1. i). Moment est.

$$EX=\int_0^\infty x\cdot\frac{\beta}{\theta}x^{\beta-1}e^{-\frac{x^\beta}{\theta}}dx = \theta^{\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)$$

let $M_1' = \tilde\theta^{\frac1\beta}\Gamma(\frac1\beta+1) \quad \Rightarrow \quad \tilde\theta = \left(\frac{\frac1n\sum x_i}{\Gamma(\frac1\beta+1)}\right)^\beta$

MLE

$$L(\theta)=\prod f_X(x_i|\theta) = \left(\frac\beta\theta\right)^n\left(\prod x_i\right)^{\beta-1}\exp\left(-\frac{\sum x_i^\beta}{\theta}\right)$$

$$\Rightarrow l(\theta)=n\log\frac\beta\theta+(\beta-1)\sum\log x_i-\frac1\theta\sum x_i^\beta$$

$$\Rightarrow \frac{\partial l(\theta)}{\partial\theta} = -\frac n\theta+\frac{\sum x_i^\beta}{\theta^2}=0 \quad\Rightarrow\quad \hat\theta=\frac{\sum x_i^\beta}{n}.$$ verify that $\hat\theta$ is MLE.

ii) $E\tilde\theta = E\left(\frac{\frac1n\sum X_i}{\Gamma(\frac1\beta+1)}\right)^\beta = \frac{1}{n[\Gamma(\frac1\beta+1)]^\beta}E(\sum X_i)^\beta$

Assume $\tilde\theta$ is unbiased. $E\tilde\theta=\theta \quad\Rightarrow\quad E(\sum X_i)^\beta = n\theta[\Gamma(\frac1\beta+1)]^\beta = n(EX_i)^\beta$

The equation holds only when $\beta=1$.

Hance $\tilde\theta$ is biased ~~at~~ unless $\beta=1$.

~~iii)~~ $E\hat\theta = E(\frac1n\sum X_i^\beta) = \frac1n\sum\int_0^\infty x_i^\beta\cdot\frac\beta\theta x_i^{\beta-1}e^{-\frac{x_i^\beta}{\theta}}dx_i = \frac1n\sum\theta=\theta$

Hence $\hat\theta$ is unbiased.

iii) $\log f_X(x|\theta) = \log\frac\beta\theta + (\beta-1)\log x - \frac1\theta x^\beta$

$\frac{\partial}{\partial\theta}\log f_X(x|\theta) = -\frac1\theta+\frac{x^\beta}{\theta^2}$

$$E\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2 = E\left[\left(-\frac1\theta+\frac{x^\beta}{\theta^2}\right)^2\right] = E\left(\frac1{\theta^2}-\frac{2x^\beta}{\theta^3}+\frac{x^{2\beta}}{\theta^4}\right)$$

$$=\frac1{\theta^2}-\frac2{\theta^3}EX^\beta+\frac1{\theta^4}EX^{2\beta}$$

$$=\frac1{\theta^2}-\frac2{\theta^3}\cdot\theta+\frac1{\theta^4}\cdot2\theta^2=\frac1{\theta^2}$$

$$\text{CRLB} = \frac{1}{nE\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2} = \frac{\theta^2}{n}$$

iv) $\hat\theta$ is unbiased. $(T(x_1,\cdots,x_n)-\theta)\cdot a(\theta)$

$$\frac{\partial}{\partial\theta}\log f_X(x|\theta) = \sum\frac{\partial}{\partial\theta}\log f(x_i|\theta) = \frac n\theta+\frac1{\theta^2}\sum x_i^\beta = (\hat\theta-\theta)\cdot\frac{n}{\theta^2}$$

$\Rightarrow$ CRLB can be achiened for $\hat\theta$

$\Rightarrow$ $\hat\theta$ is UMVUE

2. i) $f(x)=\frac{1}{2\theta}I(\theta-1<x<\theta+1)$.

$EX=\int_{\theta-1}^{\theta+1}x\cdot\frac{1}{2\theta}dx=\theta$.

