Design and Analysis of Algorithms

L25: Kruskal's Algorithm
Minimum Cost Spanning Tree

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Resources

- Text book 2: Sec 4.1, 4.3, 4.4
- Text book 1: Sec 9.1-5.4 - Levitin
- R1: Introduction to Algorithms
    - Cormen et al.
- MIT Open CourseWare
    - https://ocw.mit.edu/courses/civil-and-environmental-engineering/1-204-computer-algorithms-in-systems-engineering-spring-2010/lecture-notes/MIT1_204S10_lec11.pdf

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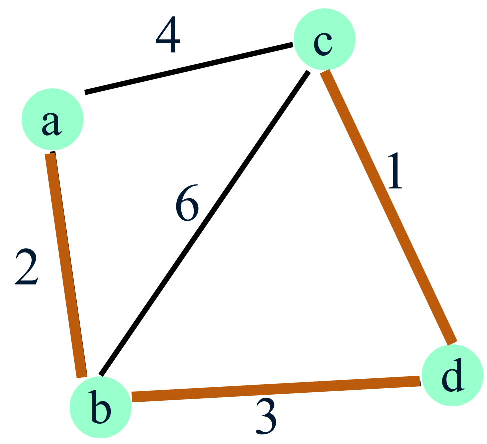
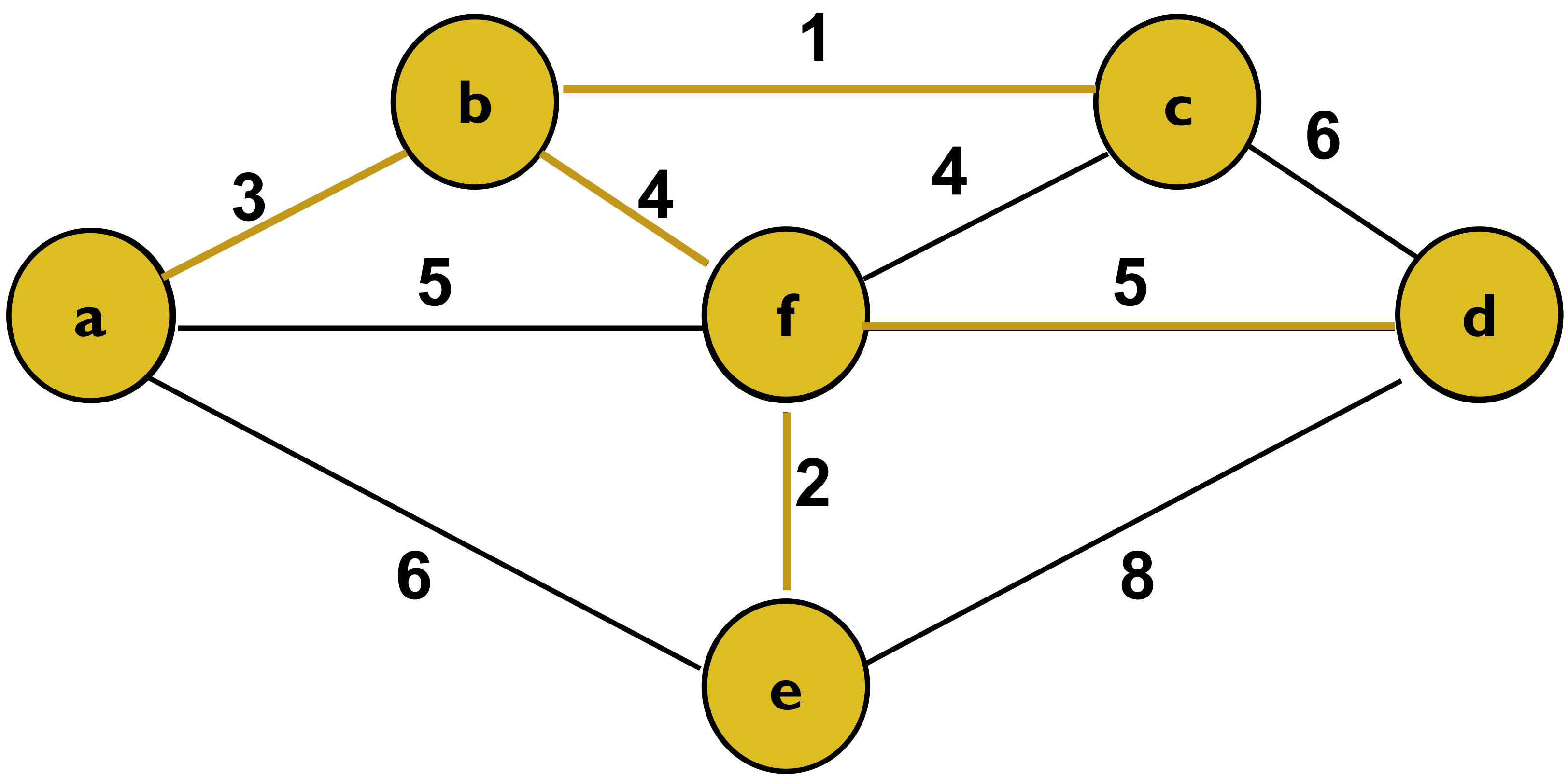
Kruskal's Algorithm

