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STAT2001 Assignment 4

Question 1

$$f(x,y) = \begin{cases} \frac{1}{36} & for\ x > y \\ \frac{7-x}{36} & for\ x = y \\ 0 & for\ x < y \end{cases}$$

$f(x,y)$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$y=1$	$\frac{6}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$y=2$	0	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$y=3$	0	0	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$y=4$	0	0	0	$\frac{3}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$y=5$	0	0	0	0	$\frac{2}{36}$	$\frac{1}{36}$
$y=6$	0	0	0	0	0	$\frac{1}{36}$

Question 2

(a) $$\begin{aligned} Cov(X+Y, X-Y) &= Cov(X, X-Y) + Cov(Y, X-Y) \\ &= Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y) \\ &= Var(X) - Var(Y) \\ &= np(1-p) - np(1-p) \\ &= 0 \end{aligned}$$

(b) $X+Y$ and $X+Y$ are not independent. Consider if $X+Y=2n$, then $X=Y=n$. We have $X-Y=0$, such that

$$P(\,X-Y=0 \mid X+Y=12\,) = \frac{P(X-Y=0 \cap X+Y=12)}{P(X+Y=12)} = 1 \neq P(X-Y=0)$$

which shows that $X+Y$ and $X-Y$ are not independent.

Question 3

$$f_X(x) = \int_x^1 2(x+y)\,dy = [2xy+y^2]_x^1 = 1+2x-3x^2\ for\ 0<x<1$$

$$f_Y(y) = \int_0^y 2(x+y)\,dx = [x^2+2xy]_0^y = 3y^2\ for\ 0<y<1$$

Since the support of joint distribution is not the product set of space of X and space of Y, X and Y are not independent.

Question 4

$$
\begin{aligned}
P(X+Y>2) &= \int_2^\infty \int_0^\infty e^{-(x+y)}\,dy\,dx + \int_0^2 \int_2^\infty e^{-(x+y)}\,dy\,dx + \int_0^2 \int_0^{2-x} e^{-(x+y)}\,dy\,dx \\
&= \int_2^\infty e^{-x}[-e^{-y}]_0^\infty\,dx + \int_0^2 e^{-x}[-e^{-y}]_2^\infty\,dx + \int_0^2 e^{-x}[-e^{-y}]_0^{2-x}\,dx \\
&= [-e^{-x}]_2^\infty + e^{-2}[-e^{-x}]_0^2 - e^{-2}[x]_0^2 + [-e^{-x}]_0^2 \\
&= 1 - e^{-4} - e^{-2} \\
&\approx 0.8463
\end{aligned}
$$

Question 5

$$f_X(x) = \int_0^x 6y\,dy = [3y^2]_0^x = 3x^2 \; for\; 0 < x < 1$$

$$f_{Y|X=0.3}(y) = \frac{6y}{3(0.3)^2} = \frac{200y}{9} \; for\; 0 < y < 0.3$$

$$E(\,Y \mid X = 0.3\,) = \int_0^{0.3} y \cdot \frac{200y}{9}\,dy = \left[\frac{200y^3}{27}\right]_0^{0.3} = 0.2$$

$$Var(\,Y \mid X = 0.3\,) = \int_0^{0.3} y^2 \cdot \frac{200y}{9}\,dy - (0.2)^2 = \left[\frac{200y^4}{36}\right]_0^{0.3} - 0.04 = 0.005$$

Question 6

$$M_{X+Y}(t) = E\left[e^{(x+y)t}\right] = E(e^{xt})E(e^{yt}) = \frac{pe^t}{1-(1-p)e^t} \cdot \frac{pe^t}{1-(1-p)e^t} = \left[\frac{pe^t}{1-(1-p)e^t}\right]^2$$

$$X + Y \sim Negative\ Binomial(2, p)$$

Question 7

$$E(X) = (1)\left(\frac{4}{15}\right) + (2)\left(\frac{7}{15}\right) + (3)\left(\frac{4}{15}\right) = 2$$

$$Var(X) = (1)^2\left(\frac{4}{15}\right) + (2)^2\left(\frac{7}{15}\right) + (3)^2\left(\frac{4}{15}\right) - (2)^2 = \frac{8}{15}$$

$$E(Y) = (1)\left(\frac{9}{15}\right) + (2)\left(\frac{4}{15}\right) + (3)\left(\frac{2}{15}\right) = \frac{23}{15}$$

$$Var(Y) = (1)^2\left(\frac{9}{15}\right) + (2)^2\left(\frac{4}{15}\right) + (3)^2\left(\frac{2}{15}\right) - \left(\frac{23}{15}\right)^2 = \frac{116}{225}$$

$$E(XY) = (1)\left(\frac{1}{15}\right) + (2)\left(\frac{4}{15}\right) + (3)\left(\frac{4}{15}\right) + (2)\left(\frac{1}{15}\right) + (3)\left(\frac{2}{15}\right) + (4)\left(\frac{3}{15}\right) = \frac{41}{15}$$

$$Cov(X,Y) = \frac{41}{15} - (2)\left(\frac{23}{15}\right) = -\frac{1}{3}$$

$$Corr(X,Y) = \frac{-1/3}{\sqrt{(8/15)(116/225)}} \approx -0.6357$$