MATH1520 Autumn 2018
Homework 5 Solution

1. Compute

(a) $\displaystyle\int_{-1}^{1} \frac{5x}{(4+x^2)^2} dx$

(b) $\displaystyle\int_{0}^{1} x\sqrt{x+1} dx$

(c) $\displaystyle\int_{2}^{4} \frac{1}{x(\ln x)^2} dx$

(d) $\displaystyle\int (2x+6)^{14}\, dx$

(e) $\displaystyle\int \sqrt{4x-1}\, dx$

(f) $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\, dx$

(g) $\displaystyle\int \frac{x}{\sqrt[3]{4-3x}}\, dx$

(h) $\displaystyle\int_{10}^{30} ve^{-v/5}\, dv$

(i) $\displaystyle\int_{2}^{1} t\ln 2t\, dt$

(j) $\displaystyle\int_{-1}^{3} (t-1)e^{1-t}\, dt$

Answer.

(a) Let $u = x^2$

$$\begin{aligned}\int_{-1}^{1} \frac{5x}{(4+x^2)^2} dx &= \int_{1}^{1} \frac{5}{2}\frac{1}{(4+u)^2} du \\ &= -\frac{5}{2}\Big[\frac{1}{4+u}\Big]_1^1 = 0\end{aligned}$$

(b) Let $u = \sqrt{x+1}$, then $x = u^2 - 1$

$$\begin{aligned}\int_{0}^{1} x\sqrt{x+1} dx &= \int_{1}^{\sqrt{2}} 2(u^4 - u^2)\, du \\ &= 2\Big[\frac{u^5}{5} - \frac{u^3}{3}\Big]_1^{\sqrt{2}} = \frac{4(\sqrt{2}+1)}{15}\end{aligned}$$

(c) Let $u = \ln x$

$$\int_2^4 \frac{1}{x(\ln x)^2} dx = \int_2^4 (\ln x)' \frac{1}{(\ln x)^2}\, dx$$
$$= \int_{\ln 2}^{\ln 4} \frac{1}{u^2}\, du = \frac{1}{2\ln 2}$$

(d)

$$\int (2x+6)^{14}\, dx = \frac{(2x+6)^{15}}{30} + C$$

(e) Let $u = \sqrt{4x-1}$,then $x = \frac{u^2+1}{4}$

$$\int \sqrt{4x-1}\, dx = \int \frac{u^2}{2}\, du = \frac{u^3}{6} = \frac{(4x-1)^{3/2}}{6}$$

or, let $u = 4x - 1$, then

$$\int \sqrt{4x-1}\, dx = \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4}\Big(\frac{u^{3/2}}{3/2}\Big) + C = \frac{1}{6}(4x-1)^{3/2} + C$$

(f) Let $u = \sqrt{x}$, then

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\, dx = \int \frac{2}{1+u}\, du = 2\ln(1+\sqrt{x}) + C$$

(g) Let $u = \sqrt[3]{4-3x}$, then $x = \frac{4-u^3}{3}$

$$\int \frac{x}{\sqrt[3]{4-3x}}\, dx = \int \frac{\frac{u^4-4u}{3}}{u}(-u^2)\, du = \int \frac{u^4-4u}{3}\, du$$
$$= \frac{u^5}{15} - \frac{2u^2}{3} + C = \frac{(4-3x)^{\frac{5}{3}}}{15} - \frac{2(4-3x)^{\frac{2}{3}}}{3} + C$$

Or, let $u = 4 - 3x$, then $x = \frac{4-u}{3}$,

$$\int \frac{x}{\sqrt[3]{4-3x}}\, dx = \int \frac{\frac{4-u}{3}}{\sqrt[3]{u}}(-\frac{1}{3})\, du = -\frac{1}{9}\int 4u^{-1/3} - u^{2/3}\, du$$
$$= -\frac{1}{9}(4\cdot\frac{u^{2/3}}{2/3} - \frac{u^{5/3}}{5/3}) + C = -\frac{2(4-3x)^{2/3}}{3} + \frac{(4-3x)^{5/3}}{15} + C$$

