Design and Analysis of Algorithms

L32: Multi-Stage Graphs
Dynamic Programming

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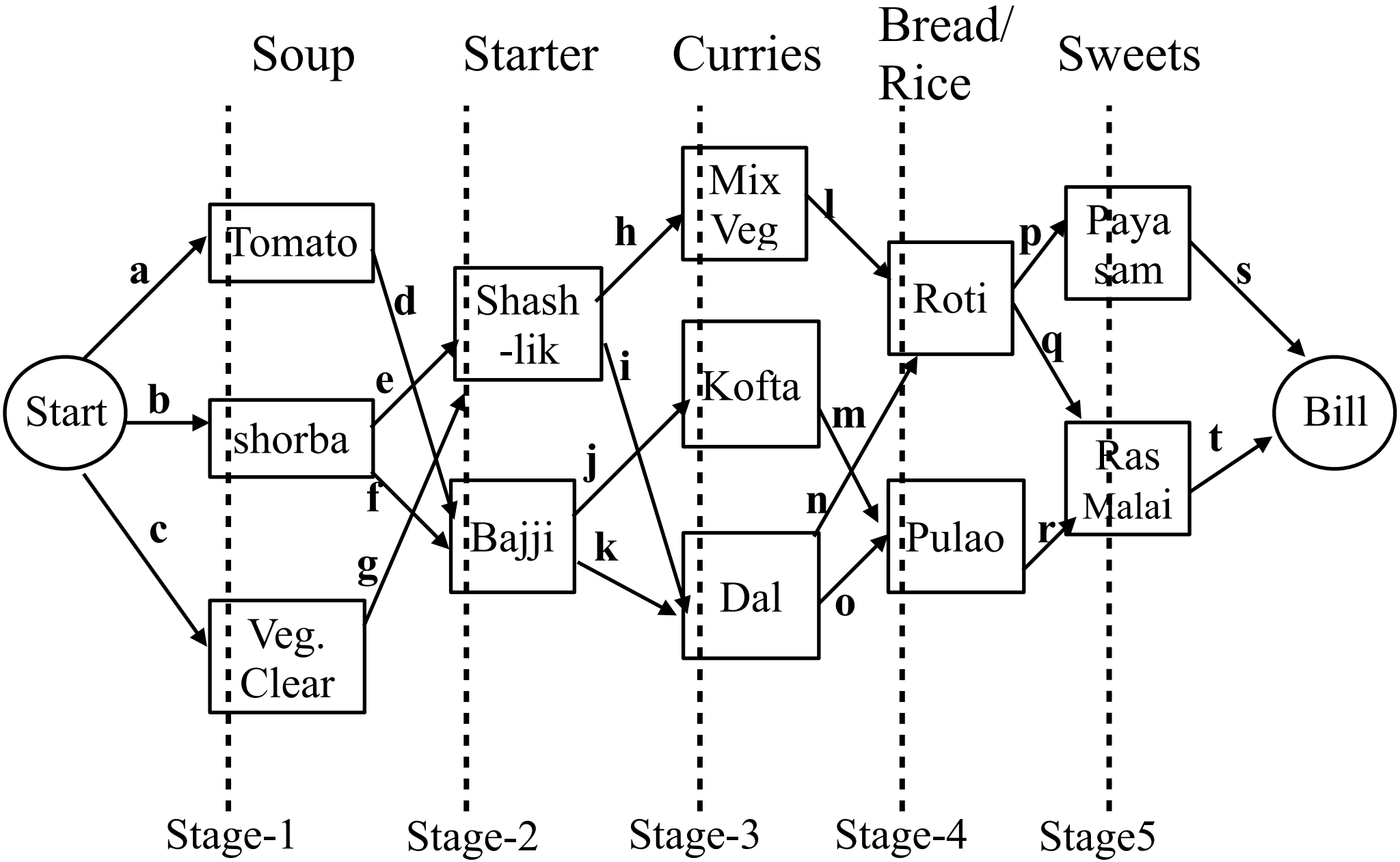
Resources

- Text book 2: Horowitz
  – Sec `5.1,5.2,5.4,5.8,5.9`
- Text book 1: Levitin
  – Sec `8.2-8.4`
- http://www.gdeepak.com/course/adslidesold/26ad.pdf
- https://ocw.mit.edu/courses/civil-and-environmental-engineering/1-204-computer-algorithms-in-systems-engineering-spring-2010/lecture-notes/MIT1_204S10_lec13.pdf
- R1: Introduction to Algorithms
  - Cormen et al.

Consider Restaurant Ordering

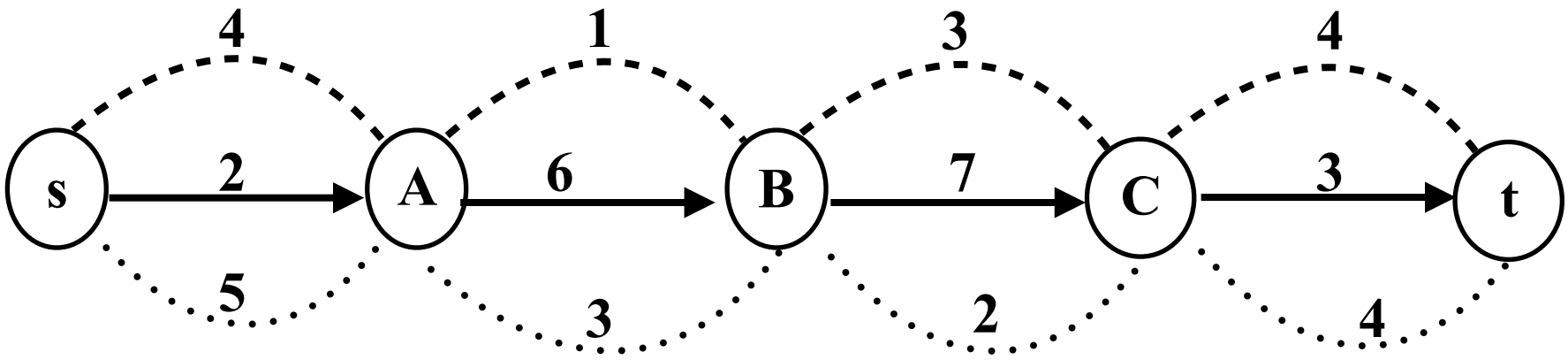
- Food order and serving
  - Soups
  - Starters
  - Main course (curries)
  - Breads/Rice
  - Sweets
  - Mouth freshners
- Each happens in stages
  - Want meal with minimum cost w/ `1` item in each stage
  - Have multiple choices in each stage.
  - Draw a multi-stage graph

