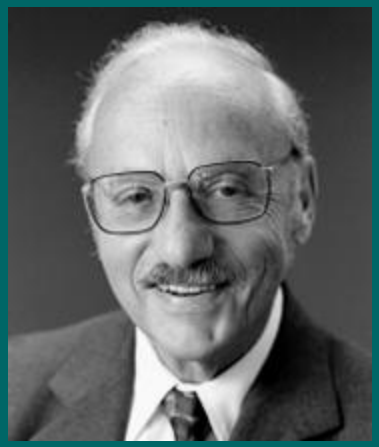

Linear Programming @Stanford

Yinyu Ye

Department of Management Science and Engineering, and by courtesy,

Electrical Engineering

Stanford University

Outline

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- History of Linear Programming at Stanford
  - George Bernard Dantzig
  - Operations Research
- Other Progresses in Linear Programming
- Few Selected Applications of Linear Programming
- More Photo Album of George

History of LP @ Stanford

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$$\begin{aligned} &\text{Minimize} && c \bullet x \\ &s.t. && a_i \bullet x = b_i, i = 1,\ldots,m, \\ & && x \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{Maxmize} && \sum_i b_i y_i \\ &s.t. && \sum_i y_i a_i + s = c, \\ & && s \geq 0 \end{aligned}$$

Childhood Years

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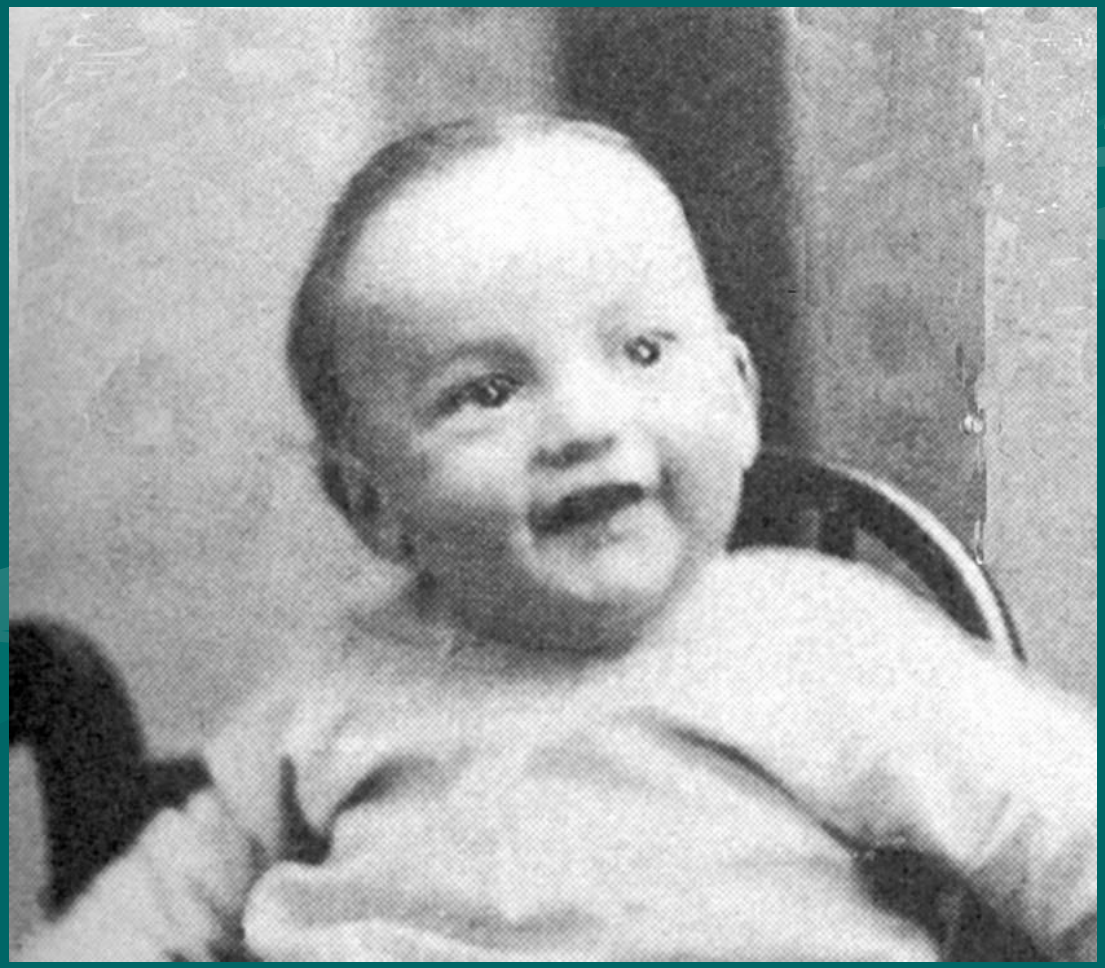
- George Bernard Dantzig was born November 8, 1914 --- a rainy Sunday --- in Portland, Oregon.



University Student Years

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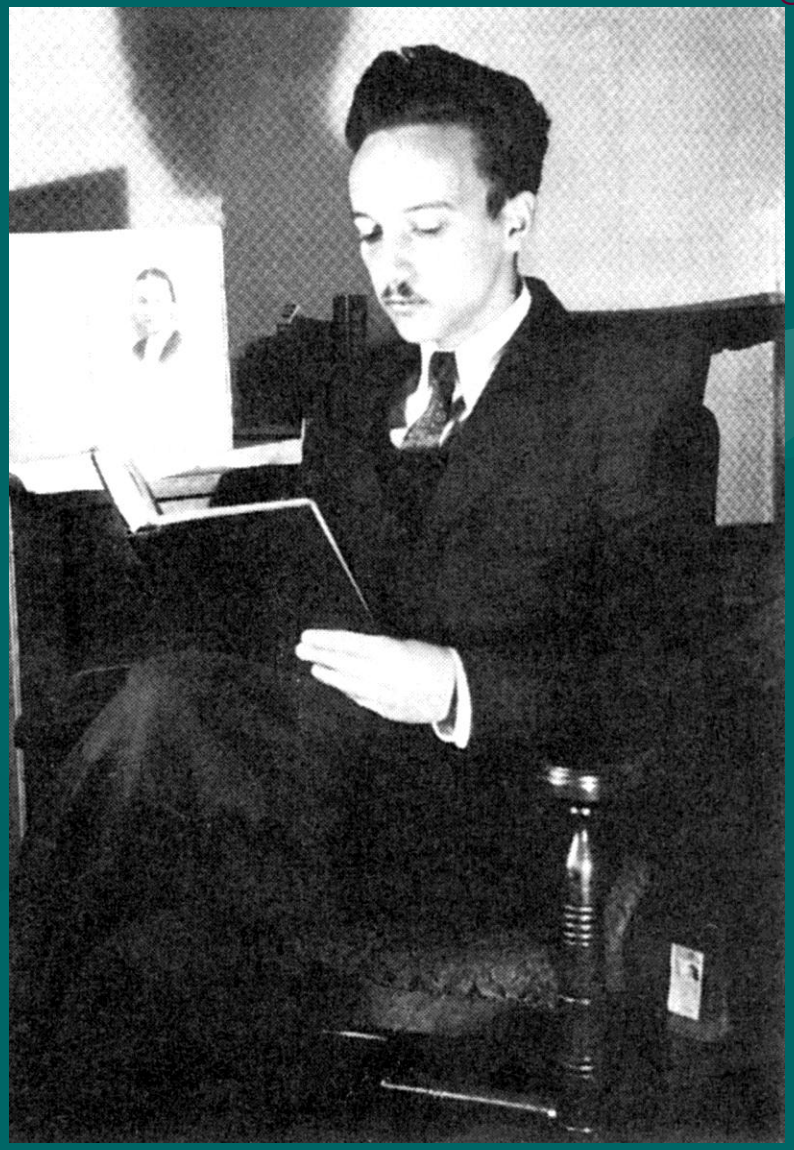
A.B. 1936 Univ. Md. (College Park) Mathematics and Physics

M.A. 1937 Univ. Mich. (Ann Arbor) Mathematics

Ph.D. 1946 Univ. Calif. (Berkeley) Mathematics, where the simplex method was invented



The Academic Years: Stanford University

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- Dantzig came to Stanford in 1966 Professor of Operations Research and Computer Science.
- At that time, Operations Research was a Program, not a Department.

1967 Stanford OR

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The Academic Years: Stanford University

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- In the Operations Research Department, Dantzig led a strong group of faculty in Mathematical Programming
  - Eaves, Glynn, Murray, Saunders, etc.
  - Linear Complementarity Problem

$$s \bullet x = 0$$

$$s = Mx + q$$

$$s, x \geq 0$$

The Academic Years: Stanford University

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- He founded the Systems Optimization Laboratory in 1973 and was its first Director
  - Gill, Murray, Saunders, Wright, Tomlin, Infanger, etc
  - Various LP and NLP software were developed
  - MINOS by Saunders: the first large-scale sparse LP solver.
  - Numerical Analysis, in particularly Numerical Linear Algebra, played a key role.

