Name: CHAN King Yeung
SID: 116511939 4

STAT 3008

Assignment 2

Question 1

a)

Variable	Coefficient	Std. Error	t-statistic	p-value
Constant	-23.4325	12.74	-1.8393	0.0842
X	1.2713	0.1528	8.32	2

t_0 for $\beta_0 = \frac{-23.4325-(0)}{12.74} \approx -1.8393$

$\beta_1 = 8.32 \times 0.1528 \approx 1.2713$

Source	df	SS	MS	F	p-value
Regression	1	1848.76	1848.76	69.2224	1.3956×10^{-7}
Residual	18	480.735	26.7075		
Total	19	2329.495			

$F_0 = (8.32)^2 = 69.2224$

$MS_{reg} = \frac{1848.76}{1} = 1848.76$

$MS_{res} = \frac{1848.76}{69.2224} \approx 26.7075$

$SS_{res} = 26.7075(18) = 480.735$

$SS_{total} = 1848.76 + 480.735 = 2329.495$

b) As the slope of the regression is positive, r_{xy} is also positive

$r_{xy} = \sqrt{\frac{1848.76}{2329.495}}$

≈ 0.8909

c) $H_0: \beta_0 = -10$ vs $H_1: \beta_0 \neq -10$

$t_0 = \frac{-23.4325-(-10)}{12.74}$ $\quad$ p-value = 0.3056

≈ -1.0544

Since p-value > 0.05, we do not reject H_0 at $\alpha = 0.05$

We do not have sufficient evidence that β_0 is different from -10

Question 2

a) $E(\hat{Y}'\hat{Y}) = E(\hat{\beta}'X'X\hat{\beta})$

$= E(Y'HY)$ where $H = X(X'X)^{-1}X'$

$= E(\beta'X'HX\beta + \beta'X'He + e'HX\beta + e'He)$

$= \beta'X'X\beta + E[tr(Hee')]$

$= \beta'X'X\beta + \sigma^2 tr[(X'X)^{-1}X'X]$

$= \beta'X'X\beta + (p+1)\sigma^2$

b) $\sum_{i=1}^{n} E(y_i^2) = E(Y'Y)$

$= E(\beta'X'X\beta + \beta'X'e + e'X\beta + e'e)$

$= \beta'X'X\beta + E[tr(ee')]$

$= \beta'X'X\beta + tr(\sigma^2 I_n)$

$= \beta'X'X\beta + n\sigma^2$

$= \beta'X'X\beta + (p+1)\sigma^2 + (n-p-1)\sigma^2$

$= E(\hat{Y}'\hat{Y}) + E(\hat{e}'\hat{e})$

$= \sum_{i=1}^{n} E(\hat{y}_i^2) + \sum_{i=1}^{n} E(\hat{e}_i^2)$

Question 4

a)

$$X'X = n\begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{u^2} & 0 \\ 0 & 0 & \overline{v^2} \end{pmatrix} \qquad (X'X)^{-1} = \begin{pmatrix} \frac{1}{n} & 0 & 0 \\ 0 & \frac{1}{SUU} & 0 \\ 0 & 0 & \frac{1}{SVV} \end{pmatrix} \qquad X'Y = \begin{pmatrix} n\bar{y} \\ SUY \\ SVY \end{pmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = \begin{pmatrix} \bar{y} \\ \frac{SUY}{SUU} \\ \frac{SVY}{SVV} \end{pmatrix} \Rightarrow \begin{aligned} \hat{\beta}_0 &= \bar{y} \\ \hat{\beta}_1 &= \frac{SUY}{SUU} \\ \hat{\beta}_2 &= \frac{SVY}{SVV} \end{aligned}$$

b) As $E(Y|U=u) = \alpha_0 + \alpha_1 u$ is a simple linear regression, we can directly state that

$$\hat{\alpha} = \begin{pmatrix} \bar{y} - \frac{SUY}{SUU}(\bar{u}) \\ \frac{SUY}{SUU} \end{pmatrix} = \begin{pmatrix} \bar{y} \\ \frac{SUY}{SUU} \end{pmatrix} \Rightarrow \begin{aligned} \hat{\alpha}_0 &= \bar{y} = \hat{\beta}_0 \\ \hat{\alpha}_1 &= \frac{SUY}{SUU} = \hat{\beta}_1 \end{aligned}$$

Question 5

a) For the true model, $E(Y) = E(X_2\beta + e) = X_2\beta$

$$\begin{aligned} E(\hat{\alpha}) &= (X'X)^{-1}X'E(Y) \\ &= (X'X)^{-1}X'X_2\beta \end{aligned}$$

b)

$$X'X = \begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix} \qquad (X'X)^{-1} = \frac{1}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{pmatrix} \qquad X'X_2 = \begin{pmatrix} n & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^3 \end{pmatrix}$$

$$(X'X)^{-1}X'X_2 = \frac{1}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}\begin{pmatrix} n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2 & (\sum_{i=1}^{n} x_i^2)^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^3 \\ 0 & -\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^2 - n\sum_{i=1}^{n} x_i^3 \end{pmatrix}$$

$$E(\hat{\alpha}) = \frac{1}{n^2(\overline{x^2} - \bar{x}^2)}\begin{pmatrix} n^2(\overline{x^2} - \bar{x}^2)\beta_0 + n^2[(\overline{x^2})^2 - \bar{x}(\overline{x^3})]\beta_1 \\ n^2[-\bar{x}(\overline{x^2}) + \overline{x^3})\beta_1 \end{pmatrix} \Rightarrow \begin{aligned} &\text{i) } E(\hat{\alpha}_0) = \beta_0 + \frac{[(\overline{x^2})^2 - \bar{x}(\overline{x^3})]\beta_1}{\overline{x^2} - \bar{x}^2} \\ &\text{ii) } E(\hat{\alpha}_1) = \frac{[-\bar{x}(\overline{x^2}) + \overline{x^3}]\beta_1}{\overline{x^2} + \bar{x}^2} \end{aligned}$$

c) i)

$$\begin{aligned} E(\lim_{n\to\infty} \hat{\alpha}_0) &= \beta_0 + \frac{\sigma_x^4 - (0)(k_x\sigma_x^3)}{\sigma_x^2 - (0)^2}\beta_1 \\ &= \beta_0 + \sigma_x^2\beta_1 \\ &\neq \beta_0 \end{aligned}$$

ii)

$$\begin{aligned} E(\lim_{n\to\infty} \hat{\alpha}_1) &= \frac{-(0)(\sigma_x^2) + k_x\sigma_x^3}{\sigma_x^2 + (0)^2}\beta_1 \\ &= k_x\sigma_x\beta_1 \\ &\neq \beta_1 \end{aligned}$$