STAT4003 Homework Assignment (#2)
(Due Monday 26 October)

1. Consider a random sample $X_1, \cdots, X_n$ from a Weibull distribution

$$f(x|\,\theta, \beta) = (\beta/\theta)x^{\beta-1}\exp(-x^\beta/\theta), \quad x > 0$$

   where $\beta > 0$ is known and θ is a parameter.

   (i) Find the method of moment estimator $\tilde{\theta}$ and the MLE $\hat{\theta}$ of θ;

   (ii) Are $\tilde{\theta}$ and $\hat{\theta}$ unbiased estimators?

   (iii) Find the Cramer-Rao lower bound;

   (iv) Is $\hat{\theta}$ UMVUE?

2. Let $X_1, X_2, \cdots, X_n$ be a random sample from a uniform distribution on the interval $(\theta - 1, \theta + 1)$.

   (i) Find the method of moments estimator for θ;

   (ii) Is your estimator in part (i) an unbiased estimator of θ?

   (iii) Given the following 5 observations,

$$6.61, \quad 7.70, \quad 6.98, \quad 8.36, \quad 7.26$$

   give a point estimate of θ;

   (iv) The method of moments estimator has greater variance than the estimator $(1/2)(X_{(1)} + X_{(n)})$. Compute the value of this estimator for the 5 observations in (iii).

3. Let $X_1, \cdots, X_n$ be a random sample from the density function

$$f(x|\theta) = \theta\, x^{\theta-1}, \quad 0 < x < 1, \theta > 0.$$

   (i) Find the MLE $\hat{\theta}_n$ of θ and show that $\hat{\theta}_n$ is a consistent estimator of θ;

   (ii) Find the method of moments estimator of θ.

4. Let $X_1, ..., X_n$ be a random sample from distribution with the following probability density function

$$f(x;\theta) = \frac{1}{2\theta^3}x^2 e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty$$

   (i) Find the MLE $\hat{\theta}$ for θ;

   (ii) Find the MLE for $\tau(\theta) = \frac{1}{\theta}$;

   (iii) Find the C-R inequality for $\tau(\theta)$;

(iv) Is $\hat{\theta}$ unbiased? Is $\tau(\hat{\theta})$ unbiased?

(v) Find the asymptotic distribution of $\sqrt{n}(\tau(\hat{\theta}) - \tau(\theta))$.

5. Let $X_1, \cdots, X_n$ be a random sample from the following distribution

$$f(x|\theta) = \frac{x^2 e^{-x/\theta}}{2\theta^3}, \quad x > 0$$

Show that the UMVU estimator of θ is

$$T = \frac{1}{3n} \sum_{i=1}^{n} X_i.$$

6. Let $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$. Prove that $(\sum_{i=1}^n X_i, \sum_{i=1}^n (X_i - \bar{X})^2)$ is minimal jointly sufficient statistics.