Mathematical Foundations of Computer Science

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Group Name: **All Right**

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V. A *partition* of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \dots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$. For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$ but $\{\{1\}, \{2, 3\}, \{1, 4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?

   4.
3. What is the size of the largest antichain?

   7.

2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2,5,4) \leq (2,6,6)$ but $(2,5,4) \not\leq (3,1,1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?

   The minimal element is $(0,0,0,\cdots,0)$.(There are n0s in the element.)

   No element is maximal.
2. Is there a minimum? A maximum?

   The minimum element is $(0,0,0,\cdots,0)$.(There are n0s in the element.)

   No element is maximum.
3. Does it have an infinite chain?

   Yes.

   There is an example: $\{(0,0,0,\cdots,0),(1,0,0,\cdots,0),(1,1,0,\cdots,0),\cdots,(1,1,1,\cdots,1)\}$
4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

   Yes. We consider an antichain like this:

   $$\{(1,0,0,\cdots,0),(0,1,0,\cdots,0),(0,0,1,\cdots,0),\cdots,(0,0,0,\cdots,1)\}$$

   For the k^{th} element, there is only one 1 in the k^{th} position, and other positions are all occupied by 0. And the antichain consists of these k elements.

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

Proof. Base case $n = 1$: Apparently, every two elements in set N_0^0 is comparable since there is only one dimension. If there exists an infinite subset $S \subseteq \mathbb{N}_0^0$, the subset S itself is an infinite chain. So, the theorem holds when $n = 1$.
Inductive hypothesis:

Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step:

Let $n = k + 1$. If there exists an infinite subset $S \subseteq \mathbb{N}_0^{k+1}$, note

$$\begin{aligned} S_1 &= \{(a_1, a_2, ..., a_k) \quad &|(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \\ S_2 &= \{a_{k+1} &|(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \end{aligned} \tag{1}$$

Since S is infinite, at least one of S_1, S_2 is infinite.

Suppose S_1 is infinite, according to inductive hypothesis, there is an infinite chain C_k for $S_1 \subseteq \mathbb{N}_0^k$.

$$\begin{aligned} C_k &= (A_1, A_2, \cdots), A_1 \leq A_2 \leq \cdots \\ A_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}) \end{aligned} \tag{2}$$

Now we construct an infinite chain C_{k+1} for $S \subseteq \mathbb{N}_0^{k+1}$. Take $b \in S_2$, we append every A_i with b to get B_i.

$$\begin{aligned} B_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}, b) \\ C_{k+1} &= (B_1, B_2, \cdots), B_1 \leq B_2 \leq \cdots \end{aligned} \tag{3}$$

So C_{k+1} is an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$.

Now suppose S_1 is finite and S_2 is infinite. Notice that S_2 itself is an infinite chain. We take $(a_1, a_2, ..., a_k) \in S_1$ and we can construct an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$ in a similar way.

So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. $\square$

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Proof. We proof it by contradiction. Suppose there is an infinite antichain which is also a subset of $\mathbb{N}_0^n$. But from Exercise 2.3, it is clear to us that every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain. So there is a contradiction. Consequently, $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. $\square$

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

$$\text{Maximum element:}(1,1,1,\cdots,1)$$
$$\text{Maximal element:}(1,1,1,\cdots,1)$$
$$\text{Minimum element:}(0,0,0,\cdots,0)$$
$$\text{Minimal element:}(0,0,0,\cdots,0)$$

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

One of the examples is as follows:

$$\{(0,0,0,\cdots,0),(1,0,0,\cdots,0),(1,1,0,\cdots,0),\cdots,(1,1,1,\cdots,1)\}$$

****Exercise 2.8.** *What is the largest antichain of $\{0,1\}^n$?*

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \ldots$. Here,$\epsilon$ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Proof. First we can proof $\{0,1\}^* \cong \mathbb{N}$:
Formally, we can define a function:

$$f_1 : \{0,1\}^* \to \mathbb{N}, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \{0,1\}^*, a_i^{(n)} = 0 or 1, i = 1,2,\ldots,n$$
$$\to 10^n + \sum_{i=1}^{n} 10^{i-1} a_i^{(n)}$$

For example:$f_1(001010) = 1001010_{10}$.
Then we can define another function:

$$f_2 : \mathbb{N} \to \{0,1\}^*, decimal \quad number \to binary$$

For example:$f_2(16) = 10000$.So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N} \tag{1}$$

Secondly, we can proof $\{0,1\}^* \cong \mathbb{N}^*$:
Define a function:

$$f_3 : \{0,1\}^* \to \mathbb{N}^*, x \to x$$

Another function:

$$f_4 : \mathbb{N}^* \to \{0,1\}^*, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots, a_n^{(n)}) \in \mathbb{N}^*, a_i^{(n)} \in \mathbb{N}, i = 1,2,\ldots,n$$
$$\to (a_1^{(n)}\%2, a_2^{(n)}\%2, a_3^{(n)}\%2, \ldots, a_n^{(n)}\%2), i = 1,2,\ldots,n$$

