Design and Analysis of Algorithms

L33: Warshall & Floyd Algo
## Dynamic Programming

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Resources

- Text book 1: Levitin
  - Sec **8.2**, 8.3, 8.4
- Text book 2: Horowitz
  - Sec 5.1, 5.2, 5.4, 5.8, 5.9
- R1: Introduction to Algorithms
    - Cormen et al.

Transitive Closure

- Computes the transitive closure of a relation
- Alternatively:
  - existence of all nontrivial paths in a digraph
- Example of transitive closure:
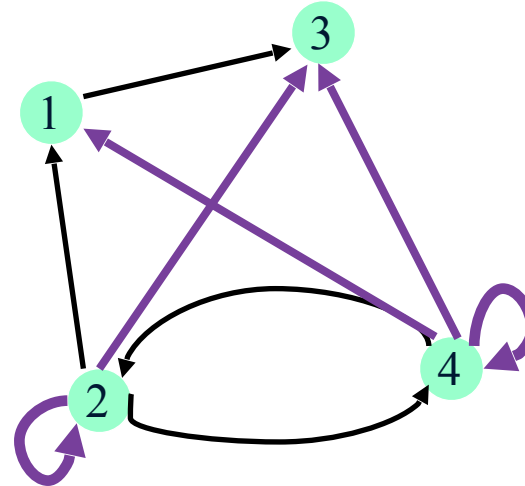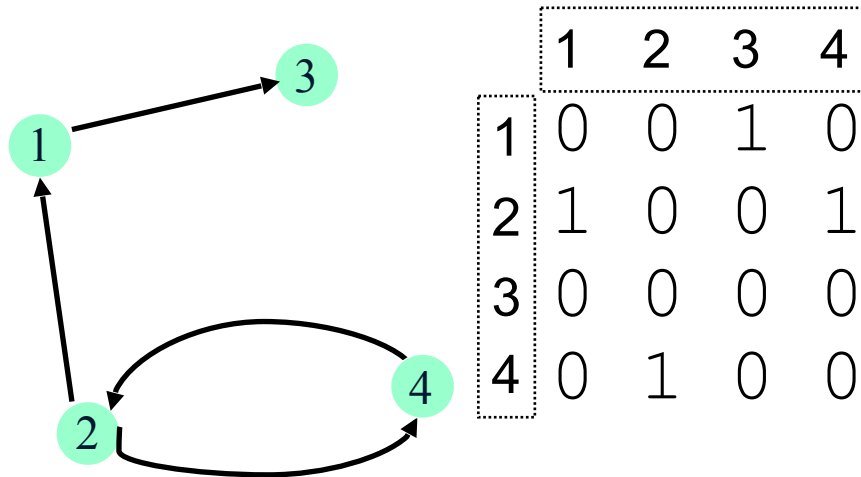
Warshall's Approach

- Constructs transitive closure T as the last matrix in the sequence of n-by-n matrices

  $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where

- $R^{(k)}[i,j]=1$ iff

  - there is nontrivial path from i to j
  - with only the first k vertices (numbered from 1 to k) are allowed as intermediate

- Note that
  - $R^{(0)} = A$ (adjacency matrix),
  - $R^{(n)} = T$ (transitive closure)

Warshall's algo: Recurrence

- On the kth iteration,
  - the algo determines for every pair of vertices i, j
  - if a path exists from i to j
    - with just vertices $1, ..., k$ allowed as intermediate

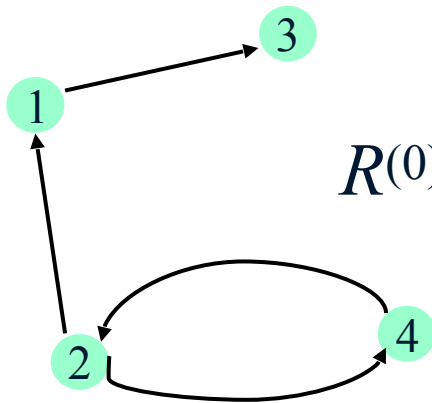$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] \quad \text{(path using just } 1, ..., k-1\text{)} \\ \textbf{or} \\ R^{(k-1)}[i,k] \textbf{ and } R^{(k-1)}[k,j] \\ \text{(path from i to k and from k to j, using just } 1, ..., k-1\text{)} \end{cases}$$
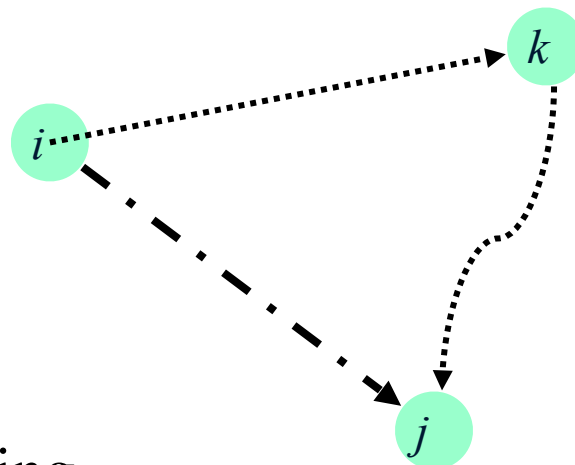
Warshall's algo: Matrix Generation

