ROBOT DYNAMICS

~~[illegible]~~ Tutorial.

WE HAVE CONSIDERED KINEMATICS
- STATIC ? POSITIONS
- VELOCITIES

WHAT FORCES ARE REQUIRED TO CAUSE MOTION?

NOW WE CONSIDER THE EQUATIONS OF MOTION FOR THE MANIPULATOR

TYPICALLY WE MODEL THESE BY A SET OF DIFFERENTIAL EQUATIONS

WE CONSIDER $\theta, \dot{\theta}, \ddot{\theta}, d, \dot{d}, \ddot{d}$

WE USE LAGRANGIAN DYNAMICS TO DERIVE THE ROBOT SYSTEM DIFFERENTIAL EQUATIONS

LAGRANGIAN FUNCTION

DIFFERENCE BETWEEN THE TOTAL KINETIC ENERGY AND THE TOTAL POTENTIAL ENERGY STORED IN THE SYSTEM

$$L = \underbrace{K(\underset{\sim}{q}, \dot{\underset{\sim}{q}})}_{\text{(KINETIC)}} - \underbrace{P(\underset{\sim}{q})}_{\text{(POTENTIAL)}}$$

$\underset{\sim}{q}$ - Vector of System States

LAGRANGIAN EQUATIONS (DIFFERENTIAL EQUATIONS OF MOTION OF THE SYSTE[M]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underset{\sim}{q}}_i}\right) - \frac{\partial L}{\partial \underset{\sim}{q}_i} = \underset{\sim}{\tau}_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underset{\sim}{q}}_i}\right) - \frac{\partial L}{\partial \underset{\sim}{q}_i} = 0$$

PRACTICALLY:

- Find equations for x, y, z etc.
- Calculate $\dot{x}$, $\dot{y}$, $\dot{z}$
- Calculate K & P

$$K = \frac{1}{2} m v^2 \qquad P = mgh$$

- Calculate Lagrangian Function (L)

$$L = K - P$$

- Calculate

$$\frac{\partial L}{\partial \theta_1} \quad \frac{\partial L}{\partial \theta_2} \quad \frac{\partial L}{\partial \theta_3} \quad \text{etc.}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} \quad \frac{\partial L}{\partial \dot{\theta}_2} \quad \frac{\partial L}{\partial \dot{\theta}_3} \quad \text{etc}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) \quad \text{etc.}$$

- Calculate

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$\tau_3 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3}$$

etc.

1. Consider a particle with mass m in the frame xyz shown in Figure 1. The length of the weightless string is r, the angle between the string and the z axis is θ, and the angle between the plane, containing both the particle and the z axis, and the plane xoz is φ. Derive the differential equations of motions.

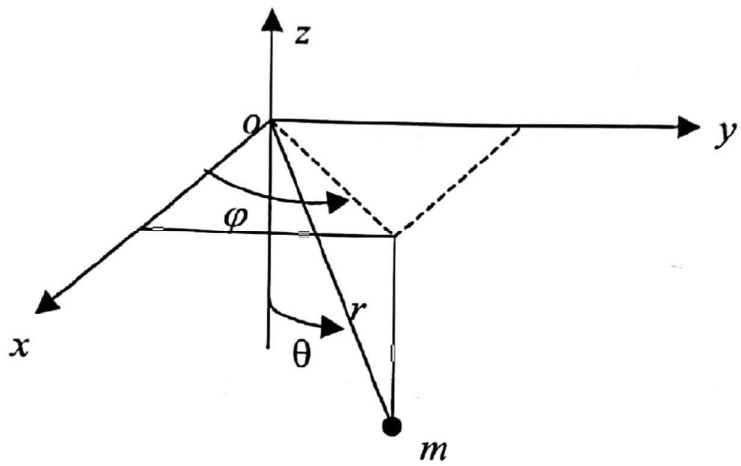
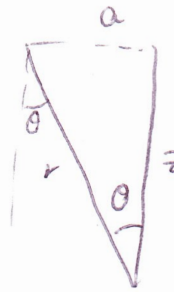


Figure 1. A particle with mass m in frame xyz

consider the triangle below the x-y plane made up of the string and the vertical.