Multi-Stage Graph: Restaurant



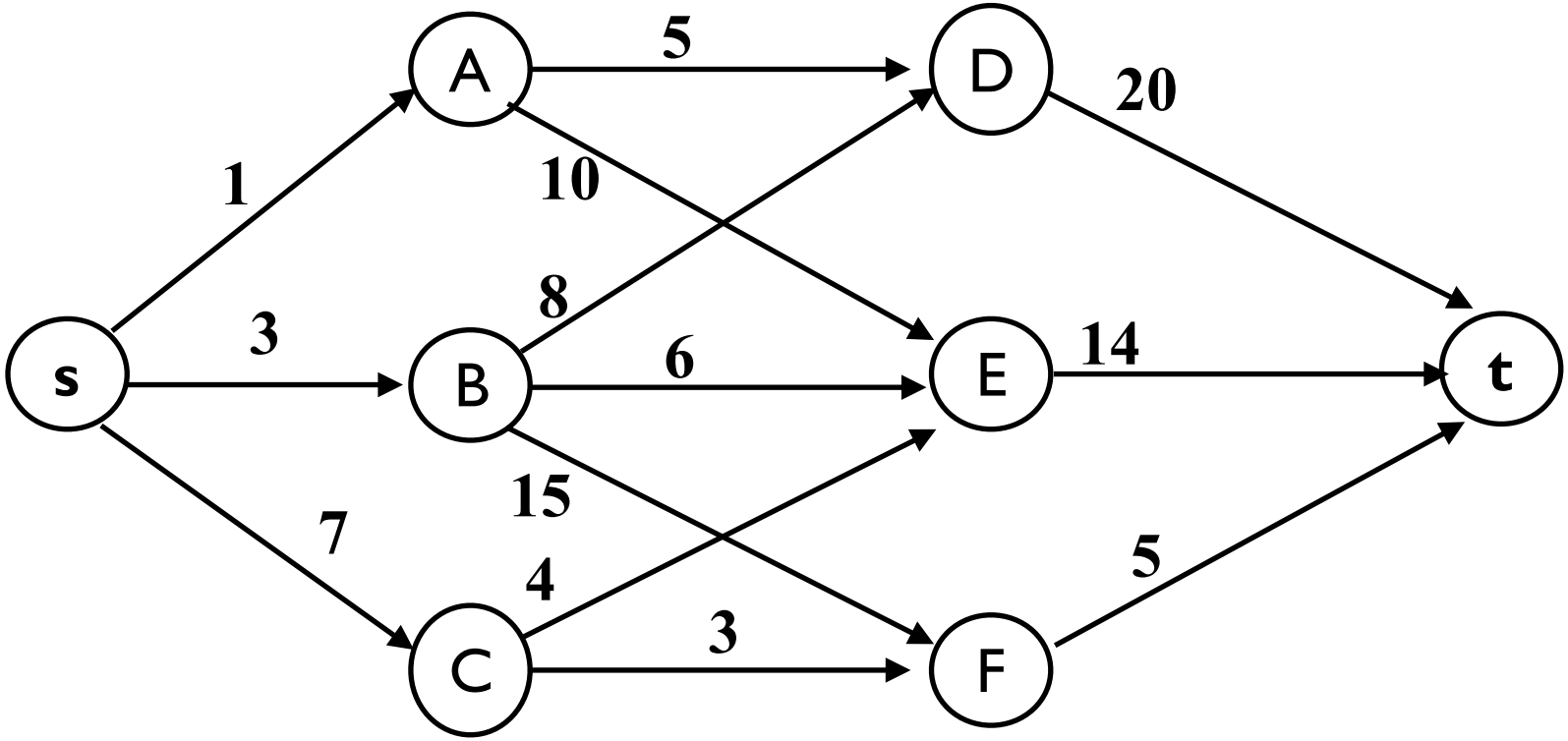
Simple Multi-Stage Graph

- Find shortest path from s to t



– Q: Does Greedy approach work?

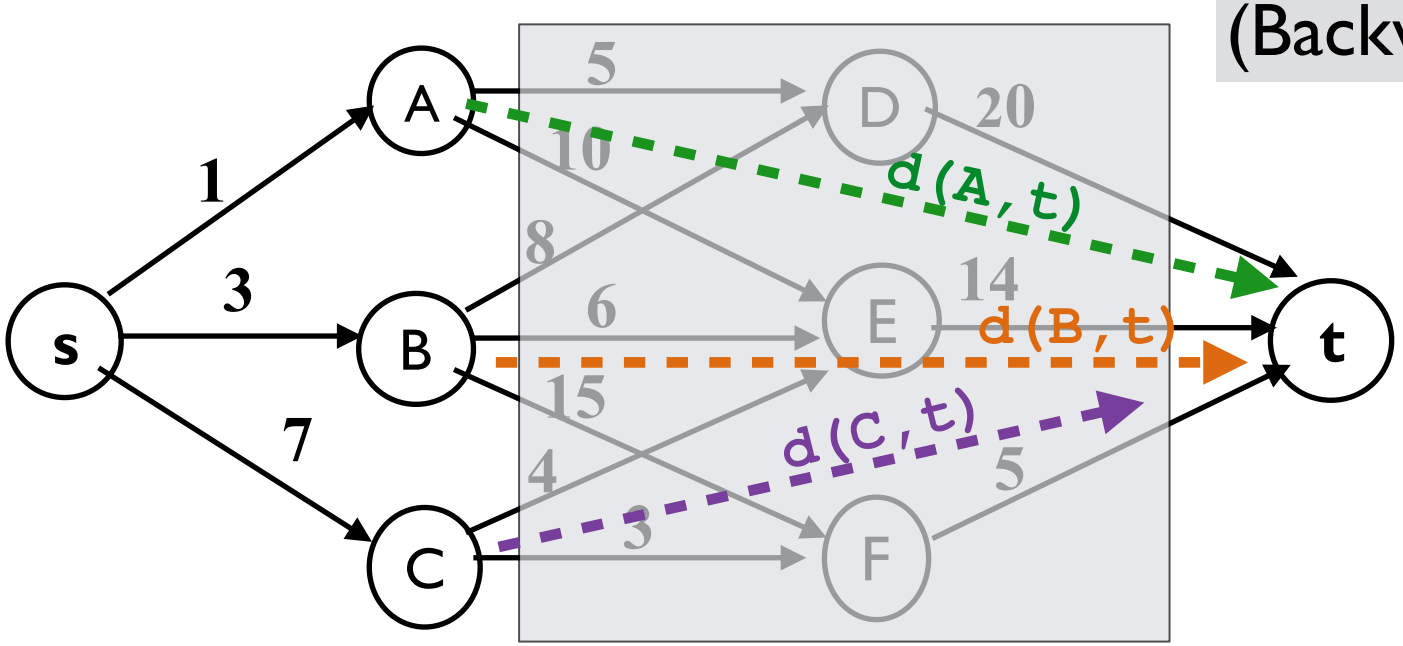
Multistage Graph: Shortest Path



- Find shortest path from s to t
  - Greedy Approach : s→A→D→t = 1+5+20=26
  - Shortest path: s→C→F→t = 7+3+5=15

