Homework 5 MATH 1520

Question 1

a) $\int_{-1}^{1} \frac{5x}{(4+x^2)^2}dx = \int_{-1}^{1} \frac{5x}{(4+x^2)^2} \frac{d(4+x^2)}{2x}$

$= \frac{5}{2}\left[\frac{-1}{4+x^2}\right]_{-1}^{1}$

$= 0$

b) $\int_0^1 x\sqrt{x+1}\,dx = \int_1^2 \sqrt{u}\,(u-1)\,du$

$= \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_1^2$

$= \frac{4(\sqrt{2}+1)}{15}$

c) $\int_2^4 \frac{1}{x(\ln x)^2}dx = \int_2^4 \frac{1}{x(\ln x)^2}\, x\, d\ln x$

$= \left[\frac{-1}{\ln x}\right]_2^4$

$= \frac{1}{2\ln 2}$

d) $\int (2x+6)^{14}dx = \int (2x+6)^{14}\frac{d(2x+6)}{2}$

$= \frac{(2x+6)^{15}}{30} + c$

e) $\int \sqrt{4x-1}\,dx = \int \sqrt{4x-1}\,\frac{d(4x-1)}{4}$

$= \frac{(4x-1)^{\frac{3}{2}}}{6} + c$

f) $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx = \int \frac{1}{\sqrt{x}(u+1)}\,2\sqrt{x}\,du$

$= 2\ln|u+1| + c$

$= 2\ln(\sqrt{x}+1) + c$

g) $\int \frac{x}{\sqrt[3]{4-3x}}dx = \int \frac{\frac{u^3-4}{-3}}{u}(-u^2)\,du$

$= \frac{1}{3}\left(\frac{u^5}{5} - 2u^2\right) + c$

$= \frac{(4-3x)^{\frac{5}{3}}}{15} - \frac{2(4-3x)^{\frac{2}{3}}}{3} + c$

h) $\int_{10}^{30} ve^{-\frac{v}{5}}dv = \left[-5ve^{-\frac{v}{5}}\right]_{10}^{30} - \int_{10}^{30} -5e^{-\frac{v}{5}}$

$= \left[-5ve^{-\frac{v}{5}} - 25e^{-\frac{v}{5}}\right]_{10}^{30}$

$= 75e^{-2} - 175e^{-6}$

i) $\int_2^1 t\ln 2t\,dt = \left[\frac{t^2\ln 2t}{2}\right]_2^1 - \int_2^1 \frac{t}{2}dt$

$= \left[\frac{t^2\ln 2t}{2} - \frac{t^2}{4}\right]_2^1$

$= -\frac{7}{2}\ln 2 + \frac{3}{4}$

j) $\int_{-1}^{3}(t-1)e^{1-t}dt = \int_2^{-2} ue^u\,du$

$= \left[ue^u\right]_2^{-2} - \int_2^{-2} e^u\,du$

$= \left[ue^u - e^u\right]_2^{-2}$

$= -3e^{-2} - e^2$

Question 2

a) $\int \frac{x^3-x+1}{x^2-1}dx = \int \left(x + \frac{1}{x^2-1}\right)dx$

$= \int \left[x + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right]dx$

$= \frac{x^2}{2} + \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2} + c$

$$\begin{array}{r|l} & x \\ x^2-1 & x^3 \quad -x+1 \\ & x^3 \quad -x \\ \hline & 1 \end{array}$$

