

# Mathematical Foundations of Computer Science

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Group Name: **All Right**

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

## 2 Partial Orderings

### 2.1 Equivalence Relations as a Partial Ordering

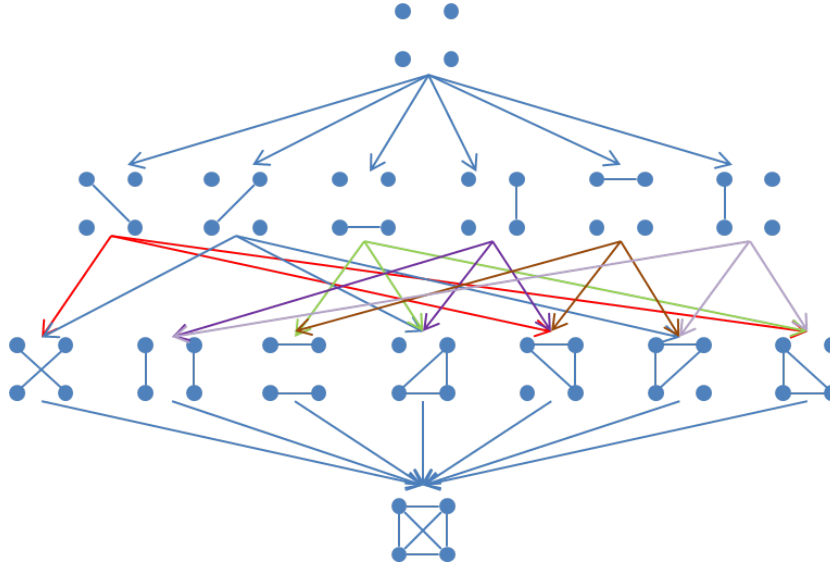
An equivalence relation  $R \subseteq V \times V$  is basically the same as a partition of  $V$ . A *partition* of  $V$  is a set  $\{V_1, \dots, V_k\}$  where (1)  $V_1 \cup \dots \cup V_k = V$  and (2) the  $V_i$  are pairwise disjoint, i.e.,  $V_i \cap V_j = \emptyset$  for  $1 \leq i < j \leq k$ . For example,  $\{\{1\}, \{2, 3\}, \{4\}\}$  is a partition of  $\{1, 2, 3, 4\}$  but  $\{\{1\}, \{2, 3\}, \{1, 4\}\}$  is not.

**Exercise 2.1.** Let  $E_4$  be the set of all equivalence relations on  $\{1, 2, 3, 4\}$ . Note that  $E_4$  is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.



2. What is the size of the largest chain?  
4.
3. What is the size of the largest antichain?  
7.

## 2.2 Chains and Antichains

Define the partially ordered set  $(\mathbb{N}_0^n, \leq)$  as follows:  $x \leq y$  if  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . For example,  $(2, 5, 4) \leq (2, 6, 6)$  but  $(2, 5, 4) \not\leq (3, 1, 1)$ .

**Exercise 2.2.** Consider the infinite partially ordered set  $(\mathbb{N}_0^n, \leq)$ .

1. Which elements are minimal? Which are maximal?  
The minimal element is  $(0, 0, 0, \dots, 0)$ . (There are  $n$  0s in the element.)  
No element is maximal.
2. Is there a minimum? A maximum?  
The minimum element is  $(0, 0, 0, \dots, 0)$ . (There are  $n$  0s in the element.)  
No element is maximum.

3. Does it have an infinite chain?

Yes. We consider an chain like this:

$$\{(1, 1, 1, \dots, 1), (2, 2, 2, \dots, 2), (3, 3, 3, \dots, 3), \dots, (k, k, \dots, k)\}$$

For every element in the chain, there are  $n$  elements in it.

4. Does it have arbitrarily large antichains? That is, can you find an antichain  $A$  of size  $|A| = k$  for every  $k \in \mathbb{N}$ ?

