BME40003 Robot Systems Design

Inverse Kinematics Questions

Questions 3-8 from J. J. Craig Introduction to Robotics: Mechanics and Control 2nd Ed., Addison-Wesley, Massachusetts, 1989.

1. Find the solution of the following set of trigonometric equations

$$\begin{cases} a\cos\alpha - b\sin\alpha = c & \text{— ①} \\ a\sin\alpha + b\cos\alpha = d & \text{— ②} \end{cases}$$

Rewrite ② to give $\sin\alpha$

$$\sin\alpha = \frac{d - b\cos\alpha}{a} \quad \text{— ③}$$

Substitute ③ into ①

$$a\cos\alpha - \frac{bd}{a} + \frac{b^2\cos\alpha}{a} = c$$

$$a^2\cos\alpha + b^2\cos\alpha - bd = ca$$

$$a^2+b^2(\cos\alpha) = ca+bd$$

$$\cos\alpha = \frac{ca+bd}{a^2+b^2} \quad \text{— ④}$$

Substitute ④ into ③

$$\sin\alpha = \left(d - \frac{bca}{a^2+b^2} - \frac{b^2d}{a^2+b^2}\right)/a$$

$$= \left(\frac{a^2d + b^2d - \cancel{bc}\,abc - b^2d}{a^2+b^2}\right)/a$$

$$= \left(\frac{a^2d - abc}{a^2+b^2}\right)a = \frac{ad-bc}{a^2+b^2}$$

So we have $\cos\alpha = \dfrac{ac+bd}{a^2+b^2}$

& $\sin\alpha = \dfrac{ad-bc}{a^2+b^2}$

We can use the solution $\left\{\begin{array}{l|l} \sin\theta = a & \\ \cos\theta = b & \theta = \mathrm{Atan2}(a,b) \end{array}\right.$

$$\alpha = \mathrm{Atan2}\left(\frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}\right)$$

As atan2 is based on a ratio

$$\alpha = \mathrm{Atan2}(ac+bd, ad-bc)$$

Q2 pg 1

2. Derive the inverse kinematics for the two degree of freedom planar manipulator shown in Figure 1.

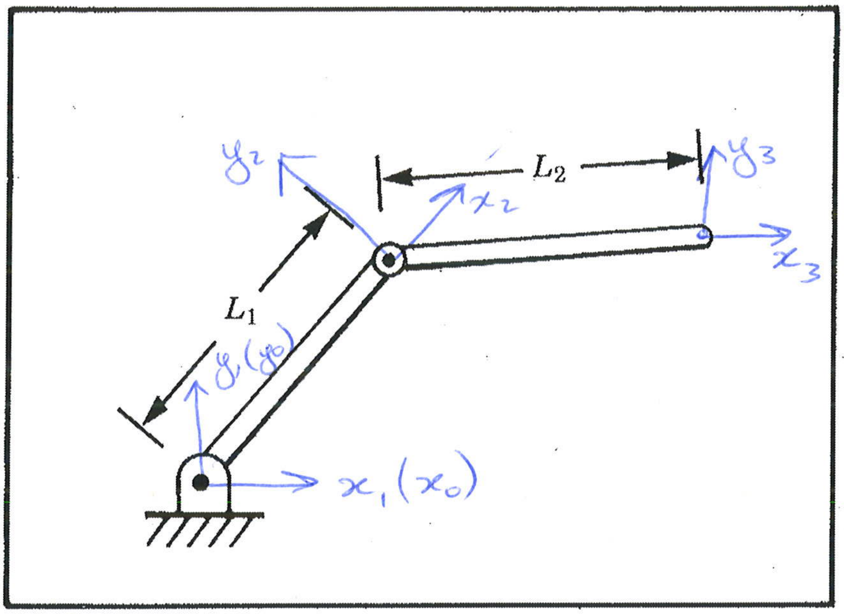


Figure 1 Two degree of freedom planar manipulator for Question 2

Forward Kinematics

Axis i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$ {}^0_1H = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2H = \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}H = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}H = {}^{0}_{1}H \; {}^{1}_{2}H \; {}^{2}_{3}H $$

$$ = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}H = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 & 0 & \cancel{\cos} L_1\cos\theta_1 \\ \sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & 0 & L_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}H = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_1\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}H = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The manipulator is planar and cannot rotate out of the page.

Therefore we can specify the rotation about $\hat{z}$ only (we'll make it β). And we can specify the position in $\hat{x}$ and $\hat{y}$ only $\therefore P_z = 0$.