- Recurrence relating elements $R^{(k)}$ to elements of $R^{(k-1)}$ is:

  $\texttt{R}^{(k)}\texttt{[i,j]=R}^{(k-1)}\texttt{[i,j]}$ **or** $(\texttt{R}^{(k-1)}\texttt{[i,k]}$ and $\texttt{R}^{(k-1)}\texttt{[k,j]})$

- It implies the following rules for generating $R^{(k)}$ from $R^{(k-1)}$:

  - Rule 1: If an element in row i and column j is 1 in $R^{(k-1)}$, it remains 1 in $R^{(k)}$

  - Rule 2: If an element in row i and column j is 0 in $R^{(k-1)}$,

    - it has to be changed to 1 in $R^{(k)}$ iff the element in its row i and column k and the element in its row k and column j are both 1's in $R^{(k-1)}$

Warshall's algo: Example

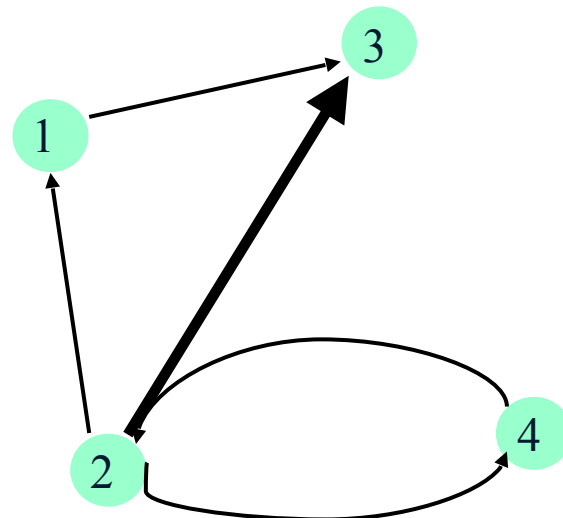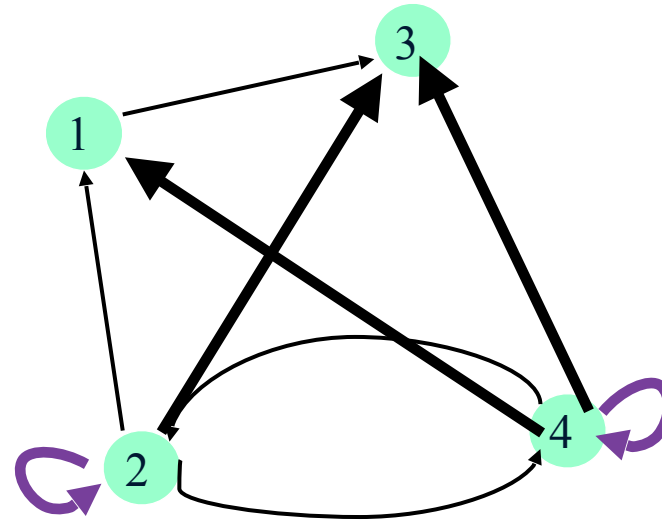$$R^{(0)} = \begin{array}{|cccc|} \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \end{array}$$

$$R^{(1)} = \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

Warshall's algo: Example

$$R^{(1)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

$$R^{(2)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

$$R^{(3)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

No Change

$$R^{(4)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Warshall's Algo: Analysis

```
Algo Warshall(A[1..n,1..n])
```
// i/p: Adjacency matrix A of a diagraph with n vertices
// o/p: Transitive closure of diagraph

