2018-19 MATH1520AB

Midterm II (2018 Nov 1)

1. (a)

$$y' = \frac{(x^2+x+3)'\sqrt{x} - (\sqrt{x})'(x^2+x+3)}{x} = \frac{1}{2}(3x^{1/2} + x^{-1/2} - 3x^{-3/2})$$

or first rewrite y in the following form:

$$y = x^{3/2} + x^{1/2} + 3x^{-1/2}$$

Then, we have

$$y' = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$$

(b)

$$\begin{aligned} s' &= (2^t)'(t^2+1)^{-1} + ((t^2+1)^{-1})'2^t \\ &= 2^t(\ln 2)(t^2+1)^{-1} - (t^2+1)^{-2}(2t)(2^t) \\ &= \frac{2^t}{(t^2+1)^2}\Big(\ln 2(t^2+1) - 2t\Big) \end{aligned}$$

(c)

$$y' = \frac{3x+5}{2x+3}\left(\frac{2x+3}{3x+5}\right)' = \frac{3x+5}{2x+3} \times \frac{2(3x+5) - 3(2x+3)}{(3x+5)^2} = \frac{1}{6x^2+19x+15}$$

(d)

$$y' = \frac{(x+\frac{c}{x}) - (1-\frac{c}{x^2})x}{(x+\frac{c}{x})^2} = \frac{2cx}{(c+x^2)^2}$$

(e)

$$y' = (e^{(1+\ln x)\ln x})' = e^{(1+\ln x)\ln x}((1+\ln x)\ln x)' = x^{\ln x}(2\ln x + 1)$$

or, take ln on both sides of the equation:

$$\ln y = (1+\ln x)\ln x$$

Differentiate both sides with respect to x:

$$\begin{aligned} \frac{1}{y}\cdot y' &= \tfrac{1}{x}\ln x + (1+\ln x)\cdot\tfrac{1}{x} \\ \Rightarrow \qquad y' &= x^{1+\ln x}\left(\tfrac{1}{x}\right)(2\ln x + 1) \\ &= x^{\ln x}(2\ln x + 1) \end{aligned}$$

(f) Take ln on both sides of the equation:

$$\ln y = 3\ln(3x+4) + \sqrt{7}\ln(5x-2) - 4\ln(2x+1) + x$$

Then take derivative with respect to x:

$$\frac{1}{y}y' = \frac{9}{3x+4} + \frac{5\sqrt{7}}{5x-2} - \frac{8}{2x+1} + 1$$

Thus

$$y' = \frac{(3x+4)^3(5x-2)^{\sqrt{7}}}{(2x+1)^4 e^x}\left(\frac{9}{3x+4} + \frac{5\sqrt{7}}{5x-2} - \frac{8}{2x+1} + 1\right)$$

2. If the function is continuous at 0, then $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$. And $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2+4x+1) = 1$ while $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(ax+b) = b$. This shows that $b = 1$.
   Then we calculate the right hand derivative and left hand derivative at $x = 0$ from first principle.

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{(0+h)^2+4(0+h)+1-(0^2+4\cdot 0+1)}{h} = \lim_{h\to 0^+}(h+4) = 4$$

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{a(0+h)+b-(a\cdot 0+b)}{h} = \lim_{h\to 0^+}(a) = a$$

   If the function is differentiable at $x = 0$, then by the calculation above $a = 4$.

   The conclusion is that if the function is continuous and differentiable at $x = 0$, then $a = 4$ and $b = 1$.

3. (a) Apply the L'Hospital rule

$$\begin{aligned}\lim_{x\to+\infty}\frac{\ln(x^2+2)}{e^{x^2+2}} &= \lim_{x\to+\infty}\frac{(\ln(x^2+2))'}{(e^{x^2+2})'}\\ &= \lim_{x\to+\infty}\frac{2x}{(x^2+2)2x(e^{x^2+2})}\\ &= \lim_{x\to+\infty}\frac{1}{(x^2+2)(e^{x^2+2})}\\ &= 0\end{aligned}$$

   (b) Apply the L'Hospital rule

$$\begin{aligned}\lim_{x\to 0}\frac{xa^x}{a^x-1} &= \lim_{x\to 0}\frac{(xa^x)'}{(a^x-1)'}\\ &= \lim_{x\to 0}\frac{a^x + x\cdot a^x\ln a}{a^x\ln a}\\ &= \lim_{x\to 0}\left(x+\frac{1}{\ln a}\right)\\ &= \frac{1}{\ln a}\end{aligned}$$