$\therefore$ The end position (desired position) can be specified by

$$
{}^{0}_{3}H = \begin{bmatrix} \cos\beta & -\sin\beta & 0 & P_x \\ \sin\beta & \cos\beta & 0 & P_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

We can equate this with the forward kinematics

$$\cos\beta = \cos(\theta_1 + \theta_2) \quad —— ①$$

$$\sin\beta = \sin(\theta_1 + \theta_2) \quad —— ②$$

$$P_x = L_1\cos\theta_1 + L_2\cos(\theta_1 + \theta_2) \quad —— ③$$

$$P_y = L_1\sin\theta_1 + L_2\sin(\theta_1 + \theta_2) \quad —— ④$$

We need to find θ_1 and θ_2

From ① and ② we know $\theta_1 + \theta_2 = \beta$

substitute this into ③ and ④

$$P_x = L_1\cos\theta_1 + L_2\cos\beta \quad —— ⑤$$

$$P_y = L_2 \overset{\sin}{\cancel{\cos}}\,\theta_1 + L_2\sin\beta \quad —— ⑥$$

We can then square ⑤ and ⑥ ~~g~~ and add them together

$$P_x^2 + P_y^2 = L_1^2 \cos^2\theta_1 + 2L_1L_2\cos\theta_1\cos\beta + L_2^2\cos^2\beta + L_1^2\sin^2\theta_1 + 2L_1L_2\sin\theta_1\sin\beta + L_2^2\sin^2\beta$$

$$= L_1^2(\cos^2\theta_1 + \sin^2\theta_2) + 2L_1L_2(\cos\theta_1\cos\beta + \sin\theta_1\sin\beta) + L_2^2(\sin^2\beta + \cos^2\beta)$$

$$= L_1^2 + 2L_1L_2(\cos(\theta_1 - \beta)) + L_2^2$$

$$= L_1^2 + L_2^2 + 2L_1L_2(\cos(\theta_1 - \beta))$$

Remember $\beta = \theta_1 + \theta_2$

$$= L_1^2 + L_2^2 + 2L_1L_2(\cos(\theta_1 - \theta_1 - \theta_2))$$

$$= L_1^2 + L_2^2 + 2L_1L_2(\cos(-\theta_2))$$

Recall $\cos\theta_2 = \cos(-\theta_2)$

$$\therefore \cos\theta_2 = \frac{P_x^2 + P_y^2 - L_1^2 - L_2^2}{2L_1L_2}$$

Now $\cos^2\theta_2 + \sin^2\theta_2 = 1$

$$\therefore \sin^2\theta_2 = \sqrt{1 - \cos^2\theta_2}$$

$$= \sqrt{1 - \frac{P_x^2 + P_y^2 - L_1^2 - L_2^2}{2L_1L_2}}$$

With $\cos\theta_2$ and $\sin\theta_2$ we can use atan2

$$\theta_2 = \operatorname{Atan2}(\sin\theta_2, \cos\theta_2)$$

Now to solve for θ_1

We can ~~w~~ rewrite ③ and ④ as

$$P_x = L_1 \cos\theta_1 + L_2(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$$
$$P_y = L_1 \sin\theta_1 + L_2(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$$

$$P_x = \cos\theta_1 (L_1 + L_2\cos\theta_2) - \sin\theta_1 (L_2 \sin\theta_2)$$
$$P_y = \sin\theta_1 (L_1 + L_2\cos\theta_2) + \cos\theta_1 (L_2 \sin\theta_2)$$

We can assign
$$\left.\begin{aligned} k_1 &= L_1 + L_2\cos\theta_2 \\ k_2 &= L_2\sin\theta_2 \end{aligned}\right\}$$
These are in terms of θ_2 which we already know.

$$P_x = k_1\cos\theta_1 - k_2\sin\theta_1 \quad \text{— ⑦}$$
$$P_y = k_1\sin\theta_1 + k_2\cos\theta_1 \quad \text{— ⑧}$$

Now we ~~w~~ rewrite k_1 and k_2 as.

$$k_1 = r\cos\gamma \qquad r = \sqrt{k_1^2 + k_2^2}$$
$$k_2 = r\sin\gamma \qquad \gamma = \mathrm{Atan2}(k_2, k_1)$$

⑦ and ⑧ become.

$$P_x = r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1$$
$$P_y = r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1$$

Giving

$$P_x = r(\cos(\gamma + \theta_1))$$
$$P_y = r(\sin(\gamma + \theta_1))$$

We can then use Atan 2

$$(\gamma + \theta_1) = \text{Atan2}\left(\frac{P_y}{r}, \frac{P_x}{r}\right)$$

Atan 2 is a ratio so r's cancel

$$\theta_1 = \text{Atan2}(P_y, P_x) - \text{Atan2}(k_2, k_1)$$

$$= \text{Atan2}(P_y, P_x) - \text{Atan2}\left(\overset{L_2\sin\theta_2}{\cancel{L_1 + L_2\cos\theta_2}}, L_1 + L_2\cos\theta_2\right)$$

3. Sketch the reachable workspace of the three-link manipulator in Figure 2 for the case $L_1 = 15.0$, $L_2 = 10.0$ and $L_3 = 3.0$.