Let $\hat{\theta}=M_1'$ $\Rightarrow$ $\hat{\theta}=\frac{1}{n}\Sigma X_i$

ii) $E(\hat{\theta})=E(\frac{1}{n}\Sigma X_i)=\frac{1}{n}\Sigma EX_i=\theta$ unbiased

iii) $\hat{\theta}=\frac{1}{n}\Sigma X_i=7.382$

iv) $X_{(1)}=6.61$. $X_{(5)}=8.36$. $\Rightarrow$ $\hat{\theta}=\frac{1}{2}(X_{(1)}+X_{(5)})=7.485$

3. i) $L(\theta)=\Pi f(x_i|\theta)=\Pi\,\theta x_i^{\theta-1}=\theta^n\Pi X_i^{\theta-1}$

$l(\theta)=\log L(\theta)=n\log\theta+(\theta-1)\Sigma\log X_i$

$\Rightarrow\frac{\partial l(\theta)}{\partial\theta}=\frac{n}{\theta}+\Sigma\log X_i=0$ $\Rightarrow$ $\hat{\theta}_n=\frac{-1}{\frac{1}{n}\Sigma\log X_i}$

consider $Y=-\log X$. for $y>0$

$P(Y\le y)=P(-\ln X\le y)=P(X\ge e^{-y})=\int_{e^{-y}}^{1}\theta x^{\theta-1}dx=1-e^{-\theta y}$

$\Rightarrow f_Y(y)=\theta e^{-\theta y}$ $y>0$. which is $Exp(\theta)$. $T=\Sigma Y_i=\Sigma\log X_i\sim\Gamma(n,\theta)$

$E(\hat{\theta}_n)=E\left(\frac{-1}{\frac{1}{n}\Sigma\log X_i}\right)=-nE\left(\frac{1}{T}\right)=-n\int_0^\infty\frac{1}{t}\cdot\frac{1}{\Gamma(\theta)\theta^n}t^{n-1}e^{-\frac{t}{\theta}}dt=\frac{n}{n-1}\theta$

$E(\hat{\theta}_n^2)=n^2E\left(\frac{1}{T^2}\right)=n^2\int_0^\infty\frac{1}{t^2}\cdot\frac{1}{\Gamma(n)\theta^n}\cdot t^{n-1}e^{-\frac{t}{\theta}}dt=\frac{n^2\theta^2}{(n-1)(n-2)}$

$MSE(\hat{\theta}_n)=E\hat{\theta}_n^2-(E\hat{\theta}_n)^2=\left[\frac{n^2}{(n-1)(n-2)}-\frac{n^2}{(n-1)^2}\right]\theta^2$

$\lim_{n\to\infty}MSE(\hat{\theta}_n)=\lim_{n\to\infty}\left[\frac{n^2}{(n-1)(n-2)}-\frac{n^2}{(n-1)^2}\right]\theta^2=0$

Hence $\hat{\theta}_n$ is consistent.

ii) $EX=\int_0^1 x\cdot\theta x^{\theta-1}dx=\frac{\theta}{\theta+1}$.

let $\frac{\hat{\theta}}{\hat{\theta}+1}=M_1'$ $\Rightarrow$ $\hat{\theta}=\frac{\bar{X}}{1-\bar{X}}$

4. i) $L(\theta)=\Pi f(x_i|\theta)=\left(\frac{1}{2\theta^3}\right)^n\Pi x_i^2\cdot e^{-\frac{\Sigma x_i}{\theta}}$

$\Rightarrow l(\theta)=\log L(\theta)=-n\log 2\theta^3+2\Sigma\log X_i-\frac{1}{\theta}\Sigma X_i$.

$\frac{\partial l(\theta)}{\partial\theta}=-\frac{3n}{\theta}+\frac{1}{\theta^2}\Sigma X_i=0$ $\Rightarrow$ $\hat{\theta}=\frac{1}{3n}\Sigma X_i=\frac{1}{3}\bar{X}$. verify that $\hat{\theta}$ is MLE.

ii) By invariance property of MLE. $\widehat{\tau(\theta)} = \tau(\hat{\theta}) = \frac{3}{\bar{X}}$

iii) $\frac{\partial}{\partial\theta}\log f(x|\theta) = -\frac{3}{\theta} + \frac{x}{\theta^2}$

$E\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2 = E\left(\frac{X}{\theta^2} - \frac{3}{\theta}\right)^2 = \frac{1}{\theta^4}(12\theta^2 - 8\theta^2 + 7\theta^2) = \frac{3}{\theta^2}$

