Design and Analysis of Algorithms

L26: Dijkstra's Algorithm
Single Source Shortest Path
+
Bellman-Ford Algorithm

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Resources

- Text book 1: Sec 9.1-5.4 - Levitin
- R1: Introduction to Algorithms
    - Cormen et al.

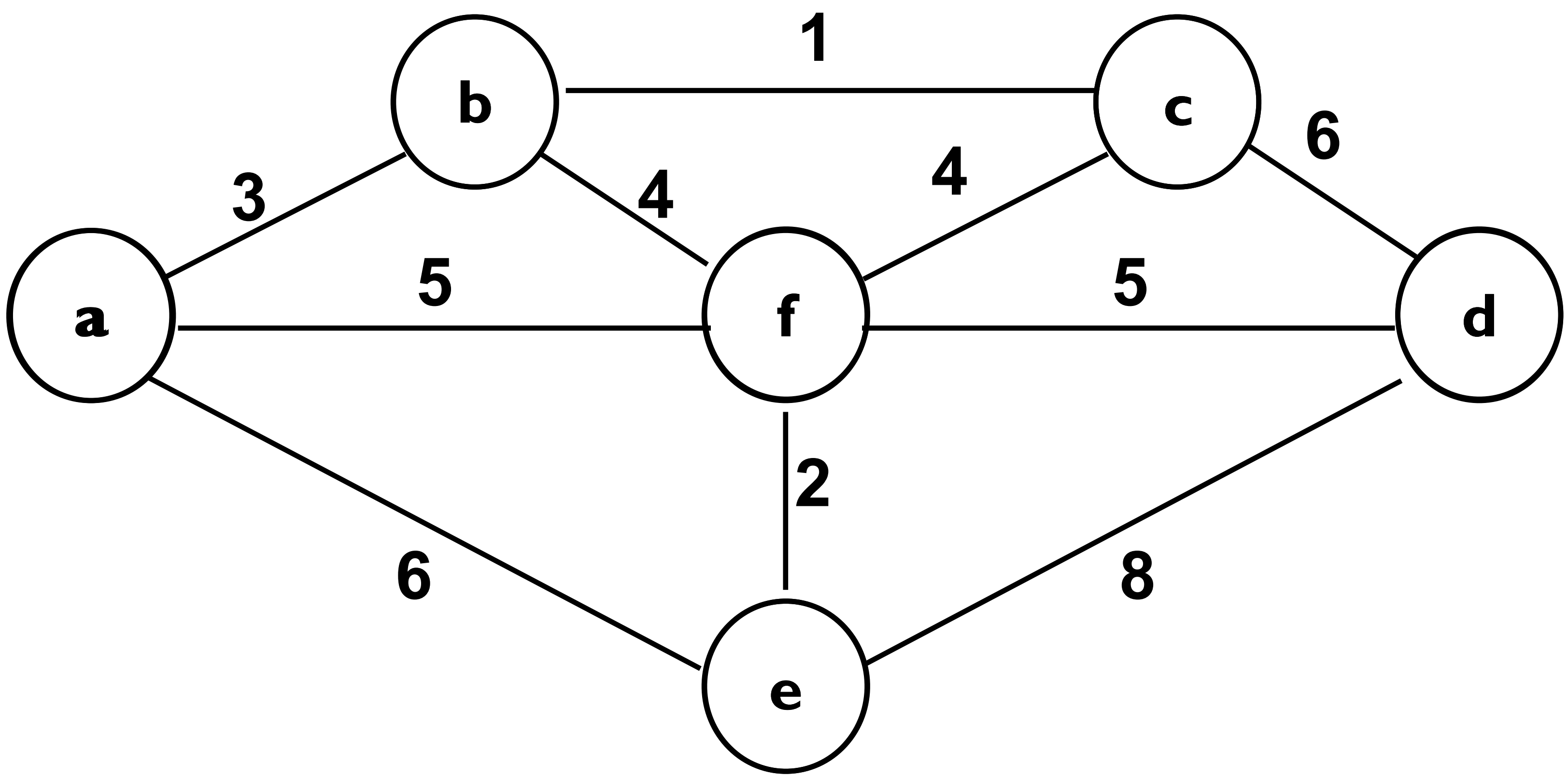
Single Source Shortest Path

- Applications
  - Supplying deliveries from a factory to various godowns
    - Minimum time/cost
  - KSIT: Moving from quadrangle to your class rooms
    - Minimum time taken

