# STAT 2006 Assignment 2 Suggested Solution

1. Note that $U = sX + tY$ is a linear combination of $X$ and $Y$ and thus it is a normal random variable. We have
$$E(U) = sE(X) + tE(Y) = s\mu_X + t\mu_Y,$$
$$Var(U) = s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y.$$
Thus
$$U \sim N(s\mu_X + t\mu_Y, s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y).$$
Note that for a normal random variable with mean $\mu$ and variance $\sigma^2$, the MGF is $e^{\mu t + \frac{\sigma^2 t^2}{2}}$. Hence
$$M_{XY}(s,t) = E(e^U) = M_U(1) = e^{\mu_U + \frac{\sigma_U^2}{2}} = e^{s\mu_X + t\mu_Y + \frac{1}{2}(s^2\sigma_X^2 + t^2\sigma_Y^2 + 2st\rho\sigma_X\sigma_Y)}.$$

2. (a) Since $0 < X_1 \le X_2 \le ... \le X_n < +\infty$ and $U_1 = X_1, U_i = X_i - X_{i-1}, i = 2, 3, ..., n$, it is easy to deduce that the support of $(U_1, U_2, ..., U_n)$ is $(0, +\infty)^n$. Also since $X_i = U_i + X_{i-1}, i = 2, 3, ..., n$ and $X_1 = U_1$, inductively we obtain $X_i = \sum_{j=1}^{i} U_j \Rightarrow \frac{\partial x_i}{\partial u_j} = \begin{cases} 1 & \text{if} \quad j = 1, 2, ..., i \\ 0 & \text{if} \quad j = i+1, i+2, ..., n \end{cases}$

   Therefore the Jacobian $J = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = 1^n = 1$

   Note that $\sum_{i=1}^{n}\sum_{j=1}^{i} u_j = \sum_{j=1}^{n}\sum_{i=j}^{n} u_j = \sum_{j=1}^{n}(n-j+1)u_j$ and we conclude that the joint pdf
$$f_{U_1,U_2,...,U_n}(u_1, u_2, ..., u_n) = \frac{n!}{\theta^n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}(n-i+1)u_i\right\}, 0 < u_i < +\infty, i = 1, 2, ..., n.$$

   (b) Note that the joint pdf can be factorized as
$$f_{U_1,U_2,...,U_n}(u_1, u_2, ..., u_n) = \prod_{i=1}^{n} \frac{n-i+1}{\theta} \exp\left\{\frac{n-i+1}{\theta} u_i\right\}, 0 < u_i < +\infty, i = 1, 2, ..., n$$
   So $U_1, U_2, ..., U_n$ are mutually independent and $U_i \sim \exp\left(\frac{\theta}{n-i+1}\right), i = 1, 2, ..., n.$

   (c) From part (a) and (b), $\mathbb{E}[X_1] = \mathbb{E}[U_1] = \frac{\theta}{n}, \mathbb{E}[X_n] = \mathbb{E}\left[\sum_{j=1}^{n} U_j\right] = \sum_{j=1}^{n}\frac{\theta}{n-j+1} = \theta\sum_{k=1}^{n}\frac{1}{k}.$

3. (a) $EX = a_X EZ_1 + b_X EZ_2 + Ec_X = a_X 0 + b_X 0 + c_X = c_X$
   $Var(X) = a_X^2 VarZ_1 + b_X^2 VarZ_2 + Varc_X = a_X^2 + b_X^2$
   We can calculate $EY$ and $VarY$ in similar way.
$$\begin{aligned} Cov(X,Y) &= EXY - EXEY \\ &= E[(a_X a_Y Z_1^2 + b_X b_Y Z_2^2 + c_X c_Y + a_X b_Y Z_1 Z_2 + a_X c_Y Z_1 + b_X a_Y Z_2 Z_1 + b_X c_Y Z_1 + c_X b_Y Z_2) - c_X c_Y] \\ &= a_X a_Y + b_X b_Y, \end{aligned}$$
   since $EZ_1^2 = EZ_2^2 = 1$, and the expectations of other terms are all zero.

   (b) Simply plug the expressions for $a_X, b_X$, etc. into the equalities in (a) and simplify.