$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$1 = A(x+1) + B(x-1)$

$A = \frac{1}{2}$; $B = -\frac{1}{2}$

$$\int \frac{x^4}{(x-1)(x-2)}\,dx = \int \left(x^2+3x+7+\frac{15x-14}{x^2-3x+2}\right)dx$$

$$= \int \left(x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}\right)dx$$

$$= \frac{x^3}{3}+\frac{3x^2}{2}+7x-\ln|x-1|+16\ln|x-2|+c$$

$$\begin{array}{r} x^2+3x+7 \\ x^2-3x+2 \,\big/\, x^4 \qquad\qquad \\ x^4-3x^3+2x^2 \\ \hline 3x^3-2x^2 \\ 3x^3-9x^2+6x \\ \hline 7x^2-6x \\ 7x^2-21x+14 \\ \hline 15x-14 \end{array}$$

$$\frac{15x-14}{x^2-3x+2} = \frac{A}{x-1}+\frac{B}{x-2}$$

$$15x-14 = A(x-2)+B(x-1)$$

$A=-1$; $B=16$

c) $$\int \frac{x+2}{x^3-x}\,dx = \int \left[\frac{-2}{x}+\frac{3}{2(x-1)}+\frac{1}{2(x+1)}\right]dx$$

$$= -2\ln|x|+\frac{3}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+c$$

$$\frac{x+2}{x^3-x} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

$A=-2$; $B=\frac{3}{2}$; $C=\frac{1}{2}$

d) $$\int_3^9 \frac{4-3x}{(x-1)^2}\,dx = \int_3^9 \left[\frac{-3}{x-1}+\frac{1}{(x-1)^2}\right]dx$$

$$= \left[-3\ln(x-1)-\frac{1}{x-1}\right]_3^9$$

$$= -6\ln 2+\frac{3}{8}$$

$$\frac{4-3x}{(x-1)^2} = \frac{A}{x-1}+\frac{B}{(x-1)^2}$$

$$4-3x = A(x-1)+B$$

$A=-3$; $B=1$

Question 3

a) $$\int e^x\sqrt{e^x-1}\,dx = \int \sqrt{u}\,du$$

$$= \frac{2u^{\frac{3}{2}}}{3}+c$$

$$= \frac{2(e^x-1)^{\frac{3}{2}}}{3}+c$$

b) $$\int \frac{x^3+2x+1}{x+1}\,dx = \int \left(x^2-x+3-\frac{2}{x+1}\right)dx$$

$$= \frac{x^3}{3}-\frac{x^2}{2}+3x-2\ln|x+1|+c$$

$$\begin{array}{r} x^2-x+3 \\ x+1 \,\big/\, x^3+\qquad 2x+1 \\ x^3+x^2 \qquad\quad \\ \hline -x^2+2x+1 \\ -x^2-x \quad \\ \hline 3x+1 \\ 3x+3 \\ \hline -2 \end{array}$$

c) $$\int (x^3-x)e^x\,dx = \int (x^3e^x-xe^x)\,dx$$

$$= \int \left(x^3e^x-\int 3x^2e^x\,dx\right)-\left(xe^x-\int e^x\,dx\right)dx$$

$$= \int x^3e^x-3\left(x^2e^x-\int 2xe^x\right)-xe^x+e^x\,dx$$

$$= \int x^3e^x-3\left(x^2e^x-2\left(xe^x-\int e^x\,dx\right)\right)-xe^x+e^x\,dx$$

$$= x^3e^x-3x^2e^x+5xe^x-5e^x+c$$

d) $$\int_2^4 \frac{e^{2x}}{1+e^x}\,dx = \int_{1+e^2}^{1+e^4} \frac{u-1}{u}\,du$$

$$= \left[u-\ln u\right]_{1+e^2}^{1+e^4}$$

$$= e^4-e^2-\ln\frac{e^4+1}{e^2+1}$$

e) $$\int_1^{10} (\ln x)^3\,dx = \int_0^{\ln 10} u^3e^u\,du$$

$$= \left[u^3e^u\right]_0^{\ln 10}-\int_0^{\ln 10} 3u^2e^u\,du$$

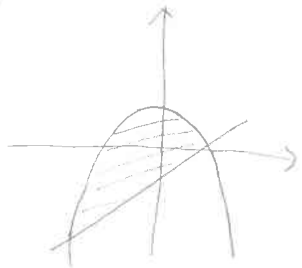
$$= \left[u^3e^u-3u^2e^u\right]_0^{\ln 10}+\int_0^{\ln 10} 2ue^u\,du$$

$$= \left[u^3e^u-3u^2e^u+6ue^u\right]_0^{\ln 10}-6\int_0^{\ln 10} e^u\,du$$

$$= \left[u^3e^u-3u^2e^u+6ue^u-6e^u\right]_0^{\ln 10}$$

$$= 10\ln^3 10-30\ln^2 10+60\ln 10-54$$

Question 4



intersection :