(h) Using integration by parts:

$$\int_{10}^{30} ve^{-v/5}\, dv = \int_{10}^{30} -5v\, de^{-\frac{v}{5}}$$
$$= \big[ve^{-\frac{v}{5}}\big]_{10}^{30} - \int_{10}^{30} -5e^{-v/5}\, dv = 75e^{-2} - 175e^{-6}$$

(i) $\int_2^1 t\ln 2t\,dt = \int_2^1 \ln 2t\, d(\frac{t^2}{2}) = [\frac{t^2}{2}\ln 2t]_2^1 - \int_2^1 \frac{t}{2}\,dt = -\frac{7}{2}\ln 2 + \frac{3}{4}$

(j) Let $u = t-1$

$$\begin{aligned}\int_{-1}^{3}(t-1)e^{1-t}\,dt &= \int_{-2}^{2} ue^{-u}\,du \\ &= \left[ue^{-u}\right]_{-2}^{2} + \int_{-2}^{2} e^{-u}\,du = -3e^{-2} - e^{2}\end{aligned}$$

2. Compute the following integral by partial fraction decomposition.

(a) $\int \frac{x^3 - x + 1}{x^2 - 1}dx$

(b) $\int \frac{x^4}{(x-1)(x-2)}dx.$

(c) $\int \frac{(x+2)}{x^3 - x}dx.$

(d) $\int_3^9 \frac{4-3x}{(x-1)^2}\,dx$

Answer.

(a)

$$\begin{array}{r} x \\ x^2-1\overline{\smash{\big)}\; x^3 - x + 1} \\ \underline{-x^3 + x} \\ 1 \end{array}$$

$$\frac{x^3 - x + 1}{x^2 - 1} = x + \frac{1}{x^2 - 1}$$

$$\frac{1}{x^2 - 1} = \frac{A}{x+1} + \frac{B}{x-1}$$

Hence,

$$1 = A(x-1) + B(x+1)$$

Substitute $x = 1$,

$$B = \frac{1}{2}$$

Substitute $x = -1$,

$$A = \frac{-1}{2}$$

Hence,

$$\begin{aligned}\int \frac{x^3 - x + 1}{x^2 - 1}dx &= \int x + \frac{1}{x^2 - 1}dx = \frac{x^2}{2} + \int \frac{1}{2}(\frac{1}{x-1} - \frac{1}{x+1})dx \\ &= \frac{x^2}{2} + \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C\end{aligned}$$

(b)

$$
\begin{array}{r}
x^2 \quad + 3x \quad + 7 \\
x^2 - 3x + 2 \overline{\big) \quad x^4 \qquad\qquad\qquad\qquad\qquad} \\
\underline{- x^4 + 3x^3 - 2x^2} \qquad\qquad\qquad \\
3x^3 - 2x^2 \qquad\qquad\qquad \\
\underline{- 3x^3 + 9x^2 \quad - 6x} \qquad\quad \\
7x^2 \quad - 6x \qquad\quad \\
\underline{- 7x^2 + 21x - 14} \\
15x - 14
\end{array}
$$

$$\frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{15x - 14}{(x-1)(x-2)}$$

$$\frac{15x - 14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Hence

$$15x - 14 = A(x-2) + B(x-1).$$

Substitute $x = 1$,

$$A = -1.$$

Substitute $x = 2$,

$$B = 16.$$

Hence

$$
\begin{aligned}
\int \frac{x^4}{(x-1)(x-2)} &= \int x^2 + 3x + 7 + \frac{15x - 14}{(x-1)(x-2)} \\
&= \int x^2 + 3x + 7 + \frac{-1}{(x-1)} + \frac{16}{(x-2)} dx \\
&= \frac{x^3}{3} + \frac{3}{2}x^2 + 7x - \ln|x-1| + 16\ln|x-2| + C
\end{aligned}
$$

(c) $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$.

