Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Theorem
Let $d = (d_1, \cdots, d_n)$ with $d - 1 \leq \cdots \leq d_n$. Define d' by

$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

Then there exists a graph with score d if and only if there exist a graph with score d'

Furthermore, if $n = 1$, then there exist a graph with score (d_1) if and only if $d_i = 1$. **Idea of Algorithm**

find-graph $(d_1, d_2, \cdots, d_n)$

sort$(d_1, d_2, \cdots, d_n)$

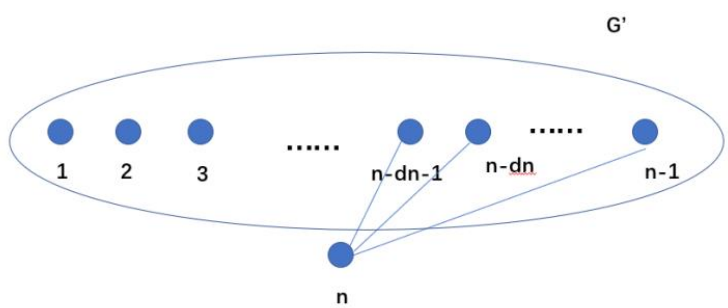
$$d'_i = \begin{cases} d_i - 1 & i = n - d_n, \cdots, n-1, n \\ d_i & i = 1, \cdots, n - d_n - 1 \end{cases}$$

$G' =$ find-graph$(d'_1, d'_2, \cdots, d'_{n-1})$

if $G' =$ NULL

return NULL

else $G = G' + vertex(n)$ **Trying to explain it over lunch:**

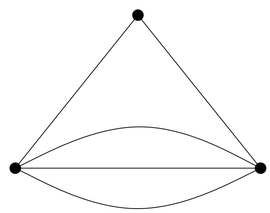


We sort the degree in a non-decreasing order. Every time we eliminate the vertex with the largest degree(suppose it is x). We subtract 1 from the other x^{th} largest degree of the vertices. If we succeed in doing it until the end with no negative numbers. We succeed in finding the graph.

7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4,4,2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs.

Theorem 7.3 (Multigraph Score Theorem). *Let* $A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$, $a_1 \geq a_2 \geq \cdots \geq a_n$. *There is a multigraph with this score if and only if* $a_1 + a_2 + \cdots + a_n$ *is even and* $a_1 \leq a_2 + \cdots + a_n$.

Furthermore, when $n = 1$, *then exists a graph with score*(a_1) *if and only if* $a_1 = 0$. *When* $n = 2$, *then exists a graph with score*(a_1, a_2) *if and only if* $a_1 = a_2$.

Remark. This is actually simpler than for graphs.

Exercise 7.4. Prove your theorem.

Proof. Suppose $v_1, v_2, \ldots, v_n$ are the corresponding vertices of $a_1, a_2, \ldots, a_n$.

Multigraph $\Rightarrow a_1 \leq a_2 + \cdots + a_n$

According to the handshake theorem, $a_1 + a_2 + \cdots + a_n$ is even.

We can split vertices into two groups: v_1 and $v_2, \ldots, v_n$.

For each edge $E(u, v)$:

If u is v_1 and v is in $v_2, \ldots, v_n$, then v_1 gains degree of 1 and $v_2, \ldots, v_n$ gain degree of 1.

If u is in $v_2, \ldots, v_n$ and v is v_1, then v_1 gains degree of 1 and $v_2, \ldots, v_n$ gain degree of 1.

If u is in $v_2, \ldots, v_n$ and v is $v_2, \ldots, v_n$, then v_1 gains no degree and $v_2, \ldots, v_n$ gain degree of 2.

Therefore, the degree of v_1 must be less or equal than the total degree of $v_2, \ldots, v_n$, namely $a_1 \leq a_2 + \cdots + a_n$.

$a_1 \le a_2 + \cdots + a_n \Rightarrow$ multigraph

Case 1. When $a_1 = a_2 + \cdots + a_n$.

Construct multigraph by connecting a_i edges between v_1 and v_i, $i \ge 2$.

v_1 gains $a_2 + \cdots + a_n$ degrees and v_i gains a_i degree(s), so there is a multigraph.

Case 2. When $a_1 < a_2 + \cdots + a_n$. Let $A = a_1$, $B = a_2 + \cdots + a_n$. $B - A = a_2 + \cdots + an - a_1$.

Construct multigraph by connecting vertices in $v_2, \ldots, v_n$ until the total left degree of $v_2, \ldots, v_n$ is equal to the degree of v_1. Then it becomes **Case 1**.