Single Source Shortest Path

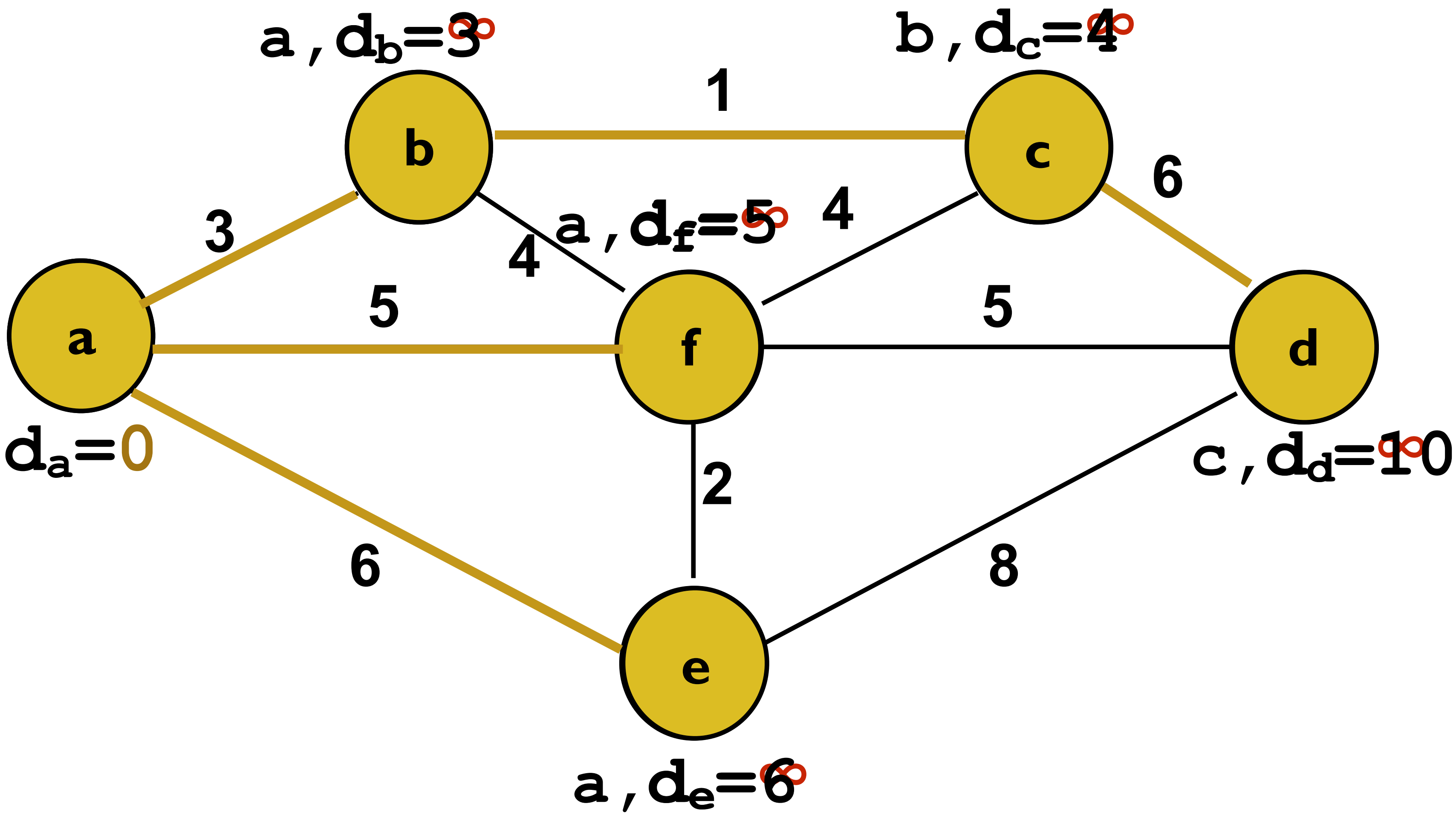
- Goal: Given a weighted connected (directed) graph G, find shortest paths from source vertex s to each of the other vertices
- Dijkstra's algorithm
  - Similar to Prim's algorithm for MST
  - Computes numerical labels differently
  - Among vertices not in the tree,
    - Find the vertex v with the smallest sum $d_v + w(u,v)$, where
  - $u \in V$ whose shortest path found in previous iteration
  - d_v is the length of shortest path from s to v
  - $w(u,v)$ is the weight of edge $u \rightarrow v$