$R^{(0)} \leftarrow A$

for $k \leftarrow 1$ **to** n **do**

    for $i \leftarrow 1$ **to** n **do**

        for $j \leftarrow 1$ **to** n **do**

            $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ **OR**

                $(R^{(k-1)}[i,k]$ **AND** $R^{(k-1)}[k,j])$
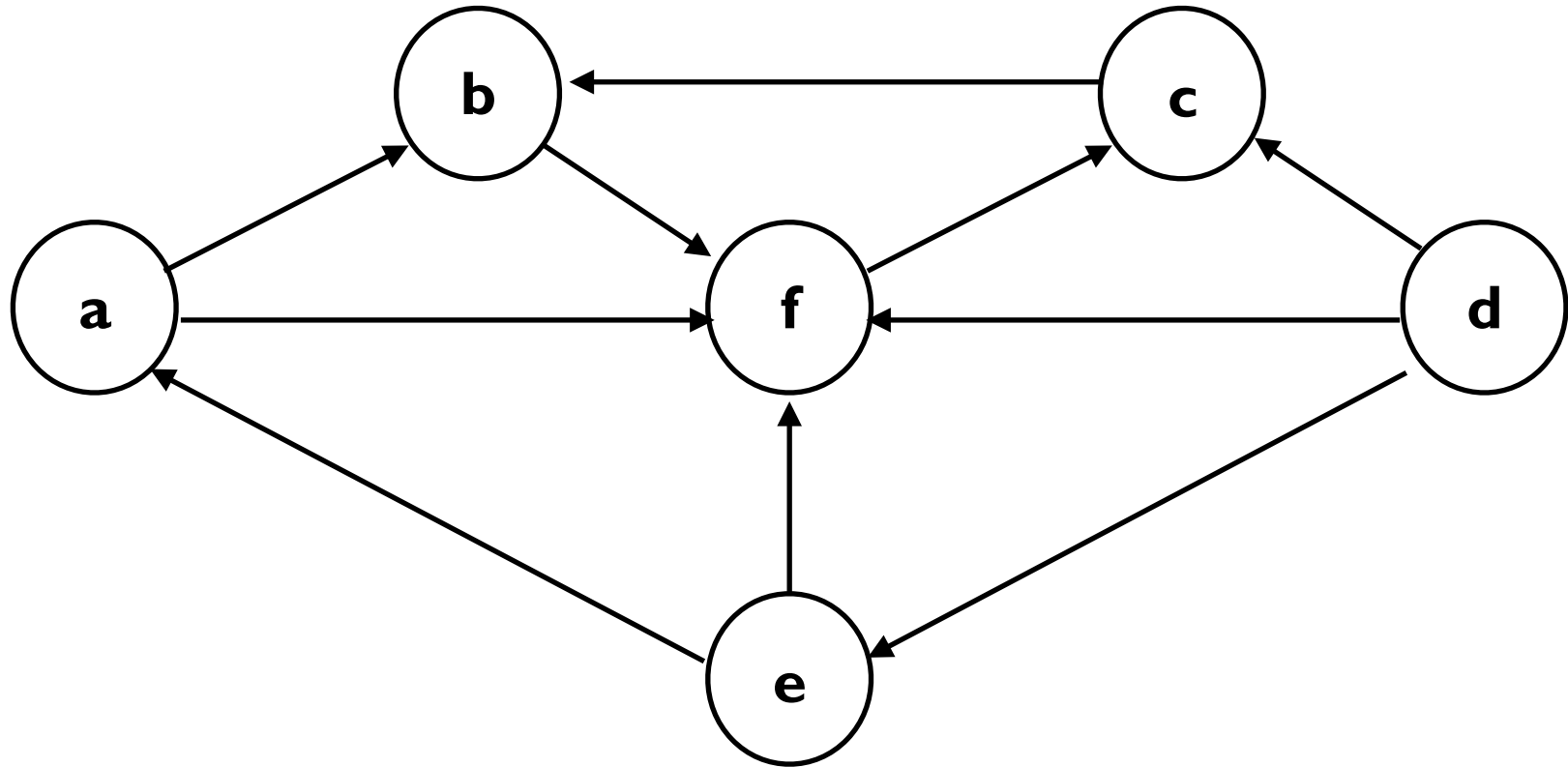
return $R^{(n)}$

