Mathematical Foundations of Computer Science

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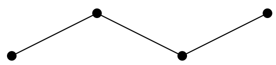
6 Graph Theory Basics

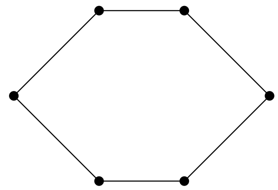
- Homework assignment published on Monday, 2018-04-02.
- Submit first solutions and questions by Sunday, 2018-04-08, 12:00, by email to dominik.scheder@gmail.com and to the TAs.
- You will receive feedback by Wednesday, 2018-04-11.
- Submit final solution by Sunday, 2018-04-15 to me and the TAs.

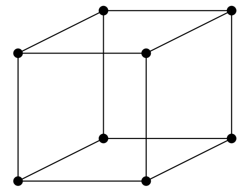
Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A *graph isomorphism* from G to H is a bijective function $f : V \to V'$ such that for all $u, v \in V$ it holds that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. If such a function exists, we write $G \cong H$ and say that G and H are *isomorphic*. In other words, G and H being isomorphic means that they are identical up to the names of its vertices.

Obviously, every graph G is isomorphic to itself, because the identity function $f(u) = u$ is an isomorphism. However, there might be several isomorphisms f from G to G itself. We call such an isomorphism from G to itself an *automorphism* of G.

Exercise 6.1. For each of the graphs below, compute the number of automorphisms it has.







Justify your answer!

Consider the n-dimensional Hamming cube H_n. This is the graph with vertex set $\{0,1\}^n$, and two vertices $x, y \in \{0,1\}^n$ are connected by an edge if they differ in exactly one edge. For example, the right-most graph in the figure above is H_3.

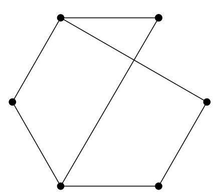
Exercise 6.2. Show that H_n has exactly $2^n \cdot n!$ automorphisms. Be careful: it is easy to construct $2^n \cdot n!$ different automorphisms. It is more difficult to show that there are no automorphisms other than those.

A graph G is called *asymmetric* if the identity function $f(u) = u$ is the only automorphism of G. That is, if G has exactly one automorphism.

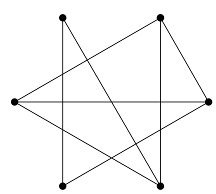
Exercise 6.3. Give an example of an asymmetric graph on six vertices.

Exercise 6.4. Find an asymmetric tree.

For a graph $G = (V, E)$, let $\bar{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote its *complement graph*.

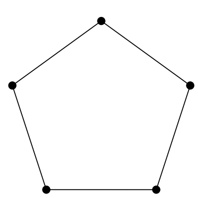


A graph H on six vertices

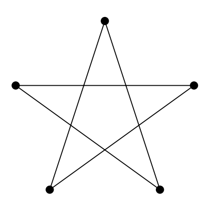


Its complement $\bar{H}$.

We call a graph *self-complementary* if $G \cong \bar{G}$. The above graph is not self-complementary. Here is an example of a self-complementary graph:



The pentagon G.



$\bar{G}$, the pentagram.

Exercise 6.5. Show that there is no self-complementary graph on 999 vertices.

Exercise 6.6. Characterize the natural numbers n for which there is a self-complementary graph G on n vertices. That is, state and prove a theorem of the form "There is a self-complementary graph on n vertices if and only if n `<put some simple criterion here>`."