "A Decomposition Theorem for Partially Ordered Sets," *Ann. Math.*, Vol. 51, No. 1, January, 1950, pp. 161–166.

Dines, L. L., and N. H. McCoy, 1933-1. "On Linear Inequalities," *Trans. Roy. Soc. Can., Sect. III*, 1933, pp. 37–70.

Doig, A. G., and M. H. Belz, 1956-1. "Report on Trim Problems for May, 1956," Department of Statistics, University of Melbourne. The report is addressed to Australia Paper Manufacturers, Melbourne, Australia, July 31, 1956.

Dorfman, R., 1951-1. *Application of Linear Programming to the Theory of the Firm*, University of California Press, Berkeley, California, 1951. 98 pp.

———, 1951-2. "Application of the Simplex Method to a Game Theory Problem," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 348–358.

Dorfman, R., P. A. Samuelson, and R. M. Solow, 1958-1. *Linear Programming and Economic Analysis*, McGraw-Hill Book Company, Inc., New York, 1958. 527 pp.

Dorn, W. S., 1961-1. "On Lagrange Multipliers and Inequalities," *Operations Research*, Vol. 9, No. 1, January-February, 1961, pp. 95–104.

Dorn, W. S., and H. J. Greenberg, 1955-1. *Linear Programming and Plastic Limit Analysis of Structures*, Technical Report No. 7, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, August, 1955. 30 pp.

Dresher, Melvin, 1961-1. *Games of Strategy: Theory and Applications*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1961. 186 pp.

Dresher, Melvin, A. W. Tucker, and Philip Wolfe (eds.), 1957-1. *Contributions to the Theory of Games*, Vol. III, Annals of Mathematics Study No. 39, Princeton University Press, Princeton, New Jersey, 1957. 448 pp.

Duffin, R. J., 1956-1. "Infinite Programs," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 157–170.

Dvoretzky, A., J. Kiefer, and J. Wolfowitz, 1952-1. "The Inventory Problem: I, Case of Known Distributions of Demand; II, Case of Unknown Distributions of Demand," *Econometrica*, Vol. 20, Nos. 2 and 3, 1952, pp. 187–222 and 450–466.

Dwyer, Paul S., 1955-1. "The Solution of the Hitchcock Transportation Problem with the Method of Reduced Matrices," *Eng. Research Inst. Report*, University of Michigan, Ann Arbor, Michigan, December, 1955.

Easterfield, Thomas E., 1946-1. "A Combinatorial Algorithm." *J. London Math. Soc.*, Vol. 21, No. 83, Part 3, July, 1946, pp. 219–226.

Edmondson, J. H., 1951-1. "Proof of the Existence Theorem of an Epsilon Transformation," class exercise dated March 28, 1951, for Department of Agriculture Graduate School course in linear programming given by George B. Dantzig. Copies may be obtained from George B. Dantzig.

Egerváry, E., 1931-1. "Matrixok Kombinatorius Tulajfonságairól," *Matematikai és Fizikai Lapok* No. 38, 1931, pp. 16–28. "On Combinatorial Properties of Matrices," translated by H. W. Kuhn, Paper No. 4, George Washington University Logistics Research Project. Published in *Logistics Papers*, Issue No. 11, Appendix I to Quarterly Progress Report No. 21, November 16, 1954-February 15, 1955, 11 pp.

Eisemann, Kurt, 1957-1. "The Trim Problem," *Management Sci.*, Vol. 3, April, 1957, pp. 279–284.

Eisemann, Kurt, and J. R. Lourie, 1959-1. "The Machine Loading Problem," IBM 704 Program, IBML. 1, International Business Machines Corporation, IBM Applications Library, New York, 1959.

El-Maghraby, S. E., 1959-1. "An Approach to Linear Programming under Uncertainty," *Operations Research*, Vol. 7, March, 1959, pp. 208–216.

———, 1960-1. "Allocation under Uncertainty When the Demand Has Continuous d.f.," *Management Sci.*, Vol. 6, No. 3, April, 1960, p. 270.

Fabian, Tibor, 1954-1. "Process Type Analysis of the Iron and Steel Industry, Part IV—Programming Open Hearth Steel Production," Discussion Paper No. 46, Management Sciences Research Project, University of California, Los Angeles, California, November 18, 1954. 12 pp.

———, 1955-1. "Process Analysis of the Iron and Steel Industry: A Model," Research Report No. 47, Management Sciences Research Project, University of California, Los Angeles, California, 1955. Published in *Econometrica*, Vol. 23, No. 3, July, 1955, pp. 347–348 (abstract).

———, 1958-1. "A Linear Programming Model of Integrated Iron and Steel Production," *Management Sci.*, Vol. 4, No. 4, July, 1958, pp. 415–449.

Farkas, J., 1902-1. "Über die Theorie der einfachen Ungleichungen," *J. Reine Angew. Math.*, Vol. 124, 1902, pp. 1–24.

Feller, William, 1957-1. *An Introduction to Probability Theory and Its Applications*, 2nd ed., Vol. 1, John Wiley & Sons, Inc., New York, 1957. 461 pp.

Fenchel, W., 1953-1. "Convex Cones, Sets, and Functions," in D. W. Blackett's *Lecture Notes*, Office of Naval Research Logistics Project Report, Department of Mathematics, Princeton University, Princeton, New Jersey, 1953, 74 pp.

Ferguson, Allen R., and G. B. Dantzig, 1954-1. "Notes on Linear Programming: Part XVI—The Problem of Routing Aircraft—a Mathematical Solution," The RAND Corporation, Research Memorandum RM-1369, September 1, 1954; also The RAND Corporation, Paper P-561, published in *Aeronaut. Eng. Rev.*, Vol. 14, No. 4, April, 1955, pp. 51–55.

———, 1956-1. "The Allocation of Aircraft to Routes—An Example of Linear Programming under Uncertain Demand," The RAND Corporation, Paper P-727, December 7, 1956, 41 pp. Published in *Management Sci.*, Vol. 3, No. 1, October, 1956, pp. 45–73.

Fisher, Walter D., and Leonard W. Schruben, 1953-1. "Linear Programming Applied to Feed-Mixing under Different Price Conditions," *J. Farm Econ.*, Vol. XXXV, No. 4, November, 1953, pp. 471–483.

Flood, Merrill M., 1953-1. "On the Hitchcock Distribution Problem," The RAND Corporation, Paper P-213, May, 1951, 26 pp. Published in *Pacific J. Math.*, Vol. 3, No. 2, 1953, pp. 369–386.

———, 1956-1. "The Traveling Salesman Problem," in J. F. McCloskey and J. M. Coppinger (eds.), *Operations Research for Management*, Vol. II, Johns Hopkins

Press, Baltimore, Maryland, 1956, pp. 340–357; also in *Operations Research*, Vol. 4, No. 1, February, 1956, pp. 61–75.

———, 1960-1. "An Alternative Proof of a Theorem of König as an Algorithm for the Hitchcock Distribution Problem," in R. Bellman and Marshall Hall, Jr. (eds.), *Proceedings of Symposia in Applied Mathematics*, Vol. X, *Combinatorial Analysis*, American Mathematical Society, Providence, Rhode Island, 1960, pp. 299–307.

Ford, Lester R., Jr., and Delbert R. Fulkerson, 1954-1. "Maximal Flow through a Network," The RAND Corporation, Research Memorandum RM-1400, November 19, 1954; also The RAND Corporation, Paper P-605, November 19, 1954. Published in *Canad. J. Math.*, Vol. 8, No. 3, 1956, pp. 399–404.

———, 1955-1. *Notes on Linear Programming: Part XXIX—A Simple Algorithm for Finding Maximal Network Flows and an Application to the Hitchcock Problem*, The RAND Corporation, Research Memorandum RM-1604, December 29, 1955, 21 pp.; also The RAND Corporation, Paper P-743, December 29, 1955.

———, 1956-1. "Notes on Linear Programming—Part XXXII: Solving the Transportation Problem," The RAND Corporation, Research Memorandum RM-1736, June 20, 1956. Published in *Management Sci.*, Vol. 3, 1956, pp. 24–32.

———, 1956-2. "Dynamic Network Flow," The RAND Corporation, Paper P-967, October, 1956, 29 pp.

———, 1957-1. "A Primal-dual Algorithm for the Capacitated Hitchcock Problem," The RAND Corporation, Paper P-827, March 23, 1956; also The RAND Corporation, Research Memorandum RM-1798, September 25, 1956. Published in *Naval Res. Logist. Quart.*, Vol. 4, No. 1, March, 1957, pp. 47–54.

———, 1958-1. "Constructing Maximal Dynamic Flows from Static Flows," *Operations Research*, Vol. 6, May, 1958, pp. 419–433.

———, 1958-2. "Suggested Computation for Maximal Multi-Commodity Network Flows," The RAND Corporation, Paper P-1114, March 27, 1958. Published in *Management Sci.*, Vol. 5, No. 1, October, 1958, pp. 97–101.

———, 1960-1. *Flows in Networks*, The RAND Corporation, Report R-375, December 20, 1960. Published by Princeton University Press, Princeton, New Jersey, 1962.

Forsythe, G. E., 1953-1. "Solving Linear Algebraic Equations Can Be Interesting," *Bull. Amer. Math. Soc.*, Vol. 59, No. 4, July, 1953, pp. 299–329.

———, 1955-1. "Computing Constrained Minima with Lagrange Multipliers," *J. Soc. Indust. Appl. Math.*, Vol. 3, No. 4, December, 1955, pp. 173–178.

Foulkes, J., 1955-1. "Linear Programming and Structural Design," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and USAF, Washington, D.C., 1955, pp. 177–184.

Fourier, Jean Baptiste Joseph, 1826-1. "Solution d'une question particulière du calcul des inégalités," 1826, and extracts from "Histoire de l'Académie," 1823, 1824, Oeuvres II, pp. 317–328.

Fox, L., 1954-1. "Practical Solution of Linear Equations and Inversion of Matrices," in Olga Taussky (ed.), *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues*, National Bureau of Standards. *Applied Mathematics Series* 39, Washington, D.C., September, 1954, pp. 1–54.

Freund, R. J., 1956-1. "The Introduction of Risk into a Programming Model," *Econometrica*, Vol. 24, No. 3, July, 1956, pp. 253–263.

Frisch, Ragnar, 1954-1. "Principles of Linear Programming with Particular Reference to the Double Gradient Form of the Logarithmic Potential Network," *Memorandum* fra Universitetets Socialokonomiske Institutt, Oslo, University Institute of Economics, October 18, 1954. 219 pp.

———, 1957-1. "Linear Dependencies and a Mechanized Form of the Multiplex Method for Linear Programming," *Memorandum*, Universitetets Socialokonomiske Institutt, Oslo, University Institute of Economics, September, 1957.

Fulkerson, D. R., 1956-1. "Note on Dilworth's Decomposition Theorem for Partially Ordered Sets," *Proc. Amer. Math. Soc.*, Vol. 7, 1956, pp. 701–702.

———, 1959-1. "A Network Flow Feasibility Theorem and Combinatorial Applications," *Canad. J. Math.*, Vol. 11, 1959, pp. 440–451.

