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STAT 4003

Assignment 2

Question 1

i) $\bar{x} = E(X)$

$= \int_0^\infty x \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{x^\beta}{\theta}} dx$

$= \int_0^\infty (\theta y)^{\frac{1}{\beta}} e^{-y} dy \quad , \quad y = \frac{x^\beta}{\theta}$

$= \theta^{\frac{1}{\beta}} \Gamma(\frac{1}{\beta}+1)$

$\tilde{\theta} = \left[\frac{\bar{x}}{\Gamma(\frac{1}{\beta}+1)}\right]^\beta$

$L(\theta) = \frac{\beta^n}{\theta^n} \prod x^{\beta-1} e^{-\frac{\sum x^\beta}{\theta}}$

$\ell(\theta) = n\ln(\beta) + (\beta-1)\ln(x) - \frac{\sum x^\beta}{\theta} - n\ln(\theta)$

$\frac{\partial \ell}{\partial \theta}\Big|_{\hat{\theta}} = 0$

$0 = \frac{\sum x^\beta}{\hat{\theta}^2} - \frac{n}{\hat{\theta}}$

$\hat{\theta} = \frac{\sum x^\beta}{n}$

$\frac{\partial^2 \ell}{\partial \theta^2}\Big|_{\hat{\theta}} = \frac{-2\sum x^\beta}{\hat{\theta}^3} + \frac{n}{\hat{\theta}^2}$

$= \frac{-\sum x^\beta}{\hat{\theta}^3}$

< 0

ii) $E(\tilde{\theta}) = \frac{1}{\Gamma(\frac{1}{\beta}+1)^\beta} \cdot \frac{1}{n^\beta} \cdot E[(\sum x)^\beta]$

$\Rightarrow \tilde{\theta}$ is unbiase if $E[(\sum x)^\beta] = [n\Gamma(\frac{1}{\beta}+1)]^\beta \theta$

$E(\hat{\theta}) = \frac{\sum E(x^\beta)}{n}$

$= \frac{n\theta}{n}$

$= \theta$

$\Rightarrow \hat{\theta}$ is unbiased

$E(x^\beta) = \int_0^\infty x^\beta \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{x^\beta}{\theta}} dx$

$= \int_0^\infty \theta y \, e^{-y} dy$

$= \theta \Gamma(2)$

$= \theta$

iii) $I(\theta) = -E\left(\frac{-2x^\beta}{\theta^3} + \frac{1}{\theta^2}\right)$

$= \frac{1}{\theta^2}$

$CRLB = \frac{1}{n \cdot \frac{1}{\theta^2}}$

$= \frac{\theta^2}{n}$

iv) $Var(\hat{\theta}) = \frac{\sum Var(x^\beta)}{n^2}$

$= \frac{1}{n}\left[E(x^{2\beta}) - E(x^\beta)^2\right]$

$= \frac{\theta^2}{n}$

$= CRLB$

$\Rightarrow \hat{\theta}$ is UMVUE

$E(x^{2\beta}) = \int_0^\infty x^{2\beta} \frac{\beta}{\theta} x^{\beta-1} e^{-\frac{x^\beta}{\theta}} dx$

$= \int_0^\infty (\theta y)^2 e^{-y} dy$

$= \theta^2 \Gamma(3)$

$= 2\theta^2$

Question 2

i) $\bar{x} = \frac{\theta+1+\theta-1}{2}$

$\tilde{\theta} = \bar{x}$

ii) $E(\tilde{\theta}) = \frac{\sum E(X)}{n}$

$= \frac{n\theta}{n}$

$= \theta$

$\Rightarrow \tilde{\theta}$ is unbiased

iii) $\tilde{\theta} = \frac{36.91}{5}$

$= 7.382$

iv) $\frac{X_{(1)} + X_{(n)}}{2} = \frac{14.97}{2}$

$= 7.485$

Question 3

i) $L(\theta) = \theta^n \prod x^{\theta-1}$

$\ell(\theta) = n\ln(\theta) + (\theta-1)\sum\ln(x)$

$\frac{\partial \ell}{\partial \theta}\Big|_{\hat{\theta}} = 0$

$0 = \frac{n}{\hat{\theta}} + \sum\ln(x)$

$\hat{\theta} = \frac{-n}{\sum \ln(x)}$

$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2}$

< 0

let $Y := -\ln(X)$, $F_Y(y) = P(-\ln(X) \leq y)$

$= 1 - P(X \leq e^{-y})$

$= 1 - [x^\theta]_0^{e^{-y}}$

$= 1 - e^{-y\theta}$

$\Rightarrow Y \sim Exp(\frac{1}{\theta})$, $Z := \sum Y \sim \Gamma(n, \frac{1}{\theta})$

$E(\frac{1}{Z}) = \int_0^\infty \frac{1}{z} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-z\theta} dz$

$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-1)}{\theta^{n-1}}$

$= \frac{\theta}{n-1}$

$E(\frac{1}{Z^2}) = \int_0^\infty \frac{1}{z^2} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-z\theta} dz$

$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-2)}{\theta^{n-2}}$

$= \frac{\theta^2}{(n-1)(n-2)}$

$Var(\frac{n}{Z}) = n^2\left[\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2}\right]$