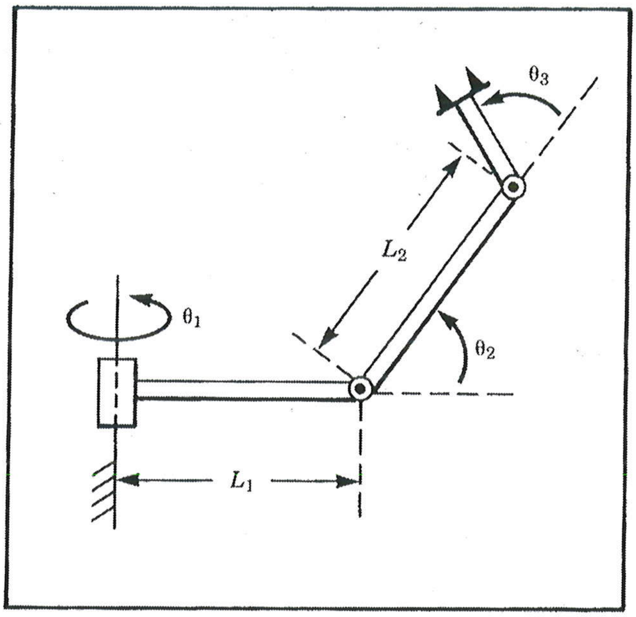
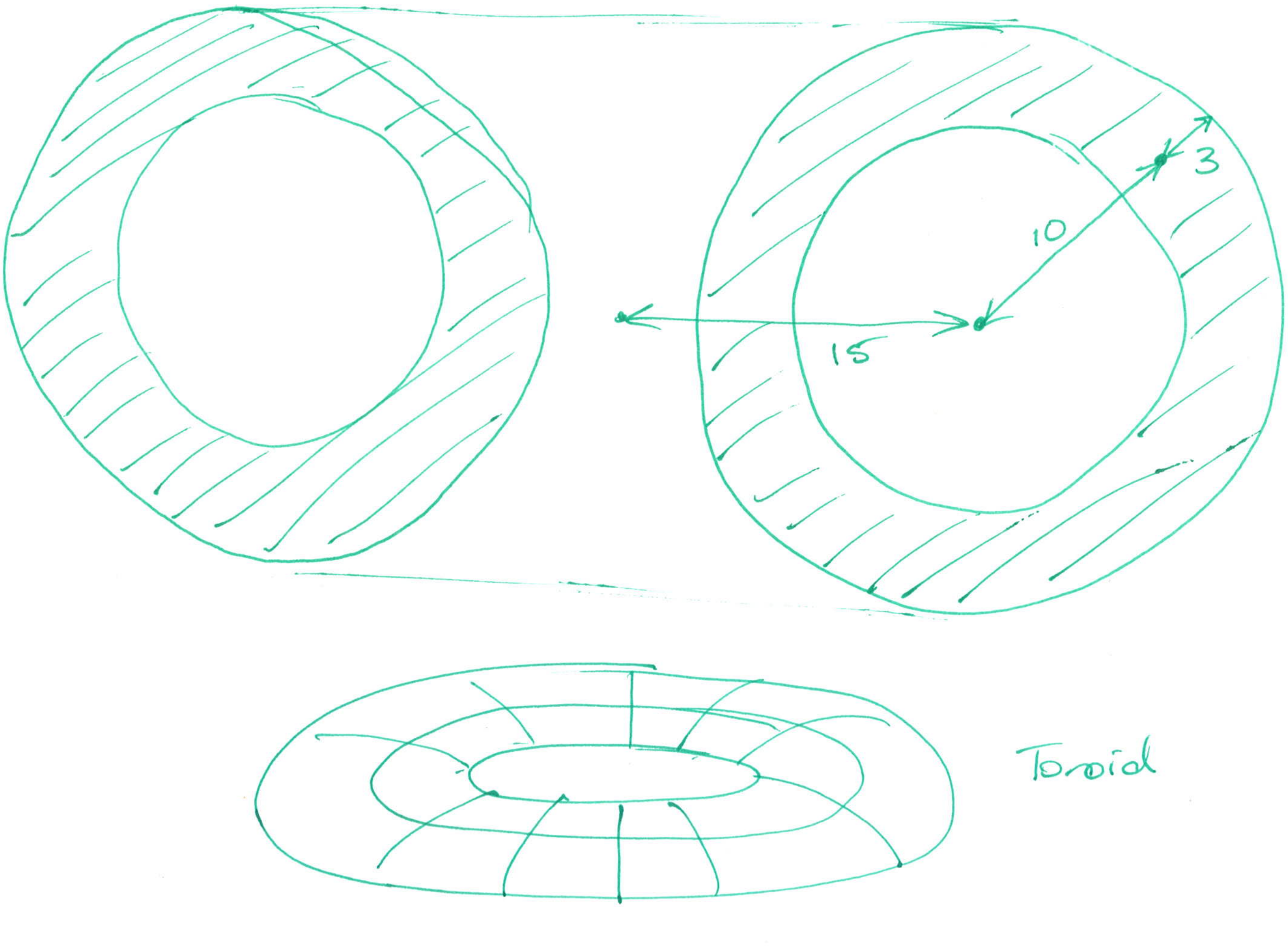


Figure 2 Three-link manipulator for Questions 3 and 4



Q4 Derive the inverse kinematics of the three-link manipulator in Figure 2.