$2 - x^2 = 2x - 1$

$x = -3$ or 1

$$\int_{-3}^{1} (2 - x^2) - (2x - 1)\, dx$$

$$= \int_{-3}^{1} (-x^2 - 2x + 3)\, dx$$

$$= \left[-\frac{x^3}{3} - x^2 + 3x\right]_{-3}^{1}$$

$$= \frac{32}{3}$$

Question 5

$$\lim_{n\to\infty} n\left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \dots + \frac{1}{(2n+n)^2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(2+\frac{i}{n})^2}$$

$$= \int_2^3 \frac{1}{x^2}\, dx$$

$$= \left[-\frac{1}{x}\right]_2^3$$

$$= \frac{1}{6}$$

Question 6

a) $$\int_0^{+\infty} x e^{-x^2} = \lim_{b\to\infty} \int_0^b x e^{-x^2} \frac{d(-x^2)}{-2x}$$

$$= \lim_{b\to\infty} \left[\frac{e^{-x^2}}{-2}\right]_0^b$$

$$= \frac{1}{2}$$

b) $$\int_0^{\infty} 2x e^{-3x} dx = 2 \lim_{b\to\infty} \left\{\left[\frac{x e^{-3x}}{-3}\right]_0^b - \int_0^b \frac{-e^{-3x}}{3} dx\right\}$$

$$= 2 \lim_{b\to\infty} \left[\frac{x e^{-3x}}{-3} - \frac{e^{-3x}}{9}\right]_0^b$$

$$= \frac{2}{9}$$

c) $$\int_{-\infty}^{0} \frac{1}{(2x-1)^2} dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{(2x-1)^2} \frac{d(2x-1)}{2}$$

$$= \frac{1}{2} \lim_{a\to-\infty} \left[\frac{-1}{2x-1}\right]_a^0$$

$$= \frac{1}{2}$$

d) $$\int_0^{\infty} x e^{1-x} dx = \lim_{b\to\infty} \left\{\left[-x e^{1-x}\right]_0^b - \int_0^b -e^{1-x} dx\right\}$$

$$= \lim_{b\to\infty} \left[-x e^{1-x} + e^{1-x}\right]_0^b$$

$$= e$$

e) $$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\, x\, d\ln x$$

$$= \lim_{b\to\infty} \left[2\sqrt{\ln x}\right]_2^b$$

$$= \infty$$

Question 7

$$I = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)} du = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)} du = \int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$$

$$2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)} dx = \int_0^a 1\, dx = a$$

$$\therefore I = \frac{a}{2}$$

Question 8

$$\int \frac{1}{x^2 - a^2}\, dx$$

for $a \neq 0$

$$\frac{1}{x^2 - a^2} = \frac{A}{x - a} + \frac{B}{x + a}$$

$$1 = A(x+a) + B(x-a)$$

$$A = \frac{1}{2a} \; ; \; B = \frac{1}{-2a}$$

$$\int \frac{1}{x^2 - a^2}\, dx = \int \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}\, dx$$

$$= \frac{\ln|x-a|}{2a} - \frac{\ln|x+a|}{2a} + c$$

for $a = 0$

$$\int \frac{1}{x^2}\, dx = -\frac{1}{x} + c$$

$$\int \frac{1}{x^2 - c^2} = \begin{cases} \dfrac{\ln|x-a|}{2a} - \dfrac{\ln|x-c|}{2a} + c & \text{for } a \neq 0 \\ -\dfrac{1}{x} + c & \text{for } a = 0 \end{cases}$$