- Sort the edges in nondecreasing order of lengths
- "Grow" tree one edge at a time
  - Produce MST through a series of expanding forests $F_1, F_2, \ldots, F_{n-1}$
- On each iteration,
  - Consider the next edge on the sorted list
    - If this edge creates a cycle,
      - Skip the edge
    - Else
      - Add the edge to spanning tree

Kruskal's Algorithm



Example 2: Kruskal Algorithm



- Q: Construct an MST using Kruskal algo

Analysis of Kruskal's Algorithm

- Algorithm looks easier to implement
  - Just sort the edge in non-increasing order of weights and consider one edge at a time
- Cycle checking is harder to implement
  - A cycle occurs when the added edge connects vertices in the same connected component
- Using Union-Find algorithm to merge the connected components
- Time complexity:
  - `O(m*lg m|)`, where `m` is number of edges.
  - The time spent is mostly on sorting.

Implementation of Kruskal's Algorithm

```
Algo Kruskal(G)
// i/p: A weighted connected graph G= ⟨V,E⟩
// o/p: Et, the set of edges composing an MST of G
sort E in non-decreasing order of edge weights i.e.
  w(e_{i_1}) ≤ w(e_{i_2}) ≤...≤ w(e_{i_m})
E_T ← Ø; edgecount ← 0; k ← 0
while edgecount < |V| - 1 do
    k ← k+1
    if E_T U {e_{i_k}} is acyclic
        E_T ← E_T U {e_{i_k}}
        edgecount ← edgecount+1
// end while
return E_T
```

Union-Find Approach

- Using set based Union-Find approach
  - It is almost `O(n)`
- Union-Find approach
- Consider the set of `n` elements which are known.
- All these elements put in an array and their id can be the array index i.e.
  - $a[i] = x_i$ # i^{th} element is x_i.
- Elements are divided into groups (sets)
  - Initially, each element is a group by itself.
- Two kinds of operations:
  - Find the group to which element belongs
  - Merge the two groups

Time Complexity Analysis

- Sorting the edges

  $O(|E| \lg |E|)$

- While loop : $|E|$ times
  - Checking for cycle formation
    - Use Union Find approach for an edge

      $\lg^* n$

- Time complexity for cycle checking for all edges
  - $O(|E| \lg^* |V|)$

- Total time complexity

  $O(|E| \lg |E|) + O(|E| \lg^* |V|)$

  $= O(|E| \lg |E|)$

Union-Find Approach

- Two operations given below are performed in arbitrary order
  - *Find*(i): return the group id containing element xi.
  - *Union*(A, B): Combine the (set) group A with (set) group B to form a new group.
    - Give a unique name to this group. All elements of this new group will have this group id.
    - This could be one of earlier groups as well i.e.
      - The new names should conflict with other names.
- Goal: Design an efficient data structure that will support any sequence of these two operations.