———, 1959-2. "Increasing the Capacity of a Network: The Parametric Budget Problem," *Management Sci.*, Vol. 5, 1959, pp. 472–483.

———, 1959-3. "An Out-of-Kilter Method for Minimal Cost Flow Problems," The RAND Corporation, Paper P-1825, 1959. Published in *J. Soc. Indust. Appl. Math.*, Vol. 9, 1961, pp. 18–27.

———, 1960-1. "Zero-one Matrices with Zero Trace," *Pacific J. Math.*, Vol. 10, 1960, pp. 831–836.

———, 1961-1. "A Network Flow Computation for Project Cost Curves," *Management Sci.*, Vol. 7, No. 2, January, 1961, pp. 167–178.

———, 1961-2. "An Out-of-Kilter Method for Minimal-Cost Flow Problems," *J. Soc. Indust. Appl. Math.*, Vol. 9, No. 1, March, 1961, pp. 18–27.

Fulkerson, D. R., and G. B. Dantzig, 1955-1. "Computations of Maximal Flows in Networks," *Naval Res. Logist. Quart.*, Vol. 2, No. 4, December, 1955, pp. 277–283.

Gaddum, J. W., 1952-1. "A Theorem on Convex Cones with Applications to Linear Inequalities," *Proc. Amer. Math. Soc.*, Vol. 3, 1952, pp. 957–960.

Gaddum, J. W., A. J. Hoffman, and D. Sokolowsky, 1954-1. "On the Solution of the Caterer Problem," *Naval Res. Logist. Quart.*, Vol. 1, No. 3, September, 1954, pp. 223–229.

Gainen, Leon, 1955-1. "Linear Programming in Bid Evaluations," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., pp. 29–38.

Gale, David, 1951-1. "Convex Polyhedral Cones and Linear Inequalities," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, Chapter XVII, pp. 287–297.

———, 1956-1. "Neighboring Vertices on a Convex Polyhedron," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 255–263.

———, 1956-2. "The Basic Theorems of Real Linear Equations, Inequalities, Linear Programming and Game Theory," *Naval Res. Logist. Quart.*, Vol. 3, No. 3, September, 1956, pp. 193–200.

———, 1957-1. "A Theorem on Flows in Networks," *Pacific J. Math.*, Vol. 7, 1957, pp. 1073–1082.

———, 1959-1. "Transient Flows in Networks," *Michigan Math. J.*, Vol. 6, 1959, pp. 59–63.

———, 1960-1. *The Theory of Linear Economic Models*, McGraw-Hill Book Company, Inc., New York, 1960. 330 pp.

Gale, David, H. W. Kuhn, and A. W. Tucker, 1950-1. "On Symmetric Games," in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 81–87.

———, 1950-2. "Reductions of Game Matrices," *Ibid.*, pp. 89–96.

———, 1951-1. "Linear Programming and the Theory of Games," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951. Also The RAND Corporation, Report R-193, June, 1951.

Garvin, Walter W., 1960-1. *Introduction to Linear Programming*, McGraw-Hill Book Company, Inc., New York, 1960, 281 pp.

Garvin, Walter W., H. W. Crandall, J. B. John, and R. A. Spellman, 1957-1. "Applications of Linear Programming in the Oil Industry," *Management Sci.*, Vol. 3, No. 4, July, 1957, pp. 407–430.

Gass, Saul I., 1958-1. *Linear Programming: Methods and Applications*, McGraw-Hill Book Company, Inc., New York, 1958. 223 pp.

Gass, Saul I., and T. L. Saaty, 1955-1. "The Computational Algorithm for the Parametric Objective Function," *Naval Res. Logist. Quart.*, Vol. 2, No. 1, June, 1955, pp. 39–45.

Gauss, Karl Friedrich, 1826-1. "Theoria Combinationis Observationum Erroribus Minimis Obnoxiae," *Supplementum*, Vol. 4, Werke, Göttingen, 1826, pp. 55–93.

Gerstenhaber, Murray, 1951-1. "Theory of Convex Polyhedral Cones," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, Chapter XVIII, pp. 298–316.

———, 1958-1. "A Solution Method for the Transportation Problem," *J. Soc. Indust. Appl. Math.*, Vol. 6, No. 4, 1958, pp. 312–316.

Goldman, A. J., 1956-1. "Resolution and Separation Theorems for Polyhedral Convex Sets," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 41–51.

Goldman, A. J., and A. W. Tucker, 1956-1. "Polyhedral Convex Cones," *Ibid.*, pp. 19–39.

———, 1956-2. "Theory of Linear Programming," *Ibid.*, pp. 53–97.

Goldstein, Leon, 1952-1. "Problem of Contract Awards," in Alex Orden and Leon Goldstein (eds.), *Symposium on Linear Inequalities and Programming*, Project SCOOP, No. 10, Planning Research Division, Director of Management Analysis Service, Comptroller, USAF, Washington, D.C., April, 1952, pp. 147–154.

Gomory, R. E., 1958-1. "Essentials of an Algorithm for Integer Solutions to Linear Programs," *Bull. Amer. Math. Soc.*, Vol. 64, No. 5, 1958.

———, 1958-2. "An Algorithm for Integer Solutions to Linear Programs," Princeton-IBM Mathematics Research Project, Technical Report No. 1, November 17, 1958.

———, 1958-3. "An Algorithm for the Mixed Integer Problem," The RAND Corporation, Paper P-1885, February 22, 1960, and Abstract 553-190, "Extension of an Algorithm for Integer-Solutions to Linear Programs," *Amer. Math. Soc. Notices*, Vol. 6, No. 1, Issue 36, February, 1959, p. 52.

Gomory, R. E., and T. C. Hu, 1960-1. "Multi-terminal Network Flows," IBM Research Report, RC318, 1960. 30 pp.

———, 1960-2. "An Application of Generalized Linear Programming to Network Flows," IBM Research Report, 1960. 50 pp.

Good, R. A., 1959-1. "Systems of Linear Relations," *J. Soc. Indust. Appl. Math.*, Vol. 1, No. 1, January, 1959, pp. 1–31.

Gordan, P., 1873-1. "Über die Anglösung linearer Gleichungen mit reellen Coefficienten," *Math. Ann.*, Vol. 6, 1873, pp. 23–28.

Greene, J. H., K. Chatto, C. R. Hicks, and C. B. Cox, 1959-1. "Linear Programming in the Packing Industry," *J. Indust. Eng.*, Vol. X, No. 5, September-October, 1959, pp. 364–372.

Gross, O., 1959-1. "The Bottleneck Assignment Problem," The RAND Corporation, Paper P-1630, March 6, 1959, presented at the RAND Symposium on Mathematical Programming (Linear Programming and Extensions), March 16–20, 1959.

Gunther, Paul, 1955-1. "Use of Linear Programming in Capital Budgeting," *JORSA*, Vol. 3, No. 2, May, 1955, pp. 219–224.

Hadley, G., 1961-1. *Linear Algebra*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961. 269 pp.

———, 1961-2. *Linear Programming*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961. 517 pp.

Halmos, P. R., and Herbert E. Vaughan, 1950-1. "The Marriage Problem," *Amer. J. Math.*, Vol. 72, No. 1, January, 1950, pp. 214–215.

Hartley, H. O., and L. D. Loftsgard, 1958-1. "Linear Programming with Variable Restraints," *J. Sci.*, Iowa State College, Ames, Iowa, Vol. 33, No. 2, November, 1958, pp. 161–172.

Heady, E. O., and W. Candler, 1958-1. *Linear Programming Methods*, Iowa State College Press, Ames, Iowa, 1958. 597 pp.

Heller, I., 1955-1. "On the Traveling Salesman's Problem," in H. A. Antosiewicz (ed.), 1955-1, *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 643–665.

———, 1957-1. "Constraint Matrices of Transportation Type Problems," *Naval Res. Logist. Quart.*, Vol. 4, No. 1, March, 1957, pp. 73–76.

Heller, I., and C. B. Tompkins, 1956-1. "An Extension of a Theorem of Dantzig's, Paper 14," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 247–254.

Henderson, A., and R. Schlaifer, 1954-1. "Mathematical Programming," *Harvard Business Rev.*, Vol. 32, May-June, 1954, pp. 73–100.

Henderson, J. M., 1955-1. "A Short-run Model for the Coal Industry," *Rev. Econ. and Statist.*, Vol. 37, No. 4, November, 1955, pp. 336–346.

Herrmann, C. C., and J. F. Magee, 1953-1. "Operations Research for Management," *Harvard Business Rev.*, Vol. 31, No. 4, July-August, 1953, pp. 100–112.

Heyman, J., 1951-1. "Plastic Design of Beams and Plane Frames for Minimum Material Consumption," *Quart. Appl. Math.*, Vol. 8, No. 4, January, 1951, pp. 373–381.

Hicks, J. R., 1960-1. "Linear Theory," *The Economic Journal* (Quarterly Journal of the Royal Economic Society), London, Vol. LXX, No. 280, December, 1960, pp. 671–709.

Hildreth, C. G., 1955-1. "Economic Implications of Some Cotton Fertilizer Experiments," *Econometrica*, Vol. 23, No. 1, January, 1955, pp. 88–98.

Hildreth, C. G., and Stanley Reiter, 1951-1. "On the Choice of Crop Rotation Plan," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 177–188.

Hirsch, Warren M., and George B. Dantzig, 1954-1. "The Fixed Charge Problem," The RAND Corporation, Paper P-648, December 1, 1954.

Hitchcock, Frank L., 1941-1. "The Distribution of a Product from Several Sources to Numerous Localities," *J. Math. Phys.*, Vol. 20, 1941, pp. 224–230.

Hoffman, A. J., 1953-1. "Cycling in the Simplex Algorithm," National Bureau of Standards Report, No. 2974, December 16, 1953. 7 pp.

———, 1956-1. "Systems of Distinct Representatives and Linear Programming," *Amer. Math. Monthly*, Vol. 63, 1956, pp. 455–460.

———, 1960-1. "Some Recent Applications of the Theory of Linear Inequalities to Extremal Combinatorial Analysis," in R. Bellman and Marshall Hall, Jr. (eds.), *Proceedings of Symposia in Applied Mathematics*, Vol. X, *Combinatorial Analysis*, American Mathematical Society, Providence, Rhode Island, 1960, pp. 113–127.

Hoffman, A. J., and J. G. Kruskal, 1956-1. "Integral Boundary Points of Convex Polyhedra," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 223–246.

Hoffman, A. J., and H. W. Kuhn, 1956-1. "On Systems of Distinct Representatives," *Ibid.*, pp. 199–206; also in *Amer. Math. Monthly*, Vol. 63, 1956, pp. 455–460.

Hoffman, A. J., M. Mannos, D. Sokolowsky, and N. Wiegmann, 1953-1. "Computational Experience in Solving Linear Programs," *J. Soc. Indust. Appl. Math.*, Vol. 1, No. 1, 1953, pp. 17–33.

Houthakker, H. S., 1955-1. "On the Numerical Solution of the Transportation Problem," *JORSA*, Vol. 3, No. 2, May, 1955, pp. 210–214.

———, 1959-1. "The Capacity Method of Quadratic Programming," in P. Wolfe (ed.), *The RAND Symposium on Mathematical Programming*, The RAND Corporation, Report R-351, March 16–20, 1959.

