

# Mathematical Foundations of Computer Science

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Group Name: **All Right**

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

## 2 Partial Orderings

### 2.1 Equivalence Relations as a Partial Ordering

An equivalence relation  $R \subseteq V \times V$  is basically the same as a partition of  $V$ . A *partition* of  $V$  is a set  $\{V_1, \dots, V_k\}$  where (1)  $V_1 \cup \dots \cup V_k = V$  and (2) the  $V_i$  are pairwise disjoint, i.e.,  $V_i \cap V_j = \emptyset$  for  $1 \leq i < j \leq k$ . For example,  $\{\{1\}, \{2, 3\}, \{4\}\}$  is a partition of  $\{1, 2, 3, 4\}$  but  $\{\{1\}, \{2, 3\}, \{1, 4\}\}$  is not.

**Exercise 2.1.** Let  $E_4$  be the set of all equivalence relations on  $\{1, 2, 3, 4\}$ . Note that  $E_4$  is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?  
4.
3. What is the size of the largest antichain?  
7.

## 2.2 Chains and Antichains

Define the partially ordered set  $(\mathbb{N}_0^n, \leq)$  as follows:  $x \leq y$  if  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . For example,  $(2, 5, 4) \leq (2, 6, 6)$  but  $(2, 5, 4) \not\leq (3, 1, 1)$ .

**Exercise 2.2.** Consider the infinite partially ordered set  $(\mathbb{N}_0^n, \leq)$ .

1. Which elements are minimal? Which are maximal?

The minimal element is  $(0, 0, 0, \dots, 0)$ . (There are  $n$  0s in the element.)

No element is maximal.

2. Is there a minimum? A maximum?

The minimum element is  $(0, 0, 0, \dots, 0)$ . (There are  $n$  0s in the element.)

No element is maximum.

3. Does it have an infinite chain?

Yes.

There is an example:  $\{(0, 0, 0, \dots, 0), (1, 0, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, 1, \dots, 1)\}$

4. Does it have arbitrarily large antichains? That is, can you find an antichain  $A$  of size  $|A| = k$  for every  $k \in \mathbb{N}$ ?

Yes. We consider an antichain like this:

$$\{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, 1, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}$$

For the  $k^{\text{th}}$  element, there is only one 1 in the  $k^{\text{th}}$  position, and other positions are all occupied by 0. And the antichain consists of these  $k$  elements.

**\*Exercise 2.3.** Does every infinite subset  $S \subseteq \mathbb{N}_0^n$  contain an infinite chain?

*Proof.* Base case  $n = 1$ : Apparently, every two elements in set  $N_0^0$  is comparable since there is only one dimension. If there exists an infinite subset  $S \subseteq N_0^0$ , the subset  $S$  itself is an infinite chain. So, the theorem holds when  $n = 1$ .

Inductive hypothesis:

Suppose the theorem holds for all values of  $n$  up to some  $k$ ,  $k \geq 1$ .

Inductive step:

Let  $n = k + 1$ . If there exists an infinite subset  $S \subseteq N_0^{k+1}$ , note

$$\begin{aligned} S_1 &= \{(a_1, a_2, \dots, a_k) \mid (a_1, a_2, \dots, a_k, a_{k+1}) \in S\} \\ S_2 &= \{a_{k+1} \mid (a_1, a_2, \dots, a_k, a_{k+1}) \in S\} \end{aligned} \quad (1)$$

Since  $S$  is infinite, at least one of  $S_1, S_2$  is infinite.

