2018-19 MATH1520AB

Midterm II (2018 Nov 1)

Name (English) Chan King Yeung (Chinese) 陳景陽

Student ID: 1155118394 Signature: Jede

Session: A / B

- There are 19 pages (including this page). Please check the number of pages.
- There are 7 questions. You have to answer all questions (Q1 - Q7). The total score is **100**.
- **Show your steps** unless otherwise stated.
- Good luck!

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
24	10	15	9	10	7	14	89

1. **(24pts)** Find the derivative of the following functions:

(a) $y = \dfrac{x^2+x+3}{\sqrt{x}}$ (b) $s = 2^t(t^2+1)^{-1}$

a) $y = x^{\frac{3}{2}} + \sqrt{x} + \frac{3}{\sqrt{x}}$

$y' = \frac{3\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} - \frac{3}{2x^{\frac{3}{2}}}$

b) $s = 2^t(t^2+1)^{-1}$

$s' = 2^t \ln 2 \, (t^2+1)^{-1} - \frac{2^t \cdot 2t}{(t^2+1)^2}$

$= \frac{2^t \ln 2}{t^2+1} - \frac{2t(2^t)}{(t^2+1)^2}$

(a) $y = \ln\left(\frac{2x+3}{3x+5}\right)$

(b) $y = \frac{x}{x + \frac{c}{x}}$, where c is a constant.

a) $y = \ln(2x+3) - \ln(3x+5)$

$y' = \frac{2}{2x+3} - \frac{3}{3x+5}$

b) $y = \frac{x^2}{x^2+c}$

$y' = \frac{2x(x^2+c) - 2x(x^2)}{(x^2+c)^2}$

$= \frac{2xc}{(x^2+c)^2}$

(a) $y = x^{(1+\ln x)}$

(b) $y = \dfrac{(3x+4)^3(5x-2)^{\sqrt{7}}}{(2x+1)^4 e^x}$

a) $\ln y = (1+\ln x)\ln x$

$y' = x^{(1+\ln x)}\left(\frac{1}{x}\cdot \ln x + \frac{1+\ln x}{x}\right)$

$= x^{(1+\ln x)}\left(\frac{1+2\ln x}{x}\right)$

b) $\ln y = 3\ln(3x+4) + \sqrt{7}\ln(5x-2) - 4\ln(2x+1) - x\ln e$

$y' = \frac{(3x+4)^3(5x-2)^{\sqrt{7}}}{(2x+1)^4 e^x}\left(\frac{9}{3x+4} + \frac{5\sqrt{7}}{5x-2} - \frac{8}{2x+1} - 1\right)$

2. **(10pts)** Let

$$f(x) = \begin{cases} x^2 + 4x + 1, & x \geq 0 \\ ax + b, & x < 0 \end{cases}$$

Find values of the constants a and b for which the function is continuous and differentiable at $x = 0$.

to be continuous at $x=0$;

$\lim_{x\to 0} f(x)$ exists and equals $f(0)$ such that

$$\lim_{x\to 0^-} ax+b = \lim_{x\to 0^+} x^2+4x+1$$

$$b = 1$$

only $b=1$ makes that $f(x)$ continuous at $x=0$

to be differentiable at $x=0$,

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists such that

$$\lim_{h\to 0^-} \frac{a(0+h)+1-1}{h} = \lim_{h\to 0^+} \frac{(0+h)^2+4(0+h)+1-1}{h}$$

$$a = 4$$

only $a=4$ makes $f(x)$ differentiable at $x=0$

15

3. **(16pts)** Find the following limits or state that it does not exist. If the limit does not exist but diverges to plus or minus infinity, please indicate so, and determine the correct sign.

