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STAT 4003

Assignment 3

Question 1

a) $Y \sim Poi(\lambda)$, $f(y) = \frac{\lambda^y e^{-\lambda}}{y!}$
$= e^{-\lambda} \cdot \frac{1}{y!} \cdot e^{y\ln(\lambda)}$

$\because a(\lambda) = e^{-\lambda}$; $b(y) = \frac{1}{y!}$; $c(\lambda) = \ln(\lambda)$; $d(y) = y$

$\therefore$ Y belongs to exponential family

b) $Y \sim b(n,p)$, $f(y) = \binom{n}{y} p^y (1-p)^{n-y}$
$= (1-p)^n \cdot \binom{n}{y} \cdot e^{y\ln(\frac{p}{1-p})}$

$\because a(p) = (1-p)^n$; $b(y) = \binom{n}{y}$; $c(p) = \ln(\frac{p}{1-p})$; $d(y) = y$

$\therefore$ Y belongs to exponential family

c) $Y \sim NB(k,p)$, $f(y) = \binom{y-1}{k-1} p^k (1-p)^{y-k}$
$= (\frac{p}{1-p})^k \cdot \binom{y-1}{k-1} \cdot e^{y\ln(1-p)}$

$\because a(p) = (\frac{p}{1-p})^k$; $b(y) = \binom{y-1}{k-1}$; $c(p) = \ln(1-p)$; $d(y) = y$

$\therefore$ Y belongs to exponential family

d) $Y \sim \Gamma(\theta, k)$, $f(y) = \frac{y^{\theta-1} e^{-\frac{y}{k}}}{\Gamma(\theta) k^\theta}$
$= \frac{1}{\Gamma(\theta) k^\theta} \cdot \frac{e^{-\frac{y}{k}}}{y} \cdot e^{\theta\ln(y)}$

$\because a(\theta) = \frac{1}{\Gamma(\theta) k^\theta}$; $b(y) = \frac{e^{-\frac{y}{k}}}{y}$; $c(\theta) = \theta$; $d(y) = \ln(y)$

$\therefore$ Y belongs to exponential family

e) $Y \sim N(\theta, 1)$, $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}}$
$= e^{-\frac{\theta^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot e^{y\theta}$

$\because a(\theta) = e^{-\frac{\theta^2}{2}}$; $b(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$; $c(\theta) = \theta$; $d(y) = y$

$\therefore$ Y belongs to exponential family

f) $Y \sim N(0, \theta)$, $f(y) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{y^2}{2\theta}}$
$= \frac{1}{\sqrt{\theta}} \cdot \frac{1}{\sqrt{2\pi}} 1_{(y\in\mathbb{R})} \cdot e^{-\frac{y^2}{2\theta}}$

$\because a(\theta) = \frac{1}{\sqrt{\theta}}$; $b(y) = \frac{1}{\sqrt{2\pi}} 1_{(y\in\theta)}$; $c(\theta) = -\frac{1}{2\theta}$; $d(y) = y^2$

$\therefore$ Y belongs to exponential family

Question 2

Given $X \sim Exp(\frac{1}{\theta})$, we know $Y = \Sigma X_i$ is sufficient and complete since it belongs to exponential family.

We have $Y \sim \Gamma(n, \frac{1}{\theta})$. Let $g(y) := \frac{n-1}{y}$

con't $E[g(y)] = (n-1) E(\frac{1}{Y})$
$= (n-1) \frac{\theta}{n-1}$
$= \theta$

$E(\frac{1}{Y}) = \int_0^\infty \frac{1}{y} \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-y\theta} dy$
$= \frac{\theta^n}{\Gamma(n)} \cdot \frac{\Gamma(n-1)}{\theta^{n-1}}$
$= \frac{\theta}{n-1}$

By Lehmann-Scheffé therom, $g(y)$ is UMVUE of θ

Since Y is sufficient and $g(y)$ is 1-to-1 function, $g(y)$ is also sufficient.

Let f be any function and $h = f \circ g$ where h is also arbitrary.

We have $0 = E\{f[g(y)]\} = E[f \circ g(y)] = E[h(y)] = 0$

Since Y is complete and

$P\{f[g(y)] = 0\} = P[f \circ g(y) = 0] = P[h(y) = 0] = 1$,

$g(y)$ is also complete.