(c) Let $D = a_X b_Y - a_Y b_X = -\sqrt{1-\rho^2}\sigma_X\sigma_Y$ and solve for $Z_1$ and $Z_2$, $Z_1 = \frac{\sigma_Y(X-\mu_X)+\sigma_X(Y-\mu_Y)}{\sqrt{2(1+\rho)}\sigma_X\sigma_Y}$, $Z_2 = \frac{\sigma_Y(X-\mu_X)+\sigma_X(Y-\mu_Y)}{\sqrt{2(1-\rho)}\sigma_X\sigma_Y}$.
Then the Jacobian is

$$J = \begin{vmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial y} \end{vmatrix} = \begin{vmatrix} b_Y/D & -b_X/D \\ -a_Y/D & a_X/D \end{vmatrix} = 1/D.$$

and we have
$f_{(X,Y)}(x,y) = \frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}\frac{(\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y))^2}{2(1+\rho)\sigma_X^2\sigma_Y^2}]\frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}\frac{(\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y))^2}{2(1-\rho)\sigma_X^2\sigma_Y^2}] \times \frac{1}{\sqrt{1-\rho^2}\sigma_X\sigma_Y}$,
after the simplification we can get the required bivariate normal pdf.

4. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_m \overset{\text{i.i.d.}}{\sim} \exp\left(\frac{1}{\theta}\right)$.

(a)

$$\mathbb{E}[T_\alpha] = \mathbb{E}[\alpha\bar{X} + (1-\alpha)\bar{Y}] = \alpha\theta + (1-\alpha)\theta = \theta$$

while

$$\begin{aligned}\text{Var}(T_\alpha) &= \text{Var}(\alpha\bar{X} + (1-\alpha)\bar{Y}) \\ &= \alpha^2\text{Var}(\bar{X}) + (1-\alpha)^2\text{Var}(\bar{Y}) + 2\alpha(1-\alpha)\text{Cov}(\bar{X},\bar{Y}) \\ &= \alpha^2\left(\frac{\theta^2}{n}\right) + (1-\alpha)^2\left(\frac{\theta^2}{m}\right) = \theta^2\left(\frac{\alpha^2}{n} + \frac{(1-\alpha)^2}{m}\right).\end{aligned}$$

(b) By Chebyshev's inequality, for any $\epsilon > 0$,

$$\mathbb{P}(|T_\alpha - \theta| > \epsilon) \le \frac{1}{\epsilon^2}\mathbb{E}\left[(T_\alpha - \theta)^2\right] = \frac{1}{\epsilon^2}\text{Var}(T_\alpha) = \frac{\theta^2}{\epsilon^2}\left(\frac{\alpha^2}{n} + \frac{(1-\alpha)^2}{m}\right) \to 0$$

as $m, n \to \infty$.

5. (a) $\mathbb{E}[\ln X_1] = \int_0^1 \ln x(1)dx = \int_0^1 \ln x dx = [(\ln x)(x)]_0^1 - \int_0^1 x d\ln x = -1.$
$\mathbb{E}[(\ln X_1)^2] = \int_0^1 (\ln x)^2 dx = \left[x(\ln x)^2\right]_0^1 - \int_0^1 xd\,(\ln x)^2 = -2\int_0^1 \ln x dx = 2.$
$\text{Var}(\ln X_1) = 2 - (-1)^2 = 1.$

(b)

$$\begin{aligned}&\mathbb{P}\left(a \le (X_1X_2\ldots X_n)^{\frac{1}{\sqrt{n}}}e^{\sqrt{n}} \le b\right) \\ =&\mathbb{P}\left(\ln a \le \frac{1}{\sqrt{n}}\sum_{i=1}^n \ln X_i + \sqrt{n} \le \ln b\right) \\ =& \mathbb{P}\left(\ln a \le \frac{\sum_{i=1}^n \ln X_i + n}{\sqrt{n}} \le \ln b\right).\end{aligned}$$

Then by CLT,

$$\lim_{n\to\infty}\mathbb{P}\left(a \le (X_1X_2\ldots X_n)^{\frac{1}{\sqrt{n}}}e^{\sqrt{n}} \le b\right) = \Phi(\ln b) - \Phi(\ln a).$$

6. (a) $L(\theta; X_1, \ldots, X_n) = \prod_{i=1}^n \theta X_i^{\theta-1} = \theta^n \prod_{i=1}^n X_i^{\theta-1}$ and $l(\theta; X_1, \ldots, X_n) = n\ln\theta + (\theta-1)\sum_{i=1}^n \ln X_i$.
$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln X_i$ and $\frac{\partial^2 l}{\partial\theta^2} = -\frac{n}{\theta^2} < 0$ for any $\theta$. Therefore,

$$\frac{\partial l}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} + \sum_{i=1}^n \ln X_i = 0 \Rightarrow \hat{\theta}_{\text{MLE}} = -\frac{n}{\sum_{i=1}^n \ln X_i}.$$

(b) By part (a), $\hat{\theta}_{\text{MLE}} = 2.238521 \approx 2.24$.

