Design and Analysis of Algorithms

L22: Knapsack Problem

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Resources

- Text book 2: Sec 4.1, 4.3, 4.4
- Text book 1: Sec 9.1-5.4 - Levitin
- R1: Introduction to Algorithms
    - Cormen et al.

Example: Knapsack Problem

- A flower street vendor procures the flowers from KR Market and sells these during the day. The quantity of flowers available are limited along with respective profits are as below.
  - Roses: 10kg with a profit of Rs 250
  - Lilies: 8kg with a profit of Rs 240
  - Daisies: 6kg with a profit of Rs 210.
  - Jasmine: 6Kg with a profit of Rs 120
- The vendor has a carrying bag with a capacity of 20kg, would like to maximize the profit for the day. The vendor can buy any quantity (from 0kg to its max limit as given above) for any flower.
- Q: Which quantity of each flower vendor should buy?

Flower Buying: Approach 1

- *Flowers: quantiy/total profit*
  - *Roses 10Kg /Rs 250, Lilies: 8Kg / Rs 240*
  - *Daisies 6Kg / Rs 210, Jasmine: 6Kg / Rs 120*
- Equal quantity of each flower:
  - Buy same quantiy of each variety of flower i.e. buy 20/4=5 kg of Rose, Daisies and Lilies and Jasmine
- The profit earned for the day is
  - Roses: 5*250/10 = Rs 125
  - Lilies: 5*240/8 = Rs 150
  - Daisies: 5*210/6 = Rs 175
  - Jasmine: 5*120/6= Rs 100
- Net profit: Rs 125+150+175+100 = Rs 550

Flower Buying: Approach 2

- *Flowers: quantiy/total profit*
  - *Roses 10Kg /Rs 250, Lilies: 8Kg / Rs 240*
  - *Daisies 6Kg / Rs 210, Jasmine: 6Kg / Rs 120*
- Buy in equal proportions of their availability
  - Roses: 20*10/30 = 20/3Kg, Lilies: 20*8/30=16/3 Kg
  - Daisies: 20*6/30 = 4Kg, Jasmine 20*6/30 = 4Kgs
- The profit earned for the day is
  - Roses: (20/3)*250/10 = Rs 500/3=Rs 166.6
  - Lilies: (16/3)*240/8 = Rs 160
  - Daisies: 4*210/6 = Rs 140
  - Jasmine: 4*120/6= Rs 80
- Net profit: Rs 166.67+160+140+80 = Rs 546.67

Flower Buying: Approach 3

- *Flowers: quantiy/total profit*
  - *Roses 10Kg /Rs 250, Lilies: 8Kg / Rs 240*
  - *Daisies 6Kg / Rs 210, Jasmine: 6Kg / Rs 120*
- Buy as per max profit known (greedy approach 1)
  - Roses: 10Kg, Lilies: 8Kg, Daisies: 2Kg, Jasmine: 0Kg
- The profit earned for the day is
  - Roses: 10*250/10 = Rs 250
  - Lilies: 8*240/8 = Rs 240
  - Daisies: 2*210/6 = Rs 70
  - Jasmine: 0*120/6= Rs 0
- Net profit: Rs 250+240+70+0 = Rs 560

Flower Buying: Approach 4

- *Flowers: quantiy/total profit*
  - *Roses 10Kg /Rs 250, Lilies: 8Kg / Rs 240*
  - *Daisies 6Kg / Rs 210, Jasmine: 6Kg / Rs 120*
- Buy as per capacity from min (greedy approach 2)
  - Jasmine: 6Kg, Daisies: 6Kg, Lilies: 8Kg, Roses: 0Kg
- The profit earned for the day is
  - Roses: 0*250/10 = Rs 0
  - Lilies: 8*240/8 = Rs 240
  - Daisies: 6*210/6 = Rs 210
  - Jasmine: 6*120/6= Rs 120
- Net profit: Rs 0+240+210+120 = Rs 570

Flower Buying: Approach 5

- *Flowers: quantiy/total profit*
  - *Roses 10Kg /Rs 250, Lilies: 8Kg / Rs 240*
  - *Daisies 6Kg / Rs 210, Jasmine: 6Kg / Rs 120*
- Greedy approach 3: get max profit per kg of flowers
  - Profits per Kg: R: Rs 25, L: Rs 30, D: 35, J: 20
  - Daisies: 6Kg, Lilies: 8Kg, Roses: 6Kg, Jasmine: 0Kg
- The profit earned for the day is
  - Roses: 6*250/10 = Rs 150
  - Lilies: 8*240/8 = Rs 240
  - Daisies: 6*210/6 = Rs 210
  - Jasmine: 0*120/6= Rs 0
- Net profit: Rs 150+240+210+0 = Rs 600