Suppose  $S_1$  is infinite, according to inductive hypothesis, there is an infinite chain  $C_k$  for  $S_1 \subseteq N_0^k$ .

$$\begin{aligned} C_k &= (A_1, A_2, \dots), A_1 \leq A_2 \leq \dots \\ A_i &= (a_{i1}, a_{i2}, \dots, a_{ik}) \end{aligned} \quad (2)$$

Now we construct an infinite chain  $C_{k+1}$  for  $S \subseteq N_0^{k+1}$ . Take  $b \in S_2$ , we append every  $A_i$  with  $b$  to get  $B_i$ .

$$\begin{aligned} B_i &= (a_{i1}, a_{i2}, \dots, a_{ik}, b) \\ C_{k+1} &= (B_1, B_2, \dots), B_1 \leq B_2 \leq \dots \end{aligned} \quad (3)$$

So  $C_{k+1}$  is an infinite chain for  $S \subseteq N_0^{k+1}$ .

Now suppose  $S_1$  is finite and  $S_2$  is infinite. Notice that  $S_2$  itself is an infinite chain. We take  $(a_1, a_2, \dots, a_k) \in S_1$  and we can construct an infinite chain for  $S \subseteq N_0^{k+1}$  in a similar way.

So, the theorem holds for  $n = k + 1$ . By the principle of mathematical induction, the theorem holds for all  $n \in \mathbb{N}$ .  $\square$

**Exercise 2.4.** Show that  $(N_0^n, \leq)$  has no infinite antichain. **Hint.** Use the previous exercise.

*Proof.* We proof it by contradiction. Suppose there is an infinite antichain which is also a subset of  $N_0^n$ . But from Exercise 2.3, it is clear to us that every infinite subset  $S \subseteq N_0^n$  contain an infinite chain. So there is a contradiction. Consequently,  $(N_0^n, \leq)$  has no infinite antichain.  $\square$

Consider the induced ordering on  $\{0, 1\}^n$ . That is, for  $x, y \in \{0, 1\}^n$  we have  $x \leq y$  if  $x_i \leq y_i$  for every coordinate  $i \in [n]$ .

**Exercise 2.5.** Draw the Hasse diagrams of  $(\{0, 1\}^n, \leq)$  for  $n = 2, 3$ .

**Exercise 2.6.** Determine the maximum, minimum, maximal, and minimal elements of  $\{0, 1\}^n$ .

Maximum element:  $(1, 1, 1, \dots, 1)$

Maximal element:  $(1, 1, 1, \dots, 1)$

Minimum element:  $(0, 0, 0, \dots, 0)$

Minimal element:  $(0, 0, 0, \dots, 0)$

**Exercise 2.7.** What is the longest chain of  $\{0, 1\}^n$ ?

One of the examples is as follows:

$$\{(0, 0, 0, \dots, 0), (1, 0, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, 1, \dots, 1)\}$$

**\*\*Exercise 2.8.** What is the largest antichain of  $\{0, 1\}^n$ ?

## 2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that  $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ , i.e., there is a bijection  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . From this, and by induction, it follows quite easily that  $\mathbb{N}^k \cong \mathbb{N}$  for every  $k$ .

**Exercise 2.9.** Consider  $\mathbb{N}^*$ , the set of all finite sequences of natural numbers, that is,  $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \dots$ . Here,  $\epsilon$  is the empty sequence. Show that  $\mathbb{N} \cong \mathbb{N}^*$  by defining a bijection  $\mathbb{N} \rightarrow \mathbb{N}^*$ .

*Proof.* First we can proof  $\{0, 1\}^* \cong \mathbb{N}$ :

Formally, we can define a function:

$$\begin{aligned} f_1 : \{0, 1\}^* &\rightarrow \mathbb{N}, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots, a_n^{(n)}) \in \{0, 1\}^*, a_i^{(n)} = 0 \text{ or } 1, i = 1, 2, \dots, n \\ &\rightarrow 10^n + \sum_{i=1}^n 10^{i-1} a_i^{(n)} \end{aligned}$$

For example:  $f_1(001010) = 1001010_{10}$ .

