Homework Assignment 4. 

1. $X \sim Bin(n, \theta)$. $\begin{cases} H_0: \theta = \frac{1}{2} \\ H_1: \theta \neq \frac{1}{2} \end{cases}$

i) Likelihood: $L(\theta) = f(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$

for $\Theta_0$: $L(\hat{\theta}_0) = \binom{n}{x}(\frac{1}{2})^x(\frac{1}{2})^{n-x}$

for $\Theta$: $\hat{\theta} = \bar{X}$. $L(\hat{\theta}) = \binom{n}{x}(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}$

$$\lambda = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} = \frac{\binom{n}{x}(\frac{1}{2})^x(\frac{1}{2})^{n-x}}{\binom{n}{x}(\frac{x}{n})^x(1-\frac{x}{n})^{n-x}} = (\frac{n}{2x})(\frac{1}{2})^{n-x}(\frac{n}{n-x})^{n-x} = (\frac{n}{2})^n(\frac{1}{x})^x(\frac{1}{n-x})^{n-x}$$

ii) Critical region: $C_1 = \{X : \lambda \le k_0\}$

$$= \{X : (\frac{n}{2})^n(\frac{1}{x})^x(\frac{1}{n-x})^{n-x} \le k_0\}$$

$$= \{X : x^x(n-x)^{n-x} \ge k_0'\}$$

$$= \{X : x\ln x + (n-x)\ln(n-x) \ge k\}$$

iii) $f'(x) = \ln x + x\cdot\frac{1}{x} - \ln(n-x) - (n-x)(\frac{1}{n-x})$

$= \ln x + 1 - \ln(n-x) - 1 = \ln x - \ln(n-x)$

let $f'(x) = 0 \Rightarrow \ln x = \ln(n-x) \Rightarrow x = \frac{n}{2}$

$f''(x) = \frac{1}{x} + \frac{1}{n-x}$. $f''(\frac{n}{2}) = \frac{4}{n} > 0$

$\Rightarrow$ $f(x)$ attains a minimum at $x = \frac{n}{2}$

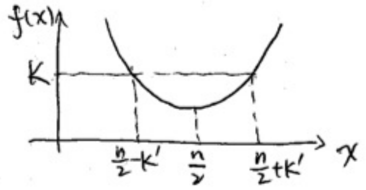
$f(n-x) = (n-x)\ln(n-x) + (n-(n-x))\ln(n-(n-x))$

$= (n-x)\ln(n-x) + x\ln x = f(x)$

$\Rightarrow$ $f(x)$ is symmetric about $x = \frac{n}{2}$

$\Rightarrow C_1 = \{X : f(x) \ge k\} = \{X : |x - \frac{n}{2}| \ge k'\}$.

where



2. likelihood : $l(\mu,\sigma^2)=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\{-\frac{1}{2\sigma^2}\Sigma(X_i-\mu)^2\}$ ②

for $\Theta_0$ : $\hat{\theta}_0=(\bar{X},\sigma_0^2)$ . $l(\hat{\theta}_0)=\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\{-\frac{1}{2\sigma_0^2}\Sigma(X_i-\bar{X})^2\}=\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\{-\frac{n\hat{\sigma}^2}{2\sigma_0^2}\}$

for $\Theta$ : $\hat{\theta}=(\bar{X},\hat{\sigma}^2)$ $l(\hat{\theta})=\left(\frac{1}{\sqrt{2\pi}\hat{\sigma}}\right)^n \exp\{-\frac{1}{2\hat{\sigma}^2}\Sigma(X_i-\bar{X})^2\}=\left(\frac{1}{\sqrt{2\pi}\hat{\sigma}}\right)^n \exp\{-\frac{n}{2}\}$

$\Rightarrow \lambda=\frac{l(\hat{\theta}_0)}{l(\hat{\theta})}=\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{\frac{n}{2}} \exp\{-\frac{n}{2}(\frac{\hat{\sigma}^2}{\sigma_0^2}-1)\}=\exp\{\frac{n}{2}\}\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\exp\{-\frac{\hat{\sigma}^2}{\sigma_0^2}\}\right)^{\frac{n}{2}}$