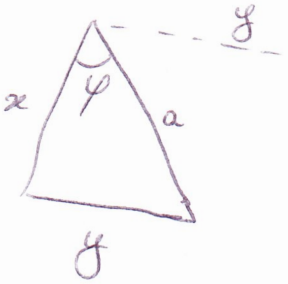
$$\sin\theta = \frac{a}{r}$$

$$a = r\sin\theta$$

$$\cos\theta = \frac{-z}{r}$$

$$z = -r\cos\theta$$

Now consider the x, y plane.



$$\sin\varphi = \frac{y}{a}$$

$$y = a\sin\varphi$$
$$= r\sin\theta\sin\varphi$$

$$\cos\varphi = \frac{x}{a}$$

$$x = a\cos\varphi$$
$$= r\sin\theta\cos\varphi$$

$$\therefore \quad x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = -r\cos\cancel{\varphi}\theta$$

Now calculate $\dot{x}, \dot{y}, \dot{z}$

Wk 9, Q1,

$$\dot{x} = (r\cos\theta\cos\varphi)\dot{\theta} - (r\sin\theta\sin\varphi)\dot{\varphi}$$

$$\dot{y} = (r\cos\theta\sin\varphi)\dot{\theta} + (r\sin\theta\cos\varphi)\dot{\varphi}$$

$$\dot{z} = \cancel{r\sin\varphi}\ (r\sin\theta)\dot{\theta}$$

Calculate K, P and L.

$$K = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{m}{2}\Big(\big((r\cos\theta\cos\varphi)\dot{\theta} - (r\sin\theta\sin\varphi)\dot{\varphi}\big)^2$$

$$+ \big((r\cos\theta\sin\varphi)\dot{\theta} + (r\sin\theta\cos\varphi)\dot{\varphi}\big)^2$$

$$+ \big((r\sin\theta)\dot{\theta}\big)^2\Big)$$

$$= \frac{m}{2}\Big(\cancel{r\cos r\dot{\theta}\cos\theta}$$

$$r^2\dot{\theta}^2\cos^2\theta\cos^2\varphi \boxed{- 2r^2\cos\theta\cos\varphi\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi}}$$

$$+ r^2\dot{\varphi}^2\sin^2\theta\sin^2\varphi$$

$$+ r^2\dot{\theta}^2\cos^2\theta\sin^2\varphi \boxed{+ 2r^2\cos\theta\sin\varphi\sin\theta\cos\varphi\,\dot{\theta}\dot{\varphi}}$$

$$+ r^2\dot{\varphi}^2\sin^2\theta\cos^2\varphi$$

$$+ r^2\dot{\theta}^2\sin^2\theta\Big)$$

Wk9, Q1, Pg

$$= \frac{m}{2}\left(r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\varphi}^2\sin^2\theta + r^2\dot{\theta}^2\sin^2\theta\right)$$

$$K = \frac{m}{2}\left(r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right)$$

$$P = mgh = mgz$$
$$= -mgr\cos\theta$$

$$L = K - P$$
$$= \frac{m}{2}\left(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + mgr\cos\theta$$

Now calculate

$$\frac{\partial L}{\partial\dot{\theta}} \quad \frac{\partial L}{\partial\theta} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) \quad \frac{\partial L}{\partial\dot{\varphi}} \quad \frac{\partial L}{\partial\varphi} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right)$$

$$\frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mr^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 - mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\theta \, \dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m r^2 \sin^2\theta \, \ddot{\varphi} + 2 m r^2 \sin\theta \cos\theta \, \dot{\varphi} \, \dot{\theta}$$

$$\frac{\partial L}{\partial \varphi} = 0$$

Substitute into $\displaystyle \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

and $\displaystyle \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad m r^2 \ddot{\theta} - m r^2 \sin\theta \cos\theta \, \dot{\varphi}^2 + m g r \sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

$$\Rightarrow \quad m r^2 \sin^2\theta \, \ddot{\varphi} + 2 m r^2 \sin\theta \cos\theta \, \dot{\varphi} \, \dot{\theta} = 0$$

2. Find the dynamics of a two-link planar RR arm shown in Figure 2, where we have assumed that the link masses are concentrated at the ends of the links. The joint variable (state variable) vector is $q = [\theta_1 \ \theta_2]^T$ and the generalized force vector is $u = [\tau_1 \ \tau_2]^T$, where τ_1 and τ_2 are torques supplied by the actuators