   (c) Since $\lim_{x\to+\infty}(e^x+x)^{1/x} = \lim_{x\to+\infty} e^{\ln(e^x+x)/x}$, we first calculate $\lim_{x\to+\infty}\frac{\ln(e^x+x)}{x}$ by L'Hospital rule.

$$\begin{aligned}\lim_{x\to+\infty}\frac{\ln(e^x+x)}{x} &= \lim_{x\to+\infty}\frac{(\ln(e^x+x))'}{x'}\\ &= \lim_{x\to+\infty}\frac{e^x+1}{e^x+x}\\ &= \lim_{x\to+\infty}\frac{1+e^{-x}}{1+xe^{-x}}\\ &= 1\end{aligned}$$

Thus $\lim_{x\to+\infty}(e^x+x)^{1/x}=e$.

(d) $\lim_{x\to1^+}(\ln(x^7-1)-\ln(x^5-1))=\lim_{x\to1^+}\ln\left(\frac{x^7-1}{x^5-1}\right)$. We only have to know $\lim_{x\to1^+}\frac{x^7-1}{x^5-1}$. By L'Hopital rule,

$$\begin{aligned}\lim_{x\to1^+}\frac{x^7-1}{x^5-1} &= \lim_{x\to1^+}\frac{7x^6}{5x^4}\\ &= \frac{7}{5}\end{aligned}$$

Thus, $\lim_{x\to1^+}(\ln(x^7-1)-\ln(x^5-1))=\ln\frac{7}{5}$.

4. (a) Substitute $x=1$, $y=0$ to the equation, we get
$LHS=1^2+0+0=1$
$RHS=e^0=1$
$\Rightarrow RHS=LHS$
so $P(0,1)$ is on the curve.

(b) Differentiate the equation with respect to x, we get

$$2x+2xy'+2y+3y^2y'=e^{xy}(xy'+y)$$

Substitute $x=1$, $y=0$, we get

$$2+2y'=y'\Longrightarrow y'=-2$$

So $\left.\frac{dy}{dx}\right|_{(1,0)}=-2$

(c) The equation of the tangent line at P is

$$\frac{y-0}{x-1}=-2\Longrightarrow y=-2x+2$$

(d) From (b), we get

$$2x+2xy'+2y+3y^2y'=e^{xy}(xy'+y)$$

Differentiate both sides, we get

$$2+2y'+2xy''+2y'+6y(y')^2+3y^2y''=e^{xy}(xy'+y)^2+e^{xy}(xy''+2y')$$

Substitute $x = 1$, $y = 0$, $y' = -2$ to the above equation, we get

$$2 - 4 - 4 + 2y'' = (-2)^2 + y'' - 4$$

Hence $\left.\dfrac{d^2y}{dx^2}\right|_{(1,0)} = 6$

5. (a) Since $f(0) = \ln 1 + 0 + 3 = 3$, g is the inverse function of f, so $g(f(0)) = g(3) = 0$.

   (b) Let $y = g(x)$. Then, $x = f(y)$.

$$f'(y) = \frac{2y}{y^2+1} + 1 = \frac{y^2+2y+1}{x^2+1}$$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{(g(x))^2+1}{(g(x))^2+2g(x)+1} = \frac{y^2+1}{y^2+2y+1}$$

   (c) $g'(3) = \dfrac{(g(3))^2+1}{(g(3))^2+2g(3)+1} = 1$

   Or,

   $g'(3) = \dfrac{1}{f'(0)} = \dfrac{0^2+1}{0^2+2(0)+1} = 1$

6. (a) f is increasing, if f' is positive. From the graph of f', we have f is increasing on $(0, 2) \cup (6, 8) \cup (10, 12)$. Similarly, f is decreasing, if f' is negative. So, f is decreasing on $(2, 6) \cup (8, 10)$.

   (b) Critical numbers: $x = 2, 6, 8, 10$ (x values at which $f'(x) = 0$ or undefined.)

   At $x = 2, 8$, f' change sign from positive to negative. Therefore, f has a relative maximum at $x = 2, 8$.

   At $x = 6, 10$, f' change sign from negative to positive. Therefore, f has a relative minimum at $x = 6, 10$.

