# STAT 2006 Suggested Solution 4

1. $\hat{\mu} = 320.1$. $s^2 = 45.5612244897959$. $s = 6.7499055171014$.

| $k$ | $b_k$ | $\Phi^{-1}(b_k)$ | $A_k$ | $Y_k$ |
|---|---|---|---|---|
| 0 | 0 | $-\infty$ | | |
| 1 | 0.125 | 312.34 | $(-\infty, 312.34]$ | 7 |
| 2 | 0.25 | 315.54 | $(312.34, 315.54]$ | 5 |
| 3 | 0.375 | 317.94 | $(315.54, 317.94]$ | 5 |
| 4 | 0.5 | 320.1 | $(317.94, 320.1]$ | 10 |
| 5 | 0.625 | 322.3 | $(320.1, 322.3]$ | 4 |
| 6 | 0.75 | 324.66 | $(322.3, 324.66]$ | 6 |
| 7 | 0.875 | 327.86 | $(324.66, 327.86]$ | 5 |
| 8 | 1 | $\infty$ | $(327.86, \infty]$ | 8 |
| | | | | 50 |

$p_{1o} = ... = p_{8o} = 0.125$. Test $H_0 : X \sim N(\hat{\mu}, s^2)$. $n \times p_{io} = 50 \times 0.125 = 6.25$.

$$Q_7 = 4.4 < 11.07 = \chi^2(5, 0.05)$$

Accept $H_0$ at 5% level of significance.

2. $k = 5$. $n = 79$. $p_{0o} = \mathbb{P}(X = 0) = C_0^4 p^0(1-p)^4 = (1-p)^4$. $p_{1o} = 4p(1-p)^3$. $p_{2o} = 6p^2(1-p)^2$. $p_{3o} = 4p^3(1-p)$. $p_{4o} = p^4$.

(a) $p = 0.5$. $p_{0o} = 0.0625$. $p_{1o} = 0.25$. $p_{2o} = 0.375$. $p_{3o} = 0.25$. $p_{4o} = 0.0625$.
Test $H_0 : X \sim b(4, 0.5)$.

$$\begin{aligned} Q_4 &= \frac{(13 - 79 \times 0.0625)^2}{79 \times 0.0625} + \frac{(22 - 79 \times 0.25)^2}{79 \times 0.25} + \frac{(24 - 79 \times 0.375)^2}{79 \times 0.375} + \\ &\quad \frac{(19 - 79 \times 0.25)^2}{79 \times 0.25} + \frac{(1 - 79 \times 0.0625)^2}{79 \times 0.0625} \\ &= 17.6582278481013 \approx 17.6582 > 9.488 = \chi^2(4, 0.05) \end{aligned}$$

Reject $H_0$ at 5% level of significance.

(b) Since $f(x|p) = C_x^4 p^x (1-p)^{4-x}$ where $x = 0, ..., 4$,

$$L(p|X_1, ..., X_{79}) = \prod_{i=1}^{79} C_{X_i}^4 p^{X_i}(1-p)^{4-X_i} = \left(\prod_{i=1}^{79} C_{X_i}^4\right) p^{\sum_{i=1}^{79} X_i}(1-p)^{356-\sum_{i=1}^{79} X_i}$$

$$l(p|X_1, ..., X_{79}) = \ln(L(p|X_1, ..., X_{79})) = \ln\left(\prod_{i=1}^{79} C_{X_i}^4\right) + \left(\sum_{i=1}^{79} X_i\right)\ln p + \left(356 - \sum_{i=1}^{79} X_i\right)\ln(1-p)$$

$$\frac{\partial l}{\partial p} = \frac{\sum_{i=1}^{79} X_i}{p} - \frac{356 - \sum_{i=1}^{79} X_i}{1-p} = 0 \Rightarrow \hat{p} = \frac{\sum_{i=1}^{79} X_i}{356} = 0.414556962025316 \approx 0.4146$$

$p_{0o} = 0.1175$. $p_{1o} = 0.3327$. $p_{2o} = 0.3534$. $p_{3o} = 0.1668$. $p_{4o} = 0.0295$.
Test $H_0 : X \sim b(4, 0.4146)$.

$$\begin{aligned} Q_4 &= \frac{(13 - 79 \times 0.1175)^2}{79 \times 0.1175} + \frac{(22 - 79 \times 0.3327)^2}{79 \times 0.3327} + \frac{(24 - 79 \times 0.3534)^2}{79 \times 0.3534} + \\ &\quad \frac{(19 - 79 \times 0.1668)^2}{79 \times 0.1668} + \frac{(1 - 79 \times 0.0295)^2}{79 \times 0.0295} \\ &= 6.07162304175273 \approx 6.0716 < 7.815 = \chi^2(3, 0.05) \end{aligned}$$

Accept $H_0$ at 5% level of significance.