Yes. We consider an antichain like this:

$$\{(0, k-1, 0, \dots, 0), (1, k-2, 0, \dots, 0), (2, k-3, 0, \dots, 0), \dots, (k-1, 0, 0, \dots, 0)\}$$

For every element in the antichain, the first element in each element of the antichain is increasing from 0 to  $k-1$ , and similarly, the second element in each element of the antichain is decreasing from  $k-1$  to 0. As for the other elements, it can be any natural numbers and in the example, we use 0 to replace them because it is an easy way to express this.

**\*Exercise 2.3.** Does every infinite subset  $S \subseteq \mathbb{N}_0^n$  contain an infinite chain?

*Proof.* Base case  $n = 1$ : Apparently, every two elements in set  $\mathbb{N}_0^1$  is comparable since there is only one dimension. If there exists an infinite subset  $S \subseteq \mathbb{N}_0^1$ , the subset  $S$  itself is an infinite chain. So, the theorem holds when  $n = 1$ .

Inductive hypothesis:

Suppose the theorem holds for all values of  $n$  up to some  $k$ ,  $k \geq 1$ .

Inductive step:

Let  $n = k + 1$ . If there exists an infinite subset  $S \subseteq \mathbb{N}_0^{k+1}$ , note

$$\begin{aligned} S_1 &= \{(a_1, a_2, \dots, a_k) \mid (a_1, a_2, \dots, a_k, a_{k+1}) \in S\} \\ S_2 &= \{a_{k+1} \mid (a_1, a_2, \dots, a_k, a_{k+1}) \in S\} \end{aligned} \tag{1}$$

Since  $S$  is infinite, at least one of  $S_1, S_2$  is infinite.

Suppose  $S_1$  is infinite, according to inductive hypothesis, there is an infinite chain  $C_k$  for  $S_1 \subseteq \mathbb{N}_0^k$ .

$$\begin{aligned} C_k &= (A_1, A_2, \dots), A_1 \leq A_2 \leq \dots \\ A_i &= (a_{i1}, a_{i2}, \dots, a_{ik}) \end{aligned} \tag{2}$$

Now we construct an infinite chain  $C_{k+1}$  for  $S \subseteq \mathbb{N}_0^{k+1}$ . Take  $b \in S_2$ , we append every  $A_i$  with  $b$  to get  $B_i$ .

$$\begin{aligned} B_i &= (a_{i1}, a_{i2}, \dots, a_{ik}, b) \\ C_{k+1} &= (B_1, B_2, \dots), B_1 \leq B_2 \leq \dots \end{aligned} \tag{3}$$

So  $C_{k+1}$  is an infinite chain for  $S \subseteq \mathbb{N}_0^{k+1}$ .

Now suppose  $S_1$  is finite and  $S_2$  is infinite. Notice that  $S_2$  itself is an infinite chain. We take  $(a_1, a_2, \dots, a_k) \in S_1$  and we can construct an infinite chain for  $S \subseteq \mathbb{N}_0^{k+1}$  in a similar way.

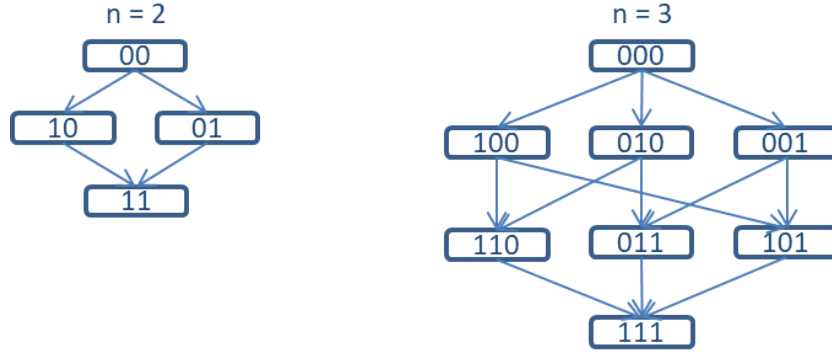
So, the theorem holds for  $n = k + 1$ . By the principle of mathematical induction, the theorem holds for all  $n \in \mathbb{N}$ .  $\square$

**Exercise 2.4.** Show that  $(\mathbb{N}_0^n, \leq)$  has no infinite antichain. **Hint.** Use the previous exercise.

*Proof.* We proof it by contradiction. Suppose there is an infinite antichain which is also a subset of  $\mathbb{N}_0^n$ . But from Exercise 2.3, it is clear to us that every infinite subset  $S \subseteq \mathbb{N}_0^n$  contain an infinite chain. So there is a contradiction. Consequently,  $(\mathbb{N}_0^n, \leq)$  has no infinite antichain.  $\square$

Consider the induced ordering on  $\{0, 1\}^n$ . That is, for  $x, y \in \{0, 1\}^n$  we have  $x \leq y$  if  $x_i \leq y_i$  for every coordinate  $i \in [n]$ .

