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STAT 4005
Assignment 2

Question 1.

a) $E(\bar{x}) = \dfrac{\Sigma E(x)}{5}$

$= \alpha$

b) $Var(\bar{x}) = \frac{1}{5^2}[5(0.8)^0 + 8(0.8)^1 + 6(0.8)^2 + 4(0.8)^3 + 2(0.8)^4]$

≈ 0.7243

Question 2

a) $ARIMA(0,0,2)$

b) $Z_t \sim ARIMA(0,0,2) = MA(2)$

$\Rightarrow Z_t$ is stationary

c) $\theta(x) = 1 + x + 0.25x^2 = 0$

$x = -2$

$\Rightarrow Z_t$ is invertible since $|x| > 1$

d)

$$r(k) = \begin{cases} 41.25 & , k=0 \\ 25 & , k=1 \\ 5 & , k=2 \\ 0 & , \text{otherwise} \end{cases}$$

$$\rho(k) = \begin{cases} 1 & , k=0 \\ \frac{20}{33} & , k=1 \\ \frac{4}{33} & , k=2 \\ 0 & , \text{otherwise} \end{cases}$$

e) $Z_t = a_t + a_{t-1} + 0.25a_{t-2}$

$= (1+0.5)^2 a_t$

$a_t = \sum_{i=0}^{\infty} (-0.5B)^{2i} Z_t$

$= \sum_{k=0}^{\infty} (k+1)(-0.5)^k Z_{t-k}$

$\Rightarrow \pi_k = (k+1)(-0.5)^k \quad \forall\ k = 0, 1, 2, \ldots$

Question 3

a) $\theta(x) = 1 - 0.5x + 0.06x^2 = 0$

$x = 5, \frac{10}{3}$

b) Z_t is stationary and casual since $|x| > 1$

c)
$$\left.\begin{aligned} r(0) &= 0.5r(1) - 0.06r(2) + 1 \\ r(1) &= 0.5r(0) - 0.06r(1) \\ r(2) &= 0.5r(1) - 0.06r(0) \end{aligned}\right\} \begin{aligned} r(0) &= 1.2908 \\ r(1) &= \tfrac{25}{53}r(0) = 0.6089 \\ r(2) &= \tfrac{233}{53}r(0) = 0.227 \end{aligned}$$

Question 4

$$\left.\begin{aligned} r(0) &= 0.7r(4) + \sigma^2 \\ r(1) &= 0.7r(3) \\ r(2) &= 0.7r(2) \\ r(3) &= 0.7r(1) \\ r(4) &= 0.7r(0) \end{aligned}\right\} \quad r(0) = \frac{\sigma^2}{0.51}$$

$$\Rightarrow r(k) = \begin{cases} 0.7^k \frac{\sigma^2}{0.51} & , k = 4n,\ n = 0,1,2,\ldots \\ 0 & , \text{otherwise} \end{cases}$$

Question 5

$Z_t = 0.6Z_{t-1} + a_t + 0.2a_{t-1}$

$Z_t(1 - 0.6B) = a_t(1 + 0.2B)$

AR representation: $a_t = \sum_{i=0}^{\infty} (-0.2B)^i (1-0.6B) Z_t$

$= [1 + \sum_{i=1}^{\infty}(-0.2B)^i - 0.6\sum_{i=1}^{\infty}(-0.2)^{i-1}B^i] Z_t$