Then we can define another function:

$$f_2 : \mathbb{N} \rightarrow \{0, 1\}^*, \text{ decimal number} \rightarrow \text{binary}$$

For example:  $f_2(16) = 10000$ . So we get the conclusion:

$$\{0, 1\}^* \cong \mathbb{N} \quad (1)$$

Secondly, we can proof  $\{0, 1\}^* \cong \mathbb{N}^*$ :

Define a function:

$$f_3 : \{0, 1\}^* \rightarrow \mathbb{N}^*, x \rightarrow x$$

Another function:

$$f_4 : \mathbb{N}^* \rightarrow \{0, 1\}^*, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots, a_n^{(n)}) \in \mathbb{N}^*, a_i^{(n)} \in \mathbb{N}, i = 1, 2, \dots, n \\ \rightarrow (a_1^{(n)} \% 2, a_2^{(n)} \% 2, a_3^{(n)} \% 2, \dots, a_n^{(n)} \% 2), i = 1, 2, \dots, n$$

For example,  $f_4((154, 3, 89, 23, 48)) = 01110$ . So we get the conclusion:

$$\{0, 1\}^* \cong \mathbb{N}^* \quad (2)$$

According (1) and (2),  $\mathbb{N} \cong \mathbb{N}^*$  is obvious.  $\square$

**Exercise 2.10.** Show that  $R \cong R \times R$ . **Hint:** Use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$  and thus show that  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ .

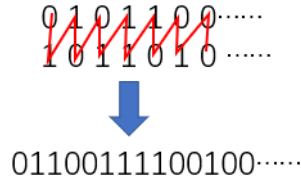
*Proof.* Obvious, there exists a function:

$$f_1 : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}, x \rightarrow (x, 0000 \dots)$$

Then, we define a function:

$$f_2 : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}, (a_1 a_2 a_3 \dots, b_1 b_2 b_3 \dots) \rightarrow (a_1 b_1 a_2 b_2 a_3 b_3 \dots)$$

Such as:



Therefore, we proof  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ , and then  $R \cong R \times R$ .  $\square$

**Exercise 2.11.** Consider  $R^{\mathbb{N}}$ , the set of all infinite sequences  $(r_1, r_2, r_3, \dots)$  of real numbers. Show that  $R \cong R^{\mathbb{N}}$ . **Hint:** Again, use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$ .

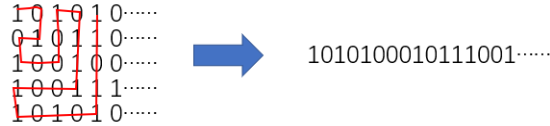
*Proof.* We only need to proof that  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$ . Firstly, we can know the following function easily:

$$f_1 : \{0, 1\}^{\mathbb{N}} \rightarrow (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}, x \rightarrow (x, 00000 \dots, 00000 \dots, 00000 \dots, \dots). \quad (4)$$



Then, define a complexer function:

$$f_2 : (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}, (x_1^1 x_2^1 x_3^1 \dots, x_1^2 x_2^2 x_3^2 \dots, x_1^3 x_2^3 x_3^3 \dots, \dots) \rightarrow (x_1^1 x_1^2 x_1^3 x_1^4 \dots, x_2^1 x_2^2 x_2^3 x_2^4 \dots, x_3^1 x_3^2 x_3^3 x_3^4 \dots, \dots), x_i^j = 0 \text{ or } 1 \quad (5)$$



Now, we can infer  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$  is true. Thus,  $R \cong R^{\mathbb{N}}$ . □

Next, let us view  $\{0, 1\}^{\mathbb{N}}$  as a partial ordering: given two elements  $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbb{N}}$ , that is, sequences  $\mathbf{a} = (a_1, a_2, \dots)$  and  $\mathbf{b} = (b_1, b_2, \dots)$ , we define  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ . Clearly,  $(0, 0, \dots)$  is the minimum element in this ordering and  $(1, 1, \dots)$  the maximum.

**Exercise 2.12.** Give a countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$ . Remember that a set  $A$  is countably infinite if  $A \cong \mathbb{N}$ .

**Exercise 2.13.** Find a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ .

**Exercise 2.14.** Find an uncountable antichain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .

**\*\*Exercise 2.15.** Find an uncountable chain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .