THE CHINESE UNIVERSITY OF HONG KONG

Department of Statistics

Subject Code:	STAT4003	Course Title:	Statistical Inference
Session:	Semester 1, 2020/2021, Final Examination		
Date:	18 December 2020	Time:	3:30 pm - 5:30 pm
Time Allowed:	120 Minutes		

This question paper has 2 pages.

Instructions to Candidates:

1. Attempt **ALL** questions
2. This paper has **4** questions.
3. Give **full details** of your working to the questions in the A4-size answer sheet.
4. A standard normal table is attached.

Subject Examiner: Professor Yuanyuan LIN

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO

1. (**25 marks**) Let X_1, X_2, ..., X_n be a random sample from Uniform(θ_1, θ_2) where θ_1 and θ_2 are unknown parameters. Let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of X_1, X_2, ..., X_n .

   (a) (**5 marks**) Find the MLE of θ_1 and θ_2.

   (b) (**5 marks**) Find the probability density function of $X_{(n)}$ and $X_{(1)}$, respectively.

   (c) (**5 marks**) Find $E(X_{(n)})$ and $E(X_{(1)})$.

   (d) (**5 marks**) Find the sufficient statistics for θ_1 and θ_2. Are they complete? Please provide detailed steps for your argument.

   (e) (**5 marks**) Find the UMVUE of $\frac{\theta_1+\theta_2}{2}$.

   Hints: The joint probability density function of $X_{(i)}$ and $X_{(j)}$ for $1 \le i < j \le n$ is

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(u)]^{i-1}[F(v)-F(u)]^{j-i-1}[1-F(v)]^{n-j}f(u)f(v),$$

   for $i < j$ and $u < v$, where $F(\cdot)$ and $f(\cdot)$ are the cumulative distribution function and probability density function of X, respectively.

2. (**25 marks**) Let $X_1, .., X_n$ be a random sample from $N(\theta, \sigma^2)$, where σ^2 is known.

   (a) (**10 marks**) Find the uniformly most powerful(UMP) test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ at the significance level α.

   (b) (**6 marks**) Is the test from part (a) a UMP test for $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$? Why? Please provide detailed steps for your argument.

   (c) (**9 marks**) For the test in part (b), find an expression for the power function of this test. The experimenter desires a Type I Error probability of 0.05 and a power of 0.8 at $\theta = \theta_0 + \sigma/2$ for this test. Find the value of sample size n to achieve this.

3. (**25 marks**) Suppose $\mathbf{X} = (X_1, ..., X_n)$ is a random sample from Exponential(θ) with probability density function $f(x) = \theta e^{-\theta x}$, $x > 0$. Consider test

$$H_0 : \ \theta = \theta_0 \ \text{ versus } \ H_1 : \theta \ne \theta_0.$$

   (a) (**8 marks**) Find the likelihood ratio $\lambda(\mathbf{X})$.

   (b) (**8 marks**) Based on part (a), construct the likelihood ratio test and find the critical region at the significance level α.

   (c) (**9 marks**) Consider a one-side test $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. Construct the likelihood ratio test and find the critical region at the significance level α.

4. (**25 marks**) Consider systems with failure times $X_1, ..., X_n$ assumed to be independent and identically exponential distribution, i.e., $f(x) = \frac{1}{\lambda}e^{-x/\lambda}$, for $x > 0$.

   (a) (**3 marks**) Find the complete and sufficient statistic for λ.

   (b) (**4 marks**) Find the maximum likelihood estimate (MLE) of λ.

   (c) (**3 marks**) Is the MLE in part(b) UMVUE? Why?

(d) (**4** marks) Find the maximum likelihood estimate of the variance of the MLE in part(b).

(e) (**3 marks**) Find the maximum likelihood estimate of the probability $\Pr(X_1 \geq 1)$ that one system will last at least a month.

(f) (**4 marks**) Find the Cramér-Rao Lower bound for variances of unbiased estimators of the probability $\Pr(X_1 \geq 1)$.

(g) (**4 marks**) Find the UMVUE of the probability $\Pr(X_1 \geq 1)$.

Appendix: Some distributions might be helpful.

- Poisson Distribution: $X \sim$Poisson (λ),

$$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \ x = 0, 1, ...; \ 0 \leq \lambda < \infty.$$

- Gamma Distribution: $X \sim$ Gamma (α, β).

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}; \ 0 \leq x < \infty, \ \alpha, \beta > 0$$

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- Beta Distribution: $X \sim$ Beta (α, β).

$$f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \ 0 \leq x \leq 1, \ \alpha > 0, \ \beta > 0.$$

*** End ***

Table E The Standard Normal Distribution

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

For z values less than −3.49, use 0.0001.

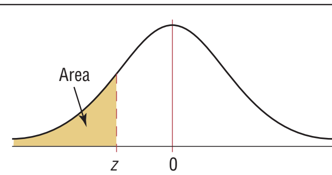

Table E *(continued)*

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

For z values greater than 3.49, use 0.9999.