# STAT4005 Solution to Assignment 3

## Question 1

Time series, ACF, and PACF of $Y$ are plotted in Figure 1.

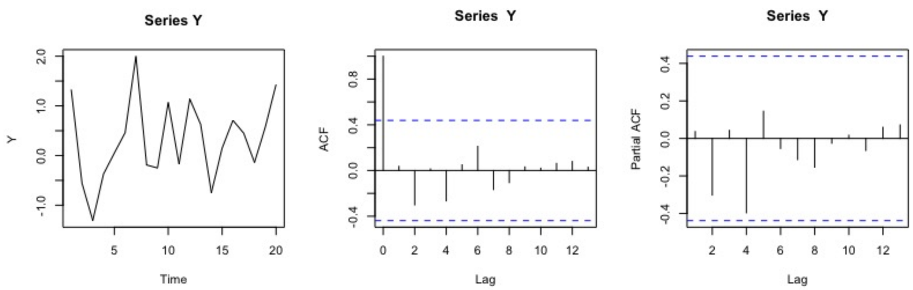


Figure 1: Values, ACF, and PACF of $Y$

## Question 2

First and second moments of the $MA(1)$ model: $Y_t = Z_t + \theta Z_{t-1}$ are

$$\begin{cases} \gamma(0) &= (1+\theta^2)\sigma^2, \\ \gamma(1) &= \theta\sigma^2 \end{cases}$$

Hence, moment estimates of $\theta$ and $\sigma^2$ satisfy

$$\begin{cases} C(0) &= (1+\hat{\theta}^2)\hat{\sigma}^2, \\ C(1) &= \hat{\theta}\hat{\sigma}^2 \end{cases}$$

where $C(k) = \frac{1}{n}\sum_{t=1}^{n-k}(Y_t - \bar{Y})(Y_{t+k} - \bar{Y})$ is the $k$-th sample moment, $k \geq 0$. Hence,

$$\frac{\hat{\theta}}{1+\hat{\theta}^2} = \frac{C(1)}{C(0)} = r(1)$$

$$r(1)\hat{\theta}^2 - \hat{\theta} + r(1) = 0$$

Hence,

$$\hat{\theta}_1 = 0.03629, \quad \hat{\theta}_2 = 27.56$$

$\hat{\theta}_2$ is rejected since $|\hat{\theta}_2| > 1$ implies non-invertibility of the model. That is, $\hat{\theta} = 0.03629$, and

$$\hat{\sigma}^2 = \frac{C(1)}{\hat{\theta}} = 0.6415$$

# Question 3

Let $\boldsymbol{Y}_t = (Y_t, Y_{t+1}, \dots, Y_{17+t})'$, $t = 1, 2, 3$, then least squares estimate of $\boldsymbol{\phi} = (\phi_2, \phi_1)$ are found from

$$\boldsymbol{Y}_3 = \boldsymbol{X}\boldsymbol{\phi} + \boldsymbol{\epsilon}$$

where $\boldsymbol{X} = (\boldsymbol{Y}_2, \boldsymbol{Y}_1)$. Hence,

$$\hat{\boldsymbol{\phi}} = \begin{pmatrix} \hat{\phi}_2 \\ \hat{\phi}_1 \end{pmatrix} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}_3 = \begin{pmatrix} 0.2354 \\ -0.2230 \end{pmatrix}$$

$$\hat{\sigma}^2 = \frac{1}{n-2}(\boldsymbol{Y}_3 - \boldsymbol{X}\hat{\boldsymbol{\phi}})'(\boldsymbol{Y}_3 - \boldsymbol{X}\hat{\boldsymbol{\phi}}) = 0.6432$$

Asymptotic distribution of $\hat{\boldsymbol{\phi}}$ is $\sqrt{n}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \sim \boldsymbol{N}(\boldsymbol{0}, \sigma^2\boldsymbol{\Gamma}_2^{-1})$, where

$$\boldsymbol{\Gamma}_2 = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}$$

Hence,

$$\widehat{var}(\hat{\boldsymbol{\phi}}) = \frac{\hat{\sigma}^2}{n}\begin{bmatrix} C(0) & C(1) \\ C(1) & C(0) \end{bmatrix}^{-1} = \frac{\hat{\sigma}^2}{n}\begin{bmatrix} 1.559 & -0.05649 \\ -0.05649 & 1.559 \end{bmatrix} = \begin{bmatrix} 0.05014 & -0.001817 \\ -0.001817 & 0.05014 \end{bmatrix}$$

$95\%$ confidence interval for $\hat{\phi}_k$ is $\hat{\phi}_k \pm 2\sqrt{\widehat{var}(\hat{\phi}_k)}$. Hence, $95\%$ C.I. for $\hat{\phi}_2$ is $[-0.2125, 0.6832]$, that for $\hat{\phi}_1$ is $[-0.6709, 0.2248]$.

