# Assignment 1 Solution --- STAT4008

1. Suppose a discrete r.v. $T$ taking value $1,3,5,7,9,12$ w.p. $\frac{1}{6},\frac{1}{3},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{16}$ respectively.

(a) Find the mean of $T$.

$\bar{T} \triangleq$ Mean of $T = 1\times\frac{1}{6}+3\times\frac{1}{3}+5\times\frac{1}{4}+7\times\frac{1}{8}+9\times\frac{1}{16}+12\times\frac{1}{16} = \frac{221}{48}$.

(b) Find the survival function of $T$.

$$S(t) \triangleq \text{survival function of } T = \mathbb{P}(T>t) = \begin{cases} 1 = 1 & \text{if } t\in[0,1); \\ 1-\frac{1}{6} = \frac{5}{6} & \text{if } t\in[1,3); \\ \frac{5}{6}-\frac{1}{3} = \frac{1}{2} & \text{if } t\in[3,5); \\ \frac{1}{2}-\frac{1}{4} = \frac{1}{4} & \text{if } t\in[5,7); \\ \frac{1}{4}-\frac{1}{8} = \frac{1}{8} & \text{if } t\in[7,9); \\ \frac{1}{8}-\frac{1}{16} = \frac{1}{16} & \text{if } t\in[9,12); \\ \frac{1}{16}-\frac{1}{16} = 0 & \text{if } t\in[12,\infty). \end{cases}$$

(c) Find $\int_0^\infty S(t)dt$.

$\int_0^\infty S(t)dt = 1\times 1+\frac{5}{6}\times 2+\frac{1}{2}\times 2+\frac{1}{4}\times 2+\frac{1}{8}\times 2+\frac{1}{16}\times 3 = \frac{221}{48}$.

(d) Compare results in (a) and (c).

The results in (a) and (c) are the same, i.e., $\bar{T} = \int_0^\infty S(t)dt$.

Note that we can actually show the equality hold in general:

$\mathbb{E}(T) = \int_0^\infty t f(t)dt = \int_0^\infty -t\,dS(t) = [-tS(t)]_0^\infty - \int_0^\infty -S(t)dt = \int_0^\infty S(t)dt \quad \ldots (\ast)$

Q2

For $t \geq 0$, the survival function of $T$ is given by

$$\begin{aligned} S_T(t) &:= \mathbb{P}\{T>t\} = \int_t^\infty \theta\exp\{-\theta u\}\mathbb{1}\{u>0\}\,du \\ &= \int_t^\infty \theta e^{-\theta u}\,du = \lim_{\tau\to\infty}\left[-e^{-\theta u}\right]_t^\tau = \lim_{\tau\to\infty}\left(-e^{-\theta\tau}\right)+e^{-\theta t} = e^{-\theta t}. \end{aligned}$$

Hence, for $s \geq 0$, we have the conditional survival function

$$\begin{aligned} S_{T|t}(s) &:= \mathbb{P}\{T>t+s|T>t\} = \frac{\mathbb{P}\{\{T>t+s\}\cap\{T>t\}\}}{\mathbb{P}\{T>t\}} \\ &= \frac{\mathbb{P}\{T>t+s\}}{\mathbb{P}\{T>t\}} = \frac{S_T(t+s)}{S_T(t)} = \frac{e^{-\theta(t+s)}}{e^{-\theta t}} = e^{-\theta s}, \end{aligned}$$

and equals one elsewhere.

## Q3

For $t > 0$, the survival function of $T$ is given by

$$S_T(t) = e^{-H_T(t)} = \exp\left\{-\int_0^t h_T(u)\,du\right\} = \exp\left\{-\int_0^t \alpha\lambda u^{\alpha-1}\,du\right\} = \exp\{-\lambda t^\alpha\}.$$

So, for $t > 0$, the probability density function of $T$ is given by

$$f_T(t) = -\frac{d}{dt}S_T(t) = -\frac{d}{dt}\exp\{-\lambda t^\alpha\} = \alpha\lambda t^{\alpha-1}e^{-\lambda t^\alpha}.$$

For the <u>one-one</u> transformation $y = \log t \ \Leftrightarrow t = e^y$, the Jacobian $J = \frac{dt}{dy} = e^y$. Hence, the p.d.f. of $Y$ is

$$f_Y(y) = f_T(e^y)|J| = \alpha\lambda t^{\alpha-1}e^{-\lambda t^\alpha}e^y = \alpha\lambda\exp\{\alpha y - \lambda e^{y\alpha}\}, \quad \text{for } y \in \mathbb{R}.$$

Note that $y = \log t$ for $t \in \mathbb{R}^+$ implies $y \in \mathbb{R}$. ■

## Q4

(c) Consider those $t$ such that $S(t) > 0$. First note that $m(t)$ can be expressed as

$$m(t) := \mathbb{E}\{T - t|T \geq t\} = \int_0^\infty \frac{\mathbb{P}\{T > u + t\}}{\mathbb{P}\{T > t\}}\,du = \frac{1}{S(t)}\int_t^\infty S(u)\,du,$$

which, together with the continuity of $S(t)$, implies $m(t) > 0$. So, we have $m(t)S(t) = \int_t^\infty S(u)du$. Assume $m(t)$ and $S(t)$ are everywhere differentiable, then differentiate the preceding equation with respect to $t$, get

$$m'(t)S(t) + m(t)S'(t) = -S(t) \qquad \Leftrightarrow \qquad \frac{-S'(t)}{S(t)} = \frac{m'(t)+1}{m(t)}.$$

In view of the definition $h(t) := f(t)/S(t) = -S'(t)/S(t)$. The result follows.