Dynamic Programming: Forward Approach

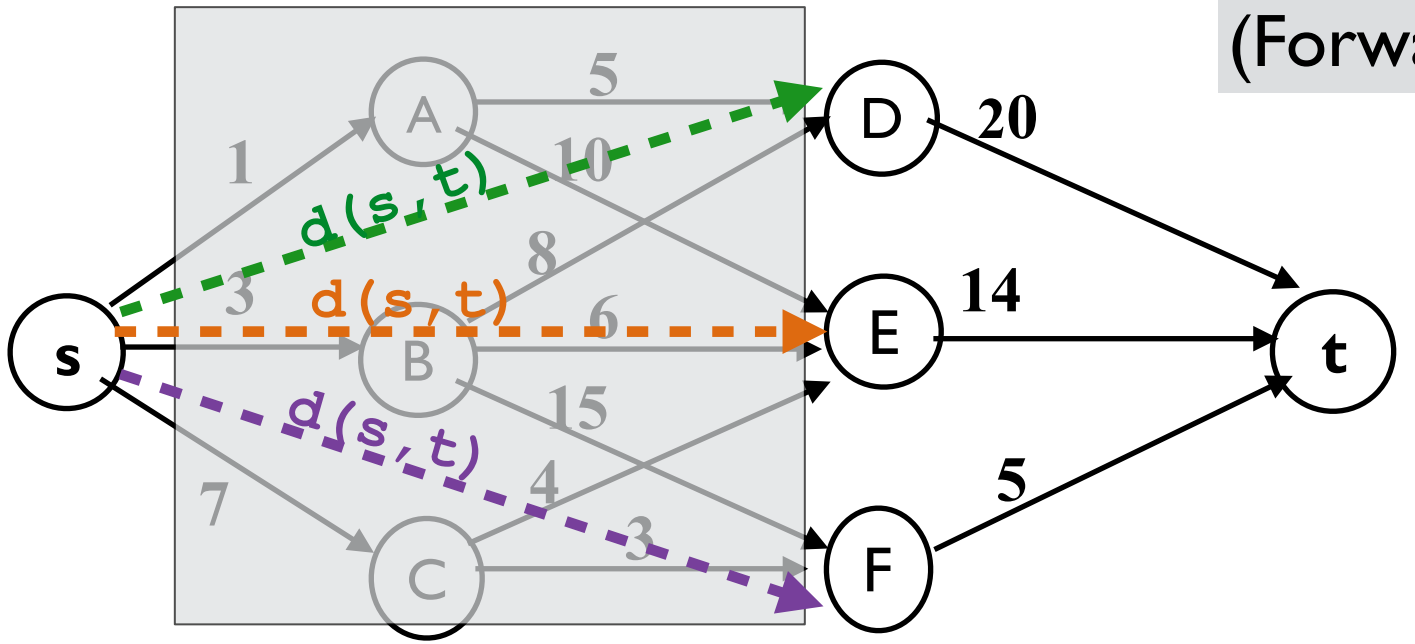
(Backward reasoning)




```
d(s,t)=min{1+d(A,t), 3+d(B,t), 7+d(C,t)}
d(A,t)=min{5+d(D,t),10+d(E,t)}=min(25,24)=24
d(B,t)=min(8+d(D,t),6+d(E,t),15+d(F,t))
      =min{8+20,6+14,15+5)=20
d(C,t)=min{4+d(E,t),3+d(F,t)}=min{18,8)=8
d(s,t)=min{1+24, 3+20, 7+8}=15
```

Dynamic Programming: Backward Approach

(Forward reasoning)




```
d(s,t)=min{d(s,D)+20, d(s,E)+14, d(s,F)+5}
d(s,D)=min{d(s,A)+5,d(s,B)+8}=min(1+5,3+8)=6
d(s,E)=min(d(s,A)+10,d(s,B)+6,d(s,C)+4)
      =min{1+10,3+6,7+4)=9
d(s,F)=min{d(s,B)+15,d(s,C)+3}=min{3+15,7+3)=10
d(s,t)=min{6+20, 9+14, 10+5}=15
```

Dynamic Programming: Applications

- Resource allocation problem
- Consider the following scenario:
  - A team of 3 students are asked to complete 4 assignments.
  - Any student can choose to complete all 4 or none.
  - At a time, only one person will do assignment.
    - First P_1, then P_2, and then P_3 (in that order)
  - However, no assignment is to be done by 2 students
    - Duplicate (wasted) efforts to be avoided
  - All the 4 assignments need to be completed.
  - Depending upon assignments completed by a students, different marks are awarded as shown next

Resource (Assignment) Allocation

- Marks evalutation for assignment done by a person

Person → Assignments↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

- Q: How to allocate assignments to team members so as to get maximum marks

Resource (Assignment) Allocation

- Possible allocations…
- P_1: 0As:
  - P_2:4A, P_3:0A:
    - Marks:0+10+0=10
  - P_2:3A, P_3:1A,
    - Marks: 0+8+5=13
  - P_2:2A, P_3:2As,
    - Marks: 0+7+5=12
  - P_2:1A, P_3:3As,
    - Marks:0+4+6=10
  - P_2:0A, P_3:4As,
    - Marks:0+0+6=6

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

Resource (Assignment) Allocation

- Possible allocations…
- P_1: 1A:
  - P_2:3A, P_3:0A,
    - Marks: 2+8+0=10
  - P_2:2A, P_3:1As,
    - Marks: 2+7+5=14
  - P_2:1A, P_3:2As,
    - Marks:2+4+5=11
  - P_2:0A, P_3:3As,
    - Marks:2+0+6=8

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

Resource (Assignment) Allocation

- Possible allocations…
- P_1: 2As.
  - P_2:2A, P_3:0A:
    - Marks:5+7+0=12
  - P_2:1A, P_3:1A,
    - Marks: 5+4+5=14
  - P_2:0A, P_3:2As,
    - Marks: 5+0+5=10

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

Resource (Assignment) Allocation

- Possible allocations
- P_1: 3As
  - P_2:1A, P_3:0A:
    - Marks=7+4+0=11
  - P_2:0A, P_3:1A:
    - Marks=7+0+5=12
- P_1:4As:
  - P_2:0A, P_3:0A
    - Marks=8