(a) $\lim\limits_{x \to +\infty} \dfrac{\ln(x^2+2)}{e^{x^2+2}}$ (b) $\lim\limits_{x \to 0} \dfrac{x a^x}{a^x - 1}$, where $a > 1$

a) $\lim\limits_{x \to +\infty} \dfrac{\ln(x^2+2)}{e^{x^2+2}}$

$= \lim\limits_{x \to +\infty} \dfrac{\frac{2x}{x^2+2}}{2x e^{x^2+2}}$

$= \lim\limits_{x \to +\infty} \dfrac{1}{e^{x^2+2}(x^2+2)}$

$= 0$

b) $\lim\limits_{x \to 0} \dfrac{x a^x}{a^x - 1}$

$= \lim\limits_{x \to 0} \dfrac{a^x + x a^x \ln a}{a^x \ln a}$

$= \lim\limits_{x \to 0} \dfrac{1 + x\ln a}{\ln a}$

$= \dfrac{1}{\ln a}$

-1

(a) $\lim_{x \to +\infty} (e^x + x)^{1/x}$

(b) $\lim_{x \to 1^+} (\ln(x^7 - 1) - \ln(x^5 - 1))$

a) $\lim_{x \to +\infty} (e^x + x)^{\frac{1}{x}}$

$= e^{\lim_{x \to +\infty} \frac{1}{x} \ln(e^x + x)}$

$= e^{\lim_{x \to +\infty} \frac{e^x + 1}{e^x + x}}$

$= e^{\lim_{x \to +\infty} \frac{e^x}{e^x + 1}}$

$= e^{\lim_{x \to +\infty} \frac{e^x}{e^x}}$

$= e$

b) $\lim_{x \to 1^+} \ln\left(\frac{x^7 - 1}{x^5 - 1}\right)$

$= \ln\left(\lim_{x \to 1^+} \frac{7x^6}{5x^4}\right)$

$= \ln\left(\frac{7}{5}\right)$

9

4. **(10pts)** Let C be the curve defined by the equation

$$x^2 + 2xy + y^3 = e^{xy}.$$

(a) Show that the curve C passes through the point $P(1, 0)$

Assume the curve C determines an implicit function $y = y(x)$ near P.

(b) Find $\left.\dfrac{dy}{dx}\right|_{(1,0)}$.

(c) Find the equation of the tangent line to the curve C at the point P.

(d) Find $\left.\dfrac{d^2y}{dx^2}\right|_{(1,0)}$.

a) $LHS = (1)^2 + 2(1)(0) + (0)^3$

$= 1$

$RHS = e^{(1)(0)}$

$= 1$

$\because LHS = RHS$

$\therefore$ curve C passes through the point $P(1,0)$

b)

$$2x + 2y + 2xy' + 3y^2y' = e^{xy}y + e^{xy}xy'$$

$$y' = \frac{ye^{xy} - 2x - 2y}{2x + 3y^2 - xe^{xy}}$$

$$y'|_{(1,0)} = -2$$

$\cancel{2x + 2y + 2xy' + 3y^2y' = e^{xy}(y + xy')}$

c) $\frac{y-0}{x-1} = -2$

$2x + y - 2 = 0$

d) $2 + 2y' + 2y' + 2xy'' + 6yy'^2 + 3y^2y'' = e^{xy}(y+xy')^2 + e^{xy}(y+xy')(y'+y'+xy'')$

substituting $(1,0)$ and $y' = -2$ into above formula

$2 + 2(-2) + 2(-2) + 2(1)y'' + 6(0)(-2)^2 + 3(0)^2y'' = e^{(1)(0)}((0)+(1)(-2))^2 + e^{(1)(0)}((0)+(1)(-2))((-2)+(-2)+(1)y'')$

$-6 + 2y'' = 12 - 2y''$

$y'' = 4.5$

~~$2 + 2y' + 2y' + 2xy'' + 6yy'^2 + 3y^2y'' = e^{xy}(y+xy')^2 + e^{xy}(y+xy')(y'+y'+xy'')$~~

5. **(10pts)** Let

$$f(x) = \ln(x^2+1) + x + 3,$$

10

and let $y = g(x)$ be the inverse of f.

(a) Show that $g(3) = 0$.

(b) Find $g'(x)$ without explicit expression of $g(x)$.