Union-Find Approach

- Approach 1: Quick Find
  - Keep ***Find***(`i`) efficient. Since all elements are accessible at the i^{th} index in array,
    - This can take `O(1)` time. Essentially, a trivial operation.
  - ***Union***(`A,B`) is expected to take more time.
    - Either change the id of all elements of A to that of B or vice versa.
    - Typically, take the smaller set and change group identity of these elements to that of larger set.
- Time complexity: $O(n\log_2 n)$
  - Each time an element's group is changed, group size at least doubles.

Union-Find Approach: Improved

- Approach 2: Quick Union
  - Make *Union*(`A,B`) efficient at the cost of *Find*(`i`).
    - Make *Union* operation takes constant time and improve upon the time taken by *Find*.
    - Use the indirect addressing for union to make it in `O(1)` time.
    - Each entry in the array has following parts
      - Identity of element i.e. group id or value.
      - Pointer which is initially `Nil`.
      - Number of elements in teh group
    - Union(A,B) is performed by making the pointer of B to point to A.
    - After several union operations, data structures becomes like a tree.

Union-Find Approach: Improved

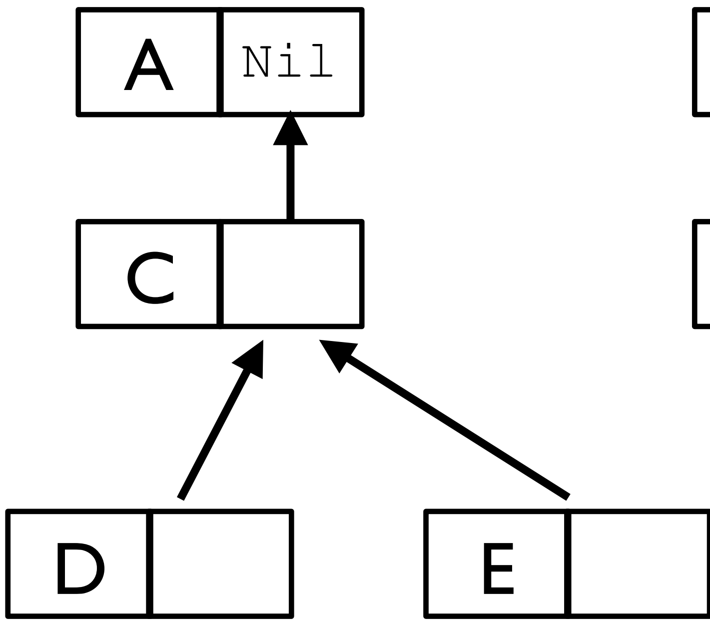
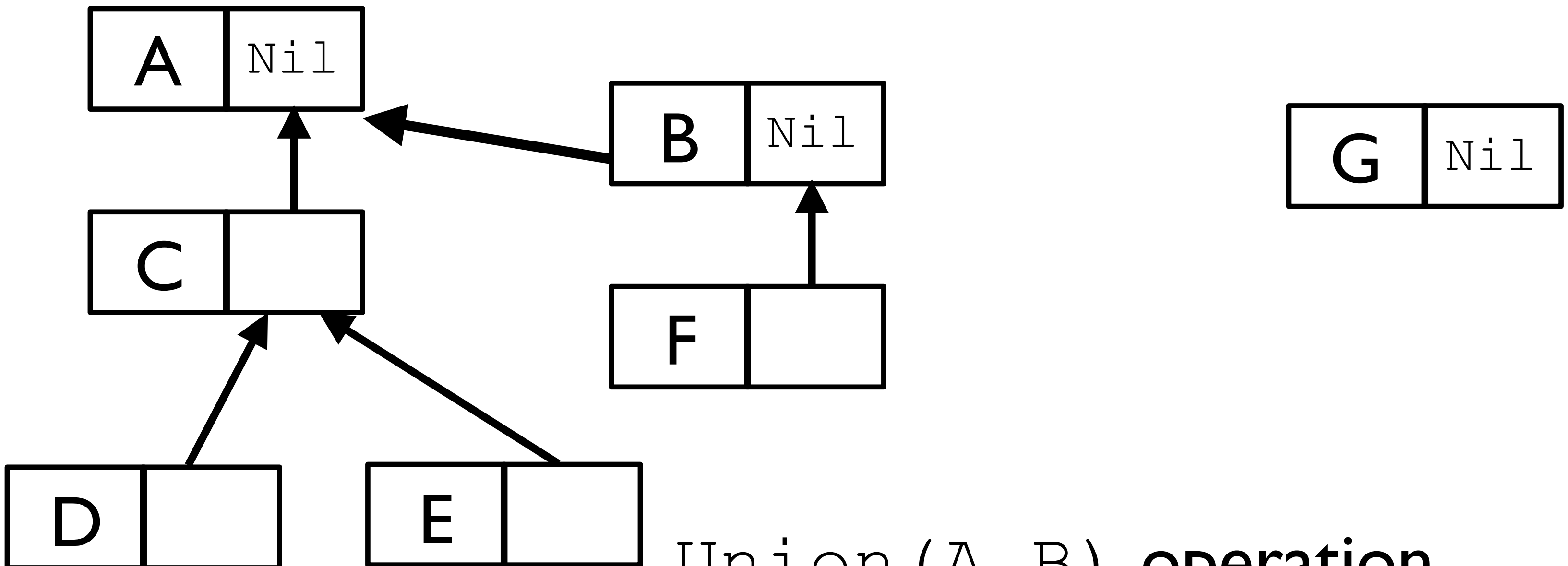