$= \frac{n^2\theta^2}{(n-1)^2(n-2)}$

con't i) $P(|\hat{\theta}-\theta| \geqslant \varepsilon) < \frac{n^2\theta^2}{(n-1)^2(n-2)\varepsilon^2} \to 0$ as $n \to \infty$

$\Rightarrow \hat{\theta}$ is consistent

ii) $\theta x^{\theta-1} = \frac{x^{\theta-1}(1-x)^{1-1}}{B(\theta,1)}$

$\Rightarrow X \sim B(\theta,1)$

$\bar{x} = \frac{\theta}{\theta+1}$

$\tilde{\theta} = \frac{\bar{x}}{1-\bar{x}}$

Question 4

i) $L(\theta) = \frac{\prod x^2 e^{-\frac{\sum x}{\theta}}}{2^n \theta^{3n}}$

$\ell(\theta) = 2\sum \ln(x) - \frac{\sum x}{\theta} - n\ln(2) - 3n\ln(\theta)$

$\frac{\partial \ell}{\partial \theta}\Big|_{\hat{\theta}} = 0$ $\qquad \frac{\partial^2 \ell}{\partial \theta^2}\Big|_{\hat{\theta}} = \frac{-2\sum x}{\theta^3} + \frac{3n}{\theta^2}$

$0 = \frac{\sum x}{\theta^2} - \frac{3n}{\theta}$ $\qquad = \frac{-27n}{\bar{x}^2}$

$\hat{\theta} = \frac{\bar{x}}{3}$ $\qquad < 0$

ii) by invariance property, $\tau(\hat{\theta}) = \frac{3}{\bar{x}}$ is MLE of $\tau(\theta)$

iii) $I(\theta) = -E\left(\frac{-2x}{\theta^3} + \frac{3}{\theta^2}\right)$

$= \frac{3}{\theta^2}$

$CRLB = \frac{\frac{1}{\theta^4}}{n \cdot \frac{3}{\theta^2}}$

$= \frac{1}{3n\theta^2}$

iv) $E(\hat{\theta}) = \frac{\sum E(X)}{n}$

$= \frac{3\theta n}{3n}$

$= \theta$

$\Rightarrow \hat{\theta}$ is unbiased

$X \sim \frac{1}{\theta} f\left(\frac{y}{\theta}\right)$

$\Rightarrow X = \theta Y$, $Y \sim Exp(1)$

$E(X) = \int_0^\infty \theta y \cdot \frac{y^2}{2} e^{-y} dy$

$= 3\theta$.

$E[\tau(\hat{\theta})] = 3n \cdot E\left(\frac{1}{\sum X}\right)$

$= 3n \cdot E\left(\frac{1}{\sum \theta Y}\right)$

$= \frac{3n}{\theta(n-1)}$

$\Rightarrow \tau(\hat{\theta})$ is biased

$\sum Y \sim \Gamma(n,1)$

$E\left(\frac{1}{\sum Y}\right) = \frac{1}{n-1}$

v) $E(X^2) = \int_0^\infty (\theta y)^2 \cdot \frac{y^2}{2} e^{-y} dy$

$= 12\theta^2$

$Var(X) = 12\theta^2 - (3\theta)^2$

$= 3\theta^2$

con't v) by CLT, $\sqrt{n}(\bar{x} - 3\theta) \to N(0, 3\theta^2)$

by delta method, we have $g(z) = \frac{3}{z}$ s.t $g'(\mu) = \frac{-1}{3\theta^2}$

$\Rightarrow \sqrt{n}(\tau(\hat{\theta}) - \tau(\theta)) \to N\left(0, 3\theta^2\left(\frac{-1}{3\theta^2}\right)^2\right)$

$\to N\left(0, \frac{1}{3\theta^2}\right)$

Question 5

$\sum \frac{\partial}{\partial \theta}\left[2\ln(x) - \frac{x}{\theta} - \ln(2) - 3\ln(\theta)\right]$

$= \sum\left(\frac{x}{\theta^2} - \frac{3}{\theta}\right)$

$= \frac{n}{3\theta^2}\left(\frac{\bar{x}}{3} - \theta\right)$

$\Rightarrow \frac{\bar{x}}{3}$ is UMVUE

Question 6

$\prod \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$= \sigma^{-n} e^{-\frac{\sum x^2 - 2\mu\sum x + n\mu^2}{2\sigma^2}} \cdot (2\pi)^{-\frac{n}{2}}$

$\Rightarrow \sum X_i$ and $\sum X_i^2$ are jointly sufficient

$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$= \sigma^{-1} e^{-\frac{x^2 - 2x\mu + \mu^2}{2\sigma^2}} \cdot (2\pi)^{-\frac{1}{2}}$

$\Rightarrow X$ belongs to exponential family

$\Rightarrow \sum X_i$ and $\sum X_i^2$ are minimal jointly sufficient statistics

$\sum X_i$ and $\sum (X_i - \bar{x})^2$ are 1 to 1 function of $\sum X_i$ and $\sum X_i^2$

$\Rightarrow \sum X_i$ and $\sum (X_i - \bar{x})^2$ are minimal jointly sufficient statistics