Mathematical Foundations of Computer Science

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- Homework assignment published on Tuesday, 2018-03-13
- Submit questions and first solutions by Sunday, 2018-03-18, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-03-21
- Revise your solution and submit your final solution by Sunday, 2018-03-25 by email to dominik.scheder@gmail.com and the TAs.

3 Basic Counting

A function $[m] \to [n]$ is *monotone* if $f(1) \leq f(2) \leq \cdots \leq f(m)$. It is *strictly monotone* if $f(1) < f(2) < \cdots < f(m)$.

Exercise 3.1. Find and justify a closed formula for the number of strictly monotone functions from $[m]$ to $[n]$.

Exercise 3.2. Find and justify a closed formula for the number of monotone functions from $[m]$ to $[n]$.

Remark. By "closed" I mean something using expressions like $\times$, $+$, $\binom{n}{k}$, $n!$, but not $\sum$ or $\prod$. For example, $\binom{n}{k^2}$ is a closed formula but $\sum_{k=0}^{n} \binom{n}{k}$ is not.

Exercise 3.3. Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ for every $n \geq 0$ by finding a combinatorial interpretation.

Exercise 3.4. [From the textbook] Find a closed formula for $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$ and prove it combinatorially, i.e., by giving an interpretation.

Exercise 3.5. Let B_n be the number of partitions of the set $[n]$ (this is the same as the number of equivalence relations on $[n]$). This is called the Bell number, thus we denote it B_n. Prove that the following recursive formula for B_n is correct:

$$B_0 = 1$$
$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \ .$$

Exercise 3.6. Let P_n be the number of ways to write the natural number n as a sum $a_1 + a_2 + \cdots + a_k$ such that $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$. For example, 3 can be written as 3, $2+1$, and $1+1+1$, so $P_3 = 3$. Find a recursive formula for P_n.

Remark. The formula might not be as simple as the above for B_n. Be creative! Start by writing a simple recursive program that computes P_n.