(a) Further assume $m(0) > 0$ and $m(u)$ is almost everywhere positive on $(0,t)$. Since $S(t) = \exp\left\{-\int_0^t h(u)\,du\right\}$, we have, for $t > 0$,

$$\begin{aligned} S(t) &= \exp\left\{-\int_0^t \frac{m'(u)+1}{m(u)}\,du\right\} = \exp\left\{-\int_0^t d\left[\ln m(u)\right] - \int_0^t \frac{du}{m(u)}\right\} \\ &= \exp\left\{-\ln\left[\frac{m(t)}{m(0)}\right] - \int_0^t \frac{du}{m(u)}\right\} = \frac{m(0)}{m(t)}\exp\left\{\int_0^t \frac{-du}{m(u)}\right\}, \end{aligned}$$

and equals one elsewhere, where I have used one **important** formula

$$\frac{d}{du}\ln m(u) = \frac{m'(u)}{m(u)}.$$

(b) By definition $f(t) := -S'(t)$, we have, for $t > 0$,

$$\begin{aligned} f(t) &= -m(0)\left\{\frac{m(t)\exp\left\{-\int_0^t \frac{du}{m(u)}\right\}\left[\frac{-1}{m(t)}\right] - m'(t)\exp\left\{-\int_0^t \frac{du}{m(u)}\right\}}{m^2(t)}\right\} \\ &= [m'(t)+1]\left(\frac{m(0)}{m^2(t)}\right)\exp\left\{\int_0^t \frac{-du}{m(u)}\right\}, \end{aligned}$$

and equals zero elsewhere. ■

Q5

For all $j \in \mathbb{N}$, hazard function is given by

$$h(j) := \frac{\mathbb{P}\{X = j\}}{\mathbb{P}\{X \geq j\}} = \frac{(1-p)^{j-1}p}{\sum_{i=j}^{\infty}(1-p)^{i-1}p} = \frac{(1-p)^{j-1}[1-(1-p)]}{(1-p)^{j-1}} = p. \blacksquare$$

Q6

For $j \in \mathbb{N} \cup \{0\}$, the hazard function is given by

$$h(j) := \frac{\mathbb{P}\{X = j\}}{\mathbb{P}\{X \geq j\}} = \frac{e^{-\lambda}\lambda^j}{j!} \frac{1}{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!} = \frac{\lambda^j}{j! \sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!},$$

which is positive for all $j$ for any $\lambda > 0$. So, we can consider for any $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\frac{h(j+1)}{h(j)} &= \frac{\lambda^{j+1}}{(j+1)!} \frac{j!}{\lambda^j} \frac{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!}{\sum_{i=j+1}^{\infty} e^{-\lambda}\lambda^i/i!} \\
&= \frac{\lambda}{j+1} \frac{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!}{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^{i+1}/(i+1)!} = \frac{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!}{\sum_{i=j}^{\infty}(j+1)e^{-\lambda}\lambda^i/(i+1)!} \\
&\geq \frac{\sum_{i=j}^{\infty} e^{-\lambda}\lambda^i/i!}{\sum_{i=j}^{\infty}(i+1)e^{-\lambda}\lambda^i/(i+1)!} = 1.
\end{aligned}$$

Therefore, $h(j+1) \geq h(j)$ for all $j \in \mathbb{N} \cup \{0\}$, i.e. $h(j)$ is monotonic increasing on $\mathbb{N} \cup \{0\}$. $\blacksquare$

Q7

(a) Since $\mathbb{P}\{T \geq 0\} = 1$, we have

$$\mathbb{E}T = \mathbb{E}\{T - 0 | T \geq 0\} = m(0) = 10.$$

(b) In view of the result of Question 4. Since $m(t)$ is differentiable, we have, for $t > 0$, that

$$h(t) = \frac{m'(t)+1}{m(t)} = \frac{1+1}{t+10} = \frac{2}{t+10}.$$

(c) We use the result of Question 4 again. First, we have, for $t > 0$

$$\int_0^t \frac{du}{m(u)} = \int_0^t \frac{du}{u+10} = \ln(u+10)\Big|_0^t = \ln\left(\frac{t+10}{10}\right).$$

Hence, for $t > 0$, yield

$$S(t) = \frac{0+10}{t+10} \exp\left\{-\ln\left(\frac{t+10}{10}\right)\right\} = \left(\frac{10}{t+10}\right)^2,$$

and equals one elsewhere. $\blacksquare$

Q8

$$\mathrm{S(t)} = \exp\left\{-\int_0^t h(u)du\right\} = \exp\left\{-\int_0^t \theta e^{\alpha u} du\right\} = \exp\{\frac{-\theta}{\alpha}(\exp(\alpha t) - 1)\}$$