MATH1520 Autumn 2018
Homework 4 Solution

1. Use the second derivative test to find the relative minimum and the relative maximum of the function

   (a) $f(x) = x^3 + 3x^2 + 1$

   (b) $f(x) = (x^2 - 9)^2$

   (c) $f(x) = x + \dfrac{1}{x}$

   (d) $f(x) = \dfrac{x^2}{x-2}$

   (e) $h(t) = \dfrac{1}{1+t^2}$

   Answer. (a) $f'(x) = 3x^2 + 6x = 3x(x+2)$. Let $f'(x) = 0$ and we have two critical numbers: $x = 0, -2$.
   $f''(x) = 6x + 6$.
   $f''(0) = 6 > 0$. This means $(0, f(0)) = (0, 1)$ is a relative minimum of f .
   $f''(-2) = -6 < 0$. This means $(-2, f(-2)) = (-2, 5)$ is a relative maximum of f.

   (b) $f'(x) = 4(x^2 - 9)x = 4x(x+3)(x-3)$. Let $f'(x) = 0$ and we have three critical numbers: $x = 0, -3, 3$.
   $f''(x) = 12x^2 - 36$.
   $f''(0) = -36 < 0$. This means $(0, f(0)) = (0, 81)$ is a relative maximum of f.
   $f''(3) = 72 > 0$. This means $(3, f(3)) = (3, 0)$ is a relative minimum of f.
   $f''(-3) = 72 > 0$. This means $(-3, f(-3)) = (-3, 0)$ is a relative minimum of f.

   (c) $f'(x) = 1 - \frac{1}{x^2}$. Let $f'(x) = 0$ and we have two critical numbers: $x = 1, -1$.
   $f''(x) = \frac{1}{2x^3}$.
   $f''(-1) < 0$. This means $(-1, f(-1)) = (-1, -2)$ is a relative maximum of f.
   $f''(1) > 0$. This means $(1, f(1)) = (1, 2)$ is a relative minimum of f.

   (d) $f'(x) = \frac{2x(x-2)-x^2}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$. Let $f'(x) = 0$ and we have two critical numbers: $x = 0, 4$.
   $f''(x) = \frac{8}{(x-2)^3}$.
   $f''(0) = -1 < 0$. Thus $(0, f(0) = (0, 0)$ is a relative maximum of f.
   $f''(4) = 1 > 0$. Thus $(4, f(4) = (4, 8)$ is a relative minimum of f.

   (e) $h'(t) = -\frac{2t}{(t^2+1)^2}$. Let $f'(x) = 0$ and we have one critical number: $t = 0$.
   $h''(t) = \frac{6t^2-2}{(t^2+1)^3}$. $h''(0) = -2 < 0$. Thus $(0, f(0)) = (0, 1)$ is a relative maximum of f.

2. The second derivative f'' of a function is given. In each case, use this information to determine where the graph of $f(x)$ is concave upward and concave downward and find

all values of x for which an inflection point occurs. [You are not required to find $f(x)$ or the y coordinates of the inflection points.]

(a) $f''(x) = x^2(x-3)(x-1)$

(b) $f''(x) = \dfrac{x^2+x-2}{x^4+2}$

Answer. (a) $f''(x)$ has 3 zeros :$0, 1, 3$. Analyze the positivity of f'', we have the following form:

x	$(-\infty, 0)$	0	$(0,1)$	1	$(1,3)$	3	$(3,\infty)$
$f''(x)$	-	0	$+$	0	-	0	$+$
concavity	down	inflection pt	up	inflection pt	down	inflection pt	up

(b)

$$f''(x) = \frac{(x-1)(x+2)}{x^4+2}.$$

f'' has two zeros : $-2, 1$.

Analyze the positivity of f'', we have the following table:

x	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,\infty)$
$f''(x)$	$+$	0	-	0	$+$
concavity	up	inflection pt	down	inflection pt	up