**Exercise 2.5.** Draw the Hasse diagrams of  $(\{0, 1\}^n, \leq)$  for  $n = 2, 3$ .



**Exercise 2.6.** Determine the maximum, minimum, maximal, and minimal elements of  $\{0, 1\}^n$ .

Maximum element:  $(1, 1, 1, \dots, 1)$   
 Maximal element:  $(1, 1, 1, \dots, 1)$   
 Minimum element:  $(0, 0, 0, \dots, 0)$   
 Minimal element:  $(0, 0, 0, \dots, 0)$

**Exercise 2.7.** What is the longest chain of  $\{0, 1\}^n$ ?

One of the examples is as follows:

$$\{(0, 0, 0, \dots, 0), (1, 0, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, 1, \dots, 1)\}$$

**\*\*Exercise 2.8.** What is the largest antichain of  $\{0, 1\}^n$ ?

## 2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that  $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ , i.e., there is a bijection  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . From this, and by induction, it follows quite easily that  $\mathbb{N}^k \cong \mathbb{N}$  for every  $k$ .

**Exercise 2.9.** Consider  $\mathbb{N}^*$ , the set of all finite sequences of natural numbers, that is,  $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \dots$ . Here,  $\epsilon$  is the empty sequence. Show that  $\mathbb{N} \cong \mathbb{N}^*$  by defining a bijection  $\mathbb{N} \rightarrow \mathbb{N}^*$ .

*Proof.* First we can proof  $\{0, 1\}^* \cong \mathbb{N}$ :

Formally, we can define a function:

$$f_1 : \{0, 1\}^* \rightarrow \mathbb{N}, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots, a_n^{(n)}) \in \{0, 1\}^*, a_i^{(n)} = 0 \text{ or } 1, i = 1, 2, \dots, n$$

$$\rightarrow 10^n + \sum_{i=1}^n 10^{i-1} a_i^{(n)}$$

For example:  $f_1(001010) = 1001010_{10}$ .

Then we can define another function:

$$f_2 : \mathbb{N} \rightarrow \{0, 1\}^*, \text{decimal number} \rightarrow \text{binary}$$

For example:  $f_2(16) = 10000$ . So we get the conclusion:

$$\{0, 1\}^* \cong \mathbb{N} \tag{1}$$

Secondly, we can proof  $\{0, 1\}^* \cong \mathbb{N}^*$ :

Define a function:

$$f_3 : \{0, 1\}^* \rightarrow \mathbb{N}^*, x \rightarrow x$$

Another function:

$$f_4 : \mathbb{N}^* \rightarrow \{0, 1\}^*, \forall a = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots, a_n^{(n)}) \in \mathbb{N}^*, a_i^{(n)} \in \mathbb{N}, i = 1, 2, \dots, n$$

$$\rightarrow (\underbrace{0, 0, \dots, 0}_{a_1^{(n)}}, 1, \underbrace{0, 0, \dots, 0}_{a_2^{(n)}}, 1, \dots, 1, \underbrace{0, 0, \dots, 0}_{a_n^{(n)}}, 1)$$

For example,  $f_4((2, 3, 1, 4)) = 00100010100001$ . So we get the conclusion:

$$\{0, 1\}^* \cong \mathbb{N}^* \quad (2)$$

According (1) and (2),  $\mathbb{N} \cong \mathbb{N}^*$  is obvious.  $\square$

**Exercise 2.10.** Show that  $R \cong R \times R$ . **Hint:** Use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$  and thus show that  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ .

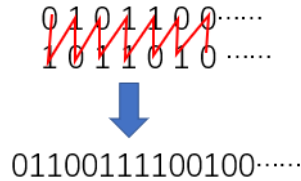
*Proof.* Obvious, there exists a function:

$$f_1 : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}, x \rightarrow (x, 0000\dots)$$

Then, we define a function:

$$f_2 : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}, (a_1 a_2 a_3 \dots, b_1 b_2 b_3 \dots) \rightarrow (a_1 b_1 a_2 b_2 a_3 b_3 \dots)$$

Such as:



Therefore, we proof  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ , and then  $R \cong R \times R$ .  $\square$

**Exercise 2.11.** Consider  $R^{\mathbb{N}}$ , the set of all infinite sequences  $(r_1, r_2, r_3, \dots)$  of real numbers. Show that  $R \cong R^{\mathbb{N}}$ . **Hint:** Again, use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$ .

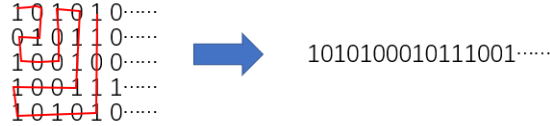
*Proof.* We only need to proof that  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$ . Firstly, we can know the following function easily:

$$f_1 : \{0, 1\}^{\mathbb{N}} \rightarrow (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}, x \rightarrow (x, 00000 \dots, 00000 \dots, 00000 \dots, \dots). \quad (4)$$



Then, define a complexer function:

$$f_2 : (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}, (x_1^1 x_2^1 x_3^1 \dots, x_1^2 x_2^2 x_3^2 \dots, x_1^3 x_2^3 x_3^3 \dots, \dots) \rightarrow (x_1^1 x_1^2 x_1^3 x_1^4 \dots, x_2^1 x_2^2 x_2^3 x_2^4 \dots, x_3^1 x_3^2 x_3^3 x_3^4 \dots, \dots), x_i^j = 0 \text{ or } 1 \quad (5)$$



Now, we can infer  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$  is true. Thus,  $R \cong R^{\mathbb{N}}$ . □

Next, let us view  $\{0, 1\}^{\mathbb{N}}$  as a partial ordering: given two elements  $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbb{N}}$ , that is, sequences  $\mathbf{a} = (a_1, a_2, \dots)$  and  $\mathbf{b} = (b_1, b_2, \dots)$ , we define  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ . Clearly,  $(0, 0, \dots)$  is the minimum element in this ordering and  $(1, 1, \dots)$  the maximum.

**Exercise 2.12.** Give a countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$ . Remember that a set  $A$  is countably infinite if  $A \cong \mathbb{N}$ .

$(0, 0, 0, \dots)$

$(1, 0, 0, \dots)$

$(1, 1, 0, \dots)$

$(1, 1, 1, \dots)$

...

Since there are countably infinite bits in every element, we can construct countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$  as showed above.

**Exercise 2.13.** Find a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ .

$(1, 0, 0, \dots)$

$(0, 1, 0, \dots)$

$(0, 0, 1, \dots)$

...

Since there are countably infinite bits in every element, we can construct countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$  as showed above.

**Exercise 2.14.** Find an uncountable antichain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .

Since  $\{0, 1\}^{\mathbb{N}} \cong \mathbb{R}$ , there is a bijection:  $x \leftrightarrow \mathbf{t}$ ,  $x \in \mathbb{R}, \mathbf{t} \in \{0, 1\}^{\mathbb{N}}$ . Let's consider  $\mathbf{t}_i$ .

$$\mathbf{t}_i = (a_1, a_2, \dots), a_k \in \{0, 1\}, k \in \mathbb{N}$$

Define  $\bar{\mathbf{t}}_i = (1 - a_1, 1 - a_2, \dots)$ . Then construct  $\hat{\mathbf{t}}_i$  as:

$$\hat{\mathbf{t}}_i = (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

Consider  $\hat{\mathbf{t}}_i, \hat{\mathbf{t}}_j, \forall i, j \in \mathbb{N}, i \neq j$ .

