

Design and Analysis of Algorithms

L15: Strassen Multiplication

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Resources

- Text book 2: Sec 3.8 - Horowitz
- Text book 1: Sec 4.5 - Levitin

Multiplication Example

- Multiply 2 numbers (of 2 digits) e.g. 98, 76

$$A = 98 = 10 * 9 + 8$$

$$B = 76 = 10 * 7 + 6$$

$$\begin{aligned} C = A * B &= 10^2 (9 * 7) + 10 (8 * 7 + 9 * 6) + 8 * 6 \\ &= 6300 + 10 (56 + 54) + 48 = 7448 \end{aligned}$$

– Multiplications : 4, additions : 3

- Alternatively

$$\begin{aligned} C &= 10^2 (9 * 7) + 8 * 6 \\ &\quad + 10 ((9 + 8) * (7 + 6) - \underline{9 * 7} - \underline{8 * 6}) \\ &= 10^2 * 9 * 7 + 10 * (8 * 7 + 9 * 6) + 8 * 6 = 7448 \end{aligned}$$

– Multiplications : 3, Additions : 6

Multiplication Example

- Multiply 2 numbers (of 2 digits)

- $A = a_2a_1, B = b_2b_1$

- $C = A * B$

- $= 100c_3 + 10c_2 + c_1$

where

$$c_3 = a_2b_2, \quad c_2 = (a_2b_1 + a_1b_2), \quad c_1 = a_1b_1$$

- Multiplications : 4 Additions : 3

- Alternatively

$$c_3 = a_2b_2,$$

$$c_1 = a_1b_1$$

$$c_2 = (a_2 + a_1) * (b_2 + b_1) - c_3 - c_1$$

- Multiplications : 3 Additions : 6

Multiplication Example

- Multiply 2 numbers (of 4 digits) e.g. 9876, 5432

$$A = 98 * 10^2 + 76, \quad B = 54 * 10^2 + 32$$

$$C = A * B = 10^4 (98 * 54)$$

$$+ 10^2 (98 * 32 + 76 * 54)$$

$$+ 76 * 32$$

– Multiplications: 4, additions: 3

- Alternatively

$$C = 10^4 (98 * 54 + 76 * 32) +$$

$$10^2 ((98 + 76) * (54 + 32) - 98 * 54 - 76 * 32))$$

– Multiplications: 3, additions: 6

Large Numbers Multiplication

- Problem:
Given two large numbers with N digits, Multiply these in efficient way
- Solution : traditional way (high school mathematics)

$$\begin{array}{r}
 a_n \ a_{n-1} \ \dots \ a_2 \ a_1 \\
 b_n \ b_{n-1} \ \dots \ b_2 \ b_1 \\
 \hline
 C_{1n+1} \ C_{1n} C_{1n-1} \ \dots C_{12} \ C_{11} \\
 C_{2n+1} \ C_{2n} C_{1n-1} \dots C_{22} \ C_{21} \\
 : \\
 C_{nn+1} C_{nn} C_{nn-1} \dots C_{n2} C_{n1} \\
 \hline
 \end{array}$$

- Efficiency : Multiplications: n^2 , Additions: $2n$
- Complexity analysis: $\Theta(n^2)$

Large Numbers Multiplication

- Given two large numbers A, B with n digits,
 - multiply these in efficient way
- Let $A = A_1A_2$, and $B=B_1B_2$,
 - where A and B are n-digit numbers, and
 - A_1, A_2, B_1 , and B_2 are $(n/2)$ -digit numbers, then

$$A*B = A_1*B_1*10^n + (A_1*B_2 + A_2*B_1) 10^{n/2} + A_2*B_2$$

- Efficiency : Multiplications: 4 each of $n/2$
- Recurrence relation

$$\begin{aligned}T(n) &= 4T(n/2) + O(n) = 2^2T(n/2) + n \\&= 2^2 [4T(n/4) + O(n/2)] + n \\&= 2^4T(n/2^2) + 2^2(n/2) + n = 2^4T(n/2^2) + 2n + n \\&= \Theta(n^2)\end{aligned}$$

Large Numbers Multiplication

- Let $A=A_1A_2$, and $B=B_1B_2$, are n -digit numbers, and
– A_1, A_2, B_1 , and B_2 are $(n/2)$ -digit numbers, then

$$A * B = 10^n * A_1 * B_1 + A_2 * B_2 \\ + 10^{n/2} ((A_1 + A_2) * (B_1 + B_2) - \underline{A_1 * B_1} - \underline{A_2 * B_2})$$

- Recurrence relation using 3 multiplications

$$\begin{aligned} T(n) &= 3T(n/2) + O(n) = 3^1 T(n/2) + cn \\ &= 3^1 [3T(n/4) + cn/2] + cn \\ &= 3^2 T(n/2^2) + c [3^1 n/2^1 + n] \\ &= \dots \\ &= 3^k T(n/2^k) + c [3^{k-1} n/2^{k-1} + \dots + n] \\ &= \Theta(3^{\log_2 n}) = \Theta(n^{\log_2 3}) \\ &\approx \Theta(n^{1.585}) \end{aligned}$$

