Design and Analysis of Algorithms

L38: Traveling Salesman Problem

Dynamic Programming

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Resources

- Text book 2: Horowitz
  – Sec 5.1, 5.2, 5.4, 5.8, **5.9**
- Text book 1: Levitin
  – Sec 8.2-8.4
- R1: Introduction to Algorithms
    - Cormen et al.
- https://onlinelibrary.wiley.com/doi/full/10.1002/net.21864
- https://www.youtube.com/watch?v=-JjA4BLQyqE
- https://www.tutorialspoint.com/design_and_analysis_of_algorithms/design_and_analysis_of_algorithms_travelling_salesman_problem.htm

Travelling Salesman Problem

- Known as Held-Karp algorithm
  - Proposed in 1962 to solve TSP
- TSP problem:
  - Find a tour of all cities in a country (assuming all cities are reachable)
    - The tour should visit each city only once
    - Tour should end at starting city, and
    - Tour should be of minimum distance. (cost)

Example 1:TSP problems

- You are organizing a function at your home and you would like to invite your friends for the same.
  - Starting from your home, you need to visit each friend's house to personally invite.
  - The route/distance from one house to another house is known.
  - The up and down time taken to travel between two houses is not same i.e. depends upon travel direction
    - e.g. some one way roads, pot-holed roads etc
- Goal: Find the shortest (time) route.

Example 2: TSP problems

- A robotic arm needs to tighten the screw/bolts on a machine.
  - There are different points where screw/bolts needs to be tightened
  - Robotic arm can reach from any screw/bolt position to another screw/bolt position.
  - The time taken to tighten to a bolt is constant so can be ignored. Time taken by robotic arm varies.
    - Interested in time taken by robotic arm when moving
- Goal: Find the optimal path for robot arm to tighten all the bolts and return to its start point.

TSP Problem

- Given directed graph $G=(V,E)$ with $n>1$ edges,
  - Cost of each directed edge (i,j) is given as $c_{ij} \geq 0$
  - Cost is considered as ∞ when edge is not defined
  - A tour of G is a directed simple cycle that includes every vertex in the graph
  - The cost of a tour is the sum of cost of edges on the tour.
  - **T**raveling **S**alesman **P**roblem is to find the tour of minimum cost.
- For simplicity, we assume tour starts at $v=1$

TSP Problem

- Brute force approach
  - Enumerate all permutations of n nodes
  - Compute the cost corresponding to each permutation
  - Find the permuation with minimum cost.
  - Time complexity: $O(n!)$
- TSP is an NP-Hard problem
  - Can we do better though still exponential, e.g. $O(2^n)$

    $O(n^n)>O(n!)>O(2^n)$
    - Subset problems are easier compared to permutations
  - k^n is always better than $n!$ (for $n>k$).
  - Subset problem leads to dynamic programming approach

TSP Problem: Dynamic Programming

- Let start vertex $s=1$, and thus tour ends at 1 too.
- Every tour consists of
  - An edge e_{1k}, for some $k \in V-\{1\}$, and
  - A path from k to 1 going thru each vertex v in V exactly once other than k and 1 i.e. $v \in V-\{1, k\}$.
- If the tour is optimal, then
  - path from k to 1 must be a shortest path going thru all vertices in $V-\{1, k\}$.
  - Essentially, principle of optimality holds

TSP Problem: Dynamic Programming

- $g(i,S)$: denotes the length of shortest path
  - starting from vertex i,
  - going thru all vertices in S
  - terminating at vertex 1.
- Goal: compute $g(1,V-\{1\})$
  - denotes the length of optimal salesperson tour
- Principle of optimality:

$$g(1,V-\{1\})=\min_{2\le k\le n}\{c_{1k}+g(k,V-\{1,k\})\} \quad \ldots\ldots(1)$$

