2018-19 MATH1520AB

Midterm I (2018 Oct 4)

1. (a) To make the function well-defined, we need

$$-\frac{x^2}{4} + 4 > 0.$$

i.e.

$$\begin{aligned} & x^2 < 16 \\ & (x+4)(x-4) < 0 \\ \Rightarrow \quad & x \in (-4, 4). \end{aligned}$$

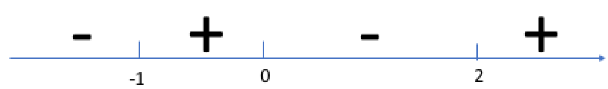
(b) To make the function well-defined, we need

$$(2x-4)(x+1) \neq 0$$

and

$$\frac{x}{(2x-4)(x+1)} \geq 0.$$

Let $x = 0$, $2x - 4 = 0$ and $x + 1 = 0$, we have $x = 0, 2$, or -1. Label all these points in the number line. And examine the sign of $\dfrac{x}{(2x-4)(x+1)}$ on each interval. We have



These two inequalities imply $x \in (2, \infty] \cup (-1, 0]$.

2. (a)

$$(f \circ g)(x) = \sqrt{\ln(4 - x^2)}$$

First consider g we have

$$4 - x^2 > 0 \quad \Rightarrow \quad -2 < x < 2$$

Then consider $f \circ g$ we have

$$\ln(4 - x^2) \geq 0 \quad \text{i.e. } 4 - x^2 \geq 1 \quad \Rightarrow \quad -\sqrt{3} \leq x \leq \sqrt{3}.$$

These two inequalities inmply $x \in [-\sqrt{3}, \sqrt{3}]$.

(b)

$$(g \circ f)(x) = \ln(2-x) - 2.$$

First consider f we have

$$x + 2 \geq 0 \quad \Rightarrow \quad x \geq -2$$

Then consider $g \circ f$ we have

$$4 - (\sqrt{x+2})^2 > 0 \quad \Rightarrow \quad x < 2$$

These two inequalities inmply $x \in [-2, 2)$.

3. (a)

$$\lim_{x \to 1} \frac{x^3 - 2x + 5}{x^2 - 2} = \frac{\lim_{x \to 1}(x^3 - 2x + 5)}{\lim_{x \to 1}(x^2 - 2)} = \frac{1 - 2 + 5}{1 - 2} = -4$$

(b)

$$\lim_{x \to 0} \frac{x}{\sqrt{cx+1} - 1} = \lim_{x \to 0} \frac{x}{\sqrt{cx+1} - 1} \cdot \frac{\sqrt{cx+1} + 1}{\sqrt{cx+1} + 1} = \lim_{x \to 0} \frac{(\sqrt{cx+1} + 1)}{c} = \frac{2}{c}$$

(c) When $x \to -2^-$, the numerator $x - 4$ tends from below to -6 while the denominator $x^2 + 2x$ is nonnegative and tends from above to 0. Thus the limit tends to $-\infty$ as $x \to -2^-$.

$$\lim_{x \to -2^-} \frac{x - 4}{x^2 + 2x} = -\infty$$

(d) Note that $|t - 3| = -(t - 3)$ as $t \to 3^-$.

$$\begin{aligned}
\lim_{t \to 3^-} \frac{t^2 - 2t - 3}{|t - 3|} &= \lim_{t \to 3^-} \frac{t^2 - 2t - 3}{-(t - 3)} \\
&= \lim_{t \to 3^-} \frac{(t-3)(t+1)}{-(t-3)} \\
&= \lim_{t \to 3^-} -(t+1) \\
&= -4
\end{aligned} \tag{1}$$

(e)

$$\begin{aligned}
\lim_{x \to +\infty} \frac{\pi + x^{2/3} - x}{x^{1/2} + x - 10} &= \lim_{x \to +\infty} \frac{\pi x^{-1} + x^{-1/3} - 1}{x^{-1/2} + 1 - 10x^{-1}} \\
&= \frac{0 + 0 - 1}{0 + 1 - 0} \\
&= -1
\end{aligned} \tag{2}$$

(f)

$$\begin{aligned}
\lim_{x\to+\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-5}\right) &= \lim_{x\to+\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-5}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-5}}{\sqrt{x^2+3x}+\sqrt{x^2-5}} \\
&= \lim_{x\to+\infty}\frac{3x+5}{\sqrt{x^2+3x}+\sqrt{x^2-5}} \\
&= \lim_{x\to+\infty}\frac{3+5x^{-1}}{\sqrt{1+3x^{-1}}+\sqrt{1-5x^{-2}}} \\
&= \frac{3}{1+1} \\
&= \frac{3}{2}
\end{aligned} \tag{3}$$

(g) Note that $\sqrt{x^2}=-x$ if x is negative.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{x-2}{\sqrt{x^2-4}} &= \lim_{x\to-\infty}\frac{1-2x^{-1}}{-\sqrt{1-4x^{-2}}} \\
&= \frac{1}{-1} \\
&= -1
\end{aligned} \tag{4}$$

(h)

$$\begin{aligned}
\lim_{x\to-\infty}\left(1-\frac{1}{x^2}\right)^x &= \lim_{x\to-\infty}\left(1-\frac{1}{x}\right)^x\left(1+\frac{1}{x}\right)^x \\
&= \lim_{x\to-\infty}\left(1-\frac{1}{x}\right)^x\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x \\
&= e^{-1}e \\
&= 1
\end{aligned} \tag{5}$$