The wrist position can be defined as...

$$ {}^{0}_{4}H = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From the Week 6 tutorial questions we have.

$$ {}^{0}_{3}H = \begin{bmatrix} c1c23 & -c1s23 & s1 & L_1c1 + L_2c1c2 \\ s1c23 & -s1s23 & -c1 & L_1s1 + L_2s1c2 \\ s23 & c23 & 0 & L_2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

We need to add a translation along $\hat{X}$ to get to the tip of the robot

$$ {}^{0}_{4}H = \begin{bmatrix} c1c23 & -c1s23 & s1 & L_1c1 + L_2c1c2 \\ s1c23 & -s1s23 & -c1 & L_1s1 + L_2s1c2 \\ s23 & c23 & 0 & L_2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Q4 pg 2

$$
{}^{0}_{4}H = \begin{bmatrix} c1c23 & -c1s23 & s1 & L_1c1 + L_2c1c2 + L_3c1c23 \\ s1c23 & -s1s23 & -c1 & L_1s1 + L_2s1c2 + L_3c1c23 \\ s23 & c23 & 0 & L_2s2 + \cancel{s23}\,L_3s23 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$P_x = L_1\cos\theta_1 + L_2\cos\theta_1\cos\theta_2 + L_3\cos\theta_1\cos(\theta_2+\theta_3) \quad — ①$$

$$P_y = L_1\sin\theta_1 + L_2\sin\theta_1\cos\theta_2 + L_3 \overset{\sin}{\cancel{\cos}}\theta_1\ \cancel{\cos}\ \cancel{\sin}\cos(\theta_2+\theta_3) \quad — ②$$

$$P_z = L_2\sin\theta_2 + L_3\sin(\theta_2+\theta_3) \quad — ③$$

① becomes

$$\frac{P_x}{\cos\theta_1} = L_1 + L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3)$$

② becomes.

$$\frac{P_y}{\sin\theta_1} = L_1 + L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3)$$

$$\therefore \quad \frac{P_x}{\cos\theta_1} = \frac{P_y}{\sin\theta_1}$$

$$P_x\sin\theta_1 = P_y\cos\theta_1$$

$$P_y\cos\theta_1 - P_x\sin\theta_1 = 0$$

Use the equation sheet to give

$$\theta_1^{(1)} = \cancel{\text{Atan2}(P_y, \pm P_x)}\ \text{Atan2}(P_y, P_x)$$

$$\theta_1^{(2)} = \text{Atan2}(-P_y, -P_x)$$

To solve for θ_3

Rearrange ① to give

$$\frac{P_x}{\cos\theta_1} - L_1 = L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3) \quad — ④$$

$$P_z = L_2 \cancel{\cos}\sin\theta_2 + L_3\sin(\theta_2+\theta_3) \quad — ⑤$$

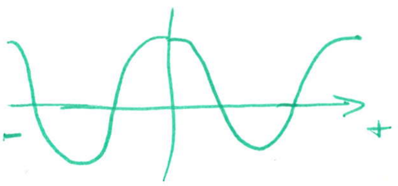
Square ④ and ⑤ to give.

$$\left(\frac{P_x}{\cos\theta_1} - L_1\right)^2 + P_z^{\,2} = L_2^{\,2}\cos^2\theta_2 + 2L_2L_3\cos\theta_2\cos(\theta_2+\theta_3)$$
$$+ L_3^{\,2}\cos^2(\theta_2+\theta_3)$$
$$+ L_2^{\,2}\sin^2\theta_2 + 2L_2L_3\sin\theta_2\sin(\theta_2+\theta_3)$$
$$+ L_3^{\,2}\sin^2(\theta_2+\theta_3)$$

$$= L_2^{\,2}(\cos^2\theta_2 + \sin^2\theta_2)$$
$$+ 2L_2L_3(\cos\theta_2\cos(\theta_2+\theta_3) + \sin\theta_2\sin(\theta_2+\theta_3))$$
$$+ L_3^{\,2}(\cos^2(\theta_2+\theta_3) + \sin^2(\theta_2+\theta_3))$$

$$= L_2^{\,2} + 2L_2L_3\cos(\theta_2 - (\theta_2+\theta_3)) + L_3^{\,2}$$
$$= L_2^{\,2} + 2L_2L_3\cos(-\theta_3) + L_3^{\,2}$$
$$= L_2^{\,2} + 2L_2L_3\cos\theta_3 + L_3^{\,2}$$

$\cos\theta = \cos(-\theta)$



$$\therefore \cos\theta_3 = \frac{\left(\frac{P_x}{\cos\theta_1} - L_1\right)^2 + P_z - L_2^{\,2} - L_3^{\,2}}{2L_2L_3}$$

2 solutions. 1 for $\theta_1^{(1)}$ and 1 for $\theta_1^{(2)}$

having $\cos\theta_1$ here is okay because we have determined it previously

Unfortunately I can't determine a way to solve for $\sin\theta_3$ (we used ① and ③ above. Solving ~~①~~ ② and ③ would produce a similar result except ~~you'll~~ you'd replace $\frac{P_x}{\cos\theta_1}$ with $\frac{P_y}{\sin\theta_1}$). Therefore we can use the 2nd solution in our equation table

You should always try to avoid this.

$$\cancel{\theta_3 = Atan2\left(\frac{\left(\frac{P_x}{\cos\theta_1} - L_1\right)^2 + P_z - L_2^2 - L_3^2}{2L_2 - L_3}\right)}$$

$$\theta_3 = Atan2\left(\pm\sqrt{1 - \frac{\left(\frac{P_x}{\cos\theta_1} - L_1\right)^2 + P_z - L_2^2 - L_3^2}{2L_2L_3}}, \frac{\left(\frac{P_x}{\cos\theta_1} - L_1\right)^2 + P_z - L_2^2 - L_3^2}{2L_2L_3}\right)$$

There are two solutions for θ_3. One for $\theta_1^{(1)}$ and the other for $\theta_1^{(2)}$

