STAT 2006 Assignment 2

Due Time and Date: 5 p.m., 26 March, 2020

1. For two random variables (X, Y), the MGF can be defined as

$$M_{XY}(s,t) = \mathbb{E}[e^{sX+tY}].$$

Find $M_{XY}(s,t)$ when X and Y are two jointly normal random variables with $\mathbb{E}(X) = \mu_X, \mathbb{E}(Y) = \mu_Y, Var(X) = \sigma_X^2, Var(Y) = \sigma_Y^2, \rho(X,Y) = \rho$.

2. Let $Y_1, Y_2, \ldots, Y_n \overset{\text{i.i.d.}}{\sim} \exp(\theta)$ are random samples. If Y_i's are sorted in ascending order, the ordered random variables $X_1, X_2, ..., X_n$ with the joint pdf

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, ..., x_n) = \frac{n!}{\theta^n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right\},$$

where $0 \le x_1 \le x_2 \le ... \le x_n < \infty$, are obtained.

   (a) Let $U_1 = X_1, U_i = X_i - X_{i-1}, i = 2, 3, ..., n$. Find the joint pdf of $(U_1, U_2, ..., U_n)$.

   (b) Are $U_1, U_2, ..., U_n$ mutually independent? What is the marginal distribution of each of the $U_i, i = 1, 2, ..., n$?

   (c) Using the result of part (a) and (b), or otherwise, find $\mathbb{E}[X_1]$ and $\mathbb{E}[X_n]$, the expectation of the sample minimum and sample maximum.

3. Let Z_1 and Z_2 be independent $N(0,1)$ random variables, and define new random variables X and Y by

$$X = a_X Z_1 + b_X Z_2 + c_X \text{ and } Y = a_Y Z_1 + b_Y Z_2 + c_Y,$$

where a_X, b_X, c_X, a_Y, b_Y and c_Y are constants.

   (a) Show that $\mathbb{E}[X] = c_X$, $\text{Var}(X) = a_X^2 + b_X^2, \mathbb{E}[Y] = c_Y$, $\text{Var}(Y) = a_Y^2 + b_Y^2$ and $\text{Cov}(X,Y) = a_X a_Y + b_X b_Y$.

   (b) If we define the constants a_X, b_X, c_X, a_Y, b_Y and c_Y by

$$a_X = \sqrt{\frac{1+\rho}{2}}\sigma_X, \quad b_X = \sqrt{\frac{1-\rho}{2}}\sigma_X, \quad c_X = \mu_X,$$

$$a_Y = \sqrt{\frac{1+\rho}{2}}\sigma_Y, \quad b_Y = -\sqrt{\frac{1-\rho}{2}}\sigma_Y, \quad c_Y = \mu_Y,$$

   where $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ are constants, $-1 \le \rho \le 1$, then show that

$$\mathbb{E}[X] = \mu_X, \quad Var(X) = \sigma_X^2, \quad \mathbb{E}[Y] = \mu_Y, \quad Var(Y) = \sigma_Y^2, \quad Corr(X,Y) = \rho.$$

   (c) Show that (X, Y) has the bivariate normal pdf with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ.

4. Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ are independent exponential distributed random samples with mean θ. Let $T_\alpha = \alpha\bar{X} + (1-\alpha)\bar{Y}$, where $0 < \alpha < 1$.

   (a) Find $\mathbb{E}[T_\alpha]$ and $Var(T_\alpha)$.

   (b) Show that, for any $\epsilon > 0, \mathbb{P}(|T_\alpha - \theta| > \epsilon) \to 0$ as $m, n \to \infty$ [Hint: using Chebyshev's inequality].

5. Let $X_1, X_2, \cdots, X_n \overset{i.i.d.}{\sim} U[0,1]$.

   (a) Find the mean and variance of $\ln(X_1)$.

   (b) Let $0 \le a < b$. Find $\lim_{n\to\infty} \mathbb{P}\left(a \le (X_1 X_2 \cdots X_n)^{n^{-1/2}} e^{n^{1/2}} \le b\right)$ in terms of a and b.

6. Let $f(x;\theta) = \theta x^{\theta-1}$ where $0 \le x \le 1$ and $0 < \theta < \infty$.

   (a) Show that the maximum likelihood estimator (MLE) of θ is $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln(X_i)}$.

   (b) Given the observed random samples be $0.55, 0.88, 0.43, 0.78, 0.66$, what is the MLE of θ?

   (c) Show that $Y_1 := -\ln(X_1) \sim \exp(\frac{1}{\theta})$.

   (d) Hence, show that $S := \sum_{i=1}^n Y_i = -\sum_{i=1}^n \ln(X_i) \sim \Gamma(n, \frac{1}{\theta})$.

   (e) Is $\hat{\theta}$ an unbiased estimator?

