STAT 2006 Assignment 1
Due Time and Date: 5 p.m., 31th Jan., 2020

1. Let X and Y are independent continuous real-valued random variables with pdf f_X, f_Y respectively.

   (a) Show that $\mathbb{E}[X] = \int_0^\infty (1 - F_X(x))dx - \int_{-\infty}^0 F_X(x)dx$.

   (b) Let $Z = X + Y$. Show that $f_Z(z) = \int_{-\infty}^\infty f_Y(z - x)f_X(x)dx$. [Remark: This is known as the convolution formula.]

2. Let X and Y are independent Poisson random variables with parameter λ and μ respectively.

   (a) Show that $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$.

   (b) What is the conditional probability $\mathbb{P}(X = x|X + Y = n)$. Show your steps.

   (c) Hence, are X and $X + Y$ independent?

3. (a) Let X be a continuous random variable with the pdf f_X, and let $Y = X^2$. By considering the CDF of Y, express the $f_Y(y)$ in terms of f_X.

   (b) In general when the transformation $Y = g(X)$ is not one-to-one in the entire support of X, we cannot directly apply the Jacobian transformation. But suppose we can partition the support of X into two (or more) sets $A_1, A_2, ..., A_k$ such that the transformation is one-to-one within each set, (like the set $\{X > 0\}$ and $\{X < 0\}$ in part a) i.e. there exist some functions $g_1, g_2, ..., g_k$ such that $Y = g_i(X)$ when $X \in A_i, i = 1, 2, ..., k$ and g_i is one-to-one, then we can extend the result as $f_Y(y) = \sum_{i=1}^{k} f_X(g_i^{-1}(y))|J_i|$ where J_i is the corresponding Jacobian of the transformation g_i.
   Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$.

      i. Find the joint pdf of $Y_1 := X_1^2 + X_2^2, Y_2 := \dfrac{X_1}{\sqrt{X_1^2 + X_2^2}}$.

      ii. Are Y_1 and Y_2 independent?

4. (a) For the hierarchical model

   $$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{Gamma}(\alpha, \beta),$$

   find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

   (b) Show that the three-stage model

   $$Y|N \sim \text{Binomial}(N, p), \quad N|\Lambda \sim \text{Poisson}(\Lambda), \quad \text{and } \Lambda \sim \text{Gamma}(\alpha, \beta)$$

   leads to the same marginal distribution of Y.

5. Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is Uniform$(0, 1)$.

   (a) Find $\mathbb{E}[Y], Var(Y)$ and $Cov(X, Y)$.

   (b) Prove that Y/X and X are independent.