Honors: 1971-1989

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- 1971: Elected to National Academy of Sciences
- 1975: Elected to American Academy of Arts and Sciences
- 1975: Received (the first) John von Neumann Theory Prize from ORSA & TIMS (INFORMS)
- 1985: Elected to National Academy of Engineering

1975 National Medal of Science

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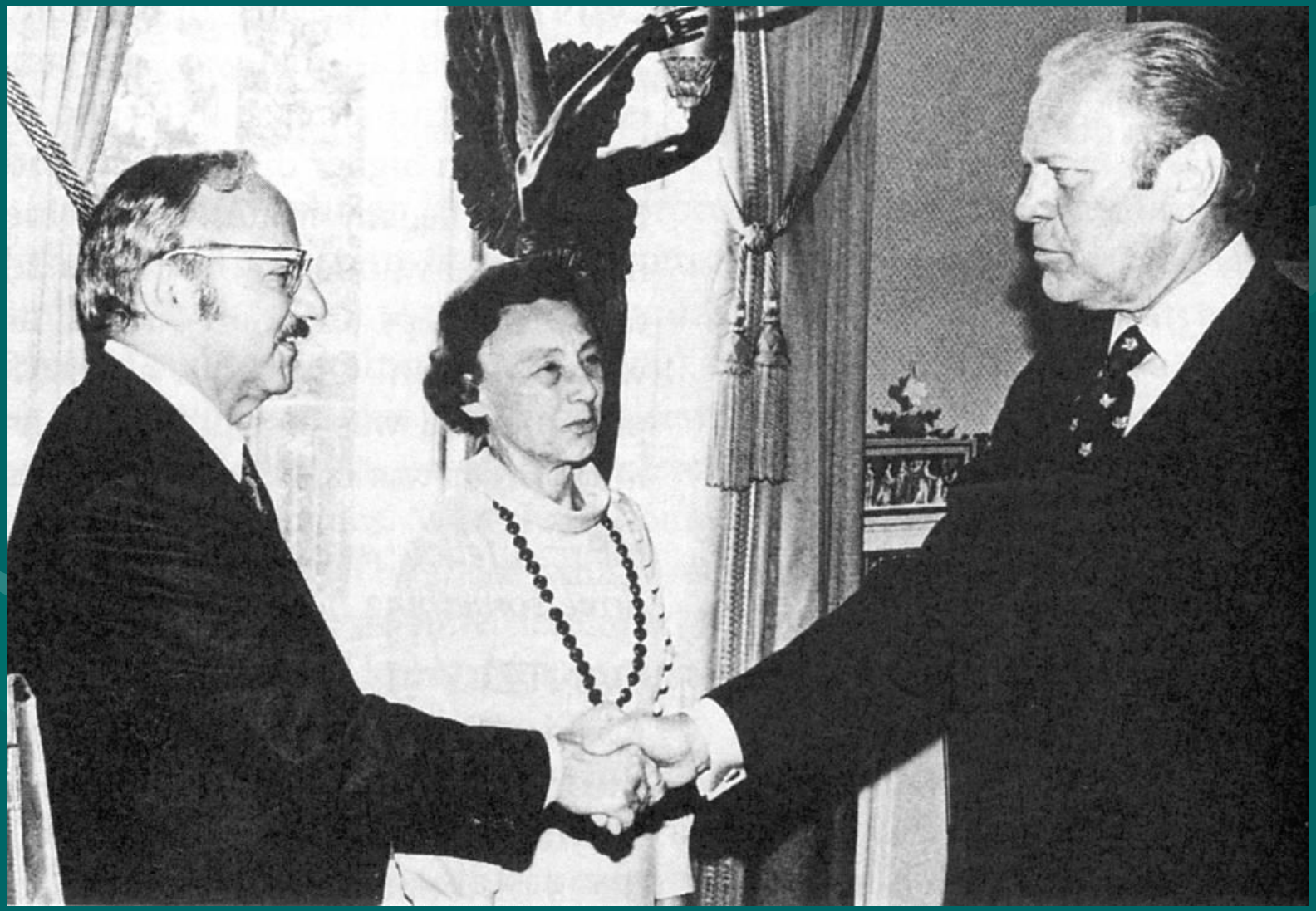

1975 Nobel Laureate

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Other Progresses in LP

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- 1979: The Ellipsoid Method solves LP with computation time in the order of n^4 .
- 1984: Karmarkar's algorithm solves LP with computation time in the order of $n^{3.5}$.
- 1986: New interior-point algorithms solve LP with computation time in the order of n^3 .
- 1992: The homogeneous and self-dual method resolved a few remaining theoretical and practical difficulties.

Other Progresses in LP

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- Commercial software, such as Cplex and Dash implementing both simplex methods and interior point algorithms; and MOSEK implementing the homogeneous and self-dual method for solving both LP and convex optimization.

- Open source software, such as SEDUMI implementing the homogeneous and self-dual method for solving both LP and convex optimization.

- An LP problem is now days solved 10^6 faster than it was 30 years ago.

Other Progresses in LP: Linear Conic Programming

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$$\begin{aligned} \text{Minimize} \quad & C \bullet X \\ s.t. \quad & A_i \bullet X = b_i, i = 1,\ldots,m, \\ & X \in K \end{aligned}$$

$$\begin{aligned} \text{Maxmize} \quad & \sum_i b_i y_i \\ s.t. \quad & \sum_i y_i A_i + S = C, \\ & S \in K^* \end{aligned}$$

Other Progresses in LP: Linear Conic Programming

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- Semidefinite programming and Second-order cone programming
- Heavily used in Control (Boyd, Vandenberghe, Lall, etc), combinatorial optimization, information network, machine learning (Boyd, Van Roy), graph realization, sensor network localization (Biswas, Jin, Saunders, So).
- Various software, SEDUMI, MOSEK, DSDP, CVX, etc were developed and maintained.

Selected LP Applications: Compressed Sensing

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Donoho, Hastie, Saunders, …

$$\text{Minimize} \quad \| Ax - b \|_2$$

$$\text{Minimize} \quad \| Ax - b \|_2 + \lambda \| x \|_1$$

$$\begin{aligned} \text{Minimize} \quad & \alpha + \lambda \bullet \beta \\ \text{s.t.} \quad & Ax + y = b \\ & (\alpha ; y) \in 2OC \\ & (\beta ; x) \in 1OC \end{aligned}$$

Compressed Sensing/Sampling

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Selected LP Applications: World Cup Betting Example

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- **Market for World Cup Winner**
  - **Assume 5 teams have a chance to win the 2006 World Cup:**

Argentina, Brazil, Italy, Germany and France

  - **We'd like to have a standard payout of $1 if a participant has a claim where his selected team won**
- **Sample Orders**

Order	Price Limit π	Quantity Limit q	Argentina	Brazil	Italy	Germany	France
1	0.75	10	1	1	1		
2	0.35	5				1	
3	0.40	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

Markets for Contingent Claims

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- **A Contingent Claim Market**
  - S possible states of the world (one will be realized)
  - N participants who, j, submit orders to a market organizer containing the following information:
    - $a_{i,j}$ - State bid (either 1 or 0)
    - q_j – Limit contract quantity
    - π_j – Limit price per contract
  - One market organizer who will determine the following:
    - p_i – State price
    - x_j – Order fill
  - Call auction mechanism is used.
  - If Order is accepted and correct state is realized, the organizer will pay the participant a fixed amount *w* for each contract.

Central Organization of the Market

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- **Belief-based**
  - Central organizer will determine prices for each state based on his beliefs of their likelihood
  - This is similar to the manner in which fixed odds bookmakers operate in the betting world
  - Generally not self-funding
- **Parimutuel**
  - A self-funding technique popular in horseracing betting
  - The winners divide the total amount bet, after deducting management expenses, in proportion to the sums they have wagered individually.

Parimutuel Market Microstructure

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Boosaerts et al. [2001], Lange and Economides [2001], Fortnow et al. [2003], Yang and Ng [2003], Peters et al. [2005], etc

$$
\begin{aligned}
Max \quad & \sum_j \pi_j x_j - w \cdot y \\
s.t. \quad & \sum_j a_{ij} x_j \le y \qquad \forall\, i \in S \\
& x_j \le q_j \qquad \forall\, j \in N \\
& x_j \ge 0 \qquad \forall\, j \in N
\end{aligned}
$$

An LP pricing for the contingent claim market

Parimutuel Market Microstructure

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Peters et al. [2005], etc

$$
\begin{aligned}
Max \quad & \sum_j \pi_j x_j - w \cdot y + \sum_i w_i \ln\left(y - \sum_j a_{ij} x_j \right) \\
s.t. \quad & \sum_j a_{ij} x_j \le y \qquad \forall\, i \in S \\
& x_j \le q_j \qquad \forall\, j \in N \\
& x_j \ge 0 \qquad \forall\, j \in N
\end{aligned}
$$

A convex pricing for the contingent claim market

World Cup Betting Results

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Orders Filled

Order	Price Limit	Quantity Limit	Filled	Argentina	Brazil	Italy	Germany	France
1	0.75	10	5	1	1	1		
2	0.35	5	5				1	
3	0.40	10	5	1		1		1
4	0.95	10	0	1	1	1	1	
5	0.75	5	5		1		1	

State Prices

	Argentina	Brazil	Italy	Germany	France
Price	0.20	0.35	0.20	0.25	0.00

Selected LP Applications: Facility Location Problem

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Input

- A set of clients or cities D
- A set of facilities F with facility cost f_i
- Connection cost C_{ij}, (obey triangle inequality)

Output

- A subset of facilities F'
- An assignment of clients to facilities in F'

Objective

- Minimize the total cost (facility + connection)