$$\frac{x+2}{x^3 - x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

$$x + 2 = A(x-1)(x+1) + Bx(x-1) + Cx(x+1).$$

Substitute $x = 0$

$$2 = -A$$

$$A = -2.$$

Substitute $x = 1$

$$3 = 2C$$

$$C = 3/2.$$

Substitute $x = -1$

$$2B = 1$$
$$B = 1/2.$$

Alternate method of finding A, B and C

$$\begin{aligned}
\frac{x+2}{x^3 - x} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\
&= \frac{A(x^2-1) + B(x^2 - x) + C(x^2 + x)}{x^3 - x} \\
&= \frac{(A+B+C)x^2 + (C-B)x - a}{x^3 - x}
\end{aligned}$$

Then we have the following equations:

$$\begin{aligned}
A + B + C &= 0, \\
C - B &= 1, \\
-A &= 2.
\end{aligned}$$

Thus we have $A = -2, B = 1/2, C = 3/2$.

$$\begin{aligned}
\int \frac{(x+2)\,dx}{x^3 - x} &= \int \left(-\frac{2}{x} + \frac{1}{2(x+1)} + \frac{3}{2(x-1)} \right) dx \\
&= -2 \ln|x| + \frac{1}{2} \ln|x+1| + \frac{3}{2} \ln|x-1| + C
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{4 - 3x}{(x-1)^2} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} \\
4 - 3x &= A(x-1) + B
\end{aligned}$$

Substitute $x = 1$,

$$B = 1$$

Substitute $x = 0$,

$$4 = -A + B = -A + 1 \quad \Rightarrow \quad A = -3$$

Hence,

$$\begin{aligned}
\int_3^9 \frac{4 - 3x}{(x-1)^2}\,dx &= \int_3^9 \frac{-3}{x-1} + \frac{1}{(x-1)^2}\,dx \\
&= \big[-3\ln(x-1) - \frac{1}{x-1} \big]_3^9 = -6 \ln 2 + \frac{3}{8}
\end{aligned}$$

3. Compute the following integrals.

(a) $\int e^x\sqrt{e^x-1}dx$;

(b) $\int \frac{x^3+2x+1}{x+1}dx$

(c) $\int (x^3-x)e^x dx$.

(d) $\int_2^4 \frac{e^{2x}}{1+e^x}\,dx$

(e) $\int_1^{10} (\ln x)^3\,dx$

Answer.

(a) $u=e^x-1, du=e^x dx$.

$$\begin{aligned}\int e^x\sqrt{e^x-1}dx &= \int \sqrt{u}du\\ &= \frac{2}{3}u^{\frac{3}{2}}+C\\ &= \frac{2}{3}(e^x-1)^{\frac{3}{2}}+C\end{aligned}$$

(b) Using long division, one has

$$\frac{x^3+2x+1}{x+1}=x^2-x+3-\frac{2}{x+1}.$$

Hence

$$\begin{aligned}\int \frac{x^3+2x+1}{x+1}dx &= \int (x^2-x+3-\frac{2}{x+1})dx\\ &= \frac{x^3}{3}-\frac{x^2}{2}+3x-2\ln|x+1|+C\end{aligned}$$

(c)

$$\begin{aligned}\int (x^3-x)e^x dx &= \int (x^3-x)d(e^x)\\ &= (x^3-x)e^x-\int (3x^2-1)e^x dx\\ &= (x^3-x)e^x-(3x^2-1)e^x+\int 6xe^x dx\\ &= (x^3-x)e^x-(3x^2-1)e^x+6xe^x-\int 6e^x dx\\ &= (x^3-3x^2+5x-5)e^x+C\end{aligned}$$

