Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Group Name: **All Right**

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V. A *partition* of V is a set $\{V_1, \ldots, V_k\}$ where (1) $V_1 \cup \cdots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$. For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$ but $\{\{1\}, \{2, 3\}, \{1, 4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2,5,4) \leq (2,6,6)$ but $(2,5,4) \not\leq (3,1,1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?
2. Is there a minimum? A maximum?
3. Does it have an infinite chain?
4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

****Exercise 2.8.** *What is the largest antichain of $\{0,1\}^n$?*

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \ldots$. Here,$\epsilon$ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Proof. First we can proof $\{0, 1\}^* \cong \mathbb{N}$:
Formally, we can define a function:

$$f_1 : \{0,1\}^* \to \mathbb{N}, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \{0,1\}^*, a_i^{(n)} = 0 or 1, i = 1, 2, \ldots, n$$
$$\to 10^n + \sum_{i=1}^{n} 10^{i-1} a_i^{(n)}$$

For example:$f_1(001010) = 1001010_{10}$.
Then we can define another function:

$$f_2 : \mathbb{N} \to \{0,1\}^*, decimal \quad number \to binary$$

For example:$f_2(16) = 10000$.So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N} \tag{1}$$

Secondly, we can proof $\{0, 1\}^* \cong \mathbb{N}^*$:
Define a function:

$$f_3 : \{0,1\}^* \to \mathbb{N}^*, x \to x$$

Another function:

$$f_4 : \mathbb{N}^* \to \{0,1\}^*, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \mathbb{N}^*, a_i^{(n)} \in \mathbb{N}, i = 1, 2, \ldots, n$$
$$\to (a_1^{(n)}\%2, a_2^{(n)}\%2, a_3^{(n)}\%2, \ldots, a_n^{(n)}\%2), i = 1, 2, \ldots, n$$

For example, $f_4((154, 3, 89, 23, 48)) = 01110$. So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N}^* \tag{2}$$

According (1) and (2), $\mathbb{N} \cong \mathbb{N}^*$ is obvious. $\square$

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

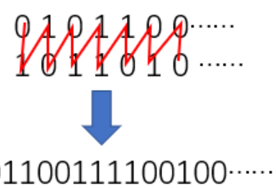
Proof. Obvious, there exits a function:

$$f_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}, x \to (x, 0000\ldots)$$

Then, we define a function:

$$f_2 : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (a_1a_2a_3\ldots, b_1b_2b_3\ldots) \to (a_1b_1a_2b_2a_3b_3\ldots)$$

Such as:

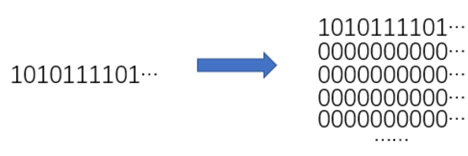


Therefore, we proof $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, and then $R \cong R \times R$. ☐

Exercise 2.11. Consider $R^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \ldots)$ of real numbers. Show that $R \cong R^{\mathbb{N}}$. **Hint:** Again, use the fact that $R \cong \{0,1\}^{\mathbb{N}}$.

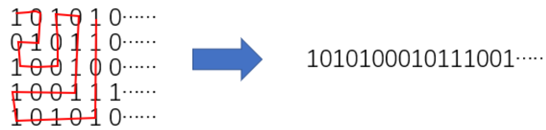
Proof. We only need to proof that $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$. Firstly, we can know the following function easily:

$$f_1 : \{0,1\}^{\mathbb{N}} \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}, x \to (x, 00000\ldots, 00000\ldots, 00000\ldots, \ldots). \quad (1)$$



Then,define a complexer function:

$$f_2 : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (x_1^1x_2^1x_3^1\ldots, x_1^2x_2^2x_3^2\ldots, x_1^3x_2^3x_3^3\ldots, \ldots) \to (x_1^1x_1^2x_2^2x_1^2x_1^3x_2^3\ldots), x_i^j = 0 \; or \; 1 \quad (2)$$



Now, we can infer $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ is true. Thus, $R \cong R^{\mathbb{N}}$. $\square$

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

$$(0, 0, 0, \dots)$$
$$(1, 0, 0, \dots)$$
$$(1, 1, 0, \dots)$$
$$(1, 1, 1, \dots)$$
$$\dots$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

$$(1, 0, 0, \dots)$$
$$(0, 1, 0, \dots)$$
$$(0, 0, 1, \dots)$$
$$\dots$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Since $\{0,1\}^{\mathbb{N}} \cong \mathbb{R}$, there is a bijection: $x \leftrightarrow \mathbf{t}$, $x \in \mathbb{R}, \mathbf{t} \in \{0,1\}^{\mathbb{N}}$. Let's consider $\mathbf{t_i}$.

$$\mathbf{t_i} = (a_1, a_2, \dots), a_k \in \{0,1\}, k \in \mathbb{N}$$

Define $\bar{\mathbf{t_i}} = (1-a_1, 1-a_2, \dots)$. Then construct $\hat{\mathbf{t_i}}$ as:

$$\hat{\mathbf{t_i}} = (a_1, 1-a_1, a_2, 1-a_2, \dots)$$

Consider $\hat{\mathbf{t_i}}, \hat{\mathbf{t_j}}, \forall i,j \in \mathbb{N}, i \neq j$.

Case 1: If $\mathbf{t_i} \not\leq \mathbf{t_j}$, obviously, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Case 2: If $\mathbf{t_i} \leq \mathbf{t_j}$

$$\mathbf{t_i} = (a_1, a_2, \dots) \quad \bar{\mathbf{t_i}} = (1-a_1, 1-a_2, \dots)$$
$$\mathbf{t_j} = (b_1, b_2, \dots) \quad \bar{\mathbf{t_j}} = (1-b_1, 1-b_2, \dots)$$

According to the definition of $\mathbf{a} \leq \mathbf{b}$, we know that $a_k \leq b_k$. So, $\bar{\mathbf{t_i}} \geq \bar{\mathbf{t_j}}$.

Compare every bit of $\hat{\mathbf{t}}$.

$\hat{\mathbf{t}}$	1	2	3	4	$\dots$
$\hat{\mathbf{t_i}}$	a_1	$1-a_1$	a_2	$1-a_2$	$\dots$
$\hat{\mathbf{t_j}}$	b_1	$1-b_1$	b_2	$1-b_2$	$\dots$

Since $a_k \leq b_k$, $1-a_k \geq 1-b_k$.

And since $i \neq j$, $\mathbf{t_i}, \mathbf{t_j}$ are not the same $\mathbf{t}$, which means that $\exists \eta, a_\eta < b_\eta, 1-a_\eta > 1-b_\eta$. So, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Therefore, $\hat{\mathbf{t_1}} \quad \hat{\mathbf{t_2}} \quad \dots$ is an uncountable antichain in $\{0,1\}^{\mathbb{N}}$.

****Exercise 2.15.** *Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.*