Set Cover Reduction

Dimitri Rusin

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1 Intro

A Set Cover instance is a tuple (S, U, k) where

- S is a finite set,
- U is a nonempty subset of the set of subsets of S,
- $k \in \mathbb{N}_0$.

The question is: Is there $U' \subseteq U$ s.t. $|U'| \leq k$ and

$$\bigcup U' = \bigcup_{u \in U'} u = S.$$

A Maximum Leaves Spanning Tree instance is a tuple (V, E, l) where

- V is a finite set,
- $E \subseteq V \times V$ is a symmetric relation (note that a set $\{v, w\}$ can be represented by two pairs $(v, w), (w, v)$ and vice versa),
- $l \in \mathbb{N}_0$.

The question is: Is there $E' \subseteq E$ s.t.

- for all $v, w \in V$ there is exactly one sequence $p_0 \dots p_m \in V^*$ with $p_0 = v$ and $p_m = w$ and $p_i p_{i+1} \in E'$ for all $i \in [0, m-1]$, and
- the number $|\{v \in V \mid \deg_{E'}(v) = 1\}|$ of leaves is at least l.

2 Reduction

We want to reduce Set Cover to the Maximum Leaves Spanning Tree problem. That is, for all instances of Set Cover we construct an instance of the Maximum Leaves Spanning Tree problem such that the two instances are equivalent, in the sense that either both answer their respective question with `Yes` or both with `No`.

Let (S, U, k) be a Set Cover instance. Let $n := |S|, m := |U|$. We construct the Maximum Leaves Spanning Tree instance in the following way:

1. We construct `V`: $V := U \cup \{\emptyset, r\} \cup S$. Hereby r is a new node.

2. We construct `E`:
   (a) *We connect everything from U and the empty set to r*: $E := \{(r,u) \mid u \in U \text{ or } u = \emptyset\}$.
   (b) *We connect S to it*: For all $u \in U$, $E := E \cup \{u\} \times u$.
   (c) *We update E with the symmetric closure of E*, that is for all $vw \in E$, we do $E := E \cup \{wv\}$.
3. We construct `l`: $l := n + m - k + 1$.

The construction is illustrated in (1).

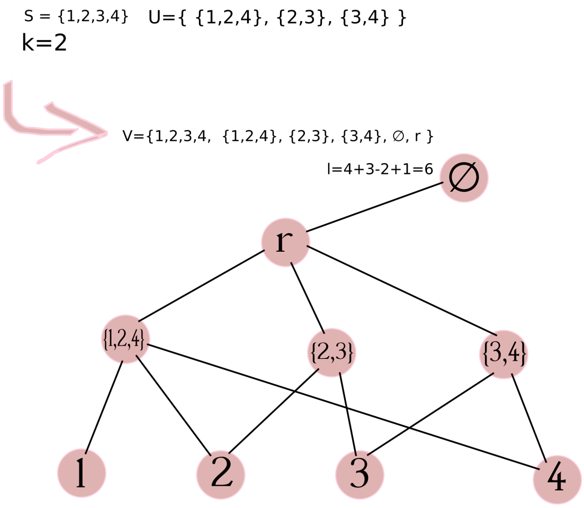

Figure 1: Conversion of the instance

Let E' be any solution to the Maximum Leaves Spanning Tree instance (V, E, l). Then we construct U' as follows: Just put in U' all elements of U whose degree in E' is at least 2. This construction can be seen in (2).

Given E', we have constructed U'. Why is U' a solution, too?

- $\bigcup U' = S$: Let $s \in S$ be s.t. $s \notin \bigcup U'$. Since E' is connected, there must be a path from s to r. But this path must go over some $u \in U$ and then u will be an internal node in that path and thus has degree at least 2. Hence $s \in u \in U'$.

- $|U'| \leq k$: There at exactly $n + m + 2$ nodes in the tree. We know, there are at least $n + m - k + 1$ leaves. So there are at most $(n + m + 2) - (n + m - k + 1) = k + 1$ inner nodes. But since $|U| > 0$ and $\emptyset \in V$, the node r has degree at least 2 so is an inner node. This leaves us with at most k inner nodes other than r. Every $u' \in U'$ is an inner node unequal r by definition, so $|U'| \leq k$.

Let U' be a solution to the Set Cover instance (S, U, k), that is the above question is answered with `Yes`. Then we construct E' as follows:

1. *Add U'*: $E' := \{r\} \times U' \cup \{\emptyset\}$.

2. *Add S*: $E' := \bigcup\{\{u'\} \times u' \mid u' \in U'\}$.

3. *Remove some edges to make it a tree*: For all $s \in S$, arbitrarily choose $u's \in E'$ and then for all $us \in E', u \neq u'$, remove us from E'.

4. Update E' to become symmetric as in step $(2c)$ of the construction of the Maximum Leaves Spanning Tree instance.

This construction is illustrated in (2).

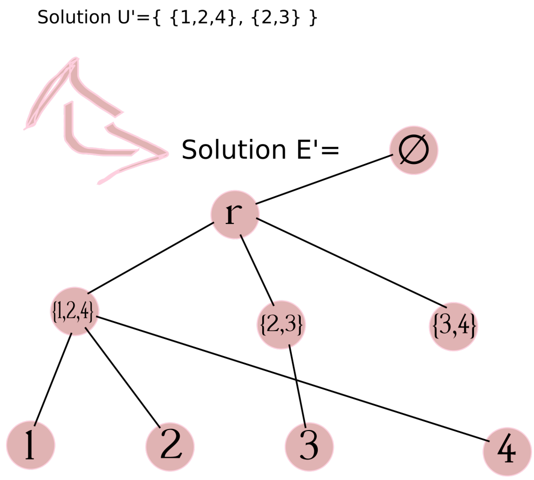

Figure 2: Conversion of the solution

Given U', we have constructed E'. Why is E' a solution, too?

In E' all elements in S are leaves and in addition all elements in U which are not in U' and also $\emptyset$. Since $|U'| \leq k$, we have $|U \setminus U'| \geq m - k$, so this makes up for at least $n + m - k + 1 = l$ leaves.

This concludes the proof.