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STAT 2006
Assignment 2 (1)

## Question 1

X and Y are jointly normal r.v.

$\Rightarrow X+Y \sim N(\mu_X+\mu_Y,\ \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2)$

$$M_{XY}(s,t) = E(e^{sX+tY})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx+ty} f_{XY}(x,y)$$
$$= e^{[s\mu_X + t\mu_Y + \frac{1}{2}(s^2\sigma_X^2 - 2\rho\sigma_X\sigma_Y st + t^2\sigma_Y^2)]}$$

## Question 2

a) $U_1 = X_1 \quad U_i = X_i - X_{i-1} \quad i=2,\dots,n$

$\Rightarrow X_i = \sum_{j=1}^{i} U_j \quad i=1,\dots,n \quad U_j > 0$

$$\frac{\partial x_i}{\partial u_j} = \begin{cases} 1 & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases}$$

$$J = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{i=1}^{n} 1 = 1$$

$$f_{U_1\dots U_n}(u_1,\dots,u_n) = |1| \cdot \frac{n!}{\theta^n} e^{\sum_{i=1}^{n}\sum_{j=1}^{i} u_j \cdot \frac{-1}{\theta}}$$
$$= \frac{n!}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}(n-i+1)u_i}$$

for $u_i > 0 \quad i=1,\dots,n$

b) we can rewrite the pdf as
$$f(u_1,\dots,u_n) = \prod_{i=1}^{n} \frac{(n-i+1)}{\theta} e^{-\frac{1}{\theta}(n-i+1)u_i}$$

$U_i$'s are independent

$\Rightarrow U_i \sim exp\left(\frac{\theta}{n-i+1}\right), \quad i=1,\dots,n$

c) $E(X_1) = E(U_1)$
$= \frac{\theta}{n}$

$E(X_n) = \sum_{i=1}^{n} E(U_i)$
$= \sum_{i=1}^{n} \frac{\theta}{n-i+1}$
$= \frac{2\theta}{n(1+n)}$

## Question 3

a) $EX = a_X EZ_1 + b_X EZ_2 + c_X = c_X$

$Var X = a_X^2 Var Z_1 + b_X^2 Var Z_1 = a_X^2 + b_X^2$

$EY = a_Y EZ_1 + b_Y EZ_2 + c_Y = c_Y$

$Var Y = a_Y^2 Var Z_1 + b_Y^2 Var Z_1 = a_Y^2 + b_Y^2$

$Cov(X,Y) = a_X a_Y EZ_1^2 + a_X b_Y EZ_1Z_2 + a_X c_Y EZ_1$
$+ a_Y b_X EZ_1Z_2 + b_X b_Y EZ_2^2 + b_X c_Y EZ_2$
$+ a_Y c_X EZ_1 + b_Y c_X EZ_2 + c_X c_Y - c_X c_Y$
$= a_X a_Y + b_X b_Y$

b) $EX = c_X = \mu_X$

$Var X = a_X^2 + b_X^2 = \frac{1+\rho}{2}\sigma_X^2 + \frac{1-\rho}{2}\sigma_X^2 = \sigma_X^2$

$EY = c_Y = \mu_Y$

$Var Y = a_Y^2 + b_Y^2 = \frac{1+\rho}{2}\sigma_Y^2 + \frac{1-\rho}{2}\sigma_Y^2 = \sigma_Y^2$

$Corr(X,Y) = \frac{a_X a_Y + b_X b_Y}{\sigma_X\sigma_Y} = \frac{\frac{1+\rho}{2}\sigma_X\sigma_Y - \frac{1-\rho}{2}\sigma_X\sigma_Y}{\sigma_X\sigma_Y} = \rho$

c) $X = a_X Z_1 + b_X Z_2 + c_X \qquad Y = a_Y Z_1 + b_Y Z_2 + c_Y$

$\Rightarrow Z_1 = \frac{b_Y(X-c_X) - b_X(Y-c_Y)}{a_X b_Y - a_Y b_X} = \frac{\sigma_Y(X-\mu_X) + \sigma_X(Y-\mu_Y)}{\sqrt{2(1+\rho)}\,\sigma_X\sigma_Y}$