(c) $\mathbb{P}(Y_1 \le y) = \mathbb{P}(-\ln X_1 \le y) = \mathbb{P}(X_1 \ge e^{-y}) = 1 - \mathbb{P}(X_1 < e^{-y}) = 1 - \int_0^{e^{-y}} \theta x^{\theta-1} dx = 1 - e^{-\theta y}$ for $0 \le y < \infty$. Therefore, $Y_1 \sim \exp\left(\frac{1}{\theta}\right)$.

(d) Since $X_i$ are independent, $Y_i := \ln X_i$ are also independent. By part (c), $Y_i \sim \exp\left(\frac{1}{\theta}\right)$. Therefore, $S = \sum_{i=1}^{n} Y_i \sim \Gamma\left(n, \frac{1}{\theta}\right)$.

(e) $\mathbb{E}\left[\hat{\theta}_{\text{MLE}}\right] = \mathbb{E}\left[\frac{n}{S}\right] = n\mathbb{E}\left[\frac{1}{S}\right] = n\frac{\theta^n}{\Gamma(n)}\int_0^\infty \frac{1}{s} s^{n-1} e^{-\theta s} ds = n\frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{n\theta}{n-1} \ne \theta.$
Therefore, $\hat{\theta}_{\text{MLE}}$ is not an unbiased estimator of $\theta$.

7. The likelihood function simplifies to

$$L(\theta) = \frac{2^n}{\theta^{2n}} \sum_{i=1}^{n} x_i \mathbb{I}(0 < x_i \le \theta).$$

But $x_i \le \theta$ for all $i = 1, \ldots, n$ if and only if $\max_{1\le i\le n} x_i \le \theta$. Hence, the likelihood can be written as

$$L(\theta) = \frac{2^n}{\theta^{2n}} \mathbb{I}\left(0 < \max_{1\le i\le n} x_i \le \theta\right) \sum_{i=1}^{n} x_i.$$

(a) It is clear from the form of the likelihood that the maximum of $L(\theta)$ occurs at the smallest value in the range of $\theta$; hence, the MLE of $\theta$ is $Y = \max_{1\le i\le n} X_i$.

(b) The cdf of $X_i$ is $F_X(x) = \frac{x^2}{\theta^2}$. Hence, the cdf and pdf of $Y$ are, respectively,

$$F_Y(y) = \frac{y^{2n}}{\theta^{2n}}, \ 0 < y \le \theta$$

$$f_Y(y) = \frac{2ny^{2n-1}}{\theta^{2n}}, \ 0 < y \le \theta.$$

So

$$E(Y) = \int_0^\theta \frac{2ny^{2n}}{\theta^{2n}} dy = \frac{2n}{2n+1}\theta.$$

So

$$c = \frac{2n+1}{2n}.$$

(c) The median is the value of $x$ which solves $x^2/\theta^2 = 1/2$, which is $\theta/\sqrt{2}$. The MLE of the median is therefore $Y/\sqrt{2}$. Note that an unbiased estimate of the median is $[(2n+1)Y]/[2n\sqrt{2}]$.

8. (a) $L(\lambda; X_1, \ldots, X_n) = \prod_{i=1}^{n} f(X_i; \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{X_i}}{X_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n X_i!}$.

$l(\lambda; X_1, \ldots, X_n) = -n\lambda + \left(\sum_{i=1}^{n} X_i\right)\ln\lambda - \ln\left(\prod_{i=1}^{n} X_i!\right)$.

$\frac{\partial l}{\partial \lambda} = -n + \frac{\sum_{i=1}^n X_i}{\lambda}$ and $\frac{\partial^2 l}{\partial \lambda^2} = -\frac{\sum_{i=1}^n X_i}{\lambda^2} < 0$ for any $\lambda$. Then

$$\frac{\partial l}{\partial \lambda} = 0 \Rightarrow -n + \frac{\sum_{i=1}^n X_i}{\lambda} = 0 \Rightarrow \hat{\lambda}_{\text{MLE}} = \frac{\sum_{i=1}^n X_i}{n} = \bar{X}.$$

(b) Let $X \sim \text{Poisson}(\lambda)$. By part $(a)$,

$$\hat{\lambda}_{\text{MLE}} = \frac{3(0)+5(1)+5(2)+8(3)+12(4)+9(5)+8(6)}{50} = 3.6.$$

