Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Theorem
Let $d = (d_1, \cdots, d_n)$ with $d - 1 \leq \cdots \leq d_n$. Define d' by

$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

Then there exists a graph with score d if and only if there exist a graph with score d'

Furthermore, if $n = 1$, then there exist a graph with score (d_1) if and only if $d_i = 1$.

Idea of Algorithm

find-graph $(d_1, d_2, \cdots, d_n)$

sort$(d_1, d_2, \cdots, d_n)$

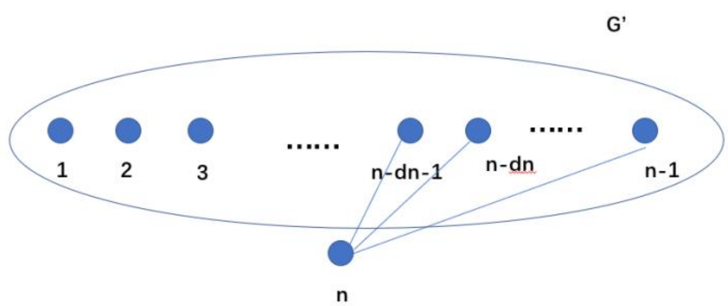
$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

$G' =$ find-graph$(d'_1, d'_2, \cdots, d'_{n-1})$

if $G' =$ NULL

return NULL

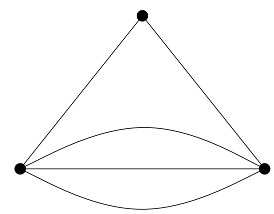
else $G = G' + vertex(n)$



7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs.

Theorem 7.3 (Multigraph Score Theorem)**.** *Let* $A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$*,* $a_1 \geq a_2 \geq \cdots \geq a_n$*. There is a multigraph with this score if and only if* $a_1 + a_2 + \cdots + a_n$ *is even and* $a_1 \leq a_2 + \cdots + a_n$*.*

Furthermore, when $n = 1$*, then exists a graph with score*(a_1) *if and only if* $a_1 = 0$*. When* $n = 2$*, then exists a graph with score*(a_1, a_2) *if and only if* $a_1 = a_2$*.*

Remark. This is actually simpler than for graphs.

Exercise 7.4. Prove your theorem.

Proof. Suppose $v_1, v_2, \ldots, v_n$ are the corresponding vertices of $a_1, a_2, \ldots, a_n$.

Multigraph $\Rightarrow a_1 \leq a_2 + \cdots + a_n$

According to the handshake theorem, $a_1 + a_2 + \cdots + a_n$ is even.

We can split vertices into two groups: v_1 and $v_2, \ldots, v_n$.

For each edge $E(u, v)$:

If u is v_1 and v is in $v_2, \ldots, v_n$, then v_1 gains degree of 1 and $v_2, \ldots, v_n$ gain degree of 1.

If u is in $v_2, \ldots, v_n$ and v is v_1, then v_1 gains degree of 1 and $v_2, \ldots, v_n$ gain degree of 1.

If u is in $v_2, \ldots, v_n$ and v is $v_2, \ldots, v_n$, then v_1 gains no degree and $v_2, \ldots, v_n$ gain degree of 2.

Therefore, the degree of v_1 must be less or equal than the total degree of $v_2, \ldots, v_n$, namely $a_1 \leq a_2 + \cdots + a_n$.

$a_1 \leq a_2 + \cdots + a_n \Rightarrow$ multigraph

Case 1. When $a_1 = a_2 + \cdots + a_n$.

Construct multigraph by connecting a_i edges between v_1 and v_i, $i \geq 2$.

v_1 gains $a_2 + \cdots + a_n$ degrees and v_i gains a_i degree(s), so there is a multigraph.

Case 2. When $a_1 < a_2 + \cdots + a_n$. Let $A = a_1$, $B = a_2 + \cdots + a_n$. $B - A = a_2 + \cdots + an - a_1$.

Construct multigraph by connecting vertices in $v_2, \ldots, v_n$ until the total left degree of $v_2, \ldots, v_n$ is equal to the degree of v_1. Then it becomes **Case 1**.

More concretely, we can do this by following process:

For v_i, $i = n, n-1, \ldots, 2$.

