STAT 6104
ASSIGNMENT 2
ANSWER

1. (a) $E[\bar{X}] = \alpha$
   (b) $Var(\bar{X}) = \frac{1}{25}(5\gamma_0 + 8\gamma_1 + 6\gamma_2 + 4\gamma_3 + 2\gamma_4) = 0.7243$

2. (a) ARIMA$(0, 0, 2)$
   (b) $\{Z_t\}$ is stationary since it is MA(2).
   (c) MA characteristic equation is $1 + x + 0.25x^2 = 0$
   $=> |x| = |-2| = 2 > 1$
   $=> \{Z_t\}$ is invertible.
   (d) $\begin{cases} \gamma_0 &= 41.25 \\ \gamma_1 &= 25 \\ \gamma_2 &= 5 \\ \gamma_k &= 0, k \geq 3 \end{cases}$
   $\begin{cases} \rho_0 &= 1 \\ \rho_1 &= \frac{20}{33} \\ \rho_2 &= \frac{4}{33} \\ \rho_k &= 0, k \geq 3 \end{cases}$
   (e) $Z_t = (1 + B + 0.25B^2)a_t = (1 + 0.5B)^2 a_t$
   $=> a_t = Z_t(\sum_{i=0}^{\infty}(-0.5)^i B^i)^2 = \sum_{k=0}^{\infty}(k+1)(-0.5)^k Z_{t-k}$
   $=> \pi_k = (k+1)(-0.5)^k, k = 0, 1, 2...$

3. (a) AR characteristic equation is $1 - 0.5x + 0.06x^2 = 0$
   $=> x_1 = 5, x_2 = \frac{10}{3}$
   (b) Since $|x_1| > 1, |x_2| > 1$
   $\{Z_t\}$ is stationary and causal.
   (c) $\begin{cases} \gamma_0 &= 0.5\gamma_1 - 0.06\gamma_2 + 1 \\ \gamma_1 &= 0.5\gamma_0 - 0.06\gamma_1 \\ \gamma_2 &= 0.5\gamma_1 - 0.06\gamma_0 \end{cases}$
   $=> \gamma_0 = 1.2908, \gamma_1 = 0.6089, \gamma_2 = 0.2270$

4. $\begin{cases} \gamma_0 &= 0.7\gamma_4 + \sigma^2 \\ \gamma_1 &= 0.7\gamma_3 \\ \gamma_2 &= 0.7\gamma_2 \\ \gamma_3 &= 0.7\gamma_1 \\ \gamma_4 &= 0.7\gamma_0 \\ \gamma_k &= 0.7\gamma_{k-4}, k \geq 5 \end{cases}$
   $=> \begin{cases} \gamma_k &= \frac{\sigma^2}{0.51} \cdot 0.7^n, \text{ if } k = 4n, n = 0, 1, 2... \\ \gamma_k &= 0, \text{ otherwise} \end{cases}$

5. $(1-0.6B)Z_t = (1+0.2B)a_t$
$=> Z_t = a_t(1+0.2B)(\sum_{i=0}^{\infty}(0.6)^i B^i) = a_t + \sum_{i=1}^{\infty}(\frac{4}{3})(0.6)^i a_{t-i}$
$=> a_t = Z_t(1-0.6B)(\sum_{i=0}^{\infty}(-0.2)^i B^i) = Z_t + \sum_{i=1}^{\infty}(4)(-0.2)^i Z_{t-i}$

6. (a) $(1-0.5B)(1-B)Z_t = (1-0.3B+0.6B^2)a_t$
$=>$ ARIMA$(1,1,2)$

(b) $(1-B)^3 Z_t = (1+0.1B)a_t$
$=>$ ARIMA$(0,3,1)$

7. (a) $(1-0.3B)^2 Z_t = (1-0.2B)a_t$
$=> a_t = Z_t(1-0.3B)^2(\sum_{i=0}^{\infty}(0.2)^i B^i) = Z_t - 0.4Z_{t-1} + \sum_{i=2}^{\infty}(\frac{1}{4})(0.2)^i Z_{t-i}$

(b) $$\begin{cases} E[a_t Z_t] &= E[a_t^2] = 1 \\ E[a_{t-1} Z_t] &= 0.6E[a_{t-1}Z_{t-1}] - 0.2E[a_{t-1}^2] = 0.4 \end{cases}$$

$$=> \begin{cases} \gamma_0 &= 0.6\gamma_1 - 0.09\gamma_2 + 0.92 \\ \gamma_1 &= 0.6\gamma_0 - 0.09\gamma_1 - 0.2 \\ \gamma_2 &= 0.6\gamma_1 - 0.09\gamma_0 \\ \gamma_k &= 0.6\gamma_{k-1} - 0.09\gamma_{k-2}, k \geq 3 \end{cases}$$

$$=> \begin{cases} \gamma_0 &= \frac{893600}{753571} = 1.1858 \\ \gamma_1 &= \frac{353620}{753571} = 0.4693 \\ \gamma_2 &= \frac{131748}{753571} = 0.1748 \end{cases}$$

$$=> \begin{cases} \rho_0 &= 1 \\ \rho_1 &= \frac{17681}{44680} = 0.3957 \\ \rho_2 &= \frac{131748}{893600} = 0.1474 \\ \rho_k &= \mathbf{0.6p_{k-1} - 0.09p_{k-2} \quad k = 3, 4, \ldots} \end{cases}$$

8. $E(Z_t) = E\left(\frac{a_t}{\phi^2} - \left(1-\frac{1}{\phi^2}\right)\sum_{k=1}^{\infty}\frac{a_{t+k}}{\phi^k}\right) = 0.$ For $k \neq 0$,

$$Cov(Z_t, Z_{t+k}) = Cov\left(\frac{a_t}{\phi^2} - \left(1-\frac{1}{\phi^2}\right)\sum_{j=1}^{\infty}\frac{a_{t+j}}{\phi^j}, \frac{a_{t+k}}{\phi^2} - \left(1-\frac{1}{\phi^2}\right)\sum_{j=1}^{\infty}\frac{a_{t+j+k}}{\phi^j}\right) = 0$$

$$Var(Z_t) = Var\left(\frac{a_t}{\phi^2} - \left(1-\frac{1}{\phi^2}\right)\sum_{k=1}^{\infty}\frac{a_{t+k}}{\phi^k}\right) = \frac{\sigma^2}{\phi^2} < \infty$$