3. Sketch the graph of a function f that has all the following properties:

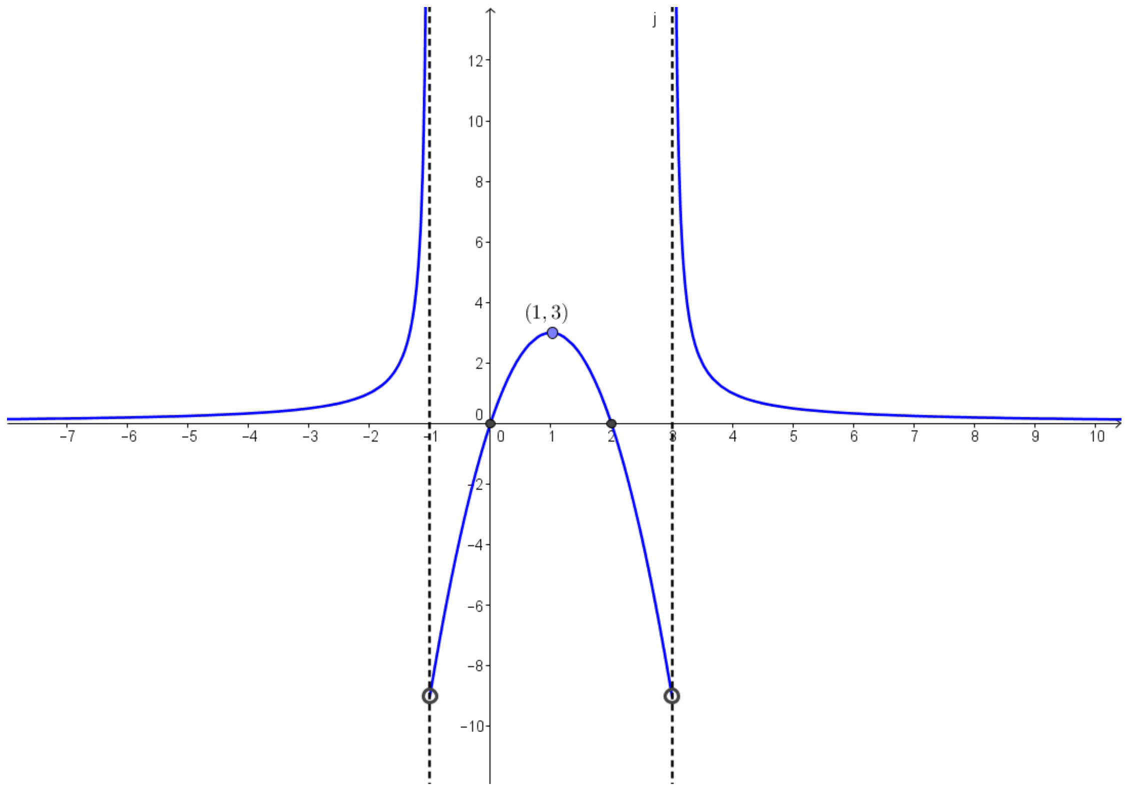
   (a) The graph has discontinuities at $x = -1$ and $x = 3$

   (b) $f'(x) > 0$ for $x < 1$, $x \neq -1$

   (c) $f'(x) < 0$ for $x > 1$, $x \neq 3$

   (d) $f''(x) > 0$ for $x < -1$ and $x > 3$ and $f''(x) < 0$ for $-1 < x < 3$

   (e) $f(0) = 0 = f(2)$, $f(1) = 3$



Answer.

4. Do the global extrema for the function

$$f(x) = x^2 - \frac{2}{x}, \; x \in [-2, -\frac{1}{2}]$$

exist? If yes, find them.

Answer. Since f is continuous on $[-2, -\frac{1}{2}]$, by the EVT, f has absolute maximum and minimum on $[-2, -\frac{1}{2}]$.

$$f'(x) = 2x + \frac{2}{x^2}.$$

Let $f'(x) = 0$ and we have one critical number: $x = -1$ in the defining domain.

$$f(-2) = 5, \quad f(-1) = -2, \quad f(-\frac{1}{2}) = 4\frac{1}{4}$$

Therefore, f has its absolute maximum at $x = -2$ and absolute minimum at $x = -1$.

5. Let $f(x) = \dfrac{x^2}{(x-2)^2}$.

(a) Find the domain of f.

(b) Find the intercepts, if any.

(c) Find the location of any vertical asymptotes of f.

(d) Find the horizontal asymptotes.

(e) Find the critical points of f.

(f) Find the intervals of increasing, decreasing.

(g) Find the possible points of inflection of f.

(h) Find the intervals of concave up and down.

(i) Sketch the graph of the function.

Answer.

(a) The denominator cannot be zero, thus the domain is $\mathbb{R}\backslash\{2\}$.

(b) $(0, 0)$.

