Mathematical Foundations of Computer Science

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Group Name: **All Right**

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

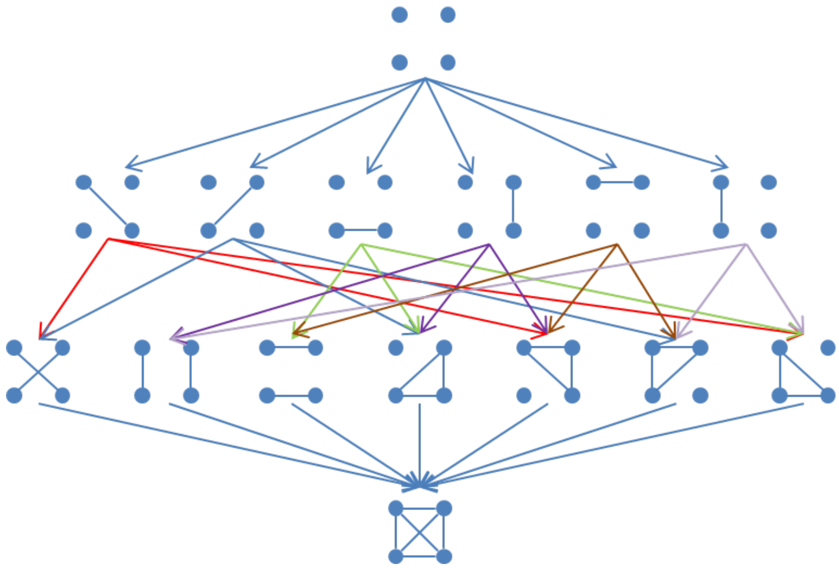
An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V. A *partition* of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \dots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \le i < j \le k$. For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$ but $\{\{1\}, \{2, 3\}, \{1, 4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.



2. What is the size of the largest chain?

   4.

3. What is the size of the largest antichain?

   7.

2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2,5,4) \leq (2,6,6)$ but $(2,5,4) \not\leq (3,1,1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?

   The minimal element is $(0,0,0,\cdots,0)$.(There are n0s in the element.)

   No element is maximal.

2. Is there a minimum? A maximum?

   The minimum element is $(0,0,0,\cdots,0)$.(There are n0s in the element.)

   No element is maximum.

3. Does it have an infinite chain?

   Yes. We consider an chain like this:

$$\{(1,1,1,\cdots,1),(2,2,2,\cdots,2),(3,3,3,\cdots,3),\cdots,(k,k,\cdots,k)\}$$

   For every element in the chain, there are n elements in it.

4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

   Yes. We consider an antichain like this:

$$\{(0,k-1,0,\cdots,0),(1,k-2,0,\cdots,0),(2,k-3,0,\cdots,0),\cdots,(k-1,0,0,\cdots,0)\}$$

   For every element in the antichain, the first element in each element of the antichain is increasing from 0 to $k-1$, and similarly, the second element in each element of the antichain is decreasing from $k-1$ to 0. As for the other elements, it can be any natural numbers and in the example, we use 0 to replace them because it is an easy way to express this.

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

Proof. Base case $n = 1$: Apparently, every two elements in set N_0^0 is comparable since there is only one dimension. If there exists an infinite subset $S \subseteq \mathbb{N}_0^0$, the subset S itself is an infinite chain. So, the theorem holds when $n = 1$.
Inductive hypothesis:

Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step:

Let $n = k+1$. Let S be an infinite subset of $\mathbb{N}_0^{k+1}$, note

$$\begin{aligned} S_1 &= \{(a_1, a_2, ..., a_k) & |(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \\ S_2 &= \{a_{k+1} & |(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \end{aligned} \tag{1}$$

Since S is infinite, at least one of S_1, S_2 is infinite.