More concretely, we can do this by following process:

For v_i, $i = n, n-1, \ldots, 2$.

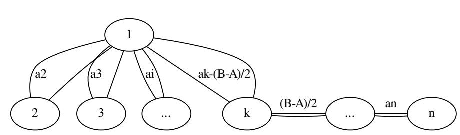
When $a_i < B - A$, then connect a_i edges between v_i and v_{i-1}. a_i becomes 0 and a_{i-1} becomes $a_{i-1} - a_i$. And B becomes $B - 2a_i$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - a_i, 0, \ldots, 0$. The new B is $B - 2a_i$.

When $a_i \ge B - A$, then connect $\frac{B-A}{2}$ edges between v_i and v_{i-1}. a_i becomes $a_i - \frac{B-A}{2}$ and a_{i-1} becomes $a_{i-1} - \frac{B-A}{2}$. And B becomes $B - (B - A) = A$.

Now the new sequence is $a_1, a_2, \ldots, a'_{i-1} = a_{i-1} - \frac{B-A}{2}, a'_i = a_i - \frac{B-A}{2}, 0, \ldots, 0$, $A = a_1 = B = a_2 + \cdots + a'_{i-1} + a'_i$. So it becomes **Case 1**, we can simply connect v_1 with v_i, $i = 2, 3, \ldots, n$.

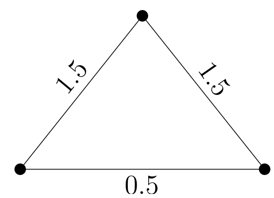
Since $A + B$ is even, $B - A$ is even, too, so $\frac{B-A}{2}$ is an integer. And $B - A < B$, so there exists v_k such that $a_k \ge B - A$.



To sum up, we can construct a multigraph if $a_1 + a_2 + \cdots + a_n$ is even and $a_1 \le a_2 + \cdots + a_n$.

$\square$

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\text{wdeg}(u) = \sum_{\{u,v\} \in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$ $a_1 \geq a_2 \geq \cdots \geq a_n$. *There is a weighted graph with this score if and only if* $a_1 \leq a_2 + \cdots + a_n$.

Remark. This is actually even simpler.

Exercise 7.7. *Proof.* Suppose $v_1, v_2, \ldots, v_n$ are the corresponding vertices of $a_1, a_2, \ldots, a_n$.

Weighted Graph $\Rightarrow a_1 \leq a_2 + \cdots + a_n$

We can split vertices into two groups: v_1 and $v_2, \ldots, v_n$.

For each edge $E(u, v)$:

If u is v_1 and v is in $v_2, \ldots, v_n$, then v_1 gains degree of the weight and $v_2, \ldots, v_n$ gain degree of the weight.

If u is in $v_2, \ldots, v_n$ and v is v_1, then v_1 gains degree of the weight and $v_2, \ldots, v_n$ gain degree of the weight.

If u is in $v_2, \ldots, v_n$ and v is $v_2, \ldots, v_n$, then v_1 gains no degree and $v_2, \ldots, v_n$ gain the degree of double weight.

Therefore, the degree of v_1 must be less or equal than the total degree of $v_2, \ldots, v_n$, namely $a_1 \leq a_2 + \cdots + a_n$.

$a_1 \leq a_2 + \cdots + a_n \Rightarrow$ Weighted Graph

Case 1. When $a_1 = a_2 + \cdots + a_n$.

Construct Weighted Graph by connecting an edge of weight a_i between v_1 and v_i, $i \geq 2$.

v_1 gains $a_2 + \cdots + a_n$ degrees and v_i gains a_i degree(s), so there is a Weighted Graph.

Case 2. When $a_1 < a_2 + \cdots + a_n$. Let $A = a_1$, $B = a_2 + \cdots + a_n$. $B - A = a_2 + \cdots + an - a_1$.

Construct Weighted Graph by connecting vertices in $v_2, \dots, v_n$ until the total left degree of $v_2, \dots, v_n$ is equal to the degree of v_1. Then it becomes **Case 1**.

More concretely, we can do this by following process:

For v_i, $i = n, n-1, \dots, 2$.

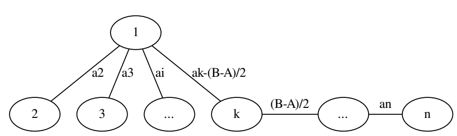
When $a_i < B - A$, then connect an edge of weight a_i between v_i and v_{i-1}. a_i becomes 0 and a_{i-1} becomes $a_{i-1} - a_i$. And B becomes $B - 2a_i$.

Now the new sequence is $a_1, a_2, \dots, a'_{i-1} = a_{i-1} - a_i, 0, \dots, 0$. The new B is $B - 2a_i$.

When $a_i \geq B - A$, then connect an edge of weight $\frac{B-A}{2}$ between v_i and v_{i-1}. a_i becomes $a_i - \frac{B-A}{2}$ and a_{i-1} becomes $a_{i-1} - \frac{B-A}{2}$. And B becomes $B - (B - A) = A$.

Now the new sequence is $a_1, a_2, \dots, a'_{i-1} = a_{i-1} - \frac{B-A}{2}, a'_i = a_i - \frac{B-A}{2}, 0, \dots, 0$, $A = a_1 = B = a_2 + \cdots + a'_{i-1} + a'_i$. So it becomes **Case 1**, we can simply connect v_1 with v_i, $i = 2, 3, \dots, n$.

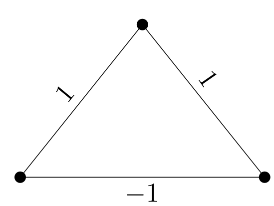
Since the sequence are sorted, there are no repeated weighted edges. And $B - A < B$, so there exists v_k such that $a_k \geq B - A$.



Therefore, we can construct a Weighted Graph if $a_1 \leq a_2 + \cdots + a_n$.

$\square$

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}^n$*. There is a graph with real edge weights with this score if and only if* $n \geq 3$ *or* $a_1 = a_2$ *when* $n = 2$ *or* $a_1 = 0$ *when* $n = 1$*.*

Exercise 7.10. Prove your theorem.

Proof. Obviously, if there is a graph with real edge weights, $n \geq 3$ or $a_1 = a_2$ when $n = 2$ or $a_1 = 0$ when $n = 1$.

Suppose $v_1, v_2, \ldots, v_n$ are the corresponding vertices of $a_1, a_2, \ldots, a_n$.

When $n = 1$, obviously, there is a graph with real edge weights if and only if $a_1 = 0$.

When $n = 2$, obviously, there is a graph with real edge weights if and only if $a_1 = a_2$.

When $n = 3$, suppose x, y, z be the weight of $E(v_2, v_3), E(v_1, v_3), E(v_1, v_2)$.

According to the degree of each vertices, there are equations as followings:

$$\begin{cases} x + y = a_3 \\ x + z = a_2 \\ y + z = a_1 \end{cases}$$

The solution is:

$$\begin{cases} x = \frac{a_2 + a_3 - a_1}{2} \\ y = \frac{a_1 + a_3 - a_2}{2} \\ z = \frac{a_1 + a_2 - a_3}{2} \end{cases}$$

Therefore, for every sequence a_1, a_2, a_3, there is a graph with real edge weights.

When $n \geq 4$.

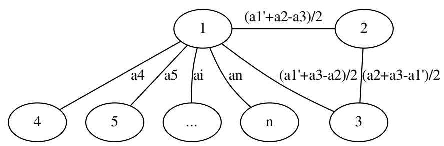
Connect an edge of weight a_i between v_i and v_1, $i \geq 4$.

Now we can image $v_1, v_4, \ldots, v_n$ as a "big" vertice v_1'. To promise the degree of v_1, the degree of the "big" vertice is $a_1' = a_1 - a_4 - \cdots - a_n$.

In fact, it becomes the cases that $n = 3$. Suppose x, y, z be the weight of $E(v_2, v_3), E(v_1', v_3), E(v_1', v_2)$.

$$\begin{cases} x + y = a_3 \\ x + z = a_2 \\ y + z = a_1' \end{cases}$$

$$\begin{cases} x = \frac{a_2+a_3-a_1'}{2} \\ y = \frac{a_1'+a_3-a_2}{2} \\ z = \frac{a_1'+a_2-a_3}{2} \end{cases}$$

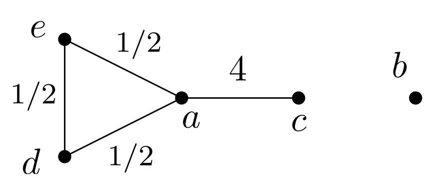


Therefore, for $a_1, a_2, \ldots, a_n$, $n \geq 4$, there is a graph with real edge weight. $\square$

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

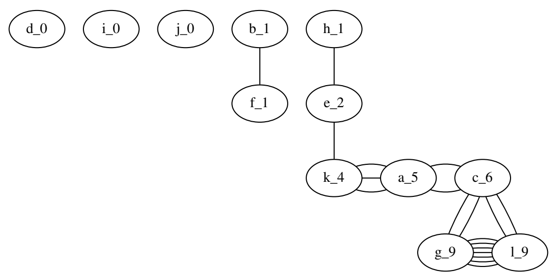
Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

Solution. I am Jiacheng Shen. My student ID is 516021910049. This is a multigraph score, as shown by this picture:



This settles (2). It is not a graph score, because there are total 9 non-zero nodes, but the max degree is 9, so no more nodes to satisfy the degree.