To solve for θ_2

$$\frac{P_x}{\cos\theta_1} - L_1 = L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3)$$

$$= L_2\cos\theta_2 + L_3\cos\theta_2\cos\theta_3 - L_3\sin\theta_2\sin\theta_3$$

$$= (L_2 + L_3\cos\theta_3)\cos\theta_2 - (L_3\sin\theta_3)\sin\theta_2$$

$$\frac{P_y}{\sin\theta_1} - L_1 = L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3)$$

$$= (L_2 + L_3\cos\theta_3)\cos\theta_2 - (L_3\sin\theta_3)\sin\theta_2 \quad \text{— (6)}$$

$$P_z = L_2\sin\theta_2 + L_3\cos\theta_2\sin\theta_3 + L_3\sin\theta_2\cos\theta_3$$

$$= (L_2 + L_3\cos\theta_3)\sin\theta_2 + (L_3\sin\theta_3)\cos\theta_2 \quad \text{— (7)}$$

We can use (6) and (7) in the 2nd last ~~equation~~ solution on the equation sheet.

~~where~~

$$\boxed{\begin{cases} a\cos\theta - b\sin\theta = c \\ a\sin\theta + b\cos\theta = d \end{cases} \implies \theta = \mathrm{Atan2}(ad-bc,\ ac+bd)}$$

where

$$\left.\begin{aligned} a &= L_2 + L_3\cos\theta_3 \\ b &= L_3\sin\theta_3 \\ c &= \frac{P_y}{\sin\theta_1} - L_1 \\ d &= P_z \end{aligned}\right\}$$

Two solutions.
One for $\theta_1^{(1)}$ and $\theta_3^{(1)}$
One for $\theta_1^{(2)}$ and $\theta_3^{(2)}$

$$\therefore\ \theta_2^{(1)} = \mathrm{Atan2}\left((L_2+L_3\cos\theta_3^{(1)})P_z - L_3\sin\theta_3^{(1)}\left(\frac{P_y}{\sin\theta_1^{(1)}} - L_1\right),\ (L_2+L_3\cos\theta_3^{(1)})\left(\frac{P_y}{\sin\theta_1^{(1)}} - L_1\right) + (L_3\sin\theta_3^{(1)})P_z\right)$$

$$\&\ \theta_2^{(2)} = \mathrm{Atan2}\left((L_2+L_3\cos\theta_3^{(2)})P_z - L_3\sin\theta_3^{(2)}\left(\frac{P_y}{\sin\theta_1^{(2)}} - L_1\right),\ (L_2+L_3\cos\theta_3^{(2)})\left(\frac{P_y}{\sin\theta_1^{(2)}} - L_1\right) + (L_3\sin\theta_3^{(2)})P_z\right)$$

To solve for θ_2 $\boxed{\text{Alternative method}}$ Q4 pg6

$$\frac{P_x}{\cos\theta_1} - L_1 = L_2\cos\theta_2 + L_3\cos(\theta_2+\theta_3)$$

$$= L_2\cos\theta_2 + L_3\cos\theta_2\cos\theta_3 - L_3\sin\theta_2\sin\theta_3$$

$$\frac{P_y}{\sin\theta_1} - L_1 = L_2\cos\theta_2 + L_3\cos\theta_2\cos\theta_3 - L_3\sin\theta_2\sin\theta_3$$

$$= (L_2 + L_3\cos\theta_3)(\cos\theta_2) - (L_3\sin\theta_3)(\sin\theta_2)$$

$$P_z = L_2\sin\theta_2 + L_3\cos\theta_2\sin\theta_3 + L_3\sin\theta_2\cos\theta_3$$

$$(L_2 + L_3\cos\theta_3)(\sin\theta_2) + (L_3\sin\theta_3)(\cos\theta_2)$$

$$k_1 = L_2 + L_3\cos\theta_3$$

$$k_2 = L_3\sin\theta_3$$

$$\therefore \frac{P_y}{\sin\theta_1} - L_1 = k_1\cos\theta_2 - k_2\sin\theta_2$$

$$P_z = k_1\sin\theta_2 + k_2\cos\theta_2$$

Let $r = +\sqrt{k_1^2 + k_2^2}$

$\gamma = \text{Atan2}(k_2, k_1)$

$\therefore \quad k_1 = r\cos\gamma$

$k_2 = r\sin\gamma$

$$\therefore \quad \frac{P_y}{\sin\theta_1} - L_1 = r\cos\gamma\cos\theta_2 - r\sin\gamma\sin\theta_2 = r\cos(\gamma+\theta_2)$$

$$P_z = r\cos\gamma\sin\theta_2 + r\sin\gamma\cos\theta_2 = r\sin(\gamma+\theta_2)$$

We can use the equation sheet

$$\begin{cases} \sin\theta = a \\ \cos\theta = b \end{cases} \Rightarrow \theta = \text{Atan2}(a,b)$$

$$\gamma + \theta_2 = \text{Atan2}\left(\frac{P_z}{r}, \left(\frac{P_y}{\sin\theta_1} - L_1\right)\frac{1}{r}\right)$$

r ~~its~~ is a common factor.