9. (a) $L(\theta; X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{\theta^4}{6} X_i^3 e^{-\theta X_i} = \frac{\theta^{4n}}{6^n} \left( \prod_{i=1}^{n} X_i^3 \right) e^{-\theta \sum_{i=1}^{n} X_i}.$

$$l(\theta; X_1, \ldots, X_n) = -n \ln 6 + 4n \ln \theta + 3 \sum_{i=1}^{n} \ln X_i - \theta \sum_{i=1}^{n} X_i.$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \frac{4}{\bar{X}}.$$

$$\left.\frac{\partial^2 l}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = \frac{-4n}{\hat{\theta}^2} < 0.$$

Therefore, $\hat{\theta}_{\text{MLE}} = \frac{4}{\bar{X}}$.

(b) When $\theta = 1$, $f(x;1) = 1$; When $\theta = 2$, $f(x;2) = \frac{1}{2\sqrt{x}}$. Therefore, $L(1; X_1, \ldots, X_n) = 1$ and $L(2; X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{1}{2\sqrt{X_i}} = \frac{1}{2^n \sqrt{\prod_{i=1}^{n} X_i}}$. Thus,

$$\hat{\theta}_{\text{MLE}} = \begin{cases} 1 & \text{if } \prod_{i=1}^{n} X_i > \frac{1}{2^{2n}}, \\ 2 & \text{if } \prod_{i=1}^{n} X_i < \frac{1}{2^{2n}}, \\ 1 \text{ or } 2 & \text{if } \prod_{i=1}^{n} X_i = \frac{1}{2^{2n}}. \end{cases}$$

Marks will not be deducted if the last condition is not written.

(c) $f(x;\theta) = \theta \mathbb{1}_{[0,\frac{1}{\theta}]}(x)$ where $\mathbb{1}_{[0,\frac{1}{\theta}]}(x) := 1$ when $0 \le x \le \frac{1}{\theta}$ and $\mathbb{1}_{[0,\frac{1}{\theta}]}(x) := 0$ when $x < 0$ or $x > \frac{1}{\theta}$.

$$L(\theta; X_1, \ldots, X_n) = \prod_{i=1}^{n} \theta \mathbb{1}_{[0,\frac{1}{\theta}]}(X_i) = \theta^n \prod_{i=1}^{n} \mathbb{1}_{[0,\frac{1}{\theta}]}(X_i) = \theta^n \mathbb{1}_{[0,\frac{1}{\theta}]}(\max X_i)$$

$$= \begin{cases} \theta^n & \text{if } \theta \le \frac{1}{\max X_i}, \\ 0 & \text{if } \theta > \frac{1}{\max X_i}. \end{cases}$$

Since $\theta^n$ is increasing as a function in $\theta$, so $\hat{\theta}_{\text{MLE}} = \frac{1}{\max X_i}$.

10. This is a uniform$(0,\theta)$ model. So $EX = (0+\theta)/2 = \theta/2$. The method of moments estimator is the solution to the equation $\tilde{\theta}/2 = \bar{X}$, that is, $\tilde{\theta} = 2X$. Because $\tilde{\theta}$ is a simple function of the sample mean, its mean and variance are easy to calculate. We have

$$E(\tilde{\theta}) = 2E\bar{X} = 2EX = 2\frac{\theta}{2} = \theta \text{ , and } Var(\tilde{\theta}) = 4Var\bar{X} = 4\frac{\theta^2/12}{n} = \frac{\theta^2}{3n}.$$

The likelihood function is

$$L(\theta|x) = \sum_{i=1}^{n} \frac{1}{\theta} \mathbb{I}_{[0,\theta]}(x_i) = \frac{1}{\theta^n} \mathbb{I}_{[0,\theta]}(x_{(n)}) \mathbb{I}_{[0,\infty)}(x_{(1)}),$$

where $x_{(1)}$ and $x_{(n)}$ are the smallest and largest order statistics. For $\theta \geq x_{(n)}$, $L = \frac{1}{\theta^n}$, a decresing function. So for $\theta \geq x_{(n)}$, $L$ is maximized at $\hat{\theta} = x_{(n)}$. $L = 0$ for $\theta < x_{(n)}$. So the overall maximum, the MLE, is $\hat{\theta} = X_{(n)}$. The pdf of $\hat{\theta} = X_{(n)}$ is $nx^{n-1}/\theta^n$, $0 \leq x \leq \theta$. This can be used to calculate

$$E\hat{\theta} = \frac{n}{n+1}\theta \ , \ E\hat{\theta}^2 = \frac{n}{n+2}\theta^2 \text{ and } Var\hat{\theta} = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

11. (a) Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} f(x;\theta)$.