# Question 4

Yule-Walker equations for the $AR(2)$ model: $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t$, are

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1)$$
$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0)$$

Hence, Yule-Walker estimates of $\boldsymbol{\phi} = (\phi_1, \phi_2)'$ is

$$\begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} C(0) & C(1) \\ C(1) & C(0) \end{pmatrix}^{-1} \begin{pmatrix} C(1) \\ C(2) \end{pmatrix} = \begin{pmatrix} 0.04718 \\ -0.3020 \end{pmatrix}$$

# Question 5

For the $ARMA(1,1)$ model of $Y_t - \phi Y_{t-1} = Z_t + \theta Z_{t-1}$, conditional on $Z_0 = 0$, $Y_0 = 0$, we have

$$\begin{aligned} Z_1 &= Y_1 \\ Z_t &= Y_t - \phi Y_t - \theta Z_{t-1}, \quad t \geq 2 \end{aligned}$$

Let $S_*(\phi, \theta) = \sum_{t=1}^{20} Z_t^2$, then conditional least squares estimate of $(\phi, \theta, \sigma^2)$ is given by

$$\begin{aligned} (\hat{\phi}, \hat{\theta}) &= \arg\min_{\phi,\theta} S_*(\phi, \theta) \\ \hat{\sigma}^2 &= \frac{1}{n-1} S_*(\hat{\phi}, \hat{\theta}) \end{aligned}$$

Hence, $\hat{\phi} = -0.5109$, $\hat{\theta} = 0.8440$, $\hat{\sigma}^2 = 0.6810$.

# Question 6

Assume $Z_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$, maximum likelihood estimate of $(\phi, \theta, \sigma^2)$ is given by

$$(\hat{\phi}, \hat{\theta}, \hat{\sigma}^2) = \arg\min_{\phi,\theta,\sigma^2} l(Y_1, Y_2, \ldots, Y_{20}) = \arg\min_{\phi,\theta,\sigma^2} \frac{1}{(2\pi)^{\frac{20}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \boldsymbol{Y'\Sigma^{-1}Y}\right)$$

where $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_{20})'$, and

$$\mathbf{\Sigma} = (\gamma(|i-j|)) = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ & & \ddots & \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{pmatrix}$$

Hence, $\hat{\phi} = -0.3883$, $\hat{\theta} = 0.9999$, $\hat{\sigma}^2 = 0.5254$, and maximized likelihood = $-23.0844$.

# Question 7

$$FPE = \hat{\sigma}^2 \frac{n+p}{n-p}$$

where $\hat{\sigma}^2$ is MLE of $\sigma^2$. According to Table 1, $AR(1)$ has the smallest FPE.

# Question 8

$$AICC = -2 \text{ log-likelihood} + \frac{2(p+q+1)n}{n-p-q-2}$$

According to Table 2, $MA(2)$ has the smallest AICC.

Table 1: FPE for $AR(p),\ p = 1, 2, 3, 4, 5$.

| model | FPE |
|---|---|
| AR(1) | 0.8042 |
| AR(2) | 0.8422 |
| AR(3) | 0.8882 |
| AR(4) | 0.9319 |
| AR(5) | 0.8477 |

Table 2: AICC for $MA(q),\ q = 1, 2, 3, 4, 5$.

| model | AICC |
|---|---|
| MA(1) | 53.3839 |
| MA(2) | 53.2022 |
| MA(3) | 56.2802 |
| MA(4) | 59.4228 |
| MA(5) | 63.5976 |

# Question 9

Fit the model by default method in R (maximum likelihood conditional on starting values), then residuals are $\hat{Z}_t = (0.9836, -1.0610, -1.4551, -0.4459, -0.2031, 0.1643, 1.6398, -0.8093, -0.4393, 0.8158, -0.6446, 0.9218, 0.1368, -1.1054, 0.0127, 0.3764, 0.0525, -0.4806, 0.3230, 1.0419)$. Now conduct the Ljung-Box test.

$H_0$ : the series of residuals exhibits no autocorrelation up to lag $10$.

$H_1$ : some autocorrelation coefficient, $\rho(k),\ k = 1, 2, \dots, 10,$ are nonzero.

The test statistic is $Q(10) = n(n+1)\sum_{j=1}^{10} \frac{\widehat{\rho}^2(j)}{n-j} = 11.7303$. Under $H_0$, $Q(10) \to \chi^2(9)$. Since $Q(10) < \chi^2_{0.95}(9) = 16.9190$, the null hypothesis of no autocorrelation up to lag $10$ cannot be rejected.

# Question 10

Consider the family of models, $ARMA(p, q)$:

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $p \in \{0, 1, 2, 3, 4, 5\}$, $q \in \{0, 1, 2, 3, 4, 5\}, p = q = 0$ excluded.
By AIC/AICC, the best model is $ARMA(0, 2)$. By BIC, the best model is $ARMA(1, 4)$. By FPE, the best model is $ARMA(2, 4)$. Since the data set is small, by consideration of parsimony, $MA(2)$ can be chosen. Analysis of residuals shows that $MA(2)$ provides a good fit to the data.