Facility Location Problem

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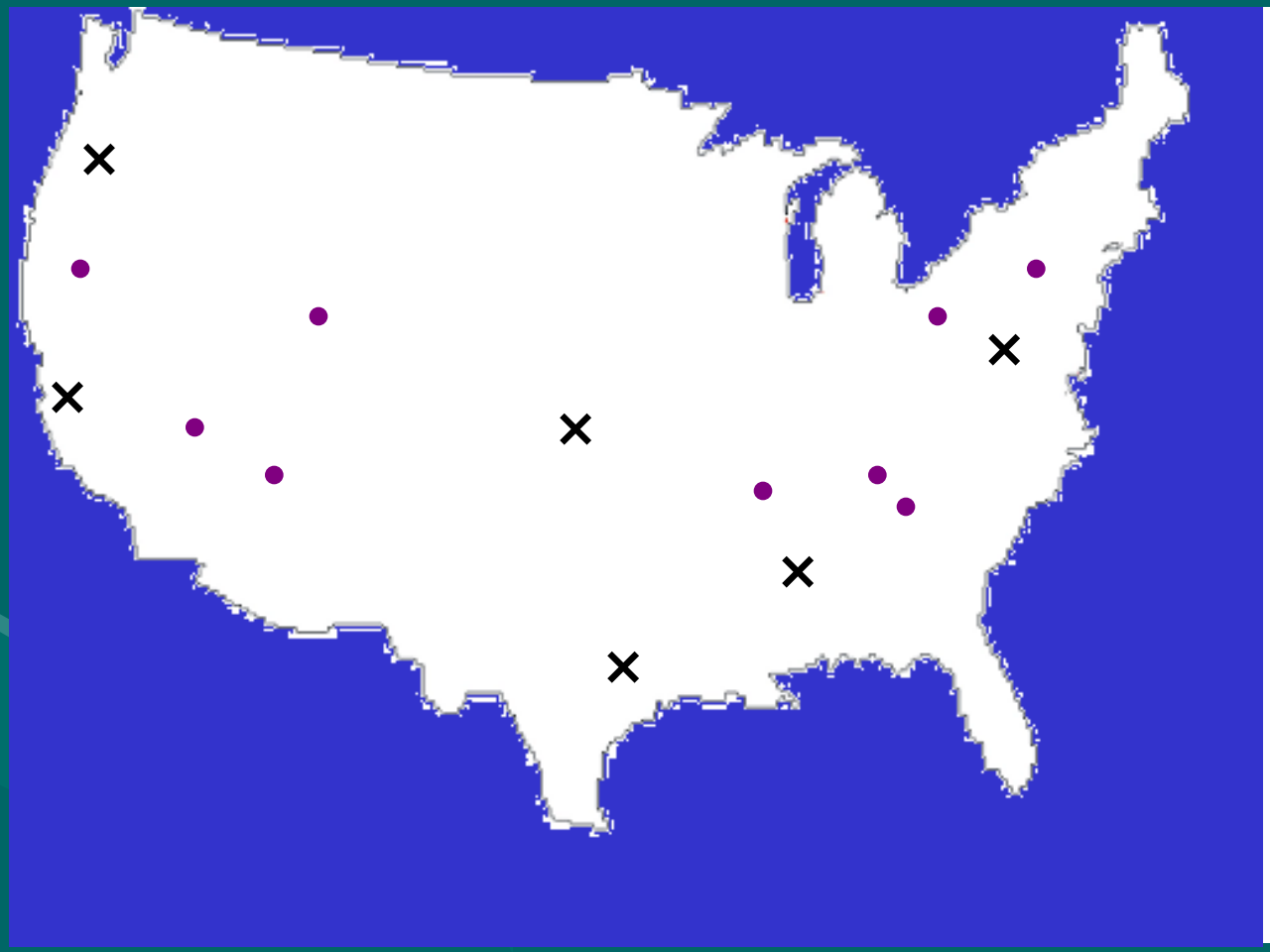

× location of a potential facility (opening cost)

• client (connection cost)

Facility Location Problem

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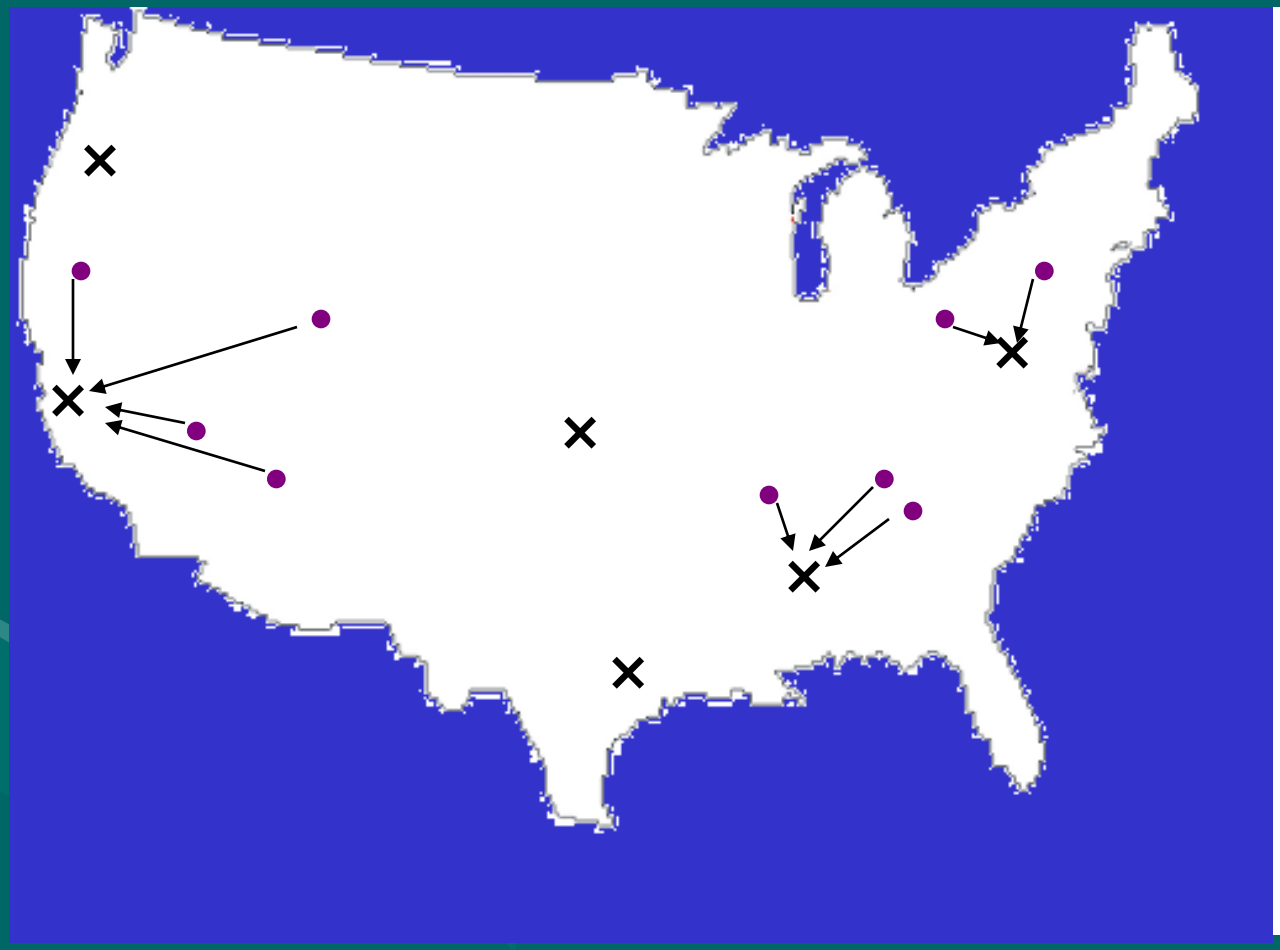

× location of a potential facility (opening cost)

• client (connection cost)

$$\min \sum \text{opening cost} + \sum \text{connection cost}$$

R-Approximate Solution and Algorithm

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An algorithm found a feasible (integral) solution of UFLP, with the total cost, $Cost$, that satisfies the following:

$$Cost \le R \bullet Cost^{*}$$

for some constant $R \ge 1$.

Hardness Results

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- NP-hard.

  Cornuejols, Nemhauser & Wolsey [1990].

- 1.463 polynomial approximation algorithm implies NP =P.

  Guha & Khuller [1998], Sviridenko [1998].

ILP Formulation

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$$
\begin{aligned}
Min \quad & \sum_{i \in F} \sum_{j \in D} C_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\
s.t. \quad & \sum_{i \in F} x_{ij} = 1 \qquad \forall\ j \in D \\
& x_{ij} \le y_i \qquad \forall\ j \in D, \forall\ i \in F \\
& x_{ij}, y_i \in \{0,1\} \quad \forall\ j \in D, \forall\ i \in F
\end{aligned}
$$

- **Each client should be assigned to one facility.**
- **Clients can only be assigned to open facilities.**

LP Relaxation and its Dual

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$$\text{Min} \quad \sum_{i \in F} \sum_{j \in D} C_{ij} x_{ij} + \sum_{i \in F} f_i y_i$$

$$\text{s.t.} \quad \sum_{i \in F} x_{ij} = 1 \qquad \forall\ j \in D$$

$$x_{ij} \le y_i \qquad \forall\ j \in D, \forall\ i \in F$$

$$x_{ij} \ge 0 \qquad \forall\ j \in D, \forall\ i \in F$$

$$\text{Max} \quad \sum_{j \in D} \alpha_j$$

$$\text{s.t.} \quad \alpha_j - \beta_{ij} \le c_{ij} \qquad \forall\ j \in D, \forall\ i \in F$$

$$\sum_{j \in D} \beta_{ij} \le f_i \qquad \forall\ i \in F$$

$$\beta_{ij} \ge 0 \qquad \forall\ j \in D, \forall\ i \in F$$

Each client should be assigned to one facility.

Clients can only be assigned to open facilities.

Interpretation: **clients share the cost to open a facility, and pay the connection cost.**

$\beta_{ij} = \max\{0, \alpha_j - c_{ij}\}$ is the contribution of client j to facility i.

