1、(a)

$$\begin{aligned} X &\sim Poisson(\theta) \\ f_X(x;\theta) &= \theta^x e^{-\theta}/x! \\ &= \exp\{x\log\theta - \log(x!) - \theta\} \\ &= \exp\{-\theta - \log(x!) + x\log\theta\} \end{aligned}$$

$$\therefore\ a(\theta) = -\theta,\ b(X) = \log(X!),\ c(\theta) = \log\theta,\ d(X) = X$$

$\therefore$ $Poisson(\theta)$ belongs to exponential family.

(b)

$$\begin{aligned} X &\sim Bin(n,\theta) \quad (n \text{ is known}) \\ f_X(x;\theta) &= \binom{n}{x}\theta^n(1-\theta)^{n-x} \\ &= \exp\left\{\log\binom{n}{x} + x\log\theta + (n-x)\log(1-\theta)\right\} \\ &= \exp\left\{n\log(1-\theta) + \log\binom{n}{x} + x\log(\frac{\theta}{1-\theta})\right\} \end{aligned}$$

$$\therefore\ a(\theta) = n\log(1-\theta),\ b(X) = \log\binom{n}{X},\ c(\theta) = \log(\frac{\theta}{1-\theta}),\ d(X) = X$$

$\therefore$ $Bin(n,\theta)$ belongs to exponential family.

(c) Note: X here means the trial number on which the rth success occurs.

$$\begin{aligned} X &\sim Neg.Bin.(r,\theta) \quad (r \text{ is known}) \\ f_X(x;\theta) &= \binom{x-1}{r-1}\theta^r(1-\theta)^{x-r} \\ &= \exp\left\{\log\binom{x-1}{r-1} + r\log\theta + (x-r)\log(1-\theta)\right\} \\ &= \exp\left\{r\log(\frac{\theta}{1-\theta}) + \log\binom{x-1}{r-1} + +x\log(1-\theta)\right\} \end{aligned}$$

$$\therefore\ a(\theta) = r\log(\frac{\theta}{1-\theta}),\ b(X) = \log\binom{X-1}{r-1},\ c(\theta) = \log(1-\theta),\ d(X) = X$$

$\therefore$ $Neg.Bin.(r,\theta)$ belongs to exponential family.

(d)

$$\begin{aligned} X &\sim gamma(k,\theta) \quad (k>0 \text{ is known}) \\ f_X(X;\theta) &= \frac{x^{k-1}e^{-x\theta}}{\Gamma(k)\theta^{-k}} \\ &= \exp\{(k-1)\log x - x\theta - \log[\Gamma(k)] + k\log\theta\} \\ &= \exp\{k\log\theta - \log[\Gamma(k)] + (k-1)\log x - x\theta\} \end{aligned}$$

$$\therefore\ a(\theta) = k\log\theta,\ b(X) = -\log[\Gamma(k)] + (k-1)\log X,\ c(\theta) = -\theta,\ d(X) = X$$

$\therefore$ $Gamma(k,\theta), k>0$ belongs to exponential family.

(e)

$$\begin{aligned} X &\sim N(\theta,1) \\ f_X(x;\theta) &= \frac{1}{\sqrt{2\pi}\cdot 1}\exp\left\{\frac{-1}{2(1)}(x-\theta)^2\right\} \\ &= \exp\left\{\frac{-1}{2}\log(2\pi) - \frac{1}{2}(x-\theta)^2\right\} \\ &= \exp\left\{\frac{-1}{2}\log(2\pi) - \frac{1}{2}x^2 + \theta x - \frac{1}{2}\theta^2\right\} \\ &= \exp\left\{-\frac{1}{2}\theta^2 - \frac{1}{2}(x^2+\log 2\pi) + \theta x\right\} \end{aligned}$$

$$\therefore\ a(\theta) = -\frac{1}{2}\theta^2,\ b(X) = -\frac{1}{2}(X^2+\log 2\pi),\ c(\theta) = \theta,\ d(X) = X$$

$\therefore$ $N(\theta,1)$ belongs to exponential family.

(f)

$$\begin{aligned} X &\sim N(0,\theta) \\ f_X(x;\theta) &= \frac{1}{\sqrt{2\pi\theta}}\exp\left\{\frac{-1}{2\theta}(x-0)^2\right\} \\ &= \exp\left\{\frac{-1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2\right\} \end{aligned}$$

$$\therefore\ a(\theta) = -\frac{1}{2}\log(2\pi\theta),\ b(X) = 0,\ c(\theta) = -\frac{1}{2\theta},\ d(X) = X^2$$

2. $X \sim \exp(\theta)$.