Suppose S_1 is infinite, according to inductive hypothesis, there is an infinite chain C_k for $S_1 \subseteq \mathbb{N}_0^k$.

$$\begin{aligned} C_k &= (A_1, A_2, \cdots), A_1 \leq A_2 \leq \cdots \\ A_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}) \end{aligned} \tag{2}$$

Now we construct an infinite chain C_{k+1} for $S \subseteq \mathbb{N}_0^{k+1}$. Take one b from S_2, we append every A_i with the same constant b to get B_i. The last number of B_i is same, so B_i can still form a chain C_{k+1}.

$$\begin{aligned} B_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}, b) \\ C_{k+1} &= (B_1, B_2, \cdots), B_1 \leq B_2 \leq \cdots \end{aligned} \tag{3}$$

So C_{k+1} is an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$.

Now suppose S_1 is finite and S_2 is infinite. Notice that S_2 itself is an infinite chain. We take $(a_1, a_2, ..., a_k) \in S_1$ and we can construct an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$ in a similar way.

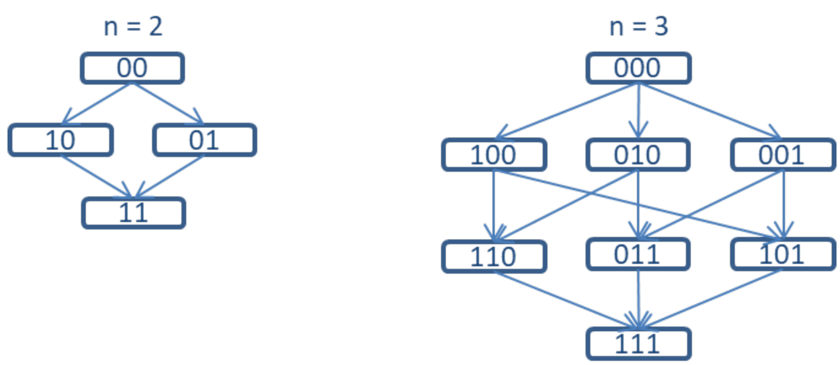
So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. $\square$

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Proof. We proof it by contradiction. Suppose there is an infinite antichain which is also a subset of $\mathbb{N}_0^n$. But from Exercise 2.3, it is clear to us that every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain. So there is a contradiction. Consequently, $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. $\square$

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.



Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

$$\text{Maximum element:}(1,1,1,\cdots,1)$$
$$\text{Maximal element:}(1,1,1,\cdots,1)$$
$$\text{Minimum element:}(0,0,0,\cdots,0)$$
$$\text{Minimal element:}(0,0,0,\cdots,0)$$

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

One of the examples is as follows:

$$\{(0,0,0,\cdots,0),(1,0,0,\cdots,0),(1,1,0,\cdots,0),\cdots,(1,1,1,\cdots,1)\}$$

****Exercise 2.8.** *What is the largest antichain of $\{0,1\}^n$?*

The largest antichain of $\{0,1\}^n$ is the middle level of the Hasse diagram, and the number of elements in the largest antichain is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. Note the ith level of the Hasse diagram of $\{0,1\}^n$ as L_i.

$$L_i = \{S | S \in \{0,1\}^n, number\ of\ 1s\ in\ S\ is\ i\} \tag{4}$$

Note a sequence $S \in \{0,1\}^n$ as

$$S = \{a_1, a_2, \cdots, a_n\}. \tag{5}$$

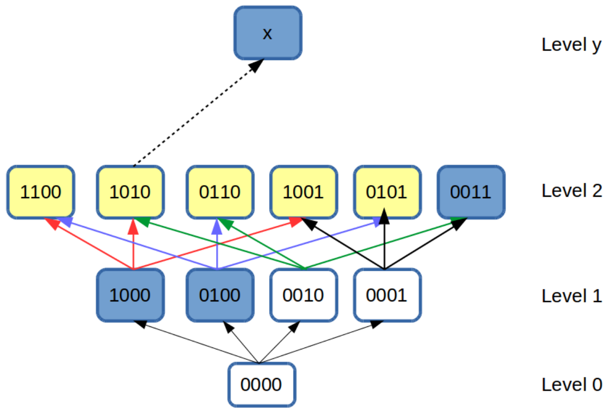
We will first prove that given an antichain A and two adjacent levels L_i and L_{i+1} in the Hasse diagram $(0 \le i < \lfloor \frac{n}{2} \rfloor)$, we can get an new antichain A' having the following properties:

- $L_i \cap A' = \emptyset$.
- $|L_{i+1} \cap A'| = |L_i \cap A| + |L_{i+1} \cap A|$.
- $L_j \cap A' = L_j \cap A, for\ j \neq i\ , i+1$.

