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Name: CHAN King Yeung

SID: 1155119394

STAT 4003

Assignment 4

Question 1

i) consider $\Theta_0 = \{\frac{1}{2}\}$, $\Theta = (0,1)$

$L(\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ where $x = \sum y_i$, $y \sim bern(\theta)$

$\ell(\theta) = \ln[\binom{n}{x}] + x\ln(\theta) + (n-x)\ln(1-\theta)$

$\frac{\partial \ell}{\partial x}\Big|_{\hat\theta} = 0$, $\hat\theta = \frac{x}{n}$ $\Rightarrow$ $\sup\{L(\theta) : \theta \in \Theta_0\} = \binom{n}{x}(\frac{1}{2})^n$

$\sup\{L(\theta) : \theta \in \Theta\} = \binom{n}{x}(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}$

$$\lambda(x) = \frac{\binom{n}{x}(\frac{1}{2})^n}{\binom{n}{x}(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}} = \frac{(\frac{1}{2})^n}{(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}}$$

ii) $\frac{(\frac{1}{2})^n}{(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}} \leq k$

$(\frac{x}{n})^x(1-\frac{x}{n})^{n-x} \geq k'$

$x\ln(x) + (n-x)\ln(n-x) \geq k$

iii) $\frac{\partial}{\partial x}[x\ln(x) + (n-x)\ln(n-x)] = 0$

$\frac{x}{n-x} = 1$

$x = \frac{n}{2}$

$\Rightarrow f(x)$ attains minimum at $x = \frac{n}{2}$

we reject H_0 if $x - \frac{n}{2} \geq k$, yet, we know binomial is symmetric at $\frac{n}{2}$. Thus, the critical region can be written as $|x - \frac{n}{2}| \geq k$

Question 2

consider $\Theta_0 = \{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 = \sigma_0^2\}$

$\Theta = \{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+\}$

$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum(x-\mu)^2}{2\sigma^2}}$

$\ell(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{\sum(x-\mu)^2}{2\sigma^2}$

$\frac{\partial \ell}{\partial \mu}\Big|_{\hat\mu} = 0$ $\qquad$ $\frac{\partial \ell}{\partial \sigma^2}\Big|_{\hat\mu, \hat\sigma^2} = 0$

$\hat\mu = \bar{x}$ $\qquad$ $\hat\sigma^2 = \frac{\sum(x-\bar{x})^2}{n}$

$$\lambda(x) = \frac{(2\pi\sigma_0^2)^{-\frac{n}{2}} e^{-\frac{\sum(x-\bar{x})^2}{2\sigma_0^2}}}{[\frac{2\pi\sum(x-\bar{x})^2}{n}]^{-\frac{n}{2}} e^{-\frac{n}{2}}}$$

$$= \left[\frac{1}{\sigma}\sqrt{\frac{\sum(x-\bar{x})^2}{n}}\right]^n e^{-\frac{1}{2}[\frac{\sum(x-\bar{x})^2}{\sigma^2} - n]}$$

$$= \left(\frac{\hat\sigma}{\sigma}\right)^n e^{-\frac{1}{2}(\frac{n\hat\sigma^2}{\sigma^2} - n)}$$

Question 3

$f_{Y_4}(y) = \frac{4y^3}{\theta^4}$, $y \in (0, \theta)$

$Q(\theta) = P(y_4 \leq \frac{1}{2} \cup y_4 > 1 \mid H_1)$

$= P(y_4 \leq \frac{1}{2} \mid H_1) + P(y_4 > 1 \mid H_1)$

$= \int_0^{\frac{1}{2}} \frac{4y^3}{\theta^4} dy + \int_1^{\theta} \frac{4y^3}{\theta^4} dy$

$= \frac{1}{16\theta^4} + 1 - \frac{1}{\theta^4}$

$= \frac{16\theta^4 - 15}{16\theta^4}$

Question 4

$$\frac{\prod f(y;\sigma_0)}{\prod f(y;\sigma_1)} = \frac{(2\pi\sigma_0^2)^{-\frac{n}{2}} e^{-\frac{\sum y^2}{2\sigma_0^2}}}{(2\pi\sigma_1^2)^{-\frac{n}{2}} e^{-\frac{\sum y^2}{2\sigma_1^2}}} \leq k$$

$(\frac{\sigma_1^2}{\sigma_0^2})^{\frac{n}{2}} e^{-\sum y^2(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2})} \leq$

$\frac{n}{2}\ln(\frac{\sigma_1^2}{\sigma_0^2}) - \sum y^2(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}) \leq k'$

$\sum y^2 \geq k$ for $\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} > 0$

$\Rightarrow$ the critical region of UMP test is $C = \{y : \sum y^2 \geq k\}$

Question 5

$f_X(x) = \frac{\theta^x e^{-\theta}}{x!}$

$= \frac{1}{x!} e^{-\theta} e^{y\ln(\theta)}$

$\Rightarrow X \sim Poi(\theta)$ belongs to exponential family

as $\ln(\theta)$ is an increasing function on θ

the critical region of UMP test is $C = \{x : \sum x \geq k\}$

given $k = 5$, $C = \{x : \sum x \geq 5\}$

denote $Y := \sum X$ and $Y \sim Poi(20\theta)$

$\alpha = P(Y \geq 5 \mid H_0)$

$= 1 - P(Y < 5 \mid H_0)$

$= 1 - \sum_{i=0}^{4} \frac{(2)^i e^{-2}}{i!}$

≈ 0.9473

$Q(\theta) = P(Y \geq 5 \mid H_1)$

$= 1 - P(Y < 5 \mid H_1)$

$= 1 - \sum_{i=0}^{4} \frac{(20\theta)^i e^{-20\theta}}{i!}$