$Z_2 = \frac{a_Y(X-c_X) - a_X(Y-c_Y)}{-(a_X b_Y - a_Y b_X)} = \frac{\sigma_X(X-\mu_X) + \sigma_Y(Y-\mu_Y)}{\sqrt{2(1-\rho)}\,\sigma_X\sigma_Y}$

$$J = \begin{vmatrix} \frac{b_Y}{a_X b_Y - a_Y b_X} & \frac{-b_X}{a_X b_Y - a_Y b_X} \\ \frac{-a_Y}{a_X b_Y - a_Y b_X} & \frac{a_X}{a_X b_Y - a_Y b_X} \end{vmatrix}$$
$$= \frac{-1}{\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}$$

$$f_{XY}(x,y) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sigma_Y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y)}{\sqrt{2(1+\rho)}\,\sigma_X\sigma_Y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\sigma_X(x-\mu_X)+\sigma_Y(y-\mu_Y)}{\sqrt{2(1-\rho)}\,\sigma_X\sigma_Y}}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

for $x,y \in R$

$\therefore (X,Y)$ follows bivariate normal pdf

## Question 4

a) $E(T_\alpha) = \frac{\alpha}{n}\sum_{i=1}^{n} E(X_i) + \frac{1-\alpha}{m}\sum_{i=1}^{m} E(Y_i)$

$= \alpha\theta + (1-\alpha)\theta$

$= \theta$

$Var(T_\alpha) = \frac{\alpha^2}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{(1-\alpha)^2}{m^2}\sum_{i=1}^{m} Var(Y_i)$

$= \theta^2\left[\frac{\alpha^2}{n} + \frac{(1-\alpha)^2}{m}\right]$

b) $P(|T_\alpha - \theta| > \varepsilon) \le \frac{MSE(T_\alpha)}{\varepsilon^2}$

$= \frac{\theta^2}{\varepsilon^2}\left[\frac{\alpha^2}{n} + \frac{(1-\alpha)^2}{m}\right]$

$\because \lim_{n,m\to\infty} \frac{\theta^2}{\varepsilon^2}\left[\frac{\alpha^2}{n} + \frac{(1-\alpha)^2}{m}\right] = 0$

## Question 5

a) $E[\ln(X_i)] = \int_0^1 \ln(x)\cdot 1\, dx$

$= [x\ln(x)]_0^1 - \int_0^1 1\, dx$

$= -1$

$Var[\ln(X_i)] = \int_0^1 \ln(x)^2 \cdot 1\, dx - (-1)^2$

$= [x\ln(x)^2]_0^1 - \int_0^1 2\ln(x)dx - 1$

$= 1$

b) $P(a \le (\prod_{i=1}^{n} X_i)^{\frac{1}{\sqrt{n}}} e^{\sqrt{n}} \le b) = P(\ln(a) \le \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \ln(X_i) + \sqrt{n} \le \ln(b))$

$= P(\ln(a) \le \frac{\sum_{i=1}^{n}\ln(X_i) - (-1)n}{(1)\sqrt{n}} \le \ln(b))$

by CLT,

$\lim_{n\to\infty} P(\ln(a) \le Z \le \ln(b)) = \Phi(\ln(b)) - \Phi(\ln(a))$

## Question 6

a) $L(\theta; x_1, \dots, x_n) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$ , $x \in [0,1]$ $\theta > 0$

$\ell(\theta; x_1, \dots, x_n) = n\ln(\theta) + (\theta - 1)\sum_{i=1}^{n} \ln(x_i)$