Bi-Factor Dual Fitting

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Suppose an algorithm found a feasible (integral) solution of UFLP, with the total cost $\sum_{j \in D} \alpha_j$, where α_j satisfies the following :

$$(1) \qquad \alpha_j - \beta_{ij} \le R_c c_{ij} \qquad \forall\ j \in D, \forall\ i \in F$$

$$(2) \qquad \sum_{j \in D} \beta_{ij} \le R_f f_i \qquad \forall\ i \in F$$

for some constant $R_c, R_f \ge 1$ and $\beta_{ij} \ge 0$, then we have :

$$F + C = \sum_{j \in D} \alpha_j \le R_f F^* + R_c C^*.$$

A bi-factor (R_f,R_c)-approximate algorithm is a *max*(R_f,R_c)-approximate algorithm

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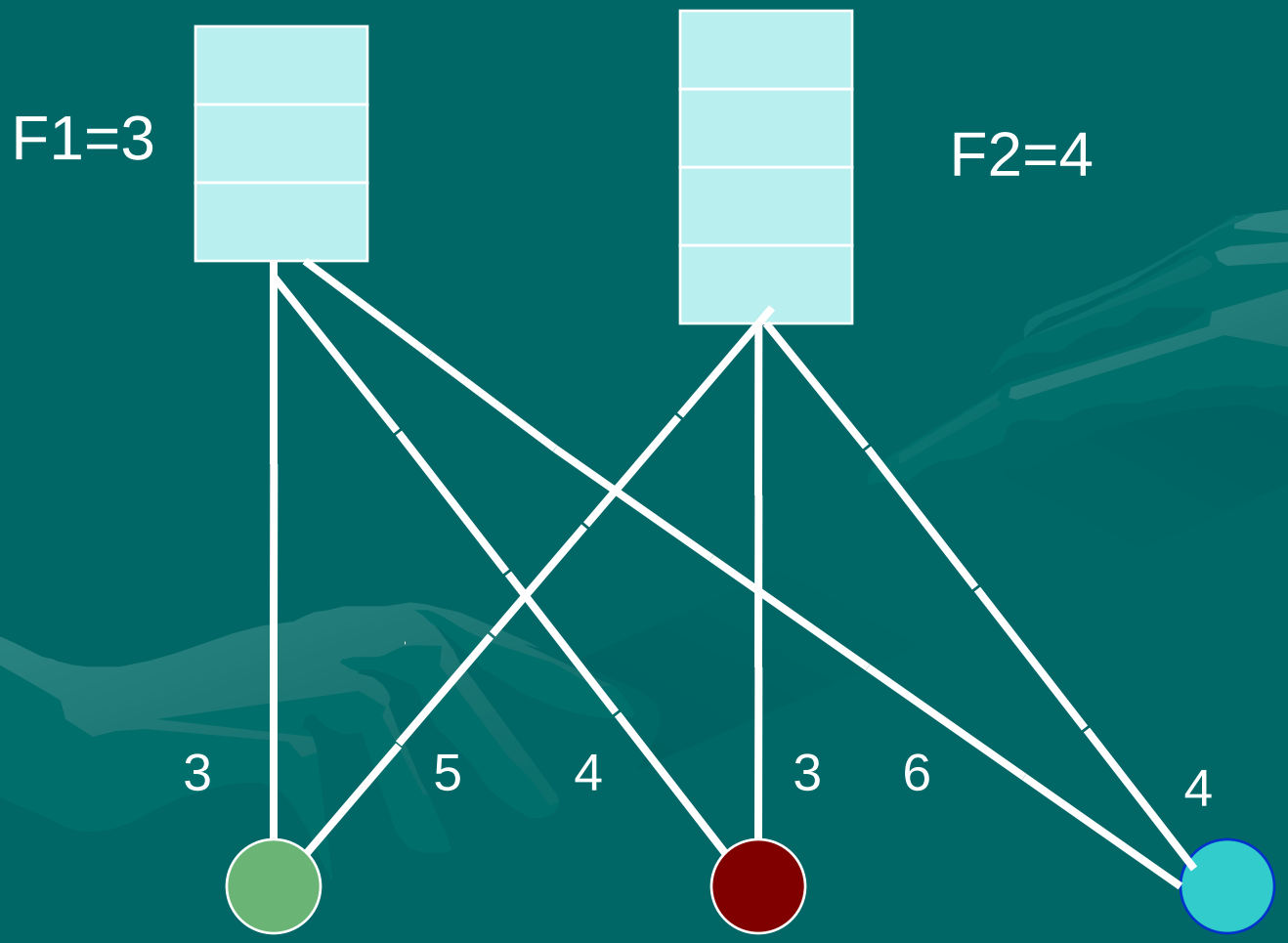
Time = 0



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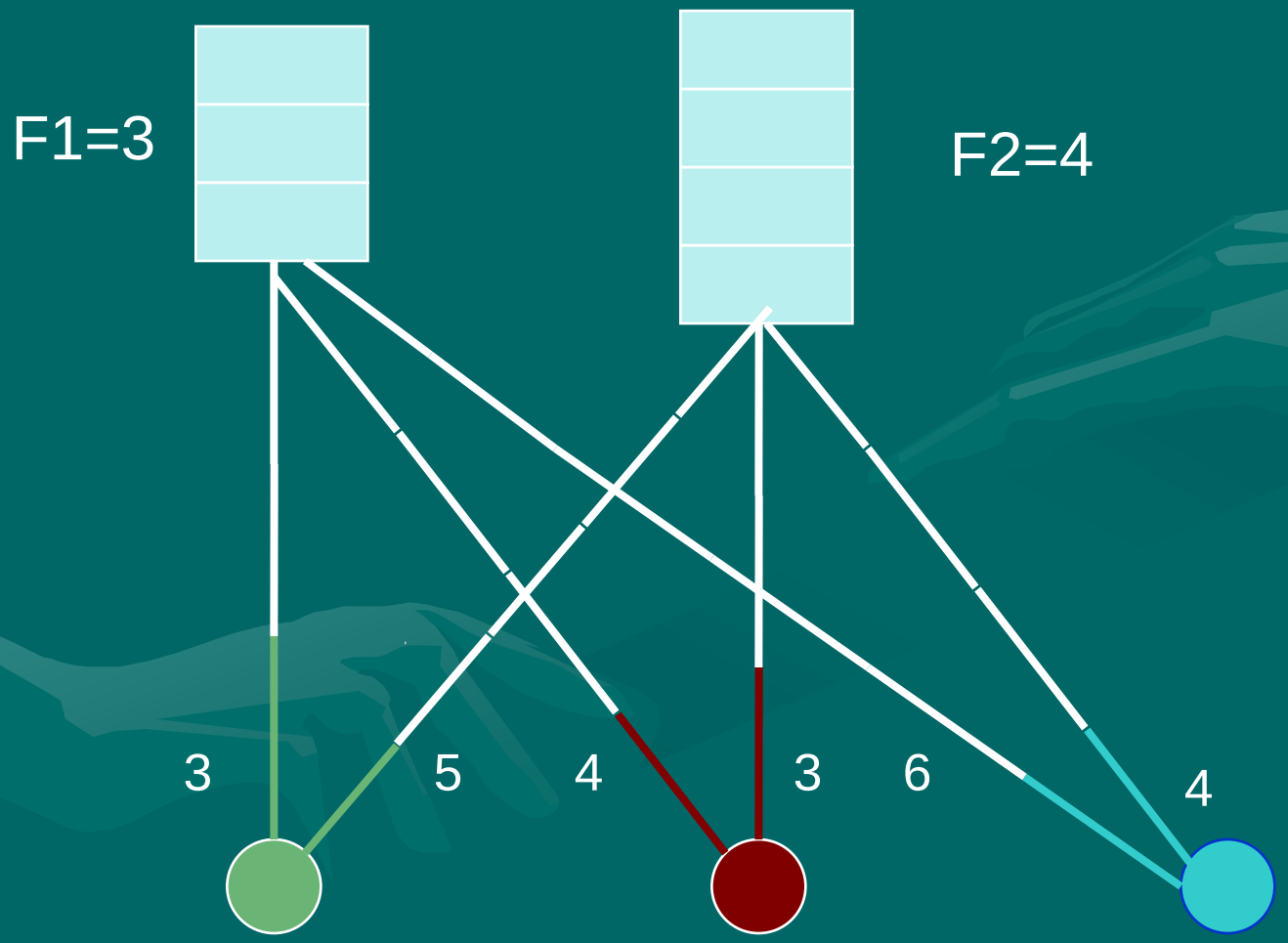
Time = 1



