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STAT 4003
Assignment 1

Question 1

a) $Y \sim B(n,p)$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) \\ &= \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} e^{ty} \\ &= \sum \binom{n}{y} (pe^t)^y (1-p)^{n-y} \\ &= (1-p+pe^t)^n, \quad t \in \mathbb{R} \end{aligned}$$

b) $Y \sim Poi(\lambda)$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) \\ &= \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} e^{ty} \\ &= e^{-\lambda} \sum \frac{(\lambda e^t)^y}{y!} \\ &= e^{\lambda(e^t-1)}, \quad t \in \mathbb{R} \end{aligned}$$

c) $Y \sim \Gamma(\alpha, \beta)$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) \\ &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} e^{ty}\, dy \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta-t}\right)^{\alpha-1} e^{-x} \frac{dx}{\frac{1}{\beta-t}} \\ &= \frac{\beta^\alpha \Gamma(\alpha)}{(\beta-t)^\alpha \Gamma(\alpha)} \\ &= \left(\frac{\beta}{\beta-t}\right)^\alpha, \quad t < \beta \end{aligned}$$

Question 2

a) $X \sim B(100, \frac{18}{38})$

$E(X) = 47.3684$, $Var(X) = 24.9307$

by CLT, $P(X > 50) = P\left(Z > \frac{50+0.5-47.3684}{\sqrt{24.9307}}\right)$

$\approx 1 - \Phi(0.63)$

$= 0.2643$

b) $X \sim Beta(3,1)$

$E(X) = \frac{3}{4}$, $Var(X) = \frac{3}{80}$

$\bar{X} \sim N(\frac{3}{4}, \frac{3}{1280})$

$P(\bar{X} < 0.5) = P\left(Z < \frac{0.5-0.75}{\sqrt{3/1280}}\right)$

$\approx \Phi(-5.16)$

≈ 0

Question 3

a) $X_1 \perp\!\!\!\perp X_2 \Rightarrow X_1 - X_2 \sim N(0,2)$

$P(X_1 - X_2 < 1) = P\left(Z < \frac{1-0}{\sqrt{2}}\right)$

$= \Phi(0.71)$

$= 0.7611$

b) $Cov(X_1-X_2, X_1+X_2) = Var(X_1) + Cov(X_1,X_2) - Cov(X_2,X_1) - Var(X_2)$

$= 0$

$X_1 - X_2$ and $X_1 + X_2$ are both joint normal and uncorrelated

$\Rightarrow X_1 - X_2 \perp\!\!\!\perp X_1 + X_2$

Question 4

let $T = \frac{X}{Y}$, $f(t) = \frac{\partial F(t)}{\partial t}$

$$\begin{aligned} &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \int_{-\infty}^{ty} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}\, dx\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot y \cdot e^{-\frac{y^2(t^2+1)}{2}}\, dy \\ &= \frac{1}{\pi} \int_0^\infty e^{-\frac{y^2(t^2+1)}{2}}\, \frac{d\, {-y^2(t^2-1)}}{t^2-1} \\ &= \frac{-1}{\pi(t^2-1)} \left[e^{-\frac{y^2(t^2-1)}{2}} \right]_0^\infty \\ &= \frac{1}{\pi(t^2-1)}, \quad t \in \mathbb{R} \\ &= \frac{\Gamma(\frac{1+1}{2})}{\sqrt{\pi(1)}\,\Gamma(\frac{1}{2})} \left(1+\frac{t^2}{1}\right)^{-\frac{1+1}{2}} \sim t(1) \end{aligned}$$

Question 5

$X_i \sim N(\mu_i, \sigma^2)$

$\Rightarrow A = \frac{S_1^2(n_1-1)}{\sigma^2} \sim \chi^2(n_1-1)$

$\Rightarrow B = \frac{S_2^2(n_2-1)}{\sigma^2} \sim \chi^2(n_2-1)$

$$\frac{S_1^2}{S_2^2} = \frac{\frac{S_1^2(n_1-1)}{\sigma^2(n_1-1)}}{\frac{S_2^2(n_2-1)}{\sigma^2(n_2-1)}}$$

$$= \frac{A/(n_1-1)}{B/(n_2-1)} \sim F(n_1-1, n_2-1)$$

Question 6

$X \sim U(0,1)$

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}(1)\,x^{i-1}(1-x)^{n-i}$$
$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)}\,x^{i-1}(1-x)^{n-i+1-1}$$
$$= \frac{x^{i-1}(1-x)^{n-i+1-1}}{B(i,n-i+1)}, \quad x \in (0,1)$$

$\because X_{(i)} \sim Beta(i, n-i+1)$

$E(X_{(i)}) = \frac{i}{n+1}$

$\therefore Var(X_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$

Question 7

a) $$\bar{X}_k + \tilde{X}_k = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{X_i-\mu}{1}\right) + \frac{1}{n-k}\sum_{i=k+1}^{n}\left(\frac{X_i-\mu}{1}\right)$$
$$= Z + Z$$

$\because \bar{X}_k + \tilde{X}_k = Z + Z$ is a joint normal and $\bar{X}_k \perp\!\!\!\perp \tilde{X}_k$

$\therefore \bar{X}_k + \tilde{X}_k \sim N(0,2)$

b) $$k\bar{X}_k^2 + (n-k)\tilde{X}_k^2 = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{X_i-\mu}{1}\right)^2 + \frac{1}{n-k}\sum_{i=k+1}^{n}\left(\frac{X_i-\mu}{1}\right)^2$$
$$= Z^2 + Z^2$$

$\because k\bar{X}_k^2 \perp\!\!\!\perp (n-k)\tilde{X}_k^2$

$\therefore k\bar{X}_k^2 + (n-k)\tilde{X}_k^2 \sim \chi^2(2)$

c) $$\frac{k\bar{X}_k^2}{(n-k)\tilde{X}_k^2} = \frac{\sum_{i=1}^{k} Z^2/k}{\sum_{i=k+1}^{n} Z^2/(n-k)}$$

$\because \sum_{i=1}^{k} Z^2 \sim \chi^2(k) \perp\!\!\!\perp \sum_{i=k+1}^{n} Z^2 \sim \chi^2(n-k)$

$\therefore \frac{k\bar{X}_k^2}{(n-k)\tilde{X}_k^2} \sim F(k, n-k)$