$\tau'(\theta) = \left(\frac{1}{\theta}\right)' = \frac{-1}{\theta^2}$

$CRLB = \frac{\{\tau'(\theta)\}^2}{E\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2} = \frac{\frac{1}{\theta^4}}{n \cdot 3 \cdot \frac{1}{\theta^2}} = \frac{1}{3n\theta^2}$

$\Rightarrow Var\left[\widehat{\tau(\theta)}\right] \geq \frac{1}{3n\theta^2}$

iv). $EX = \int_0^\infty x \cdot \frac{1}{2\theta^3} x^2 e^{-\frac{x}{\theta}} dx = 3\theta.$

$E\hat{\theta} = \frac{1}{3} E\bar{X} = \frac{1}{3}(3\theta) = \theta \quad \Rightarrow \hat{\theta}$ is unbiased.

$E\left(\widehat{\tau(\theta)}\right) = E\left(\frac{3}{\bar{X}}\right) = 3E\left(\frac{1}{\bar{X}}\right).$

Assume $\widehat{\tau(\theta)}$ is unbiased, then

$E\left(\widehat{\tau(\theta)}\right) = \frac{1}{\theta} = 3E\left(\frac{1}{\bar{X}}\right) \Rightarrow E\left(\frac{1}{\bar{X}}\right) = \frac{1}{3\theta}$

Since $g(t) = \frac{1}{t}$ is convex. $0 < \theta < \infty$.

By Jensen's inequality

$E\left(\frac{1}{\bar{X}}\right) > \frac{1}{E\bar{X}} = \frac{1}{3\theta} \Rightarrow E\left(\frac{1}{\bar{X}}\right) \neq \frac{1}{3\theta}. \Rightarrow E\left(\widehat{\tau(\theta)}\right) \neq \frac{1}{\theta} \Rightarrow$ biased.

v). Since $\widehat{\tau(\theta)}$ is MLE of $\tau(\theta)$. By asymptotic normality of MLE. for $\tau(\theta)$

$CRLB = \frac{1}{3n\theta^2}. \Rightarrow \sqrt{n}\left(\widehat{\tau(\theta)} - \tau(\theta)\right) \xrightarrow{d} N\left(0, \frac{1}{3\theta^2}\right).$

5. $EX = \int_0^\infty x \cdot \frac{1}{2\theta^3} x^2 e^{-\frac{x}{\theta}} dx = 3\theta. \quad ET = \frac{1}{3n}\sum X_i = \frac{1}{3n}(3n\theta) = \theta. \Rightarrow$ unbiased

$\frac{\partial}{\partial\theta}\log f(\underline{x}|\theta) = \sum \frac{\partial}{\partial\theta}\log f(x_i|\theta) = \sum \frac{\partial}{\partial\theta}\left(2\log x_i - \frac{x_i}{\theta} - \log 2 - 3\log\theta\right)$

$= \sum\left(\frac{x_i}{\theta^2} - \frac{3}{\theta}\right) = \frac{1}{\theta^2}\left(\sum x_i - 3n\theta\right) = \underbrace{\frac{3n}{\theta^2}}_{a(\theta)}(T - \theta)$

$\Rightarrow$ CRLB can be achieved

$\Rightarrow T$ is UMVUE

6. From some previous results we know

$(\sum X_i, \sum X_i^2)$ is minimal sufficient for (μ, σ^2).

$\sum(X_i - \bar{X})^2 = \sum X_i^2 - n\bar{X}^2$.

Since $(\sum X_i, \sum X_i^2) \longrightarrow (\sum X_i, \sum(X_i-\bar{X})^2)$ is 1-1 map.

$\Rightarrow (\sum X_i, \sum(X_i - \bar{X})^2)$ is jointly sufficient

Since $(\sum X_i, \sum X_i^2)$ is minimal sufficient

Then it's a function of any other jointly sufficient statistic

Since T is a 1-1 map.

Then $(\sum X_i, \sum(X_i-\bar{X})^2)$ is also a function of any other jointly sufficient statistic.

$\Rightarrow (\sum X_i, \sum(X_i-\bar{X})^2)$ is minimal sufficient