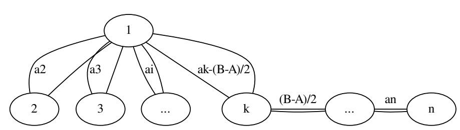
When $a_i < B - A$, then connect a_i edges between v_i and v_{i-1}. a_i becomes 0 and a_{i-1} becomes $a_{i-1} - a_i$. And B becomes $B - 2a_i$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - a_i, 0, \ldots, 0$. The new B is $B - 2a_i$.

When $a_i \geq B - A$, then connect $\frac{B-A}{2}$ edges between v_i and v_{i-1}. a_i becomes $a_i - \frac{B-A}{2}$ and a_{i-1} becomes $a_{i-1} - \frac{B-A}{2}$. And B becomes $B - (B-A) = A$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - \frac{B-A}{2}, a'_i = a_i - \frac{B-A}{2}, 0, \ldots, 0$, $A = a_1 = B = a_2 + \cdots + a'_{i-1} + a'_i$. So it becomes **Case 1**, we can simply connect v_1 with v_i, $i = 2, 3, \ldots, n$.

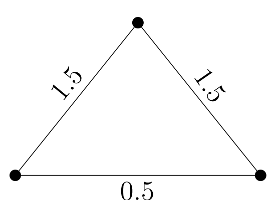
Since $A + B$ is even, $B - A$ is even, too, so $\frac{B-A}{2}$ is an integer. And $B - A < B$, so there exists v_k such that $a_k \geq B - A$.



To sum up, we can construct a multigraph if $a_1 + a_2 + \cdots + a_n$ is even and $a_1 \leq a_2 + \cdots + a_n$.

$\square$

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\operatorname{wdeg}(u) = \sum_{\{u,v\} \in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$ $a_1 \geq a_2 \geq \cdots \geq a_n$. *There is a weighted graph with this score if and only if* $a_1 \leq a_2 + \cdots + a_n$.

Remark. This is actually even simpler.

Exercise 7.7. *Proof.* Suppose $v_1, v_2, \ldots, v_n$ are the corresponding vertices of $a_1, a_2, \ldots, a_n$.

Weighted Graph $\Rightarrow a_1 \leq a_2 + \cdots + a_n$

We can split vertices into two groups: v_1 and $v_2, \ldots, v_n$.

For each edge $E(u, v)$:

If u is v_1 and v is in $v_2, \ldots, v_n$, then v_1 gains degree of the weight and $v_2, \ldots, v_n$ gain degree of the weight.

If u is in $v_2, \ldots, v_n$ and v is v_1, then v_1 gains degree of the weight and $v_2, \ldots, v_n$ gain degree of the weight.

If u is in $v_2, \ldots, v_n$ and v is $v_2, \ldots, v_n$, then v_1 gains no degree and $v_2, \ldots, v_n$ gain the degree of double weight.

Therefore, the degree of v_1 must be less or equal than the total degree of $v_2, \ldots, v_n$, namely $a_1 \leq a_2 + \cdots + a_n$.

$a_1 \leq a_2 + \cdots + a_n \Rightarrow$ Weighted Graph

Case 1. When $a_1 = a_2 + \cdots + a_n$.

Construct Weighted Graph by connecting an edge of weight a_i between v_1 and v_i, $i \geq 2$.

v_1 gains $a_2 + \cdots + a_n$ degrees and v_i gains a_i degree(s), so there is a Weighted Graph.

Case 2. When $a_1 < a_2 + \cdots + a_n$. Let $A = a_1$, $B = a_2 + \cdots + a_n$. $B - A = a_2 + \cdots + an - a_1$.

Construct Weighted Graph by connecting vertices in $v_2, \ldots, v_n$ until the total left degree of $v_2, \ldots, v_n$ is equal to the degree of v_1. Then it becomes **Case 1**.

More concretely, we can do this by following process:

For v_i, $i = n, n-1, \ldots, 2$.

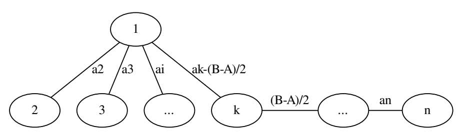
When $a_i < B - A$, then connect an edge of weight a_i between v_i and v_{i-1}. a_i becomes 0 and a_{i-1} becomes $a_{i-1} - a_i$. And B becomes $B - 2a_i$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - a_i, 0, \ldots, 0$. The new B is $B - 2a_i$.

