FlorianHomework2

November 13, 2021

1 Homework Set 2

Please **submit this Jupyter notebook through Canvas** no later than **Mon Nov. 15, 9:00**.

Homework is in **groups of two**, and you are expected to hand in original work. Work that is copied from another group will not be accepted.

2 Exercise 0

Write down the names + student ID of the people in your group.

Write your answer, using LaTeX, in this box.

Run the following cell to import the necessary packages.

[1]:
```
import numpy as np
import scipy.linalg as la
import matplotlib.pyplot as plt
import random
```

3 Exercise 1

The goal of this problem is to show that apparently harmless looking systems of linear equations may be very difficult to solve. Some functions that may be useful are `numpy.triu`, `numpy.tril`, `numpy.eye`, `random.randrange`. ## (a) Generate an $n \times n$ matrix B with random integer elements in the range $b_{ij} \in [-10, 10]$. Choose for instance $n = 20$.

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3.1 (b)

Remove the diagonal of B, save the upper triangular part in U and the lower triangular part in L, and put ones on the diagonals $l_{ii} = u_{ii} = 1$.

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3.2 (c)

Compute $A = L \cdot U$. What is the value of $\det(A)$ and why? Compute the determinant using the appropriate python command and confirm your prediction. In case that you have doubts about the result, compute separately $\det(L)$ and $\det(U)$.

Write your answer, using $\LaTeX$, in this box.

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[ ]:
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3.3 (d)

Choose now an exact solution, for instance $x_e =$ `numpy.ones(n)`, and compute the corresponding right hand side $b = Ax_e$.

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```

3.4 (e)

Solve $Ax = b$ using `scipy.linalg.lu_factor` and `scipy.linalg.lu_solve` and compare the solution with the exact x_e.

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3.5 (f)

Explain the bad results by computing the condition number of A.

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[ ]:
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Write your answer, using $\LaTeX$, in this box.

4 Exercise 2

(N.B. this is a theory exercise.) Suppose we write a $(p+q) \times (p+q)$ matrix M in block form where A, B, C, D are respectively $p \times p, p \times q, q \times p$ and $q \times q$ matrices

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

4.1 (a)

Verify that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ CA^{-1} & I_q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & A^{-1}B \\ 0 & I_q \end{bmatrix}.$$

Write your answer, using $\LaTeX$, in this box.

4.2 (b)

Describe how a system $Mx = b$, with x and b in $\mathbb{R}^{p+q}$, can be solved by applying matrix-vector products with C and B and solves with A and $(D - CA^{-1}B)$.

Write your answer, using LaTeX, in this box.

4.3 (c)

What is the cost, to highest order, of LU-factorizing A and of computing and LU-factorizing $D - CA^{-1}B$?

Write your answer, using LaTeX, in this box.

Remark: Although in this case no savings were obtained, the decomposition above is very useful for solving linear systems with many zero coefficients, in other words where M is a sparse matrix. After applying a permutation of the indices such a matrix is written in the above form, where q is as small as possible and A is blockdiagonal, i.e. $A = \begin{bmatrix} E & O \\ O & F \end{bmatrix}$. This blockdiagonal form then causes big savings in computational cost. Moreover, the procedure can be applied recursively to E and F.