Given $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

$\Rightarrow Y = \sum_{i=1}^{n} X_i \sim gamma(n, \theta)$

$\Rightarrow f_Y(y) = \frac{\theta^n}{\Gamma(n)} y_{n-1} e^{-\theta y}$

$f_X(x;\theta) = \theta e^{-\theta x} = \exp(\log\theta - \theta x)$

$$\therefore\ a(\theta) = \log\theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$$\therefore\ f_X(x;\theta) \text{ belongs to the exponential family.}$$

$$\text{and } \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i \text{ is complete and sufficient.}$$

$$\begin{aligned}
\therefore\ E\left(\frac{n-1}{Y}\right) &= (n-1)\int_0^\infty \frac{1}{y} f_Y(y)\, dy \\
&= (n-1)\int_0^\infty \frac{1}{y}\frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y}\, dy \\
&= \frac{(n-1)\theta}{\Gamma(n)} \int_0^\infty \theta^{n-1} y^{n-2} e^{-\theta y}\, dy \\
&= \frac{(n-1)\theta}{\Gamma(n)}\Gamma(n-1) \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y}\, dy \\
&= (n-1)\theta \cdot \frac{1}{n-1} \cdot 1 \\
&= \theta
\end{aligned}$$

$$\because\ \Gamma(n) = (n-1)\Gamma(n-1) \quad \text{and} \quad \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} \text{ is pdf of gamma}(n-1, \frac{1}{\theta})$$

Since $\frac{n-1}{Y}$ is function of complete sufficient statistic, $\frac{n-1}{Y}$ is UMVUE for θ.

3. 1) $X \sim Exp(\frac{1}{\theta})$. $\Rightarrow \sum X_i \sim \Gamma(n, \frac{1}{\theta}) \Rightarrow \frac{2\sum X_i}{1/\theta} \sim \chi^2_{2n}$ is pivotal quantity

Choose $a<b$. $P(a < 2\theta\sum X_i < b) = 1-\alpha$.

one of the choice $\begin{cases} a = \chi^2_{2n, 1-\frac{\alpha}{2}} \\ b = \chi^2_{2n, \frac{\alpha}{2}} \end{cases}$ $\Rightarrow P\left(\frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{2\sum X_i} < \theta < \frac{\chi^2_{2n,\frac{\alpha}{2}}}{2\sum X_i}\right) = 1-\alpha$

$\Rightarrow \mu = \frac{1}{\theta}$. C.I. for μ: $\left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}}, \frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right)$

2) $\sigma^2 = \frac{1}{\theta^2}$ $\Rightarrow P(a < 2\theta\sum X_i < b) = 1-\alpha$.

Similarly C.I. for σ^2 $\left(\left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}}\right)^2, \left(\frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right)^2\right)$

3) $P\left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}} < \mu < \frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}},\ \left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}}\right)^2 < \sigma^2 \left(\frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right)^2\right)$

$= P\left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}} < \frac{1}{\theta} < \frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}},\ \left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}}\right)^2 < \frac{1}{\theta^2} < \left(\frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right)^2\right)$

$= P\left(\frac{2\sum X_i}{\chi^2_{2n,\frac{\alpha}{2}}} < \frac{1}{\theta} < \frac{2\sum X_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}\right) = 1-\alpha$ (equivalent)

4) Similar to 1). $P\left(\frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{2\sum X_i} < \theta < \frac{\chi^2_{2n,\frac{\alpha}{2}}}{2\sum X_i}\right) = 1-\alpha$

$\Rightarrow P\left(\exp\left(-\frac{\chi^2_{2n,\frac{\alpha}{2}}}{2\sum X_i}\right) < e^{-\theta} < \exp\left(-\frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{2\sum X_i}\right)\right) = 1-\alpha$

$\Rightarrow 100(1-\alpha)\%$ C.I. for $\tau = e^{-\theta}$: $\left(\exp\left(-\frac{\chi^2_{2n,\frac{\alpha}{2}}}{2\sum X_i}\right),\ \exp\left(-\frac{\chi^2_{2n,1-\frac{\alpha}{2}}}{2\sum X_i}\right)\right)$

4 (a)

$$\begin{aligned}
f(x;p) &= p^x(1-p)^{1-x} \\
L(p) &= f_{\underset{\sim}{X}}(\underset{\sim}{x};p) \\
&= \prod_{i=1}^{n} f_{X_i}(x_i;p) \\
&= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \\
&= p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}
\end{aligned}$$

$$\begin{aligned}
\log L(p) &= \left(\sum_{i=1}^n x_i\right)\log p + \left(n - \sum_{i=1}^n x_i\right)\log(1-p) \\
\frac{\partial}{\partial p}\log L(p) &= \left(\sum_{i=1}^n x_i\right)\frac{1}{p} - \left(n-\sum_{i=1}^n x_i\right)\frac{1}{1-p}
\end{aligned}$$