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\theta x f(x;\theta)dx = \int_0^{\frac{\theta}{2}} x\left(\frac{4}{\theta^2}x\right)dx + \int_{\frac{\theta}{2}}^\theta x\left[-\frac{4}{\theta^2}x + \frac{4}{\theta}\right]dx \\ &= \frac{4}{\theta^2}\int_0^{\frac{\theta}{2}} x^2 dx - \frac{4}{\theta^2}\int_{\frac{\theta}{2}}^\theta x^2 dx + \frac{4}{\theta}\int_{\frac{\theta}{2}}^\theta x dx \\ &= \frac{\theta}{2}.\end{aligned}$$

Therefore, $\mathbb{E}[X] = \bar{X} \Rightarrow \hat{\theta}_{\text{MM}} = 2\bar{X}$.

(b) By part (a), $\hat{\theta}_{\text{MM}} = 2\bar{X} = 0.74654$.

12. (a) $M_X(t) = \mathbb{E}(e^{tX}) = (1 - t\theta)^{-1}$, so
$M_W(t) = \mathbb{E}(e^{tW}) = \mathbb{E}(e^{2t/\theta\sum_{i=1}^n X_i}) = \mathbb{E}(e^{2t/\theta X_1})\cdots\mathbb{E}(e^{2t/\theta X_n}) = (1-2t)^{-n}$.

(b) $[\frac{2n\bar{x}}{\chi^2_{\alpha/2}(2n)}, \frac{2n\bar{x}}{\chi^2_{1-\alpha/2}(2n)}]$.

(c) $[39.671, 131.029]$.

13. (a) The density of $X$ is

$$f(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\beta^\alpha} & \text{if } 0 \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

The likelihood function is

$$L(\alpha, \beta; x_1, x_2, ..., x_n) = (\frac{\alpha}{\beta^\alpha})^n(\prod_{i=1}^n x_i)^{\alpha-1}\mathbf{1}\{X_{(n)} \leq \beta\}\mathbf{1}\{X_{(1)} \geq 0\}$$

because the likelihood function is decreasing with respect to $\beta$, $X_{(n)}$ is the MLE of $\beta$.

$$\begin{aligned}\left.\frac{\partial \ln L}{\partial \alpha}\right|_{\beta = x_{(n)}} &= \frac{n}{\alpha} - n\ln x_{(n)} + \sum_{i=1}^n \ln x_i = 0 \\ \Longrightarrow \hat{\alpha} &= \frac{n}{n\ln x_{(n)} - \sum_{i=1}^n \ln x_i}\end{aligned}$$

Since

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} < 0$$

$\hat{\alpha}$ is the MLE of $\alpha$.

(b) $x_{(n)} = 26.0$ , $\sum_{i=1}^n \ln x_i = 44.03$, from (a), $\hat{\alpha}_{\text{MLE}} = 8.84$ , $\hat{\beta}_{\text{MLE}} = 26.0$.

(c)

$$0.05 = P_\beta(X_{(n)}/\beta \le c) = P_\beta(\text{all } X_i \le c\beta) = \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n} = c^{\alpha_0 n}$$

which implies that $c = 0.05^{\frac{1}{\alpha_0 n}}$. Thus

$$0.95 = P_\beta(X_{(n)}/\beta > c) = P_\beta(X_{(n)}/c > \beta)$$

So $\{\beta : \beta < X_{(n)}/(0.05^{1/\alpha_0 n})\}$ is a 95% upper confidence limit for $\beta$.

(d) From (b), $\hat{\alpha}_{\text{MLE}} = 8.84$ and $X_{(n)} = 26.0$, so the confidence interval is $[26, 26/[0.05^{1/(8.84\times 14)}]) = [26, 26.63)$.

