STAT 4005 Time Series
Assignment 4
Due date: 21 Apr 2021; 5pm

1. Given a data set $Y_1, \ldots, Y_{20}$.

$$\begin{aligned} Y = (&1.33, -0.56, -1.31, -0.37, 0.05, 0.46, 2.00, -0.19, -0.25, 1.07, \\ &-0.17, 1.14, 0.63, -0.75, 0.15, 0.71, 0.45, -0.14, 0.57, 1.43). \end{aligned}$$

   (a) Fit an MA(2) model to $\{Y_t\}$, find the k-step ahead forecast and the 95% prediction intervals for $k = 1, 2, 3, \ldots$.

   (b) With the MA(2) model fitted in (a), find the partial autocorrelations ϕ_{11}, ϕ_{22} and ϕ_{33} using the first principle.

   (c) Fit an AR(1) model to $\{Y_t\}$, find the k-step ahead forecast and the 95% prediction intervals for $k = 1, 2, 3, \ldots$.

   (d) With the AR(1) model fitted in (c), find $\mathrm{Cov}(e_{20}(k), e_{20}(l))$, where $k \neq l$ are positive integers.

   (e) Fit an ARMA(1,1) model to $\{Y_t\}$, find the 1st and 2nd-step ahead forecast and the 95% prediction intervals.

   (f) Fit an ARIMA(1,1,0) model to $\{Y_t\}$, find the 1st and 2nd-step ahead forecast and the 95% prediction intervals.

   Note: You could use the R function `arima()` for model fitting.

2. Consider the GARCH(1,1) model

$$\begin{aligned} X_t &= \sigma_t \epsilon_t, \quad \epsilon_t \overset{\text{iid}}{\sim} N(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{aligned}$$

   where $\alpha_0, \alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$.

   (a) Express X_{t+1} and X_{t+2} in terms of X_t, σ_t, ϵ_{t+1} and ϵ_{t+2}.

   (b) Given observed values of σ_1, X_1, X_2 and X_3, express the likelihood function $L(\alpha_0, \alpha_1, \beta_1)$ in terms of σ_1, X_1, X_2 and X_3.

3. Prove the following result.

   Theorem 1 *If X_t is a GARCH(p,q) process, then X_t^2 is an ARMA(m,p) process with noise $\nu_t = \sigma_t^2(\epsilon_t^2 - 1)$, where $m = \max(p,q)$.*