con't a) $\frac{\partial \ell}{\partial \theta}\Big|_{\hat\theta} = 0$

$0 = \frac{n}{\hat\theta} - \sum_{i=1}^{n}\ln(x_i)$

$\hat\theta = \frac{-n}{\sum_{i=1}^{n}\ln(x_i)}$

$\frac{\partial^2 \ell}{\partial\theta^2} = -\frac{n}{\theta^2}$

$< 0$ for any $\theta$

$\therefore \hat\theta_{MLE} = \frac{-n}{\sum_{i=1}^{n}\ln(X_i)}$

b) $\hat\theta = \frac{-5}{-2.2336}$ $\qquad \sum_{i=1}^{n}\ln(x_i) = -2.2336$

$\approx 2.2385$

c) $P(Y \le y) = P(-\ln(X) \le y)$ , $y > 0$

$= 1 - P(X \le e^{-y})$

$= 1 - \int_0^{\infty} \theta x^{\theta-1} dx$

$= 1 - e^{-\theta y}$

$\therefore Y \sim exp(\frac{1}{\theta})$

d) $M_S(t) = \prod_{i=1}^{n} E(e^{tY_i})$

$= \prod_{i=1}^{n} M_Y(t)$

$= (1 - \frac{1}{\theta}t)^{-n}$

$\therefore S \sim \Gamma(n, \frac{1}{\theta})$

e) $E(\hat\theta) = nE(\frac{1}{S})$

$= \frac{n\Gamma(n-1)(\frac{1}{\theta})^{n-1}}{\Gamma(n)(\frac{1}{\theta})^n}\int_0^\infty \frac{s^{(n-1)-1}e^{-s\theta}}{\Gamma(n-1)(\frac{1}{\theta})^{n-1}} ds$

$= \frac{n\theta}{n-1}$

$\because E(\hat\theta) \ne \theta$

$\therefore$ $\hat\theta_{MLE}$ is NOT an unbiased estimator

## Question 7

c) $f(x;\theta) = \frac{2x}{\theta}\mathbb{1}(x<\theta)$

$L(\theta; x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{2x_i}{\theta}\mathbb{1}(x_i < \theta)$

$= \frac{2^n \prod_{i=1}^{n} x_i}{\theta^n}\mathbb{1}(max(x) < \theta)$

$\because L \propto \frac{1}{\theta^n}$ is a decreasing function

$\therefore \hat\theta_{MLE} = \max_{1\le i\le n}(X_i)$

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b) $Y = \max(X_i) \quad y \in (0, \theta)$

$$P(Y \le y) = P(X \le y)^n = \left[\int_0^y \frac{2x}{\theta^2} dx\right]^n = \frac{y^{2n}}{\theta^{2n}}$$

$$f_Y(y) = \frac{2n y^{2n-1}}{\theta^{2n}} 1_{(0<y<\theta)}$$

$$E(\hat\theta) = E(Y) = \int_0^\theta y \frac{2n y^{2n-1}}{\theta^{2n}} dy = \frac{2n}{\theta^{2n}}\left[\frac{y^{2n+1}}{2n+1}\right]_0^\theta = \frac{2n\theta}{2n+1}$$

$$E(c\hat\theta) = \theta$$
$$\theta = c \cdot \frac{2n\theta}{2n+1}$$
$$c = \frac{2n+1}{2n}$$

c) $\int_0^x \frac{2t}{\theta^2} dt = \frac{1}{2}$

$$\frac{x^2}{\theta^2} =$$
$$x = \frac{\theta}{\sqrt2}$$

$\Rightarrow$ MLE for the median $= \frac{\hat\theta}{\sqrt2}$

$\Rightarrow$ unbiased estimator $= \frac{(2n+1)\hat\theta}{2n\sqrt n}$

Question 8

a) $L(\lambda; x_1, \dots, x_n) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}, \quad x, \lambda > 0$

$$= \frac{\lambda^{\sum_{i=1}^n x_i} e^{-\lambda n}}{\prod_{i=1}^n x_i!}$$

$$\ell(\lambda; x_1, \dots, x_n) = \sum_{i=1}^n x_i \ln(\lambda) - \lambda n - \sum_{i=1}^n \ln(x_i)$$

$\frac{\partial \ell}{\partial \lambda}\Big|_{\hat\lambda} = 0 \qquad \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{\sum_{i=1}^n x_i}{\lambda^2}$