3. $\begin{cases} H_0: \theta=1 \\ H_1: \theta\neq 1 .\end{cases}$ $C_1=\{\underline{x} : X_{(4)}\leq\frac{1}{2} \text{ or } X_{(4)}>1\}$

$Q(\theta)=P(\underline{X}\in C_1)=P(X_{(4)}\leq\frac{1}{2} \text{ or } X_{(4)}>1)$

$f_{X_{(4)}}(x)=4f_X(x)[F_X(x)]^3=4(\frac{1}{\theta})\cdot(\frac{x}{\theta})^3=\frac{4}{\theta^4}x^3$ . $0<x<\theta$ .

$\Rightarrow Q(\theta)=P(X_{(4)}\leq\frac{1}{2} \text{ or } X_{(4)}>1)$

$=\int_0^{\frac{1}{2}}\frac{4}{\theta^4}x^3dx+\int_1^{\theta}\frac{4}{\theta^4}x^3dx = 1-\frac{15}{16\theta^4}$

4. $\begin{cases} H_0: \sigma=\sigma_0 \\ H_1: \sigma=\sigma_1 \end{cases}$ $f(x|\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$ . $\sigma_1>\sigma_0 \Rightarrow \frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}>0$ ⑤

$\frac{f(\underline{x}|\sigma_0)}{f(\underline{x}|\sigma_1)}=\frac{\Pi f(x_i|\sigma_0)}{\Pi f(x_i|\sigma_1)}=\left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\{-\frac{1}{2}(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2})\Sigma x_i^2\}$

By N-P Lemma

$C_1=\{\underline{x}: \frac{f(\underline{x}|\sigma_0)}{f(\underline{x}|\sigma_1)}\leq k_0\}=\{\underline{x}: c_1\exp\{c_2\Sigma x_i^2\}\leq k_0\}=\{\underline{x}: \Sigma x_i^2\geq k\}$. $k>0$

$P_{\sigma_0}(\underline{x}\in C_1)=\alpha \Rightarrow P_{\sigma_0}(\Sigma X_i^2\geq k)=P_{\sigma_0}(\frac{\Sigma X_i^2}{\sigma_0^2}\geq\frac{k}{\sigma_0^2})=\alpha$.

$\frac{\Sigma X_i^2}{\sigma_0^2}\sim\chi_n^2 \Rightarrow \frac{k}{\sigma_0^2}=\chi^2_{n,\alpha}$. $\Rightarrow k=\sigma_0^2\chi^2_{n,\alpha}$

$\Rightarrow C_1=\{\underline{x}: \Sigma X_i^2\geq\sigma_0^2\chi^2_{n,\alpha}\}$. UMP test.

11. $\begin{cases} H_0 : \theta = 0.1 \\ H_1 : \theta > 0.1 \end{cases}$ $\quad f(x|\theta) = \frac{\theta^x}{x!} e^{-\theta} = e^{-\theta} \frac{1}{x!} e^{x \ln \theta} \Rightarrow \begin{array}{l} c(\theta) = \ln \theta \nearrow \\ d(x) = x \end{array}$

$\Rightarrow C_1 = \{ \underline{x} : \sum d(x_i) = \sum x_i \geq k \}$ is UMP test

let $k=5$. $\quad C_1 = \{ \underline{x} : \sum x_i \geq 5 \}$ is UMP test

$\sum X_i \sim Poi(n\theta) = Poi(2)$ under $H_0$.

$\alpha = P_{H_0}(\underline{x} \in C_1) = P_{H_0}(\sum X_i \geq 5) = 1 - P_{H_0}(\sum X_i < 5) = 1 - \sum_{y=1}^{4} \frac{2^y e^{-2}}{y} = 0.0527$

$\sum X_i \sim Poi(20\theta)$.

$Q(\theta) = P(\sum X_i \geq 5) = 1 - P(\sum X_i < 5) = 1 - \sum_{y=1}^{4} \frac{(20\theta)^y e^{-20\theta}}{y!}$