Reference solution for Assignment 2

October 25, 2019

1. (15')

   As there are 5 prizes to choose, the most valuable prize, denoted by X, should range from 5 to 100.

   If the highest one is $X = k(5 \leq k \leq 100)$, the other 4 should be smaller than it, which means there are $\binom{k-1}{4}$ combinations.

   There are altogether $\binom{100}{5}$ combinations , thus

$$P(X = k) = \frac{\binom{k-1}{4}}{\binom{100}{5}}, \quad 5 \leq k \leq 100.$$

2. (15')

   By definition, for discrete distributions,

$$E(X) = \sum_{k=1}^{\infty} kf(k) = \sum_{k=1}^{\infty} k\frac{cp^x}{k} = \sum_{k=1}^{\infty} cp^k = c\frac{p}{1-p}, 0 < p < 1.$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 f(k) = \sum_{k=1}^{\infty} k^2 \frac{cp^x}{k} = \sum_{k=1}^{\infty} ckp^k. \tag{1}$$

   Thus,

$$pE(X^2) = \sum_{k=1}^{\infty} ckp^{k+1} = \sum_{k=2}^{\infty} (k-1)cp^k. \tag{2}$$

   (1)-(2), we get

$$(1-p)E(X^2) = cp + \sum_{k=2}^{\infty} cp^k = c\frac{p}{1-p}. \tag{3}$$

   Alternatively, recall that for random variable Y following a Geometric distribution with success probability $1-p$, we have

$$E(Y) = \sum_{k=1}^{\infty} k(1-p)p^{k-1} = \frac{1}{1-p}. \tag{4}$$

Then, either from (3) or (4), we can get that

$$E(X^2) = \frac{cp}{1-p}\sum_{k=1}^{\infty} k(1-p)p^{k-1} = c\frac{p}{(1-p)^2}.$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{cp - c^2p^2}{(1-p)^2}.$$

3. (10')

   The PMF for $Poisson(\lambda)$ distribution is $p(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \cdots$.

   We already know that the MGF is

$$M(t) = e^{\lambda(e^t-1)}.$$

   Then, $g(t) = \ln(M(t)) = \lambda(e^t - 1)$.

   The j-th cumulant is thus

$$\kappa_j = g^j(0) = \lambda, \quad j \geq 1.$$

4. (15')

   By linear properties of expectations,

$$E(X+3) = E(X) + 3 = 9, \quad E(X) = 6.$$

$$E((X+3)^2) = E(X^2 + 6X + 9) = E(X^2) + 6E(X) + 9 = 112,$$

   then,

$$E(X^2) = 112 - 9 - 6 \times 6 = 67.$$

   (a) $Var(X-4) = Var(X) = E(X^2) - (E(X))^2 = 67 - 36 = 31$.

(b) $E(X) = 6$.

(c) $Var(X) = E(X^2) - (E(X))^2 = 67 - 36 = 31$.

5. (15')

For each round of selection, the probability of choosing 2 black, 2 white is

$$P(stop) = \frac{\binom{7}{2}\binom{5}{2}}{\binom{12}{4}} = \frac{14}{33}.$$

To make at least 8 selections means that for the first 7 selections, the event of 2 black and 2 white doesn't happen, then we know the corresponding probability is

$$(1 - P(stop))^7 = (\frac{19}{33})^7.$$

6. (15')

As the MGF is $M(t) = (0.2 + 0.8e^t)^{12}$, we know X follows a binomial distribution with $n = 12, p = 0.8$, i.e. $X \sim B(12, 0.8)$.

Thus, for binomial distributions

$$E(X) = np = 12 \times 0.8 = 9.6.$$

$$E(X^2) = (E(X))^2 + Var(X) = n^2p^2 + np(1-p) = 94.08.$$

$$\begin{aligned} P(X < 3) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \binom{12}{0}(1-p)^{12} + \binom{12}{1}p(1-p)^{11} + \binom{12}{2}P^2(1-p)^{10} \\ &= 0.00000452608. \end{aligned}$$

7. (15')

(a) Since $f(x)$ is a PMF, then we know that $\sum_{k=4}^{\infty} f(k) = 1$, which means that

$$\sum_{k=4}^{\infty} c(\frac{1}{5})^k = 1 = c\frac{(\frac{1}{5})^4}{1 - \frac{1}{5}} = \frac{c}{4 \times 5^3},$$

thus, $c = 500$.

(b) The PMF is $f(x) = 500(\frac{1}{5})^x, x = 4, 5, 6, \cdots$, then the MGF is

$$\begin{aligned} M(t) = E(e^{tX}) &= \sum_{k=4}^{\infty} e^{tk} 500(\frac{1}{5})^k = 500\sum_{k=4}^{\infty}(\frac{e^t}{5})^k \\ &= 500\frac{(\frac{e^t}{5})^4}{1 - \frac{e^t}{5}} = \frac{500e^{4t}}{625 - 125e^t}. \end{aligned}$$

But we should pay attention that $-1 < \frac{e^t}{5} < 1$, i.e. $t < \ln(5)$.