$0 = \frac{\sum_{i=1}^n x_i}{\hat\lambda} - n \qquad < 0$ for any $\lambda$

$\hat\lambda = \bar x$

$\therefore \hat\lambda_{MLE} = \bar x$

b) $\hat\lambda = \frac{180}{50}$

$= 3.6$

Question 9

a) $L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n \frac{\theta^4}{6} x_i^3 e^{-\theta x_i}, \quad x, \theta > 0$

$$= \frac{\theta^{4n}}{6^n} \prod_{i=1}^n x_i^3 e^{-\theta \sum_{i=1}^n x_i}$$

$$\ell(\theta; x_1, \dots, x_n) = 4n\ln(\theta) - 3\sum_{i=1}^n \ln(x_i) - \theta\sum_{i=1}^n x_i - n\ln(6)$$

$\frac{\partial \ell}{\partial \theta}\Big|_{\hat\theta} = 0 \qquad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{4n}{\theta^2}$

$0 = \frac{4n}{\hat\theta} - \sum_{i=1}^n x_i \qquad < 0$ for any $\theta$

$\hat\theta = \frac{4}{\bar x}$

$\therefore \hat\theta_{MLE} = \frac{4}{\bar x}$

b) $L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n 1$ for $\theta = 1$ $\qquad L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n \frac{1}{2\sqrt{x_i}}$ for $\theta = 2$

$= 1 \qquad = \frac{1}{2^n \sqrt{\prod_{i=1}^n x_i}}$

$$\hat\theta_{MLE} = \begin{cases} 1 & \prod_{i=1}^n X_i > \frac{1}{2^{2n}} \\ 1 \text{ or } 2 & \prod_{i=1}^n X_i = \frac{1}{2^{2n}} \\ 2 & \prod_{i=1}^n X_i < \frac{1}{2^{2n}} \end{cases}$$

c) $f(x; \theta) = \theta \, 1_{(x \le \frac{1}{\theta})}$

$$L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n \theta \, 1_{(x \le \frac{1}{\theta})} = \theta^n 1_{(\max(x_i) \le \frac{1}{\theta})}$$

$\because L = \theta^n$ is an increasing function in $\theta$

$\therefore \hat\theta_{MLE} = \frac{1}{\max_{1\le i \le n}(X_i)}$

Question 10

$\because X|\theta \sim U(0, \theta)$

$\therefore E(X) = \frac{\theta}{2} \qquad Var(X) = \frac{\theta^2}{12}$

con't $\bar{x} = \frac{\theta}{2}$

$\Rightarrow \tilde{\theta}_{MM} = 2\bar{x}$

$\therefore E(\tilde{\theta}_{MM}) = \frac{2}{n}\sum_{i=1}^{n} E(X_i) = \theta$ $\quad Var(\tilde{\theta}_{MM}) = \frac{4}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\theta^2}{3n}$

$f(x|\theta) = \frac{1}{\theta} 1_{(x \le \theta)}$

$L(\theta|x) = \prod_{i=1}^{n} \frac{1}{\theta} 1_{(x_i \le \theta)} = \frac{1}{\theta^n} 1_{(max(x_i) \le \theta)}$

$\because L = \frac{1}{\theta^n}$ is a decreasing function in $\theta$

$\therefore \tilde{\theta}_{MLE} = \max_{1 \le i \le n}(X_i)$

$Y := max(X_i)$ $\quad y \in (0, \theta)$

$P(Y \le y) = P(X \le y)^n = [\int_0^y \frac{1}{\theta} dy]^n = \frac{y^n}{\theta^n}$

$f_Y(y) = \frac{n y^{n-1}}{\theta^n}$, $y \in (0, \theta)$

$E(\hat{\theta}_{MLE}) = E(Y) = \int_0^\theta y \frac{n y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n}[\frac{y^{n+1}}{n+1}]_0^\theta = \frac{n\theta}{n+1}$

$Var(\hat{\theta}_{MLE}) = E(Y^2) - (\frac{n\theta}{n+1})^2 = \frac{n}{\theta^n}[\frac{y^{n+2}}{n+2}]_0^\theta - (\frac{n\theta}{n+1})^2 = \frac{n\theta^2}{n+2} - (\frac{n\theta}{n+1})^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$

Question 11

a) $E(X) = \int_0^{\frac{\theta}{2}} x \cdot \frac{4x}{\theta^2} dx + \int_{\frac{\theta}{2}}^{\theta} -x \cdot \frac{4x}{\theta^2} + x \cdot \frac{4}{\theta} dx$