Hu, T. C., and W. Prager, 1958-1. "Network Analysis of Production Smoothing," Division of Applied Mathematics, Brown University, Providence, Rhode Island, March, 1958.

Iri, M., 1960-1. "A New Method of Solving Transportation Network Problems," *J. Operations Res. Soc. Japan*, Vol. 3, No. 1 and 2, October, 1960, pp. 27–87.

Jackson, J. R., 1957-1. "Simulation Research on Job Shop Production," *Naval Res. Logist. Quart.*, Vol. 1, No. 4, December, 1957, p. 287.

Jacobs, W. W., 1954-1. "The Caterer Problem," *Naval Res. Logist. Quart.*, Vol. 1, No. 2, June, 1954, pp. 154–165.

———, 1955-1. "Military Applications of Linear Programming," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 1–27.

Jaeger, A., 1961-1. *Introduction to Analytic Geometry and Linear Algebra*, Holt, Rinehart, and Winston, Inc., New York, 1961. 299 pp.

Jewell, W. S., 1957-1. "Warehousing and Distribution of a Seasonal Product," in E. H. Bowman and R. B. Fetter (eds.), *Analyses of Industrial Operations*, 1959, pp. 129–135; also in *Naval Res. Logist. Quart.*, Vol. 4, No. 1, March, 1957, pp. 29–34.

———, 1958-1. "Optimal Flow Through Networks," Interim Technical Report No. 8 on Fundamental Investigations in Methods of Operations Research, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1958.

———, 1960-1. "Optimal Flow Through Networks with Gains," presented at Second International Conference on Operations Research, Aix-en-Provence, France, 1960.

John, F., 1948-1. "Extremum Problems with Inequalities as Subsidiary Conditions," *Studies and Essays* (Courant Anniversary Volume), Interscience Publishers, Inc., New York, 1948, pp. 187–204.

Johnson, S., 1954-1. "Optimal Two-and-Three-Stage Production Schedules with Setup Times Included," *Naval Res. Logist. Quart.*, Vol. 1, 1954, pp. 61–68.

———, 1958-1. "Discussion: Sequencing *n* Jobs on Two Machines with Arbitrary Time Lags: Alternate Proof and Discussion of General Case," The RAND Corporation, Paper P-1526, October, 1958. Published in *Management Sci.*, Vol. 5, No. 3, April, 1959.

Jordan, W., 1920-1. *Handbuch der Vermessungskunde*, Vol. 1, J. B. Metclersche Buchhandlung, Stuttgart, 7th ed., 1920, p. 36; also in 5th ed., 1904, Vol. 1, pp. 81–83, 100–105.

Kalaba, R. E., and M. L. Juncosa, 1956-1. "Optimal Design and Utilization of Communication Networks," *Management Sci.*, Vol. 3, No. 1, 1956, pp. 33–44.

Kantorovich, L. V., 1939-1. "Mathematical Methods in the Organization and Planning of Production," Publication House of the Leningrad State University, 1939. 68 pp. Translated in *Management Sci.*, Vol. 6, 1960, pp. 366–422.

———, 1942-1. "On the Translocation of Masses," *Compt. Rend. Acad. Sci. U.R.S.S.*, 37, 1942, pp. 199–201.

———, 1959-1. *Economicheskii Raschet Nailuchshego Ispol'Zovaniia Resursov*, Akademii Nauk SSSR, Moscow, 1959. 343 pp.

Kantorovich, L. V., and M. K. Gavurin, 1949-1. "The Application of Mathematical Methods to Problems of Freight Flow Analysis" (translation), *Akademii Nauk SSSR*, 1949.

Kaplansky, Irving, 1945-1. "A Contribution to von Neumann's Theory of Games," *Ann. Math.*, Vol. 46, No. 3, July, 1945, pp. 474–479.

Karlin, Samuel, 1959-1. *Mathematical Methods and Theory in Games, Programming, and Economics*, Vols. 1 and 2, Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts, 1959.

Kawaratani, T. K., R. J. Ullman, and G. B. Dantzig, 1960-1. "Computing Tetraethyl-lead Requirements in the Linear Programming Format," The RAND Corporation, Research Memorandum RM-2425, April 1, 1960; also The RAND Corporation, Paper P-1545, June 26, 1959. Published in *Operations Research*, Vol. 8, No. 1, January-February, 1960.

Kelley, J. E., Jr., 1960-1. "The Cutting Plane Method for Solving Convex Programs," *J. Soc. Indust. Appl. Math.*, Vol. 8, No. 4, December, 1960, pp. 703–712.

———, 1961-1. "Critical-Path Planning and Scheduling; Mathematical Basis," *Operations Research*, Vol. 9, May, 1961, pp. 296–320.

Kemeny, J. G., H. Mirkil, J. C. Snell, and G. L. Thompson, 1959-1. *Finite Mathematical Structures*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1959, 487 pp.

Kemeny, J. G., J. C. Snell, and G. L. Thompson, 1957-1. *Introduction to Finite Mathematics*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1957. 372 pp.

Kendall, David G., 1960-1. "On Definite Doubly-Stochastic Matrices and Birkhoff's Problem III," *J. London Math. Soc.*, Vol. 35, Part 1, No. 137, January, 1960. pp. 81–84.

König, Dines, 1936-1. *Theorie der Endlichen und Unendlichen Graphen*, Akad. Verl. M.B.H., Leipzig, 1936, and Chelsea Publishing Company, New York, 1950. 258 pp.

Koopmans, T. C., 1947-1. "Optimum Utilization of the Transportation Sytsem," *Proceedings of the International Statistical Conferences*, 1947, Washington, D.C. (Vol. 5 reprinted as Supplement to *Econometrica*, Vol. 17, 1949.)

——— (ed), 1951-1. *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951. 404 pp.

———, 1951-2. "Analysis of Production as an Efficient Combination of Activities," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 33–97.

———, 1954-1. "Uses of Prices," *Proceedings of the Conference on Operations Research in Production and Inventory Control*, Case Institute of Technology, Cleveland, Ohio, 1954, pp. 90–96.

Koopmans, T. C., and S. Reiter, 1951-1. "A Model of Transportation," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 222–259.

Kruskal, J. B., Jr., 1956-1. "On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem." *Proc. Amer. Math. Soc.*, Vol. 7, 1956, pp. 48–50.

Kuhn, H. W., 1955-1. "The Hungarian Method for the Assignment Problem," *Naval Res. Logist. Quart.*, Vol. 2, 1955, pp. 83–97.

———, 1956-1. "Solvability and Consistency for Linear Equations and Inequalities," *Amer. Math. Monthly*, Vol. 63, No. 4, April, 1956, pp. 217–232.

———, 1957-1. "A Note on Prager's Transportation Problem," *J. Math. Phys.*, Vol. 36, No. 2, 1957, pp. 107–111.

Kuhn, H. W., and A. W. Tucker (eds.), 1950-1. *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950.

———, 1950-2. "Nonlinear Programming," in Jerzy Neyman (ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, California, 1950, pp. 481–492; also in *Econometrica*, Vol. 19, No. 1, January, 1951, pp. 50–51 (abstract).

——— (eds.), 1953-1. *Contributions to the Theory of Games*, Vol. II, Annals of Mathematics Study No. 28, Princeton University Press, Princeton, New Jersey, 1953.

———, 1955-1. "Games, Theory of," *Encyclopaedia Britannica*, Vol. 10, 1958, pp. 5–10.

——— (eds.), 1956-1. *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956. 322 pp.

——— (eds.), 1956-2. "A Bibliography on Linear Inequalities and Related Systems," *Ibid.*, pp. 305–322.

———, 1958-1. "John von Neumann's Work in the Theory of Games and Mathematical Economics," *Bull. Amer. Math. Soc.*, Vol. 64, No. 3, Part 2, May, 1958, pp. 100–122.

Kunz, K. S., 1957-1. *Numerical Analysis*, McGraw-Hill Book Company, Inc., New York, 1957.

Lambert, F., 1960-1. "Programmes linéaires mixtes," *Cahiers centre d'études rech. oper.*, Brussels, 1960.

Land, A. H., and A. G. Doig, 1957-1. "An Automatic Method of Solving Discrete Programming Problems" (unpublished work, prepared as part of a project for the British Trading Company, Ltd., at the London School of Economics and Political Science).

LaVallee, R. S., 1955-1. "The Application of Linear Programming to the Problem of Scheduling Traffic Signals," *JORSA*, Vol. 3, No. 4, November, 1955, p. 562 (abstract).

Lemke, C. E., 1954-1. "The Dual Method of Solving the Linear Programming Problem," *Naval Res. Logist. Quart.*, Vol. 1, 1954, pp. 36–47.

Leontief, Wassily, 1951-1. *The Structure of American Economy, 1919–1939*, Oxford University Press, New York, 1951.

Leontief, Wassily, et al., 1953-1. *Studies in the Structure of the American Economy*, Oxford University Press, New York, 1953. 561 pp.

Lesourne, J., 1960-1. *Technique Economique Gestion Industrielle*, 2nd edition, Dunod, Paris, France, 1960, pp. 408–476.

Lewis, Robert E., 1955-1. "Top Management Looks at Linear Programming and Inventory Management," *Proceedings of the Linear Programming and Inventory Management Seminar*, Methods Engineering Council, Pittsburgh, Pennsylvania, September 15–16, 1955, pp. B-1 to B-8.

Luce, R. Duncan, and Howard Raiffa, 1957-1. *Games and Decisions, Introduction and Critical Survey*, John Wiley & Sons, Inc., New York, 1957. 509 pp.

MacDonald, J., 1950-1. *Strategy in Poker, Business and War*, W. W. Norton & Company, Inc., New York, 1950.

Madansky, Albert, 1959-1. "Some Results and Problems in Stochastic Linear Programming," The RAND Corporation, Paper P-1596, January 19, 1959.

———, 1960-1. "Inequalities for Stochastic Linear Programming Problems," *Management Sci.*, Vol. 6, No. 2, January, 1960.

———, 1960-2. "Methods of Solution of Linear Programs under Uncertainty," The RAND Corporation, Paper P-2132, November 8, 1960.

Manne, Alan S., 1956-1. *Scheduling of Petroleum Refinery Operations*, Harvard University Press, Cambridge, Massachusetts, 1956. 185 pp.

———, 1956-2. "An Application of Linear Programming to the Procurement of Transport Aircraft," *Management Sci.*, Vol. 2, No. 2, January, 1956, pp. 190–191 (abstract).

Marcus, M., 1960-1. "Some Properties of Doubly Stochastic Matrices," *Amer. Math. Monthly*, Vol. 67, No. 3, March, 1960, pp. 215–221.

Markowitz, Harry M., 1952-1. "Portfolio Selection," *The Journal of Finance*, Vol. VII, No. 1, March, 1952.

———, 1954-1. *The Nature and Applications of Process Analysis*, The RAND Corporation, Research Memorandum RM-1254, May 24, 1954.

———, 1954-2. "Concepts and Computing Procedures for Certain X_{ij} Programming Problems," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 509–565; also The RAND Corporation, Paper P-602, November 19, 1954.