Matrix Multiplication

- Conventional matrix multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

where the element c_{ij} is computed as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$$

- computations required for c_{ij}
 - Multiplications: n
 - additions: n
- Total computations required for matrix multiplication:
 - $2n^3$ i.e.
 - $\Theta(n^3)$

Matrix Multiplication

- Conventional matrix multiplication

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} * \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$
$$= \begin{bmatrix} A_1B_1+A_2B_3 & A_1B_2+A_2B_4 \\ A_3B_1+A_4B_3 & A_3B_2+A_4B_4 \end{bmatrix}$$

- Multiplications: 8 ($=2^3$), Additions: 4
- Recurrence relation:
$$T(n) = 8T(n/2) + 4(n/2) = 2^3T(n/2) + 2^2(n/2)$$
$$= \Theta(n^3)$$
- Regular matrix multiplication takes n^3 computations

Strassen's Matrix Multiplication

- Strassen discovered that product of two matrices can be computed as follows

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
$$= \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}$$

- $M_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$
- $M_2 = (A_{21} + A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12} - B_{22})$
- $M_4 = A_{22} * (B_{21} - B_{11})$
- $M_5 = (A_{11} + A_{12}) * B_{22}$
- $M_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$
- $M_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

Strassen's Matrix Multiplication

- $M_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$
- $M_2 = (A_{21} + A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12} - B_{22})$
- $M_4 = A_{22} * (B_{21} - B_{11})$
- $M_5 = (A_{11} + A_{12}) * B_{22}$
- $M_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$
- $M_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}$$

$$\begin{aligned} C_{11} &= M_1 + M_4 - M_5 + M_7 = (A_{11} + A_{22}) * (B_{11} + B_{22}) + A_{22} * (B_{21} - B_{11}) \\ &\quad - (A_{11} + A_{12}) * B_{22} + (A_{12} - A_{22}) * (B_{21} + B_{22}) \\ &= \cancel{A_{11}B_{11}} + \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} \\ &\quad - \cancel{A_{22}B_{11}} - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} \\ &\quad - \cancel{A_{22}B_{21}} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{22}} \\ &= A_{11}B_{11} + A_{12}B_{21} \end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$
- $M_2 = (A_{21} + A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12} - B_{22})$
- $M_4 = A_{22} * (B_{21} - B_{11})$
- $M_5 = (A_{11} + A_{12}) * B_{22}$
- $M_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$
- $M_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}$$

$$\begin{aligned} C_{12} &= M_3 + M_5 = A_{11} * (B_{12} - B_{22}) + (A_{11} + A_{12}) * B_{22} \\ &= A_{11}B_{12} - \cancel{A_{11}B_{22}} + \cancel{A_{11}B_{22}} + A_{12}B_{22} \\ &= A_{11}B_{12} + A_{12}B_{22} \end{aligned}$$

$$\begin{aligned} C_{21} &= M_2 + M_4 = (A_{21} + A_{22}) * B_{11} + A_{22} * (B_{21} - B_{11}) \\ &= A_{21}B_{11} + \cancel{A_{22}B_{11}} + A_{22}B_{21} - \cancel{A_{22}B_{11}} \\ &= A_{21}B_{11} + A_{22}B_{21} \end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$
- $M_2 = (A_{21} + A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12} - B_{22})$
- $M_4 = A_{22} * (B_{21} - B_{11})$
- $M_5 = (A_{11} + A_{12}) * B_{22}$
- $M_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$
- $M_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}$$

$$\begin{aligned} C_{22} &= M_1 + M_3 - M_2 + M_6 = (A_{11} + A_{22}) * (B_{11} + B_{22}) + A_{11} * (B_{12} - B_{22}) \\ &\quad - (A_{21} + A_{22}) * B_{11} + (A_{21} - A_{11}) * (B_{11} + B_{12}) \\ &= \cancel{A_{11}B_{11}} + \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{22}} + A_{22}B_{22} + \cancel{A_{11}B_{12}} - \cancel{A_{11}B_{22}} \\ &\quad - \cancel{A_{21}B_{11}} - \cancel{A_{22}B_{11}} + \cancel{A_{21}B_{11}} - \cancel{A_{11}B_{11}} + A_{21}B_{12} - \cancel{A_{11}B_{12}} \\ &= A_{22}B_{22} + A_{21}B_{12} \\ &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$
- $M_2 = (A_{21} + A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12} - B_{22})$
- $M_4 = A_{22} * (B_{21} - B_{11})$
- $M_5 = (A_{11} + A_{12}) * B_{22}$
- $M_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$
- $M_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}$$

- **Count number of Multiplications and Additions**

- **Multiplications : 7, Additions : 18**

- **Recurrence equation**

$$T(n) = 7T(n/2) + 18(n/2); \quad T(1) = 1$$

$$= \Theta(7^{\log_2 n})$$

$$= \Theta(n^{\log_2 7})$$

$$= \Theta(n^{2.807}) \text{ vs. } \Theta(n^3) \text{ of brute force}$$

- **There exist algos with better efficiency, but more complexity**

Summary

- Large number multiplication
- Matrix Multiplication