MATH1520 Autumn 2018
Homework 1

1. Determine the domain of the following functions.

   (a) $f(t) = \sqrt{3-t} - \sqrt{2+t}$

   (b) $f(x) = \dfrac{\log(x^2-1)}{\sqrt{4-x^2}}$

   (c) $g(x) = \dfrac{1}{\log\sqrt{5-x}}$

   (d) $h(x) = \dfrac{\ln x}{x^2-2x-15}$

   (e) $f(u) = \dfrac{u+1}{1+\dfrac{1}{u+1}}$

   Answer.

   (a) Because square root only makes sense when $3-t \geq 0$ and $2+t \geq 0$, so we have $-2 \leq t \leq 3$.

   (b) Because log function only makes sense when $x^2-1 > 0$, so we have $x > 1$ or $x < -1$. And we also need to male sure $4-x^2 > 0$,because the denominator has a square root. This results $-2 < x < 2$. Thus the domain is$(-2,-1) \cup (1,2)$.

   (c) Because the denominator is a square root, we have $5-x > 0$ and $5-x \neq 1$. So the domain is $(4,5) \cup (-\infty,4)$.

   (d) Because log function only makes sense when $x > 0$. We also need to make sure $x^2-2x-15 \neq 0$. Thus the domain is $(0,5) \cup (5,+\infty)$.

   (e) We need to make sure all the denominators are nonzero, so $u+1 \neq 0$ and $1+\dfrac{1}{u+1} \neq 0$. Thus the domain is $(-\infty,-2) \cup (-2,-1) \cup (-1,+\infty)$

2. Find the domain and sketch the graph of the following function.

   (a) $F(x) = |2x+1|$

   (b) $g(x) = |x| - x$

   (c) $h(x) = \dfrac{3x+|x|}{x}$

   (d) $f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ 1-x, & \text{if } x \geq 0 \end{cases}$

   Answer.

   (a) Domain: $x \in \mathbb{R}$.

$$F(x) = \begin{cases} 2x+1, & \text{if } x \geq -\frac{1}{2} \\ -2x-1, & \text{if } x < -\frac{1}{2}. \end{cases}$$

(b) Domain: $x \in \mathbb{R}$.

$$g(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0. \end{cases}$$

(c) Domain: $x \in (-\infty, 0) \cup (0, +\infty)$.

$$h(x) = \begin{cases} 4, & \text{if } x > 0 \\ 2, & \text{if } x < 0. \end{cases}$$

(d) Domain: $x \in \mathbb{R}$.

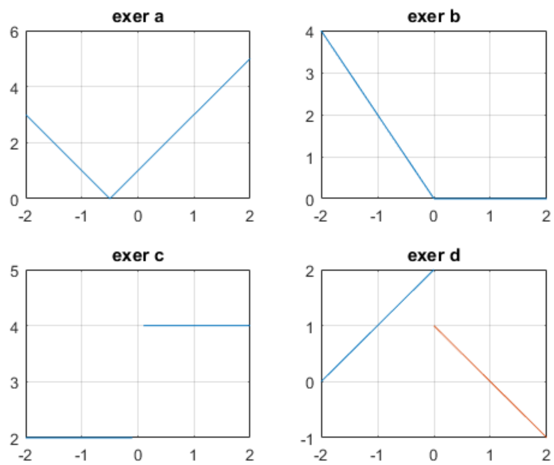

Figure 1: exercise 2

3. Suppose

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0\ , \\ \sqrt{x} + 4 & \text{if } 0 \leq x < 1, \\ 3x - 1 & \text{if } x \geq 1. \end{cases}$$

Compute $f(2.5)$,$f(0)$,$f(-4)$.
Answer. $f(2.5) = 3 \times 2.5 - 1 = 6.5$. $f(0) = \sqrt{0} + 4 = 4$. $f(-4) = (-4)^2 + 1 = 17$.