(c)

$$\lim_{x\to 2^+} \frac{x^2}{(x-2)^2} = +\infty \qquad \lim_{x\to 2^-} \frac{x^2}{(x-2)^2} = +\infty$$

Therefore, f has one vertical asymptote: $x = 2$.

(d)

$$\lim_{x\to +\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to +\infty} \frac{1}{(1-\frac{2}{x})^2} = 1 \qquad \lim_{x\to -\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to -\infty} \frac{1}{(1-\frac{2}{x})^2} = 1$$

f has one horizontal asymptote: $y = 1$.

(e)

$$f'(x) = \frac{-4x}{(x-2)^3}.$$

Let $f'(x) = 0$ and we get one critical numbers: $x = 0$.
Therefore, there is one critical point: $(0, f(0)) = (0, 0)$.

(f) Analyze the derivative, we have the following table :

x	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	-	+	-
monotonicity	decrease	increase	decrease

Besides, $(0, 0)$ is a relative minimum point.

(g)

$$f''(x) = \frac{8(x+1)}{(x-2)^4}.$$

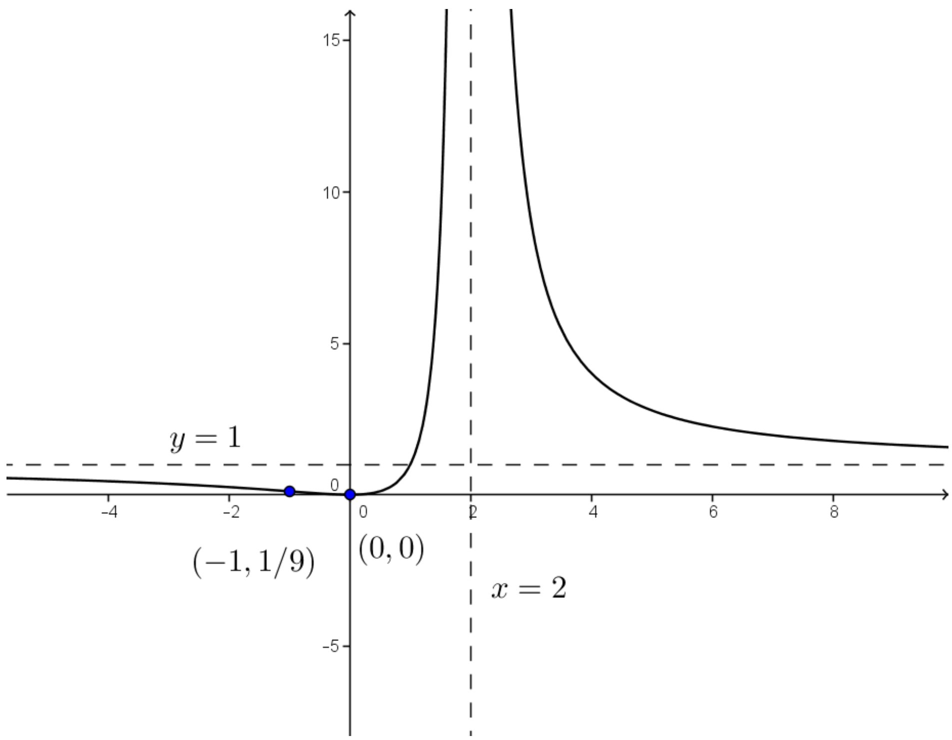
The only zero of f'' is $x = -1$. This is a possible point of inflection.

(h) Analyze the second derivative:

x	$(-\infty, -1)$	$(-1, 2)$	$(2, +\infty)$
$f''(x)$	-	+	+
concavity	down	up	up

$(-1, f(-1)) = (-1, 1/9)$ is a point of inflection.

(i)



6. Suppose the graph $y = f(x)$ is concave upward. Show that the graph $y = f(x)$ lies above the tangent to it at $x = a$.

   Hint.

   (a) Find the equation of the tangent in terms of a, $f(a)$ and $f'(a)$.

   (b) Let $g(x) = f(x) - f(a) - f'(a)(x - a)$. Show that $g(x)$ is minimum at $x = a$.

   Answer.

   (a) The slope at the point $(a, f(a))$ is $f'(a)$. Therefore the equation of the tangent line is $y = f'(a)(x - a) + f(a)$.