Time efficiency: $\Theta(n^3)$

Space efficiency:

    Matrices can be written over their predecessors (with some care), so it's $\Theta(n^2)$.
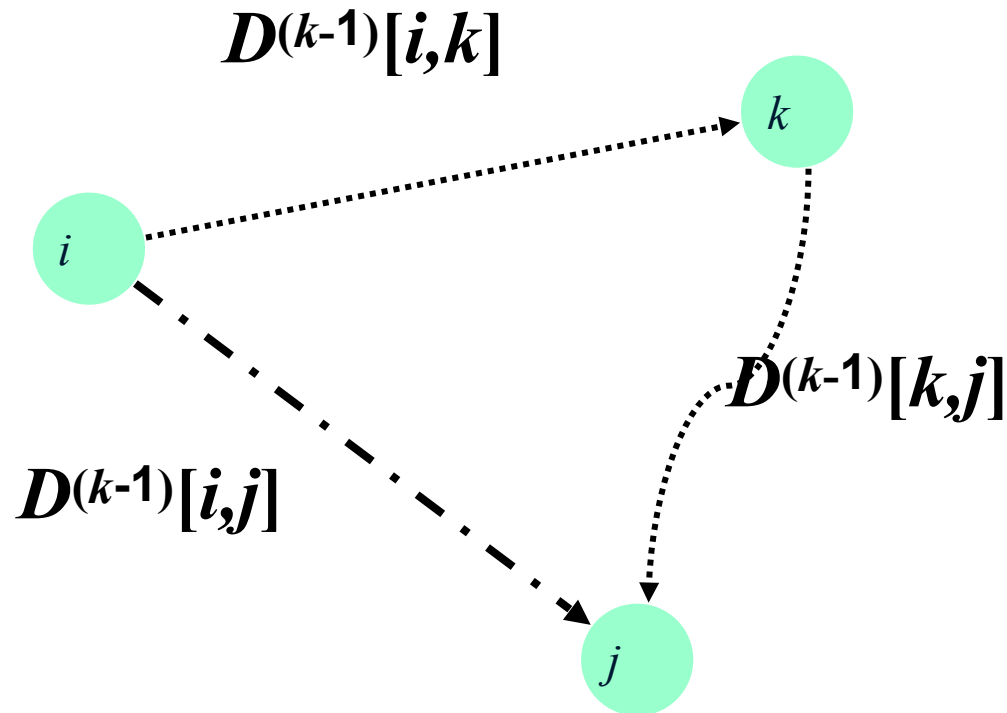
Exercise:

- Ex: Construct transitive closure for below graph

Floyd's Algorithm: Matrix Generation
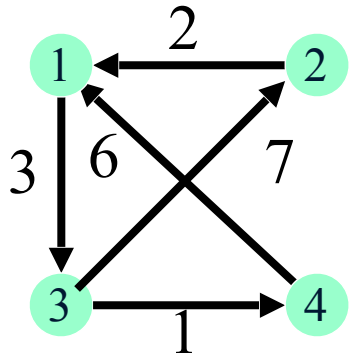
- On the $k^{\texttt{th}}$ iteration,
  - the algorithm determines shortest paths between every pair of vertices i, j that use only vertices among $1, \dots, k$ as intermediate

$$D^{(k)}[i,j] = \min \{D^{(k-1)}[i,j], \ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\}$$

$$\boldsymbol{D^{(k\text{-}1)}[i,k]}$$

$$\boldsymbol{D^{(k\text{-}1)}[k,j]}$$

$$\boldsymbol{D^{(k\text{-}1)}[i,j]}$$

Example: Floyd Algo

$$D^{(0)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix}$$

$$D^{(1)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{bmatrix}$$

$$D^{(2)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix}$$

$$D^{(3)} = \begin{bmatrix} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{bmatrix}$$

$$D^{(4)} = \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix}$$

DAA/Dynamic Programming

Floyd Algo: Analysis

```
Algo Floyd(A[1..n,1..n])
```
// i/p: Weight matrix W of a diagraph A with `n` vertices
// o/p: Distance matrix of shortest path lengths
```
D ← W
```
for `k←1` **to** `n` **do**
    for `i←1` **to** `n` **do**
        for `j←1` **to** `n` **do**
            `D[i,j]←min{D[i,j],D[i,k]+D[k,j]}`
            if `D[i,k]+D[k,j] < D[i,j]` **then**
              `P[i,j] ← k`
   `return D`

Time efficiency: $\Theta(n^3)$

Space efficiency: Matrices can be written over their predecessors (with some care), so it's $\Theta(n^2)$.