Now $\gamma = \text{Atan2}(k_2, k_1)$

$= \text{Atan2}(L_2 + L_3\cos\theta_3, L_3\sin\theta_3)$

$$\therefore \quad \theta_2 = \text{Atan2}\left(P_z, \frac{P_y}{\sin\theta_1} - L_1\right) - \text{Atan2}(L_2 + L_3\cos\theta_3, L_3\sin\theta_3)$$

We know θ_1 and θ_3 so this is okay.

5. Sketch the reachable workspace of the three degree of freedom manipulator in Figure 3.

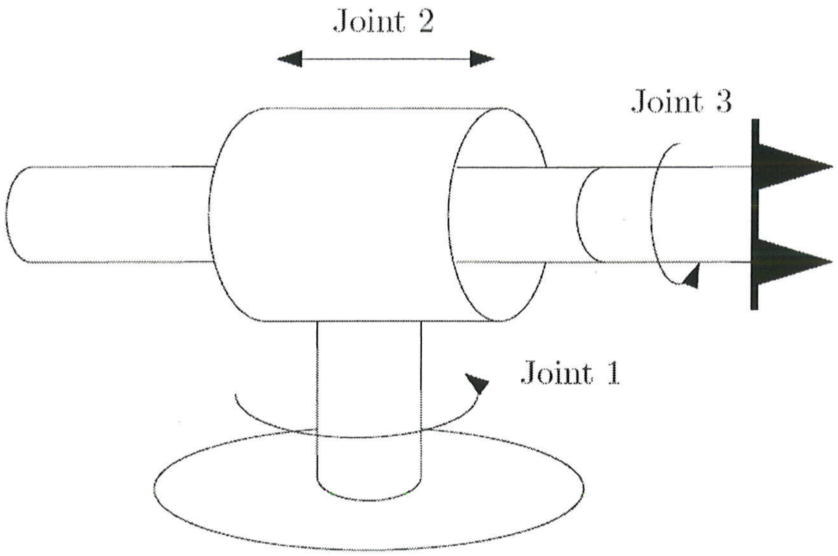
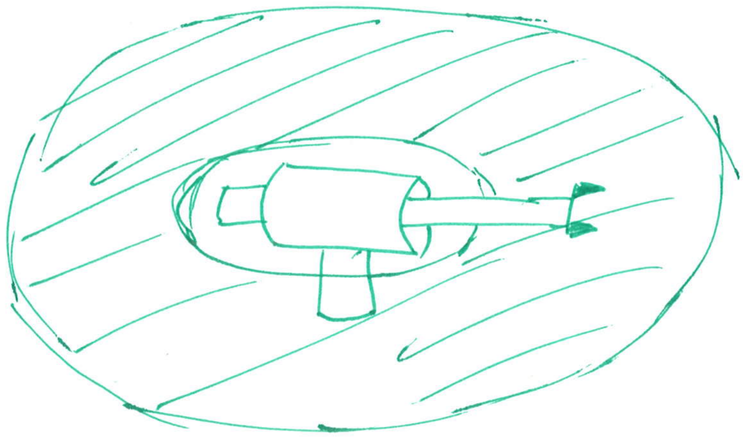


Figure 3 Three degree of freedom manipulator for Questions 5 and 6



Flat disk.

Inner radius equal to minimum extension of joint 2.

Outer radius equal to maximum extension of joint 2

6. Derive the inverse kinematics of the three degree of freedom manipulator in Figure 3.

From last week's lecture.

$$ {}^0_3H = {}^0_1H\,{}^1_2H\,{}^2_3H = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} c1 & 0 & s1 & d_2 s1 \\ s1 & 0 & -c1 & -d_2 c1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c3 & -s3 & 0 & 0 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} \cos\theta_1\cos\theta_3 & -\cos\theta_1\sin\theta_3 & \sin\theta_1 & d_2\sin\theta_1 + l_2\sin\theta_1 \\ \sin\theta_1\cos\theta_3 & -\sin\theta_1\sin\theta_3 & -\cos\theta_1 & -d_2\cos\theta_1 - l_2\cos\theta_1 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

To take it to the tip of the robot add an extra frame at the end and multiply ${}^0_3H$ by ${}^3_4H$

$$ {}^0_4H = \begin{bmatrix} c1c3 & -c1s3 & s1 & d_2 S1 + l_2 S1 \\ s1c3 & -s1s3 & -c1 & -d_2 c1 - l_2 c1 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_4H = \begin{bmatrix} c1c3 & -c1s3 & s1 & d_2 s1 + l_2 s1 + l_3 s1 \\ s1c3 & -s1s3 & -c1 & -d_2 c1 - l_2 c1 - l_3 c1 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Equate ${}^{0}_{4}H$ to $\begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad P_z = 0$ — this makes sense. the workspace is a flat disk.

$$P_x = \cancel{S1}\, d_2 s1 + l_2 s1 + l_3 s1$$
$$= \sin\theta_1 (d_2 + l_2 + l_3)$$

$$P_y = -d_2 c1 - l_2 c1 - l_3 c1$$
$$= -\cos\theta_1 (d_2 + l_2 + l_3)$$

$$\therefore \quad d_2 + l_2 + l_3 = \frac{P_x}{\sin\theta_1} = \frac{P_y}{-\cos\theta_1}$$

$$\therefore \quad \sin\theta_1 P_y = -\cos\theta_1 P_x$$
$$P_y \sin\theta_1 \cancel{P} + P_x \cos\theta_1 = 0$$