When $a_i \geq B - A$, then connect an edge of weight $\frac{B-A}{2}$ between v_i and v_{i-1}. a_i becomes $a_i - \frac{B-A}{2}$ and a_{i-1} becomes $a_{i-1} - \frac{B-A}{2}$. And B becomes $B - (B - A) = A$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - \frac{B-A}{2}, a'_i = a_i - \frac{B-A}{2}, 0, \ldots, 0$, $A = a_1 = B = a_2 + \cdots + a'_{i-1} + a'_i$. So it becomes **Case 1**, we can simply connect v_1 with v_i, $i = 2, 3, \ldots, n$.

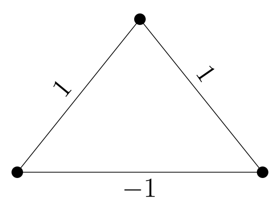
Since the sequence are sorted, there are no repeated weighted edges. And $B - A < B$, so there exists v_k such that $a_k \geq B - A$.



Therefore, we can construct a Weighted Graph if $a_1 \leq a_2 + \cdots + a_n$.

□

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if $n \geq 3$ or $a_1 = a_2$ when $n = 2$ or $a_1 = 0$ when $n = 1$.*

Exercise 7.10. Prove your theorem.

Proof. Obviously, if there is a graph with real edge weights, $n \geq 3$ or $a_1 = a_2$ when $n = 2$ or $a_1 = 0$ when $n = 1$.

Suppose $v_1, v_2, \dots, v_n$ are the corresponding vertices of $a_1, a_2, \dots, a_n$.

When $n = 1$, obviously, there is a graph with real edge weights if and only if $a_1 = 0$.

When $n = 2$, obviously, there is a graph with real edge weights if and only if $a_1 = a_2$.

When $n = 3$, suppose x, y, z be the weight of $E(v_2, v_3), E(v_1, v_3), E(v_1, v_2)$.

According to the degree of each vertices, there are equations as followings:

$$\begin{cases} x + y = a_3 \\ x + z = a_2 \\ y + z = a_1 \end{cases}$$

The solution is:

$$\begin{cases} x = \frac{a_2 + a_3 - a_1}{2} \\ y = \frac{a_1 + a_3 - a_2}{2} \\ z = \frac{a_1 + a_2 - a_3}{2} \end{cases}$$

Therefore, for every sequence a_1, a_2, a_3, there is a graph with real edge weights.

When $n \geq 4$.

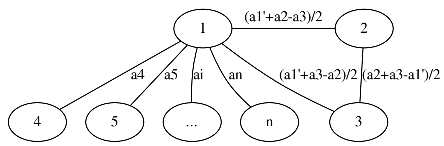
Connect an edge of weight a_i between v_i and v_1, $i \geq 4$.

Now we can image $v_1, v_4, \dots, v_n$ as a "big" vertice v_1'. To promise the degree of v_1, the degree of the "big" vertice is $a_1' = a_1 - a_4 - \cdots - a_n$.

In fact, it becomes the cases that $n = 3$. Suppose x, y, z be the weight of $E(v_2, v_3), E(v_1', v_3), E(v_1', v_2)$.

$$\begin{cases} x + y = a_3 \\ x + z = a_2 \\ y + z = a_1' \end{cases}$$

$$\begin{cases} x = \frac{a_2 + a_3 - a_1'}{2} \\ y = \frac{a_1' + a_3 - a_2}{2} \\ z = \frac{a_1' + a_2 - a_3}{2} \end{cases}$$

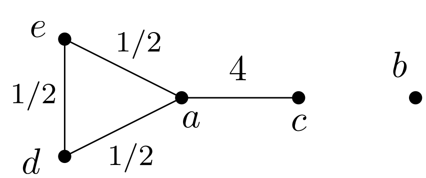


Therefore, for $a_1, a_2, \ldots, a_n$, $n \geq 4$, there is a graph with real edge weight. $\square$

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

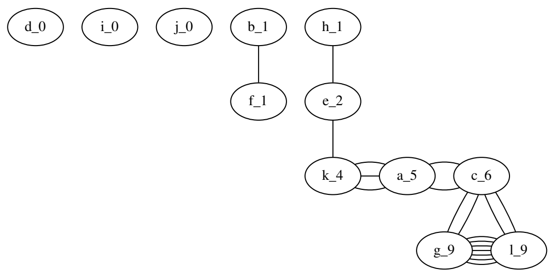
Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

Solution. My student ID is 516021910049. This is a multigraph score, as shown by this picture:



This settles (2). It is not a graph score, because there are total 9 non-zero nodes, but the max degree is 9, so no more nodes to satisfy the degree.