Mathematical Foundations of Computer Science

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- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V. A *partition* of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \dots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$. For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$ but $\{\{1\}, \{2, 3\}, \{1, 4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2, 5, 4) \leq (2, 6, 6)$ but $(2, 5, 4) \not\leq (3, 1, 1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?

   The minimal element is $(0, 0, 0, \cdots, 0)$.(There are n0s in the element.)

   No element is maximal.

2. Is there a minimum? A maximum?

   The minimum element is $(0, 0, 0, \cdots, 0)$.(There are n0s in the element.)

   No element is maximum.

3. Does it have an infinite chain?

   Yes.

   There is an example: $\{(0, 0, 0, \cdots, 0), (1, 0, 0, \cdots, 0), (1, 1, 0, \cdots, 0), \cdots, (1, 1, 1, \cdots, 1)\}$

4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

   Yes. We consider an antichain like this:

   $$\{(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), (0, 0, 1, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1)\}$$

   For the k^{th} element, there is only one 1 in the k^{th} position, and other positions are all occupied by 0. And the antichain consists of these k elements.

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

Proof. Base case $n = 1$: Apparently, every two elements in set N_0^0 is comparable since there is only one dimension. If there exists an infinite subset $S \subseteq \mathbb{N}_0^0$, the subset S itself is an infinite chain. So, the theorem holds when $n = 1$.
Inductive hypothesis:
Suppose the theorem holds for all values of n up to some k, $k \geq 1$.
Inductive step:
Let $n = k + 1$. If there exists an infinite subset $S \subseteq \mathbb{N}_0^{k+1}$, note

$$\begin{aligned} S_1 &= \{(a_1, a_2, ..., a_k) \quad &&|(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \\ S_2 &= \{a_{k+1} &&|(a_1, a_2, ..., a_k, a_{k+1}) \in S\} \end{aligned} \tag{1}$$

Since S is infinite, at least one of S_1, S_2 is infinite.

Suppose S_1 is infinite, according to inductive hypothesis, there is an infinite chain C_k for $S_1 \subseteq \mathbb{N}_0^k$.

$$\begin{aligned} C_k &= (A_1, A_2, \cdots), A_1 \leq A_2 \leq \cdots \\ A_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}) \end{aligned} \tag{2}$$

Now we construct an infinite chain C_{k+1} for $S \subseteq \mathbb{N}_0^{k+1}$. Take $b \in S_2$, we append every A_i with b to get B_i.

$$\begin{aligned} B_i &= (a_{i1}, a_{i2}, \cdots, a_{ik}, b) \\ C_{k+1} &= (B_1, B_2, \cdots), B_1 \leq B_2 \leq \cdots \end{aligned} \tag{3}$$

So C_{k+1} is an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$.

Now suppose S_1 is finite and S_2 is infinite. Notice that S_2 itself is an infinite chain. We take $(a_1, a_2, ..., a_k) \in S_1$ and we can construct an infinite chain for $S \subseteq \mathbb{N}_0^{k+1}$ in a similar way.

So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$. □

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Proof. We proof it by contradiction. Suppose there is an infinite antichain which is also a subset of $\mathbb{N}_0^n$. But from Exercise 2.3, it is clear to us that every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain. So there is a contradiction. Consequently, $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. □

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

$$\begin{gathered}\text{Maximum element:}(1,1,1,\cdots,1)\\ \text{Maximal element:}(1,1,1,\cdots,1)\\ \text{Minimum element:}(0,0,0,\cdots,0)\\ \text{Minimal element:}(0,0,0,\cdots,0)\end{gathered}$$

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

One of the examples is as follows:

$$\{(0,0,0,\cdots,0),(1,0,0,\cdots,0),(1,1,0,\cdots,0),\cdots,(1,1,1,\cdots,1)\}$$

****Exercise 2.8.** *What is the largest antichain of* $\{0,1\}^n$*?*

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$. Here, ϵ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Exercise 2.11. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \dots)$ of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$.

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0,0,\dots)$ is the minimum element in this ordering and $(1,1,\dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

****Exercise 2.15.** *Find an uncountable chain in* $\{0,1\}^{\mathbb{N}}$. *That is, an antichain* A *with* $A \cong \mathbb{R}$.