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

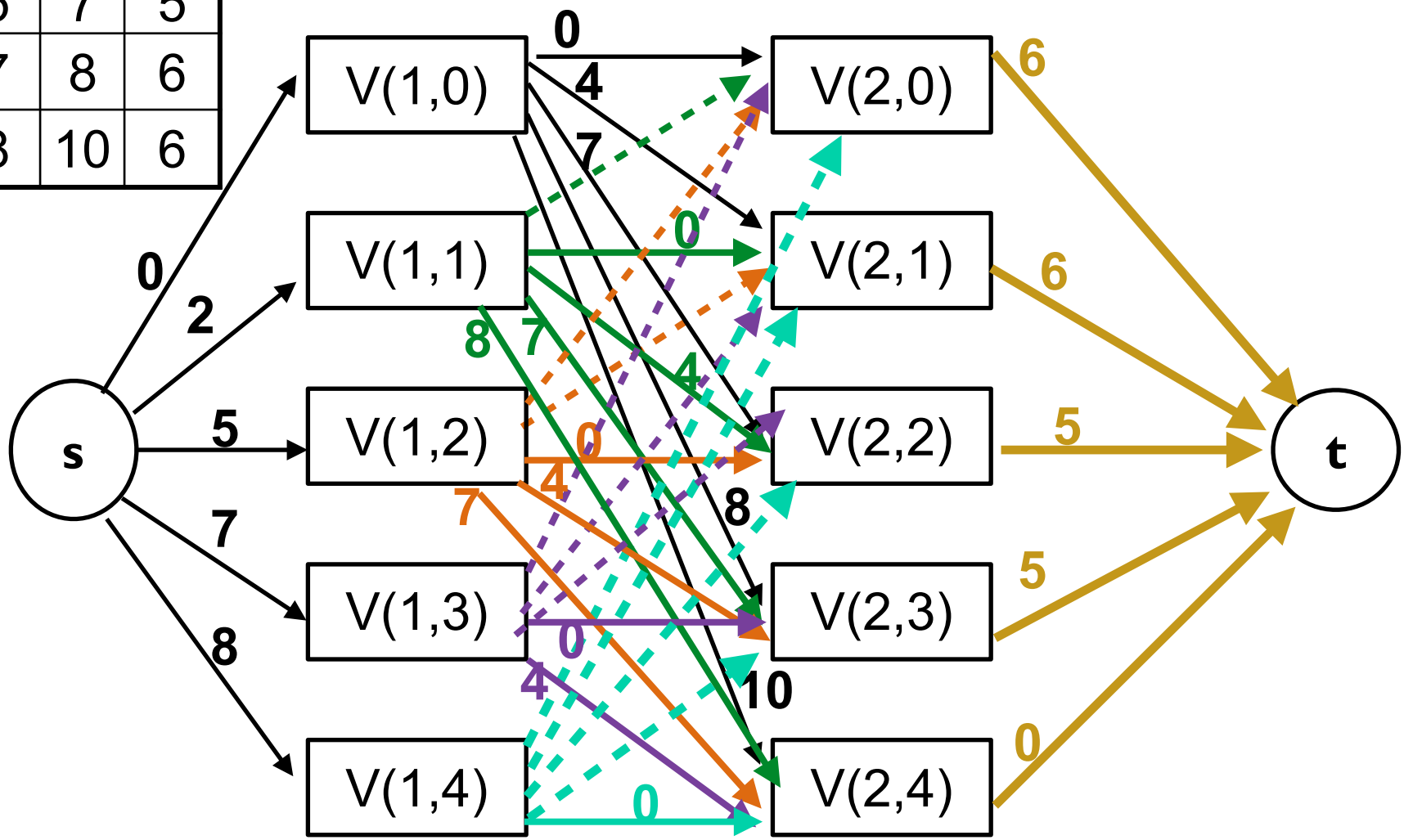
Resource (Assignment) Allocation

- Construction of Multistage graph
- The graph has 4 stages
  - Stage 1: Start
  - Stage 2: P_1 does some assignments
  - Stage 3:P_2 - some remaining assignments
  - Stage 4:P_3 - all remaining assignments
    - The end stage: all assignments are done
- From each stage to next stage
  - Draw edge with allowed possibilities
- Each stage (except start, end) has 5 vertices
  - `V(i,j)`: Person P_i, `j` num of assignments done.
    - $1 \le i < 3$; and $0 \le j \le 4$
- Start, end stage has one vertex each

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6

Multistage Graph

P → A↓	P1	P2	P3
1	2	4	5
2	5	7	5
3	7	8	6
4	8	10	6



Q: Find max marks using DP Forward approach?

Q: Find max marks using DP Backward approach?

DP Forward approach: Steps

- Generate multi-stage graph in forward direction
  - Start at source node `s`
  - Compute `V(i,j)` and edge cost as graph is built
  - Keep track of predecessor `P(i)` of each node that yields highest `V(i,j)`
    - Eliminates non-optimal subsequences (pruning)
  - Eliminate infeasible edges/nodes as graph is built
  - Construct solution by tracing back from sink `t` to source `s` using predecessor `P(i)` variable

DP Forward approach: Algo

Algo: `FGraph(Graph G, int k, int p[])`

// i/p `k`-stage graph `n` vertices indexed in order of stages.
// edge `c(i,j)` is cost of edge $v_i \rightarrow v_j$
// `p[1:k]` is a minimum cost path

```
float cost[maxsize]; int d[maxsize], r;
cost[n]=0.0
for j=n-1 to 1 // compute cost[j]
    Let r be a vertex such that vj→vr is an edge, and
    c(j,r)+cost[r] is minimum
    cost[j] = c[j,r) + cost(r)
    d[j]=r
p[1]=1; p[k]=n;
for j=2 to k-1
    p[j]=d[p[j-1]]
```

DP Backward approach: Algo

Algo: `BGraph(Graph G, int k, int p[])`

// i/p `k`-stage graph `n` vertices indexed in order of stages.
// edge `c(i,j)` is cost of edge $v_i \rightarrow v_j$
// `p[1:k]` is a minimum cost path

```
float bcost[maxsize]; int d[maxsize], r;
bcost[n]=0.0
for j=2 to n // compute bcost[j]
    Let r be a vertex such that v_r → v_j is an edge, and
    bcost[r]+c(r,j) is minimum
    bcost[j] = bcost(r) + c[r,j)
    d[j]=r
p[1]=1; p[k]=n;
for j=k-1 to 2
    p[j]=d[p[j+1]]
```