———, 1956-1. "The Optimization of a Quadratic Function Subject to Linear Constraints," The RAND Corporation, Research Memorandum RM-1438, February 21, 1955; also The RAND Corporation, Paper P-637, June 27, 1955. Published in *Naval Res. Logist. Quart.*, Vol. 3, 1956.

———, 1959-1. *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons, Inc., New York, 1959. 344 pp.

Markowitz, Harry M., and A. S. Manne, 1957-1. "On the Solution of Discrete Programming Problems," *Econometrica*, Vol. 25, No. 1, January, 1957, p. 19.

Massé, Pierre, 1946-1. "Les réserves et la regulation de l'avenir dans la vie économique," Vol. I, *Avenir determiné*, Vol. II, *Avenir aléatoire*, Librairie scientifique, Hermann, Paris, 1946.

Massé, Pierre, and R. Gibrat, 1957-1. "Applications of Linear Programming to Investments in the Electric Power Industry," *Management Sci.*, Vol. 3, No. 1, January, 1957, pp. 149–166.

Maynard, Harold B., 1955-1. "Putting New Management Tools to Work," *Proceedings of the Linear Programming and Inventory Management Seminar*, Methods Engineering Council, Pittsburgh, Pennsylvania, September 15–16, 1955.

McKinsey, J. C. C., 1952-1. *Introduction to the Theory of Games*, The RAND Corporation, Report R-228, July, 1952; also published by McGraw-Hill Book Company, Inc., New York, 1952. 371 pp.

Miller, C. E., 1960-1. "The Simplex Method for Local Separable Programming," Standard Oil Company of California, August, 1960.

Miller, C. E., A. W. Tucker, and R. A. Zemlin, 1960-1. "Integer Programming Formulation of Traveling Salesman Problems," *J. Assoc. Comput. Mach.*, Vol. 7, No. 4, October, 1960.

Mills, Harlan D., 1956-1. "Marginal Values of Matrix Games and Linear Programs," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 183–193.

Minkowski, H., 1896-1. *Geometrie der Zahlen*, B. G. Teubner, Leipzig and Berlin, 1910. 256 pp. First printing, 1896; also reprinted by Chelsea Publishing Company, New York, 1953.

Monroe, Arthur Eli, 1924-1. *Early Economic Thought; Selections from Economic Literature Prior to Adam Smith*, Harvard University Press, Cambridge, Massachusetts, 1924. 400 pp.

Moore, E. F., 1957-1. "The Shortest Path Through a Maze," unpublished mimeographed report, Bell Telephone Laboratories, 16 pp.

Morgenstern, Oskar, 1949-1. "The Theory of Games," *Sci. American*, Vol. 180, No. 5, May, 1949, pp. 22–25.

——— (ed.), 1954-1. *Economic Activity Analysis*, John Wiley & Sons, Inc., New York, 1954. 554 pp.

Morin, George, 1955-1. "More Effective Production Planning with Linear Programming," Paper F from *Proceedings of the Linear Programming and Inventory Management Seminar*, Methods Engineering Council, Pittsburgh, September 15–16, 1955.

Motzkin, T. S., 1936-1. *Beiträge zur Theorie der Linearen Ungleichungen*, Jerusalem, 1936 (Doctoral Thesis, University of Zurich).

———, 1952-1. "The Multi-index Transportation Problem," *Bull. Amer. Math. Soc.*, Vol. 58, No. 4, 1952, p. 494 (abstract).

———, 1956-1. "The Assignment Problem," *Proceedings of the Sixth Symposium in Applied Mathematics*, McGraw-Hill Book Company, Inc., 1956, pp. 109–125.

Motzkin, T. S., H. Raiffa, G. L. Thompson, and R. M. Thrall, 1953-1. "The Double Description Method," in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. II, Annals of Mathematics Study No. 28, Princeton University Press, Princeton, New Jersey, 1953, pp. 51–73.

Motzkin, T. S., and I. J. Schoenberg, 1954-1. "The Relaxation Method for Linear Inequalities," *Canad. J. Math.*, Vol. 6, 1954, pp. 393–404.

Munkres, James, 1957-1. "Algorithms for the Assignment and Transportation Problems," *J. Soc. Indust. Appl. Math.*, Vol. 5, No. 1, March, 1957, pp. 32–38.

Natrella, Joseph V., 1955-1. "New Applications of Linear Programming," presented at the Annual Meeting of the Association for Computing Machinery, Philadelphia, Pennsylvania, September 14–16, 1955; also in *Computers and Automation*, Vol. 4, No. 11, p. 22, Item 48, 1955 (abstract).

Nelson, Richard R., 1957-1. "Degeneracy in Linear Programming: A Simple Geometric Interpretation," *Rev. Econ. Statist.*, Vol. 39, 1957, pp. 402–407.

Newman, P., 1955-1. "Some Calculations on Least-Cost Diets Using the Simplex Method," *Bulletin*, Oxford University Institute of Statistics, Oxford, England, September, 1955, Vol. 17, pp. 303–320.

New York Times, 1959-1. "Russian Assails Soviet Economics," by Harry Schwartz, *The New York Times*, May 29, 1959.

Neyman, J., 1950-1. *First Course in Probability and Statistics*, Vol. 1, Henry Holt & Company, Inc., New York, 1950. 350 pp.

Neyman, J., and E. S. Pearson, 1936-1. "Contributions to the Theory of Testing Statistical Hypotheses," *Statist. Res. Mem.*, Parts I and II, 1936, 1938.

Norman, R. Z., 1955-1. "On the Convex Polyhedra of the Symmetric Traveling Salesman Problem," *Bull. Amer. Math. Soc.*, Vol. 61, No. 6, November, 1955, p. 559 (abstract).

Orchard-Hays, W., 1954-1. *A Composite Simplex Algorithm—II*, The RAND Corporation, Research Memorandum RM-1275, May, 1954.

———, 1955-1. *RAND Code for the Simplex*, The RAND Corporation, Research Memorandum RM-1440, February 7, 1955.

———, 1956-1. "Evolution of Computer Codes for Linear Programming," The RAND Corporation, Paper P-810, March 14, 1956, pp. 22–24.

Orchard-Hays, W., Leola Cutler, and Harold Judd, 1956-1. "Manual for the RAND IBM Code for Linear Programming on the 704," The RAND Corporation, Paper P-842, May 16, 1956, pp. 24–26.

Orden, Alex, 1952-1. "Application of the Simplex Method to a Variety of Matrix Problems," in Alex Orden and Leon Goldstein (eds.), *Symposium on Linear Inequalities and Programming*, Project SCOOP, No. 10, Planning Research Division, Director of Management Analysis Service, Comptroller, USAF, Washington, D.C., April, 1952.

———, 1955-1. "Survey of Research on Mathematical Solutions of Programming Problems," *Management Sci.*, Vol. 1, No. 2, January, 1955, pp. 170–172.

———, 1956-1. "The Transshipment Problem," *Management Sci.*, Vol. 2, No. 3, April, 1956, pp. 276–285.

Orden, Alex, and L. Goldstein (eds.), 1952-1. *Symposium on Linear Inequalities and Programming*, Project SCOOP, No. 10, Planning Research Division, Director of Management Analysis Service, Comptroller, USAF, Washington, D.C., April, 1952.

Paull, A. E., and John R. Walter, 1955-1. "The Trim Problem: An Application of Linear Programming to the Manufacture of Newsprint Paper," *Econometrica*, Vol. 23, No. 3, July, 1955, p. 336 (abstract).

Phipps, C. G., 1952-1. "Maxima and Minima Under Restraint," *Amer. Math. Monthly*, Vol. 59, No. 4, April, 1952, pp. 230–235.

Poussin, M. Ch. J. de la Vallée, 1911-1. "Sur la méthode de l'approximation minimum," *Ann. soc. sci. de Bruxelles*, 35, 1911, pp. 1–16.

Prager, William, 1956-1. "On the Caterer Problem," *Management Sci.*, Vol. 3, No. 1, October, 1956, pp. 15–23; *Management Sci.*, Vol. 3, No. 2, January, 1957, p. 209 (abstract).

———, 1957-1. "A Generalization of Hitchcock's Transportation Problem," *J. Math. Phys.* (M.I.T.), Vol. 36, No. 2, July, 1957, pp. 99–106.

Pyne, Insley B., 1956-1. "Linear Programming on an Electronic Analogue Computer," AIEE Transactions Annual, 1956; also Reeves Instrument Corporation, Technical Article Reprint No. 110.

Radner, Roy, 1955-1. "The Linear Team: An Example of Linear Programming Under Uncertainty," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 381–396.

Raiffa, H., G. L. Thompson, and R. M. Thrall, 1952-1. "An Algorithm for the Determination of all Solutions of a Two-person Zero-sum Game with a Finite Number of Strategies (Double Descriptive Method)," in A. Orden and Leon Goldstein (eds.), *Symposium on Linear Inequalities and Programming*, Project SCOOP, No. 10, Planning Research Division, Director of Management Analysis Service, Comptroller, USAF, Washington, D.C., April, 1952, pp. 100–114.

Reisch, E., and L. Eisgruber, 1960-1. "Bibliography of Linear Programming and Its Application to Agricultural Economic Problems," mimeographed report (probably available from Purdue University), 1960, 30 pp.

Reiter, S., 1957-1. "Surrogates for Uncertain Decision Problems: Minimal Information for Decision Making," *Econometrica*, Vol. 25, 1957, pp. 339–345.

Richardson, Moses, 1958-1. *Fundamentals of Mathematics*, The Macmillan Company, New York, 1958.

Riley, Vera, and Saul I. Gass, 1958-1. *Linear Programming and Associated Techniques, a Comprehensive Bibliography on Linear, Nonlinear and Dynamic Programming*, Johns Hopkins Press, Baltimore, Maryland, 1958. 613 pp.

Robinson, Julia, 1951-1. "An Iterative Method of Solving a Game," *Ann. of Math.*, Vol. 54, No. 2, September, 1951, pp. 296–301.

Rosen, J. B., 1960-1. "Gradient Projection Method for Non-Linear Programming, Part I, Linear Constraints," *J. Soc. Indust. Appl. Math.*, Vol. 8, No. 1, 1960, pp. 181–217. "Part II, Non-Linear Constraints," preprint Shell Development Company, New York, Paper P-954, January, 1961. Published in *J. Soc. Indust. Appl. Math*, Vol. 9, No. 4, December, 1961, pp. 514–532.

Ryser, H. J., 1961-1. "Matrices of Zeros and Ones," *Bull. Amer. Math. Soc.*, Vol. 66, 1960, pp. 442–464.

Saaty, T. L., 1955-1. "The Number of Vertices of a Polyhedron," *Amer. Math. Monthly*, Vol. 65, No. 5, May, 1955, pp. 327–331.

———, 1959-1. *Mathematical Methods of Operative Research*, McGraw-Hill Book Company, Inc., New York, 1959. 421 pp.

Salveson, M. E., 1953-1. "A Computational Technique for the Fabrication Scheduling Problem," Management Sciences Research Project, University of California, Los Angeles, California, 1953.

Samuelson, P. A., 1955-1. "Linear Programming and Economic Theory," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 251–272.

Sasieni, M. W., A. Yaspan, and L. Friedman, 1959-1. *Operations Research: Methods and Problems*, John Wiley & Sons, Inc., New York, 1959. 316 pp.

