2018-19 MATH1520AB

Midterm I (2018 Oct 4)

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Session: ~~A~~ / B

- There are 17 pages (including this page). Please check the number of pages.
- There are 8 questions (7 main questions + 1 bonus question). You have to answer all the main questions (Q1 - Q7). The total score is **100** (+15).
- **Show your steps** unless otherwise stated.
- Good luck!

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
6	8	26	10	9	9	20	0	88+0

8

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1. **(8pts)** Find the natural domain of the following functions:

(a) $F(x) = \ln\left(-\frac{x^2}{4}+4\right)$ (b) $g(x) = \sqrt{\frac{x}{(2x-4)(x+1)}}$

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a) $F(x)$ is well defined only when $-\frac{x^2}{4}+4 > 0$

$-\frac{x^2}{4} > -4$

$x^2 < 16$

$x < 4 \cup x > -4$

domain = $(-4, 4)$

b) $g(x)$ is well defined when

$\frac{x}{(2x-4)(x+1)} \geq 0 \cap (2x-4)(x+1) \neq 0$

$x \geq 0$ $x \neq -1 \cup 2$

2

domain = $[0, \infty) \setminus \{2\}$

2. **(12pts)** Let $f(x) = \sqrt{x+2}$ and $g(x) = \ln(4-x^2)-2$. Expand and simplify the following expression. State the domain of each new function.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

a) $(f \circ g)(x)$

$= \sqrt{[\ln(4-x^2)-2]+2}$

$= \sqrt{\ln(4-x^2)}$

domain $= (-2, 2)$

4

b) $(g \circ f)(x)$

$= \ln[4-(\sqrt{x+2})^2]-2$

$= \ln(2-x)-2$

domain $= (-2, 2)$

4

3. (**32pts**) Without using L'Hôpital's rule, evaluate the following limits or state that it does not exist. If the limit does not exist but diverges to plus or minus infinity, please indicate so, and determine the correct sign.

(a) $\lim_{x\to 1} \dfrac{x^3 - 2x + 5}{x^2 - 2}$

(b) $\lim_{x\to 0} \dfrac{x}{\sqrt{cx+1} - 1}$, where c is a nonzero constant.

a) $\lim_{x\to 1} \dfrac{x^3-2x+5}{x^2-2}$

$= \dfrac{(1)^3-2(1)+5}{(1)^2-2}$

$= -4$

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b) $\lim_{x\to 0} \dfrac{x}{\sqrt{cx+1}-1}$

$= \lim_{x\to 0} \dfrac{x(\sqrt{cx+1}+1)}{c x}$

$= \dfrac{\sqrt{c(0)+1}+1}{c}$

$= \dfrac{2}{c}$

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(c) $\lim_{x \to -2^-} \frac{x-4}{x^2+2x}$

(d) $\lim_{t \to 3^-} \frac{t^2-2t-3}{|t-3|}$

c) $\lim_{x \to -2^-} \frac{x-4}{x^2+2x}$

$= \lim_{x \to -2^-} \frac{x-4}{x(x+2)}$

$= -\infty$

d) $\lim_{t \to 3^-} \frac{t^2-2t-3}{|t-3|}$

$= \lim_{t \to 3^-} \frac{(t-3)(t+1)}{-(t-3)}$

$= -[(3)+1]$

$= -2$

-2

8

(e) $\lim_{x \to +\infty} \frac{\pi + x^{2/3} - x}{x^{1/2} + x - 10}$

(f) $\lim_{x \to +\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-5}\right)$

e) $\lim_{x \to +\infty} \frac{\pi + x^{\frac{2}{3}} - x}{x^{\frac{1}{2}} + x - 10}$

$= \lim_{x \to +\infty} \frac{\frac{\pi}{x} + \frac{1}{x^{\frac{1}{3}}} - 1}{\frac{1}{x^{\frac{1}{2}}} + 1 - \frac{10}{x}}$

$= -1$

f) $\lim_{x \to +\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-5}\right)$

