Mathematical Foundations
of
Computer Science

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7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Theorem
Let $d = (d_1, \cdots, d_n)$ with $d-1 \leq \cdots \leq d_n$. Define d' by

$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

Then there exists a graph with score d if and only if there exist a graph with score d'

Furthermore, if $n = 1$, then there exist a graph with score (d_1) if and only if $d_i = 1$.

Idea of Algorithm

find-graph $(d_1, d_2, \cdots, d_n)$

sort$(d_1, d_2, \cdots, d_n)$

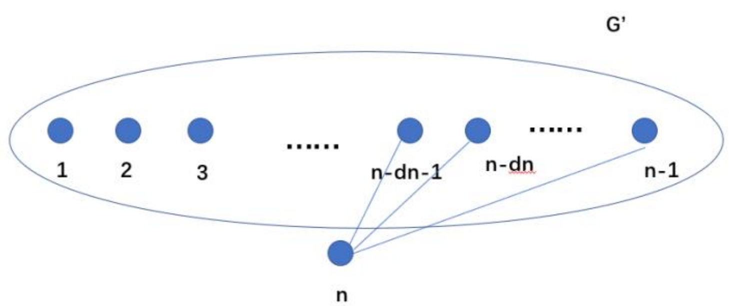
$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

$G' =$ find-graph$(d'_1, d'_2, \cdots, d'_{n-1})$

if $G' =$ NULL

return NULL

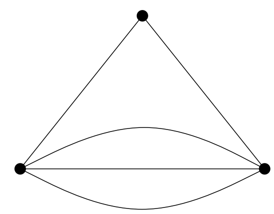
else $G = G' + vertex(n)$



7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs.

Theorem 7.3 (Multigraph Score Theorem)**.** *Let* $A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$*,* $a_1 \geq a_2 \geq \cdots \geq a_n$*. There is a multigraph with this score if and only if for* $A' = (a_1 - 1, a_2 - 1, a_3, \ldots, a_n)$*has nonnegative integers and is multigraphic.*

Furthermore, if $n = 1$*, then exists a graph with score*(a_1) *if and only if* $a_1 = 0$*. If* $n = 2$*, then exists a graph with score*(a_1, a_2) *if and only if* $a_1 = a_2$ *and* a_1, a_2 *are even.*

Remark. This is actually simpler than for graphs.

Exercise 7.4. Prove your theorem.

Proof. $A' \Rightarrow A$

Obviously, it is true since if A' is a multigraph, a edge is added between the largest two vertices, so A is also a multigraph.

$A \Rightarrow A'$

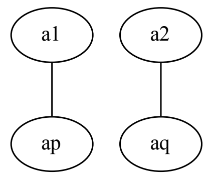
A has corresponding multigraph and we want to prove that A' also has one.

Let's prove that in A, there always exists a multigraph corresponding to A that it has an edge between the vertices of a_1, a_2.

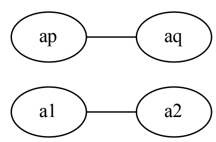
If the multigraph corresponding to A has an edge between the vertices of a_1, a_2, it's okay.

Else if there is no edge between the vertices a_1, a_2. Then there must exist another two vertices connect to them. Suppose the vertice of a_p is connected to a_1 and the vertice of a_q is connected to a_2.

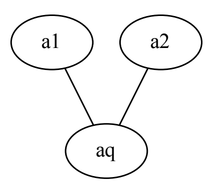
Case 1: a_p **and** a_q **are not the same corresponding vertice.**



We can transform edges (a_1, a_p) and (a_2, a_q) to (a_1, a_2) and (a_p, a_q) ((a_i, a_j) means the edge between the vertices of a_i and a_j). It is a new multigraph and it doesn't change the degree of every vertices. So this multigraph is the one we want to find.



Case 2: a_p and a_q are the same corresponding vertice.



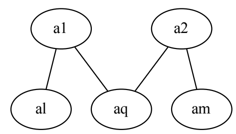
Suppose there are total u edges between the vertices of a_1, a_p and v edges between the vertices of a_2, a_p. $u, v \geq 1$

Then $a_1 \geq a_2 \geq a_p \geq u + v$ since the vertice of a_p is connected to at least two vertices.

So the vertices of a_1, a_2 have another v, u vertices to connect.

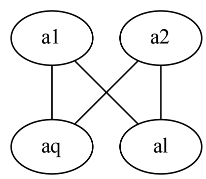
Suppose the vertice of a_1 is connected to the vertices of a_l and the vertice of a_2 is connected to the vertices of a_m.

Case 2(a): a_l and a_m are not the same corresponding vertices.



In fact, it is the same as **Case 1**. a_l, a_m are the new a_p, a_q.

Case 2(b): a_1 and a_m are the same corresponding vertice.

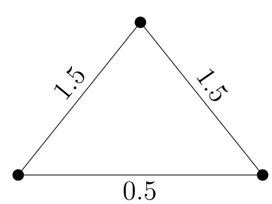


In fact, it is the same as **Case 1**. a_p, a_l are the new a_p, a_q.

So, in A, there always exists a multigraph corresponding to A that it has an edge between the vertices of a_1, a_2. And in A', we just reduce one edge between the vertices of a_1, a_2. Therefore, A' is also multigraphic.

□

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\mathrm{wdeg}(u) = \sum_{\{u,v\}\in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

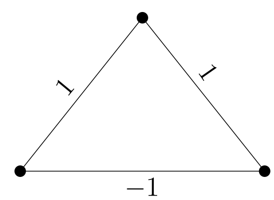
Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$*. There is a weighted graph with this score if and only if* `<fill in some simple criterion here>`.

Remark. This is actually even simpler.

Exercise 7.7. Prove your theorem.

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

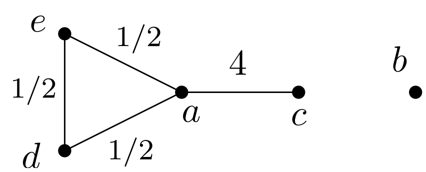
Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}^n$*. There is a graph with real edge weights with this score if and only if* `<fill in some simple criterion here>`.

Exercise 7.10. Prove your theorem.

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).