STAT4005 Solution to Assignment 4

Question 1

(a)

For an $MA(2)$ model of

$$Y_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad Z_t \sim WN(0, \sigma^2)$$

which is in R notation, we have k-step ahead forecasts

$$\begin{aligned}
&Y_{n+1}^n = \mathbb{E}(Y_{n+1}|Y_n, \ldots, Y_1) = \mathbb{E}(Z_{n+1} + \theta_1 Z_n + \theta_2 Z_{n-1}|Y_n, \ldots, Y_1) = \theta_1 Z_n + \theta_2 Z_{n-1} \\
&Y_{n+2}^n = \theta_2 Z_n \\
&Y_{n+k}^n = 0, \quad k \geq 3
\end{aligned}$$

and forecast error

$$\begin{aligned}
e_n(1) &= Y_{n+1} - Y_{n+1}^n = Z_{n+1} \\
e_n(2) &= Z_{n+2} + \theta_1 Z_{n+1} \\
e_n(k) &= Z_{n+k} + \theta_1 Z_{n+k-1} + \theta_2 Z_{n+k-2}, \quad k \geq 3
\end{aligned}$$

Hence, variance of forecast error is given by

$$\begin{aligned}
P_{n+1}^n &= Var(e_n(1)|Y_n, \ldots, Y_1) = \sigma^2 \\
P_{n+2}^n &= (1 + \theta_1^2)\sigma^2 \\
P_{n+k}^n &= (1 + \theta_1^2 + \theta_2^2)\sigma^2, \quad k \geq 3
\end{aligned}$$

95% prediction interval for k-step forecast is $Y_{n+k}^n \pm 1.96\sqrt{P_{n+k}^n}$.
Estimation of the model by default method in R yields

$$Y_t = Z_t + 0.5806 Z_{t-1} - 0.4194 Z_{t-2}, \quad Z_t \sim WN(0, 0.5061)$$

Hence,

$$\begin{aligned}
&Y_{21}^{20} = 0.2377, \qquad Y_{22}^{20} = -0.3979, \qquad Y_{20+k}^{20} = 0, \quad k \geq 3 \\
&P_{21}^{20} = 0.5061, \qquad P_{22}^{20} = 0.6767, \qquad P_{20+k}^{20} = 0.7657, \quad k \geq 3
\end{aligned}$$

95% confidence intervals are

$$\begin{aligned}
Y_{21}^{20} &\in [-1.1567, 1.6320] \\
Y_{22}^{20} &\in [-2.0102, 1.2145] \\
Y_{20+k}^{20} &\in [-1.7151, 1.7151], \quad k \geq 3
\end{aligned}$$

(b)

For the $MA(2)$ model, we have following autocorrelation function

$$\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \\
\rho(2) &= \frac{\theta_2}{1+\theta_1^2+\theta_2^2} \\
\rho(k) &= 0, \quad k \geq 3
\end{aligned}$$

By the first principle, ϕ_{kk} can be found from

$$\begin{pmatrix} \phi_{k1} \\ \vdots \\ \phi_{kk} \end{pmatrix} = \begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(k-1) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(0) \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \vdots \\ \rho(k) \end{pmatrix}$$

Substitute into estimated coefficient values, we find

$$\phi_{11} = 0.2228, \quad \phi_{22} = -0.3439, \quad \phi_{33} = 0.1905$$

(c)

For the $AR(1)$ model of

$$Y_t = \alpha Y_{t-1} + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

we can find by induction that, for $k \geq 1$,

$$\begin{aligned}
Y_{n+k}^n &= \alpha^k Y_n \\
e_n(k) &= \sum_{i=0}^{k-1} \alpha^i Z_{n+k-i} \\
P_{n+k}^n &= \sigma^2 \sum_{i=0}^{k-1} \alpha^{2i} = \frac{\sigma^2(1-\alpha^{2k})}{1-\alpha^2}
\end{aligned}$$

95% prediction interval for k-step forecast is $Y_{n+k}^n \pm 1.96\sqrt{P_{n+k}^n}$.
Estimation of the model by default method in R yields

$$Y_t = 0.1410 Y_{t-1}, \quad Z_t \sim WN(0, 0.7276)$$

Hence, k-step forecast is $Y_{20+k}^{20} = 1.43(0.1410^k)$, k-step forecast error is $P_{20+k}^{20} = 0.7423(1 - 0.0199^k)$, 95% prediction interval is $1.43(0.1410^k) \pm 1.6887\sqrt{1-0.0199^k}$.