**Case 1:** If  $\mathbf{t}_i \not\leq \mathbf{t}_j$ , obviously,  $\hat{\mathbf{t}}_i \not\leq \hat{\mathbf{t}}_j$ .

**Case 2:** If  $\mathbf{t}_i \leq \mathbf{t}_j$

$$\mathbf{t}_i = (a_1, a_2, \dots) \quad \bar{\mathbf{t}}_i = (1 - a_1, 1 - a_2, \dots)$$

$$\mathbf{t}_j = (b_1, b_2, \dots) \quad \bar{\mathbf{t}}_j = (1 - b_1, 1 - b_2, \dots)$$

According to the definition of  $\mathbf{a} \leq \mathbf{b}$ , we know that  $a_k \leq b_k$ . So,  $\bar{\mathbf{t}}_i \geq \bar{\mathbf{t}}_j$ . Compare every bit of  $\hat{\mathbf{t}}$ .

$\hat{\mathbf{t}}$	1	2	3	4	...
$\hat{\mathbf{t}}_i$	$a_1$	$1 - a_1$	$a_2$	$1 - a_2$	...
$\hat{\mathbf{t}}_j$	$b_1$	$1 - b_1$	$b_2$	$1 - b_2$	...



Since  $a_k \leq b_k$ ,  $1 - a_k \geq 1 - b_k$ .

And since  $i \neq j$ ,  $\mathbf{t}_i, \mathbf{t}_j$  are not the same  $\mathbf{t}$ , which means that  $\exists \eta, a_\eta < b_\eta, 1 - a_\eta > 1 - b_\eta$ . So,  $\hat{\mathbf{t}}_i \not\leq \hat{\mathbf{t}}_j$ .

Therefore,  $\hat{\mathbf{t}}_1 \hat{\mathbf{t}}_2 \dots$  is an uncountable antichain in  $\{0, 1\}^{\mathbb{N}}$ .

**\*\*Exercise 2.15.** Find an uncountable chain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .

*Proof.* As we have learnt in our class,  $\exists f : \mathbb{N} \leftrightarrow \mathbb{Q}$ . So we can arrange  $\mathbb{Q}$  in the sequence of  $\mathbb{N}$ , such as:

$$\begin{array}{ccccccc} \mathbb{N} & 1 & 2 & 3 & 4 & 5 & \dots \\ \mathbb{Q} & f(1) & f(2) & f(3) & f(4) & f(5) & \dots \end{array}$$

Construct a set  $\mathbb{S}$ ,

$$\mathbb{S} = \{(i_1, i_2, \dots, i_k, \dots) \in \{0, 1\}^{\mathbb{N}} \mid i_k = \begin{cases} 1 & f(k) \leq x \\ 0 & f(k) > x \end{cases} (x \in \mathbb{R})\}$$

Because between any two real numbers  $x_1, x_2$ , there must be a  $q \in \mathbb{Q}$ , that obviously we can find it in the sequence of  $\mathbb{Q}$ , it will only be 1 in one of the sequence generated from  $x_1, x_2$ , so for any different  $x$ , a sequence is corresponding to it, so  $g : x \rightarrow s \in \mathbb{S}$  is injective, and cardinality of  $\mathbb{S}$  is larger or equal to  $\mathbb{R}$  where  $x$  belongs to. What's more,

$$\forall x_1 < x_2, (x_1, x_2 \in \mathbb{R}), s_1 = g(x_1) = (i_1^1, i_2^1, i_3^1, \dots), s_2 = g(x_2) = (i_1^2, i_2^2, \dots),$$

$$\begin{array}{c} i_k^1 \\ i_k^2 \end{array} \left| \begin{array}{c} f(k) \leq x_1 \\ 1 \\ 1 \end{array} \right| \begin{array}{c} x_1 < f(k) \leq x_2 \\ 0 \\ 1 \end{array} \left| \begin{array}{c} f(k) > x_2 \\ 0 \\ 0 \end{array} \right.$$

So  $\forall x_1 < x_2, s_1 < s_2$ , which means  $\mathbb{S}$  is a chain. So  $\mathbb{S}$  is an uncountable chain of  $\{0, 1\}^{\mathbb{N}}$   $\square$

**Question:**

1. We are wondering if there is an easier or another way to solve 2.14.
2. We are wondering how to illustrate the obvious conclusion in 2.8.