Shapley, L. S., 1959-1. "On Network Flow Functions," The RAND Corporation, Research Memorandum RM-2338, March, 1959, and The RAND Corporation, Paper P-2185, January 9, 1961; also in *Naval Res. Logist. Quart.*, Vol. 8, No. 2, June, 1961.

Shapley, L. S., and R. N. Snow, 1950-1. "Basic Solutions of Discrete Games," in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 27–35.

Sherman, H. C., H. L. Campbell, and C. S. Lanford, 1939-1. "Experiments on the Relation of Nutrition to the Composition of the Body and the Length of Life," *Proc. Nat. Acad. of Sci., U.S.*, Vol. 25, 1939, pp. 16–20.

Shettz, C. M., 1959-1. "A Solution to the Transportation Problem with Non-Linear Costs," *Operations Research*, Vol. 7, No. 5, September-October, 1959, pp. 571–580.

Shubik, Martin, 1959-1. *Strategy and Market Structure; Competition, Oligopoly, and the Theory of Games*, John Wiley & Sons, Inc., New York, 1959.

Simon, H. A., 1956-1. "Dynamic Programming Under Uncertainty with a Quadratic Criterion Function," *Econometrica*, Vol. 24, 1956, pp. 74–81.

Slater, M., 1950-1. "Lagrange Multipliers Revisited," *Cowles Commission Discussion Papers*, Math. 403, November, 1950; The RAND Corporation, Research Memorandum RM-676, August, 1951.

Smith, L. W., Jr., 1956-1. "Current Status of the Industrial Use of Linear Programming," *Management Sci.*, Vol. 2, No. 2, January, 1956, pp. 156–158.

Smith, V. E., 1959-1. "Linear Programming Models for the Determination of Palatable Human Diets," *J. of Farm Econ.*, Vol. 41, No. 2, May, 1959, pp. 272–283.

Solow, R. M., 1952-1. "On the Structure of Linear Models," *Econometrica*, Vol. 20, No. 1, January, 1952, pp. 29–46.

Stiefel, E., 1960-1. "Note on Jordan Elimination, Linear Programming and Tchebycheff Approximation," *Numer. Math.*, Vols. 2 and 4, 1960.

Stiemke, E., 1915-1. "Über positive Lösungen homogener linearer Gleichungen," *Math. Ann.*, Vol. 76, 1915, pp. 340–342.

Stigler, George J., 1945-1. "The Cost of Subsistence," *J. of Farm Econ.*, Vol. 27, No. 2, May, 1945, pp. 303–314.

Stokes, R. W., 1931-1. "A Geometric Theory of Solution of Linear Inequalities," *Trans. Amer. Math. Soc.*, Vol. 33, 1931, pp. 782–805.

Swanson, E. R., 1955-1. "Integrating Crop and Livestock Activities in Farm Management Activity Analysis," *J. of Farm Econ.*, Vol. 37, No. 5, December, 1955, pp. 1249–1258.

———, 1955-2. "Solving Minimum-Cost Feed Mix Problems," *J. of Farm Econ.*, Vol. 37, 1955, pp. 135–139.

Symonds, G. H., 1955-1. *Linear Programming: The Solution of Refinery Problems*, Esso Standard Oil Company, New York, 1955.

———, 1955-2. "Mathematical Programming as an Aid to Decision Making," *Advanced Management*, Vol. 20, No. 5, May, 1955, pp. 11–17.

Talacko, J. V., 1959-1. "On Stochastic Linear Inequalities," *Trabajos Estadist.*, Vol. 10, 1959, pp. 89–112.

Theil, H., 1957-1. "A Note on Certainty Equivalence in Dynamic Planning," *Econometrica*, Vol. 25, 1957, pp. 346–349.

Thrall, R. M., 1955-1. "Some Results in Non-linear Programming," The RAND Corporation, Research Memorandum RM-909, August 6, 1952, 17 pp.; also in H. A.

Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comp troller, USAF, Washington, D.C., 1955, pp. 471–493.

Thrall, R. M., and L. Tornheim, 1957-1. *Vector Spaces and Matrices*, John Wiley & Sons, Inc., New York, 1957. 318 pp.

Tintner, G., 1955-1. "Stochastic Linear Programming with Applications to Agricultural Economics," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 1, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 197–228.

Tompkins, C. B., 1955-1. "Projection Methods in Calculation," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 425–448.

———, 1957-1. "Some Methods of Computational Attack on Programming Problems, Other Than the Simplex Method," *Naval Res. Logist. Quart.*, Vol. 4, No. 1, March, 1957, pp. 95–96.

Tucker, A. W., 1950-1. "Linear Programming and Theory of Games," *Econometrica*, Vol. 18, No. 2, April, 1950, p. 189.

———, 1950-2. "Analogues of Kirchoff's Laws," *George Washington University Logistics Papers*, Vol. 3, 1950, 4 pp.

———, 1955-1. "Linear Inequalities and Convex Polyhedral Sets," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 569–602.

———, 1955-2. *Game Theory and Programming*, National Science Foundation, Summer Mathematics Institute Notes, Department of Mathematics, Oklahoma A and M College, Stillwater, Oklahoma, 1955.

———, 1956-1. "Dual Systems of Homogeneous Linear Relations," in H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, Annals of Mathematics Study No. 38, Princeton University Press, Princeton, New Jersey, 1956, pp. 3–18.

———, 1957-1. "Linear and Nonlinear Programming," *Operations Research*, Vol. 5, No. 2, April, 1957, pp. 244–257.

———, 1960-1. "On Directed Graphs and Integer Programs," IBM Mathematical Research Project, Technical Report, Princeton University Press, Princeton, New Jersey, 1960.

———, 1960-2. "A Combinatorial Equivalence of Matrices," in R. Bellman and Marshall Hall, Jr. (eds.), *Proceedings of Symposia in Applied Mathematics*, Vol. X, *Combinatorial Analysis*, American Mathematical Society, Providence, Rhode Island, 1960, pp. 129–140.

———, 1960-3. "Solving a Matrix Game by Linear Programming," *IBM J. Res. Develop.*, Vol. 4, No. 5, November, 1960, pp. 507–517.

Tutte, W. T., 1946-1. "On Hamiltonian Circuits," *J. London Math. Soc.*, Vol. 21, Part 2, No. 82, April, 1946, pp. 98–101.

USAF, 1954-1, Hq., Comptroller, Computational Division. "The Appreciation of Linear Programming Techniques to Air Force Problems," December 17, 1954, 27 pp.

Uzawa, Hirofumi, 1958-1. "The Kuhn-Tucker Theorem in Concave Programming," in Kenneth J. Arrow, Leonid Hurwicz, and Hirofumi Uzawa (eds.), *Studies in Linear and Non-Linear Programming*, Stanford University Press, Stanford, California, 1958.

———, 1960-1. "Market Mechanisms and Mathematical Programming," *Econometrica*, Vol. 28, No. 4, October, 1960, pp. 872–881.

Vajda, S., 1956-1. *The Theory of Games and Linear Programming*, John Wiley & Sons, Inc., New York, 1956. 106 pp.

———, 1958-1. *Readings in Linear Programming*, John Wiley & Sons, Inc., New York, 1958, and Sir Isaac Pitman & Sons Ltd., London, 1958. 99 pp.

———, 1958-2. "Inequalities in Stochastic Linear Programming," *Bull. Internat. Statist. Inst.*, Vol. 36, 1958, pp. 357–363.

———, 1961-1. *Mathematical Programming*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961. 310 pp.

Vazsonyi, Andrew, 1955-1. "Optimizing a Function of Additively Separated Variables Subject to a Simple Restriction," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, USAF, Washington, D.C., 1955, pp. 453–469.

———, 1958-1. *Scientific Programming in Business and Industry*, John Wiley & Sons, Inc., New York, 1958. 474 pp.

Ville, Jean A., 1938-1. "Sur la théorie générale des jeux où intervient l'habilité des jouers," *Applications aux jeux de hasard* by Emile Borel and Jean Ville, Tome 4, Fascicule 2, in *Traité du calcul des probabilités et de ses applications*, by Emile Borel, 1938, pp. 105–113.

von Neumann, John, 1928-1. "Zur Theorie der Gesellschaftsspiele," *Math. Ann.*, Vol. 100, 1928, pp. 295–320. Translated by Sonya Bargmann in A. W. Tucker and R. D. Luce (eds.), *Contributions to the Theory of Games*, Vol. IV, Annals of Mathematics Study No. 40, Princeton University Press, Princeton, New Jersey, 1959, pp. 13–42.

———, 1937-1. "Über un okononisches Gleichungsystem und ein Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines Mathematischen Kolloquiums*, No. 8, 1937. Translated in *Rev. Econ. Studies*, Vol. 13, No. 1, 1945–46, pp. 1–9.

———, 1947-1. "On a Maximization Problem" (manuscript), Institute for Advanced Study, Princeton, New Jersey, November, 1947.

———, 1948-1. "A Numerical Method for Determination of the Value and Best Strategies of a Zero-Sum, Two-Person Game," Institute for Advanced Study, Princeton, New Jersey, 1948.

———, 1953-1. "A Certain Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problem," in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 2, Annals of Mathematics Study No. 28, Princeton University Press, Princeton, New Jersey, 1953, pp. 12–15.

———, 1954-1. "A Numerical Method to Determine Optimum Strategy," *Naval Res. Logist. Quart.*, Vol. 1, No. 2, 1954.

von Neumann, John, and Oskar Morgenstern, 1944-1. *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944; 2nd ed., 1947; 3rd ed., 1953.

Votaw, D. F., Jr., 1952-1. "Methods of Solving Some Personnel Classification Problems," *Psychometrika*, Vol. 17, No. 3, September, 1952, pp. 255–266.

———, 1955-1. "Programming Under Conditions of Uncertainty," in H. A. Antosiewicz (ed.), *Proceedings of the Second Symposium in Linear Programming*, Vol. 2, National Bureau of Standards and Directorate of Management Analysis, DCS/Comptroller, Washington, D.C., 1955, pp. 187–195.

Votaw, D. F., Jr., and A. Orden, 1952-1. "Personnel Assignment Problem," in Alex Orden and Leon Goldstein (eds.), *Symposium on Linear Inequalities and Programming*, Project SCOOP, No. 10, Planning Research Division, Director of Management Analysis Service, Comptroller, USAF, Washington, D.C., April, 1952, pp. 155–163.

Wagner, H. M., 1957-1. "A Comparison of the Original and Revised Simplex Methods," *Operations Research*, Vol. 5, No. 3, June, 1957, pp. 361–369.

———, 1957-2. "A Linear Programming Solution to Dynamic Leontief Type Models," *Management Sci.*, Vol. 3, No. 3, April, 1957, pp. 234–254.

———, 1958-1. "The Dual Simplex Algorithm for Bounded Variables," *Naval Res. Logist. Quart.*, Vol. 5, No. 3, September, 1958, pp. 257–261.

———, 1958-2. "On the Distribution of Solutions in Linear Programming Problems," *J. Amer. Statist. Assoc.*, Vol. 53, March, 1958, pp. 161–163.