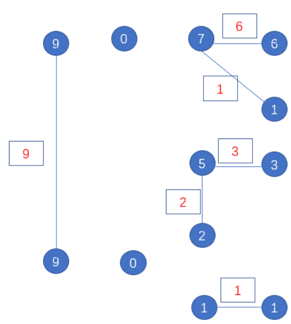
Solution. Ny name is YimingLiu. My student ID is 516021910379. We sort it in an non-increasing order (997653211100). And we set aside the 2 ”0”s so the sequence is d:(9976532111)

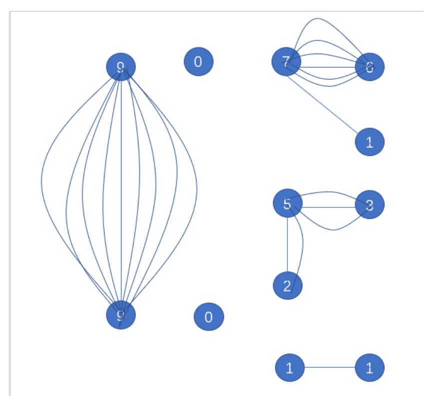
It is not a graph score because d’:(865421000) is not a score.

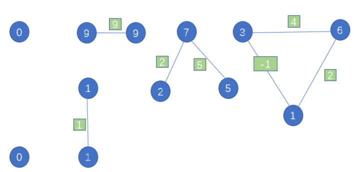
It is a multigraph score because the sum of degrees is an even number and the largest degree of the vertices is smaller than the sum of other degree of other vertices. As is shown in the picture:

It is a weighted graph score. As is shown in the picture:

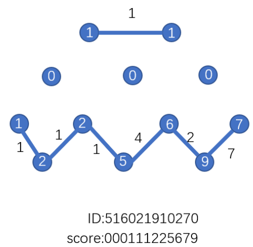
It is the score of a graph with real edge weights. As is shown in the picture:







I’m Zhou Yiyuan, my student ID is 516021910270. This is a multigraph score, as shown by this picture:

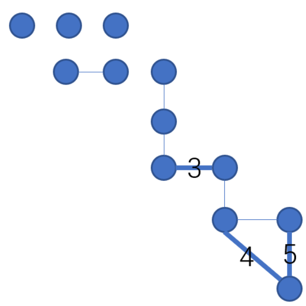


This settles(2). It's not a graph score, because the score of my ID is (000111225679), and the largest degree is 9, which is larger than 8, the quantity of verticals with a non-zero degree except the largest one.

Solution. My ID is 515021910404. The corresponding score is $(0, 0, 0, 1, 1, 1, 2, 4, 4, 5, 5, 9)$.

There is no graph for this score. As $d' = (0, 0, -1, 0, 0, 0, 1, 3, 3, 4, 4)$, it is impossible to have a negative degree for a vertex of graph.

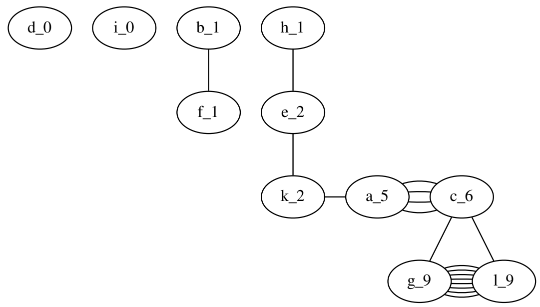
There is a multigraph for this score as the following picture shows. The integers on the edges stands for the number of edges between two vertices.



There is a weighted graph because there is already a multigraph.

There is a real weighted graph becaues there is already a multigraph.

Solution. My student ID is 516021910229. This is a multigraph score, as shown by this picture:



This settles (2). It is not a graph score, because $d' = (0, 0, 0, 0, 0, 1, 1, 1, 4, 5, 8)$, there are total 6 non-zero nodes, but the max degree is 8, so no more nodes to satisfy the degree.