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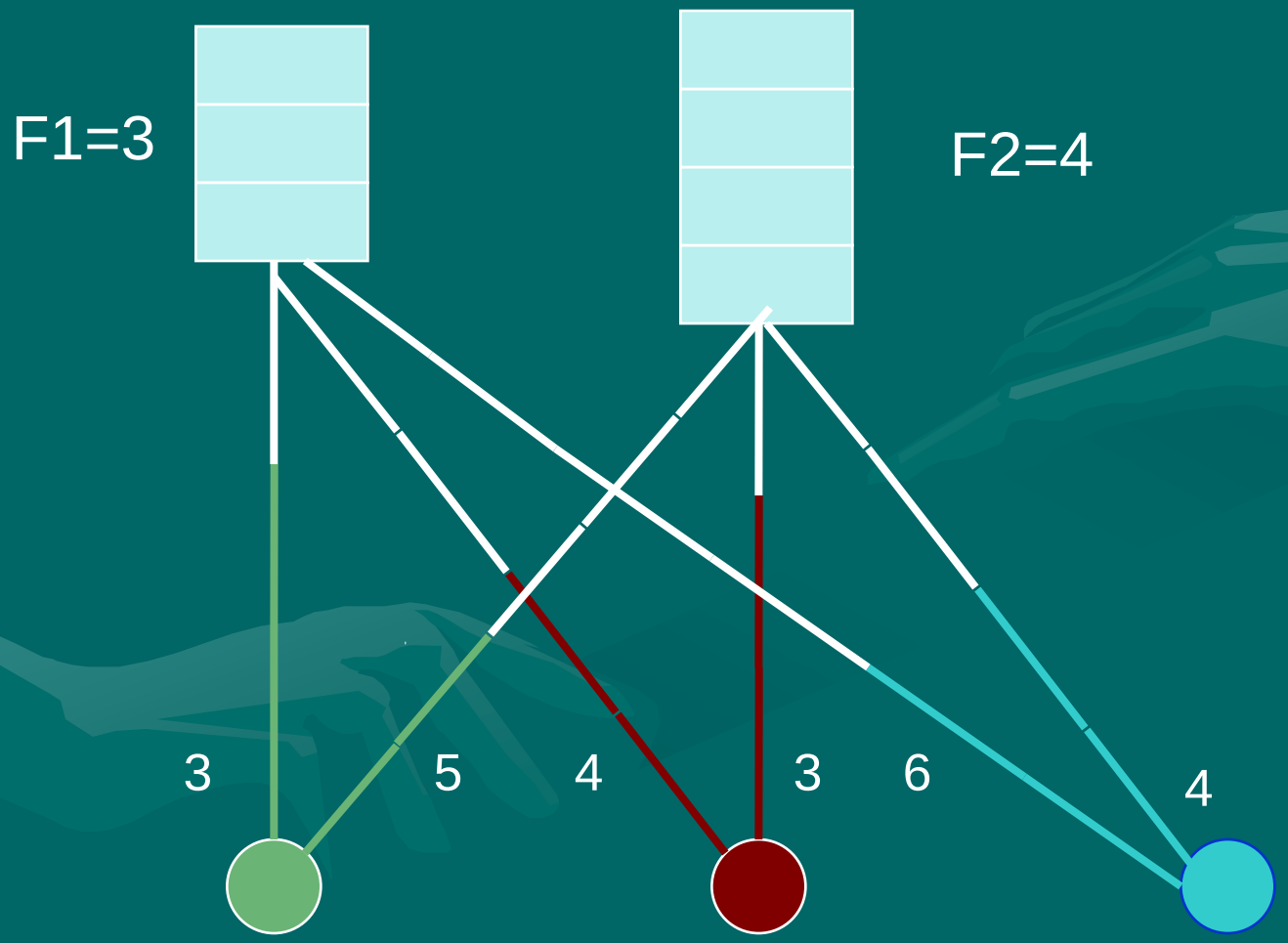
Time = 2



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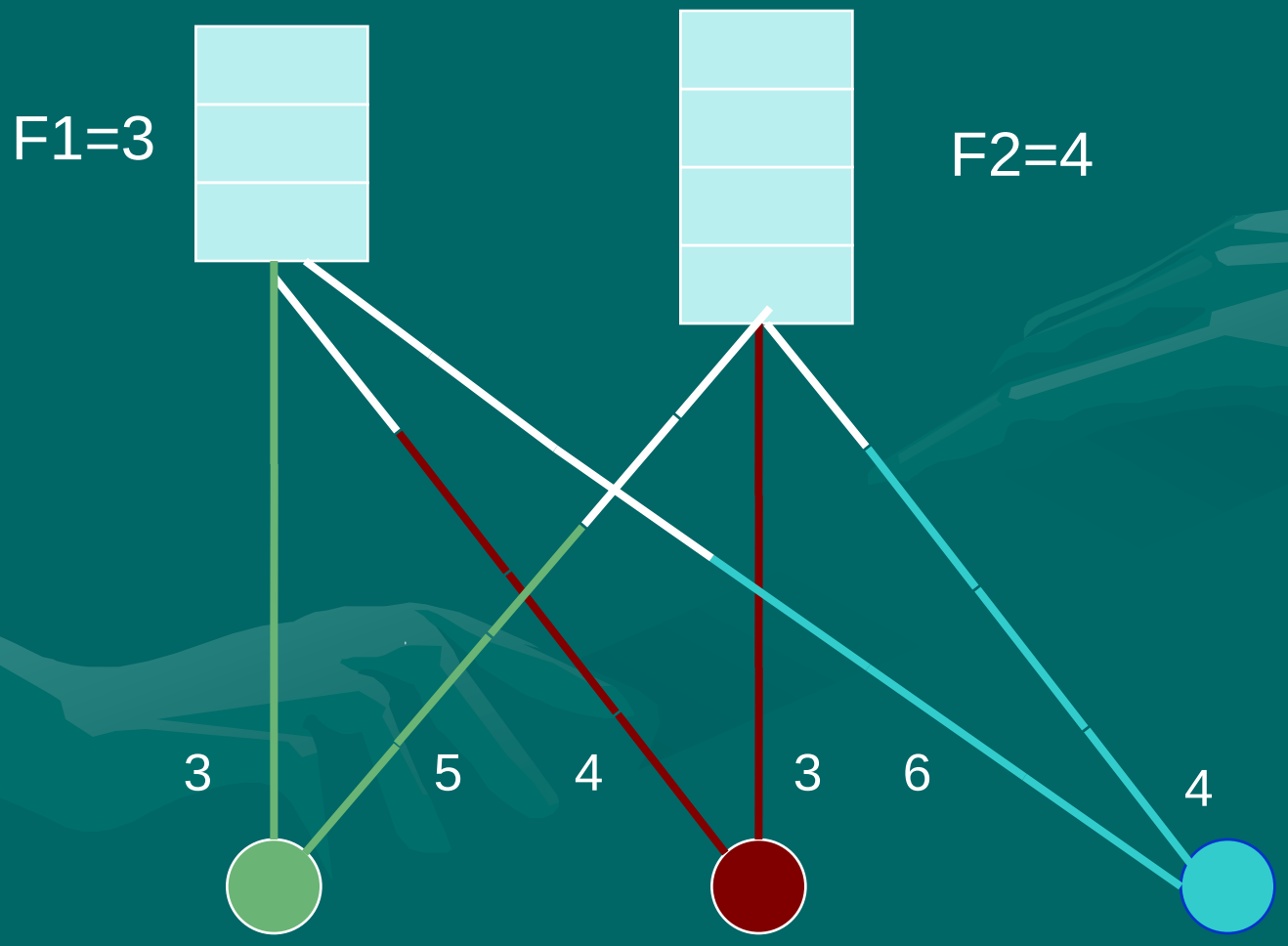
Time = 3



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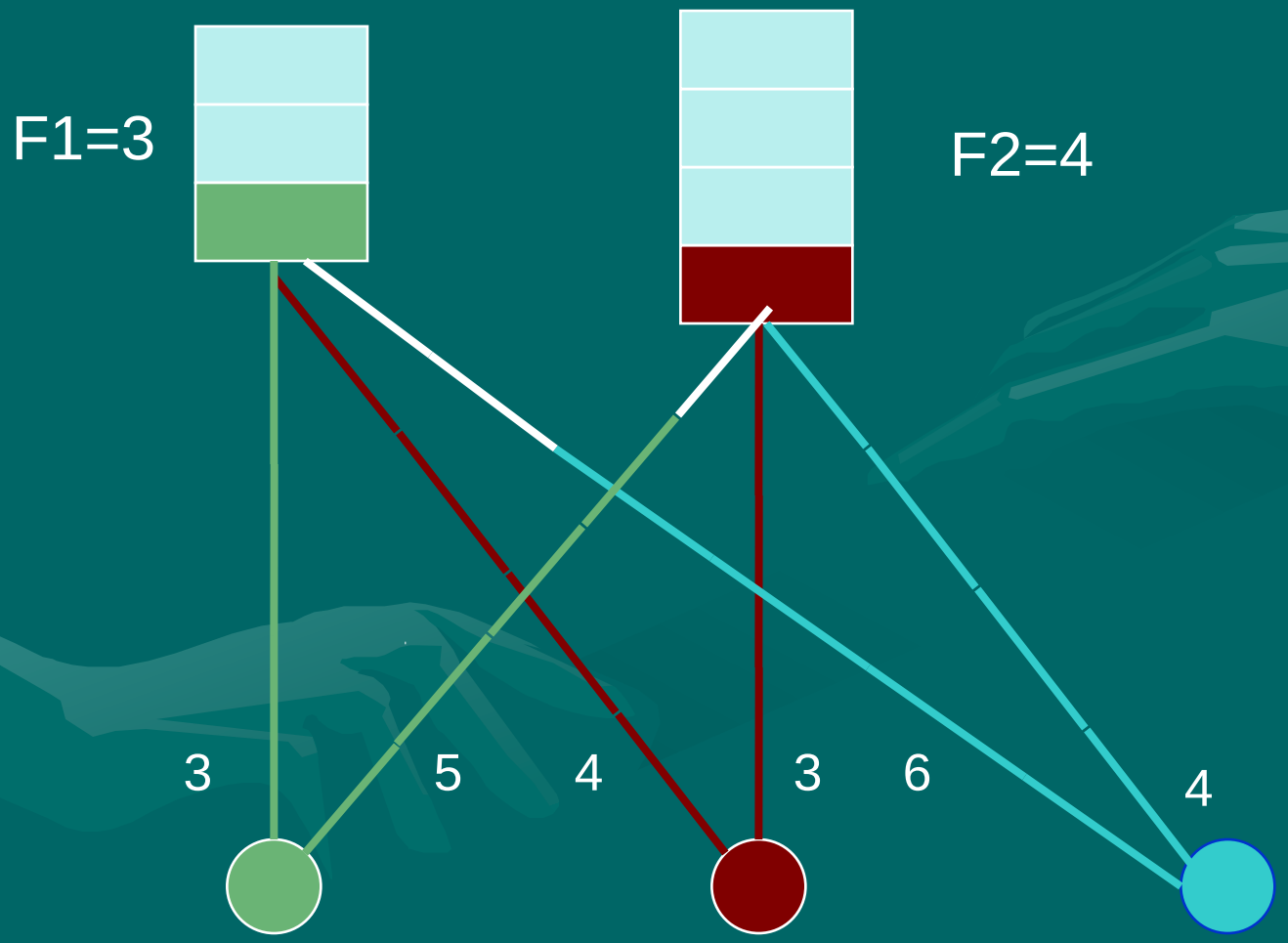
Time = 4