Setting equal to 0 $\Rightarrow$ $\hat{p} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$

$\therefore$ By invariant property of MLE, MLE for $\theta = (1-\hat{p})^2 = (1-\bar{x})^2$

(b)

$$X_1 + X_2 \sim Bin(2,p)$$

$$\begin{aligned}
\therefore\ E(\hat{\theta}) &= 1\cdot P(X_1+X_2=0) + 0\cdot P(X_1+X_2\neq 0) \\
&= \binom{2}{0} p^0(1-p)^{2-0} \\
&= (1-p)^2 \\
&= \theta
\end{aligned}$$

$\therefore$ $\hat{\theta}$ is an unbiased estimator of θ.

(c)

$$\begin{aligned} f(x;p) &= p^x(1-p)^{1-x} \\ &= \exp\{x\log p + (1-x)\log(1-p)\} \\ &= \exp\left\{\log(1-p) + x\log\left(\frac{p}{1-p}\right)\right\} \end{aligned}$$

$$\therefore\ a(p) = \log(1-p), b(X) = 0, c(p) = \left(\frac{p}{1-p}\right), d(X) = X$$

$\therefore$ $f(X;p)$ belongs to exponential family and $\sum_{i=1}^n X_i$ is a complete and sufficient statistic for p.

Let $S = \sum_{i=1}^n X_i \sim Bin(n;p)$

By Rao-Blackwell theorem, UMVUE for $\theta = E(\hat\theta|S=s)$, $\hat\theta$ is unbiased, S is sufficient.

$$\begin{aligned} \Rightarrow E(\hat\theta|S=s) &= 1\cdot P(X_1+X_2=0|S=s) + 0\cdot P(X_1+X_2\neq 0|S=s) \\ &= \frac{P(X_1+X_2=0, \sum_{i=1}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} \\ &= \frac{P(X_1+X_2=0, \sum_{i=3}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} \\ &= \frac{P(X_1+X_2=0)\cdot P(\sum_{i=3}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} \\ &= \frac{\binom{2}{0}p^0(1-p)^{2-0}\binom{n-2}{s}p^s(1-p)^{n-2-s}}{\binom{n}{s}p^s(1-p)^{n-s}} \\ &= \frac{(n-2)!}{s!(n-2-s)!}\cdot\frac{s!(n-s)!}{n!} \\ &= \frac{(n-s)(n-s-1)}{n(n-1)} \end{aligned}$$

5. two different types $\Rightarrow \sigma_1^2 \neq \sigma_2^2$

90% C.I. $(\bar{X}-\bar{Y}) \pm t_{k,0.05}\sqrt{\frac{S_X^2}{n_1}+\frac{S_Y^2}{n_2}}$

$k = \dfrac{(S_1^2/n_1 + S_2^2/n_2)^2}{\frac{1}{n_1-1}(S_1^2/n_1)^2 + \frac{1}{n_2-1}(S_2^2/n_2)^2} \approx 94$ is large

$\Rightarrow t_{k,0.05} \approx z_{0.05}$

6. $\sigma_1 = \sigma_2$ 95% C.I. for $\mu_1 - \mu_2$. $\bar{x} - \bar{y} \pm t_{n_1+n_2-2, \frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ ④

$$S_p = \frac{(n_1-1)S_x^2 + (n_2-1)S_y^2}{n_1+n_2-2}$$

under Normal assumption $\frac{(n_1-1)S_x^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$, $\frac{(n_2-1)S_y^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$

$\overset{\text{indep}}{\Longrightarrow}$ $\frac{S_y^2}{\sigma_2^2} \Big/ \frac{S_x^2}{\sigma_1^2} = \frac{S_y^2}{S_x^2} \cdot \frac{\sigma_1^2}{\sigma_2^2} \sim F_{n_2-1, n_1-1}$

$\Rightarrow P\left(F_{n_2-1, n_1-1, 0.95} < \frac{S_y^2}{S_x^2} \times \frac{\sigma_1^2}{\sigma_2^2} < F_{n_2-1, n_1-1, 0.05} \right) = 0.9$

$\Rightarrow P\left(\sqrt{\frac{S_x^2}{S_y^2} F_{n_2-1, n_1-1, 0.95}} < \frac{\sigma_1}{\sigma_2} < \sqrt{\frac{S_x^2}{S_y^2} F_{n_2-1, n_1-1, 0.05}} \right) = 0.9$