We can use the equation sheet

$$\boxed{a\cos\theta - b\sin\theta = 0 \implies \begin{array}{l} \theta^{(1)} = \mathrm{Atan2}(\cancel{P_x}^{a}, \cancel{P_y}^{b}) \\ \theta^{(2)} = \mathrm{Atan2}(-a, -b) \cancel{\pm} \pi + \theta^{(1)} \end{array}}$$

$a = \cancel{m} P_x \qquad b = -P_y$

$$\therefore \quad \theta^{(1)} = \text{Atan2}(P_x, -P_y)$$

$$\theta^{(2)} = \text{Atan2}(-P_x, P_y) = \pi + \theta^{(1)}$$

Now solve for d_2

$$d_2^{(1)} = \frac{P_x}{\sin \theta_1^{(1)}} - l_2 - l_3$$

$$d_2^{(2)} = \frac{P_x}{\sin \theta_1^{(2)}} - l_2 - l_3$$

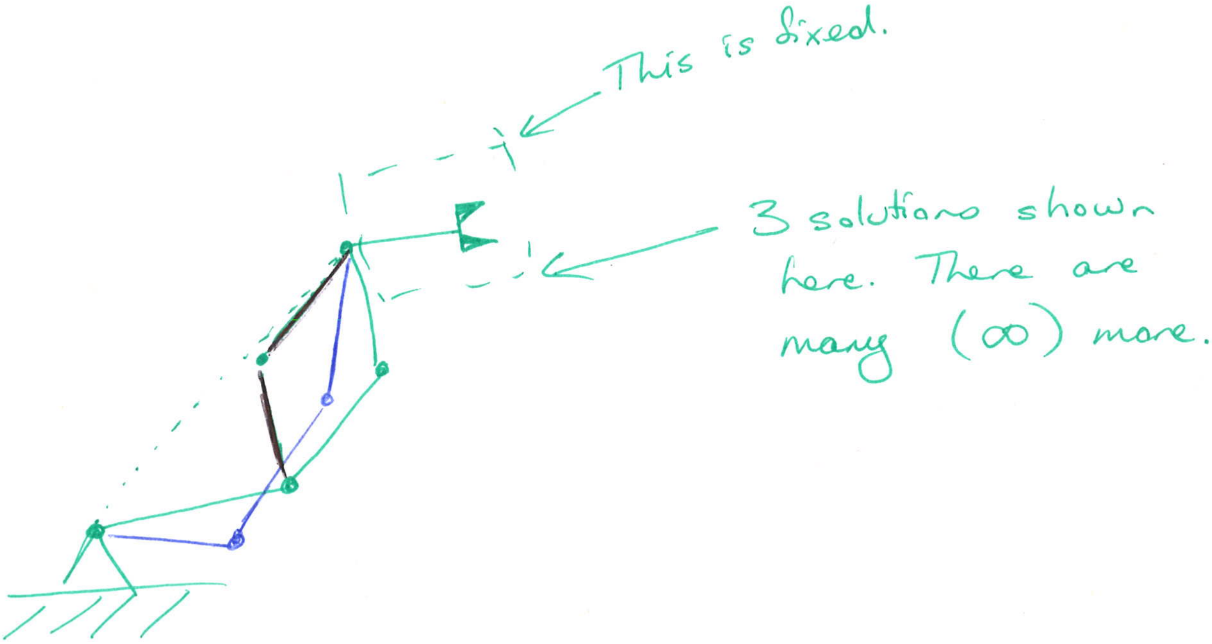
To solve for θ_3 we can't use P_x, P_y or P_z

$$\sin \theta_3 = R_{31}$$
$$\cos \theta_3 = R_{32}$$

$$\therefore \quad \theta_3 = \text{Atan2}(R_{31}, R_{32})$$

7. Given the desired position and orientation of the hand of a three-link planar rotary jointed manipulator, there are two possible solutions. If we add one more rotational joint (in such a way that the arm is still planar), how many solutions are there?

There are an infinite number of solutions.



The three remaining links make up a four bar linkage (with the base), and hence can take on an infinite number of positions.

8. Figure 4 shows a two-link planar arm with rotary joints. For this arm, the second link is half as long as the first, that is: $L_1 = 2L_2$. The joint range limits in degrees are

$$0 < \theta_1 < 180$$
$$-90 < \theta_2 < 180$$

Sketch the approximate reachable workspace (an area) of the tip of link 2.