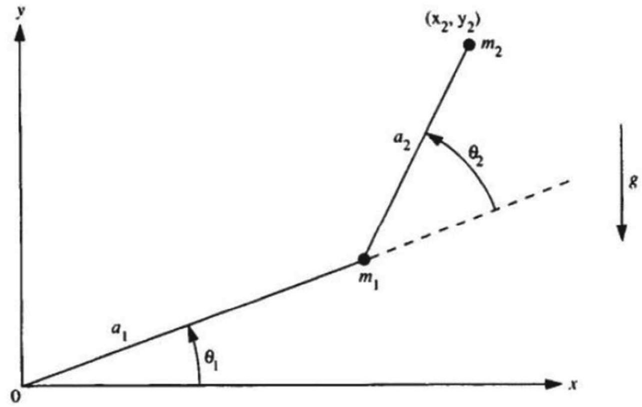
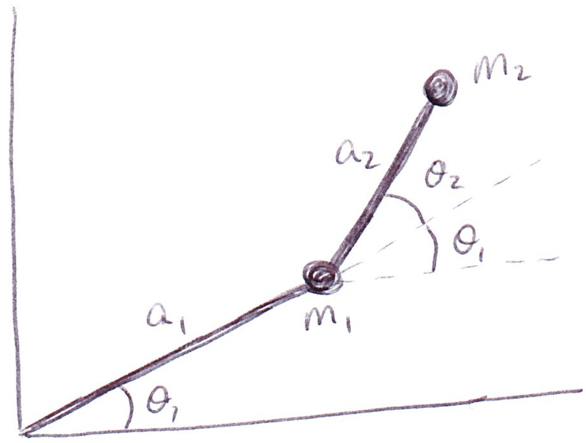


Figure 2. Two-link planar RR



Find x_1, y_1, x_2, y_2, $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_1$, $\dot{y}_2$

$x_1 = a_1 \cos\theta_1$ $\quad$ $x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$

$y_1 = a_1 \sin\theta_1$ $\quad$ $y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$

$$\dot{x}_1 = -a \sin\theta_1 \dot{\theta}_1$$

$$\dot{y}_1 = a_1 \cos\theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = -a_1 \sin\theta_1 \dot{\theta}_1 - a_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{y}_2 = a_1 \cos\theta_1 \dot{\theta}_1 + a_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

Calculate

K_1 , K_2, P_1 , P_2

$$K_1 = \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2)$$

$$= \frac{m_1}{2} \left(a_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2 + a_1 \cos^2\theta_1 \, \dot{\theta}_1^2 \right)$$

$$= \frac{m_1}{2} \left(a_1^2 \dot{\theta}_1^2 \right)$$

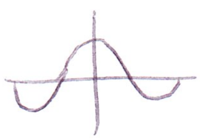
$$K_2 = \frac{1}{2} m_2 v_2^2$$

$$= \frac{m_2}{2} \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

$$= \frac{m_2}{2} \Big(a_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2 + 2 a_1 a_2 \boxed{\sin\theta_1 \sin(\theta_1+\theta_2)} \dot{\theta}_1 (\dot{\theta}$$
$$+ a_2^2 \sin^2(\theta_1+\theta_2) (\dot{\theta}_1 + \dot{\theta}_2)^2$$
$$+ a_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + 2 a_1 a_2 \boxed{\cos\theta_1 \cos(\theta_1+\theta_2)} \dot{\theta}_1 (\dot{\theta}$$
$$+ a_2^2 \cos^2(\theta_1+\theta_2) (\dot{\theta}_1 + \dot{\theta}_2)^2 \Big)$$

$$= \frac{m_2}{2} \Big(a_1^2 \dot{\theta}_1^2 + a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 a_1 a_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)$$
$$\left(\cos(\theta_1 - (\theta_1 + \theta \right.$$

$\cos(-\theta) = \cos(\theta)$



$$= \frac{m_2}{2} \Big(a_1^2 \dot{\theta}_1^2 + a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 a_1 a_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2 \Big)$$

$$P_1 = m_1 g y_1$$
$$= m_1 g a_1 \sin\theta_1$$

$$P_2 = \cancel{m_2} m_2 g y_2$$
$$= m_2 g (a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2))$$

$$L = K - P = K_1 + K_2 - P_1 - P_2$$

$$= \frac{m_1}{2}(a_1^2 \dot\theta_1^2) + \frac{m_2}{2}(a_1^2 \dot\theta_1^2) + \frac{m_2}{2}(a_2^2(\dot\theta_1 + \dot\theta_2)^2)$$
$$+ \frac{m_2}{2}(2 a_1 a_2 \dot\theta_1 (\dot\theta_1 + \dot\theta_2) \cos\theta_2)$$
$$- m_1 g a_1 \sin\theta_1 - m_2 g a_1 \sin\theta_1 - m_2 g a_2 \sin(\theta_1 + \theta_2)$$

$$= \frac{m_1 + m_2}{2}(a_1^2 \dot\theta_1^2) + \frac{m_2}{2} a_2^2 (\dot\theta_1 + \dot\theta_2)^2$$
$$+ m_2 a_1 a_2 \dot\theta_1 (\dot\theta_1 + \dot\theta_2) \cos\theta_2 - (m_1 + m_2) a_1 g \sin\theta_1$$
$$- m_2 g a_2 \sin(\theta_1 + \theta_2)$$