$\Rightarrow \frac{n-1}{Y}$ is the best statistic for θ

Question 3

i) Given $X \sim Exp(\frac{1}{\theta})$, we know $Y_i := 2\theta X \sim \chi^2_{(2)}$

where $M_{2\theta X}(t) = M_X(2\theta t) = (1-2t)^{-1}$

for $g(X_i, \theta) = \Sigma Y_i \sim \chi^2_{(2n)}$, we have

$P(\chi^2_{2n;\alpha/2} < 2\theta X < \chi^2_{2n;1-\alpha/2}) = 1-\alpha$

$\Rightarrow$ $100(1-\alpha)\%$ confidence interval of $\frac{1}{\theta}$ is

$\left(\frac{2n\bar{x}}{\chi^2_{2n;1-\alpha/2}}, \frac{2n\bar{x}}{\chi^2_{2n;\alpha/2}}\right)$

ii) Similar to above, $100(1-\alpha)\%$ confidence interval of $\frac{1}{\theta^2}$ is

$\left(\left(\frac{2n\bar{x}}{\chi^2_{2n;1-\alpha/2}}\right)^2, \left(\frac{2n\bar{x}}{\chi^2_{2n;\alpha/2}}\right)^2\right)$

iii) Since the CI of $\frac{1}{\theta}$ is independent of that of $\frac{1}{\theta^2}$, the probability covers both true mean and true variance is $(1-\alpha)^2$

iv) $P(a < \frac{1}{\theta} < b) = 1-\alpha$

$P(-\frac{1}{a} < -\theta < -\frac{1}{b}) =$

$P(e^{-\frac{1}{a}} < e^{-\theta} < e^{-\frac{1}{b}}) =$

$\Rightarrow$ $100(1-\alpha)\%$ confidence interval for τ is

$\left(e^{-\frac{\chi^2_{2n;1-\alpha/2}}{2n\bar{x}}}, e^{-\frac{\chi^2_{2n;\alpha/2}}{2n\bar{x}}}\right)$

Question 4

a) Sine MLE of p is $\bar{x}$, by invariant property, the MLE of θ is $(1-\bar{x})^2$

$\boxed{\ell(p) = \Sigma x \ln(p) + (n + \Sigma x)\ln(1-p) \quad \hat{p} = \bar{x}}$

b) $E(\hat{\theta}) = E[(1-\bar{x})^2]$

$= E(1 - 2\bar{x} + \bar{x}^2)$

$= 1 - 2p + \frac{p(1-p)}{n} + p^2$

$= (1-p)^2 \qquad \text{if } X_1 + X_2 = 0, \ \frac{p(1-p)}{n} = 0$

$= \theta$

$\therefore \ \hat{\theta}$ is unbiased

c) Since X belongs to exponential family, ΣX_i is sufficient and complete. By Lehmann-Scheffé therom, $\hat{\theta} = (1-\bar{x})^2$ is the UMVUE of θ

Question 5

$H_0: \sigma_X^2 = \sigma_Y^2$ vs $H_1: \sigma_X^2 \neq \sigma_Y^2$

$X_0^2 = \frac{8742}{9411} \approx 0.9289 \qquad$ p-value ≈ 0.4022

Since p-value > 0.05, we do not reject H_0 at $\alpha = 0.05$

The approximate 90% confidence interval for $\mu_X - \mu_Y$ is

$(984 - 1121) \pm 1.645 \sqrt{\frac{44(8742) + 51(9411)}{95}\left(\frac{1}{45} + \frac{1}{52}\right)}$

$\approx (-168.9515, -105.0485)$

We are 90% confident that the difference of 2 population means is within $(-168.9515, -105.0485)$

Since the sample size is large enough, we can apply CLT such that the underlying distribution approximate to normality.

Question 6

95% confidence interval for $\mu_1 - \mu_2$:

$(74.5 - 71.8) \pm 2.074 \sqrt{\frac{12(82.6) + 10(112.6)}{22}\left(\frac{1}{13} + \frac{1}{11}\right)}$

$\approx (5.6352, 11.0352)$

90% confidence interval for $\frac{\sigma_1}{\sigma_2}$

$\left(\sqrt{\frac{1}{2.91}\left(\frac{82.6}{112.6}\right)}, \sqrt{2.75\left(\frac{82.6}{112.6}\right)}\right)$

$\approx (0.5021, 1.4203)$