(d) Let $e^x + 1 = t$, then $x = \ln(t-1)$

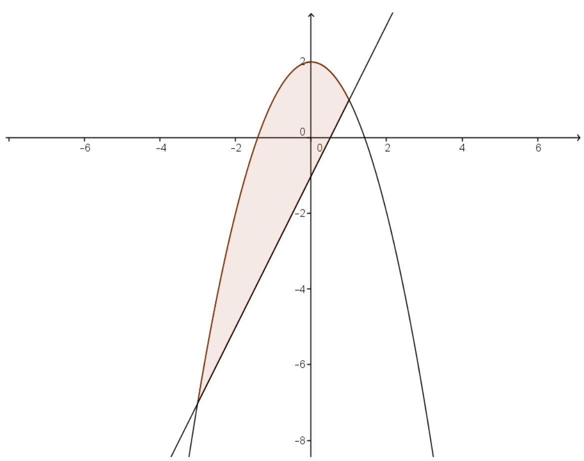
$$\int_2^4 \frac{e^{2x}}{1+e^x}\,dx = \int_{e^2+1}^{e^4+1} \frac{(t-1)}{t}\,dt$$
$$= \big[t - \ln t\big]_{e^2+1}^{e^4+1} = e^4 - e^2 - \ln\frac{e^4+1}{e^2+1}$$

(e) Let$t = \ln x$, then $x = e^t$

$$\int_1^{10} (\ln x)^3\,dx = \int_0^{\ln 10} t^3 e^t\,dt = \int_0^{\ln 10} t^3 de^t$$
$$= \big[t^3 e^t\big]_0^{\ln 10} - \int_0^{\ln 10} 3t^2 e^t dt$$
$$= 10(\ln 10)^3 - \big[3t^2 e^t\big]_0^{\ln 10} + \int_0^{\ln 10} 6te^t\,dt$$
$$= 10(\ln 10)^3 - 30(\ln 10)^2 + 60\ln 10 - 54$$

4. Find the area between the region enclosed by $y = 2 - x^2$ and $y = 2x - 1$.

Answer.



Solve $2 - x^2 = 2x - 1$, $x = 1$ or $x = -3$. So we just need to compute the following.

$$\int_{-3}^{1} (2 - x^2 - 2x + 1)\,dx = \int_{-3}^{1} 3 - 2x - x^2\,dx = \Big[3x - x^2 - \frac{x^3}{3}\Big]_{-3}^{1} = \frac{83}{3}$$

5. Evaluate the given limit using appropriate definite integral.

$$\lim_{n\to\infty} n\left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \cdots + \frac{1}{(2n+n)^2}\right]$$

Answer.

$$\begin{aligned}
&\lim_{n\to\infty} n\left[\frac{1}{(2n+1)^2}+\frac{1}{(2n+2)^2}+\cdots+\frac{1}{(2n+n)^2}\right]\\
&=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{(2+1/n)^2}+\frac{1}{(2+2/n)^2}+\cdots+\frac{1}{(2+n/n)^2}\right] \qquad (1)\\
&=\int_2^3 \frac{1}{x^2}\,dx=\left[-\frac{1}{x}\right]_2^3=\frac{1}{6}
\end{aligned}$$

(1) is the right Riemann sum.

6. Evaluate the following improper integrals.

   (a) $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$

   (b) $\displaystyle\int_0^{+\infty} 2xe^{-3x}\,dx$

   (c) $\displaystyle\int_{-\infty}^{0} \frac{1}{(2x-1)^2}\,dx$

   (d) $\displaystyle\int_0^{+\infty} xe^{1-x}\,dx$

   (e) $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

   Answer.

   (a) First, we consider $\displaystyle\int_0^{b} xe^{-x^2}\,dx$. Using $u=x^2$ and $du=2x\,dx$, we have the following.

   $$\begin{aligned}
   \int_0^b xe^{-x^2}\,dx &= \int_0^{b^2}\frac{1}{2}e^{-u}\,du\\
   &=\left[-\frac{1}{2}e^{-u}\right]_0^{b^2}=\frac{1}{2}-\frac{1}{2}e^{-b^2}
   \end{aligned}$$

   Hence, we have the following.

   $$\int_0^{+\infty} xe^{-x^2}\,dx=\lim_{b\to+\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-b^2}\right)=\frac{1}{2}$$

(b) First, we consider $\int_0^b 2xe^{-3x}\,dx$. Using integration by parts, we have the following.

