Mathematical Foundations of Computer Science

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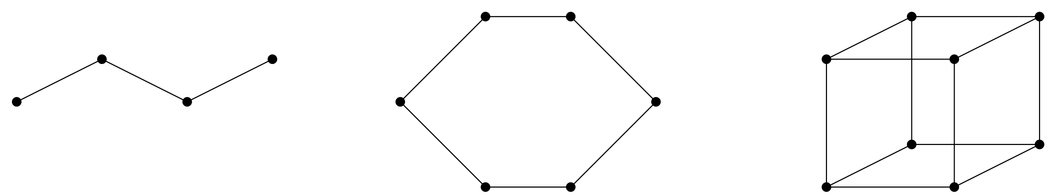
6 Graph Theory Basics

- Homework assignment published on Monday, 2018-04-02.
- Submit first solutions and questions by Sunday, 2018-04-08, 12:00, by email to dominik.scheder@gmail.com and to the TAs.
- You will receive feedback by Wednesday, 2018-04-11.
- Submit final solution by Sunday, 2018-04-15 to me and the TAs.

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A *graph isomorphism* from G to H is a bijective function $f : V \to V'$ such that for all $u, v \in V$ it holds that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. If such a function exists, we write $G \cong H$ and say that G and H are *isomorphic*. In other words, G and H being isomorphic means that they are identical up to the names of its vertices.

Obviously, every graph G is isomorphic to itself, because the identity function $f(u) = u$ is an isomorphism. However, there might be several isomorphisms f from G to G itself. We call such an isomorphism from G to itself an *automorphism* of G.

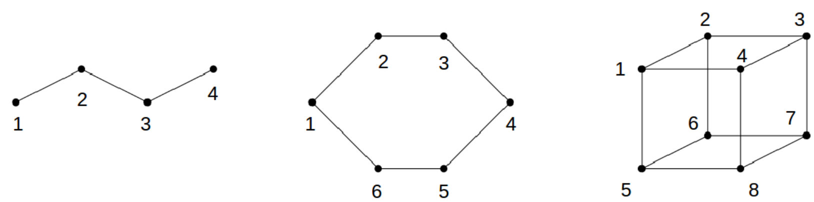
Exercise 6.1. For each of the graphs below, compute the number of automorphisms it has.



Justify your answer!

Answer: 2, 12, 48

Proof. First we give each vertex of graphs an ID for convenience.



In the first graph, there are two vertex $1, 4$ with only one degree, which means their corresponding vertices in automorphism have only one degree. Therefore we have

$$f(1) = 1, f(4) = 4$$

or

$$f(1) = 4, f(4) = 1$$

Either case the automorphism can be determined. There are 2 automorphic graphs. The functions are

$$f_1 = \{\{1, 1\}, \{2, 2\}, \{3, 3\}, \{4, 4\}\}$$

$$f_2 = \{\{1, 4\}, \{2, 3\}, \{3, 2\}, \{4, 1\}\}$$

In the second graph, we take vertex 1 and 2 and the edge between them e_{12}. The corresponding edge in the automorphism can be $e_{12}, e_{21}, e_{23}, e_{32}, ..., e_{61}, e_{16}$. Once the corresponding edge of e_{12} is determined, the automorphism is determined. So there are 12 automorphisms.

In the third graph, we will illustrate our methods by an example first. If we take e_{12} and choose e_{43} as its mapping in automorphism, there are 4 choices left for e_{14}, as e_{14} can be $e_{14}, e_{23}, e_{48}, e_{37}$. Once the mappings of e_{12}

and e_{14} are determined, the automorphism is determined. We have 12 choices for e_{12} and 4 choices for e_{14}, so there are 48 automorphisms.

Consider the n-dimensional Hamming cube H_n. This is the graph with vertex set $\{0,1\}^n$, and two vertices $x, y \in \{0,1\}^n$ are connected by an edge if they differ in exactly one edge. For example, the right-most graph in the figure above is H_3.

Exercise 6.2. Show that H_n has exactly $2^n \cdot n!$ automorphisms. Be careful: it is easy to construct $2^n \cdot n!$ different automorphisms. It is more difficult to show that there are no automorphisms other than those.

Proof. First we choose a vertex in G to be corresponding to our first vertex, say vertex 1. There is 2^n ways. Note that n adjacent vertexes to it can uniquely form a hyperplane and they are all symmetrical. So we arrange them to the adjacent vertexes and there is only $n!$ ways since all the hyperplanes are unique. As a result, the number of automorphism is $2^n n!$.

We prove there are no automorphisms other than those by contradiction. We assume that there is more than $2^n \times n$ ways automorphisms. So for the first vertex, there is more than $n!$ ways for the next adjacent vertices to be put. So there must be a repeat for it. But as we state above, for each way to put the adjacent vertices, the next level of adjacent vertices are unique. So there is a contradiction.