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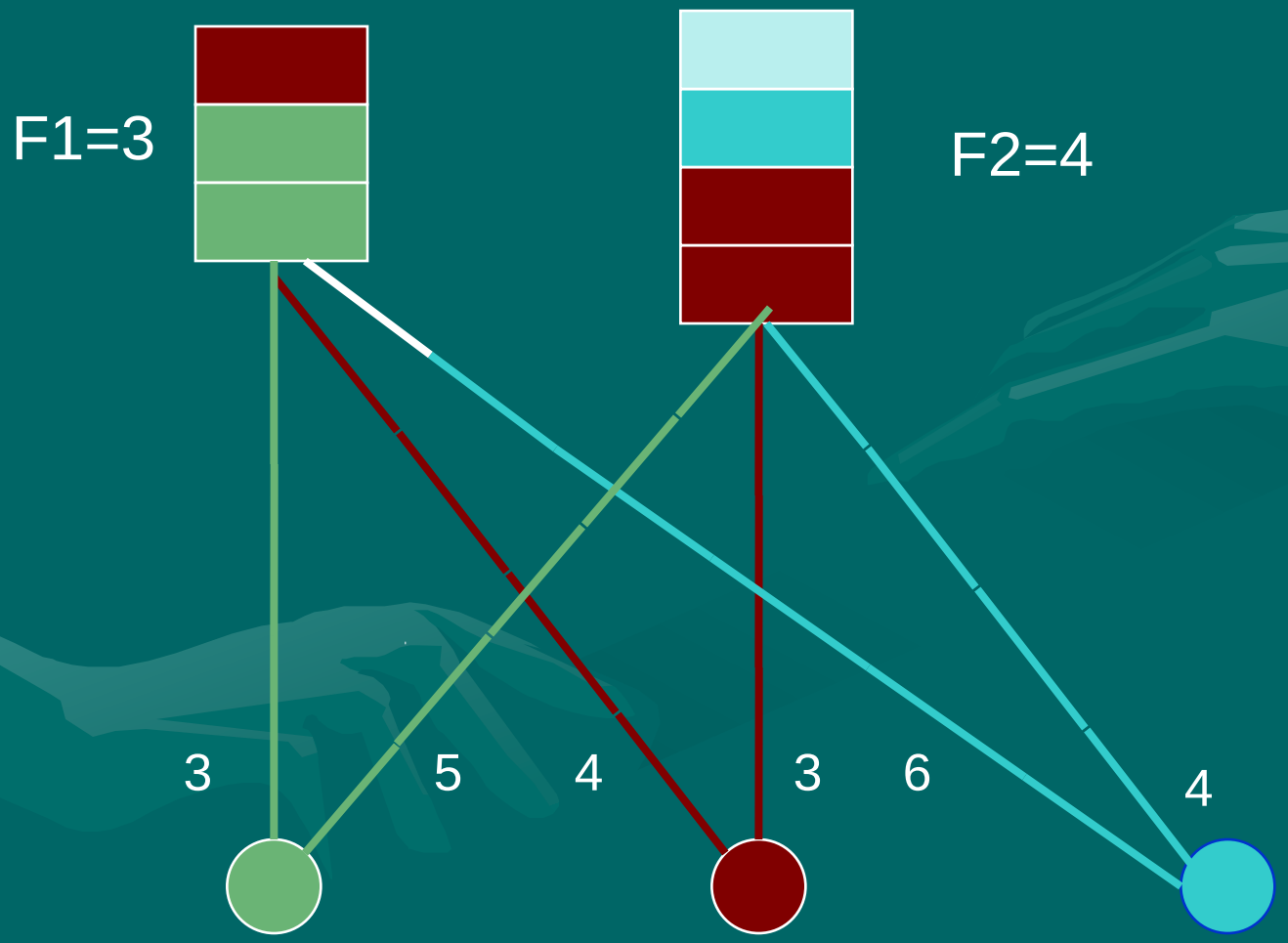
Time = 5



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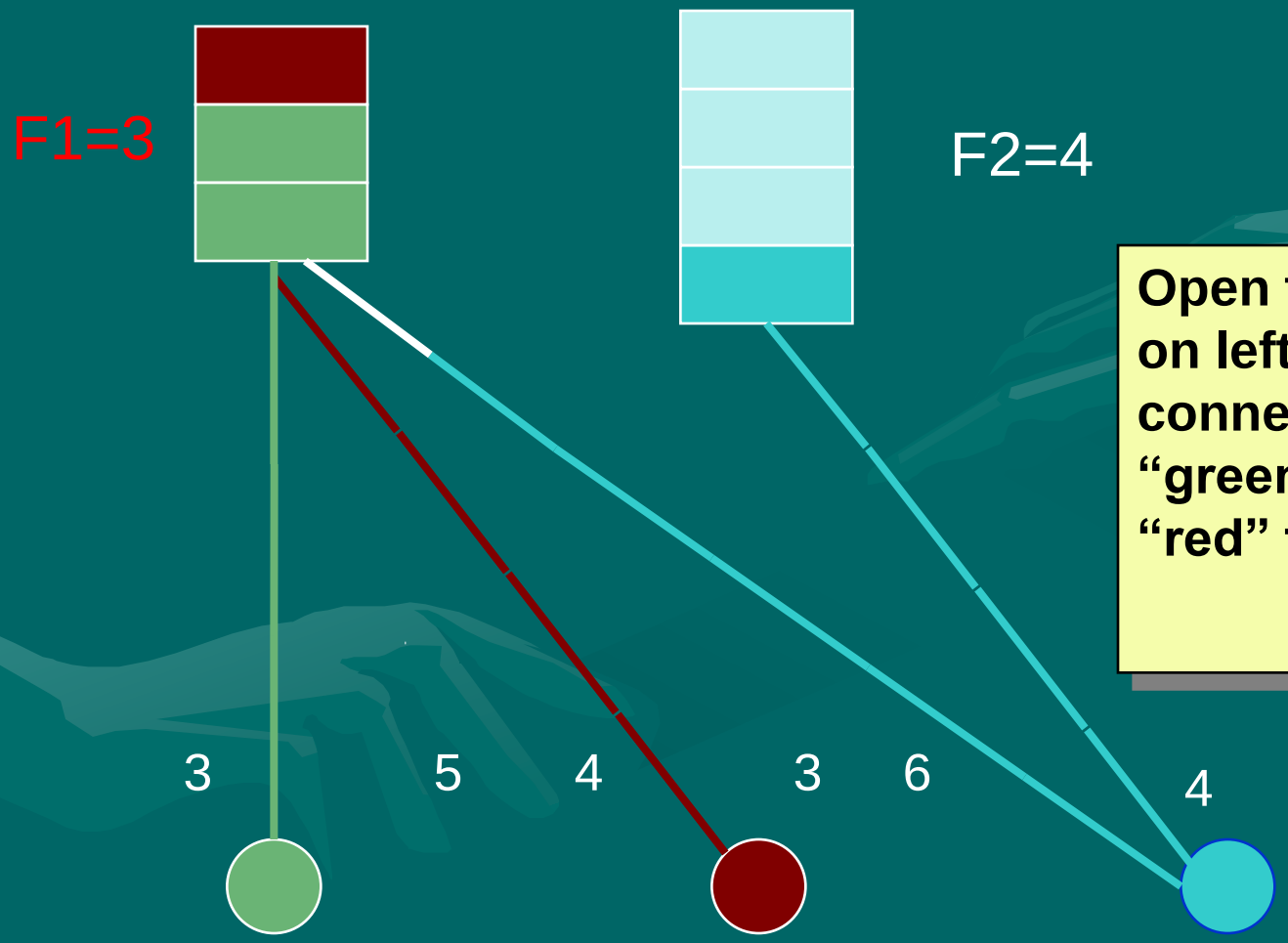
Time = 5



Open the facility on left, and connect clients "green" and "red" to it.