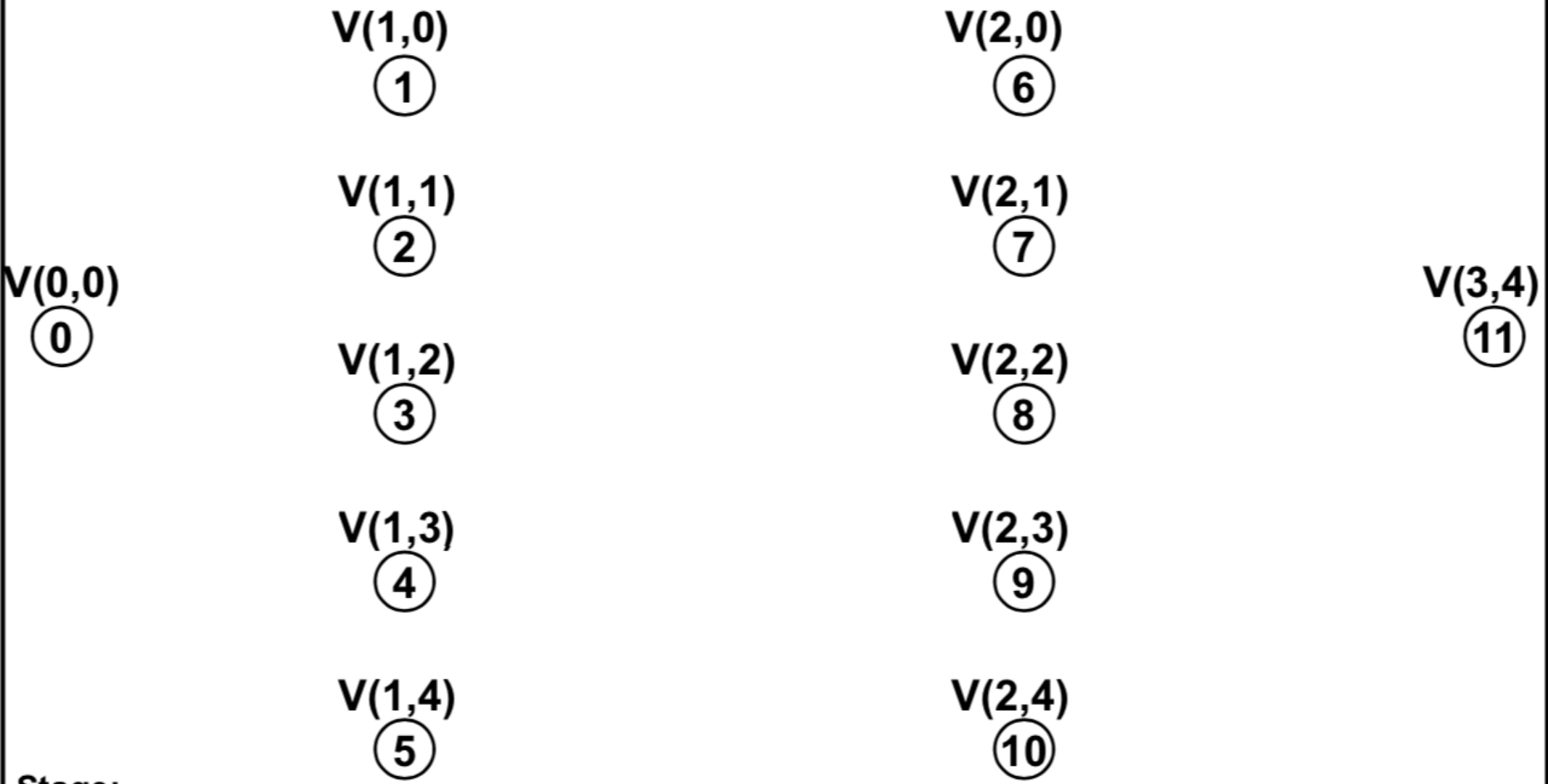
Ex: Build Multistage graph

src: https://ocw.mit.edu/courses/civil-and-environmental-engineering/1-204-computer-algorithms-in-systems-engineering-spring-2010/lecture-notes/MIT1_204S10_lec13.pdf

Proj	0	1	2
Investment	Benefit	Benefit	Benefit
1	6	5	1
2	8	11	4
3	8	16	5
4	10	17	6

src: https://ocw.mit.edu/courses/civil-and-environmental-engineering/1-204-computer-algorithms-in-systems-engineering-spring-2010/lecture-notes/MIT1_204S10_lec13.pdf

Dynamic programming graph: feasible



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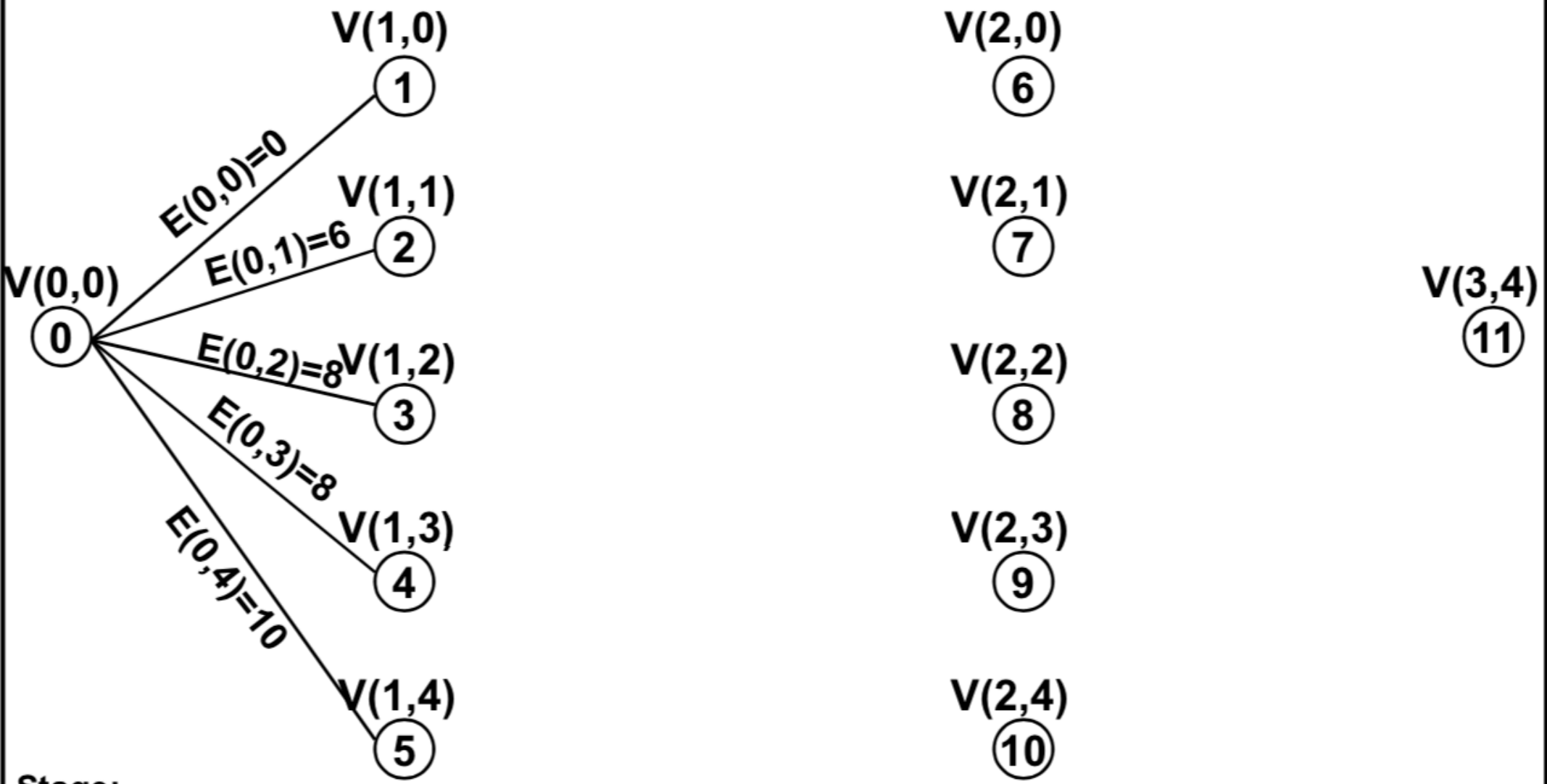
Dynamic programming graph: feasible