$= \lim_{x \to +\infty} \frac{3x + 5}{\sqrt{x^2+3x} + \sqrt{x^2-5}}$

$= \lim_{x \to +\infty} \frac{3 + \frac{5}{x}}{\sqrt{1 + \frac{3}{x}} + \sqrt{1 - \frac{5}{x^2}}}$

$= \frac{3}{2}$

(g) $\lim_{x \to -\infty} \frac{x-2}{\sqrt{x^2-4}}$

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(h) $\lim_{x \to -\infty} \left(1 - \frac{1}{x^2}\right)^x$

g) $\lim_{x \to -\infty} \frac{x-2}{\sqrt{x^2-4}}$

$= \lim_{x \to -\infty} \frac{1 - \frac{2}{x}}{-\sqrt{1 - \frac{4}{x^2}}}$

$= -1$

h) $\lim_{x \to -\infty} \left(1 - \frac{1}{x^2}\right)^x$ let $\frac{1}{y} = -\frac{1}{x^2}$

$x = \sqrt{-y}$

$= \lim_{x \to -\infty} \left(1 + \frac{1}{y}\right)^{\sqrt{-y}}$

$= \lim_{x \to -\infty} \left(1 + \frac{1}{y}\right)^{y \cdot \frac{\sqrt{-y}}{y}}$

$= \lim_{x \to -\infty} e^{\frac{z}{\sqrt{y}}}$

$= e^0$

$= 1$

4. **(10pts)** Let

$$f(x) = \begin{cases} \ln(x^2+1)+2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ |3^x - 3|, & \text{if } x < 0. \end{cases}$$

Evaluate the following limits. Furthermore, if the limit doesn't exist but diverges to $\pm\infty$, please indicate so and determine the correct sign.

(a) $\lim_{x\to-1} f(x)$

(b) $\lim_{x\to0^+} f(x)$

(c) $\lim_{x\to0^-} f(x)$

(d) $\lim_{x\to0} f(x)$

a) $\lim_{x\to-1} f(x)$

$= |3^{(-1)} - 3|$

$= -(-\frac{8}{3})$

$= \frac{8}{3}$

b) $\lim_{x\to0^+} f(x)$

$= \ln[(0)^2+1]+2$

$= 2$

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c) $\lim_{x\to0^-} f(x)$

$= |3^{(0)} - 3|$

$= -(-2)$

$= 2$

d) $\because \lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 2$

$\lim_{x\to0} f(x)$ exists

$\therefore \lim_{x\to0} f(x) = 2$

5. (**9pts**) Show that there is a real number a such that $e^a + 2a = 3$.

let $f(x) = 3 - e^x - 2x$

since $f(x)$ is continuous on $\mathbb{R}$ as it is polynomial, we can apply intermediate value theorem,

$f(-\infty) = 3 - e^{(-\infty)} - 2(-\infty) = \infty > 0$; and

$f(\infty) = 3 - e^{(\infty)} - 2(\infty) = -\infty < 0$

thus, there is a number $a \in \mathbb{R}$ such that $f(a) = 0$, which mean there is a real number a such that $e^a + 2a = 3$

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6. (**9pts**) Using the first principle, find the derivative of $f(x) = x + \sqrt{x}$.

$f'(x)$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{x+h + \sqrt{x+h} - x - \sqrt{x}}{h}$

$= \lim_{h \to 0} \frac{h + \sqrt{x+h} - \sqrt{x}}{h}$

$= \lim_{h \to 0} \frac{h(\sqrt{x+h} + \sqrt{x}) + h}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h \to 0} \left(1 + \frac{1}{\sqrt{x+h} + \sqrt{x}}\right)$

$= 1 + \frac{1}{2\sqrt{x}}$

9'

7. (**20pts**) Let

$$f(x) = \begin{cases} ax, & \text{if } x < -2, \\ -ax^2 + bx - 4, & \text{if } x \geq -2. \end{cases}$$

(a) Find (i) $\lim_{x \to -2^-} f(x)$ and (ii) $\lim_{x \to -2^+} f(x)$.

(b) Given that $f(x)$ is continuous at $x = -2$, show that $a + b = -2$.

