MATH1520 Autumn 2018
Homework 2 Solution

1. Determine the points of discontinuity of the function:

$$f(x) = \frac{x^2 - 7x + 1}{x^2 - 2x}.$$

Answer. $f(x) = \dfrac{x^2 - 7x + 1}{x^2 - 2x}$. is continuous everywhere except at x=0 and x=2.

2. Suppose $f(x)$ and $g(x)$ are continuous at $x = 1$ with $f(1) = 1$, $g(1) = 10$. Compute

$$\lim_{x\to 1} \left| \frac{f(x)^2 - g(x)}{f(x) + 2g(x)} \right|. \tag{1}$$

Answer.

$$\begin{aligned} \lim_{x\to 1} \left| \frac{f(x) - g(x)}{f(x) + 2g(x)} \right| &= \left| \frac{1^2 - 10}{1 + 2 \times 10} \right| \\ &= \frac{3}{7}. \end{aligned}$$

3. For what values of a and b is

$$f(x) = \begin{cases} -2 & x \le -1 \\ ax - b & -1 < x < 1 \\ 3 & x \ge 1 \end{cases} \tag{2}$$

continuous at every x?

Answer. Since $f(x)$ is continuous at $x = -1$ and $x = 1$,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax - b = -a - b = f(-1) = -2$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax - b = a - b = f(1) = 3$$

. After solving the equations, we get $a = \frac{5}{2}$ and $b = -\frac{1}{2}$.

4. Determine whether $f(x)$ is continuous at $x = 0$:

$$f(x) = \begin{cases} \dfrac{x(x+1)}{|x|}, & \text{if } x \ne 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Answer.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x(x+1)}{x} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} \frac{x(x+1)}{-x} = -1$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

This shows that $\lim_{x\to 0} f(x)$ doesn't exist. Hence $f(x)$ is not continuous at $x = 0$

5. Let $f(x) = x^3 - \dfrac{3}{x}$. Show that there exists $c \in [1, 2]$ such that $f(c) = 3$.

   Answer. Since $f(1) = -2 < 3$ and $f(2) = \frac{13}{2} > 3$, we see that 3 is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there exists $c \in [1, 2]$ such that $f(c) = 3$.

6. Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

   Answer. Let $f(x) = x^3 - x - 1$. Since $f(1) = -1 < 0$ and $f(2) = 5 > 0$, we see that 0 is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

7. Use the first principle to find the derivative of $f(x) = x^2 + 2x + x^{-1}$.

   Answer.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) + (x+h)^{-1} - x^2 - 2x - x^{-1}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 2h + (x+h)^{-1} - x^{-1}}{h} \\
&= \lim_{h\to 0} \left(2x + h + 2 + \frac{1}{h} \cdot \left(\frac{1}{x+h} - \frac{1}{x}\right)\right) \\
&= 2x + 2 + \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= 2x + 2 + \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= 2x + 2 - \frac{1}{x^2}
\end{aligned}$$

8. Use the first principle to find the derivative of $f(x) = \dfrac{x^2}{x+1}$.

Answer.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{(x+h)^2}{x+h+1} - \frac{x^2}{x+1}}{h} \\
&= \lim_{h\to 0} \frac{(x+1)(x+h)^2 - x^2(x+h+1)}{h(x+1)(x+h+1)} \\
&= \lim_{h\to 0} \frac{(x^3+2x^2h+xh^2+x^2+2xh+h^2)-(x^3+x^2h+x^2)}{h(x+1)(x+h+1)} \\
&= \lim_{h\to 0} \frac{x^2h+xh^2+2xh+h^2}{h(x+1)(x+h+1)} \\
&= \lim_{h\to 0} \frac{x^2+xh+2x+h}{(x+1)(x+h+1)} \\
&= \frac{x^2+2x}{(x+1)^2}
\end{aligned}$$

9. Use the first principle to find the derivative of $f(x) = \sqrt{x^2+1}$.

Answer. First note that

$$\begin{aligned}
&\left(\sqrt{(x+h)^2+1} - \sqrt{x^2+1}\right) \cdot \left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right) \\
&= (x+h)^2 + 1 - (x^2+1) \\
&= 2xh + h^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{2xh+h^2}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
&= \frac{x}{\sqrt{x^2+1}}
\end{aligned}$$

10. Use the first principle to find the derivative of $f(x) = x^{1/4}$.
Hint: Use $a^4 - b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$.

Answer. Let $a = (x+h)^{1/4}, b = x^{1/4}$. Using this formula, we have

$$\begin{aligned}
&\left[(x+h)^{1/4} - x^{1/4}\right] \cdot \left[(x+h)^{3/4} + (x+h)^{1/2}x^{1/4} + (x+h)^{1/4}x^{1/2} + x^{3/4}\right] \\
&= (x+h) - x \\
&= h.
\end{aligned}$$

Thus we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h}{(x+h)^{3/4} + (x+h)^{1/2}x^{1/4} + (x+h)^{1/4}x^{1/2} + x^{3/4}} \\ &= \lim_{h\to 0} \frac{1}{(x+h)^{3/4} + (x+h)^{1/2}x^{1/4} + (x+h)^{1/4}x^{1/2} + x^{3/4}} \\ &= \frac{1}{4}x^{-3/4}. \end{aligned}$$

11. Find the value of a that makes the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0, \\ x^2 - 5x, & \text{if } x \geq 0. \end{cases}$$

Answer.

$$\lim_{h\to 0^+} \frac{g(h) - g(0)}{h} = \lim_{h\to 0^+} \frac{h^2 - 5h}{h} = -5$$

$$\lim_{h\to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h\to 0^-} \frac{ah}{h} = a$$

Since $f(x)$ is differentiable at $x = 0$, $a = -5$. Clearly $f(x)$ is differentiable if $x > 0$ or $x < 0$, so when $a - -5$, $f(x)$ is differentialble everywhere.

12. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

(a) $\dfrac{d}{dx}(uv)$

(b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$

(d) $\dfrac{d}{dx}(7v - 2u)$

Answer. When $x = 1$

(a)

$$\begin{aligned} \frac{d}{dx}(uv) &= v\frac{d}{dx}(u) + u\frac{d}{dx}(v) \\ &= v(1)u^{'}(1) + u(1)v^{'}(1) \\ &= 10 \end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{u^{'}(1)v(1)-v^{'}(1)u(1)}{v(1)^2}\\ &= \frac{0\times 5-(-1)\times 2}{5^2}\\ &= \frac{2}{25}\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dx}\left(\frac{v}{u}\right) &= \frac{v^{'}(1)u(1)-u(1)^{'}v(1)}{u(1)^2}\\ &= \frac{-1\times 2-0\times 5}{2^2}\\ &= -\frac{1}{2}\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dx}(7v-2u) &= 7\frac{dv}{dx}-2\frac{du}{dx}\\ &= 7v^{'}(1)-2u^{'}(1)\\ &= -7\end{aligned}$$

13. Compute the derivatives of the following functions.

(a) $f(x)=3x^2+\sqrt{x}$

(b) $g(x)=e^{4x^3}$

(c) $h(x)=\sqrt{x^2+1}$

(d) $p(x)=(1+e^x)(x^2+1)$

(e) $q(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

(f) (new starting from this one) $y(x)=\dfrac{2x+5}{3x-2}$

(g) $w(x)=(2x-7)^{-1}(x+5)$

(h) $r(t)=2\left(\frac{1}{\sqrt{t}}+\sqrt{t}\right)$

(i) $y(x)=\sqrt[3]{x^{8.6}}+2e^{2.3}$

(j) $w(z)=3z^2e^{3z}$

(k) $w(x)=\left(\dfrac{1+3x}{3x}\right)(3-x)$

(l) $f(t)=\dfrac{t^2+3}{(t-1)^3+(t+1)^3}$

Answer.

(a) For $f(x)=3x^2+\sqrt{x}$, we simply use the power rule.

$$\begin{aligned}f'(x) &= (3x^2)'+(x^{\frac{1}{2}})'\\ &= 6x+\frac{1}{2}x^{-\frac{1}{2}}\\ &= 6x+\frac{1}{2\sqrt{x}}\end{aligned}$$

(b) For $g(x) = e^{4x^3}$, we use the chain rule.

$$\begin{aligned} g'(x) &= (e^{4x^3})' \\ &= e^{4x^3}(4x^3)' \\ &= 12x^2 e^{4x^3} \end{aligned}$$

(c) For $h(x) = \sqrt{x^2+1}$, we again use the chain rule.

$$\begin{aligned} h'(x) &= (\sqrt{x^2+1})' \\ &= \frac{1}{2\sqrt{x^2+1}}(x^2+1)' \\ &= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

(d) For $p(x) = (1+e^x)(x^2+1)$, we apply the product rule.

$$\begin{aligned} p'(x) &= (1+e^x)'(x^2+1) + (1+e^x)(x^2+1)' \\ &= (e^x)(x^2+1) + (1+e^x)(2x) \\ &= (x+1)^2 e^x + 2x \end{aligned}$$

(e) For $q(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, we use the quotient rule.

$$\begin{aligned} q'(x) &= \frac{(e^x - e^{-x})'(e^x + e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\ &= \frac{(2e^x)(2e^{-x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} \end{aligned}$$

14. By using the logarithmic differentiation, compute $\dfrac{dy}{dx}$:

(a) $y = (2x+1)^3(x-1)^4\sqrt{(3x+2)^5}$.

(b) $y = x^{x^2}$.

(c) $y = (\ln x + 1)^{\ln x}$.

Answer.

(a) Take ln, $\ln y = 3\ln(2x+1) + 4\ln(x-1) + \frac{5}{2}\ln(3x+2)$.

Take $\dfrac{d}{dx}$, $\dfrac{y'}{y} = \dfrac{6}{2x+1} + \dfrac{4}{x-1} + \dfrac{15}{2(3x+5)}$.

$\dfrac{dy}{dx} = (2x+1)^3(x-1)^4\sqrt{(3x+2)^5}\left(\dfrac{6}{2x+1} + \dfrac{15}{2(3x+5)} + \dfrac{4}{x-1}\right)$

(b) Take ln, $\ln y = x^2 \ln x$.

Take $\dfrac{d}{dx}$, $\dfrac{y'}{y} = 2x \ln x + x$.

$$\frac{dy}{dx} = x^{x^2}\left(2x \ln x + x\right)$$

(c) Take ln, $\ln y = \ln x(\ln x + 1)$.

Take $\dfrac{d}{dx}$, $\dfrac{y'}{y} = \dfrac{1}{x}(\ln x + 1) + (\ln x)\dfrac{1}{x} = \dfrac{2\ln x + 1}{x}$.

$$\frac{dy}{dx} = (\ln x + 1)^{\ln x}\left(\frac{2\ln x + 1}{x}\right)$$