- Generalizing above for $i\notin S$

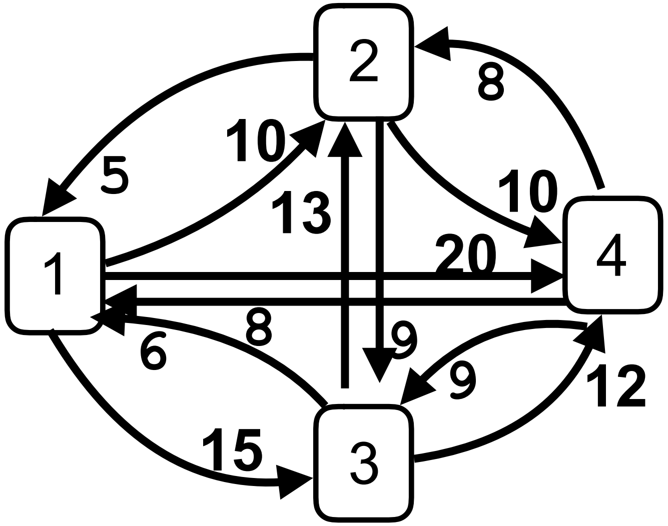
$$g(i,S)=\min_{j\in S}\{c_{ij}+g(j,S-\{j\})\} \quad \ldots\ldots(2)$$

- Solving $g(1,V-\{1\})$ requires to solve $g(k,V-\{1,k\})$ for all $k\neq 1$

TSP Problem: Dynamic Programming

- $g(i, \emptyset)$ implies shortest path from node i to 1
  - thru an empty set of vertices in $\emptyset$, i.e.
  - without going thru any vertex
- Thus, $g(i, \emptyset) = c_{i1},\ 1 \le i \le n$.
- Using eq (2), we can compute $g(i, S)$ for all S of size 1.
- Thus, then we can compute $g(i, S)$ for all S with $|S| = 2$, and so on
- When, $|S| < n-1$, then the values of i and S for which $g(i, S)$ is needed are such that $i \neq 1$, $1 \notin S$, and $i \notin S$.
- Tour construction requires that we maintain node j that minimizes $g(i, S)$ i.e. $\min_{j \in S}\{c_{ij} + g(j, S-\{j\})\}$
  - Let $J(i, S)$ denote this value

TSP Example: Computation



	1	2	3	4
1	0	10	15	20
2	5	0	9	10
3	6	13	0	12
4	8	8	9	0

- Goal: $g(1,V-\{1\})$
- Power set of $\{2,3,4\}$

$\emptyset,\{2\},\{3\},\{4\},$
$\{2,3\},\{2,4\},\{3,4\}$
$\{2,3,4\}$

$g(1,\emptyset)=c_{11}=0$

$g(2,\emptyset)=c_{21}=5$

$g(3,\emptyset)=c_{31}=6$

$g(4,\emptyset)=c_{41}=8$

Compute $g(i,S)$, for $|S|=1$

$g(2,\{3\})=c_{23}+g(3,\emptyset)=9+6=15$

$g(2,\{4\})=c_{24}+g(4,\emptyset)=10+8=18$

$g(3,\{2\})=c_{32}=g(2,\emptyset)=13+5=18$

$g(3,\{4\})=c_{34}=g(4,\emptyset)=12+8=20$

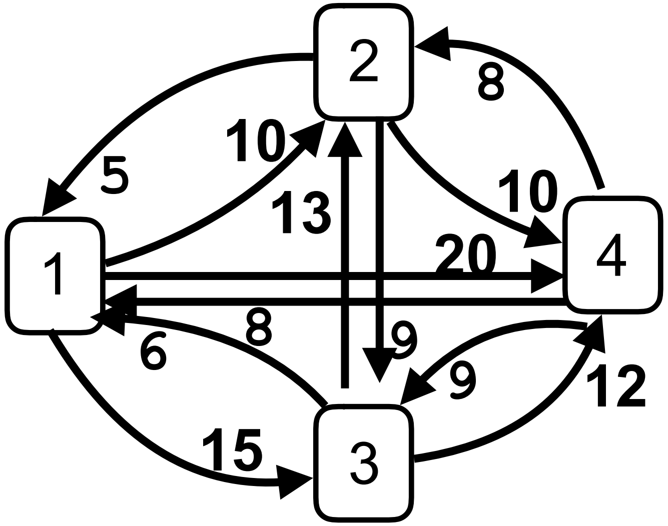
$g(4,\{2\})=c_{42}=g(2,\emptyset)=8+5=13$

