STAT 2006 Assignment 3

Due Time and Date: 5 p.m., 23 April, 2020

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$, then the pivotal quantity $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, and we can make use of its quantiles a, b to construct a $100(1-\alpha)\%$ confidence interval for σ. The quantiles a, b need to satisfy the constraint

$$G(b) - G(a) = Pr\left\{a \leq \frac{(n-1)S^2}{\sigma^2} \leq b\right\} = 1 - \alpha$$

where G is the CDF of $\chi^2(n-1)$. Obviously there are many possible choices for a and b.

   (a) Construct the $100(1-\alpha)\%$ confidence interval for σ in terms of the quantiles a, b defined above. Let k be the length of the confidence interval. Express k in terms of n, s^2, a and b.

   (b) Show that the k is minimized when a, b also satisfy

$$a^{\frac{n}{2}} e^{-\frac{a}{2}} - b^{\frac{n}{2}} e^{-\frac{b}{2}} = 0.$$

   Combining with the constraint above, we can numerically solve for the optimal pair of quantiles a, b to minimize the length of the confidence interval.

2. It is reported that in a telephone poll of 2000 adult, 1325 of them are nonsmokers. Also, $y_1 = 650$ of nonsmokers and $y_2 = 425$ of smokers said yes to a particular question. Let p_1, p_2 equal the proportions of nonsmokers and smokers that would say yes to this question respectively

   (a) Find a two-sided 95% confidence interval for $p_1 - p_2$.

   (b) Find a two-sided 95% confidence interval for p, the proportion of adult who would say yes to this question.

3. Let Y be Binomial$(50, p)$. To test $H_0 : p = 0.08$ against $H_1 : p < 0.08$, we reject H_0 if and only if $Y \leq 7$.

   (a) Determine the significance level α of the test

   (b) Calculate the value of the power function if in fact $p = 0.05$.

4. The mean birth weight in the United States is $\mu = 3320$ grams, with a standard deviation of $\sigma = 580$. Let X equal the birth weight in Rwanda. Assume that the distribution of X is $N(\mu, \sigma^2)$. We shall test the hypothesis $H_0 : \sigma = 580$ against the alternative hypothesis $H_1 : \sigma < 580$ at an $\alpha = 0.05$ significance level.

   (a) What is your decision if a random sample of size $n = 81$ yields $\bar{X} = 2989$ and $s = 516$?

   (b) What is the approximate p-value of this test?

5. Assume that IQ scores for a certain population are approximately $N(\mu, 100)$. To test

$$H_0 : \mu = 110 \quad \text{against} \quad H_1 : \mu > 110$$

we take random sample of size $n = 16$ from this population and observe $\bar{X} = 114$

   (a) Do we accept or reject H_0 at the 1% significance level?

   (b) Do we accept or reject H_0 at the 5% significance level?

   (c) What is the p-value of this test?

6. The following text was shown to a large class of students for 30 seconds, and they were told to report the number of F's that they found:

   IN FINANCIAL TRANSACTIONS, SIMPLE INTEREST IS OFTEN USED FOR FRACTIONS OF AN INTEREST PERIOD FOR CONVENIENCE.

   Let p equal the proportion of students who find 6F's. We shall test the null hypothesis

   $$H_0 : p = 0.5 \quad \text{against} \quad H_1 : p < 0.5$$

   (a) Given a sample size of $n = 230$, define a critical region with an approximate significance level of $\alpha = 0.05$.

   (b) If $y = 110$ students report that they found 6F's, what is your conclusion?

   (c) what is the p-value of this test?

7. In 1000 tosses of a coin, 560 heads and 440 tails appear. Using direct calculation or normal approximation, test whether the coin is fair, at the 5% significance level.

8. For a random sample $X_1, \cdots, X_n$ of Bernoulli(p) variables, it is desired to test

   $$H_0 : p = 0.49 \quad \text{against} \quad H_1 : p = 0.51$$

   Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about 0.01. Use a test function that rejects H_0 if $\sum_{i=1}^n X_i$ is large. Find the critical value as well.

9. Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on $(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test

   $$\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq k,$$

   where k is a constant, $Y_1 = \min\{X_1, \cdots, X_n\}, Y_n = \max\{X_1, \cdots, X_n\}$. Determine k, in terms of n and α, so that the test would have significance level α.

10. In a given city it is assumed that the number of automobile accidents in a given year follows a Poisson distribution. In past years the average number of accidents per year was 15, and this year it was 10. Test whether the accident rate has dropped, at the 5% significance level.