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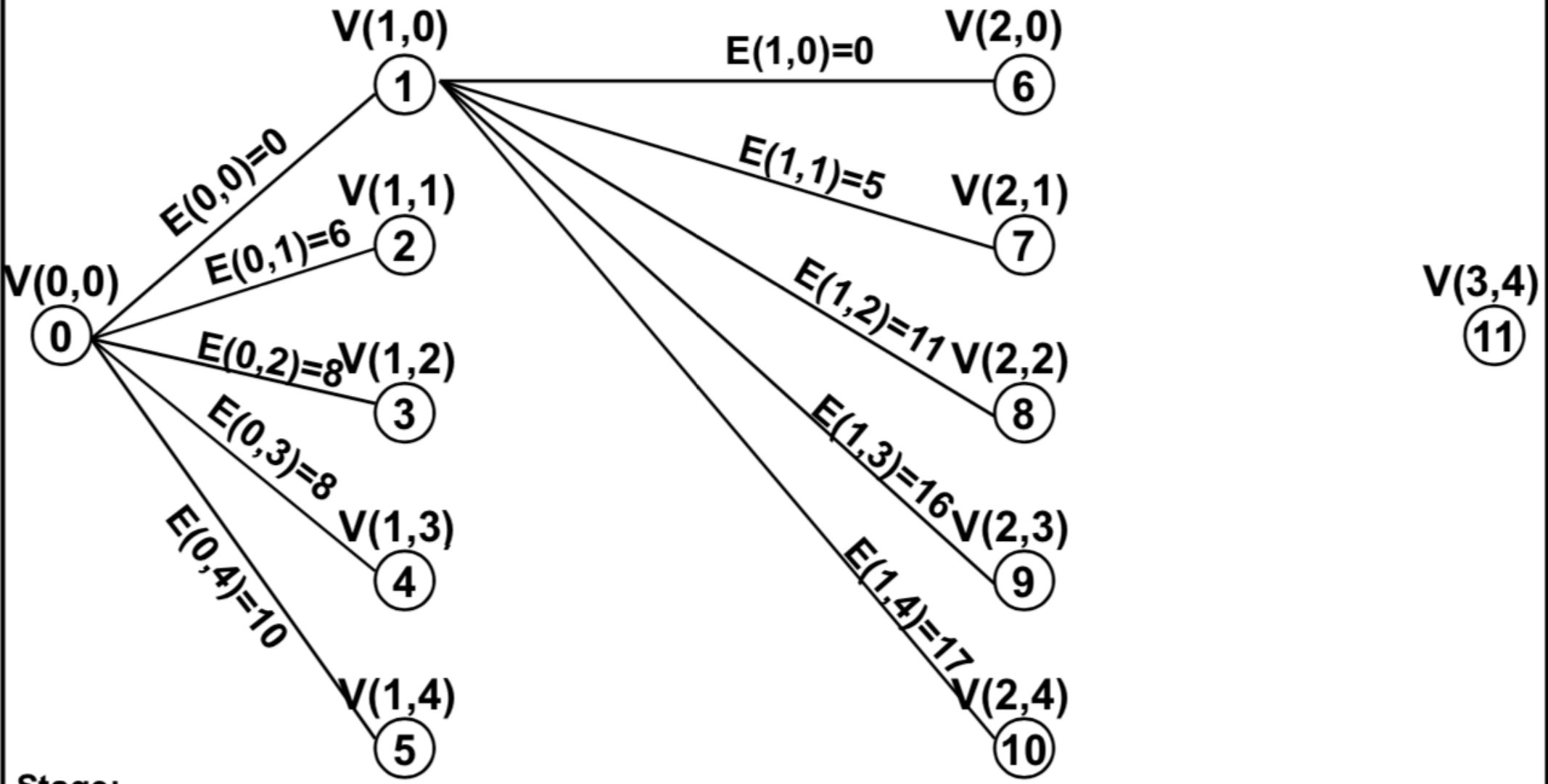
Dynamic programming graph: feasible



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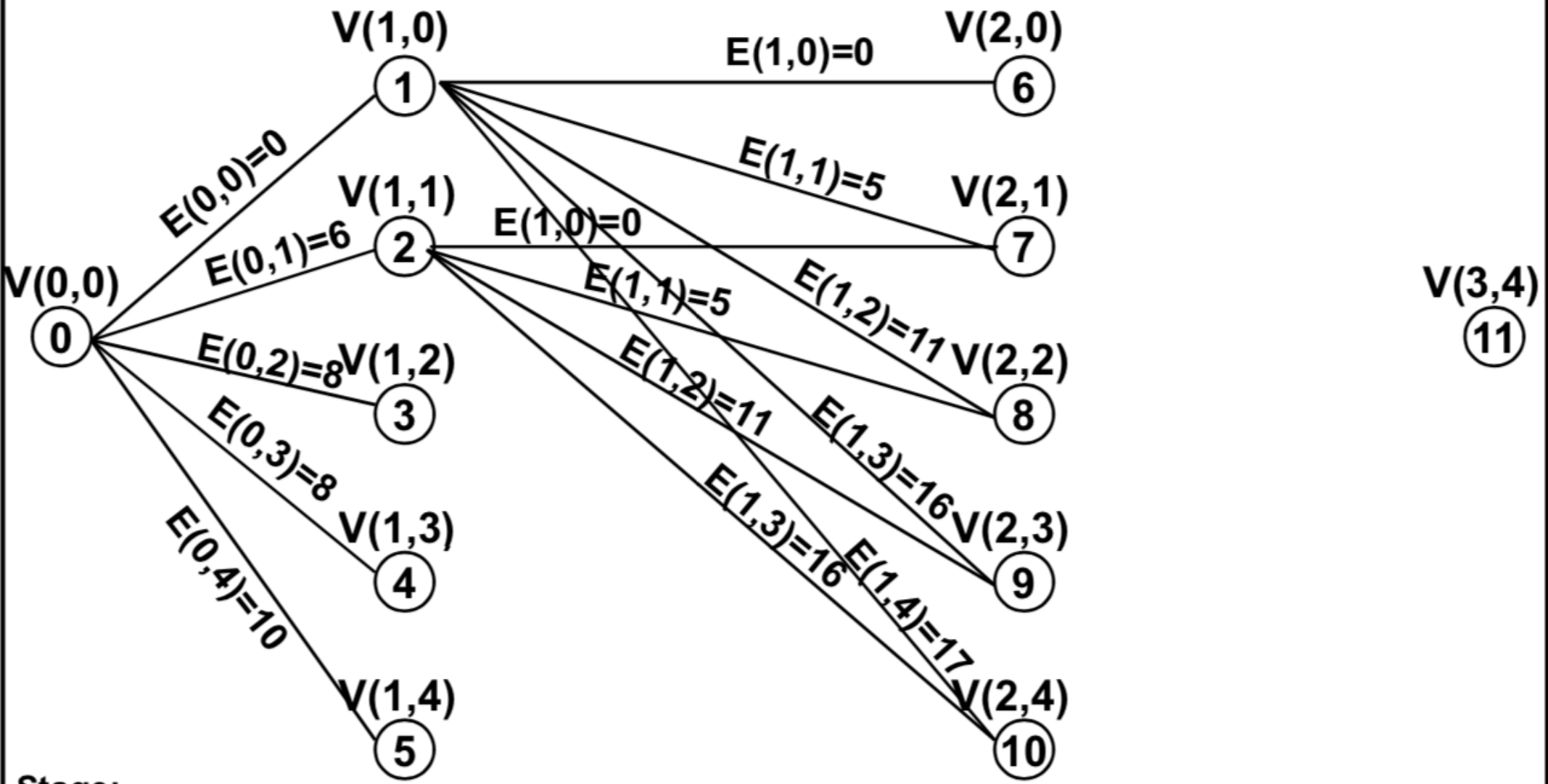