$= \frac{4}{\theta}[\frac{x^3}{3}]_0^{\frac{\theta}{2}} + [-\frac{4x^3}{3\theta^2} + \frac{2x^2}{\theta}]_{\frac{\theta}{2}}^{\theta}$

$= \frac{\theta}{2}$

$\bar{x} = \frac{\theta}{2}$

$\Rightarrow \tilde{\theta}_{MM} = 2\bar{x}$

b) $\tilde{\theta} = 2(\frac{3.7327}{10}) \approx 0.7465$ $\qquad \sum_{i=1}^{n} x_i = 3.7327$

Question 12

a) $M_W(t) = \prod_{i=1}^{n} E(e^{\frac{2}{\theta}tX_i}) = \prod_{i=1}^{n} M_X(\frac{2}{\theta}t) = (1-2t)^{-n}$

$\therefore W \sim \chi^2(2n)$

b) $[\frac{2\sum_{i=1}^{n} x_i}{\chi^2_{\alpha/2}(2n)}, \frac{2\sum_{i=1}^{n} x_i}{\chi^2_{1-\alpha/2}(2n)}]$

c) $[\frac{2(8)(66.2)}{26.3}, \frac{2(8)(65.2)}{7.962}] \approx [39.6654, 131.0224]$

Question 13

a) $f_X(x|\alpha,\beta) = \frac{\alpha x^{\alpha-1}}{\beta^\alpha} 1_{(x \le \beta)} 1_{(x \ge 0)}$

$L(\alpha,\beta|x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{\alpha x_i^{\alpha-1}}{\beta^\alpha} 1_{(x \le \beta)} 1_{(x \ge 0)} = \frac{\alpha^n \prod_{i=1}^{n} x_i^{\alpha-1}}{\beta^{\alpha n}} 1_{(max(x_i) \le \beta)} 1_{(min(x) \ge 0)}$

$\because L \propto \frac{1}{\beta^{\alpha n}}$ is a decreasing function in $\beta$, given $\alpha > 0$

$\therefore \hat{\beta}_{MLE} = \max_{1 \le i \le n}(X_i)$

$\ell(\alpha,\beta|x_1,\ldots,x_n) = n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) - \alpha n \ln(\beta)$

$\frac{\partial \ell}{\partial \alpha}\Big|_{\hat{\alpha},\hat{\beta}} = 0$ $\qquad \frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} < 0$ for any $\alpha$.

$0 = \frac{n}{\hat{\alpha}} + \sum_{i=1}^{n}\ln(x_i) - n\ln(\hat{\beta})$

$\hat{\alpha} = \frac{n}{n\ln(\hat{\beta}) - \sum_{i=1}^{n}\ln(x_i)}$

$\therefore \hat{\alpha}_{MLE} = \frac{n}{n\ln(\max_{1 \le i \le n}(X_i)) - \sum_{i=1}^{n}\ln(X_i)}$

b) $\hat{\beta} = 26$ $\qquad \max_{1 \le i \le n}(x_i) = 26$

$\hat{\alpha} = \frac{14}{(14)\ln(26) - 44.0327} \approx 8.8571$ $\qquad \sum_{i=1}^{n}\ln(x_i) = 44.0327$

c) $P\left(\frac{\max(X_i)}{\beta} \le c\right) = P(X \le c\beta)^n$

$= \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n}$

$= c^{\alpha_0 n}$

$0.05 = c^{\alpha_0 n}$

$c = 0.05^{\frac{1}{\alpha_0 n}}$

$\frac{\max(X_i)}{0.05^{\frac{1}{\alpha_0 n}}}$ is the 95% upper limit for $\beta$

d) $\left[\max(X_i), \frac{\max(X_i)}{0.05^{\frac{1}{\alpha_0 n}}}\right]$

$= \left[26, \frac{26}{0.05^{\frac{1}{(10)(8.857)}}}\right]$

$\approx [26, 26.6358]$

Question 14

a) $f_X(x) = \frac{d}{dx} \int_0^{\frac{x-\theta_1}{\theta_2}} \frac{1}{\lambda} e^{-\frac{1}{\lambda}y} dy$

