STAT4003 Homework Assignment (#1)
(Due 30 Sept 2020)

1. Find the moment generating function of the following random variables

   (i) Binomial(n, p);

   (ii) Poisson(λ);

   (iii) Gamma(α, β), i.e., the pdf is given by

   $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

2. (a) In the casino game roulette, the probability of winning with a bet on red is $p = 18/38$. Let X be the number of winning bets out of 100 independent bets that are placed. Find $P(X > 50)$ approximately.

   (b) Let $\{X_i, 1 \leq i \leq 16\}$ be a random sample form a distribution with pdf $f(x) = 3x^2, 0 < x < 1$. Approximate $P(\bar{X} < 0.5)$.

3. Let X_1 and X_2 be a random sample from $N(\mu, 1)$.

   (i) Find $P(X_1 - X_2 < 1)$;

   (ii) Prove that $X_1 - X_2$ and $X_1 + X_2$ are independent.

4. If X and Y are independent standard normal random variables. Show that X/Y has a t-distribution with 1 degree of freedom, which is also called the Cauchy distribution.

5. Let $\{X_{i1}, \cdots, X_{in_i}\}$ be a random sample from $N(\mu_i, \sigma^2)$, $i = 1, 2$. Assume that the random samples are independent. Prove that

   $$\mathcal{S}_1^2/\mathcal{S}_2^2$$

   has an F-distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, where $\mathcal{S}_i^2, i = 1, 2$ are the sample variance of the random samples.

6. Let $X_1, X_2, \cdots, X_n$ be a random sample from the uniform distribution $U(0,\ 1)$. Find the pdf of the ith smallest order statistic $X_{(i)}$ and its expectation and variance.

7. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, 1)$. Define

   $$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k}(X_i - \mu) \text{ and } \tilde{X}_k = \frac{1}{n-k}\sum_{i=k+1}^{n}(X_i - \mu)$$

   For $1 \leq k \leq n - 1$,

   (i) What is the distribution of $\bar{X}_k + \tilde{X}_k$?

   (ii) What is the distribution of $k\bar{X}_k^2 + (n-k)\tilde{X}_k^2$?

   (iii) What is the distribution of $k\bar{X}_k^2/((n-k)\tilde{X}_k^2)$?