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STAT 4005

Assignment 4

Question 1

a) from .R, we have

$Y_t = a_t + 0.5806 a_{t-1} - 0.4194 a_{t-2}$, $a_t \sim WN(0, 0.5061)$

$a_{19} = 1.6554$ $a_{20} = 0.0447$, assuming $a_{-1} = a_1 = 0$

$\Rightarrow$ 1-step ahead

$Y_{21}^{20} = 0.5806(0.0447) - 0.4194(1.6554)$ $e_{20}(1) = a_{21}$

≈ -0.6683 $P_{21}^{20} = 0.5061$

$CI: -0.6683 \pm 1.96\sqrt{0.5061}$

$\approx [-2.0627, 0.7261]$

$\Rightarrow$ 2-step ahead

$Y_{22}^{20} = -0.4194(0.0447)$ $e_{20}(2) = a_{22} + 0.5806 a_{21}$

≈ -0.0187 $P_{22}^{20} \approx 0.6767$

$CI: -0.0187 \pm 1.96\sqrt{0.6767}$

$\approx [-1.631, 1.5936]$

$\Rightarrow$ k-step ahead, $k \geq 3$

$Y_{20+k}^{20} = 0$ $e_{20}(k) = a_{20+k} + 0.5806 a_{20+k-1} - 0.4194 a_{20+k-2}$

$P_{20+k}^{20} \approx 0.7657$

$CI: \pm 1.96\sqrt{0.7657}$

$\approx [-1.7151, 1.7151]$

b) $r(0) = 1.513\sigma^2$ $r(1) = 0.3371\sigma^2$ $r(2) = -0.4194\sigma^2$

$\rho(1) \approx 0.2228$ $\rho(2) \approx -0.2772$ $\rho(k) = 0 \quad \forall k \geq 3$

$\phi_{11} = 0.2228$ $\phi_{22} = \dfrac{-0.2772 - (0.2228)^2}{1 - (0.2228)^2}$ $\phi_{33} = \dfrac{(0.2228)^3 + (0.2228)(-0.2772)^2 - 2(0.2228)(-0.2772)}{2(0.2228)^2(-0.2772) - 2(0.2228)^2 - (-0.2772)^2 + 1}$

≈ -0.3439 ≈ 0.1905

c) from .R, we have

$Y_t = 0.141 Y_{t-1} + a_t$, $a_t \sim WN(0, 0.7276)$

$\Rightarrow$ k-step ahead, $k \geq 1$

$Y_{20+k}^{20} = (0.141)^k(1.43)$ $e_{20}(k) = \sum_{i=1}^{k} (0.141)^{k-i} a_{20+i}$

$P_{20+k}^{20} = 0.7276\left(\dfrac{1 - 0.141^{2k}}{1 - 0.141^2}\right)$

$CI: (0.141)^k(1.43) \pm 1.96\sqrt{0.7276\left(\dfrac{1-0.141^{2k}}{1-0.141^2}\right)}$

$\approx (0.141)^k(1.43) \pm 1.6887\sqrt{1 - 0.141^{2k}}$

d) $a_t = \sum 0.141^i a_{t+i}$

$Cov(e_{20}(k), e_{20}(\ell)) = \sigma^2 \sum_{i=0}^{\ell-1} (0.141)^i (0.141)^{i+k-\ell}$