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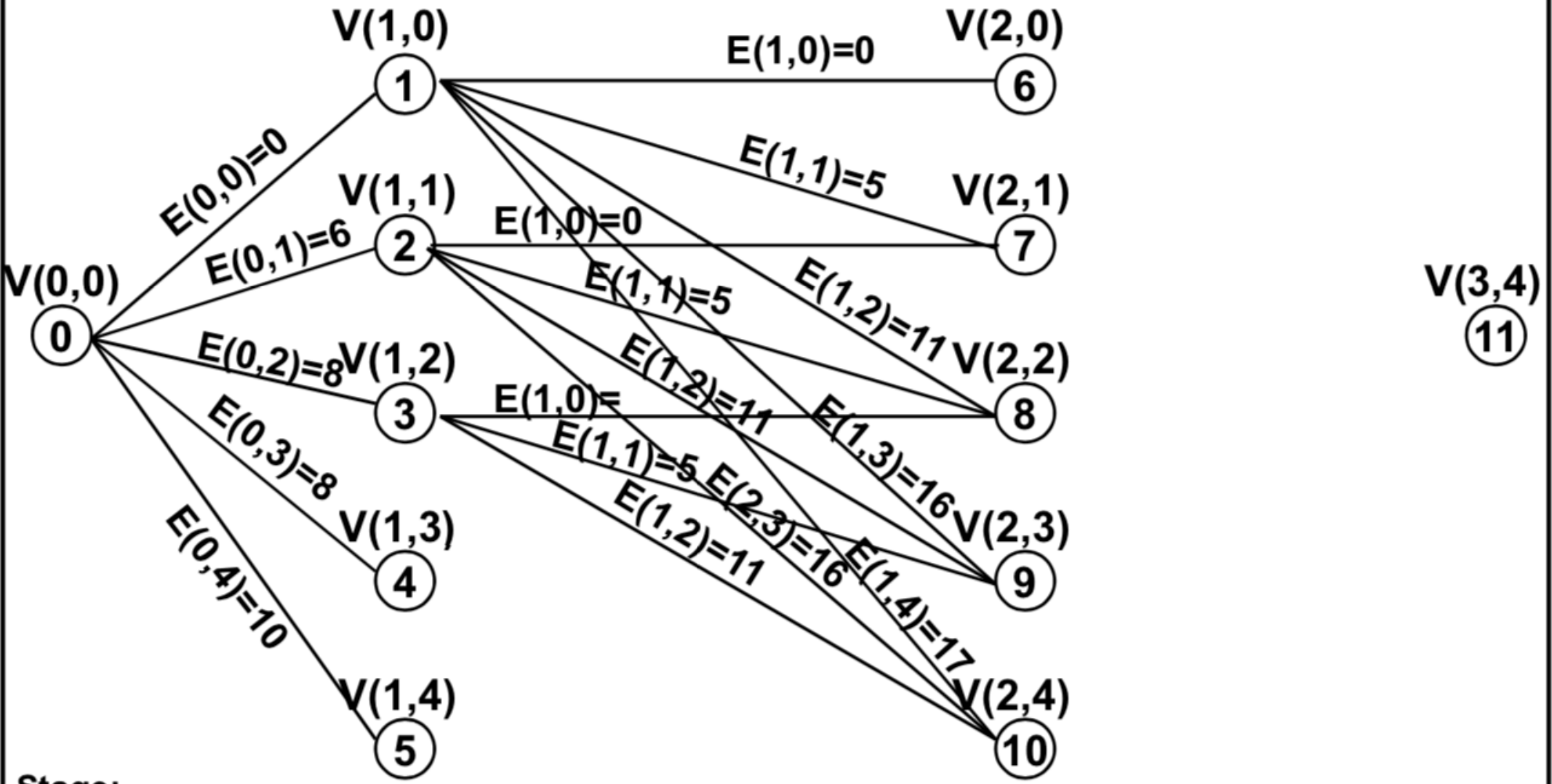
Dynamic programming graph: feasible



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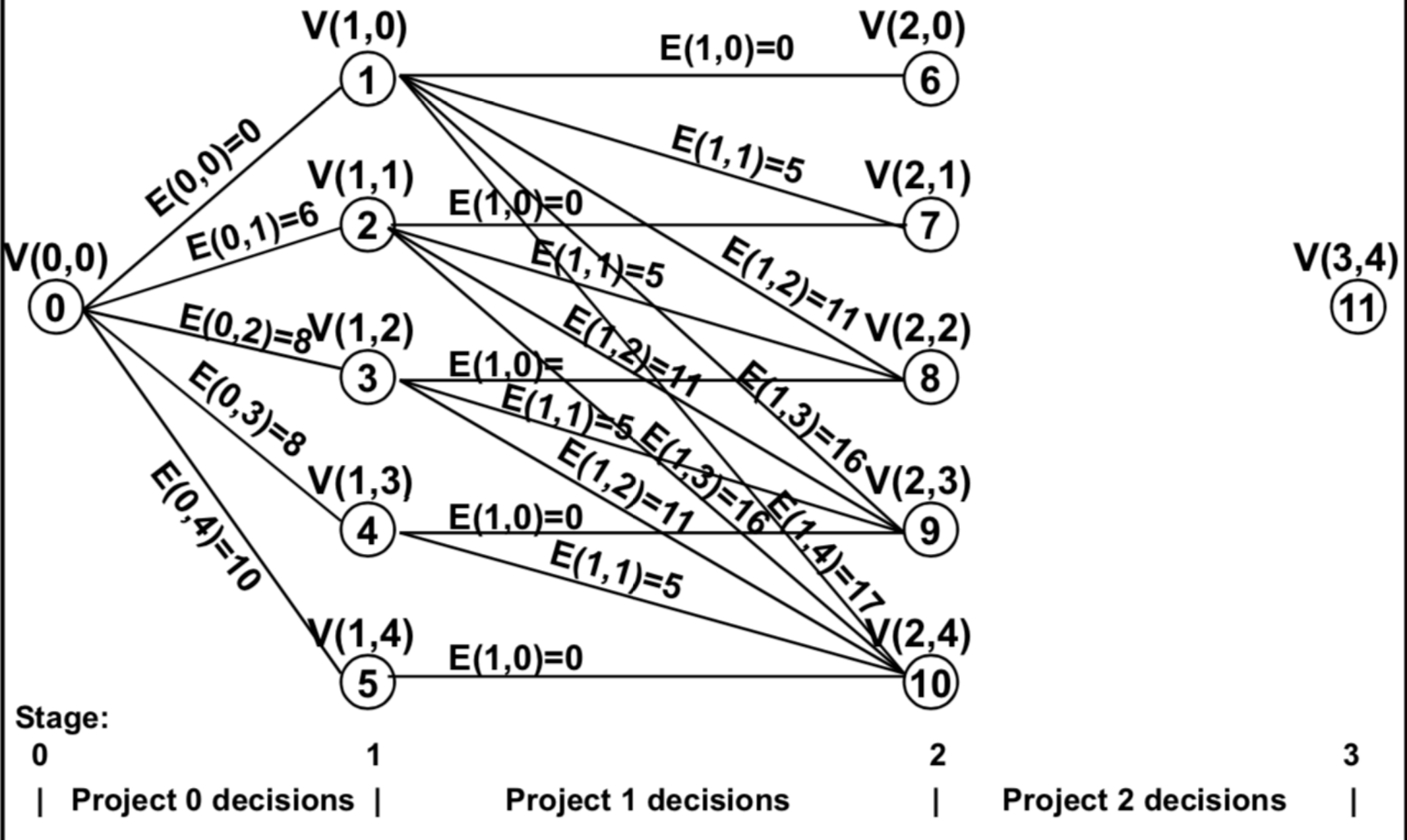
Dynamic programming graph: feasible