$g(4,\{3\})=c_{43}=g(3,\emptyset)=9+6=15$

$J(2,\{3\})=3, J(2,\{4\})=4, J\{3,\{2\}=2$

$J(3,\{4\})=4, J(4,\{2\})=2, J(4,\{3\}=3$

TSP Example: Computation



	1	2	3	4
1	0	10	15	20
2	5	0	9	10
3	6	13	0	12
4	8	8	9	0

g(2,{3})=15, g(2,{4})=18,g(3,{2})=18,
g(3,{4})=20, g(4,{2})=13, g(4,{3})=15

Compute g(i,S),for |S|=2

$g(2,\{3,4\})=\min\{c_{23}+g(3,\{4\}),c_{24}+g(4,\{3\})$

$=\min\{9+20,\ 10+15\} = 25$ J(2,{3,4})=4

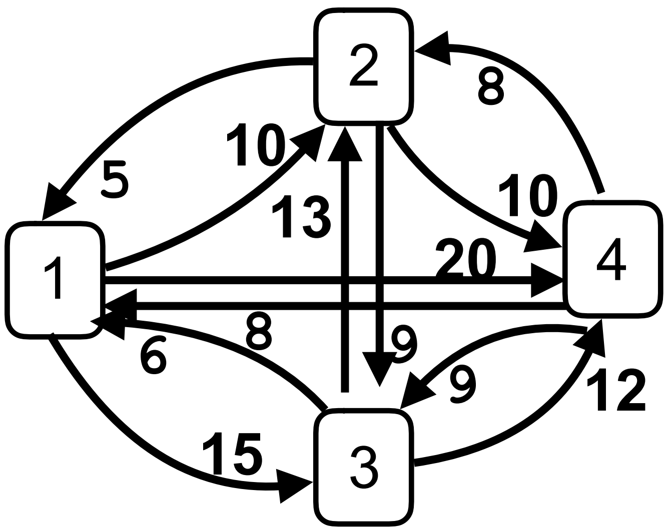
$g(3,\{2,4\})=\min\{c_{32}+g(2,\{4\}),c_{34}+g(4,\{2\})$

$=\min\{13+18,\ 12+13\} = 25$ J(3,{2,4})=4

$g(4,\{2,3\})=\min\{c_{42}+g(2,\{3\}),c_{43}+g(3,\{2\})$

$=\min\{8+15,\ 9+18\} = 23$ J(2,{3,4})=3

TSP Example: Computation



	1	2	3	4
1	0	10	15	20
2	5	0	9	10
3	6	13	0	12
4	8	8	9	0

g(2,{3,4})=25, g(3,{2,4})=25, g(4,{2,3})=23,

Compute g(i,S), for |S|=3

g(1,{2,3,4})=

min{c_{12}+g(2,{3,4}),c_{13}+g(3,{2,4}),c_{14}+g(4,{2,3})}

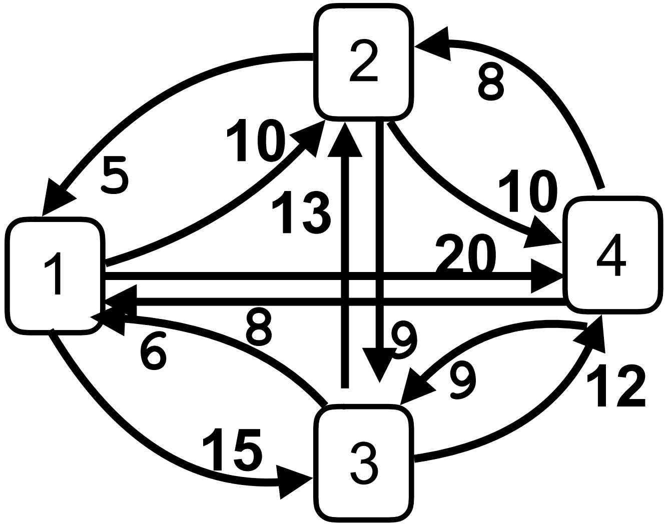
=min{10+25, 15+25, 20+23}

=35

J(1,{2,3,4})=2

- Thus, the optimal tour has length 35.

TSP Example: Tour Construction



	1	2	3	4
1	0	10	15	20
2	5	0	9	10
3	6	13	0	12
4	8	8	9	0

Knowing `J(1,{2,3,4})=2,`
`J(2,{3,4})=4`, and
`J(4,{3})=3`

The optimal tour is `1,2,4,3,1.`

Complexity Analysis

- For the n vertices in the graph,
  - There are 2^n subsets.
- For each subset, two kind of work is done
  - Addition (costs), comparison (to find minimum).
- Computation for each subset
  - go thru each vertex once to find the min cost path
  - for each vertex, check which is the right vertex before it.
  - Thus, work done n^2.

– Total time complexity: $O(n^2 2^n)$.

Summary

- Understanding TSP problem
- Application of Dynamic Programming
- Complexity analysis