———, 1959-1. "On a Class of Capacitated Transportation Problems," *Management Sci.*, Vol. 5, No. 3, April, 1959, pp. 304–318.

Wald, Abraham, 1935-1. "Über die eindeutige positive Lösbarkeit der neunen Produktions gleichungen"; also "Über die Produktions gleichungen der ökonomischen

Wertlehre," *Ergebnisse eines Mathematischen Kolloquiums*, No. 6, 1935, pp. 12–18, and No. 7, 1936, pp. 1–6.

———, 1950-1. *Statistical Decision Functions*, John Wiley & Sons, Inc., New York, 1950. 179 pp.

Walras, Leon, 1874-1. "Eléments d'économie politique pure ou théorie de la richesse sociale," 1st fasc, Vol. 8, Lausanne, Paris, and Basel, 1874, 208 pp. For other references on Walras, see Dorfman, Samuelson, and Solow, 1958-1.

Waugh, F. V., 1951-1. "The Minimum-Cost Dairy Feed," *J. of Farm Econ.*, Vol. 33, No. 3, August, 1951, pp. 299–310.

———, 1958-1. "Alligations, Forerunner of Linear Programming," *J. of Farm Econ.*, Vol. 40, 1958, pp. 89–101.

Weyl, H., 1935-1. "Elementare Theorie der konvexen Polyeder," *Comment. Math. Helv.*, Vol. 7, 1935, pp. 290–306. Translated in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. 1, Annals of Mathematics Study No. 24, Princeton University Press, Princeton, New Jersey, 1950, pp. 3–18.

White, W. B., S. M. Johnson, and G. B. Dantzig, 1958-1. "Chemical Equilibrium in Complex Mixtures," *J. Chem. Phys.*, Vol. 28, No. 5, May, 1958, pp. 751–755. See also Dantzig, Johnson and White, "A Linear Programming Approach to the Chemical Equilibrium Problem," *Management Sci.*, Vol. 5, No. 1, October, 1958, pp. 38–43.

Williams, A. C., 1960-1. "The Method of Continuous Coefficients," Socony Mobile Oil Company, Part I, Report No. ECC 60.3, March 1, 1960; Part II, Report No. ECC 60.4, March 21, 1960.

Williams, John D., 1954-1. *The Compleat Strategyst*, The RAND Corporation Research Studies, McGraw-Hill Book Company, Inc., New York, 1954.

Wolfe, Philip, 1957-1. "Computational Techniques for Non-Linear Programs," Princeton University Conference, Princeton, New Jersey, March, 1957.

———, 1959-1. "The Simplex Method for Quadratic Programming," *Econometrica*, Vol. 27, No. 3, July, 1959.

——— (ed.), 1959-2. *The RAND Symposium on Mathematical Programming*, The RAND Corporation, Report R-351, March 16–20, 1959.

———, 1960-1. "Accelerating the Cutting Plane Method for Non-Linear Programming," The RAND Corporation, Paper P-2010, August, 1960; also in the *J. Soc. Indust. Appl. Math.*, Vol. 9, No. 3, September, 1961, pp. 481–488.

Wood, Marshall K., 1948-1. "Scientific Planning Techniques," Project SCOOP, Discussion Paper Number 1-DU, Planning Research Division, Directorate of Program Standards and Cost Control, Comptroller, USAF, Washington, D.C., August, 1948, p. 6A.

Wood, Marshall K., and G. B. Dantzig, 1949-1. "The Programming of Interdependent Activities: General Discussion," in T. C. Koopmans, *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 15–18; also in *Econometrica*, Vol. 17, Nos. 3 and 4, July-October, 1949, pp. 193–199.

Wood, Marshall K., and M. A. Geisler, 1951-1. "Development of Dynamic Models for Program Planning," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, Inc., New York, 1951, pp. 189–192.

Zimmern, B., 1957-1. "Résolution des programmes linéaires de transport par la méthode de séparation en étoile," *Revue de recherche opérationnelle*, Vol. 1, No. 3, 1957, pp. 117–132.

Zoutendijk, G., 1959-1. "Maximizing a Function in a Convex Region," *J. Roy. Statist. Soc. Ser. B*, Vol. 21, No. 2, 1959, pp. 338–355.

———, 1960-1. *Methods of Feasible Directions*, Elsevier Publishing Company, New York, 1960. 126 pp.

ADDENDUM:

Graves, Robert L., and Wolfe, Philip, eds., *Recent Advances in Mathematical Programming*, McGraw-Hill Book Company, Inc., New York, 1963.

SUBJECT INDEX

Activity, 2, 6, 433
  definition of, 32
  level of, 2, 32
  permanently feasible, 499
  variable coefficients of, 433
    with one control parameter, 440–444
    with several control parameters, 444–445
Activity analysis, 6, 19
  (*See also* Linear programming assumptions)
Allocation of aircraft to routes, 568–591
  fixed-demand case in the problem of, 570–574
  formulation of the problem of, 568–578
  uncertain-demand case in the problem of, 574–577
    numerical solution of, 580–590
Allocation with surplus and deficit, 322–329
  example of, with slack, 326–329
  as a transportation problem, 323
Alternative
  k-fold, 538–540
  theorem of, for a matrix, 21, 139
  (*See also* Dichotomies)
Ambitious industrialist example, 260–262
Arc of a graph, 352
  slack, 421
Assignment problem, 247, 310, 316–322, 515, 517
  definition of, 316
  degeneracy in, 318–319
  equivalence of transportation problem and, 319–321
  reduction of, to a linear program, 318n
  typical uses of the, 321–322
  (*See also* Allocation of aircraft to routes; Distribution problems)
Automation, 1
  relation of, to mathematical programming, 10–11
Average
  weighted, 47

Basic solution, 22
  definition of, 81
  degenerate, 81, 99
  feasible, optimal, 95
    to the transportation problem, 302–303, 386–387
    to the transshipment problem, 346, 386–387
    in using the upper-bounding technique, 372
  initial feasible, to the capacitated transportation problem, 378–379
    to the weighted transportation problem, 415–417
  (*See also* Feasible solutions)
Basis
  complementary, in quadratic programming, 493
  complementary primal and dual, 241–242
  inverse of, 198, 210
    product form of, 200
  of a linear program, 81
  number of, 235
  relationship between tree and, 356
  triangular, 303, 325, 340
  of a vector space, 181
  (*See also* Matrix; Systems of linear equations; Vector)
Blending problem examples, 42–50, 63–64, 160–161
  application of the simplex method to, 110
  gasoline, 443–444
Block-pivoting, 201–202

Cannery example, 2–3, 35–42
  (*See also* Transportation problem)
Canonical form (*see* Systems of linear equations)
Capacitated system, 370
Caterer problem, 366
Center of gravity, 47, 92, 161
Chain-decomposition theorem, 342, 388
Chain in a graph, 353
Changes in constraint constants, 269
Changes in cost coefficients
  for basic activities, 270
  for nonbasic activities, 267
Changes in input-output coefficients
  for basic activities, 271–272
  for nonbasic activities, 267–268
Chemical equilibrium problem, 481–482
Circling, 25, 100n, 210, 228, 231
  examples of, using simplex algorithm, 228–230
  in transportation problems, 307–308
  (*See also* Degeneracy)
Coefficients
  input-output, 35

Combination
linear, 21
of equations, 70
of vectors, 179
weighted, 151
(*See also* Extreme points)
Combinatorial problems, 317, 514
examples of, 515
Complementary primal and dual bases, 241–242
Complementary slackness
application to matrix games of, 287
in primal and dual systems, 135
Tucker's theorem on, 139
Composite algorithm, 241
Computational techniques, 25–26
Computer
code, 13, 25, 200, 216, 231
used for solution of Stigler's nutrition problem, 557–558
Concave function, 471
global minimum for, 543–545
Cone, 168
Conservative player (*see* Player)
Convex function, 153–154
definition of, 153, 189
Convex hull, 161
Convex polyhedron, 152, 171
edge of, 152, 155, 160
Convex programming, 9, 471–498
convergence of iterative procedure in, 474–478
general theory of, 471–478
Kuhn-Tucker conditions for, 471–472
Kuhn-Tucker problem of, 439–440, 471
reduced to generalized program, 439–440
nondegeneracy assumption in, 472–473
separable convex objective function in, 482–490
(*See also* Chemical equilibrium problem; Kuhn-Tucker conditions; Quadratic programming)
Convex regions, 147–156, 166–169
Convex sets, 148
closed, 148
intersection of, 148–149
polyhedral, represented in terms of their vertices, 437–438, 449
unbounded polyhedral, 453
Cost, 34, 61
expected, 500
(*See also* Objective function)
Cost factors
relative, 95
Cut, 393–397
definition of, 393
equality of maximum flow value and minimum value, 394
value of, 393
Cutting plane, 519–520
(*See also* Discrete programming; Integer forms)
Cycling (*see* Circling)

Decomposition principle, 27, 433n, 448–470
animated, 455–461
iterative process in, 452
master program for, 450
restricted, 452
unbounded subprogram for, 453–454
(*See also* Generalized program of Wolfe)
Degeneracy, 25, 100, 121, 222, 228, 231–237
in the assignment problem, 318–319
in the primal or the dual, 246
in the transportation problem, 307–308
(*See also* Circling; Lexicographic rule; Random choice rule)
Degenerate solutions (*see* Basic solution)
Detached coefficient form of a linear program, 71
Dichotomies, 537–538
Discrete problems, 9
(*See also* Nonlinear programming)
Discrete programming, 514–550
iterative procedure for, 530–532
problems solvable by, 535–550
Distribution problems, 316–334, 368–369
with predetermined values, 330–332
and primal-dual method, 247
weighted, 22, 413–432
example of, 573–574
graph structure of bases to, 420–424
initial basic solution to, 415–417
near triangularity of bases to, 419–420
standard form of, 413
subclass of, with triangular optimum bases, 424–431
(*See also* Assignment problem)
Double description method, 25
Dual linear program, 49–50, 62, 123–127
the ambitious industrialist example and, 260
complementary primal and dual bases, and the, 241
feasibility of the, 128
method of Lagrange multipliers, and the, 144
of a mixed program, 125–127
of a standard program, 62, 127
Tucker diagram for, 124, 138, 241–242
Dual simplex method, 241, 243–244

Dual system, 21
Duality, 23–24
  basic theorem on, 134–140
Duality theorem, 23, 120, 125
  matrix games and, 287
  proof of, 128–135
  weak form of, 130
  (*See also* Complementary slackness; Simplex method)
Dynamic programming, 9–10, 510
  approach of, to knapsack problem, 519
  principle of optimality in, 10

Economic models
  relation of, to linear programming, 16–20, 254
Elementary operations, 73–75
  (*See also* Pivot operations; Systems of linear equations)
Elimination method for solving inequality systems, 84–85, 93, 254
Equation
  material balance, 35
  vector, 179–180
  (*See also* Linear equations)
Equivalent systems (*see* Systems of linear equations)
Expected value, 500
Extreme points
  definition of, 154
  fractional, 517
  linear combination of, 172
  (*See also* Convex sets)
Extreme solution, 22
  (*See also* Extreme points)