14. (a) Note that the pdf of $Y$ is $f_Y(y) = \frac{1}{\lambda}e^{-\frac{1}{\lambda}y}, y \ge 0, \lambda > 0$ and $Y = \dfrac{X-\theta_1}{\theta_2}$. Therefore $\dfrac{\partial y}{\partial x} = \dfrac{1}{\theta_2}$ and the pdf of $X$, $f_X(x) = \dfrac{1}{\lambda\theta_2}\exp\left\{-\dfrac{x-\theta_1}{\lambda\theta_2}\right\}, x \ge \theta_1$

(b) Note that $E[Y] = \lambda, Var[Y] = \lambda^2$. Therefore, $E[X] = E[\theta_1 + \theta_2 Y] = \theta_1 + \theta_2 E[Y] = \theta_1 + \lambda\theta_2, Var[X] = Var[\theta_1 + \theta_2 Y] = \theta_2^2 Var[Y] = \lambda^2\theta_2^2$. Denote $\bar{X} \triangleq \dfrac{1}{n}\sum_{i=1}^n X_i, V \triangleq \dfrac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ and equate them with the theoretical moments:

$$\begin{cases} \tilde{\theta}_1 + \lambda\tilde{\theta}_2 = \bar{X} \\ \lambda^2\tilde{\theta}_2^2 = V \end{cases} \Rightarrow \begin{cases} \tilde{\theta}_1 = \bar{X} - \sqrt{V} \\ \tilde{\theta}_2 = \sqrt{V}/\lambda \end{cases}$$

i.e. The method-of-moments estimators for $\theta_1, \theta_2$ are $\tilde{\theta}_1 = \bar{X} - \sqrt{V}, \tilde{\theta}_2 = \sqrt{V}/\lambda$ respectively.

(c) As usual we can rewrite the likelihood function as

$$L(\theta_1, \theta_2; x_1, x_2, ..., x_n) = \frac{1}{\lambda^n\theta_2^n}\exp\left\{-\frac{1}{\lambda\theta_2}\left(\sum_{i=1}^n x_i - n\theta_1\right)\right\}\mathbf{1}\{\theta_1 \le x_{(1)}\}$$

Therefore, when $\theta_1 > x_{(1)}$, the indicator $\mathbf{1}\{\theta_1 \le x_{(1)}\} = 0 \Rightarrow L(\theta_1, \theta_2) = 0$; when $\theta_1 \le x_{(1)}$, $\dfrac{\partial L}{\partial \theta_1} = \dfrac{n}{\lambda^{n+1}\theta_2^{n+1}}\exp\left\{-\dfrac{1}{\lambda\theta_2}\left(\sum_{i=1}^n x_i - n\theta_1\right)\right\} > 0$ for any $\theta_2 > 0$, i.e. it is strictly increasing in $\theta_1$. Hence for any fixed $\theta_2 > 0$, $L$ is maximized when $\theta_1 = x_{(1)}$ and thus the MLE of $\theta_1$, $\hat{\theta}_1 = X_{(1)}$. On the other hand, note that $\ln L(\theta_1, \theta_2) = -n\ln\lambda - n\ln\theta_2 - \dfrac{1}{\lambda\theta_2}\left(\sum_{i=1}^n x_i - n\theta_1\right)$ for $\theta_1 \le x_{(1)}$. Differentiate the log-likelihood with respect to $\theta_2$ and evaluate at $\theta_1 = x_{(1)}$, we have

$$\left.\frac{\partial \ln L}{\partial \theta_2}\right|_{\theta_1 = x_{(1)}} = -\frac{n}{\theta_2} + \frac{1}{\lambda\theta_2^2}\left(\sum_{i=1}^n x_i - nx_{(1)}\right) = \frac{n}{\lambda\theta_2^2}\left(\frac{1}{n}\sum_{i=1}^n x_i - x_{(1)} - \lambda\theta_2\right)$$

$$\begin{cases} > 0 & \text{if } \; 0 < \theta_2 < (\bar{x} - x_{(1)})/\lambda \\ = 0 & \text{if } \; \theta_2 = (\bar{x} - x_{(1)})/\lambda \\ < 0 & \text{if } \; \theta_2 > (\bar{x} - x_{(1)})/\lambda \end{cases}$$

Therefore the MLE of $\theta_2$, $\hat{\theta}_2 = (\bar{X} - X_{(1)})/\lambda$.

15. Note that $\left[\bar{x} - z_{0.05}\frac{s}{\sqrt{n}}, \bar{x} + z_{0.05}\frac{s}{\sqrt{n}}\right]$ is an approximate 90% confidence interval for $\mu$. Therefore, $[\bar{x} - \epsilon, \bar{x} + \epsilon]$ is an approximate 90% confidence for $\mu$ if and only if $\epsilon = z_{0.05}\frac{s}{\sqrt{n}}$. Since $z_{0.05} \approx 1.645$, $s = 58$, $\epsilon = 10$, we have the required sample size $n \approx 91.03$. As n is an integer, the minimal required sample size is 92.