

Design and Analysis of Algorithms

L05: Analysis Framework

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Resources

- Text book 1: Levitin
- <https://brainly.com/>

Algorithm Analysis

- Analysis
 - Detailed examination of the elements or structure of something, typically as a basis for discussion or interpretation.
 - *Mathematics*: the part of mathematics concerned with the theory of functions and the use of limits, continuity, and the operations of calculus.
- Analysis of algorithms:
 - Investigation of an algorithm's efficiency w.r.t. running time and memory space.

Algorithm Analysis

- Issues:
 - Correctness
 - Time efficiency
 - Space efficiency
 - Optimality
- Approaches:
 - Theoretical analysis
 - Empirical analysis
- From practical point of view:
 - Efficiency concerns are primary
 - Space (memory) is no more an issue today
 - Difference: secondary, primary, cache
 - We will mostly study time efficiency

Time vs Space Efficiency

- Consider multiplication of any two digit numbers
 - Example: Multiply $79 * 67$
- Time efficiency requires computation
 - Number of multiplications (single digit): 4
 - Number of additions (single digit): 5 (or 4)
 - Total mathematical operations: 9
 - Total space requirement: 14 digits
- Space (memory) efficiency
 - Store all multiplication values in an 100×100 array.
 - Then do a lookup.
 - Time requirement: 2 lookup
 - Space requirement: 10000 memory locations

Measuring Input Size

- Representing a polynomial:
 - $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
 - l/p size: $n+1$
 - or n (+1 is fixed and ignored, inconsequential)
- At times choice of parameter specifying the input size is important
 - Consider $n \times n$ matrix multiplication
 - Is input size n or n^2 ?
 - If considering matrix order: size is n
 - If consider number of elements: size is n^2
 - Latter is preferred as it works for $n \times m$ matrix too

Input Size and Operation Examples

Problem	Input size measure	Basic operation
Searching for key in a list of n items	Number of list's items, i.e. n	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Checking primality of a given integer n	number of digits (in binary representation)	Division
Typical graph problem	number of vertices and/or edges	Visiting a vertex or traversing an edge

Empirical Analysis of Time Efficiency

- Select a specific (typical) sample of inputs
- Choices:
 - Physical unit of time (e.g., milliseconds), or
 - Count actual number of basic operation's executions
- Physical unit of time varies depending upon computer being used, operations available
 - Count of basic operations is considered.
 - This is further approximated, need not be actual value
- Analyze the empirical data

Theoretical Analysis: Time Efficiency

- Time efficiency is analyzed by determining the number of repetitions of the basic operation as a function of input size
- Basic operation: the operation that contributes the most towards the running time of the algorithm
 - $T(n) \approx c_{op} C(n)$
 - c_{op} is cost of operation
 - $C(n)$ represents number of operations

Time Efficiency

- Consider sum of n numbers
 - Total sum operations $C(n) = n-1 \approx n$
 - $T(n) = c_{op}C(n) = c_{op}n$
 - $T(2n) = c_{op}C(2n) = c_{op}(2n) = c_{op}2n$
 - $T(2n)/T(n) = 2$
 - $T(10n)/T(n) = 10$
- Summary: c_{op} does not play a role when comparing the performance on two inputs.
 - **Order of growth** is more important.
 - Lower order terms and constants multiple are not important
 - These are subsumed in **order of growth**.

Best, Worst and Average case

- For some algorithms, efficiency depends on form of input:
- Worst case: $C_{\text{worst}}(n)$ – max over inputs size n
- Best case: $C_{\text{best}}(n)$ – min over inputs of size n
- Avg case: $C_{\text{avg}}(n)$ – “average” over inputs of size n
 - Number of times the basic operation will be executed on typical input
 - NOT the average of worst and best case
 - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs.
 - So, avg = expected under uniform distribution.