That means we can move antichain elements in level L_i to level L_{i+1} without changing the rest antichain elements.
We can select a subset D from L_i and subset T from L_{i+1} so that

$$\begin{aligned} &D = L_i \cap A \\ &T = \{S | S \in L_{i+1},\ S\ is\ a\ direct\ succession\ of\ D\} \end{aligned} \tag{6}$$

Following is an example of D and T. Elements in antichain A is shown as blue boxes. $D = \{\{1000\},\{0100\}\}$. T is shown as yellow boxes.



Part of Hasse diagram (n=4)

First we will prove that if we delete D from antichain A and then add T to get A'. A' is still an antichain after moving operation.

Suppose that after this operation, there are new relations between elements in T and origin antichain x in level L_y, shown as the dot line in the picture. Since T is direct succession of D, there must be element in D that have relation with x, which means there are two comparable elements in the origin antichain A. It is against the definition of antichain, so A' is still an antichain after moving operation.

Then we will prove after the moving operation,

$$|A| < |A'| \tag{7}$$

Observe that the sum of in-degree of T is at least the sum of out-degree of D, because there are in-degrees provided by non-antichains nodes in level L_i.

$$indegree(T) \geq outdegree(D) \tag{8}$$

We can calculate in-degree and out-degree of an element $S_i \in L_i$. Out-degree of S_i is $n-i$ since out-degree equals to the number of 0s in S_i. In-degree of S_i is i since in-degree equals to the number of 1s in S_i.

$$\begin{aligned} outdegree(S_i) &= n - i \\ indegree(S_i) &= i \end{aligned} \tag{9}$$

Rewrite inequality (7) as

$$\begin{aligned}|T| \cdot indegree(S_{i+1}) &\geq |D| \cdot outdegree(S_i) \\ \frac{|T|}{|D|} &\geq \frac{outdegree(S_i)}{indegree(S_{i+1})} \\ \frac{|T|}{|D|} &\geq \frac{n-i}{i+1} = \frac{\binom{n}{i+1}}{\binom{n}{i}}\end{aligned} \tag{10}$$

If $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,

$$\frac{|T|}{|D|} \geq \frac{\binom{n}{i+1}}{\binom{n}{i}} > 1 \tag{11}$$

Therefore we prove that there are enough "space" in level L_{i+1} for moving antichain in level L_i to L_{i+1}.

So far we have proved that it is feasible to move antichain elements from level L_i to L_{i+1} and get a larger antichain if $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Notice that given an antichain configuration, we can always ajust the antichain to the middle level $L_{\left\lfloor \frac{n}{2} \right\rfloor}$ by adjusting two adjacent levels. So the size of largest antichain must be equal or less than the size of middle level $L_{\left\lfloor \frac{n}{2} \right\rfloor}$.

However, the middle level itself is an antichain. So the largest antichain is the middle level $L_{\left\lfloor \frac{n}{2} \right\rfloor}$.