   (c) As for the concavity of f, besides of using the sign of f'', we can also use monotonicity of f' to determine it.

   f is concave upward, if f'' is positive, or equivalently f' is increasing. From the graph, we can see f' is increasing on $(4, 8) \cup (8, 12)$. So, f is concave upward on $(4, 8) \cup (8, 12)$.

   Similarly, f is concave downward, if f'' is negative, or f' is decreasing. From the graph, we can see f' is decreasing on $(0, 4)$. So, f is concave downward on $(0, 4)$.

   (d) At $x = 4$, f changes concavity from concave downward to concave upward. Therefore, f has an inflection point at $x = 4$. It is the only one inflection point of f.

7. (a) i. $\mathbb{R}$

   ii. when $x = 0$, $y = 0$
   when $y = 0$, $x = 0$ or $x = 5$
   So x-intercept: $(0, 0)$, $(5, 0)$; y-intercept:$(0, 0)$

iii.

$$\lim_{x\to+\infty} x^{5/3} - 5x^{2/3} = +\infty$$
$$\lim_{x\to-\infty} x^{5/3} - 5x^{2/3} = -\infty$$

No vertical or horizontal asymptotes.

(b) i.

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{5}{3}x^{-\frac{1}{3}}(x-2) = \frac{5}{3}\Big(\frac{x-2}{x^{1/3}}\Big)$$

ii. $f'(x) = 0$, $x = 2$; $f'(x)$ undefined, $x = 0$.

x	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	$+$	$-$	$+$

So increasing intervals: $(-\infty, 0)$, $(2, +\infty)$; decreasing interval: $(0, 2)$

iii. relative minimal: $(2, f(2)) = (2, -3\cdot 2^{\frac{2}{3}})$;

relative maximal: $(0, f(0)) = (0, 0)$.

(c) i.

$$f''(x) = \frac{5}{3}(-\frac{1}{3}(x^{-\frac{4}{3}}(x-2) + x^{-\frac{1}{3}}) = \frac{-5(x-2)+15x}{9x^{\frac{4}{3}}}$$
$$= \frac{10(x+1)}{9x^{\frac{4}{3}}}$$

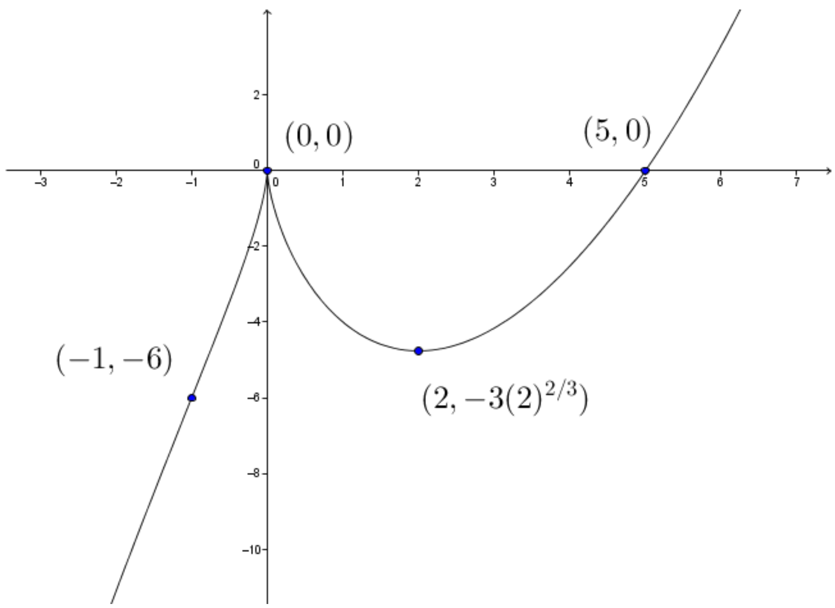
ii. Let $f''(x) = 0$, $x = -1$; $f''(x)$ undefined, $x = 0$.

x	$(-\infty, -1)$	$(-1, 0)$	$(0, +\infty)$
$f''(x)$	$-$	$+$	$+$
$f(x)$	$\cap$	$\cup$	$\cup$

concave upward intervals: $(-1, 0) \cup (0, +\infty)$;
concave downward intervals: $(-\infty, -1)$

iii. inflection point:$(-1, -6)$

(d)


$(0,0)$
$(5,0)$
$(-1,-6)$
$(2,-3(2)^{2/3})$
-3
-2
-1
0
1
2
3
4
5
6
7
2
0
-2
-4
-6
-8
-10