Example: Sequential Search

- **Algorithm: SequentialSearch($A[0 \dots n-1], K$)**
// searches for a value K in a given array by sequential searching
// Input: an array $A[0 \dots n-1]$, and key K
// Output: Index of the element that matches K
// -1 , if element is not found
 $i \leftarrow 0$
 while $i < n$ **and** $A[i] \neq K$ **do**
 $i \leftarrow i + 1$
 if $i < n$ **then**
 return i
 else
 return -1
- **What is Worst, Best and Average case analysis?**

Sequential Search: Avg Case Analysis

- Let p be the probability of finding key K
 - Thus, probability of not finding K is $(1-p)$
- Probability of finding K at position i is same i.e. p/n
- The expected number of searches $C_{avg}(n)$ is given by
$$\begin{aligned} & 1 * p/n + 2 * p/n + \dots + n * p/n + (1-p) * n \\ &= (p/n) (1+2+\dots+n) + (1-p) * n \\ &= (p/n) * n(n+1) / 2 + (1-p) * n \\ &= p * (n+1) / 2 + (1-p) * n \end{aligned}$$
- When $p=1$ i.e. search is always successful
$$C_{avg}(n) = (n+1) / 2 \approx n/2$$
- When $p=0$ i.e. search is failure
$$C_{avg}(n) = n$$

Sequential Search Analysis

- Computation of average case analysis is lot more complex than best case and worst case.

- For this specific problem

$$C_{\text{avg}}(n) = (C_{\text{worst}}(n) + C_{\text{best}}(n)) / 2$$

- This is not true in general and can't be simplified this way. Not a legitimate way.

Order of Growth

n	$\log_2 n$	\sqrt{n}	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	3.16	10	33.2	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10	10^2	664	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	31.6	10^3	9965	10^6	10^9		
10^4	13.2	100	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	16.5	316.2	10^5	$1.6 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	1000	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Amortised Efficiency

- So far, efficiency is related to a single run of algorithm.
- In some cases, single run can be very expensive, but subsequent run can be much cheaper
 - Real life example:
 - To drink water, dig a well.
 - First time very costly, subsequently minimal cost
 - Giving a lecture first time on new topic
 - Subsequent lectures on same topic much easier
- Thus, amortise the cost over n operations

Exercises - A

- For the following algorithms (problems), identify
 - Natural input size metric
 - Basic operation
 - Count of basic operation (average case)
- P01: Computing sum of n numbers
- P02: Computing factorial(n)
- P03: Finding largest element of n numbers
- P04: List all prime numbers $\leq n$ using Sieve method
- P05: Multiplying 2 numbers each of n digits

Exercises-B

- Glove selection: There are 22 gloves in a drawer: 5 pairs of red gloves, 4 pairs of yellow, and 2 pairs of green.
- You select two gloves in the dark and can check them only after a selection has been made. What is the smallest number of gloves you need to select to have (guarantee) the following:

(Hint: Gloves are left and right side)

- At least one matching pair? (worst case)
- At least one matching pair in the best case?
- At least one matching pair of each color? (worse case)
- Ans:
 - one matching pair: worst case: 12, best case: 2
 - one matching pair of each color: 21

Exercises-C

Missing socks: Imagine that after washing 5 distinct pairs of socks, you discover that two socks are missing. Of course, you would like to have the largest number of complete pairs remaining. Thus, you are left with 4 complete pairs in the best-case scenario and with 3 complete pairs in the worst case. Assuming that the probability of disappearance for each of the 10 socks is the same, find the probability of

- Q C_1 : The best-case scenario;
- Q C_2 : The worst-case scenario;
- Q C_3 : The number of pairs you should expect in the average case.
- Answers:
 - C_1 : $5/45 = 1/9$
 - C_2 : $40/45 = 8/9$
 - C_3 : $4 \cdot 1/9 + 3 \cdot 8/9 = 28/9 = 3\frac{1}{9}$

Summary :Analysis Framework

- Both time and space efficiencies are measured as functions of the algorithm's input size
- Time efficiency is measured by counting the number of times the base operation of algorithm is executed.
- Space efficiency is measured by counting the number of extra memory units consumed by algo.
- Efficiency for same algorithm may vary significantly for inputs of same size.
 - Worst case, Best case, and Average case
- Framework primary interest in order of growth
 - Running time of algorithm