□

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \ldots$. Here,$\epsilon$ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Proof. First we can proof $\{0,1\}^* \cong \mathbb{N}$:

Formally, we can define a function:

$$f_1 : \{0,1\}^* \to \mathbb{N}, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \{0,1\}^*, a_i^{(n)} = 0 or 1, i = 1, 2, \ldots, n$$
$$\to 10^n + \sum_{i=1}^{n} 10^{i-1} a_i^{(n)}$$

For example:$f_1(001010) = 1001010_{10}$.
Then we can define another function:

$$f_2 : \mathbb{N} \to \{0,1\}^*, decimal \quad number \to binary$$

For example:$f_2(16) = 10000$.So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N} \tag{1}$$

Secondly, we can proof $\{0,1\}^* \cong \mathbb{N}^*$:
Define a function:

$$f_3 : \{0,1\}^* \to \mathbb{N}^*, x \to x$$

Another function:

$$f_4 : \mathbb{N}^* \to \{0,1\}^*, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \mathbb{N}^*, a_i^{(n)} \in \mathbb{N}, i = 1, 2, \ldots, n$$
$$\to (\underbrace{0,0,\ldots,0}_{a_1^{(n)}}, 1, \underbrace{0,0,\ldots,0}_{a_2^{(n)}}, 1, \ldots, 1, \underbrace{0,0,\ldots,0}_{a_n^{(n)}}, 1)$$

For example, $f_4((2,3,1,4)) = 00100010100001$. So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N}^* \tag{2}$$

According (1) and (2), $\mathbb{N} \cong \mathbb{N}^*$ is obvious. □

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

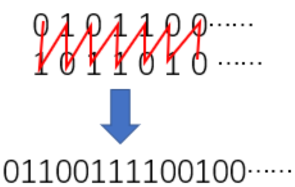
Proof. Obvious, there exits a function:

$$f_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}, x \to (x, 0000\ldots)$$

Then, we define a function:

$$f_2 : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (a_1a_2a_3\ldots, b_1b_2b_3\ldots) \to (a_1b_1a_2b_2a_3b_3\ldots)$$

Such as:

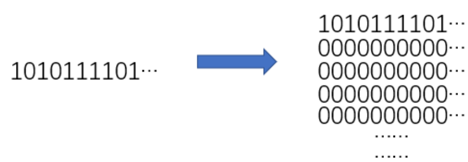


Therefore, we proof $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, and then $R \cong R \times R$. $\square$

Exercise 2.11. Consider $R^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \ldots)$ of real numbers. Show that $R \cong R^{\mathbb{N}}$. **Hint:** Again, use the fact that $R \cong \{0,1\}^{\mathbb{N}}$.

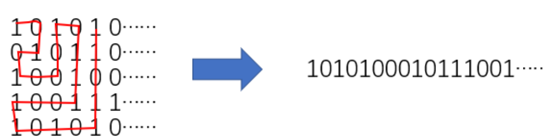
Proof. We only need to proof that $\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$. Firstly, we can know the following function easily:

$$f_1 : \{0,1\}^{\mathbb{N}} \to \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}, x \to (x, 00000\ldots, 00000\ldots, 00000\ldots, \ldots). \quad (12)$$



Then,define a complexer function:

$$f_2 : \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (x_1^1 x_2^1 x_3^1 \ldots, x_1^2 x_2^2 x_3^2 \ldots, x_1^3 x_2^3 x_3^3 \ldots, \ldots) \to (x_1^1 x_1^2 x_2^2 x_1^2 x_1^3 x_2^3 \ldots), x_i^j = 0 \; or \; 1 \quad (13)$$



Now, we can infer $\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ is true. Thus, $R \cong R^{\mathbb{N}}$. $\square$

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

$$(0, 0, 0, \dots)$$
$$(1, 0, 0, \dots)$$
$$(1, 1, 0, \dots)$$
$$(1, 1, 1, \dots)$$
$$\dots$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

$$(1, 0, 0, \dots)$$
$$(0, 1, 0, \dots)$$
$$(0, 0, 1, \dots)$$
$$\dots$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Since $\{0,1\}^{\mathbb{N}} \cong \mathbb{R}$, there is a bijection: $x \leftrightarrow \mathbf{t}$, $x \in \mathbb{R}, \mathbf{t} \in \{0,1\}^{\mathbb{N}}$. Let's consider $\mathbf{t_i}$.

$$\mathbf{t_i} = (a_1, a_2, \dots), a_k \in \{0,1\}, k \in \mathbb{N}$$

Define $\bar{\mathbf{t_i}} = (1 - a_1, 1 - a_2, \dots)$. Then construct $\hat{\mathbf{t_i}}$ as:

$$\hat{\mathbf{t_i}} = (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

Consider $\hat{\mathbf{t_i}}, \hat{\mathbf{t_j}}, \forall i, j \in \mathbb{N}, i \neq j$.

