Learning model parameters under partial observability

1. Model

A system with state x_t evolves overtime according the recursive equation

1. $P(x_{t+1} = x \mid x_t, \theta) = f(x \mid x_t, \theta)$

where θ is a set of parameters that govern the systems dynamics.

Each year an observers makes a measurement Y_t that contains information about the value of x_t. The observer knows the distribution of Y_t conditional on x_t

2. $P(Y_t \mid x_t) = h(Y_t \mid x_t)$.

At $t = 0$ the observer has beliefs over the true state of the system and the parts that govern its dynamics expressed as a joint probability density

3. $P(x_0, \theta) = b(x_0, \theta)$.

2. Belief dynamics

Our goal will be to derive the observers beliefs over x_{t+1} and θ after they observe Y_{t+1}, given their current beliefs $b(x_t, \theta)$.

To start I define the joint distribution over x_{t+1}, x_t and θ before they observe Y_{t+1} as

4. $P(x_{t+1}, x_t, \theta) = P(x_{t+1} \mid x_t, \theta)P(x_t, \theta) = f(x_{t+1} \mid x_t, \theta)b(x_t, \theta)$

This distribution can be updated using Baye's rule which states

5. $P(x_{t+1}, x_t, \theta \mid Y_{t+1}) \propto P(Y_{t+1} \mid x_{t+1}, x_t, \theta)P(x_{t+1}, x_t, \theta)$

where the left hand side is the updated joint distribution, the first term on the right hand side is the probability of observing Y_{t+1} and the second term is the joint distribution of x_t, x_{t+1} and θ from equation 4.

We can rewrite this expression in terms given by the model by substituting the first term on the right hand side with equation 2, and the second term with equation 4. Making these substitutions yields

6. $P(x_{t+1}, x_t, \theta \mid Y_{t+1}) \propto h(Y_{t+1} \mid x_{t+1})f(x_{t+1} \mid x_t, \theta)b(x_t, \theta)$.

Finally we can obtain the updated belief state by integrating over x_t

7. $b(x_{t+1}, \theta) = \int_X P(x_{t+1}, x_t, \theta \,|\, Y_t) dx_t.$

Combining equations size and seven, and writing out the normalizing constant gives a recursive formula for the observers beliefs.

8. $b'(x_{t+1}, \theta) = \int_X \frac{h(Y_{t+1} \,|\, x_{t+1}) f(x_{t+1} \,|\, x_t, \theta) b(x_t, \theta)}{\int_\Theta \int_X \int_X h(Y_{t+1} \,|\, u) f(u \,|\, v, w) b(v, w) du dv dw} dx_t.$

3. Remark 1

Equation six provides useful intuition for the updating process. First, the beliefs are updated by making a prediction based on current knowledge (multiplication by f) and then they are updated by making the observation (multiplication by h).

4. Remark 2

In order to fully capture the dynamics of the observers beliefs we need to model them as a joint distribution over the state variable x_t and parameters θ. In other word the belief state $b(x_t, \theta)$ cannot be separated into two indtepend distribution or in equation form

9. $b(x_t, \theta) \neq b_x(x_t) b_\theta(\theta)$ for any probability densities b_x and b_θ .

This statement may be clear from equation six and eight, but we can show this more explicitly by supposing that $b(x_t, \theta) = b_x(x_t) b_\theta(\theta)$ and $b'(x_{t+1}, \theta) = b'_x(x_{t+1}) b'_\theta(\theta)$

Now consider the simple case where the observation Y_{t+1} is not informative in this case equation eight simplified to

10. $b'(x_{t+1}, \theta) = \int_X f(x_{t+1} \,|\, x_t, \theta) b(x_t, \theta) dx_t.$

Furthermore, by hypothesis

11. $b'_x(x_{t+1}) b'_\theta(\theta) = \int_X f(x_{t+1} \,|\, x_t, \theta) b_x(x_t) b_\theta(\theta) dx_t.$

Integrating the right hand side yields

12. $b'_x(x_{t+1}) b'_\theta(\theta) = P(x_{t+1} \,|\, \theta) b_\theta(\theta)$

but this is a contradiction because the first term on the right hand side states a dependence between θ and x_{t+1}. $\square$