Eigenvalue decomposition and singular value decomposition

Eigenvalue decomposition (EVD) (spectral decomposition)

Given square, symmetric, positive-definite (p x p) matrix A (in statistics often Σ, S, ρ, R):

$$A = \sum_{i=1}^{p} \lambda_i e_i e_i^T = E\Lambda E^T$$

$$E = \begin{bmatrix} \uparrow & \dots & \uparrow \\ e_1 & \dots & e_p \\ \downarrow & \dots & \downarrow \end{bmatrix}$$ and E is orthogonal ($E^T = E^{-1}$)

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p \end{bmatrix}$$

$\lambda_i,\ i = 1..p$ the eigenvalues are solutions to characteristic equation $|A - \lambda I| = 0$
$e_i,\ i = 1..p$ the eigenvectors are solutions to eigenvalue problems $Ae_i = \lambda_i e_i,\ \ ||e_i|| = 1$

Singular value decomposition (SVD)

Given general (n x p) matrix B where $n > p = \mathrm{rank}(B)$ (in statistics often data matrix X, Z):

$$B = \sum_{i=1}^{p} \lambda_i u_i v_i^T = U\Lambda V^T$$ (rank-reduced, "economy" SVD form)

$$U = \begin{bmatrix} \uparrow & \dots & \uparrow \\ u_1 & \dots & u_p \\ \downarrow & \dots & \downarrow \end{bmatrix}$$ and $U^T U = I$ (U^T is pseudo-inverse of U)

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p \end{bmatrix}$$

$$V = \begin{bmatrix} \uparrow & \dots & \uparrow \\ v_1 & \dots & v_p \\ \downarrow & \dots & \downarrow \end{bmatrix}$$ and V is orthogonal ($V^T = V^{-1}$)

$\lambda_i,\ i = 1..p$ the singular values are **square roots** of the p eigenvalues of $B^T B$
$u_i,\ i = 1..p$ are the eigenvectors corresponding to the p non-zero eigenvalues of BB^T
$v_i,\ i = 1..p$ are the eigenvectors corresponding to the p eigenvalues of $B^T B$