(c) Given that $f(x)$ is differentiable at $x = -2$, show that $3a + b = 0$.

(d) Hence, find the values of a and b such that $f(x)$ is continuous and differentiable at $x = -2$, and find $f'(-2)$.

*Use the values of a and b from (d) to answer the following questions. *

(e) Find $f'(x)$ for $x \neq -2$.

(f) Explain whether $f'(x)$ is continuous at $x = -2$.

(g) Explain whether $f'(x)$ is differentiable at $x = -2$.

a) i, $\lim_{x \to -2^-} f(x)$

$= -2a$

ii, $\lim_{x \to -2^+} f(x)$

$= -4a - 2b - 4$

b) given that $f(x)$ is continuous at $x = -2$

$\lim_{x \to -2} f(x)$ exists, where

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$

$-2a = -4a - 2b - 4$

$4 = -2a - 2b$

$a + b = -2$

c) given that $f(x)$ is differentiable at $x=2$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, where

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0^-} \frac{a(-2+h)+4a+2b+4}{h} = \lim_{h\to 0^+} \frac{-a(-2+h)^2+b(-2+h)-4+4a+2b+4}{h}$$

$$\lim_{h\to 0^-} \frac{-2a+4a+2b+4}{h}+a = \lim_{h\to 0^+} -ah+4a+b$$

$$a = 4a+b$$

$$3a+b = 0$$

d) to make $f(x)$ differentiable at $x=-2$,

both $\lim_{x\to -2} f(x)$ and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, such that

$$a+b = -2 \quad \ldots (1)$$
$$3a+b = 0 \quad \ldots (2)$$

thus, $a=1$, $b=-3$

$f'(-2)$

$= 4(1)+(-3)$

$= 1$

e) for $x < -2$

$f'(x)$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{x+h-x}{h}$

$= \lim_{h \to 0} \frac{h}{h}$

$= 1$

for $x > -2$

$f'(x)$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-(x+h)^2 - 3(x+h) - 4 + x^2 + 3x + 4}{h}$

$= \lim_{h \to 0} (-2xh - h^2 - 3h)\frac{1}{h}$

$= -2x - 3$

$f'(x) = \begin{cases} 1 & \text{for } x < -2 \\ -2x-3 & \text{for } x > -2 \end{cases}$

f) $\lim_{x \to -2^-} f(x)$

$= -2$

$\lim_{x \to -2^+} f(x)$

$= -(-2)^2 - 3(-2) - 4$

$= -2$

$\because \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = -2$

$\therefore \lim_{x \to -2} f(x) = -2$ exists

$f(-2)$

$= -(-2)^2 - 3(-2) - 4$

$= -2$

$\because \lim_{x \to -2} f(x) = f(-2)$

$\therefore f(x)$ is continuous at $x = -2$

g) at $x = -2$

$\lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h}$

$= \lim_{h \to 0^-} \frac{-2+h+2}{h}$

$= 1$

$\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$

$= \lim_{h \to 0^+} \frac{-(-2+h)^2 - 3(-2+h) - 4 + (-2)^2 + 3(-2) + 4}{h}$

$= 1$

$\because \lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$

$\therefore \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$ exists $\Rightarrow f$

answer on P17

Question 7

f) $\lim_{x \to -2^-} f'(x) = 1$

$\lim_{x \to -2^+} f(x) = 1$

$\because \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = 1$

$\therefore \lim_{x \to -2} f(x)$ exists

$f'(x)$ is continuous at $x = -2$

2

g) ~~$\lim_{h \to 0} f'(x+h$~~

at $x = -2$

$$\lim_{h \to 0^-} \frac{f'(-2+h) - f(-2)}{h} = 0$$

$$\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} = -2$$

$$\because \lim_{h \to 0^-} \frac{f'(-2+h) - f(-2)}{h} \neq \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}$$

$\therefore f(x)$ not differentiable at $x = -2$

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8. **Bonus Question** (**15pts**) Show that the function defined by $f(x) = x^2 - (a+b-1)x + ab$ takes on the value $\dfrac{a+b}{2}$, where a and b are any two real numbers.