Farkas' lemma, 21, 123, 137, 145
Feasible program, 34
Feasible solutions, 155
  convex sets of, 152–155
    (*See also* Extreme points)
  definition of, 61
  for the dual and the primal, 128
  initial basic, 22, 95, 101
    (*See also* Phase I of the simplex method)
  (*See also* Basic solution)
Finiteness
  of algorithm solving discrete programs, 532–535
  of algorithm solving generalized programs, 438
  of quadratic algorithm, 496
  of simplex algorithm, 100, 120, 122–123, 235
Firm, 19
Fixed-charge problem, 545

Flow
  chain, 388
  equations of conservation of, 385–386
  exogeneous, 32, 35
  (*See also* Network)
Flow capacity, 385
Four-color problem
  two formulations of the, 548–550
Fourier-Motzkin elimination method (*see* Elimination method for solving inequality systems)

Game
  linear program and, 277–298
  matrix, constructive solution to a, 291–297
    definition of, 277–278
    equivalent linear program to a, 287–288
    solution of a, 287
    value of a, 285, 287
  Morra, 279
  strictly determined, 281
  symmetric matrix, definition of, 288
    reduction of a linear program to a, 290–291
    reduction of a matrix game to a, 288–290
  theory of, 20, 24
  two-person zero-sum, 9
  (*See also* Minimax theorem; Player)
Generalized program of Wolfe, 433–440
  definition of, 434
  equivalent formulation of a, 438–439
  master program for a, 435
  restricted master program for a, 437–438
  seen as a linear program, 434–435
  special cases of a, 440–445
  subprogram of a, 437–438
  used for solving the Kuhn-Tucker problem, 472–473
Gradient, 159, 171
Graph
  linear, 23, 352
  (*See also* Network)

Half-space, 151
Hirsch's conjecture, 160, 168
Homogeneous systems (*see* Systems of linear equations)
Hungarian method, 247, 404
  (*See also* Assignment problem)
Hyperplane, 151

Inequality (*see* Linear inequalities)
Infeasibility form, 102, 235
Infeasibility theorem, 129, 137

Integer forms
generation of, for the integer case, 523–527
for the mixed-integer case, 527–529
Gomory's method of, 521–535
(*See also* Discrete programming)
Integer programming, 27
(*See also* Discrete programming)
Inventory control, 8
example of, 503–507
(*See also* Scheduling)
Item, 32, 60
Iterations
number of, 99n, 160, 247, 368

Knapsack problem, 517–520
Kuhn-Tucker conditions, 471–472
(*See also* Convex programming)

Lagrange multipliers, 140–144, 210, 471
definition of, 141
Latin-square problem
orthogonal, 547–548
Leontief system, 209
Lexicographic ordering, 231n, 294–295
(*See also* Perturbed problem; Vector)
Lexicographic rule
proof of, 234–237
in the simplex method using multipliers, 221–222
Line segment, 149
Linear equations, 2
inconsistent, 71, 101
material balance, 33, 35
redundant, dependent, 71, 101
vacuous, 71
(*See also* Systems of linear equations)
Linear inequalities, 2
systems of, 20–21, 81–89, 91–92
(*See also* Convex regions; Linear equations)
Linear programming
application of, to games, 277–298
assumptions in, 6–7, 32
central mathematical problem of, 35
definition of, 60–62
history of, 20–27
concept of, 1
economic models in, 16–20
equivalence of matrix games and, 286–291
examples of, problems, 2–6
generalized (*see* Generalized program of Wolfe)
geometry of, 147–172
(*See also* Convex regions)
graphical methods in, 34
inequality form of a, problem, 85–89, 93
method of Lagrange multipliers in, 143–144
methods other than simplex method in, 25, 119
origins of, 12
parametric, 241
algorithm for, 245–247
problem, as a center of gravity problem, 161, 171
with upper-bounded variables, 368–384
(*See also* Program; Simplex method)
Linear programming model, 6, 20
formulation of a, 32–68, 551–556
Loop in a graph, 353

Manager of the machine tool plant (example of the), 254–260
Markov system, 209
Marriage game, 321, 332, 517
Mathematical programming
definition of, 2
relation of, to automation, 10–11
(*See also* Programming problems)
Matrix
definition of, 183
doubly stochastic, 317
elementary, 198
of a game, 277–278
(*See also* Game)
identity, 193
input-output, 18
inverse of a, 189–194
permutation, 317–318
singular, 189
skew-symmetric, 288
technological, 27
theorem of alternative for a, 21, 139
of a transportation problem, 301
transpose of a, 183, 193
(*See also* Vector)
Matrix algebra, 187–189
Matrix operations, 183–187
Minimax theorem, 21, 24
proof of the, using the duality theorem, 287
using the simplex algorithm, 291–297
Model
building, 6
economic, 16
Leontief, 17, 19
linear programming, 6–7
building a, 34–35
formulation of a, 32–68
tableau form of a, 41

mathematical, 2
definition of, 34
multistage, 9
nutrition (*see* Nutrition model)
problems arising in formulating a, 551–557
Walrasian, 17
Multiplex method, 25
Multipliers, 70
(*See also* Lagrange multipliers; Simplex multipliers)
Multistage models, 9
(*See also* Model; Scheduling, of activities through time)
Multistage programs
decomposition principle applied to, 466–469
with general linear structure, 509–510
two-stage case of, 507–509

Network, 27, 299
capacitated, 23
connected, 346
cut in a (*see* Cut)
definition of, 352
maximal flow problem in a, 385–403
Ford-Fulkerson algorithm for, 405–411
Ford-Fulkerson theory for, 385–397
tree method for the, 398–403
planar, 393
shortest route in a (*see* Shortest-route problem)
(*See also* Transportation problem; Transshipment problem)
Network flow problem, 27
Node, 352
Nonlinear programming, 8
(*See also* Convex programming; Discrete programming)
Nutrition model, 114, 117-118
applicable to the housewife's problem, 3
of Stigler, 551–567
of Thrall, 64
Nutrition pill manufacturer (example of the), 262–264

Objective function
homogeneous, of first degree, 479–480
linear, 32, 61
definition of, 61
nonlinear, and discrete programming, 540–542
parametric, 241
separable convex, 482–490
unbounded, 120, 129
On-the-job training example, 4–6, 57–59, 67
(*See also* Scheduling, of activities through time)

Parametric linear programming (*see* Linear programming)
Payoff, 278
expected, 282
Permutation, 316
Perturbed problem, 231, 234–237
for phase II, 235–237
transportation, 307
(*See also* Lexicographic ordering; Lexicographic rule)
Phase I of the simplex method, 94, 101–104
alternative criterion for, 252–253
characteristics of the, 101–102
Phase II of the simplex method, 94, 103
perturbed problem for, 235
Pivot, 76
Pivot operations
properties of, 173
Pivot theory, 173–176
Pivotal subsystem, 174
Pivoting, 79–80, 173–176, 201–202
(*See also* Block-pivoting)
Planning
central, without complete information at the center, 462–465
organization of, 22
Player
column, 278
problem of the, 286
conservative, 284
row, 278
problem of the, 286
Point, 149
Polyhedron (*see* Convex polyhedron)
Prices, 22, 210
implicit (*see* Simplex multipliers)
mechanism of, in the simplex method, 254–276
sign convention on, 264–265
(*See also* Simplex multipliers)
Pricing-out
definition of, 276
Primal-dual correspondence, 242
Primal-dual method, 241, 247–252
for transportation problems, 404–412
Primal system, 21, 49, 124
initial restricted, 250
restricted, 247
Product-mix problem, 50, 64–65, 160–161, 163–164
Production problems, 22

Program
classification, 7–10
convex, 433–447, 471–498
definition of, 2
feasible, 34
infinite, 23
large-scale, 25, 433n
(*See also* Decomposition principle)
mathematical, 2, 10
multistage (*see* Multistage programs)
relation to automation, 10–11
with variable coefficients, 433–447
(*See also* Generalized program of Wolfe)
(*See also* Linear programming)
Programming problems, 1–2
classification of, 7–9
deterministic, 7
linear, 32–34, 60
(*See also* Convex regions; Simplex method)
probabilistic, 8–9
(*See also* Programming under uncertainty)
(*See also* Linear programming; Mathematical programming)
Programming under uncertainty, 27, 499–513
minimization, of expected costs, 500–501
of variance of costs, 501–503
(*See also* Allocation of aircraft to routes; Scheduling)
Projection method, 25

Quadratic form
convexity of a, 491
positive definite, 491
positive semi-definite, 491
Quadratic programming, 27, 490–496
algorithm for, 496
applications of, 490
complementary property of bases in, 493
optimality conditions in, 492
problem, 491

Random choice rule, 99–100, 123, 221, 231
proof of, 123
(*See also* Circling)
Rank
of a matrix, 188–189
of a system, 176–177
of a vector space, 181–182
Ray, 168
Relaxation method, 25

Saddle-point, 281, 283
Scheduling
of activities through time, 5
to meet an uncertain demand, 503–507
to meet variable costs, 499–503
Scheduling problems
and discrete programming, 514
examples of, 515
Sensitivity analysis, 265–275
importance of, 266–267
Separable convex objective functions, 482–490
Shortest-route problem, 335, 361–366, 515, 517
direct solution of, 363–366
iterative solution of, 361–363
(*See also* Transshipment problem)
Simplex, 160, 163
definition of, 164
Simplex algorithm, 1, 94–100, 111–114, 234
finiteness of, 100, 120–123, 235
inductive proof of, 120–123
in matrix form, 195–202
variants, 240–253
(*See also* Simplex method)
Simplex method, 15, 21, 24–25, 94–119
applied to proof of duality theorem, 129–135
applied in upper-bounding technique, 372–375
and the decomposition principle, 451
dual, 241, 243-244
finiteness under perturbation of, 228–239
flow diagram for, 104
geometrical interpretation of, 156–160
the method of Lagrange multipliers and the, 144
price mechanism of, 254–276
revised (*see* Simplex method using multipliers)
simplex interpretation of, 160–166
solving the transportation problem by, 301–308
solving the transshipment problem by, 346–350
(*See also* Phase I of the simplex method; Phase II of the simplex method; Simplex algorithm)
Simplex method using multipliers, 106n, 210–227
computational advantages of, 210
computational remarks about, 215–217
using upper-bounding technique, 375–376
Simplex multipliers, 197, 210, 212, 215, 236

definitions, 220, 433
interpretation of, as prices, 254
use of, for the allocation problem, 325
for the transportation problem, 305
for the transshipment problem, 342
for the weighted distribution problem, 417–418
(*See also* Prices)
Space (*see* Vector space)
Standard form of a linear program, 60, 86–88
Stochastic programming (*see* Programming under uncertainty)
Strategy
mixed, definition of, 284
optimum mixed, 285
pure, definition of, 278
(*See also* Game)
Systems of linear equations, 2, 20, 69–93
canonical, 75–81, 89–91, 94, 174, 190, 217
definition of, 73
equivalent, 174–176
dependent, 71
diagonal, 77
equivalent, 73, 174
homogeneous, 22, 189
definition of, 136
Gordan's theorem for, 136
Stiemke's theorem for, 138
unbounded convex sets and, 453–454
independent, 71
rank of a, 176–177
reduced, 79
redundancy tracing theorem for, 175
solvability of, 80
square, 75–77
triangular, 77
trivial solution of, 136
(*See also* Linear equations)
System of linear inequalities (*see* Systems of linear equations)