4. Let $f(u) = u^2 + 4u + 8$ and $g(x) = x^2 - \sqrt{x} + 1$. Find $(f \circ g)(x)$, $(g \circ f)(x)$ and determine their domains.
Answer.

$$\begin{aligned} f(g(x) &= (g(x))^2 + 4g(x) + 8 \\ &= (x^2 - \sqrt{x} + 1)^2 + 4(x^2 - \sqrt{x} + 1) + 8 \\ &= x^4 - 2x^2\sqrt{x} + 6x^2 - 6\sqrt{x} + x + 13 \end{aligned}$$

And the domain is $[0, +\infty)$

$$\begin{aligned} g(f(x) &= (f(x))^2 - \sqrt{f(x)} + 1 \\ &= (x^2 + 4x + 8)^2 - \sqrt{x^2 + 4x + 8} + 1. \end{aligned}$$

Because the discriminant of u^2+4u+8 is less than 0,$u^2+4u+8>0$ for all u. The domain is $\mathbb{R}$.

5. Find the difference quotient function of the following functions.

   (a) x^3+x^2+1.

   (b) $2x^2-4x+3$.

   (c) $\dfrac{x+3}{x+6}$

   Answer.

   (a)

$$\begin{aligned}\frac{(x+h)^3+(x+h)^2+1-(x^3+x^2+1)}{h} &= \frac{h^3+3xh^2+3x^2h+h^2+2xh}{h}\\ &= h^2+3xh+3x^2+h+2x\end{aligned}$$

   (b)

$$\frac{2(x+h)^2-4(x+h)+3-(2x^2-4x+3)}{h}=\frac{2(h^2+2xh)-4h}{h} \quad =2h+4x-4$$

   (c)

$$\frac{\dfrac{x+h+3}{x+h+6}-\dfrac{x+3}{x+6}}{h}=\frac{3}{(x+6+h)(x+6)}$$

6. Find the limit. If it doesn't exist, state whether it is $+\infty$, $-\infty$ or neither.

   (a) $\displaystyle\lim_{x\to 2}\frac{x^2+6x-16}{14x-2x^2-20}$.

   (b) $\displaystyle\lim_{x\to 1}\frac{x+x^2+x^3-3}{x^2-1}$.

   (c) $\displaystyle\lim_{x\to+\infty}\sqrt{x+8}-\sqrt{x-4}$.

   (d) $\displaystyle\lim_{x\to+\infty}\frac{2x+3}{5x+7}$

   (e) $\displaystyle\lim_{x\to+\infty}\frac{10x^5+x^4+31}{x^6}$

   (f) $\displaystyle\lim_{x\to+\infty}\sqrt{x^2+3x}-\sqrt{x^2-2x}$

   (g) $\displaystyle\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

   (h) $\displaystyle\lim_{x\to 1}\frac{2-x}{(x-1)^2}$

   (i) $\displaystyle\lim_{x\to 2^+}\frac{x^2+2x-8}{x^2-5x+6}$

(j) $\lim_{x\to 2^-} \dfrac{x^2-2x}{x^2-4x+4}$

(k) $\lim_{x\to 0} \dfrac{5}{2x}$

(l) $\lim_{x\to 0} \dfrac{\sqrt{x^2+9}-3}{x^2}$

(m) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2+x}\right)$

(n) $\lim_{x\to -2} \dfrac{2-|x|}{2+x}$

Answer.

(a) $\lim_{x\to 2} \dfrac{x^2+6x-16}{14x-2x^2-20} = \lim_{x\to 2} \dfrac{(x+8)(x-2)}{-2(x-2)(x-5)} = \lim_{x\to 2} \dfrac{x+8}{-2(x-5)} = \dfrac{5}{3}$

(b) $\lim_{x\to 1} \dfrac{x+x^2+x^3-3}{x^2-1} = \lim_{x\to 1} \dfrac{(x-1)(x^2+2x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{3+2x+x^2}{x+1} = 3$

(c) $\lim_{x\to +\infty} \sqrt{x+8} - \sqrt{x-4} = \lim_{x\to +\infty} \dfrac{(x+8)-(x-4)}{\sqrt{x+8}+\sqrt{x-4}} = \lim_{x\to +\infty} \dfrac{12}{\sqrt{x+8}+\sqrt{x-4}} = 0$

(d) $\lim_{x\to +\infty} \dfrac{2x+3}{5x+7} = \lim_{x\to +\infty} \left(\dfrac{2+\frac{3}{x}}{5+\frac{7}{x}}\right) = \dfrac{2}{5}$