3. Under $H_0$, the expected count is $224/4 = 56$. The test statistic is

$$Q = \frac{(42-56)^2}{56} + \frac{(64-56)^2}{56} + \frac{(53-56)^2}{56} + \frac{(65-56)^2}{56} = 6.25 < 7.815 = \chi^2_{0.05}(3).$$

Do not reject the null hypothesis at $\alpha = 0.05$.

4. (a) $\mathbb{E}\left[\bar{X}\right] = \dfrac{\sum \mathbb{E}[X_i]}{n} = \dfrac{n\mathbb{E}[X_1]}{n} = \mathbb{E}[X_1] = p;$

$Var\left(\bar{X}\right) = \dfrac{Var\left(\sum X_i\right)}{n^2} = \dfrac{Var(X_1)}{n} = \dfrac{p(1-p)}{n}$

(b)

$$\begin{aligned} & f(X|p) = p^X(1-p)^{1-X} \\ \Rightarrow\ & \ln f(X|p) = X\ln p + (1-X)\ln(1-p) \\ \Rightarrow\ & \frac{\partial \ln f}{\partial p} = \frac{X}{p} + \frac{X-1}{1-p} \\ \Rightarrow\ & \frac{\partial^2 \ln f}{\partial p^2} = -\frac{X}{p^2} + \frac{X-1}{(1-p)^2} \\ \Rightarrow\ & \mathbb{E}\left[\frac{X}{p^2} - \frac{X-1}{(1-p)^2}\right] = \frac{p}{p^2} - \frac{p-1}{(1-p)^2} = \frac{1}{p(1-p)} \end{aligned}$$

Therefore, the Rao-Cramér lower bound $= \dfrac{p(1-p)}{n}$.

(c) Efficiency of $\bar{X} = \dfrac{\frac{p(1-p)}{n}}{\frac{p(1-p)}{n}} = 1$. Therefore, $\bar{X}$ is the best unbiased estimator of $p$.

5. (a)

$$\begin{aligned} L(\sigma^2|X_1,...,X_n) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{X_i^2}{2\sigma^2}} \\ &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum_{i=1}^n X_i^2}{2\sigma^2}} \\ &= (2\pi)^{-\frac{n}{2}}(\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum_{i=1}^n X_i^2}{2\sigma^2}} \end{aligned}$$

$$l(\sigma^2|X_1,...,X_n) = \ln L(\sigma^2|X_1,...,X_n) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{\sum_{i=1}^n X_i^2}{2\sigma^2}$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2}\frac{\sum_{i=1}^n X_i^2}{(\sigma^2)^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^n X_i^2}{n}$$

$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}\left[\frac{\sum_{i=1}^n X_i^2}{n}\right] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^n X_i^2\right] = \frac{1}{n} n \mathbb{E}\left[X_1^2\right] = \sigma^2$$

(b) $Var(\hat{\sigma}^2) = \dfrac{1}{n}Var(X_1^2)$. But $\dfrac{X_1}{\sigma} \sim N(0,1)$, then $\left(\dfrac{X_1}{\sigma}\right)^2 \sim \chi^2(1)$. $Var\left(\left(\dfrac{X_1}{\sigma}\right)^2\right) = 2$

Therefore, $Var(\hat{\sigma}^2) = \dfrac{2\sigma^4}{n}$.

(c) $\dfrac{\partial^2 \ln f}{\partial(\sigma^2)^2} = \dfrac{1}{2\sigma^4} - \dfrac{X^2}{\sigma^6} \Rightarrow \mathbb{E}\left[-\dfrac{\partial^2 \ln f}{\partial(\sigma^2)^2}\right] = \dfrac{1}{2\sigma^4}$

Therefore, the Rao-Cramér lower bound $= \dfrac{2\sigma^4}{n}$.

6. (a) If $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

So

$$E(S^2) = E\left[\frac{\sigma^2}{n-1}\frac{(n-1)S^2}{\sigma^2}\right] = \frac{\sigma^2}{n-1}(n-1) = \sigma^2.$$

(b) Note that

$$\frac{\partial^2}{\partial(\sigma^2)^2} \ln\left[\frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-\mu)^2/(2\sigma^2)}\right] = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}$$

and

$$-E\left[\frac{1}{2\sigma^4} - \frac{(X-\mu)^2}{\sigma^6}\right] = -\frac{1}{2\sigma^4} + \frac{E(X-\mu)^2}{\sigma^6} = -\frac{1}{2\sigma^4} + \frac{\sigma^2}{\sigma^6} = \frac{1}{2\sigma^4}.$$

Therefore, the Rao-Cramér lower bound $= \dfrac{2\sigma^4}{n}$. Then

$$\mathrm{Var}(S^2) = \mathrm{Var}\left[\frac{\sigma^2}{n-1}\frac{(n-1)S^2}{\sigma^2}\right] = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n}.$$

So $S^2$ does not attain the Rao-Cramér lower bound.