$$\begin{aligned}\int_0^b 2xe^{-3x}\,dx &= \int_0^b -\frac{2x}{3}\,de^{-3x}\\ &= \left[-\frac{2x}{3}e^{-3x}\right]_0^b + \int_0^b \frac{2}{3}e^{-3x}\,dx\\ &= -\frac{2b}{3}e^{-3b} + \left[-\frac{2}{9}e^{-3x}\right]_0^b\\ &= -\frac{2b}{3}e^{-3b} - \frac{2}{9}e^{-3b} + \frac{2}{9}\end{aligned}$$

Hence, we have the following.

$$\int_0^{+\infty} 2xe^{-3x}\,dx = \lim_{b\to+\infty}\left(-\frac{2b}{3}e^{-3b} - \frac{2}{9}e^{-3b} + \frac{2}{9}\right) = \frac{2}{9}$$

(c) First, we consider $\int_b^0 \frac{1}{(2x-1)^2}\,dx$. Using $u = 2x-1$ and $du = 2\,dx$, we have the following.

$$\begin{aligned}\int_b^0 \frac{1}{(2x-1)^2}\,dx &= \int_{2b-1}^{-1} \frac{1}{2u^2}\,du\\ &= \left[-\frac{1}{2u}\right]_{2b-1}^{-1} = \frac{1}{2} + \frac{1}{2(2b-1)}\end{aligned}$$

Hence, we have the following.

$$\int_{-\infty}^0 \frac{1}{(2x-1)^2}\,dx = \lim_{b\to-\infty}\left(\frac{1}{2} + \frac{1}{2(2b-1)}\right) = \frac{1}{2}$$

(d) First we consider $\int_0^b xe^{1-x}\,dx$

$$\begin{aligned}\int_0^b xe^{1-x}\,dx &= \int_0^b -x\,de^{1-x} = \left[-xe^{1-x}\right]_0^b + \int_0^b e^{1-x}dx\\ &= -be^{1-b} - e^{1-b} + e\end{aligned}$$

Hence,we have the following. $\int_0^{+\infty} xe^{1-x}\,dx = \lim_{b\to+\infty} -be^{1-b} - e^{1-b} + e = e$

(e) First, we consider $\int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$. Using $u = \ln x$ and $du = \frac{1}{x}dx$

$$\int_2^b \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\ln b} \frac{1}{\sqrt{u}}du = \left[2u^{\frac{1}{2}}\right]_{\ln b}^{\ln 2} = 2(\sqrt{\ln 2} - \sqrt{\ln b})$$

Hence,

$$\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to+\infty} 2(\sqrt{\ln 2} - \sqrt{\ln b}) = -\infty$$

7. Suppose that if f is continuous, find the value of the integral $I = \displaystyle\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ by making the substitution $u = a - x$ and adding the resulting integral to I.

Answer.

$$I = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$$

$$2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx = \int_0^a 1\,dx = a$$

$$\therefore I = \frac{a}{2}$$

8. Compute

$$\int \frac{1}{x^2-a^2}\,dx,\ \ a \in \mathbb{R}.$$

Solution:

(i) If $a \neq 0$,

$$\frac{1}{x^2-a^2} = \frac{A}{x+a} + \frac{B}{x-a} \Rightarrow \begin{cases} A+B=0 \\ -aA+aB=1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{2a} \\ B = \frac{1}{2a} \end{cases}$$

$$\int \frac{1}{x^2-a^2}dx = -\frac{1}{2a}\int \frac{1}{x+a}\,dx + \frac{1}{2a}\int \frac{1}{x-a}\,dx = -\frac{1}{2a}\ln|x+a| + \frac{1}{2a}\ln|x-a| + C$$

(ii) If $a = 0$,

$$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{x^2}dx = -\frac{1}{x} + C.$$

In conclusion,

$$\int \frac{1}{x^2-a^2}dx = \begin{cases} \ln|x+a| + \frac{1}{2a}\ln|x-a| + C & ,\text{if } a \neq 0 \\ -\frac{1}{x} + C & ,\text{if } a = 0 \end{cases}$$