STAT 2006 Assignment 1 Suggested Solution

1. (a) Since

$$\begin{aligned}
&\int_0^\infty (1 - F_X(x))dx \\
=& \int_0^\infty \int_x^\infty f(y)dydx \\
=& \int_0^\infty \int_0^y f(y)dxdy \\
=& \int_0^\infty yf(y)dy,
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^0 F_X(x)dx \\
=& \int_{-\infty}^0 \int_{-\infty}^x f(y)dydx \\
=& \int_{-\infty}^0 \int_y^0 f(y)dxdy \\
=& \int_{-\infty}^0 -yf(y)dy,
\end{aligned}$$

Hence,

$$\int_0^\infty (1 - F_X(x))dx - \int_{-\infty}^0 F_X(x)dx = E[X]$$

(b)
- The Jacobian approach: Note $Z = X + Y, W = X \Rightarrow X = W, Y = Z - W$. Hence the Jacobian

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

Therefore, the joint pdf $f_{Z,W}(z, w) = f_{X,Y}(x = w, y = z - w)|J| = f_X(w)f_Y(z - w)$ and hence integrating with respect to w gives the marginal pdf of Z:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(w)f_Y(z - w)dw = \int_{-\infty}^{+\infty} f_Y(z - x)f_X(x)dx.$$

- The CDF approach:

$$\begin{aligned}
f_Z(z) &= \frac{\partial}{\partial z}F_Z(z) = \frac{\partial}{\partial z}\mathbb{P}(Z \le z) = \frac{\partial}{\partial z}\mathbb{P}(X + Y \le z) \\
&= \frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \mathbb{P}(x + Y \le z | X = x)f_X(x)dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial z}\mathbb{P}(Y \le z - x)f_X(x)dx \\
&= \int_{-\infty}^{+\infty} f_Y(z - x)f_X(x)dx
\end{aligned}$$

2. (a) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Then $M_X(t) = e^{\lambda(e^t-1)}$ and $M_Y(t) = e^{\mu(e^t-1)}$. By independence,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\lambda(e^t-1)}e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$$

Therefore, $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(b) For $x = 0, 1, \ldots, n$,

$$\mathbb{P}(X = x | X + Y = n) = \frac{\mathbb{P}(X = x, Y = n - x)}{\mathbb{P}(X + Y = n)} = \frac{\mathbb{P}(X = x)\mathbb{P}(Y = n - x)}{\mathbb{P}(X + Y = n)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!}\frac{e^{-\mu}\mu^{n-x}}{(n-x)!}}{\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}} = C_x^n \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{n-x}.$$

For $x \neq 0, 1, \ldots, n$, $\mathbb{P}(X = x|X + Y = n) = 0$. Therefore, $X|X + Y = n \sim \text{Bin}\left(n, \frac{\lambda}{\lambda+\mu}\right)$.

(c) Since $X|X + Y = n$ does not follow Poisson distribution, they are not independent.

3. (a) Obviously $Y = X^2 > 0$, so the pdf $f_Y(y) = 0$ for any $y < 0$. We only need to consider $y \geq 0$. Note the CDF of Y, $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y})$
$= \mathbb{P}(X \leq \sqrt{y}) - \mathbb{P}(X < -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$
Hence, $f_Y(y) = \frac{\partial}{\partial y}[F_X(\sqrt{y}) - F_X(-\sqrt{y})] = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\left(\frac{-1}{2\sqrt{y}}\right)$
$= \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})]$.

(b) Note $Y_1 = X_1^2 + X_2^2 > 0, Y_2 = \frac{X_1}{\sqrt{X_1^2 + X_2^2}} \in (-1, 1)$. Therefore,
$X_1 = Y_2\sqrt{X_1^2 + X_2^2} = Y_2\sqrt{Y_1} \Rightarrow X_2^2 = Y_1 - X_1^2 = Y_1 - Y_2^2 Y_1 = Y_1(1 - Y_2^2)$
$\Rightarrow X_2 = \pm\sqrt{Y_1(1 - Y_2^2)}$, i.e. The transformation is not one-to-one. However, it is one to one in the set $\{X_2 > 0\}$ and $\{X_2 < 0\}$ respectively. (As X_2 is a continuous random variable, $\Pr\{X_2 = 0\} = 0$ and thus we can safely ignore this set.) Therefore, when $x_2 > 0$,
$\frac{\partial x_1}{\partial y_1} = \frac{y_2}{2\sqrt{y_1}}, \frac{\partial x_1}{\partial y_2} = \sqrt{y_1}, \frac{\partial x_2}{\partial y_1} = \frac{\sqrt{1 - y_2^2}}{2\sqrt{y_1}}, \frac{\partial x_2}{\partial y_2} = \frac{-\sqrt{y_1}y_2}{\sqrt{1 - y_2^2}}$ and the corresponding Jacobian

$$J_+ = \left(\frac{y_2}{2\sqrt{y_1}}\right)\left(\frac{-\sqrt{y_1}y_2}{\sqrt{1 - y_2^2}}\right) - (\sqrt{y_1})\left(\frac{\sqrt{1 - y_2^2}}{2\sqrt{y_1}}\right) = -\frac{y_2^2}{2\sqrt{1 - y_2^2}} - \frac{\sqrt{1 - y_2^2}}{2} = -\frac{1}{2\sqrt{1 - y_2^2}}$$