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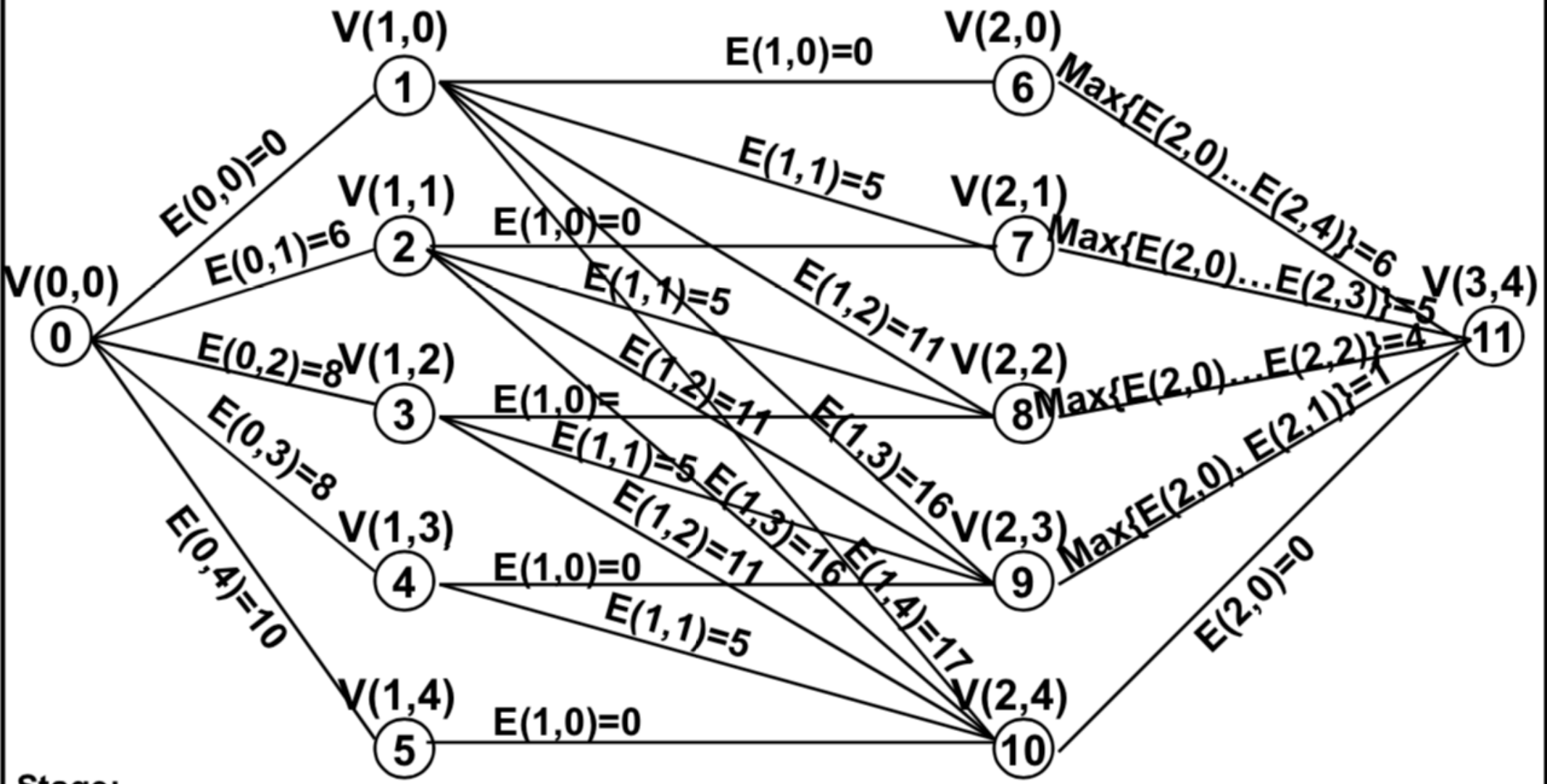
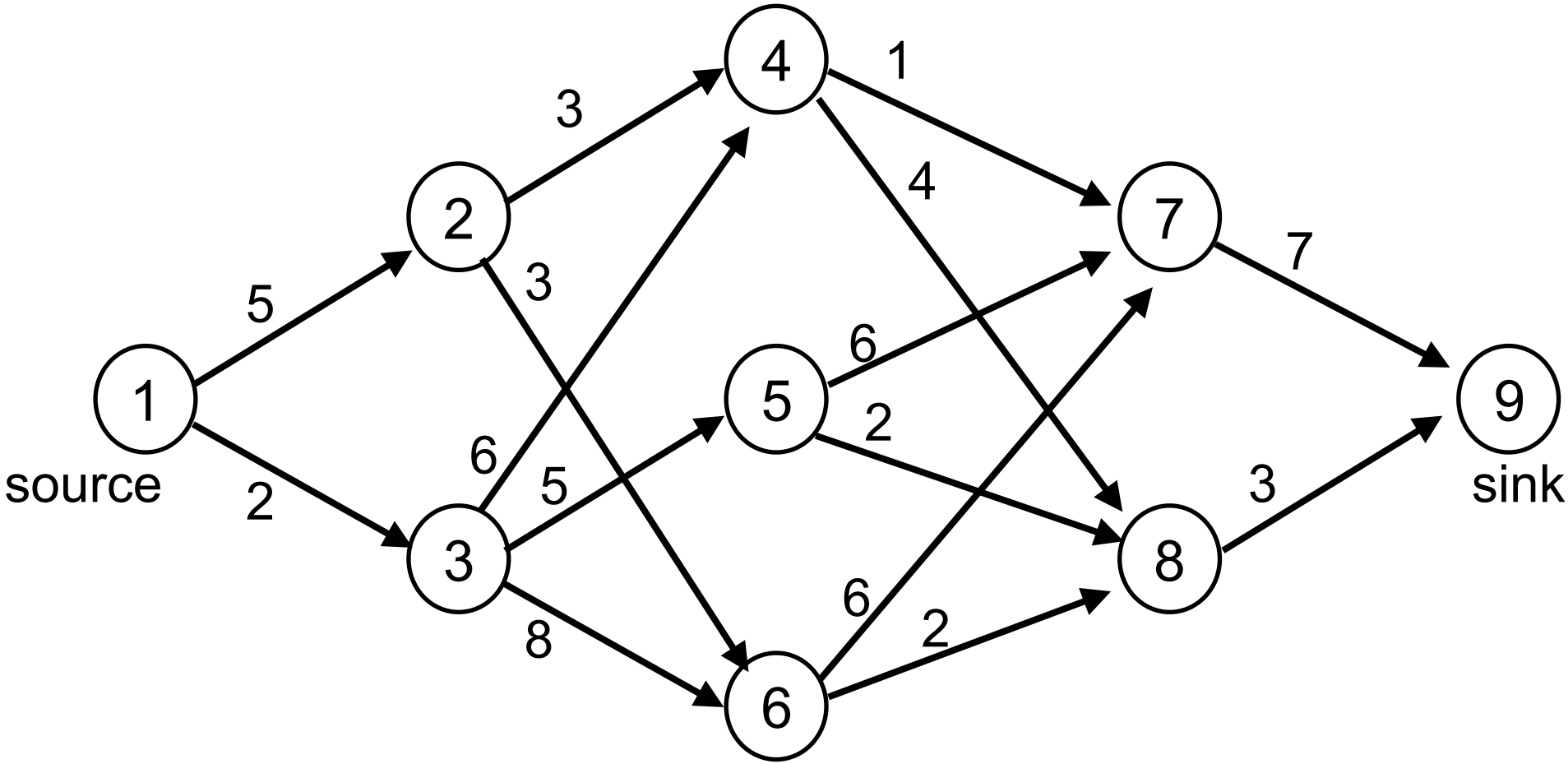


Stage:

0 | Project 0 decisions | 1 | Project 1 decisions | 2 | Project 2 decisions | 3

Ex 02: Find min cost path

- Using forward approach
- Using backward approach



Summary

- Multi stage graph
- Forward approach
- Backward approach