$= (0.7276)(0.141)^{k-\ell}\left(\dfrac{1 - 0.141^{2\ell}}{1 - 0.141^2}\right)$

$\approx 0.7424(0.141^{k-\ell} - 0.141^{k+\ell})$

e) from .R, we have

$Y_t = -0.3883 Y_{t-1} + a_t + a_{t-1}$, $a_t \sim WN(0, 0.5254)$

$a_{20} = 1.4876$, assuming $a_1 = 0$

$\Rightarrow$ 1-step ahead

$Y_{21}^{20} = -0.3883(1.43) + 1.4876$ $e_{20}(1) = a_{21}$

≈ 0.9323 $P_{21}^{20} = 0.5254$

$CI: 0.9323 \pm 1.96\sqrt{0.5254}$

$\approx [-0.4884, 2.353]$

$\Rightarrow$ 2-step ahead

$Y_{22}^{20} = -0.3883(0.9323)$ $e_{20}(2) = a_{22} - 0.6117 a_{21}$

≈ -0.362 $P_{22}^{20} \approx 0.722$

$CI: -0.362 \pm 1.96\sqrt{0.722}$

$\approx [-2.0274, 1.3034]$

f) from .R, we have

$Y_t(1-B) = -0.2673 Y_{t-1}(1-B) + a_t$, $a_t \sim WN(0, 1.106)$

$Y_t = Y_{t-1} - 0.2673 Y_{t-1}(1-B) + a_t$

$\Rightarrow$ 1-step ahead

$Y_{21}^{20} = 1.43 - 0.2673(1.43 - 0.57)$ $e_{20}(1) = a_{21}$

≈ 1.2001 $P_{21}^{20} = 1.106$

$CI: 1.2001 \pm 1.96\sqrt{1.106}$

$\approx [-0.8612, 3.2614]$

$\Rightarrow$ 2-step ahead

$Y_{22}^{20} = 1.2001 - 0.2673(1.2001 - 1.43)$

≈ 1.2616

$e_{20}(2) = a_{22} + 0.7327 a_{21}$

$P_{22}^{20} \approx 1.6988$

$CI: 1.2616 \pm 1.96\sqrt{1.6988}$

$\approx [-1.293, 3.8162]$

Question 2

a) $X_{t+1}^2 = \sigma_{t+1}^2 \varepsilon_{t+1}^2$

$= (\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2)\, \varepsilon_{t+1}^2$

$X_{t+1} = \sqrt{(\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2)\, \varepsilon_{t+1}^2}$

$X_{t+2}^2 = \sigma_{t+2}^2 \varepsilon_{t+2}^2$

$= (\alpha_0 + \alpha_1 X_{t+1}^2 + \beta_1 \sigma_{t+1}^2)\, \varepsilon_{t+2}^2$

$= [\alpha_0 + (\alpha_1 \varepsilon_{t+1}^2 + \beta_1)(\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2)]\, \varepsilon_{t+2}^2$

$X_{t+2} = \sqrt{[\alpha_0 + (\alpha_1 \varepsilon_{t+1}^2 + \beta_1)(\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2)]\, \varepsilon_{t+2}^2}$

b) $L(\alpha_i, \beta_i ; x) = f(X_1 | \sigma_1^2) f(X_2 | X_1, \sigma_1^2) f(X_3 | X_2, X_1, \sigma_1^2)$

$= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{X_1^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{X_2^2}{2\sigma_2^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_3^2}} e^{-\frac{X_3^2}{2\sigma_3^2}}$

$= \frac{1}{\sqrt{(2\pi)^3 \sigma_1^2 \sigma_2^2 \sigma_3^2}} e^{-\frac{1}{2}\sum_{i=1}^{3} \frac{X_i^2}{\sigma_i^2}}$

where $\sigma_2^2 = \alpha_0 + \alpha_1 X_1^2 + \beta_1 \sigma_1^2$,

$\sigma_3^2 = \alpha_0(1+\beta_1) + \alpha_1 (X_2^2 + \beta_1 X_1^2) + \beta_1^2 \sigma_1^2$

Question 3

$X_t \sim GARCH(p, q)$

$\Rightarrow \quad X_t^2 = X_t^2 + \sigma_t^2 - \sigma_t^2$

$= \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2 + X_t^2 - \sigma_t^2$

$= \alpha_0 + \Sigma \beta_i (\sigma_{t-i}^2 + X_{t-i}^2 - X_{t-i}^2) + \Sigma \alpha_j X_{t-j}^2 + X_t^2 - \sigma_t^2$

$= \alpha_0 + \sum_{l=1}^{m} r_l X_{t-l}^2 + \Sigma(-\beta_i)(X_{t-i}^2 - \sigma_{t-i}^2) + X_t^2 - \sigma_t^2$

where $r_l = \beta_l + \alpha_l \quad \forall\ l \leq \min(p, q)$,

$r_l = \alpha_l \quad \forall \quad p < q$

$r_l = \beta_l \quad \forall \quad p > q$

$\Rightarrow X_t^2 \sim ARMA(m, p)$