Name: CHAN King Yeung
SID: 1155119394
STAT2006 Assignment 1

Question 1

a) $\int_0^\infty \left(1-F_X(x)\right) dx - \int_{-\infty}^0 F_X(x)\, dx = \left[x\left(1-F_X(x)\right)\right]_0^\infty + \int_0^\infty x f_X(x)\, dx - [xF_X(x)]_{-\infty}^0 + \int_{-\infty}^0 x f_X(x)\, dx$

$= \int_{-\infty}^\infty x f_X(x)\, dx$

$= E(X)$

b) $f_{XY}(x,y) = f_X(x) f_Y(y)$

$Z = X + Y$ $\qquad W = X$

$Y = Z - W$ $\qquad X = W$

$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$f_{WZ}(w,z) = f_X(w) f_Y(z-w)|1|$

$f_Z(z) = \int_{-\infty}^\infty f_X(w) f_Y(z-w)\, dw$

$= \int_{-\infty}^\infty f_X(x) f_Y(z-x)\, dx$

Question 2

a) $M_{X+Y}(t) = E\left[e^{(X+Y)t}\right]$

$= E(e^{Xt})E(e^{Yt})$

$= e^{\lambda(t-1)} \cdot e^{\mu(t-1)}$

$= e^{(\lambda+\mu)(t-1)}$

$X + Y \sim Poisson(\lambda + \mu)$

b) $P(X = x | X + Y = n) = \frac{P(X=x, Y=n-x)}{P(X+Y=n)}, x = 0, 1, \dots, n$

$= \frac{P(X=x)P(Y=n-x)}{P(X+Y=n)}$

$= \frac{\frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\mu^{n-x} e^{-\mu}}{(n-x)!}}{\frac{(\lambda+\mu)^n e^{-(\lambda+\mu)}}{n!}}$

$= \binom{n}{x} \frac{\lambda^x \mu^{n-x}}{(\lambda+\mu)^n} \cdot \left(\frac{\lambda+\mu}{\lambda+\mu}\right)^x$

$= \binom{n}{x} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$

$P(X = x | Y + X = n) = 0, x \neq 0, 1, \dots, n$

$X | X + Y = n \sim Binomial\left(n, \frac{\lambda}{\lambda+\mu}\right)$

c) Since $X | X + Y$ does not follow Poisson distribution, they are not independent

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Question 3

a) $F_Y(y) = P(Y \le y)$
$= P(X^2 \le y)$
$= P(-\sqrt{y} \le X \le \sqrt{y})$
$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$
$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$f_Y(y) = \frac{\partial}{\partial y}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$
$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$
$$= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, y \ge 0$$

$f_Y(y) = 0, y < 0$

b)

i. $f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x_1^2+x_2^2)}{2\sigma^2}}$

$Y_1 = X_1^2 + X_2^2 \qquad Y_2 = \frac{X_1}{\sqrt{X_1^2+X_2^2}}$

$X_1 = Y_2\sqrt{Y_1} \qquad X_2 = \pm\sqrt{Y_1(1-Y_2^2)}$

$$J_{X_2>0} = \begin{vmatrix} \frac{Y_2}{2\sqrt{Y_1}} & \sqrt{Y_1} \\ \frac{\sqrt{1-Y_2^2}}{2\sqrt{Y_1}} & -\frac{Y_2\sqrt{Y_1}}{\sqrt{1-Y_2^2}} \end{vmatrix} = -\frac{1}{2\sqrt{1-Y_2^2}}$$

$$J_{X_2<0} = \begin{vmatrix} \frac{Y_2}{2\sqrt{Y_1}} & \sqrt{Y_1} \\ -\frac{\sqrt{1-Y_2^2}}{2\sqrt{Y_1}} & \frac{Y_2\sqrt{Y_1}}{\sqrt{1-Y_2^2}} \end{vmatrix} = \frac{1}{2\sqrt{1-Y_2^2}}$$

