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STAT 4008

Assignment 1

Question 1

a) $E(T) = 1(\frac{1}{6}) + 3(\frac{1}{3}) + 5(\frac{1}{4}) + 7(\frac{1}{8}) + 9(\frac{1}{16}) + 12(\frac{1}{16})$

$= \frac{221}{48}$

b)

$$S(t) = \begin{cases} 1 & \text{for } t < 1 \\ \frac{5}{6} & \text{for } 1 \leq t < 3 \\ \frac{1}{2} & \text{for } 3 \leq t < 5 \\ \frac{1}{4} & \text{for } 5 \leq t < 7 \\ \frac{1}{8} & \text{for } 7 \leq t < 9 \\ \frac{1}{16} & \text{for } 9 \leq t < 12 \\ 0 & \text{for } t \geq 12 \end{cases}$$

c) area $= (1-0)(1) + (3-1)(\frac{5}{6}) + (5-3)(\frac{1}{2}) + (7-5)(\frac{1}{4}) + (9-7)(\frac{1}{8})$

$+ (12-9)(\frac{1}{16}) + (\infty - 12)(0)$

$= \frac{221}{48}$

d) both results in (a) and (c) are the same,

we have $E(T) = \text{area}$

Question 2

$S(t) = 1 - F(t)$

$= 1 - \int_0^t \theta e^{-\theta y} dy$

$= e^{-t\theta}$

$P(T > t+s \mid T \geq t) = \frac{P(T > t+s)}{P(T \geq t)}$

$= P(T > s)$ by memoryless property

$= S(s)$, $s \geq 0$

$= e^{-t\theta}$

$\therefore P(T > t+s \mid T \geq t) = P(T > s) = e^{-t\theta}$

Question 3

$f_T(t) = h(t)\, S(t)$

$= \alpha \lambda t^{\alpha - 1} e^{-H(t)}$

$= \alpha \lambda t^{\alpha-1} e^{-\lambda t^{\alpha}}$

$f_Y(y) = \frac{\partial}{\partial y} P(\log T \leq y)$

$= \frac{\partial}{\partial y} \int_0^{e^y} \alpha \lambda t^{\alpha-1} e^{-\lambda t^{\alpha}} dt$

$= \alpha \lambda e^{\alpha y - \lambda e^{\alpha y}}$, $y \in \mathbb{R}$

Question 4

a) $m(t) = \int_0^{\infty} \frac{P(T > t+y)}{P(T > t)} dy$

$= \frac{1}{S(t)} \int_t^{\infty} S(y)\, dy$

$\Rightarrow m(t) S(t) = \int_t^{\infty} S(y)\, dy$

$m'(t) S(t) + m(t) S'(t) = -S(t)$

$-\frac{S'(t)}{S(t)} = \frac{m'(t)+1}{m(t)}$

$S(t) = e^{-\int_0^t \frac{m'(y)+1}{m(y)} dy}$

$= e^{-\ln\left[\frac{m(t)}{m(0)}\right] - \int_0^t \frac{1}{m(y)} dy}$

$= \frac{m(0)}{m(t)} e^{-\int_0^t \frac{1}{m(y)} dy}$

b) $f(t) = -\frac{\partial}{\partial t}\left[\frac{m(0)}{m(t)} e^{-\int_0^t \frac{1}{m(y)} dy}\right]$

$= -m(0) \cdot \frac{-m(t) e^{-\int_0^t \frac{1}{m(y)} dy} \cdot \frac{1}{m(t)} - m'(t) e^{-\int_0^t \frac{1}{m(y)} dy}}{m^2(t)}$

$= [m'(t)+1]\left[\frac{m(0)}{m^2(t)}\right] e^{-\int_0^t \frac{1}{m(y)} dy}$

c) $h(t) = \frac{f(t)}{S(t)}$

$= \frac{m'(t)+1}{m(t)}$

Question 5

$h(t) = \frac{P(T=t)}{P(T \geq t)}$

$= \frac{(1-p)^{t-1} p}{\sum_{i=t}^{\infty} (1-p)^{i-1} p}$

$= \frac{(1-p)^{t-1}[1-(1-p)]}{(1-p)^{t-1}}$

$= p$

Question 6

$$h(t) = \frac{P(T=t)}{P(T \geq t)}$$

$$= \frac{e^{-\lambda}\lambda^t}{t!} \Big/ \sum_{i=t}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= \frac{\lambda^t}{t! \sum \frac{\lambda^i}{i!}}$$

$$\frac{h(t+1)}{h(t)} = \frac{\dfrac{\lambda^{t+1}}{(t+1)! \sum \frac{\lambda^{i+1}}{(i+1)!}}}{\dfrac{\lambda^t}{t! \sum \frac{\lambda^i}{i!}}}$$

$$= \frac{\lambda}{t+1} \frac{\sum \frac{\lambda^i}{i!}}{\sum \frac{\lambda^{i+1}}{(i+1)!}}$$

$$= \frac{\sum \frac{\lambda^i}{i!}}{\sum \frac{\lambda^i (t+1)}{(i+1)!}}$$

$$\geq \frac{\sum \frac{\lambda^i}{i!}}{\sum \frac{\lambda^i (i+1)}{(i+1)!}} = 1$$

Since $h(t+1) \geq h(t)$, $h(t)$ is monotone increasing

Question 7

a) given $P(T \geq 0) = 1$, we have

$$E(T) = m(0)$$

$$= 10$$

b) $$h(t) = \frac{m'(t)+1}{m(t)}$$

$$= \frac{2}{t+10}$$

c) $$S(t) = \frac{m(0)}{m(t)} e^{-\int_0^t m(y) dy}$$

$$= \frac{10}{t+10} e^{-\ln\left(\frac{t+10}{10}\right)}$$

$$= \left(\frac{10}{t+10}\right)^2$$

Question 8

$$S(t) = e^{-H(t)}$$

$$= e^{-\int_0^t \theta e^{\alpha y} dy}$$

$$= e^{-\frac{\theta}{\alpha}(e^{\alpha t}-1)}$$