Flower Buying

- Profit comparisons:
  - Approach 1 (equal quantity): Rs 550/-
  - Approach 2 (in equal ratios): Rs 546.67
  - Approach 3 (Max highest profit): Rs 560/-
  - Approach 4 (Smallest capacities): Rs 570/-
  - Approach 5 (Greedy): Rs 600/-
- Does the Greedy approach always works?
  - Yes (for fractional knapsack)
  - No (for 0-1 knapsack)
    - 0-1 knapsack: can not buy partial quantities
- Can there be multiple optimal solutions?
  - Consider that both Roses, Lilies have profit of Rs 25/Kg

Overview: Knapsack Problem

- Knapsack problem (fractional):
  - Given n objects, and a knapsack (bag) with a capacity m, fill the knapsack to maximize the value as follows
    - Each object i has weight w_i (+ve number)
    - Each object i has +ve profit p_i (+ve number)
    - If a fraction x_i ($0 \le x_i \le 1$) of the object i is placed in the knapsack, the profit $p_i x_i$ is earned.
  - **Objective**: Obtain a filling of the knapsack that maximizes the total profit earned. Mathematically

Maximize $\sum_{1 \le i \le n} p_i x_i$

Subject to $\sum_{1 \le i \le n} w_i x_i \le m$ and $0 \le x_i \le 1, \ 1 \le i \le n$

Knapsack Problem

- Lemma 1:
  - In case the sum of all quantities is $\leq m$, then
    $x_i = 1,\ 1 \leq i \leq n$ is an optimal solution.
  - So, let us consider that sum of weights exceed m.
- Lemma 2:
  - All optimal solutions will fill the knapsack exactly.
  - Note: we can always increase the quantity of some object i by a fractional amount till the total weight becomes exactly m.
- Analysis: Does it fit the subset paradigm?
  - Yes: we are selecting a subset of objects.

Algorithm: Knapsack Problem

```
Void GreedyKnapsack(float m, int n) {
//p[1:n] and w[1:n] contain the profits and weights
//The indices are ordered as per following criteria
// p[i]/w[i]≥p[i+1]/w[i+1] , 1≤i<n.
// m is knapsack size, and x[1:n] is the solution vector
    initialize x[i] to 0.0
    float U=m
    for i=1 to n
        if w[i] >U
            break
        x[i]=1.0
        U=U-[xi]
    if i≤n
        x[i] = U/w[i]
```

Theorem: Knapsack Problem

Theorem:

<u>If $p_1/w_1 \geq p_2/w_2 \geq \ldots \geq p_n/w_n$, then `GreedyKnapsack` generates an optimal solution to the given instance of the knapsack problem.</u>

Metholodology to be used for proof:

- Compare the greedy solution with any optimal solution.
- If the two solutions differ, then first x_i at which they differ.
- Then show that x_i in the optimal solution equal to that in the greedy solution without any loss in total value.
- Repeated use of this transformation shows that greedy solution is optimal

Proof: Greedy Approach is Optimal

- Let $x=(x_1, ..., x_n)$ be the solution generated by `GreedyKnapsack`.
- If all the x_i equal one, the solution is optimal.
- Let j be the least index such that $x_j \neq 1$.
- From the algorithm, we know that
  $0 \leq x_j < 1$, and
  $x_i = 1$ for $1 \leq i < j$, and
  $x_i = 0$ for $j < i \leq n$.
- Let $y=(y_1, ..., y_n)$ be the optimal solution, Thus
  $\Sigma w_i y_i = m$
- Let k be the index such that $y_k \neq x_k$
- Since two solutions differ, such k must exist. Since all x_k prior to x_j's are 1, clearly $y_k < x_k$.