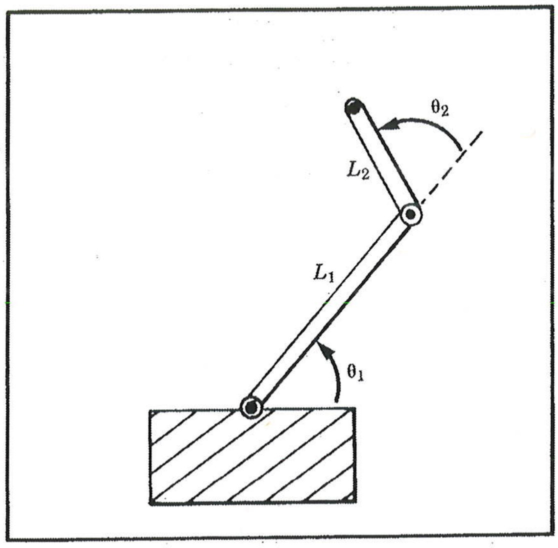
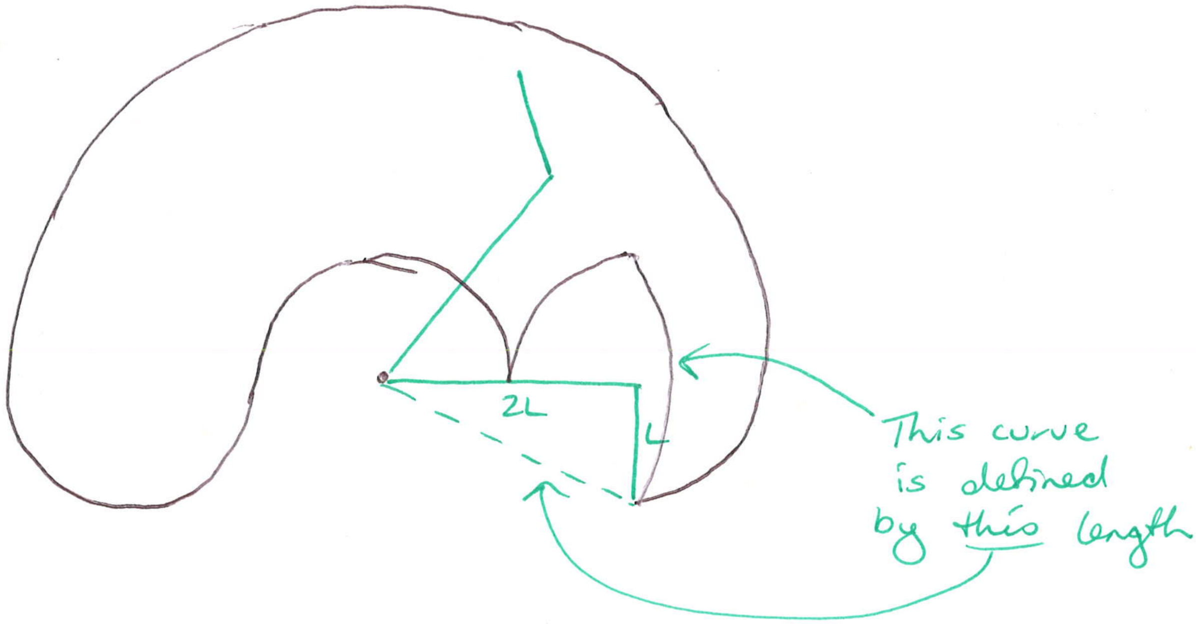
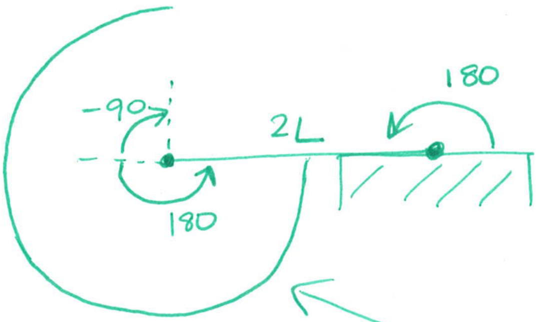


Figure 4 Two degree of freedom planar manipulator for Question 8

More difficult than it looks.

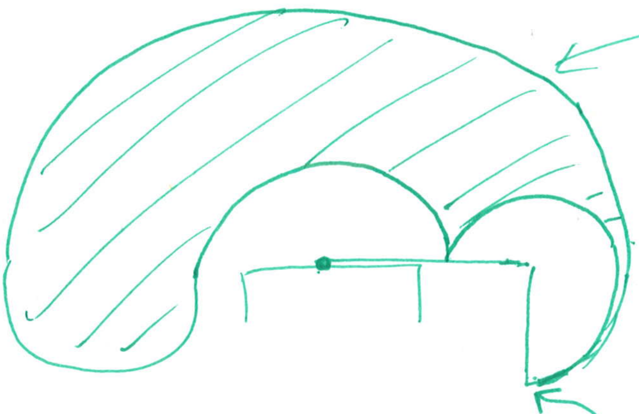


Consider the link at one extreme.



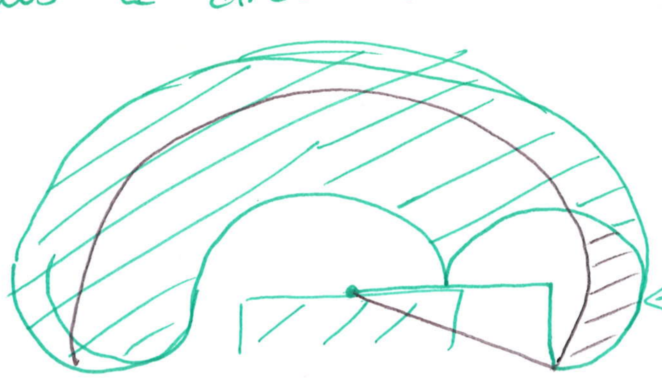
Can reach anywhere on this line

Now if we sweep the first link. back to 180 we get



This shape.

But on the way around this point draws a circle of radius $\sqrt{2L^2 + L^2}$



So the robot can reach this space as well.