Fig: Representation for Union Find problem

- The element at root of the Tree is the identity of group.
- To find the group of an element, follow the path till the root of the tree.

Union-Find Approach: Improved



`Union(A,B)` operation

- Take the tree with smaller number of nodes to point to root of the tree of larger number of nodes.
- This requires storing the number of elements in the tree at the root as well.
- On Union operation, update the pointer of smaller tree and count at the larger tree.
  - Break the tie arbitrarily

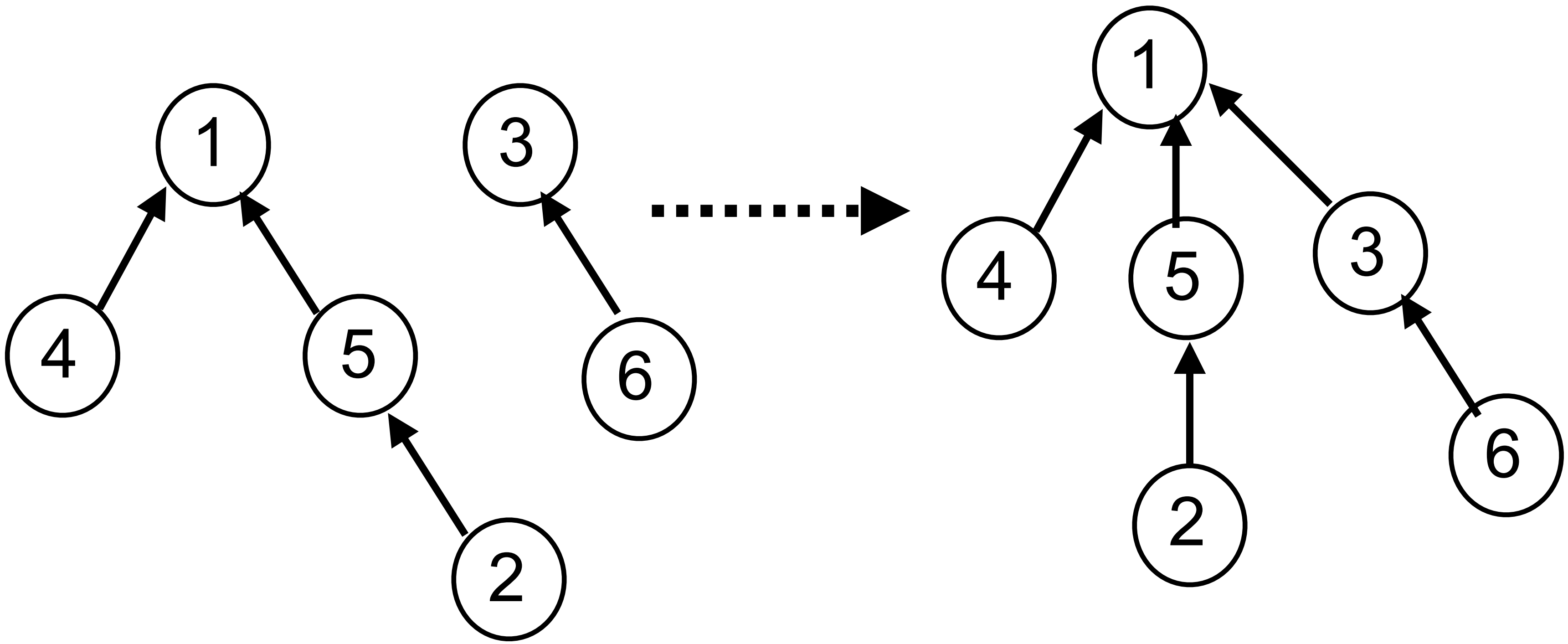
Union-Find Approach: Improved

- Basic idea: Balance and collapse the tree
  - Union operation still takes `O(1)` time
    - Changing the pointer `O(1)` time
    - Updating the count `O(1)` time
  - Thus, `O(1) + O(1) = O(1)`
    - i.e. Union operation takes `O(1)` time

Union-Find Approach: Quick Union

- Theorem: When balancing is used, the tree of height h will contain at least 2^h nodes.
- Proof outline
  - First union operation results in tree of height 1 with two elements.
  - Consider A is of height h_A and B is of height h_B.
    - Let A is larger tree. Thus, on merging B, root of B points to root of A.
    - If $h_A > h_B$, then A's height remains h_A i.e. unchanged
    - Otherwise, height of tree becomes h_B+1
    - Thus, with increase in height of 1, the size (number of nodes) of tree has at least doubled.
- Thus, time taken for a *Find(*X) operation is $O(\log_2 n)$