(c) Find $g'(3)$.

a) $g(x) = y \Rightarrow x = f(y)$

$\because g(3) = 0 \Rightarrow f(0)$

$= \ln((0)^2+1) + (0) + 3$

$= 3$

$\therefore g(3) = 0$

b) $g'(x) = \dfrac{1}{f'(y)}$

$= \dfrac{1}{\frac{2y}{y^2+1}+1}$

$= \dfrac{y^2+1}{y^2+2y+1}$

c) $g'(3) = \dfrac{(0)^2+1}{(0)^2+2(0)+1}$

$= 1$

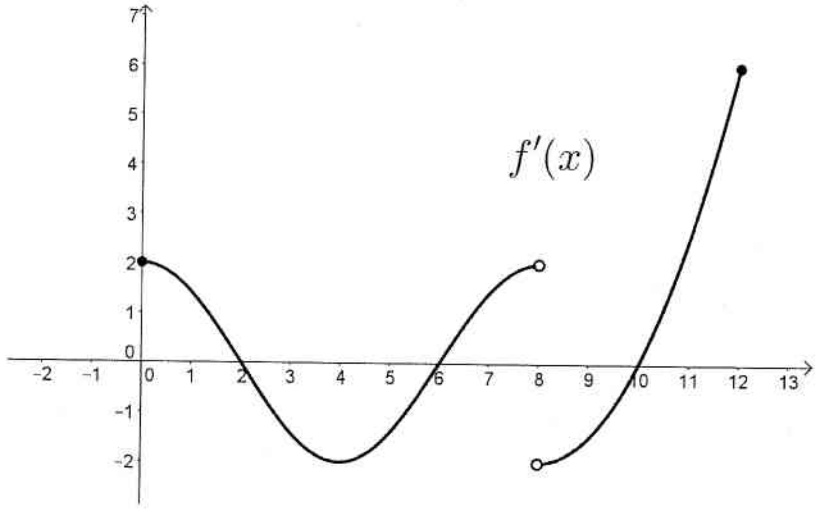
6. (**10pts**) Assume f is a continuous function on $[0, 12]$, even at the points where f' is undefined. Here is the graph of its DERIVATIVE, $f'(x)$. Please answer the following questions with an explanation.

(a) Find the open interval(s) on which f is increasing or decreasing.

(b) Find the x-coordinates of all relative extrema of f, if any, and classify them as relative maximum or relative minimum, or neither.

(c) Find the open interval(s) on which f is concave upward or downward.

(d) Find the x-coordinates of all inflection points, if any.



a) increasing intervals : $(0,2) \cup (6,8) \cup (10,12)$
$\therefore f'(x) > 0$

decreasing intervals : $(2,6) \cup (8,10)$
$\therefore f'(x) < 0$

b) relative maximum at $x = 2, 8$
$\because$ $f'(x) > 0$ for $x < 2$ $f'(x) > 0$ for $x < 8$
$f'(x) < 0$ for $x > 2$ $f'(x) < 0$ for $x > 8$

relative minimum at $x = 6, 10$
$\because$ $f'(x) < 0$ for $x < 6$ $f'(x) < 0$ for $x < 10$
$f'(x) > 0$ for $x > 6$ $f'(x) > 0$ for $x > 10$

c)

concave upward intervals = $(0, 2) \cup (6, 8)$ X

X

$\because$ the tangent line of $f'(x)$ always above the curve of $f'(x)$, which $f''(x) > 0$

concave downward intervals = $(2, 6) \cup (8, 12)$ ✓

X

$\because$ the tangent line of $f'(x)$ always below the curve of $f'(x)$, which $f''(x) < 0$

d) inflection points at $x = 2, 6, 8$ X

$\because$ $f''(x)$ changes sign at $x = 2, 6, 8$ respectly. where

$$\begin{cases} f''(x) > 0 & \text{for } x < 2 \\ f''(x) < 0 & \text{for } x > 2 \end{cases}$$

$$\begin{cases} f''(x) > 0 & \text{for } x < 8 \\ f''(x) < 0 & \text{for } x > 8 \end{cases}$$

7

$$\begin{cases} f''(x) < 0 & \text{for } x < 6 \\ f''(x) > 0 & \text{for } x > 6 \end{cases}$$

7. (**20pts**) For the following function:

$$y = f(x) = x^{5/3} - 5x^{2/3}$$

(a) i. State the domain of $f(x)$.
 ii. Find x, y intercepts, if any.
 iii. Find all vertical and horizontal asymptotes, if any.

(b) i. Compute $f'(x)$.
 ii. Find interval(s) on which $f(x)$ is increasing or decreasing.
 iii. Find and classify the relative extrema of $f(x)$.

