# MATH1550 Mid-term examination

1. Let
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

   (a) Find $A^TA$ and $AA^T$.

   (b) Find $\begin{bmatrix} A & 0 \\ I_2 & A^T \end{bmatrix} \begin{bmatrix} A^T & I_2 \\ 0 & A \end{bmatrix}$ where $I_2$ is the identity matrix of order 2.

   **Answer.** (a)
$$A^TA = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}.$$
$$AA^T = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

   (b) $$\begin{pmatrix} A & 0 \\ I & A^T \end{pmatrix} \begin{pmatrix} A^T & I \\ 0 & A \end{pmatrix} = \begin{pmatrix} AA^T & A \\ A^T & I + A^TA \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 0 & 2 & 6 & -1 \\ -1 & 1 & 0 & -1 & 3 \end{pmatrix}.$$

   $\square$

2. Consider the system of linear equations
$$\begin{cases} 2x_1 - 4x_2 + x_3 + 5x_4 = 7 \\ x_1 - 2x_2 + 2x_4 = 3 \\ -3x_1 + 6x_2 + 4x_3 - 2x_4 = -5 \end{cases}$$

   (a) Write down the augmented matrix of the system.

   (b) Show that the reduced row echelon form of the augmented matrix is
$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   (c) Write down the solution set of the system.

   **Answer.** (a) The augmented matrix is $\begin{pmatrix} 2 & -4 & 1 & 5 & 7 \\ 1 & -2 & 0 & 2 & 3 \\ -3 & 6 & 4 & -2 & -5 \end{pmatrix}$.

   (b) Omitted.

   (c) The solution set is $\{(2a - 2b + 3, a, -b + 1, b) | a, b \in \mathbb{R}\}$.
   $\square$

3. Given that the systems of linear equations

$$\begin{cases} x_1 + 2x_2 + ax_3 = 3 \\ 3x_1 - x_2 + 5x_3 = -5 \\ -x_1 + 4x_2 + 2x_3 = b \end{cases}$$

has infinitely many solutions.

(a) Show that $a = 4$.

(b) Find the value of $b$.

(c) Write down a solution to the system with $x_1 = 5$.

**Answer.** (a) (b)

Do row reduction, we get $\begin{pmatrix} 1 & 2 & a & 3 \\ 0 & 1 & \frac{3a-5}{7} & 2 \\ 0 & 0 & -\frac{11}{7}a + \frac{44}{7} & b-9 \end{pmatrix}$. To make this system has infinitely many solutions, we must have

$$-\frac{11}{7}a + \frac{44}{7} = b - 9 = 0.$$

i.e. $a = 4$ and $b = 9$.

(c) Substitute $a, b$ and to row reduction, we have $\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The solution set is $\{(-2t-1, -t+2, t) | t \in \mathbb{R}\}$. When $x_1 = 5$, $t = -3$. Thus the solution is $(5, 5, -3)$.

□

4. Determine whether the following subsets of $\mathbb{R}^3$ are vector subspaces.

(a) $\{[x_1, x_2, x_3]^T | x_1 + 2x_3 = 0\}$

(b) $\{[x_1, x_2, x_3]^T | x_1 = x_2 \text{ or } x_3 = 0\}$

**Answer.** (a) This is a vector space.
$0 = (0, 0, 0)$ is in this set.
Choose $u, v$ in this set and $a, b \in \mathbb{R}$. We have

$$u_1 + 2u_3 = 0$$

and

$$v_1 + 2v_3 = 0.$$

Therefore we have

$$(au_1 + bv_1) + 2(au_3 + bu_3) = a(u_1 + 2u_3) + b(v_1 + 2v_3) = 0.$$

This shows $au + bv$ is in this set.

(b) This is not a vector space.
Consider $u = (1, 1, 1)$ and $v = (1, 0, 0)$. We have $u + v = (2, 1, 1)$ which is not in this set.

$\square$

5. Determine whether the following sets of vectors are linearly independent. If the vectors are linearly dependent, express the zero vector as a linear combination of the vectors in a non-trivial way.

   (a) $\mathbf{v}_1 = [2, 1, 1]^T$, $\mathbf{v}_2 = [1, 2, 1]^T$, $\mathbf{v}_3 = [1, 1, 2]^T$
   (b) $\mathbf{v}_1 = [1, 2, 3]^T$, $\mathbf{v}_2 = [2, 4, 6]$, $\mathbf{v}_3 = [5, -2, 7]^T$

**Answer.** (a) Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Do row elimination and we get the row reduced form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which shows $A$ is of full rank. Therefore, $v_1, ..., v_3$ are linearly independent.

(b) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & -2 \\ 3 & 6 & 7 \end{pmatrix}$. Do row elimination and we get the row reduced form $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This matrix is not of full rank, i.e. $v_1, ..., v_3$ are linearly dependent.
The equation $Ax = 0$ has one solution $(-2, 1, 0)$.Thus we have a nontrivial relation $-2v_1 + v_2 = 0$.
$\square$

6. Let $\mathbf{v}_1 = [4, -1, 1]^T$, $\mathbf{v}_2 = [3, 0, 2]^T$, $\mathbf{v}_3 = [1, 5, a]^T$ be vectors in $\mathbb{R}^3$. Let $\mathbf{u}_1 = [3, -3, 8]^T$ and $\mathbf{u}_2 = [2, 1, 3]^T$.

   (a) Find the value of $a$ if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not constitute a basis for $\mathbb{R}^3$.
   (b) Suppose $a = -2$.
      (i) Express $\mathbf{u}_1$ and $\mathbf{u}_2$ as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
      (ii) Express $-3\mathbf{u}_1 + \mathbf{u}_2$ as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

**Answer.** (a) Consider the matrix $A = \begin{pmatrix} 4 & 3 & 1 \\ -1 & 0 & 5 \\ 1 & 2 & a \end{pmatrix}$. Do row elimination and we get $\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \\ 0 & 0 & a-2 \end{pmatrix}$. To make $v_1, v_2, v_3$ not a basis, i.e. to make $Ax = 0$ have nonzero solution, we only need $a = 2$.

(b) $A = \begin{pmatrix} 4 & 3 & 1 \\ -1 & 0 & 5 \\ 1 & 2 & -2 \end{pmatrix}$.

(i) Solve the equation $Ax = u_1$, we have $x = (-2, 4, -1)$. Thus $u_1 = -2v_1 + 4v_2 - v_3$. Solve the equation $Ax = u_2$, we have $x = (-1, 2, 0)$. Thus $u_1 = -v_1 + 2v_2$.

(ii) $-3u_1 + u_2 = -3(-2v_1 + 4v_2 - v_3) + (-v_1 + 2v_2) = 5v_1 - 10v_2 + 3v_3$.

□

7. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be 3 vectors in $\mathbb{R}^n$.
   (a) Prove that if $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}$ is a linear combination of $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$.
   (b) Prove that if $\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

**Answer.** (a) Proof. Suppose $v = a_1v_1 + a_2v_2$. We can write it as $v = \frac{a_1+a_2}{2}(v_1 + v_2) + \frac{a_1-a_2}{2}(v_1 - v_2)$, which is a linear combination of $v_1 + v_2$ and $v_1 - v_2$.

(b) We write $(v_1, v_1+v_2, v_1+v_2+v_3) = (v_1, v_2, v_3)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Here the transition matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is of full rank. Thus $v_1, v_2, v_3$ are linearly independent.

□