Mathematical Foundations
of
Computer Science

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- Homework assignment published on Tuesday, 2018-03-13
- Submit questions and first solutions by Sunday, 2018-03-18, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-03-21
- Revise your solution and submit your final solution by Sunday, 2018-03-25 by email to dominik.scheder@gmail.com and the TAs.

3 Basic Counting

A function $[m] \to [n]$ is *monotone* if $f(1) \leq f(2) \leq \cdots \leq f(m)$. It is *strictly monotone* if $f(1) < f(2) < \cdots < f(m)$.

Exercise 3.1. Find and justify a closed formula for the number of strictly monotone functions from $[m]$ to $[n]$.

Solution. We choose m elements in $[n]$, and match each one with the element in $[n]$ with the same order. Note that if $n < m$, there is no feasible matching like that, and the solution is 0.

$$N_{m,n} = \binom{n}{m}$$

Exercise 3.2. Find and justify a closed formula for the number of monotone functions from $[m]$ to $[n]$.

Solution. We choose 1 to m elements in $[n]$, and then we need to allocate elements in $[m]$ to them. If we choose i elements in $[n]$, $[m]$ needs to be divided into i parts. Note that elements in each part is continuous in order, so we can insert $i-1$ plates into intervals of $[m]$, where there are $m-1$ intervals. Then we can use the formula $\binom{n}{m} = \binom{n}{n-m}$. At last we can use a combinatorial interpretation to get the final formula for the number of monotone functions from $[m]$ to $[n]$.

$$N_{m,n} = \sum_{i=0}^{m}(\binom{n}{i} \times \binom{m-1}{i-1}) = \sum_{i=1}^{m}(\binom{n}{i} \times \binom{m-1}{m-i}) = \binom{m+n-1}{m}$$

Remark. By "closed" I mean something using expressions like $\times$, $+$, $\binom{n}{k}$, $n!$, but not $\sum$ or $\prod$. For example, $\binom{n}{k^2}$ is a closed formula but $\sum_{k=0}^{n}\binom{n}{k}$ is not.

Exercise 3.3. Prove that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ for every $n \geq 0$ by finding a combinatorial interpretation.

Solution. Obviously $\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k} \times \binom{n}{n-k}$. Assume there are two sets each of which has n elements, and we need to choose n elements from them. One way is to choose k $(0 \leq k \leq n)$ elements from one set and then choose $n-k$ elements from the other, which is $\sum_{k=0}^{n}\binom{n}{k}^2$. Another way is to see them as a whole which has $2n$ elements, and then we choose n elements from it, which is $\binom{2n}{n}$.

Exercise 3.4. [From the textbook] Find a closed formula for $\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}$ and prove it combinatorially, i.e., by giving an interpretation.

Exercise 3.5. Let B_n be the number of partitions of the set $[n]$ (this is the same as the number of equivalence relations on $[n]$). This is called the Bell number, thus we denote it B_n. Prove that the following recursive formula for B_n is correct:

$$B_0 = 1$$
$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k \ .$$

Exercise 3.6. Let P_n be the number of ways to write the natural number n as a sum $a_1 + a_2 + \cdots + a_k$ such that $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$. For example, 3 can be written as 3, $2+1$, and $1+1+1$, so $P_3 = 3$. Find a recursive formula for P_n.

Remark. The formula might not be as simple as the above for B_n. Be creative! Start by writing a simple recursive program that computes P_n.