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi\sigma^2} e^{\frac{-y_1}{2\sigma^2}} \left|-\frac{1}{2\sqrt{1-Y_2^2}}\right| + \frac{1}{2\pi\sigma^2} e^{\frac{-y_1}{2\sigma^2}} \left|\frac{1}{2\sqrt{1-Y_2^2}}\right|$$
$$= \frac{1}{2\pi\sigma^2} e^{\frac{-y_1}{2\sigma^2}} \cdot \frac{1}{\sqrt{1-y_2^2}}, y_1 > 0, y_2 \in (-1,1)$$

ii. Since the support of the joint distribution is the product set of space of Y_1 and space of Y_2; and $f_{Y_1Y_2}(y_1, y_2)$ can be rewrite as $\left[g(y_1) := \frac{1}{2\sigma^2} e^{\frac{-y_1}{2\sigma^2}}\right]\left[h(y_2) := \frac{1}{\pi\sqrt{1-y_2^2}}\right]$, where $Y_1 \sim Exponential(2\sigma^2)$ and $Y_2 \sim \frac{1}{\pi\sqrt{1-y_2^2}} \Longrightarrow \int_{-1}^{1} \frac{1}{\pi\sqrt{1-y_2^2}} dy_2 = \frac{1}{\pi}[sin^{-1}(y_2)]_{-1}^{1} = \frac{\pi}{\pi} = 1$, both are valid pdfs

Y_1 and Y_2 are independent

Question 4

a) $f_Y(y) = \int_0^\infty f_Y(y|\lambda) f_\Lambda(\lambda)\, d\lambda$

$= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \cdot \frac{\lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, d\lambda$

$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{y+\alpha-1} e^{\frac{-\lambda}{\left(\frac{\beta}{1+\beta}\right)}} \left(\frac{\frac{\beta}{1+\beta}}{\frac{\beta}{1+\beta}}\right)^{y+\alpha} d\lambda$

$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{1+\beta}\right)^{y+\alpha}, y = 0, 1, \ldots$

$= \binom{y+\alpha-1}{y} \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha, \alpha \text{ is a positive integer}$

$Y \sim Negative\ Binomial\left(\alpha, \frac{1}{1+\beta}\right)$

$E(Y) = E[E(Y|\Lambda)]$
$= E(\Lambda)$
$= \alpha\beta$

$Var(Y) = Var[E(Y|\Lambda)] + E[Var(Y|\Lambda)]$
$= Var(\Lambda) + E(\Lambda)$
$= \alpha\beta(\beta+1)$

b) $P(Y = y|\lambda) = \sum_{n=y}^\infty P(Y = y|N = n, \lambda) P(N = n|\lambda)$

$= \sum_{n=y}^\infty \binom{n}{y} p^y (1-p)^{n-y} \frac{\lambda^n e^{-\lambda}}{n!}$

$= \frac{e^{-\lambda}}{y!} [(1-p)\lambda]^y \sum_{m=0}^\infty \frac{[(1-p)\lambda]^m}{m!} \cdot \frac{e^{-(1-p)\lambda}}{e^{-(1-p)\lambda}}, m = n - y$

$= \frac{(p\lambda)^y e^{-p\lambda}}{y!}, y = 0, 1, \ldots$

$P(Y = y) = \int_0^\infty \frac{(p\lambda)^y e^{-p\lambda}}{y!} \cdot \frac{\lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, d\lambda$

$= \frac{p^y}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{1+\beta p}\right)^{\alpha+y}, y = 0, 1, \ldots$

$= \binom{y+\alpha-1}{y} \left(\frac{\beta p}{1+\beta p}\right)^y \left(\frac{1}{1+\beta p}\right)^\alpha, \alpha \text{ is a positive integer}$

$Y \sim Negative\ Binomial\left(\alpha, \frac{1}{1+\beta p}\right)$

Question 5

a) $E(Y) = E[E(Y|X)]$
$= E(X)$
$= \frac{1}{2}$

$Var(Y) = Var[E(Y|X)] + E[Var(Y|X)]$
$= Var(X) + E(X^2)$
$= \frac{1}{12} + \frac{1}{3}$
$= \frac{5}{12}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$
$= E[E(XY|X)] - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
$= E[XE(Y|X)] - \frac{1}{4}$
$= E(X^2) - \frac{1}{4}$
$= \frac{1}{3} - \frac{1}{4}$
$= \frac{1}{12}$

b) Since $Y|X = x \sim N(1,1)$, $\frac{Y}{X}$ and X are independent