7. Suppose $X_1, \cdots, X_n$ are i.i.d. with pdf $f(x;\theta) = 2x/\theta^2, 0 < x < \theta$, zero elsewhere. Note this is a non-regular case. Find:

   (a) The MLE $\hat{\theta}$ for θ.

   (b) The constant c so that $\mathbb{E}(c\hat{\theta}) = \theta$.

   (c) The MLE for the median of the distribution.

8. A random sample $X_1, X_2, \cdots, X_n$ of size n is taken from a Poisson distribution with a mean of $\lambda, 0 < \lambda < \infty$.

   (a) Show that the maximum likelihood estimator for λ is $\hat{\lambda} = \bar{X}$.

   (b) Let X equal the number of flaws per 100 feet of a used computer tape. Assume that X has a Poisson distribution with a mean of λ. If 50 observations of X yielded 3 zeros, 5 ones, 5 twos, 8 threes, 12 fours, 9 five and 8 six, fnd the maximum likelihood estimate of λ.

9. Let $X_1, X_2, \cdots, X_n$ be random samples from distributions with the given probability density functions $f(x;\theta)$. In each case, find the maximum likelihood estimator $\hat{\theta}$.

   (a) $f(x;\theta) = \frac{\theta^4}{6} x^3 e^{-\theta x}$ where $0 < x < \infty$ and $0 < \theta < \infty$.

   (b) When $\theta = 1$, $f(x;\theta) = 1$ where $0 < x < 1$. When $\theta = 2$, $f(x;\theta) = \frac{1}{2\sqrt{x}}$ where $0 < x < 1$.

   (c) $f(x;\theta) = \theta$ where $0 \le x \le \frac{1}{\theta}$ and $\theta > 0$.

10. Let $X_1, X_2, \cdots, X_n$ be i.i.d. with pdf

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$$

   Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators.

11. Let the pdf of X be defined by

$$f(x;\theta) = \begin{cases} (\frac{4}{\theta^2})x & \text{for } 0 < x \le \frac{\theta}{2}, \\ -(\frac{4}{\theta^2})x + \frac{4}{\theta} & \text{for } \frac{\theta}{2} < x \le \theta, \\ 0 & \text{otherwise,} \end{cases}$$

   where $0 < \theta \le 2$.

(a) Find the method-of-moment estimator of θ.

(b) For the following observations of X, give a point estimate of θ:

0.3209	0.2412	0.2557	0.3544	0.4168
0.5621	0.0230	0.5442	0.4552	0.5592

12. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from an exponential distribution with unknown mean θ.

    (a) Show that the distribution of the random variable $W = (2/\theta)\sum_{i=1}^{n} X_i$ is $\chi^2(2n)$.

    (b) Use W to construct a $100(1-\alpha)\%$ confidence interval for θ.

    (c) If $n = 8$ and $\bar{x} = 65.2$, give the endpoints for a 90% confidence interval for the mean θ.

13. The independent random variables $X_1, \cdots, X_n$ have the common distribution

$$P(X_i \le x|\alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \le x \le \beta \\ 1 & \text{if } x > \beta \end{cases}$$

    where the parameters α and β are positive.

    (a) Assume α and β are both unknown, find the MLEs of α and β.

    (b) The length of cuckoos's eggs found in hedge sparrow nests can be modeled with this distribution. For the data

$$22.0, 23.9, 20.9, 23.8, 26.0, 25.0, 21.7, 23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0$$

    find the MLEs of α and β.

    (c) If α is a known constant, α_0, find an upper confidence limit for β with confidence coefficient 0.95.

    (d) Use the data in (b) to construct an interval estimate for β. Assume that α is known and equal to its MLE.

14. (a) Let Y be an exponential random variable with mean λ and $X \triangleq \theta_1 + \theta_2 Y, \theta_2 > 0$. Find the pdf of X and remember to state the support of X. X is said to follow a shifted exponential distribution with location parameter θ_1 and scale parameter θ_2.

    (b) Let $X_1, X_2, \cdots, X_n$ be a random sample which X_i are identically distributed as X. Find the method-of-moments estimator for θ_1 and θ_2.

    (c) When θ_2 is fixed, show that the likelihood function is strictly increasing in θ_1 when $\theta_1 \le x_{(1)}$ and is equal to zero when $\theta_1 > x_{(1)}$, where $x_{(1)} \triangleq \min\{x_1, x_2, \cdots, x_n\}$ is the sample minimum. Hence find the maximum likelihood estimator of θ_1 and θ_2.

15. A manufacturer sells a light bulb that has a mean life of 1580 hours with a standard deviation of 58 hours. A new manufacturing process is being tested, and there is interest in knowing the mean life μ of the new bulbs. How large a sample is required so that $[\bar{x} - 10, \bar{x} + 10]$ is an approximate 90% confidence interval for μ? You may assume that the change in the standard deviation is minimal.