Proof: Greedy Approach is Optimal…

x_1	x_2	…	x_{j-1}	$\mathbf{x_j}$	x_{j+1}	…			x_n
1	1	1	1	$\mathbf{x_j}$	0	0	0		0

Solution by Greedy Approach

First index where x_j is not 0

y_1	y_2	…	$\mathbf{y_k}$	…	y_j	…			x_n
1	1	1	$\mathbf{y_k}$	…	y_j	0	0		0

An optimal solution found some way

First index where y_k differs from x_k

Proof: Greedy Approach is Optimal…

case `1: k<j, xk=1, hence yk<xk` ($k<j$, $x_k=1$, hence $y_k<x_k$)

x_1	x_2	…	x_k	…	x_{j-1}	$\boldsymbol{x_j}$	x_{j+1}	…		x_n
1	1	1	1	1	1	$\boldsymbol{x_j}$	0	0		0

Solution by Greedy Approach

First index where x_j is not 0

y_1	y_2	…	$\boldsymbol{y_k}$	…	y_{j-1}	y_j	…			y_n
1	1	1	$\boldsymbol{y_k}$	…	y_{j-1}	y_j	…			

An optimal solution found some way

First index where y_k differs from x_k

Proof: Greedy Approach is Optimal…

case 2 : $k=j$

if $y_k \not\leq x_k$, then $\Sigma w_i y_i > m$, because $\Sigma w_i x_i = m$

x_1	x_2	…	…	…	x_{j-1}	$\mathbf{x_j}$	x_{j+1}	…		x_n
1	1	1	1	1	1	$\mathbf{x_j}$	0	0		0

Solution by Greedy Approach

First index where x_j is not

y_1	y_2	…	…	…	…	$\mathbf{y_k}$	…			y_n
1	1	1	…	…	1	$\mathbf{y_k}$	…			

An optimal solution found some way

First index where y_k differs from x_k

Proof: Greedy Approach is Optimal…

case 3 : $k>j$, This is not possible since $\Sigma w_i y_i > m$

x_1	x_2	…	…	…	x_{j-1}	$\mathbf{x_j}$	x_{j+1}	…		x_n
1	1	1	1	1	1	$\mathbf{x_j}$	0	0		0

Solution by Greedy Approach

First index where x_j is not 0

y_1	y_2	…	…	…	…	…	…	$\mathbf{y_k}$		y_n
1	1	1	…	…	1	1	…	$\mathbf{y_k}$		

An optimal solution found some way

First index where y_k differs from x_k

Proof: Greedy Approach is Optimal…

- To show that $y_k < x_k$, there exists 3 possibilities

  i. $k<j$: since $x_k=1$, and $y_k \neq x_k$, and so $y_k<y_k$

  ii. $k=j$: since $\Sigma w_i x_i = m$, and $y_i = x_i$ for $1 \leq i < j$,

  then either $y_k < x_k$ or $\Sigma w_i y_i > m$

  iii. $k>j$: then $\Sigma w_i y_i > m$, which is not possible

- To show that $x = (x_1, \ldots, x_n)$ is optimal solution.
  - Increase y_k to x_k and decrease as many of $(y_{k+1}, \ldots, y_n)$ as necessary so that total capacity is still m.
  - This gives a new solution $z = (z_1, \ldots, z_n)$ such that

    $z_i = x_i, 1 \leq i \leq k$; and

    $\Sigma_{k<i\leq n}\ w_i (y_i - z_i) = w_k (z_k - y_k)$

Proof: Greedy Approach is Optimal…

- Thus, we have

$$\sum_{1\le i\le n} p_i z_i = \sum_{1\le i\le n} (p_i y_i) + (z_k - y_k) w_k \frac{p_k}{w_k} - \sum_{k<i\le n} (y_i - z_i) w_i \frac{p_i}{w_i}$$

$$\ge \sum_{1\le i\le n} (p_i y_i) + \left[(z_k - y_k) w_k - \sum_{k<i\le n} (y_i - z_i) w_i \right] \frac{p_k}{w_k}$$

$$= \sum_{1\le i\le n} (p_i y_i)$$

since $p_k/w_k \ge p_{k+1}/w_{k+1} \ge \ldots \ge p_n/w_n$

- Thus, if $\Sigma p_i z_i > \Sigma p_i y_i$, then y could not have been optimal solution.
- If $\Sigma p_i z_i = \Sigma p_i y_i$, then either $z = x$ and x is optimal, or $z \neq x$.
- If $z \neq x$, then repeat the process to show that y is not optimal or transform y to x and hence x is optimal.

Summary

- Greedy approach (fractional) knapsack