For example, $f_4((154, 3, 89, 23, 48)) = 01110$. So we get the conclusion:

$$\{0,1\}^* \cong \mathbb{N}^* \tag{2}$$

According (1) and (2), $\mathbb{N} \cong \mathbb{N}^*$ is obvious. □

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

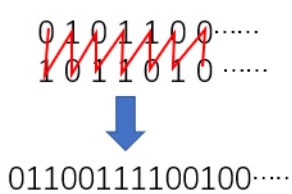
Proof. Obvious, there exits a function:

$$f_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}, x \to (x, 0000\ldots)$$

Then, we define a function:

$$f_2 : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (a_1a_2a_3\ldots, b_1b_2b_3\ldots) \to (a_1b_1a_2b_2a_3b_3\ldots)$$

Such as:



Therefore, we proof $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, and then $R \cong R \times R$. □

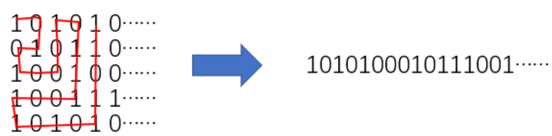
Exercise 2.11. Consider $R^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \dots)$ of real numbers. Show that $R \cong R^{\mathbb{N}}$. **Hint:** Again, use the fact that $R \cong \{0,1\}^{\mathbb{N}}$.

Proof. We only need to proof that $\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$. Firstly, we can know the following function easily:

$$f_1 : \{0,1\}^{\mathbb{N}} \to \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}, x \to (x, 00000\dots, 00000\dots, 00000\dots, \dots). \tag{4}$$

Then,define a complexer function:

$$f_2 : \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, (x_1^1 x_2^1 x_3^1 \dots, x_1^2 x_2^2 x_3^2 \dots, x_1^3 x_2^3 x_3^3 \dots, \dots) \to (x_1^1 x_1^2 x_2^2 x_2^1 x_1^3 x_2^3 \dots), x_i^j = 0 \text{ } or \text{ } 1 \tag{5}$$



Now, we can infer $\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ is true. Thus, $R \cong R^{\mathbb{N}}$. □

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

$$(0, 0, 0, \dots)$$

$$\begin{gathered}(1,0,0,\dots)\\(1,1,0,\dots)\\(1,1,1,\dots)\\\dots\end{gathered}$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

$$\begin{gathered}(1,0,0,\dots)\\(0,1,0,\dots)\\(0,0,1,\dots)\\\dots\end{gathered}$$

Since there are countably infinite bits in every element, we can construct countably infinite chain in $\{0,1\}^{\mathbb{N}}$ as showed above.

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Since $\{0,1\}^{\mathbb{N}} \cong \mathbb{R}$, there is a bijection: $x \leftrightarrow \mathbf{t}$, $x \in \mathbb{R}, \mathbf{t} \in \{0,1\}^{\mathbb{N}}$. Let's consider $\mathbf{t_i}$.

$$\mathbf{t_i} = (a_1, a_2, \dots), a_k \in \{0,1\}, k \in \mathbb{N}$$

Define $\bar{\mathbf{t_i}} = (1-a_1, 1-a_2, \dots)$. Then construct $\hat{\mathbf{t_i}}$ as:

$$\hat{\mathbf{t_i}} = (a_1, 1-a_1, a_2, 1-a_2, \dots)$$

Consider $\hat{\mathbf{t_i}}, \hat{\mathbf{t_j}}, \forall i,j \in \mathbb{N}, i \neq j$.

Case 1: If $\mathbf{t_i} \not\leq \mathbf{t_j}$, obviously, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Case 2: If $\mathbf{t_i} \leq \mathbf{t_j}$

$$\begin{aligned}\mathbf{t_i} &= (a_1, a_2, \dots) \quad \bar{\mathbf{t_i}} = (1-a_1, 1-a_2, \dots)\\ \mathbf{t_j} &= (b_1, b_2, \dots) \quad \bar{\mathbf{t_j}} = (1-b_1, 1-b_2, \dots)\end{aligned}$$

According to the definition of $\mathbf{a} \leq \mathbf{b}$, we know that $a_k \leq b_k$. So, $\bar{\mathbf{t_i}} \geq \bar{\mathbf{t_j}}$.

Compare every bit of $\hat{\mathbf{t}}$.

$\hat{\mathbf{t}}$	1	2	3	4	...
$\hat{\mathbf{t_i}}$	a_1	$1-a_1$	a_2	$1-a_2$	...
$\hat{\mathbf{t_j}}$	b_1	$1-b_1$	b_2	$1-b_2$	...

Since $a_k \leq b_k$, $1-a_k \geq 1-b_k$.

And since $i \neq j$, $\mathbf{t_i}, \mathbf{t_j}$ are not the same $\mathbf{t}$, which means that $\exists \eta, a_\eta < b_\eta, 1-a_\eta > 1-b_\eta$. So, $\hat{\mathbf{t_i}} \not\leq \hat{\mathbf{t_j}}$.

Therefore, $\hat{\mathbf{t_1}} \quad \hat{\mathbf{t_2}} \quad \dots$ is an uncountable antichain in $\{0,1\}^{\mathbb{N}}$.

****Exercise 2.15.** *Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.*

Question:

1. We are wondering if there is an easier or another way to solve 2.14.

2. We are wondering how to illustrate the obvious conclusion in 2.8.