$= [1 - 0.8\sum_{i=1}^{\infty}(-0.2)^{i-1}B^i] Z_t$

$= Z_t - 0.8\sum_{i=1}^{\infty}(-0.2)^{i-1} Z_{t-i}$

MA representation: $Z_t = \sum_{i=0}^{\infty}(0.6B)^i(1+0.2B)a_t$

$= [1 + \sum_{i=1}^{\infty}(0.6B)^i + 0.2\sum_{i=1}^{\infty}(0.6)^{i-1}B^i] a_t$

$= [1 + 0.8\sum_{i=1}^{\infty}(0.6)^{i-1}B^i] a_t$

$= a_t + 0.8\sum_{i=1}^{\infty}(0.6)^{i-1} a_{t-i}$

Question 6

a) $Z_t = 1.5 Z_{t-1} - 0.5 Z_{t-2} + a_t - 0.3 a_{t-1} + 0.6 a_{t-2}$

$(1 - \frac{1}{2}B)(1 - B) Z_t = (1 - 0.3B + 0.6B^2) a_t$

$\Rightarrow ARIMA(1,1,2)$

$Z_t = 3 Z_{t-1} - 3 Z_{t-2} + Z_{t-3} + a_t + 0.1 a_{t-1}$

$(1 - B)^3 Z_t = (1 + 0.1B) a_t$

$\Rightarrow ARIMA(0,3,1)$

Question 8

$$Var(Z_t) = Var\left[\frac{a_t}{\phi^2} - \left(1 - \frac{1}{\phi^2}\right)\sum_{k=1}^{\infty} \frac{a_{t+k}}{\phi^k}\right]$$

$$= \sigma^2\left[\frac{1}{\phi^4} + \left(1 - \frac{1}{\phi^2}\right)^2\left(\frac{1}{\phi^2} + \frac{1}{\phi^4} + \frac{1}{\phi^6} + \cdots\right)\right]$$

$$= \sigma^2\left[\frac{1}{\phi^4} + \left(1 - \frac{1}{\phi^2}\right)^2 \frac{\frac{1}{\phi^2}}{1 - \frac{1}{\phi^2}}\right]$$

$$= \frac{\sigma^2}{\phi^2}$$

$\Rightarrow Z_t$ is a white noise process with $Var[Z_t] = \frac{\sigma^2}{\phi^2}$

Question 7

a) $Z_t = 0.6 Z_{t-1} - 0.09 Z_{t-2} + a_t - 0.2 a_{t-1}$

$(1 - 0.6B + 0.09B^2) Z_t = (1 - 0.2B) a_t$

$$a_t = \sum_{i=0}^{\infty} (0.2B)^i (1 - 0.6B + 0.09B^2) Z_t$$

$$= \left[1 + 0.2B + \sum_{i=2}^{\infty}(0.2B)^i - 0.6B - 0.6\sum_{i=2}^{\infty}(0.2)^{i-1}B^i + 0.09\sum_{i=2}^{\infty}(0.2)^{i-2}B^i\right] Z_t$$

$$= \left[1 - 0.4B + 0.01\sum_{i=2}^{\infty}(0.2)^{i-2}B^i\right] Z_t$$

$$= Z_t - 0.4 Z_{t-1} + 0.01\sum_{i=2}^{\infty}(0.2)^{i-2} Z_{t-i}$$

b) $E(a_t Z_t) = E(0.6 Z_{t-1} a_t - 0.09 Z_{t-2} a_t + a_t^2 - 0.2 a_{t-1} a_t)$

$= 1$

$E(a_{t-1} Z_t) = E(0.6 Z_{t-1} a_{t-1} - 0.09 Z_{t-2} a_{t-1} + a_t a_{t-1} + 0.2 a_{t-1}^2)$

$= 0.4$

$$\left.\begin{aligned} r(0) &= 0.6\, r(1) - 0.09\, r(2) + 0.92 \\ r(1) &= 0.6\, r(0) - 0.09\, r(1) - 0.2 \\ r(2) &= 0.6\, r(1) - 0.09\, r(0) \end{aligned}\right\} \quad \begin{aligned} r(0) &= 1.18582058 \\ r(1) &= \frac{60}{109} r(0) - \frac{20}{109} = 0.46925034 \\ r(2) &= \frac{2619}{10900} r(0) - \frac{12}{109} = 0.174831568 \end{aligned}$$

$r(k) = 0.6\, r(k-1) - 0.09\, r(k-2)$, $k \geq 3$

$$\rho(k) = \begin{cases} 1, & k = 0 \\ 0.3957, & k = 1 \\ 0.1474, & k = 2 \\ 0.6\rho(k-1) - 0.09\,(k-2), & k \geq 3 \end{cases}$$