(d)

Without loss of generality, assume $l \geq k$, then

$$cov(e_{20}(l), e_{20}(k)) = \sigma^2 \sum_{i=0}^{k-1} \alpha^i \alpha^{i+l-k} = \sigma^2 \alpha^{l-k} \frac{1-\alpha^{2k}}{1-\alpha} = \frac{\sigma^2}{1-\alpha^2}(\alpha^{l-k} - \alpha^{l+k})$$

Substitute into estimated values, we have $cov(e_{20}(l), e_{20}(k)) = 0.7423(0.1410^{l-k} - 0.1410^{l+k})$.

(e)

For the $ARMA(1,1)$ model of

$$Y_t = \alpha Y_{t-1} + Z_t + \theta Z_{t-1}, \quad Z_t \sim WN(0, \sigma^2)$$

we have

$$\begin{aligned}
Y_{n+1}^n &= \mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \alpha Y_n + \theta Z_n \\
Y_{n+2}^n &= \mathbb{E}(Y_{n+2}|\mathcal{F}_n) = \alpha Y_{n+1}^n = \alpha^2 Y_n + \alpha\theta Z_n \\
e_n(1) &= Y_{n+1} - Y_{n+1}^n = Z_{n+1} \\
e_n(2) &= Y_{n+2} - Y_{n+2}^n = \alpha e_n(1) + Z_{n+2} + \theta Z_{n+1} = Z_{n+2} + (\alpha+\theta) Z_{n+1} \\
P_{n+1}^n &= var(e_n(1)|\mathcal{F}_n) = \sigma^2 \\
P_{n+2}^n &= var(e_n(2)|\mathcal{F}_n) = (1 + (\alpha+\theta)^2)\sigma^2
\end{aligned}$$

Estimation of the model by default method in R yields

$$Y_t = -0.3883 Y_t + Z_t + 0.9999 Z_{t-1}, \quad Z_t \sim WN(0, 0.5254)$$

Hence,

$$\begin{aligned}
Y_{21}^{20} &= 0.3810, \quad Y_{22}^{20} = -0.1479 \\
P_{21}^{20} &= 0.5254, \quad P_{22}^{20} = 0.7220
\end{aligned}$$

95% confidence intervals are

$$\begin{aligned}
Y_{21}^{20} &\in [-1.0397, 1.8017] \\
Y_{22}^{20} &\in [-1.8134, 1.5175]
\end{aligned}$$

(f)

For the $ARIMA(1,1,0)$ model of

$$Y_t - Y_{t-1} = \alpha(Y_{t-1} - Y_{t-2}) + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

we have

$$Y_t = (1+\alpha)Y_{t-1} - \alpha Y_{t-2} + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

Hence,

$$\begin{aligned}
Y_{n+1}^n &= (1+\alpha)Y_n - \alpha Y_{n-1} \\
Y_{n+2}^n &= (1+\alpha)Y_{n+1}^n - \alpha Y_n = (\alpha^2+\alpha+1)Y_n - (\alpha^2+\alpha)Y_{n-1} \\
e_n(1) &= Z_{n+1} \\
e_n(2) &= (1+\alpha)e_n(1) + Z_{n+2} = (1+\alpha)Z_{n+1} + Z_{n+2} \\
P_{n+1}^n &= \sigma^2 \\
P_{n+2}^n &= (\alpha^2+2\alpha+2)\sigma^2
\end{aligned}$$

Estimation of the model by default method in R yields

$$Y_t = 0.7327 Y_{t-1} + 0.2673 Y_{t-2} + Z_t, \quad Z_t \sim WN(0, 1.1058)$$

Hence,

$$\begin{aligned}
Y_{21}^{20} &= 1.2001, \quad Y_{22}^{20} = 1.2616 \\
P_{21}^{20} &= 1.1058, \quad P_{22}^{20} = 1.6995
\end{aligned}$$