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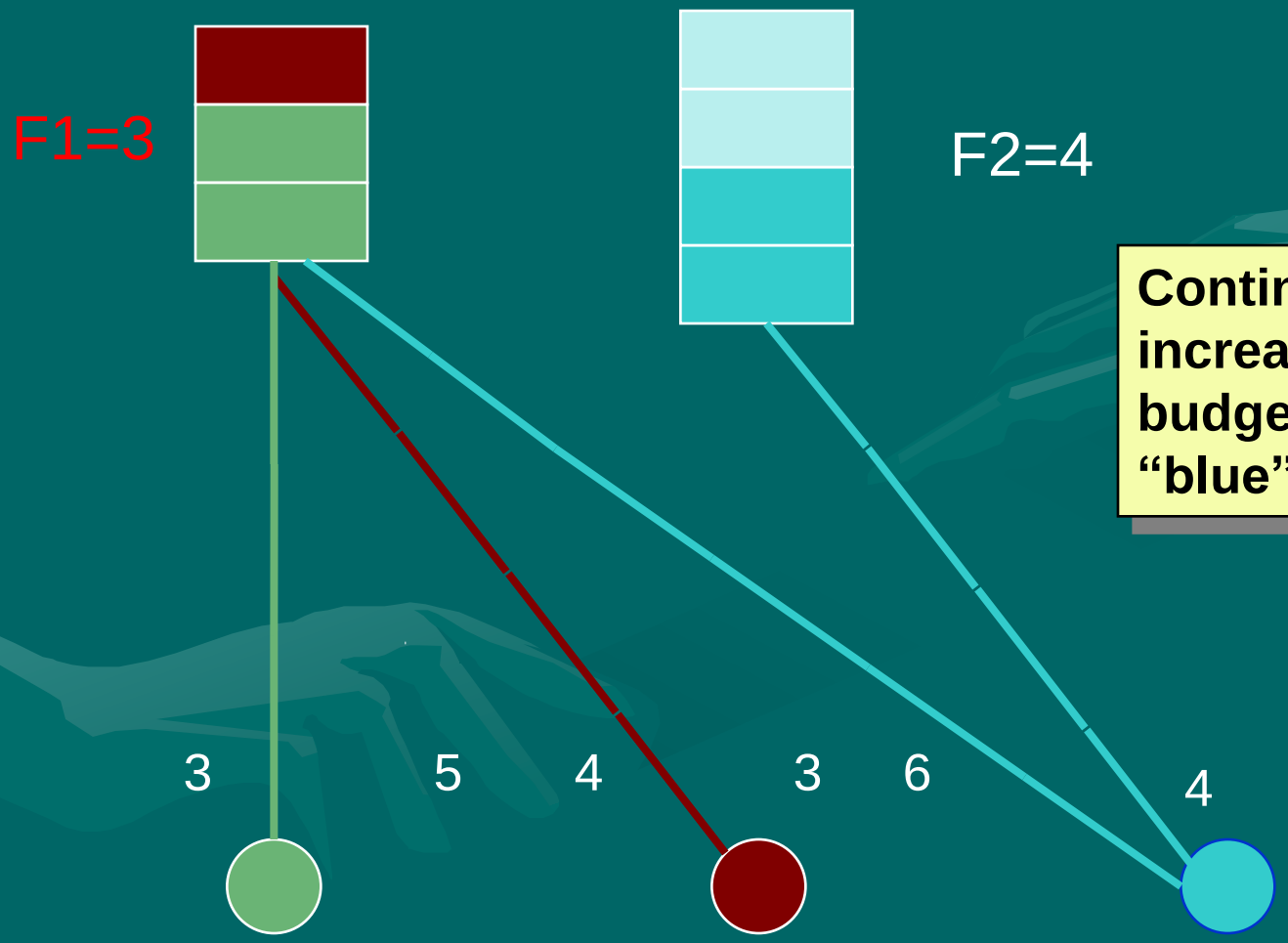
Time = 6



Continue increase the budget of client "blue"

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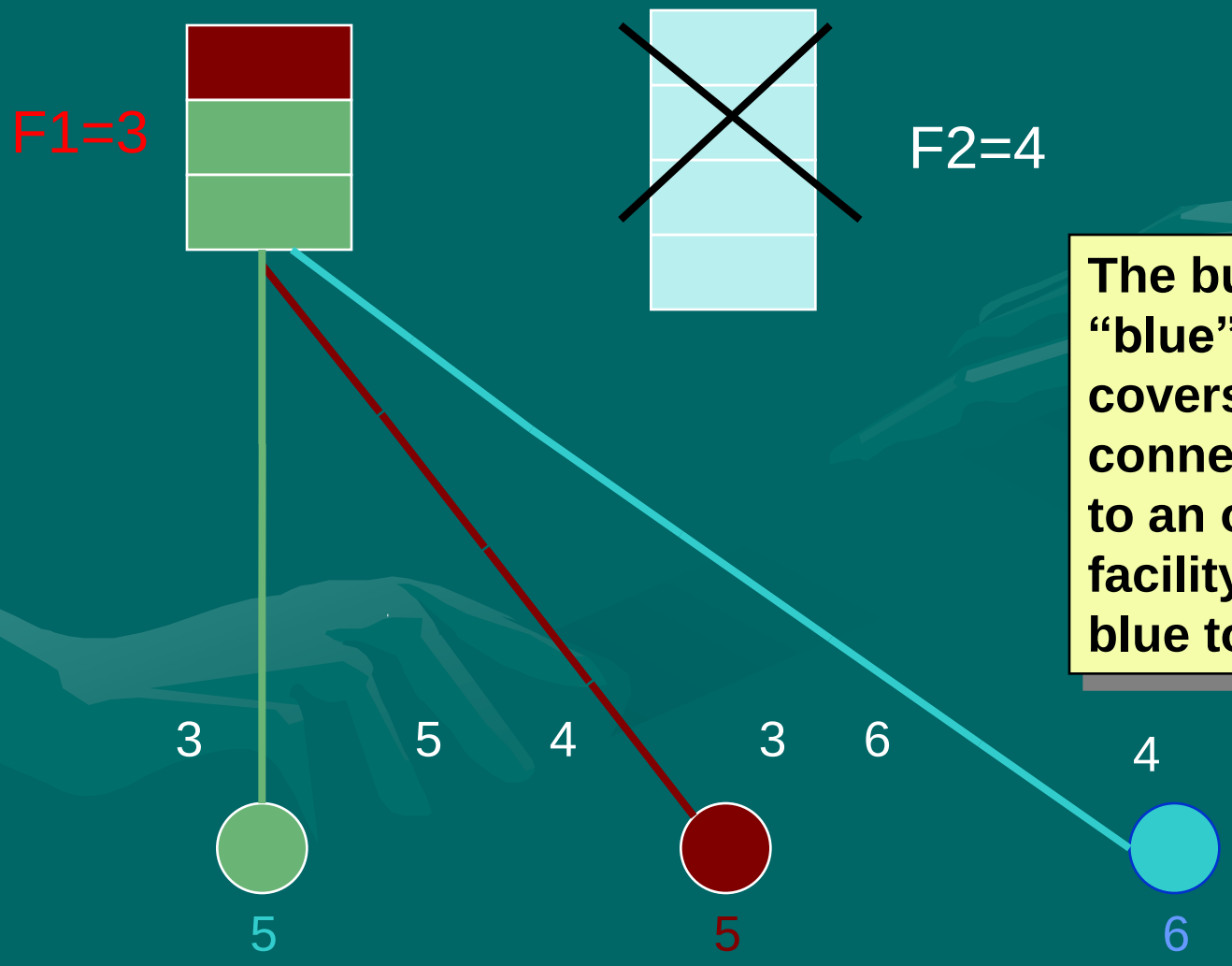
Time = 6



The budget of "blue" now covers its connection cost to an opened facility; connect blue to it.

The Bi-Factor Revealing LP

Jain et al [2003], Mahdian et al [2006]

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Given R_f, R_c is bounded above by

$$\max \frac{\sum_{j=1}^{k} \alpha_j - R_f f_i}{\sum_{j=1}^{k} c_{ij}}$$

Subject to:

$$\alpha_1 \le \alpha_2 \le \ldots \le \alpha_{|D|}$$

$$\sum_{l \ge j} \max(0, \alpha_j - c_{il}) \le f_i$$

$$\alpha_j \le \alpha_l + c_{ij} + c_{il}$$

In particular, if

$R_f = 1.861$, then $R_c \le 1.861$. We got a 1.861 - appr. alg.

Approximation Results

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Ratio	Reference	Algorithm
$1+\ln(\lvert D\rvert)$	Hochbaum (1982)	**Greedy algorithm**
3.16	Shmoys et.al (1997)	**LP rounding**
2.408	Guha and Kuller (1998)	**LP rounding + Greedy augmentation**
1.736	Chudak (1998)	**LP rounding**
1.728	Charika and Guha (1999)	**LP + P-dual + Greedy augmentation**
1.61	Jain et.al (2001)	**Greedy algorithm**
1.582	Svridenko (2002)	**LP rounding**
1.517	Mahdian et.al (2002)	**Revised Greedy algorithm**

Selected LP Applications: Walras-Arrow-Debreu Equilibrium

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The problem was first formulated by Leon Walras in 1874, Elements of Pure Economics, or the Theory of Social Wealth

n players, each with

- an initial endowment of a divisible good
- utility function for consuming all goods—own and others.

Every player

1. sells the entire initial endowment
2. uses the revenue to buy a bundle of goods such that his or her utility function is maximized.

Walras asked,

Can prices be set for all the goods such that the market clears?

Answer by Arrow and Debreu in 1954:

yes, under mild conditions if the utility functions are concave.