Case 1: If $\mathbf{t_i} \not\leq \mathbf{t_j}$, obviously, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Case 2: If $\mathbf{t_i} \leq \mathbf{t_j}$

$$\mathbf{t_i} = (a_1, a_2, \dots) \quad \bar{\mathbf{t_i}} = (1 - a_1, 1 - a_2, \dots)$$

$$\mathbf{t_j} = (b_1, b_2, \dots) \quad \bar{\mathbf{t_j}} = (1 - b_1, 1 - b_2, \dots)$$

According to the definition of $\mathbf{a} \leq \mathbf{b}$, we know that $a_k \leq b_k$. So, $\bar{\mathbf{t_i}} \geq \bar{\mathbf{t_j}}$.

Compare every bit of $\hat{\mathbf{t}}$.

$\hat{\mathbf{t}}$	1	2	3	4	$\dots$
$\hat{\mathbf{t_i}}$	a_1	$1 - a_1$	a_2	$1 - a_2$	$\dots$
$\hat{\mathbf{t_j}}$	b_1	$1 - b_1$	b_2	$1 - b_2$	$\dots$

Since $a_k \leq b_k$, $1 - a_k \geq 1 - b_k$.

And since $i \neq j$, $\mathbf{t_i}, \mathbf{t_j}$ are not the same $\mathbf{t}$, which means that $\exists \eta, a_\eta < b_\eta, 1 - a_\eta > 1 - b_\eta$. So, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Therefore, $\hat{\mathbf{t_1}} \quad \hat{\mathbf{t_2}} \quad \dots$ is an uncountable antichain in $\{0, 1\}^{\mathbb{N}}$.

****Exercise 2.15.** *Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.*

Proof. As we has learnt in our class, $\exists f : \mathbb{N} \leftrightarrow \mathbb{Q}$. So we can arrange $\mathbb{Q}$ in the sequence of $\mathbb{N}$, such as:

$\mathbb{N}$	1	2	3	4	5	$\dots$
$\mathbb{Q}$	f(1)	f(2)	f(3)	f(4)	f(5)	$\dots$

Construct a set $\mathbb{S}$,

$$\mathbb{S} = \{(i_1, i_2, \dots, i_k, \dots) \in \{0, 1\}^{\mathbb{N}} | i_k = \begin{cases} 1 & f(k) \leq x \\ 0 & f(k) > x \end{cases} \ (x \in \mathbb{R})\}$$

Because between any two real number x_1, x_2, there must be a $q \in \mathbb{Q}$, that obviously we can find it in the sequence of $\mathbb{Q}$, it well only be 1 in one of the sequence generated from x_1, x_2, so for any different x, a sequence is corresponding to it , so $g : x \to s \in \mathbb{S}$ is injective,and cardinality of $\mathbb{S}$ is larger or equal to $\mathbb{R}$ where x belongs to. What's more,

$$\forall x_1 < x_2, (x_1, x_2 \in \mathbb{R}), s_1 = g(x_1) = (i_1^1, i_2^1, i_3^1, \dots), s_2 = g(x_2) = (i_1^2, i_2^2, \dots),$$

	$f(k) \leq x_1$	$x_1 < f(k) \leq x_2$	$f(k) > x_2$
i_k^1	1	0	0
i_k^2	1	1	0

So $\forall x_1 < x_2, s_1 < s_2$, which means $\mathbb{S}$ is a chain. So $\mathbb{S}$ is an uncountable chain of $\{0,1\}^{\mathbb{N}}$ $\square$

Question:

1. We are wondering if there is an easier or another way to solve 2.14.
2. We are wondering how to illustrate the obvious conclusion in 2.8.