HW 1 Solution

1. 1) $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^n (1-p)^{n-x} = (1-p+pe^t)^n$

2) $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = e^{\lambda(e^t-1)}$

3) $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$

$= \left(\frac{\beta}{\beta-t}\right)^\alpha$ (Hint: $\int_0^\infty x^{\alpha-1}e^{-\beta x}dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$)

2. 1) $X \sim Bin(100, \frac{18}{38})$. $np = 100\times\frac{18}{38} > 5$

$\Rightarrow$ $X \rightarrow X_c \sim N(np, npq)$ $np = 100\times\frac{18}{38} = \frac{900}{19}$.

$npq = 100\times\frac{18}{38}\times(1-\frac{18}{38}) = \frac{9000}{361}$

$P(X>50) = 1-P(X\le 50)$

$= 1-P(X_c \le 50+0.5) = 1-P\left(Z \le \frac{50+0.5-\frac{900}{19}}{\sqrt{\frac{9000}{361}}}\right)$

$= 1-P(Z\le 0.63) = 1-0.7357 = 0.2643$

2) $EX = \int_0^1 x\cdot 3x^2dx = \frac{3}{4}$ $EX^2 = \int_0^1 x^2\cdot 3x^2 dx = \frac{3}{5}$

Since $\mu = \frac{3}{4} < \infty$. $\sigma^2 = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{80} \in (0,\infty)$

$\Rightarrow$ apply CLT $\bar{X} \longrightarrow N(\frac{3}{4}, \frac{1}{16}\times\frac{3}{80})$

$P(\bar{X}<0.5) = P\left(Z < \frac{0.5-0.75}{\sqrt{0.0375/16}}\right) = P(Z<-5.16) \approx 0$

3. 1) X_1, X_2 is r.s. from $N(\mu,1)$. Hence they're independent

$\Rightarrow X_1 - X_2 \sim N(0,2)$

$\Rightarrow P(X_1-X_2<1) = P(Z<\frac{1-0}{\sqrt{2}}) = P(Z<0.71) = 0.7611$

2) Since X_1, X_2 are independent.

$\begin{pmatrix}X_1\\X_2\end{pmatrix} \sim N(\underset{\sim}{\mu}, \Sigma)$. where $\underset{\sim}{\mu} = \begin{pmatrix}\mu\\\mu\end{pmatrix}$ $\Sigma = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

Let $A = \begin{pmatrix}1&-1\\1&1\end{pmatrix}$ then $AX \sim N(A\underset{\sim}{\mu}, A\Sigma A^T)$

Therefore $AX = \begin{pmatrix}X_1-X_2\\X_1+X_2\end{pmatrix} \sim N\left(\begin{pmatrix}0\\2\mu\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}\right)$.

Since $\begin{pmatrix}X_1-X_2\\X_1+X_2\end{pmatrix}$ follows multivariate normal, and $Cov(X_1-X_2, X_1+X_2)=0$

Then X_1-X_2, X_1+X_2 are independent.

4. X and Y are independent, and both are from $N(0,1)$

Method 1: $\forall t \in \mathbb{R}$

$$P(\tfrac{X}{Y} \le t) = P(\tfrac{X}{Y} \le t, Y>0) + P(\tfrac{X}{Y} \le t, Y<0) + P(\tfrac{X}{Y} \le t, Y=0)$$

Note that $0 \le P(\tfrac{X}{Y} \le t, Y=0) \le P(Y=0) = 0 \Rightarrow P(\tfrac{X}{Y} \le t, Y=0) = 0$

$$\Rightarrow P(\tfrac{X}{Y} \le t) = P(\tfrac{X}{Y} \le t, Y>0) + P(\tfrac{X}{Y} \le t, Y<0)$$

$$= \int_0^\infty P(\tfrac{X}{Y} \le t \mid Y=y) f_Y(y)\,dy + \int_{-\infty}^0 P(\tfrac{X}{Y} \le t \mid Y=y) f_Y(y)\,dy$$

$$= \int_0^\infty P(X \le ty) f_Y(y)\,dy + \int_{-\infty}^0 P(X \ge ty) f_Y(y)\,dy$$

$$\frac{\partial}{\partial t} P(\tfrac{X}{Y} \le t) = f_{\frac{X}{Y}}(t) = \int_0^\infty y f_X(ty) f_Y(y)\,dy + \int_{-\infty}^0 -y f_X(ty) f_Y(y)\,dy$$

$$= \int_0^\infty y f_X(ty) f_Y(y)\,dy + \int_0^\infty y f_X(ty) f_Y(y)\,dy \quad \begin{pmatrix} f_X(-ty) = f_X(ty) \\ f_Y(-y) = f_Y(y) \end{pmatrix}$$

$$= 2\int_0^\infty y f_X(ty) f_Y(y)\,dy = 2\int_0^\infty y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy$$

$$= \frac{1}{\pi}\int_0^\infty y e^{-\frac{1}{2}(t^2+1)y^2}\,dy = \frac{1}{\pi(1+t^2)}, \quad \forall t \in \mathbb{R}$$

Which is the density function of t-distribution with 1 degree of freedom

Method 2: We already know that $P(\tfrac{X}{Y} \le t, Y=0) = 0$

$$P(\tfrac{X}{Y} \le t) = P(\tfrac{X}{Y} \le t, Y>0) + P(\tfrac{X}{Y} \le t, Y<0)$$

$$= P(\tfrac{X}{|Y|} \le t, Y>0) + P(\tfrac{X}{-|Y|} \le t, Y<0)$$

$$= P(\tfrac{X}{|Y|} \le t, Y>0) + P(\tfrac{-X}{|Y|} \le t, Y<0)$$

Since X has a standard normal distribution, $-X$ also follow $N(0,1)$ by the symmetry of the standard normal distribution.