   Let $g(x) = f(x) - f(a) - f'(a)(x - a)$.

   $$g'(x) = f'(x) - f'(a).$$

   Let $g'(x) = 0$. We have $f'(x) = f'(a)$. Since f is concave upward, we have $f'' > 0$ for any x. Therefore, f' is an increasing function, which implies $x = a$ is the only solution to $f'(x) = f'(a)$. Hence, the only critical number of $g(x)$ is $x = a$.
   Also, $g''(a) = f''(a) > 0$. Thus $x = a$ is a relative minimum. By analyzing the derivative, it is actually a global minimum. $g(a) = 0$. This means $g(x) > g(a) = 0$ for any x, i.e., $f(x) > f'(a)(x-a)+f(a)$, i.e. the graph of f lies above the tangent line.

7. Find a point on the curve $y = x^2$ that is closest to the point $(18, 0)$.

Answer. The squared distance between (x, x^2) and $(18, 0)$ is $f(x) = (x - 18)^2 + x^4$ for $x \in \mathbb{R}$.

$$f'(x) = 2(x - 2)\left(2x^2 + 4x + 9\right).$$

The only critical number is $x = 2$.

$f''(x) = 2\left(6x^2 + 1\right) > 0$ for any x. Thus f has a relative minimum at $x = 2$. By analyzing the derivative, it is actually a global minimum. Thus $(2, 4)$ is closest to $(18, 0)$ on the curve .

8. When a resistor of R ohms is connected across a battery with electromotive force U volts and internal resistance r ohms, a current of I amperes will flow, generating P watts of power, where

$$I = \frac{U}{r + R} \quad \text{and} \quad P = I^2 R$$

Assuming r, U are constants, what choice of R results in maximum power?

Answer.

$$P = I^2 R = \frac{U^2}{(R + r)^2} R \quad (R \geq 0)$$

$$\frac{dP}{dR} = \frac{U^2(r - R)}{(r + R)^3}$$

Let $P' = 0 \Longrightarrow R = r$,

x	$(0, r)$	r	$(r, +\infty)$
$f'(x)$	$+$	0	$-$
Monotonicity	$\nearrow$		$\searrow$

Thus when $R = r$,the system generates the maximal power: $\dfrac{U^2}{4r}$

9. When the price of a certain commodity is p dollars per unit, the manufacturer is willing to supply x hundred units, where

$$3p^2 - x^2 = 12.$$

How fast is the supply changing when the price is \$4 per unit and is increasing at the rate of 87 cents per month?

Answer.

In this case $p = 4$ dollars and $\frac{dp}{dt} = 0.87$ dollars per month, thus $x = 6$ hundred units. Take the derivative of both sides of the equation wrt t.

$$6p\frac{dp}{dt} - 2x\frac{dx}{dt} = 0$$

i.e. $\dfrac{dx}{dt} = 3\dfrac{p}{x}\dfrac{dp}{at} = 1.74$ dollars per month.

10. A storm at sea has damaged an oil rig. Oil spills from the rupture at the constant rate of 60 ft^3/min, forming a slick that is roughly circular in shape and 3 inches thick.

(a) How fast is the radius of the slick increasing when the radius is 70 feet?

(b) Suppose the rupture is repaired in such a way that the flow is shut off instantaneously. If the radius of the slick is increasing at the rate of 0.2 ft/min when the flow stops, what is the total volume of oil that spilled onto the sea?

Answer.

(a) We can think of the slick as a cylinder of oil of radius r feet and thickness $h = \frac{3}{12} = 0.25$ feet. Such a cylinder will have volume

$$V = \pi r^2 h = 0.25\pi r^2 \text{ ft}^3$$

Differentiating implicitly with respect to t, we get

$$\frac{dV}{dt} = 0.25\pi\left(2r\frac{dr}{dt}\right) = 0.5\pi r\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi r}\frac{dV}{dt}$$

Since $\dfrac{dV}{dt} = 60$ all the time, when $r = 70$, we obtain

$$\frac{dr}{dt} = \frac{2}{\pi(70)}(60) \approx 0.55 \text{ ft/min}$$

Thus, when the radius is 70 feet, it is increasing at about 0.55 ft/min.

(b) We can compute the total volume of oil in the spill if we know the radius of the slick at the instant the flow stops. Since $\dfrac{dr}{dt} = 0.2$ at that instant, we have

$$60 = 0.5\pi r(0.2) \quad \Rightarrow \quad r \approx 191 \text{ feet}$$

Therefore, the total amount of oil spilled is

$$V = 0.25\pi(191)^2 \approx 28,652 \text{ ft}^3$$