95% confidence intervals are

$$\begin{aligned}
Y_{21}^{20} &\in [-0.8609, 3.2612] \\
Y_{22}^{20} &\in [-1.2935, 3.8167]
\end{aligned}$$

Question 2

(a)

$$X_{t+1} = \sigma_{t+1}\epsilon_{t+1} = \sqrt{\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2}\,\epsilon_{t+1}$$

$$\begin{aligned}
X_{t+2} &= \sigma_{t+2}\epsilon_{t+2} \\
&= \sqrt{\alpha_0 + \alpha_1 X_{t+1}^2 + \beta_1 \sigma_{t+1}^2}\,\epsilon_{t+2} \\
&= \sqrt{\alpha_0 + (\alpha_1 \epsilon_{t+1}^2 + \beta_1)\sigma_{t+1}^2}\,\epsilon_{t+2} \\
&= \sqrt{\alpha_0 + (\alpha_1 \epsilon_{t+1}^2 + \beta_1)(\alpha_0 + \alpha_1 X_t^2 + \beta_1 \sigma_t^2)}\,\epsilon_{t+2}
\end{aligned}$$

(b)

$$
\begin{aligned}
L(\alpha_0, \alpha_1, \beta_1) &= log[f(X_3|X_2, X_1)\, f(X_2|X_1)\, f(X_1)] \\
&= log[\phi(0, \sigma_3^2)\, \phi(0, \sigma_2^2)\, \phi(0, \sigma_1^2)] \\
&= -\frac{3}{2}log(2\pi) - \frac{1}{2}(log\sigma_3^2 + log\sigma_2^2 + log\sigma_1^2) - \frac{1}{2}(\frac{X_3^2}{\sigma_3^2} + \frac{X_2^2}{\sigma_2^2} + \frac{X_1^2}{\sigma_1^2})
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_2^2 &= \alpha_0 + \alpha_1 X_1^2 + \beta_1 \sigma_1^2 \\
\sigma_3^2 &= \alpha_0 + \alpha_1 X_2^2 + \beta_1 \sigma_2^2 = \alpha_0 + \alpha_0\beta_1 + \alpha_1\beta_1 X_1^2 + \alpha_1 X_2^2 + \beta_1^2 \sigma_1^2
\end{aligned}
$$

and ϕ is the pdf for standard normal distribution.

Question 3

Proof. Given the $GARCH(p, q)$ model of the form

$$
\begin{aligned}
X_t &= \sigma_t \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} N(0, 1) \\
\sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2
\end{aligned}
$$

we have

$$
\begin{aligned}
X_t^2 &= \sigma_t^2 + X_t^2 - \sigma_t^2 \\
&= \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2 - \sigma_t^2 \\
&= \alpha_0 + \sum_{i=1}^{p} \beta_i (\sigma_{t-i}^2 - X_{t-i}^2) + \sum_{i=1}^{p} \beta_i X_{t-i}^2 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2 + X_t^2 - \sigma_t^2 \\
&= \alpha_0 + \sum_{j=1}^{min(p,q)} (\alpha_j + \beta_j) X_{t-j}^2 + \sum_{k=min(p,q)+1}^{max(p,q)} \theta_k X_{t-k}^2 + \sigma_t^2(\epsilon_t^2 - 1) - \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 (\epsilon_{t-i}^2 - 1)
\end{aligned}
$$

where $\theta_k = \beta_k I(p \geq q) + \alpha_k I(p < q)$. Let $\nu_t = \sigma_t^2(\epsilon_t^2 - 1)$, it can be shown that $\{\nu_t\}$ is a white noise process under some conditions of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Hence,

$$
X_t^2 = \alpha_0 + \sum_{j=1}^{min(p,q)} (\alpha_j + \beta_j) X_{t-j}^2 + \sum_{k=min(p,q)+1}^{max(p,q)} \theta_k X_{t-k}^2 + \nu_t - \sum_{i=1}^{p} \beta_i \nu_{t-i}
$$

is an $ARMA(m, p)$ process, where $m = max(p, q)$. □