Wk 9, Q2, P

$$\frac{\partial L}{\partial \dot\theta_1} = (m_1+m_2)a_1^2\dot\theta_1 + m_2a_2^2(\dot\theta_1+\dot\theta_2)$$

$$+ 2m_2a_1a_2\cos\theta_2\,\dot\theta_1 + m_2a_1a_2\dot\theta_2\cos\theta_2$$

$$= (m_1+m_2)a_1^2\dot\theta_1 + m_2a_2^2(\dot\theta_1+\dot\theta_2) + m_2a_1a_2(2\dot\theta_1+\dot\theta_2)c$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = (m_1+m_2)a_1^2\ddot\theta_1 + m_2a_2^2(\ddot\theta_1+\ddot\theta_2)$$

$$+ 2m_2a_1a_2\cos\theta_2\,\ddot\theta_1 - 2m_2a_1a_2\,\cancel{\cos}\sin\theta_2(\dot\theta_1\dot\theta_2)$$

$$+ m_2a_1a_2\ddot\theta_2\cos\theta_2 - m_2a_1a_2\dot\theta_2\sin\theta_2\,\dot\theta_2$$

$$= (m_1+m_2)a_1^2\ddot\theta_1 + m_2a_2^2(\ddot\theta_1+\ddot\theta_2)$$

$$+ m_2a_1a_2\cos\theta_2(2\ddot\theta_1+\ddot\theta_2)$$

$$- m_2a_1a_2\sin\theta_2(2\dot\theta_1\dot\theta_2+\dot\theta_2^2)$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1+m_2)ga_1\cos\theta_1 - m_2ga_2\cos(\theta_1+\theta_2)$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2a_2^2(\dot\theta_1+\dot\theta_2) + m_2a_1a_2\dot\theta_1\cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2a_2^2(\ddot\theta_1+\ddot\theta_2) + m_2a_1a_2\ddot\theta_1\cos\theta_2$$

$$- m_2a_1a_2\dot\theta_1\dot\theta_2\sin\theta_2$$

Wk9, Q2, Pg

$$\frac{\partial L}{\partial \theta_2} = -m_2 a_1 a_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) \sin\theta_2 - m_2 g a_2 \cos(\theta_1 + \theta_2)$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$\tau_1 = (m_1 + m_2) a_1^2 \ddot{\theta}_1 + m_2 a_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 a_1 a_2 \cos\theta_2 \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)$$

$$- m_2 a_1 a_2 \sin\theta_2 \left(2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2\right) + (m_1 + m_2) g a_1 \cos\theta_1$$

$$+ m_2 g a_2 \cos(\theta_1 + \theta_2)$$

$$= \left((m_1 + m_2) a_1^2 + m_2 a_2^2 + 2 m_2 a_1 a_2 \cos\theta_2\right) \ddot{\theta}_1$$

$$+ \left(m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2\right) \ddot{\theta}_2$$

$$- m_2 a_1 a_2 \sin\theta_2 \left(2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2\right)$$

$$+ (m_1 + m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2)$$

$$\tau_2 = m_2 a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 \ddot{\theta}_1 \cos\theta_2$$
$$- m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 + m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \sin\theta_2$$
$$+ m_2 g a_2 \cos(\theta_1 + \theta_2)$$

$$= (m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2)\ddot{\theta}_1 + m_2 a_2^2 \ddot{\theta}_2$$
$$+ m_2 a_1 a_2 \dot{\theta}_1^2 \sin\theta_2 + m_2 g a_2 \cos(\theta_1 + \theta_2)$$

We can extend this into matrix form

$$\underset{\sim}{u} = M(q)\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + H(q, \dot{q})\dot{q} + G(q)$$

$\underset{\sim}{u}$: Vector of control inputs

$M(q)$: Mass matrix of manipulator

$H(q,\dot{q})\dot{q}$: Vector of velocity terms

$G(q)$: Vector of gravity terms

q: Vector of system states

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2)a_1^2 + m_2 a_2^2 + 2 m_2 a_1 a_2 \cos\theta_2 & m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 \\ m_2 a_2^2 + m_2 a_1 a_2 \cos\theta_2 & m_2 a_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} -2 m_2 a_1 a_2 \sin\theta_2 \dot{\theta}_2 & - m_2 a_1 a_2 \sin\theta_2 \dot{\theta}_2 \\ m_2 a_1 a_2 \sin\theta_2 \dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} (m_1 + m_2) g a_1 \cos\theta_1 + m_2 g a_2 \cos(\theta_1 + \theta_2) \\ m_2 g a_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$