(c) i. Verify that $f''(x) = \frac{10}{9}\left(\frac{x+1}{x^{4/3}}\right)$.
 ii. Find the interval(s) on which $f(x)$ is concave upward or downward.
 iii. Find the point(s) of inflection of $f(x)$.

(d) Sketch the graph of $f(x)$. (Label all intercepts, vertical/horizontal asymptotes, relative extrema, point(s) of inflection.)

14

a) i, domain = $\mathbb{R}$

ii, x intercept = $(0, 0)$

y intercept = $(0, 0)$

iii, Since $x \in \mathbb{R}$, vertical asymptote does not exist.

$\lim_{x \to -\infty} f(x) = -\infty$ $\lim_{x \to +\infty} f(x) = +\infty$

Since $\lim_{x \to \pm\infty} f(x) = \pm\infty$, horizontal asymptote does not exist.

b) i, $f'(x) = \frac{5x^{\frac{2}{3}}}{3} - \frac{10x^{-\frac{1}{3}}}{3}$

$= \frac{5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}}}{3}$

ii, $f'(x) = 0$

$x = 0$

↗ 0 ↘

increasing interval = $(-\infty, 0)$
decreasing interval = $(0, \infty)$

iii, Since $f'(x)$ changes sign at $x=0$, relative maximum is at $x=0$

c) i, $f''(x) = \frac{5}{3}\left(\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}x^{-\frac{4}{3}}\right)$

$$= \frac{10}{9}\left(\frac{x+1}{x^{\frac{4}{3}}}\right)$$

ii, $f''(x) = 0$ $\quad$ $f''(x)$ undefines at $x = 0$

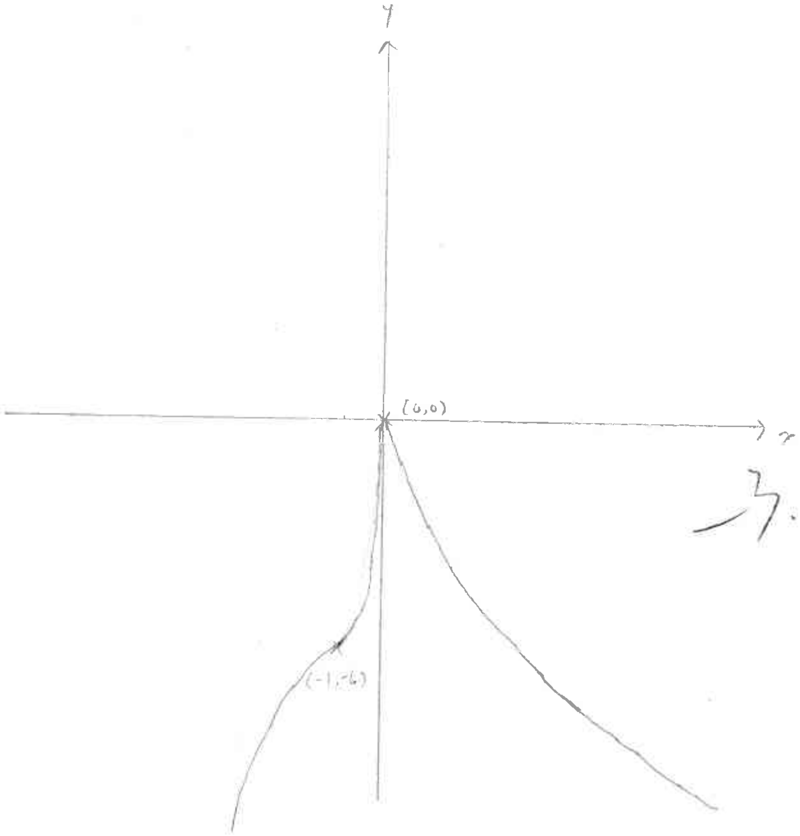
$x = -1$

$\cap$ $\quad -1 \quad$ $\cup$ $\quad 0 \quad$ $\cup$

concave upward interval $= (-1, \infty)$

concave downward interval $= (-\infty, -1)$

iii, Since only at $x=-1$ changes sign of $f''(x)$, point of inflection is $(-1, -6)$



-3.