Exercise:

- Ex: Find all pair shortest distance for below graph
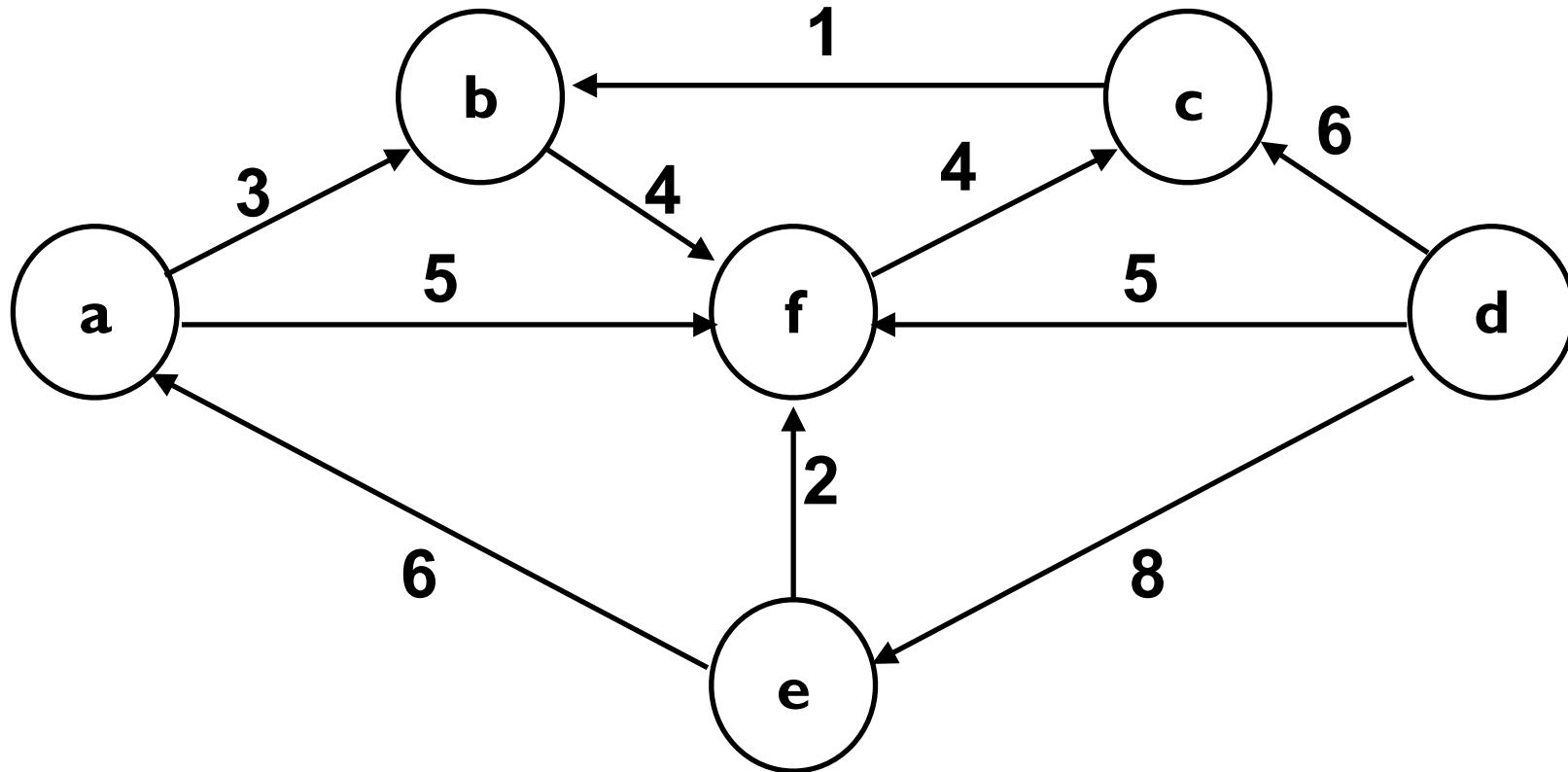
Summary

- Transitive closure
- Warshall Algorithm
- Floyd Algorithm