STAT2001 midterm exam solution for 2019-20 T1

1. (a) Suppose that in a hospital, number of babies born follows a Poisson process with a rate of 5 births per day. What is the probability that at most 2 babies are born during the next 2 hours?
   (b) Bowl I contains 7 red chips and 3 blue chips. 5 of these 10 chips are chosen at random and without replacement and put in bowl II, which was originally empty. 1 chip is then chosen at random from bowl II. Given that this chip is blue, find the conditional probability that 2 red chips and 3 blue chips are transferred from bowl I to bowl II.

   Answer
   (a) Let X be the number of babies born in the next 2 hours. X~Poisson(5/12). So
$$P(X \le 2) = \sum_{k=0}^{2} \frac{e^{-\frac{5}{12}}\left(\frac{5}{12}\right)^k}{k!} = 0.9912.$$

   (b) Let A denote {0 blue from I to II}, B denote {1 blue from I to II}. C denote {2 blue from I to II}, D denote {3 blue from I to II}, F denote {Blue from II}
$$P(D|F) = \frac{P(F|D)P(D)}{P(F)} = \frac{P(F|D)P(D)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C) + P(F|D)P(D)}$$
$$= \frac{\frac{3}{5}\frac{\binom{3}{3}\binom{7}{2}}{\binom{10}{5}}}{0 + \frac{1}{5}\frac{\binom{3}{1}\binom{7}{4}}{\binom{10}{5}} + \frac{2}{5}\frac{\binom{3}{2}\binom{7}{3}}{\binom{10}{5}} + \frac{3}{5}\frac{\binom{3}{3}\binom{7}{2}}{\binom{10}{5}}} = \frac{1}{6}$$

2. (a) Three distinct integers are chosen at random from the first 20 positive integers (i.e. 1,2,3,…,20). Compute the probability that the product of the chosen integers is even.
   (b) Let C_1, C_2 and C_3 be mutually independent events with probabilities $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Compute $P(C_1 \cup C_2 \cup C_3)$.
   Answer:
   (a)
$$P(product\ is\ even) = 1 - P(product\ is\ odd) = 1 - \frac{\binom{10}{3}}{\binom{20}{3}} = 0.8947$$
   (b) $P(C_1 \cup C_2 \cup C_3) = P(C_1) + P(C_2) + P(C_3) - P(C_1 \cap C_2) - P(C_2 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3) = P(C_1) + P(C_2) + P(C_3) - P(C_1)P(C_2) - P(C_2)P(C_3) - P(C_1)P(C_3) + P(C_1)P(C_2)P(C_3) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2}\frac{1}{3} - \frac{1}{3}\frac{1}{4} - \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{3}\frac{1}{4} = 0.75$

3. Let X be a discrete random variable having the following probability mass function.
$$f(x) = \frac{3^x e^{-3}}{3(x-1)!}, \quad x = 1,2,3,\ldots$$
   (a) Find the moment generating function of X.
   (b) Find the expected value of X.

(c) Find the variance of *X*.

Answer:

(a) $M(t) = E(e^{tX}) = \sum_{x=1}^{\infty} \frac{3^{x-1}e^{-3}}{(x-1)!} e^{tx} = \sum_{y=0}^{\infty} \frac{3^y e^{-3}}{y!} e^{t(y+1)} = e^{t-3} \sum_{y=0}^{\infty} \frac{(3e^t)^y}{y!} = e^{t-3} e^{(3e^t)}$
$= \exp\left(t + 3(e^t - 1)\right)$

(b) $R(t) = lnM(t) = t + 3(e^t - 1)$

$R'(t) = 1 + 3e^t \Rightarrow EX = R'(0) = 1 + 3 = 4$

(c) $R''(t) = 3e^t \Rightarrow \mathrm{Var}(X) = R''(0) = 3$

Remark:

1, For (b) and (c), of course you can differentiate on M(t) to get the same results.

2, If you can notice that X=Y+1 where Y is Poisson(3), you can finish the whole question very quickly.

4. A continuous random variable *X* has the following probability density function.

$$f(x) = 6x(1-x) \qquad 0 < x < 1.$$

(a) Find the mean of *X.*
(b) Find the variance of *X*.
(c) Let μ and σ be the mean and standard deviation of *X* respectively.
Find *P(μ-3σ<X< μ+3σ)*.
(d) Find the cumulative distribution function of *X*.

Answer:

(a) $EX = \int_0^1 6x^2(1-x)dx = \int_0^1 (6x^2 - 6x^3)dx = 2 - \frac{3}{2} = \frac{1}{2}$

(b) $EX^2 = \int_0^1 6x^3(1-x)dx = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}$

$$Var(X) = EX^2 - (EX)^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

(c) *μ-3σ=-0.1708, μ+3σ=1.1708. P(-0.1708<X<1.1708)=1.*

(d)

$$F(x) = \int_0^x 6t - 6t^2 dt = 3x^2 - 2x^3 \quad for\ 0 < x < 1;$$
$$F(x) = 0\ for\ x \le 0; \quad F(x) = 1\ for\ x \ge 1$$

5. Let *X* be a continuous random variable following Exponential distribution with parameter θ. Find $E(|X-\theta|)$.

Answer:

$E(|X-\theta|)$

$$= \int_0^\infty |x-\theta| \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) dx$$

$$= \int_0^\theta (\theta - x)\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)dx + \int_\theta^\infty (x-\theta)\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)dx$$
$$= \theta P(X < \theta) + \int_0^\theta x d\exp\left(-\frac{x}{\theta}\right) - \int_\theta^\infty x d\exp\left(-\frac{x}{\theta}\right) - \theta P(X > \theta)$$
$$= \theta(1 - e^{-1}) + [x\exp\left(-\frac{x}{\theta}\right)]_0^\theta - \int_0^\theta \exp\left(-\frac{x}{\theta}\right)dx$$
$$-[x\exp\left(-\frac{x}{\theta}\right)]_\theta^\infty + \int_\theta^\infty \exp\left(-\frac{x}{\theta}\right)dx - \theta e^{-1}$$

$$= \theta(1 - 2e^{-1}) + \theta e^{-1} + \theta(e^{-1} - 1) + \theta e^{-1} + \theta e^{-1}$$
$$= 2\theta e^{-1}$$