A new method for solving a quartic equation

Direct solution of a quartic equation is a task that is rich in possibilities and which, although already solved - still attracts the interest of mathematicians. The problem, in itself, is a very complicated one[[1]](#footnote-1) and the fact which makes things more difficult is that, in various stages of solving, one cannot avoid extracting square roots – which, because of ambiguity, leads to branching out, to special cases and awkward discussions on discriminants. It is worth mentioning that this is the highest degree of an algebraic equation for which a closed solution can be found. More concretely, mathematicians *Niels Abel*and *Évariste Galois*, independently demonstratedi,ii that a direct method, in quintic equations (and higher), is not at all possible.

By all accounts, most frequently applied and referenced in practice is the *Ferrari* *method*iii, which arose by solving an equation originally proposed by his mentor, renaissance mathematician and philosopher - *Gerolamo Cardano*. This method implies the absence of a cubic polynomial (*depressed quartic*), which doesn't pose a problem, considering that it can always be eliminated by elementary algebraic transformations. Also, this method reduces a quartic equation to a cubic one (*cubic resolvent*), which is a frequent motif in subsequent methods.

***A proposal of a solution of a quartic equation***

Here we will offeranother way for solving a quartic equation, in the most general form. It should be the simplest, the most intuitive and the most unambiguous[[2]](#footnote-2) one. More importantly, that method is certainly more efficient from a programmer’s standpoint! Namely, there are savings made in the method in the use of *expensive* processor’s functions и (i.e. )[[3]](#footnote-3), which makes it both faster and (on a certain level) numerically more stable.

Every quartic polynomial can be represented as a product of two square trinomials, i.e.:

(3)

The basic idea lies in that one should first find coefficients *p1,q1,p2,q2*,, and then, through a separate solving of square equations (on the right), should practically determine the square roots of the initial quartic equation! The coefficients can be determined by solving the system of equations stemming from (3):

(4)

Through the introduction of substitute *y=q1+q2* and through the application of *Vieta's* formulasv,vi, we come to the system of quadratic equations:

(5)

By solving *p,q* (in symbolic form) and by including *p1,2* and *q1,2* in the third equation of the system (4), we get the final form of a cubic equation (cubic resolvent), by *у*:

(6)

Solving a cubic equation is already much simpler[[4]](#footnote-4) and it is performed in a standard way. With such determined value - *у*, quadratic equations (5) become independent and they are solved directly. With *p1,2* and *q1,2* calculated, one can, in a trivial way, solve quadratic equations from (3). This fully solves the initial problem, given in the form of a quartic equation!

*Discussion*: Generally, a cubic equation can have three distinct real roots or, one real and two complex conjugate roots. In the second case, the selection of a real *у* guarantees solving of the equations (3) and (5), without any problems. As for the first case, – it is the only situation which requires caution and necessary discussion. Three different real solutions (*у1, у2, у3*) correspond to different combinations of coefficients of quadratic trinomials – a) p1q1\_p2q2; b) p2q2\_p1q1; c) p1q2\_p2q1. The first two cases are practically the same and they lead to analogous solutions. It is necessary however, to avoid the case of '*intertwined coefficients*'i.e. variation (c). This is achieved in a highly simple way - through a selection of solutions with maximum absolute value i.e. . In this way the possibility of having '*intertwined coefficients*' is excluded! The facts given in this discussion are a novelty and it is them that make this method a simple and efficient one.

From a programmer’s standpoint, additional (and not in the least insignificant) saving of time, is achieved by an additional little trick. Namely, in order to get *p1* and *p2*, it is not necessary to solve a classic quadratic equation. Considering the last equation of the system (4) and assumption that *y=q1+q2*, one simply gets *q1* and *q2*:

Finally, considering the first and the third equations of the system (4), through a simple application of Cramer's rule, we get *p1* and *p2*.

With *p1,q1,p2,q2* known, solutions for an initial quartic equation are found trivially by means of solving the corresponding quadratic equations.

References:

i Beweis der Unmöglichkeit, algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen - *H.N.Abel*, (*J. reine angew. Math. 1, 65*, 1826)

ii OEuvres mathématiques d'Évariste Galois - (*Journal des mathématiques pures et appliquées XI*, 1846)

iii ru.wikipedia.org/wiki/Метод\_Феррари

iv A Universial Method of Solving Quartic Equations - *Sergei L. Shmakov* (*International Journal of Pure and Applied Mathematics*, 2011)

v Opera mathematica - *F. Viète*. (1579; *Reprinted Leiden, Netherlands*, 1646).

vi ru.wikipedia.org/wiki/Формулы\_Виета

vii en.wikipedia.org/wiki/Cubic\_function

viii ru.wikipedia.org/wiki/Тригонометрическая\_формула\_Виета

ix Ars magna or The Rules of Algebra *- Cardano, Gerolamo* (1545)

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1. Allegedlyxxiv, Spanish mathematician Valmes was burned at the stake for claiming to have a solution to this problem. Grand Inquisitor Torquemada told him that, by God’s will, such a problem lies beyond human comprehension! [↑](#footnote-ref-1)
2. i.e. with less branching and fewer discussions. [↑](#footnote-ref-2)
3. In this way, the cube root has been implemented in a number of software solutions [↑](#footnote-ref-3)
4. There are several popular methods of which the authors single out the one based on *Vieta's substitute*xxxi (for simplicity of mathematical execution) and the one based on *Vieta's* trigonometric formulasxxxii (for finished and efficient programs in which it has been implemented). [↑](#footnote-ref-4)