$= \frac{1}{\lambda\theta_2} e^{-\frac{x-\theta_1}{\lambda\theta_2}}, \quad x \ge \theta_1$

b) $E(X) = \theta_1 + \theta_2 E(Y)$

$= \theta_1 + \lambda\theta_2$

$Var(X) = \theta_2^2 Var(Y)$

$= \lambda^2\theta_2^2$

$\bar{x} = \frac{\Sigma x_i}{n} = \theta_1 + \lambda\theta_2 \qquad V = \frac{\Sigma(x_i - \bar{x})^2}{n} = \lambda^2\theta_2^2$

$\Rightarrow \tilde{\theta}_{1MM} = \bar{x} - \sqrt{V} \qquad \tilde{\theta}_{2MM} = \frac{\sqrt{V}}{\lambda}$

c) $L(\theta_1, \theta_2; x_1, \dots, x_n) = \prod_{i=1}^n \frac{1}{\lambda\theta_2} e^{-\frac{x_i-\theta_1}{\lambda\theta_2}} 1_{(x_i \ge \theta)}$

$= \frac{1}{\lambda^n\theta_2^n} e^{-\frac{\sum_{i=1}^n x_i - n\theta_1}{\lambda\theta_2}} 1_{(\min(x_i) \ge \theta)}$

for $\theta_1 \le \min_{1\le i\le n}(x_i)$, $\frac{\partial L}{\partial\theta_1} = \frac{1}{\lambda^n\theta_2^n} e^{-\frac{\Sigma x_i - n\theta_1}{\lambda\theta_2}} \cdot \frac{n}{\lambda\theta_2}$

$= \frac{n}{\lambda^{n+1}\theta_2^{n+1}} e^{-\frac{\sum_{i=1}^n x_i - n\theta_1}{\lambda\theta_2}}$

$> 0$ for any $\theta_2$

$L$ is strictly increasing in $\theta_1$

con't c) for $\theta_1 > \min_{1\le i\le n}(x_i)$, $L \propto 1_{(\min(x_i) \ge \theta)}$

$= 0$

$\therefore$ $L$ is an increasing function in $\theta_1$

$\therefore \hat{\theta}_{1MLE} = \min_{1\le i\le n}(X_i)$ for any $\theta_2$

$\ell(\theta_1, \theta_2; x_1, \dots, x_n) = -\frac{\sum_{i=1}^n x_i - n\theta_1}{\lambda\theta_2} - n\ln(\lambda) - n\ln(\theta_2)$

$\frac{\partial\ell}{\partial\theta_2}\Big|_{\hat{\theta}_1, \hat{\theta}_2} = 0$

$0 = \frac{\sum_{i=1}^n x_i - n\hat{\theta}_1}{\lambda\hat{\theta}_2^2} - \frac{n}{\hat{\theta}_2}$

$\hat{\theta}_2 = \frac{\bar{x} - \hat{\theta}_1}{\lambda}$

$\frac{\partial^2\ell}{\partial\theta_2^2}\Big|_{\hat{\theta}_1, \hat{\theta}_2} = \frac{-2\left(\sum_{i=1}^n x_i - n\hat{\theta}_1\right)}{\lambda\left(\frac{\sum_{i=1}^n x_i - n\hat{\theta}_1}{n\lambda}\right)^3} + \frac{n}{\left(\frac{\sum_{i=1}^n x_i - n\hat{\theta}_1}{n\lambda}\right)^2}$

$= \frac{-n\lambda^2}{(\bar{x} - \hat{\theta}_1)^2} < 0$, for any $\theta_2 > 0$

$\therefore \hat{\theta}_{2MLE} = \frac{\bar{x} - \min_{1\le i\le n}(X_i)}{\lambda}$

Question 15

$ME = z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$

$n = \left(\frac{z_{\alpha/2}\,\sigma}{ME}\right)^2$

$= \left[\frac{(1.645)(58)}{10}\right]^2$

$\approx 91.0307$

$\therefore$ the minimal sample size is 92