Example: Dijkstra's Algorithm



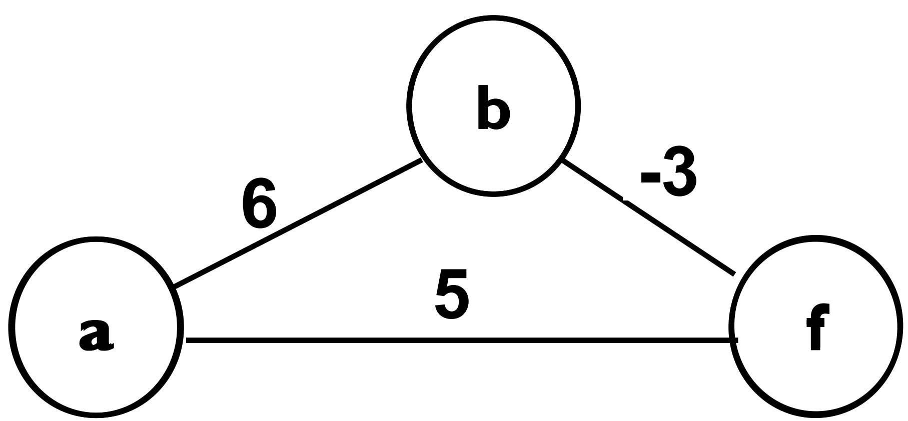
- Q: Construct an SSSP using Dijkstra's algo starting from vertex `a`

Example: Dijkstra's Algorithm



Notes on Dijkstra's Algorithm

- Proof of correctness:
  - Using induction
- Works with graph with +ve weights only
  - Build a counter example with -ve weight where Dijkstra's algorithm does not work
- Works for both directed and undirected graphs



Algorithm: Dijkstra's Algorithm

```
Algo Dijkstra(G,s)
```

// i/p: a weighted connected graph G=(V,E), and src s

// all edges are non-negative weights

//o/p: Length dv of a shortest path from s to v.

// along with it predecessor vertex from v to s.

Initialize(Q) // priority queue of vertices is empty initially

for each vertex $v \in V$, do

$d_v \leftarrow \infty$; $p_v \leftarrow$ Null;

Insert(Q, v, d_v) // initialize vertex priority in priority Q

$d_s \leftarrow 0$;

Decrease (Q, s, d_s)

$p_s \leftarrow$ Null;

$V_T \leftarrow \emptyset$

Algorithm: Dijkstra's Algorithm…

```
Algo Dijkstra(G,s)...
    for i=0 to |V|-1 do
        u = DeleteMin(Q)  //time implemementation based
        VT= VT={u}
        for every vertex w∈V-VT adjacent to w, do
            if du+weight(u,w)< dw, then
                dw←du+weight(u,w)
                pw←u
                Decrease(Q,w,dw) //time implementation based
            fi
        end //for w∈V-VT
    end //for i=0
end //algo
```

Analysis: Dijkstra's Algorithm…

- Implementation using Adjacency matrix
  - priority Q using unsorted array
  - Outer for loop (`i=0 to |V|-1`): $O(|V|)$ times
  - DeleteMin takes $O(|V|)$ times
    - Total time for all vertices: $O(|V|^2)$
  - Decrease(Q, w, d_w) takes $O(1)$ time
    - Total time for all vertices: $O(|E|)$
  - Time Complexity: $O(|V|^2)$

Analysis: Dijkstra's Algorithm

- Implementation using Adjacency List
  - priority Q using Heap
  - Outer for loop (`i=0 to |V|-1`): `O(|V|)` times
  - DeleteMin takes `O(lg|V|)` times
    - Total time for all vertices: `O(|V|lg|V|)`
  - Decrease(`Q`, `w`, d_w) takes `O(lg|V|)` time
    - Total time for all vertices: `O(|E|lg|V|)`
  - Time Complexity: `O(|E|lg|V|)`

Questions

- Q1: what adjustments if any need to be made in Dijkstra's algorithm to solve the single-source shortest-paths problem for directed weighted graphs.
- Ans:
  - Do we need any changes? Just follow the directed edges.

Questions

- Q2: Find a shortest path between two given vertices of a weighted graph or digraph. (This variation is called the single-pair shortest-path problem.)
- Ans:
  - Start from one vertex as source
  - Iterate the for loop till you find 2nd vertex.

Questions

- Q3: Find the shortest paths to a given vertex from each other vertex of a weighted graph. (This variation is called the single destination shortest-paths problem.)
- Ans: Undirected graph
  - Start from the destination vertex as source
  - Find the shortest path from this to all other vertices
  - Reverse the path
- Ans: directed graph
  - Reverse the direction of all edges.
  - Start from the destination vertex as source
  - Find the shortest path from this src to all other vertices
  - Reverse the path

Questions

- Q4: Solve the single-source shortest-path problem in a graph with non-negative numbers assigned to its vertices (and the length of a path defined as the sum of the vertex numbers on the path).
- Hint:
  - The weight of the edge is sum of non-negative numbers assigned to vertices of the corresponding edge.

Summary

- Dijkstra's algorithm
  - Keeps shortest length for each vertex from source `s`
  - Keep predecessor with each vertex towards `s`
  - Different from Prim's algorithm
    - Dijkstra: Chooses vertex with min shortest length
    - Prim: chooses edges with minimum weight.