Similarly, when $x_2 < 0$, the corresponding Jacobian $J_- = \frac{1}{2\sqrt{1 - y_2^2}}$

Combining together, the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$
$= f_{X_1,X_2}\left(y_2\sqrt{y_1}, \sqrt{y_1(1 - y_2^2)}\right)|J_+| + f_{X_1,X_2}\left(y_2\sqrt{y_1}, -\sqrt{y_1(1 - y_2^2)}\right)|J_-|$
$= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{y_1}{2\sigma^2}\right\}\frac{1}{2\sqrt{1 - y_2^2}} + \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{y_1}{2\sigma^2}\right\}\frac{1}{2\sqrt{1 - y_2^2}}$
$= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{y_1}{2\sigma^2}\right\}\frac{1}{\sqrt{1 - y_2^2}}, 0 < y_1 < +\infty, -1 < y_2 < 1$

Note the joint pdf can be factorized as $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\sigma^2}\exp\left\{-\frac{y_1}{2\sigma^2}\right\} \times \frac{1}{\pi\sqrt{1 - y_2^2}}$ and the support of (Y_1, Y_2) is rectangular, so they are independent. It is easy to recognize as the former factor as the pdf of an exponential distribution, $Y_1 \sim \exp(2\sigma^2)$. We can also verify that the latter factor is a valid pdf: $\int_{-1}^{1}\frac{1}{\pi\sqrt{1 - y_2^2}}dy_2 = \frac{1}{\pi}\sin^{-1}(y_2)\Big|_{-1}^{1} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$

For your interest: Geometrically, if we view (X_1, X_2) as the Cartesian coordinates, then (Y_1, Y_2) is related to the corresponding polar coordinates (R, θ) because $Y_1 = R^2, Y_2 = \cos\theta$. It shows that the radius R and the polar angle θ are independent. (Recall the joint pdf of a pair of iid normal random variables has a set of concentric circular contours)

4. (a) The pmf of Y, for $y = 0, 1, \cdots$, is

$$\begin{aligned} f_Y(y) &= \int_0^{+\infty} f_Y(y|\lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^{+\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^{+\infty} \lambda^{y+\alpha-1} \exp\left\{\frac{-\lambda(1+\beta)}{\beta}\right\} d\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{1+\beta}\right)^{y+\alpha}. \end{aligned}$$

$EY = E(E(Y|\Lambda)) = E\Lambda = \alpha\beta$

$Var(Y) = Var(E(Y|\Lambda)) + E(Var(Y|\Lambda)) = \alpha\beta^2 + \alpha\beta$.

If α is a positive integer, $f_Y(y) = \binom{y+\alpha-1}{y}\left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha$, or the negative binomial $(\alpha, 1/(1+\beta))$ pmf.

(b) For $y = 0, 1, \cdots$, we have

$$\begin{aligned} P(Y = y|\lambda) &= \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda) P(N = n|\lambda) \\ &= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!} \left(\frac{p}{1-p}\right)^y [(1-p)\lambda]^n e^{-\lambda} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{y!m!} \left(\frac{p}{1-p}\right)^y [(1-p)\lambda]^{m+y} \\ &= \frac{e^{-\lambda}}{y!} \left(\frac{p}{1-p}\right)^y [(1-p)\lambda]^y \left[\sum_{m=0}^{\infty} \frac{[(1-p)\lambda]^m}{m!}\right] \\ &= e^{-\lambda}(p\lambda)^y e^{(1-p)\lambda} \\ &= \frac{(p\lambda)^y e^{-p\lambda}}{y!}. \end{aligned}$$

the Poisson$(p\lambda)$ pmf. Thus, $Y|\Lambda \sim$ Poisson$(p\lambda)$.

Follow the similar calculations in (a) yield the pmf of Y, for $Y = 0, 1, \cdots$, is $\frac{1}{y!\Gamma(\alpha)(p\beta)^\alpha} \Gamma(y+\alpha) \left(\frac{p\beta}{1+p\beta}\right)^{y+\alpha}$.

If α is a positive integer, $Y \sim$ negative binomial$(\alpha, 1/(1+p\beta))$.

5. (a) $EY = E[E[Y|X]] = EX = \frac{1}{2}$.

$Var(Y) = Var(E(Y|X)) + E(Var(Y|X)) = Var(X) + EX^2 = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$.

$E(XY) = E[E(XY|X)] = E[XE(Y|X)] = EX^2 = \frac{1}{3}$.

$Cov(X, Y) = E(XY) - EXEY = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$.

(b) The quick proof is to note that the distribution of $Y|X = x$ is $n(1, 1)$, hence is independent of X. The bivariate transformation $t = y/x, u = x$ will also show that the joint density factors.