Quick Union of 2 Trees



a: Forest representation of two sets used by quick union

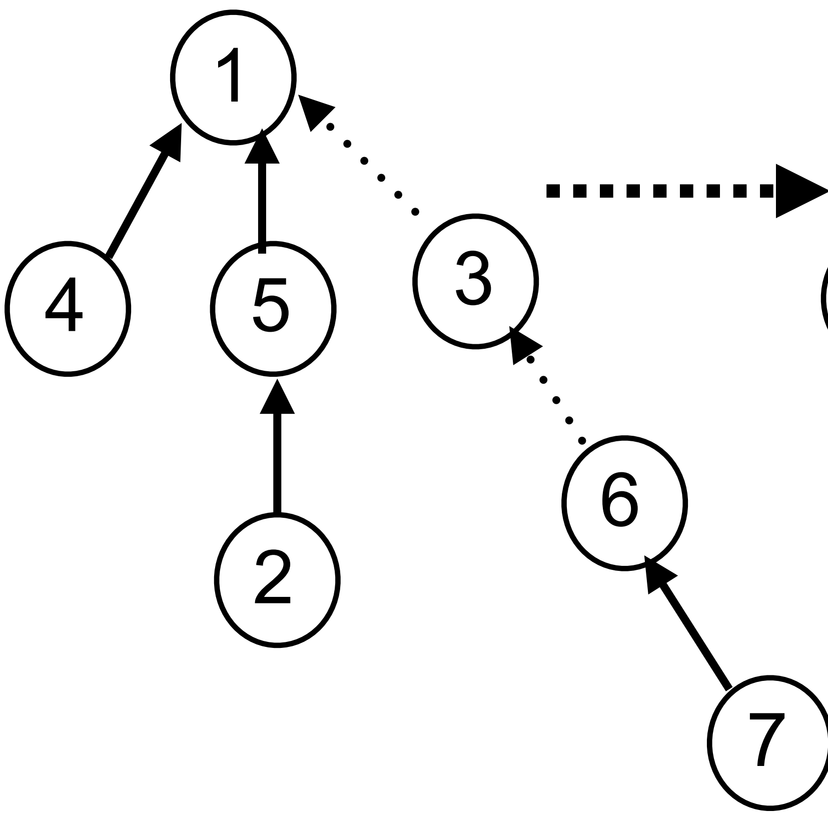
b: Result of quick union

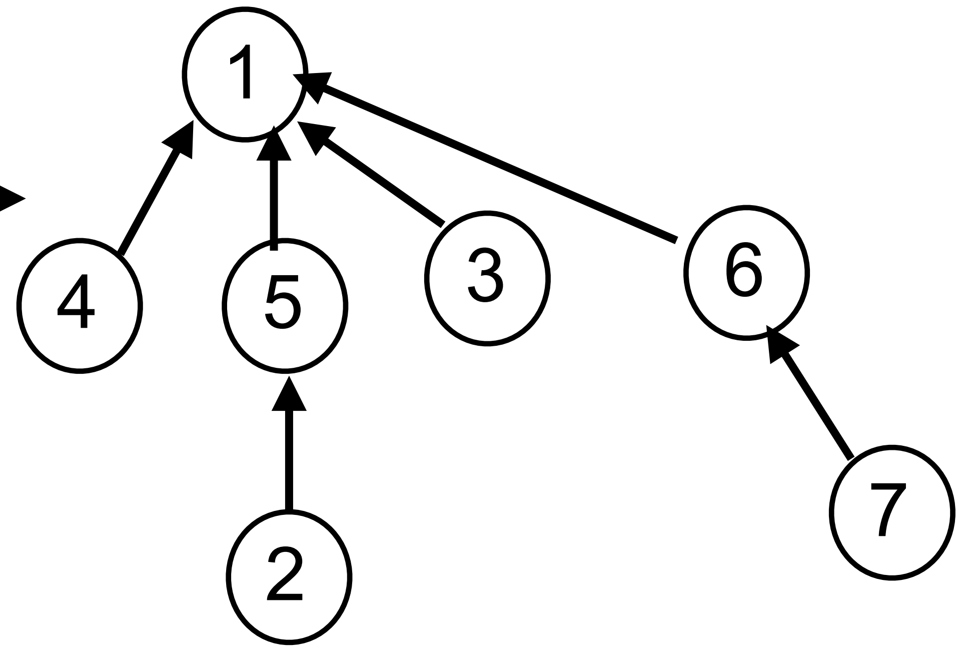
Union-Find Approach: Quick Union

- Further improvement
- Any time we do a *Find* operation, change the pointers of all the nodes in the path to directly point to the root of the tree.
  - This is called **path compression**.
- Traversing the path takes only double the number of steps, and thus
  - Time complexity of *Find* remains the same.
- Time complexity with path compression for `m` operations is given by

  `O(mlog*n)`, where `log*n` is iterated logarithm function

Quick Union of 2 Trees





b: Result of path compression
All nodes in the path from 6
to root 1 point to root 1

a: Representation of Tree
before finding node 6

Iterated Logarithm

- Iterated logarithm function is defined as

$\log^* n = 1 + \log^*(\lceil \log_2 n \rceil)$

$\log^* 2 = 1$ (Given)

$\log^* 4 = \log^* 2^2$

$= 1 + \log^*(\lceil \log_2 2^2 \rceil)$

$= 1 + \log^* 2 = 1 + 1 = 2$

$\log^* 16 = \log^* 2^4$

$= 1 + \log^*(\lceil \log_2 2^4 \rceil)$

$= 1 + \log^* 4 = 1 + 2 = 3$

$\log^* 65536 = \log^* 2^{16}$

$= 1 + \log^*(\lceil \log_2 2^{16} \rceil)$

$= 1 + \log^* 16 = 1 + 3 = 4$

$\log^* 2^{65536} = 1 + \log^* 65536 = 5$ (very large n)

Summary

- Kruskal Algo
  - Sort the edges in non-decreasing order of weights
  - take one edge at a time and check for cycle
  - Cycle check is done by using Union-Find algo
  - Time complexity: $O(|E| \lg |E|)$