Tableau économique, 13, 16, 18
(*See also* Matrix, input-output)
Transportation problem, 18, 25, 27, 63, 101, 299–315
array, 301
with bounded partial sums of variables, 382–383
cannery example of, 2–3, 35
capacitated, 377–382
equivalence of the, and the transportation problem, 380–382
computational algorithm for, 308–313
definition of, 299–300
degeneracy in the, 307–308
feasible solutions to the (*see* Basic solution)
historical summary of the, 299–300
integrality of basic variables in the, 305
least-cost rule for the, 309, 330
predetermined values in the, 330
primal-dual method for, 404–412
solution of, by the simplex method, 301–308
solving a bounded-variable, 377–380
triangularity of bases in the, 303
(*See also* Assignment problem; Network flow problem; Transshipment problem)
Transposition theorem, 21, 138–139
Transshipment problem, 22, 335–351
direct shipment procedure for, 338
equivalence of the, and the transportation problem, 342–345
feasible solutions to (*see* Basic solution)
formulation of, 335–336
graph associated with a, 355
network representation of the, 337
solution of the, using the associated graph, 357–361
using the simplex method, 346–350
standard form of the, 340–341
triangularity of bases for the, 340
(*See also* Shortest-route problem)
Traveling-salesman problem, 520
different formulations for the, 545–547
Tree, 353
(*See also* Basis)
Triangular model, 16, 26
Trivial solution, 136
(*See also* Systems of linear equations)
Tucker diagram, 124–125, 138, 241
(*See also* Dual linear program)

Uncertainty, 8, 499–513
(*See also* Programming under uncertainty)
Upper-bounding technique, 368–384

Variables
artificial, 94, 102–103, 130, 218, 235
in the dual simplex method, 243–244
(*See also* Phase I of the simplex method)
basic, 73
discrete, 514–550
fractional, 521
method of leading variables, 241
slack, 60
transshipment, 340
with upper bounds, 368–384
(*See also* Upper-bounding technique)

Vector
definition of, 177
lexico-positive, 238, 294
linearly independent, 179, 189
operations, 177–178
product of a matrix by a, 184
representation in terms of the basis of a, 181
unit, 179
Vector space, 180–181
Vertex, 21
solution, 22
(*See also* Extreme points)

Warehouse problem, 55, 67, 412
(*See also* Multistage models)
Weight, 70, 210
(*See also* Multipliers; Prices)

NAME INDEX

Ablow, C. N., 26
Almon, C., viii, 254, 462
Arrow, K. J., 9, 19, 27

Balinski, M., viii
Barankin, E. W., 27, 490
Barth, E., ix
Baumol, W. J., viii, 20
Beale, E. M. L., 27, 100n, 121, 228, 237, 241, 521
Beckenbach, E. F., ix
Beckman, M. J., 20
Bell, E. J., ix
Bellman, R., 10, 363, 510
Belz, M. H., 29, 515
Berge, C., 27
Birkhoff, G., 27, 318
Blattner, W. O., ix, 265n
Boelter, L. M. K., 10
Borden, J., viii
Borel, E., 24, 277
Bose, R. C., 548
Brigham, G., 26
Brown, G. W., 19, 25, 290
Burns, R., ix

Cahn, A., 15, 25, 412
Charnes, A., 13, 25, 27–28, 231, 238, 412, 483, 542
Chatto, K., 118
Clark, J., ix
Classen, R., 404
Cooper, W. W., 13, 25, 28, 412
Cornfield, J., 18, 551
Cottle, R. W., ix
Cox, C. B., 118
Crandall, H. W., 28
Curtiss, J. H., 15, 25
Cutler, L., ix, 241

Dalkey, N. C., 24
Dantzig, A. S., v, x
Dantzig, G. B., 2, 6n, 13, 23–24, 26–29, 120, 210, 231, 231n, 238, 241, 247, 252, 287, 290, 300, 303, 363, 368, 398, 412, 444, 449n, 471, 481, 483, 490, 499, 509, 520, 536, 542, 545–546, 569
Dantzig, T., v
Diehm, C., 26
Dines, L. L., 20
Doig, A. G., 29, 515
Dorfman, R., viii, 13, 15, 19–20, 27, 262, 291n, 490
Dresher, M., ix, 24, 277n
Duffin, R. J., 23
Dvoretsky, A., 10

Edmondson, J. H., 231
Egerváry, E., 27, 247, 300, 404
Einstein, H., ix
Eisemann, K., 29, 515
Elmaghaby, S. E., 501
Euler, L., 547
Evans, W. D., 18–19

Fabian, T., viii, 20, 29
Farkas, J., 13, 21, 123
Ferguson, A. R., 29, 499, 569
Fisher, R. A., 18
Fisher, W. D., 29
Flood, M. M., 19, 404, 546
Ford, L. R., 13, 27, 241, 247, 252, 300, 385, 393, 404, 449
Fourier, J. B., 12–13, 21, 84
Frisch, R., 25
Fulkerson, D. R., viii, 13, 27, 29, 241, 247, 252, 300, 363, 385, 393, 398, 404, 449, 520, 546

Gaddum, J. W., 367
Gainen, L., 29
Gale, D., 13, 19, 24, 123, 168, 277, 287–288, 363, 548
Garvin, W., 28
Gass, S., 241, 246
Gauss, K., 13, 20, 75
Gavurin, M. K., 22
Geisler, M. A., 12–13, 15–16, 19
Georgescu-Roegen, N., 19
Gibrat, R., 13, 20, 29
Glicksberg, I., 10, 508
Goldman, A. J., 24, 124
Goldstein, L., 13, 29
Gomory, R., viii, 13, 27, 433, 521, 532, 535, 550
Gordan, P., 13, 21, 136
Greene, J. H., 118
Gross, O., 10
Gunther, P., 580

Halmos, P. R., 321, 332
Harary, F., 421
Harris, T. E., ix, 9
Hartley, H. O., 471
Haydon, B., ix
Helmer, O., 24

Henderson, A., 25, 29
Hermann, C. C., 6
Hicks, C. R., 118
Hirsch, W. M., 160, 168, 545
Hitchcock, F. L., 13, 300, 310, 335, 404
Hoffenberg, M., 18–19
Hoffman, A. J., viii, 25–27, 100n, 121, 228, 231, 237, 367, 517
Hogan, J., ix
Houthakker, H. S., 27
Hu, T. C., 433
Hurwicz, L., 19, 24, 27

Jackson, J. R., 28
Jacobs, W. W., 16, 366
John, J. B., 28
Johnson, S., 13, 27, 481, 490, 515, 520, 546
Jordan, W., 20
Judd, H., 241
Juncosa, M. L., 29

Kalaba, R. E., 29
Kantorovich, L. V., 12–13, 22–23, 65, 299
Kawarantani, T. K., 444
Kelley, J. E., 171
Kemeny, J. G., 67
Kiefer, J., 10
Koenig, D., 404
Koopmans, T. C., 1, 6, 13, 15, 17–20, 24–25, 300, 337, 342, 352
Kruskal, J. B., Jr., 517
Kuhn, H. W., viii, 13, 19, 24, 27, 123, 247, 277n–278, 287–288, 300, 404, 439, 471

Laderman, J., 13, 25, 551
Lagrange, J. L., 140–146
Land, A. H., 29, 515
Learned, E. P., 14
Lemke, C. E., 26–27, 241, 252, 483, 542
Leontief, W., 13, 15, 17
Lerner, A., 19
Lewis, R. E., 29
Little, J., 404
Luce, R. D., 277n

Madansky, A., ix, 27, 499, 507
Magee, J. F., 6
Manne, A. S., 13, 28, 382, 515, 542
Mannos, M., 25
Marckx, M. R., ix
Marcus, M., 318
Markowitz, H., 13, 20, 27, 252–253, 424, 490, 501, 515, 542
Marschak, J., 10, 19
Massé, P., 9, 13, 20, 29
Maynard, H. B., 29
McCoy, L. H., 20
McKinsey, J. C. C., 24
Mellon, B., 13, 28
Minkowski, H., 13, 22, 171
Monroe, A. E., 16
Montalbano, M., 26
Moore, E. F., 363
Morgenstern, O., 13, 19, 21, 24, 277, 288
Morin, G., 29
Motzkin, T., 13, 20–21, 25, 84, 138
Munkres, J., 404

Nachtigal, E., ix
Natrella, J. V., 16
Neyman, J., 13, 18, 23
Norton, J., 15

Orchard-Hays, W., ix, 13, 26, 67, 99n, 200, 210, 241, 246
Orden, A., 13, 25–26, 120, 200, 231, 231n, 238, 308, 314, 335, 403

Parker, E. T., 548
Paull, A. E., 29, 515
Pearson, E. S., 23
Pearson, K., 18
Prager, W., 367, 412
Pyne, I. B., 26

Quade, E. S., ix

Raiffa, H., 25, 277n
RAND Corporation, 404
Randolph, P., viii
Rawlings, E. W., 14
Reiter, S., 19, 507
Robacker, T., 112
Rosen, J. B., 27
Ryan, M., ix

Saaty, T. L., 241, 246
Salveson, M. E., 28
Samuelson, P., 13, 15, 19–20, 262
Scarf, H., 506
Schlaifer, R., 29
Schoenberg, I. J., 25
Schruben, L. W., 29
Shapiro, M., ix
Shapley, L. S., ix, 24, 297
Shaw, C., 368n
Shukhande, S. S., 548
Simon, H., 19, 507
Slater, M., 471
Sokolowsky, D., 25, 367
Solow, R. M., 13, 20, 262
Spellman, R. A., 28
Steinberg, D. A., ix

Steinitz, E., 172
Stewart, D. G., ix
Stiemke, E., 13, 21, 138
Stigler, G. J., 4, 13, 25, 500, 552
Stokes, R. W., 20
Sussman, B., ix
Symonds, G. H., 28

Theil, H., 507
Thompson, G. L., 25
Thornton, C. B., 14n
Thrall, R. M., 25, 64
Tompkins, C. B., 19, 25
Trinkl, F., ix
Tucker, A. W., v, viii, 13, 15, 19, 21, 24, 73n, 123, 125, 136, 139, 156, 173, 201, 277n, 278, 287–288, 290, 294n, 439, 471, 547

Ullman, R. J., 444
Uzawa, H., 27, 471

Vajda, S., 25
Vallée Poussin, A. de la, 21
Van Slyke, R. M., ix
Vaughan, H. E., 321, 332
Ville, J., 21, 23–24, 139
von Neumann, J., 6n, 13, 15–16, 21, 23–24, 123, 277, 285, 288, 318

Wade, B. L., ix
Wald, A., 9, 13, 23
Walras, L., 16
Walter, J. R., 29, 515
Waugh, F. V., 114, 227
Wets, R., ix
Weyl, H., 20
White, W. B., 481, 490
Wiegmann, N., 25
Williams, J. D., v, vii, 13, 25, 277n, 278, 279, 471
Wolfe, P., viii, 13, 27, 32n, 120, 171, 200, 231, 231n, 238, 298, 433, 449, 471, 490, 503, 543
Wolfowitz, J., 10
Wood, M. K., 2, 12–13, 15–16, 19