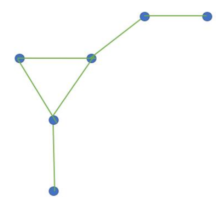
there is an example. For vertex$(1,1,1,1,1)$, the adjacent vertices are

$$(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),(1,0,1,1,1),(0,1,1,1,1),$$

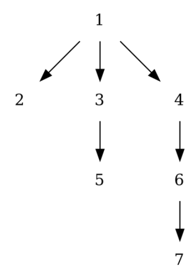
with 5! ways to put them. We can know that the only vertice $(1,1,1,1,0),(1,1,1,0,1)$ are connected to is$(1,1,1,0,0)$, so if there is a repeat for the way the n adjacent vertices to put, it must be the same, which is a contradiction. $\square$

A graph G is called *asymmetric* if the identity function $f(u) = u$ is the only automorphism of G. That is, if G has exactly one automorphism.

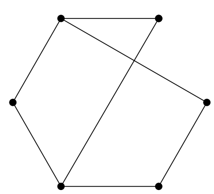
Exercise 6.3. Give an example of an asymmetric graph on six vertices.



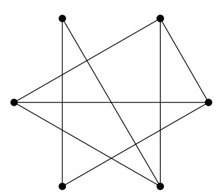
Exercise 6.4. Find an asymmetric tree.



For a graph $G = (V, E)$, let $\bar{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote its *complement graph*.

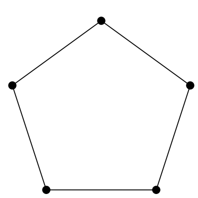


A graph H on six vertices

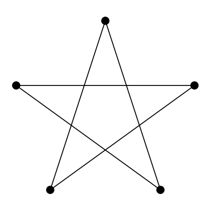


Its complement $\bar{H}$.

We call a graph *self-complementary* if $G \cong \bar{G}$. The above graph is not self-complementary. Here is an example of a self-complementary graph:



The pentagon G.



$\bar{G}$, the pentagram.

Exercise 6.5. Show that there is no self-complementary graph on 999 vertices.

Proof. Since the edges of a graph and its automorphism are same, total edges of the complete graph be composed of both of them must be even.However, a graph consists of 999 vertices has $C_{999}^{2} = 498501$ edges, which is odd. So there is no self-complementary graph on 999 vertices. $\square$

Exercise 6.6. Characterize the natural numbers n for which there is a self-complementary graph G on n vertices. That is, state and prove a theorem of the form "There is a self-complementary graph on n vertices if and only if n `<put some simple criterion here>`."

The criterion we find is:

There is a self-complementary graph on n vertices if and only if $n = 4k$ or $n = 4k + 1$ where $k \in \mathbb{N}$.

Proof. There are $\binom{n}{2}$ edges in a complete graph of n vertices, which means the sum of the edges of a graph and its complementary graph is $\binom{n}{2}$.

$$|E| + |E'| = \binom{n}{2}$$

For a self-complementary graph, there is $|E| = |E'|$.

$$|E| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$$

Obviously, the edges of a graph should be an integer and as n and $n-1$ must be one odd and one even, so

$$n \equiv 0 \mod 4$$

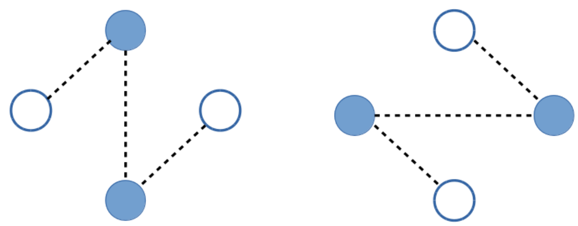
or

$$n - 1 \equiv 0 \mod 4$$

these two conditions can be rewritten as

$$n = 4k \textit{ or } 4k + 1, \textbf{where } k \in \mathbb{N}$$

Then we will prove the condition is sufficient by finding a self-complementary graph for $n = 4k$ and $n = 4k + 1$.



For $n = 4k$, we split the vertices to 4 groups with k vertices in each group, as the following picture shows.

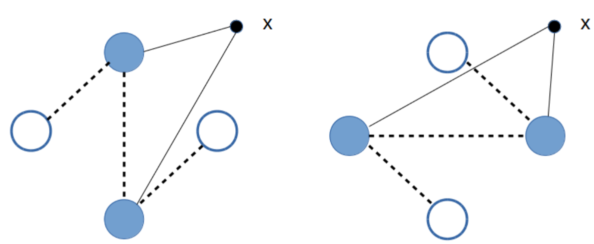
A group in blue stands for the vertices in group is "fully connected", that is, $\{u, v\} \in E$ for any two $u, v \in V_{blue}$.

A group in white stands for there is no edge between any two vertices in white group. $\{u, v\} \notin E$ for any two $u, v \in V_{white}$.

The dot lines stands for there is an edge for any two vertices from groups the line connecting to. $\{u, v\} \in E$ for $u \in V_1$, $v \in V_2$.

The complementary graph of the left graph is the right graph. And we can easily see that they are isomorphic. So there is a self-complementary graph for $n = 4k$.

For $n = 4k + 1$, we add one more vertex x to the graph we found for $n = 4k$. Then we put edges between x and vertices in blue groups. The



black line stands for there is an edge between the vertex and any verterx in the group. $\{x, u\} \in E$ for any $u \in V_{blue}$.

The graph's complementary graph is shown as well. We observe that they are isomorphic. So there is also a self-complementary graph for $n = 4k + 1$.

So there is a self-complementary graph on n vertices if and only if $n = 4k$ or $n = 4k + 1$ where $k \in \mathbb{N}$. $\square$