Walras-Arrow-Debreu Equilibrium

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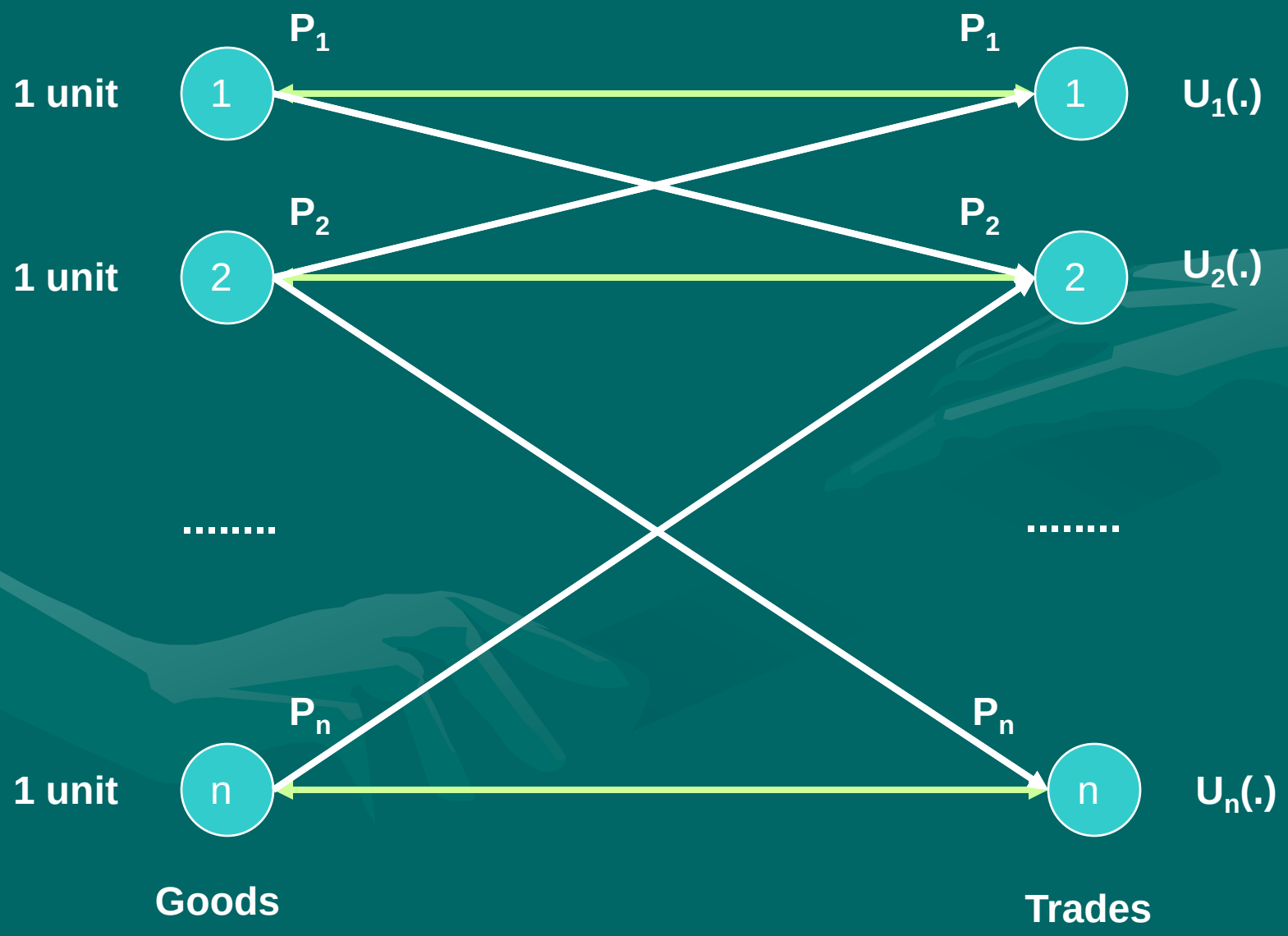

Fisher Equilibrium

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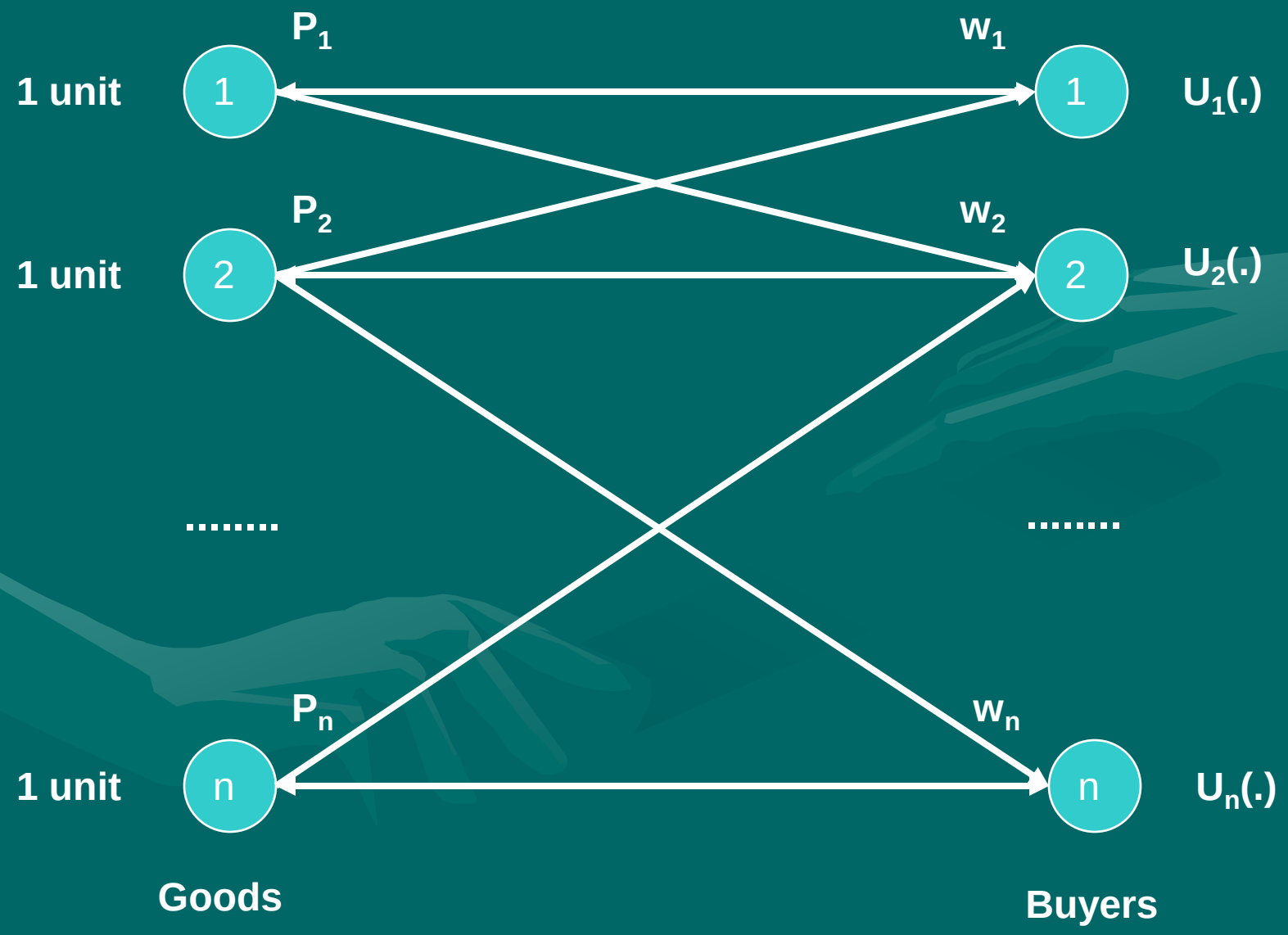

Utility Functions

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Linear Utility : $u_i(x_i) = \sum_j a_{ij} x_{ij}$

Leontief Utility : $u_i(x_i) = \min_j \left\{ \frac{x_{ij}}{a_{ij}} \right\}$

Fisher Equilibrium Example

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$$\text{Player } 1: \max \quad u_1(x_1, y_1)$$

$$s.t. \quad p_x x_1 + p_y y_1 \le 5$$

$$\text{where} \quad u_1(x_1, y_1) = 2x_1 + y_1 \text{ or } \min\left\{\frac{x_1}{2}, \frac{y_1}{1}\right\}$$

$$\text{Player } 2: \max \quad u_2(x_2, y_2)$$

$$s.t. \quad p_x x_2 + p_y y_2 \le 8$$

$$\text{where} \quad u_2(x_2, y_2) = 3x_2 + y_2 \text{ or } \min\left\{\frac{x_2}{3}, \frac{y_2}{1}\right\}$$

$$p_x = \frac{26}{3}, p_y = \frac{13}{3}, x_1 = \frac{1}{13}, y_1 = 1, x_2 = \frac{12}{13}, y_2 = 0$$

WAD Equilibrium Example

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$$\text{Player } 1: \max \quad u_1(x_1, y_1)$$

$$s.t. \quad p_x x_1 + p_y y_1 \le p_x$$

$$\text{where} \quad u_1(x_1, y_1) = 2x_1 + y_1 \text{ or } \min\left\{\frac{x_1}{2}, \frac{y_1}{1}\right\}$$

$$\text{Player } 2: \max \quad u_2(x_2, y_2)$$

$$s.t. \quad p_x x_2 + p_y y_2 \le p_y$$

$$\text{where} \quad u_2(x_2, y_2) = 3x_2 + y_2 \text{ or } \min\left\{\frac{x_2}{3}, \frac{y_2}{1}\right\}$$

$$p_x = 2,\ p_y = 1,\ x_1 = \frac{1}{2},\ y_1 = 1,\ x_2 = \frac{1}{2},\ y_2 = 0$$

Equilibrium Computation

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Eisenberg and Gale [1959] , Scarf [1973], Eaves [1976,1985]

Utility\Model	Fisher	WAD
Linear	Convex Opt.	LCP
Leontief	Convex Opt.	

Equilibrium Computation

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Nenakhov and Primak [1983], Jain [2004]

Utility\Model	Fisher	WAD
Linear	Convex Opt.	Convex Opt
Leontief	Convex Opt.	

Equilibrium Computation

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Y [2004, 2005]

Utility\Model	Fisher	WAD
Linear	LP-class	LP-class
Leontief	LP-class*	

Equilibrium Computation

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Codenotti et al. [2005], Y [2005]
Deng et al. [2005, 2006],

Utility\Model	Fisher	WAD
Linear	LP-class	LP-class
Leontief	LP-class*	NP-Hard

1987 Student Graduation

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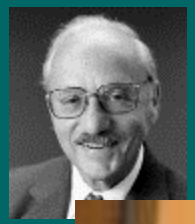

2004 90th Birthday Party

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Organized by MS&E, Stanford, November 12, 2004 (Thapa, Infanger, etc)

2004 90th Birthday Party

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2004 90th Birthday Party

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LP/Dantzig Legacy Continues …

Happy Birthday to Numerical Analysis and Computational Math at Stanford