4. (a) $\lim_{x\to-1} f(x) = \lim_{x\to-1}|3^x-3| = \lim_{x\to-1}(3-3^x) = \frac{8}{3}$

   (b) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\ln(x^2+1)+2 = 2$

   (c) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}|3^x-3| = \lim_{x\to 0^-}(3-3^x) = 2$

   (d) Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 2$,

$$\lim_{x\to 0} f(x) = 2$$

5. Let $f(x)=e^x+2x-3$. Then

$$\begin{aligned}
f(0) &= e^0-3=-2<0, \\
f(1) &= e+2-3=e-1>0.
\end{aligned}$$

Since $f(x)$ is continuous, we can apply intermediate value theorem. So there exists $a\in(0,1)$ such that $f(a)=0$, which implies $e^a+2a=3$.

6.

$$\begin{aligned}
\frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{x+h+\sqrt{x+h}-x-\sqrt{x}}{h} = \lim_{h\to 0}\frac{h+(\sqrt{x+h}-\sqrt{x})}{h} \\
&= 1+\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = 1+\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= 1+\frac{1}{2\sqrt{x}}
\end{aligned}$$

7. (a)

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} ax = -2a$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} -ax^2+bx-4 = -4a-2b-4$$

(b) Since $f(x)$ is continuous at $x=-2$, we have

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = f(-2)$$

From (a), we know $-2a = -4a-2b-4$, hence $a+b=-2$.

(c) Since $f(x)$ is differentiable at $x=-2$, it should be continuous at $x=-2$. From (b), we know that

$$f(x) = -4a-2b-4 = -2a$$

$$\lim_{h\to 0^-}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0^-}\frac{a(-2+h)-(-2a)}{-2} = a$$

$$\begin{aligned}
\lim_{h\to 0^+}\frac{f(-2+h)-f(-2)}{h} &= \lim_{h\to 0^+}\frac{-a(-2+h)^2+b(-2+h)-4-(-4a-2b-4)}{-2} \\
&= 4a+b
\end{aligned}$$

Since $f(x)$ is differentiable,

$$\lim_{h\to 0^-}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0^+}\frac{f(-2+h)-f(-2)}{h}$$

hence $4a+b=a$, $3a+b=0$.

(d) Solve the equation

$$\begin{cases} a+b=-2 \\ 3a+b=0 \end{cases}$$

we get

$$\begin{cases} a=1 \\ b=-3 \end{cases}$$

From (c), $f(-2) = \lim_{h\to 0}\dfrac{f(-2+h)-f(-2)}{h} = a = 1$

(e) Use $a = 1$, $b = -3$, we get

$$f(x) = \begin{cases} x, & \text{if } x < -2, \\ -x^2 - 3x - 4, & \text{if } x \geq -2. \end{cases}$$

$$f^{'}(x) = \begin{cases} 1, & \text{if } x < -2, \\ 1, & \text{if } x = 1 \\ -2x - 3, & \text{if } x > -2. \end{cases}$$

(f) $\lim_{x \to -2^-} f^{'}(x) = \lim_{x \to -2^+} f^{'}(x) = 1$

$\Rightarrow \quad \lim_{x \to -2} f'(x) = 1 = f'(-2)$, hence $f'(x)$ is continuous at $x = -2$.

(g)

$$\begin{aligned} \lim_{h \to 0^-} \frac{f'(-2+h) - f'(-2)}{h} &= \lim_{h \to 0^-} \frac{1-1}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{f'(-2+h) - f'(-2)}{h} &= \lim_{h \to 0^-} \frac{-2(-2+h) - 3 - 1}{h} \\ &= -2 \end{aligned}$$

$$\lim_{h \to 0^-} \frac{f'(-2+h) - f'(-2)}{h} \neq \lim_{h \to 0^+} \frac{f'(-2+h) - f'(-2)}{h}$$

So $\lim_{h \to 0} \frac{f'(-2+h) - f'(-2)}{h}$ doesn't exist. $f'(x)$ is not differentiable at $x = -2$.

8. (a) If $a = b$, then

$$f(x) = x^2 - (a + a - 1)x + a^2$$

when $x = a$,

$$f(a) = 2a^2 - 2a^2 + a = a = \frac{a+b}{2}.$$

(b) If $a < b$, then

$$\begin{aligned} f(a) &= a^2 - (a + b - 1)a + ab = a < \frac{a+b}{2} \\ f(b) &= b^2 - (a + b - 1)b + ab = b > \frac{a+b}{2} \end{aligned}$$

Since $f(x)$ is a quadratic function, which is continuous on $[a, b]$, by intermediate value theorem, there exists some $c \in (a, b)$, such that $f(c) = \frac{a+b}{2}$.

(c) If $a > b$, then

$$\begin{aligned} f(a) &= a^2 - (a+b-1)a + ab = a > \frac{a+b}{2} \\ f(b) &= b^2 - (a+b-1)b + ab = b < \frac{a+b}{2} \end{aligned}$$

Since $f(x)$ is a quadratic function, which is continuous on $[b, a]$, by intermediate value theorem, there exists some $c \in (b, a)$, such that $f(c) = \frac{a+b}{2}$.

Therefore, $f(x)$ takes on the value $\frac{a+b}{2}$, where a and b are any two real numbers.