MATH1550 Methods of Matrices and Linear Algebra

Suggested Answer for Assignment 1

1-1: Let

$$\begin{aligned} x+2y+\ 3z &= 1 && ① \\ 2x+4y+\ 7z &= 2 && ② \\ 3x+7y+11z &= 8 && ③ \end{aligned}$$

$(-2)\times①+②$ and $(-3)\times①+③$:

$$\begin{aligned} x+2y+3z &= 1 && ① \\ z &= 0 && ② \\ y+2z &= 5 && ③ \end{aligned}$$

By substitution, we have $z=0$, $y=5$ and $x=-9$. So, the solution set is $\{(-9,5,0)\}$.

1-2:

$$\begin{aligned} f(1) &= a+\ b+\ c = -1 && ① \\ f(2) &= a+2b+4c = \ \ 3 && ② \\ f(3) &= a+3b+9c = 13 && ③ \end{aligned}$$

$(-1)\times①+②$ and $(-1)\times①+③$:

$$\begin{aligned} a+\ b+\ c &= -1 && ① \\ b+3c &= 4 && ② \\ 2b+8c &= 14 && ③ \end{aligned}$$

$(-2)\times②+③$:

$$\begin{aligned} a+b+\ c &= -1 && ① \\ b+3c &= 4 && ② \\ 2c &= 6 && ③ \end{aligned}$$

So we have $c=3$ and then $b=-5$ and $a=1$. So the polynomial is $f(t)=1-5t+3t^2$.

1-3: (a)

$$\begin{aligned} x+\ y-\ \ z &= -2 && ① \\ 3x-5y+13z &= 18 && ② \\ x-2y+\ 5z &= k && ③ \end{aligned}$$

$(-3)\times①+②$ and $(-1)\times①+③$:

$$\begin{aligned} x+\ y-\ \ z &= -2 && ① \\ -8y+16z &= 24 && ② \\ -3y+\ 6z &= k+2 && ③ \end{aligned}$$

Divide ② by -8:

$$\begin{aligned} x+\ y-\ z &= \quad -2 & ① \\ y-2z &= \quad -3 & ② \\ -3y+6z &= k+2 & ③ \end{aligned}$$

$3\times ② + ③$:

$$\begin{aligned} x+y-\ z &= \quad -2 & ① \\ y-2z &= \quad -3 & ② \\ 0 &= k-7 & ③ \end{aligned}$$

Thus, the system has solution only if $k=7$.

(b) When $k=7$. The system becomes

$$\begin{aligned} x+y-\ z &= -2 & ① \\ y-2z &= -3 & ② \end{aligned}$$

By substituting $y=2z-3$ into ① we have $x+z=1$ or equivalent to $x=-z+1$.

Thus, there infinitely many solutions which are $(-a+1, 2a-3, a)$, where $a\in\mathbb{R}$.

1-4: Let x_1 be the hundreds digit, x_2 the tens digit, and x_3 the ones digit. Then the first condition says that $x_2+x_3=5$. The original number is $100x_1+10x_2+x_3$, while the reversed number is $100x_3+10x_2+x_1$. So the second condition is

$$792=(100x_1+10x_2+x_3)-(100x_3+10x_2+x_1)=99x_1-99x_3.$$

So we have the system of equations

$$\begin{cases} \quad\ \ x_2 + x_3 = 5 \\ 99x_1 \quad - 99x_3 = 792 \end{cases}$$

Multiplying the last equation by $1/99$ we have the equivalent system

$$\begin{cases} \quad x_2 + x_3 = 5 \\ x_1 \quad - x_3 = 8 \end{cases}$$

Thus, we have $x_1=a+8$, $x_2=5-a$ and $x_3=a$, where $a\in\mathbb{R}$.

However, x_3 must be a digit, restricting us to ten values (0-9).

Furthermore, if $c>1$, then the first equation forces $a>9$ which is impossible.

Setting $c=0$, yields 850 as a solution, and setting $c=1$ yields 941 as another solution.

1-5: Note that J^TB is an $n\times n$ matrix and BJ^T is an $m\times m$ matrix.

(a) $(J^TB)_{ij}=\sum\limits_{k=1}^{m}(J^T)_{ik}(B)_{kj}=\sum\limits_{k=1}^{m}(J)_{ki}(B)_{kj}=\sum\limits_{k=1}^{m}(B)_{kj}=\sum\limits_{k=1}^{m}j=jm$, $1\le i,j\le n$. That is,

$$J^TB=\begin{pmatrix} m & 2m & 3m & \cdots & (n-1)m & nm \\ m & 2m & 3m & \cdots & (n-1)m & nm \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & \cdots & (n-1)m & nm \end{pmatrix}.$$

(b) $(BJ^T)_{ij} = \sum\limits_{k=1}^{n} (B)_{ik}(J^T)_{kj} = \sum\limits_{k=1}^{n} (B)_{ik}(J)_{jk} = \sum\limits_{k=1}^{n} (B)_{ik} = \sum\limits_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $1 \le i, j \le m$. That is,

$$BJ^T = \frac{1}{2}n(n+1)J_m,$$

where J_m is an $m \times m$ matrix whose entries are 1.

1-6: $(AA^T)^T = (A^T)^T A^T = AA^T$. Thus AA^T is symmetric.

1-7: Let $X = \frac{1}{2}(A + A^T)$ and $Y = \frac{1}{2}(A - A^T)$. Clearly, $X + Y = A$ and it is easy to check that $X^T = X$ and $Y^T = -Y$.