(e) $\lim_{x\to +\infty} \dfrac{10x^5+x^4+31}{x^6} = \lim_{x\to +\infty} \dfrac{10}{x} + \dfrac{1}{x^2} + \dfrac{31}{x^6} = 0$

(f) $\lim_{x\to +\infty} \sqrt{x^2+3x} - \sqrt{x^2-2x} = \lim_{x\to +\infty} \dfrac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$
$= \lim_{x\to +\infty} \dfrac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}} = \dfrac{5}{2}$

(g) $\lim_{x\to -\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to -\infty} \dfrac{-3+\frac{4}{x^3}}{-\sqrt{1+\frac{9}{x^6}}} = 3 \qquad (\text{for } x<0,\ x^3 = -\sqrt{|x|^6})$

(h) This limit does not exist because the denominator tends to zero but the numerator is finite and the limit is $+\infty$.

(i) $\lim_{x\to 2^+} \dfrac{x^2+2x-8}{x^2-5x+6} = \lim_{x\to 2^+} \dfrac{(x+4)(x-2)}{(x-2)(x-3)} = \lim_{x\to 2^+} \dfrac{x+4}{x-3} = -6$

(j) This limit does not exist because the denominator tends to zero but the numerator is finite and the limit is $-\infty$. $\lim_{x\to 2^-} \dfrac{x^2-2x}{x^2-4x+4} = \lim_{x\to 2^-} \dfrac{x}{x-2} = -\infty$

(k) This limit does not exist because $\lim_{x\to 0^+} \dfrac{5}{2x} = +\infty$ but $\lim_{x\to 0^-} \dfrac{5}{2x} = -\infty$

(l) $\lim_{x\to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} \cdot \dfrac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x\to 0} \dfrac{x^2}{x^2(\sqrt{x^2+9}+3)} = \dfrac{1}{6}$

(m) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2+x}\right) = \lim_{x\to 0} \dfrac{x}{x^2+x} = \lim_{x\to 0} \dfrac{1}{x+1} = 1$

(n) $\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2+x}{2+x} = 1$

7. Suppose we have

$$\lim_{x\to +\infty} \frac{ax^2+x-1}{bx+4} = 2.$$

Find a, b.

Answer.

$$\begin{aligned}\lim_{x\to +\infty} \frac{ax^2+x-1}{bx+4} &= \lim_{x\to +\infty} \frac{ax+1-\frac{1}{x}}{b+\frac{4}{x}} \\ &= \lim_{x\to +\infty} \frac{ax+1}{b} = 2\end{aligned}$$

Obviously, $a = 0$ otherwise the numerator will tend to infinity. Thus $b = \frac{1}{2}$.

8. Use the following figure to estimate the limits if they exist:

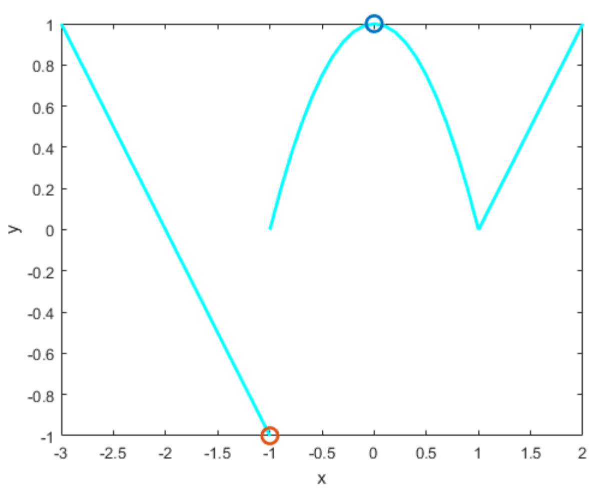

Figure 2: exercise 8

(a) $\lim_{x\to -1^+} f(x)$

(b) $\lim_{x\to 0} f(x)$

(c) $\lim_{x\to 2^-} f(x)$

Answer.

(a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to 0} f(x) = 1$

(c) $\lim_{x\to 2^-} f(x) = 1$