$$\Rightarrow P(\tfrac{X}{Y} \le t) = P(\tfrac{X}{|Y|} \le t, Y>0) + P(\tfrac{X}{|Y|} \le t, Y<0)$$

$$= P(\tfrac{X}{|Y|} \le t) \quad \forall t \in \mathbb{R}$$

Since $Y \sim N(0,1) \Rightarrow Y^2 \sim \chi^2(1)$, X and Y are independent.

$$\frac{X}{|Y|} = \frac{X}{\sqrt{Y^2}} \sim t(1)$$

Since $P(\tfrac{X}{Y} \le t) = P(\tfrac{X}{|Y|} \le t)$, $\tfrac{X}{Y}$ and $\tfrac{X}{|Y|}$ have the same distribution.

$$\frac{X}{Y} \sim t(1)$$

5. The 2 random variable share the same variance σ^2., $X_i \sim N(\mu_i, \sigma^2)$

$$S_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_{1i}-\bar{X}_1)^2 \quad S_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(X_{2i}-\bar{X}_2)^2$$

then $\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2(n_1-1)$, $\frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2(n_2-1)$

and they're independent.

$$\Rightarrow F = \frac{\frac{(n_1-1)S_1^2}{\sigma^2}/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma^2}/(n_2-1)} = \frac{S_1^2}{S_2^2} \sim F(n_1-1, n_2-1)$$

6. Since $X_i \sim U(0,1)$, then $f_X(t) = \begin{cases} 1, & t\in(0,1) \\ 0, & \text{otherwise} \end{cases}$ $F_X(t) = \begin{cases} 1, & t \geq 1 \\ t, & t\in(0,1) \\ 0, & t \leq 0 \end{cases}$

Thus pdf of $X_{(i)}$ at $t\in(0,1)$

$$f_{X_{(i)}}(t) = \frac{n!}{(i-1)!\,1!\,(n-i)!}\left[P(X_1\leq t)\right]^{i-1} f_{X_1}(t)\left[P(X_1>t)\right]^{n-i}$$

$$= \frac{n!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}, \quad i=1,\cdots,n, \quad t\in(0,1)$$

Since $\int_0^1 \frac{n!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}dt = 1 \Rightarrow \int_0^1 t^{i-1}(1-t)^{n-i}dt = \frac{(i-1)!(n-i)!}{n!}$

$$EX_{(i)} = \int_0^1 t f_{X_{(i)}}(t)dt = \int_0^1 t\frac{n!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}dt$$

$$= \frac{n!}{(i-1)!(n-i)!}\int_0^1 t^{(i+1)-1}(1-t)^{(n+1)-(i+1)}dt$$

$$= \frac{n!}{(i-1)!(n-i)!}\frac{((i+1)-1)!\,((n+1)-(i+1))!}{(n+1)!} = \frac{i}{n+1}$$

$$EX_{(i)}^2 = \int_0^1 t^2 f_{X_{(i)}}(t)dt = \frac{n!}{(i-1)!(n-i)!}\int_0^1 t^{(i+2)-1}(1-t)^{(n+2)-(i+2)}dt$$

$$= \frac{n!}{(i-1)!(n-i)!}\frac{((i+2)-1)!\,((n+2)-(i+2))!}{(n+2)!} = \frac{i(i+1)}{(n+1)(n+2)}$$

Therefore.

$$\text{Var}X_{(i)} = EX_{(i)}^2 - (EX_{(i)})^2$$

$$= \frac{i(i+1)}{(n+1)(n+2)} - \left(\frac{i}{n+1}\right)^2 = \frac{i(n+1-i)}{(n+1)^2(n+2)}$$

7. 1). Since $X_1, \cdots, X_n$ are independent, $X_i \sim N(\mu, 1)$

$\Rightarrow X_1, \cdots, X_k$ and $X_{k+1}, \cdots, X_n$ are independent

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k}(X_i - \mu) = \frac{1}{k}\sum_{i=1}^{k} X_i - \mu \sim N(0, \tfrac{1}{k})$$

$$\tilde{X}_k = \frac{1}{n-k}\sum_{i=k+1}^{n}(X_i - \mu) = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i - \mu \sim N(0, \tfrac{1}{n-k})$$

Since $\bar{X}_k$ and $\tilde{X}_k$ are independent normal

$$\Rightarrow \bar{X}_k + \tilde{X}_k \sim N(0, \frac{n}{k(n-k)})$$

2) $\bar{X}_k \sim N(0, \frac{1}{k}) \Rightarrow \sqrt{k}\bar{X}_k \sim N(0,1) \Rightarrow k\bar{X}_k^2 \sim \chi^2(1)$

$\tilde{X}_k \sim N(0, \frac{1}{n-k}) \Rightarrow \sqrt{n-k}\tilde{X}_k \sim N(0,1) \Rightarrow (n-k)\tilde{X}_k^2 \sim \chi^2(1)$

Since $\bar{X}_k$ and $\tilde{X}_k$ are independent $\Rightarrow k\bar{X}_k^2$ and $(n-k)\tilde{X}_k^2$ are independent

$$\Rightarrow k\bar{X}_k^2 + (n-k)\tilde{X}_k^2 \sim \chi^2(2)$$

3) $k\bar{X}_k^2 \sim \chi^2(1)$. $(n-k)\tilde{X}_k^2 \sim \chi^2(1)$ and they're independent

$$\Rightarrow F = \frac{k\bar{X}_k^2/1}{(n-k)\tilde{X}_k^2/1} = \frac{k\bar{X}_k^2}{(n-k)\tilde{X}_k^2} \sim F(1,1)$$