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STAT 2006

Assignment 3

Question 1

a) from the constraint, the $100(1-\alpha)\%$ CI is

$$a \leq \frac{(n-1)s^2}{\sigma^2} \leq b$$

$$\frac{1}{b} \leq \frac{\sigma^2}{(n-1)s^2} \leq \frac{1}{a}$$

$$\sqrt{\frac{(n-1)s^2}{b}} \leq \sigma \leq \sqrt{\frac{(n-1)s^2}{a}}$$

the lenght is

$$k = \sqrt{(n-1)s^2}\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right)$$

b) by Lagrange optimisation,

$$L(a,b,\lambda) = \sqrt{(n-1)s^2}\left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}\right) - \lambda[G(b)-G(a)-(1-\alpha)]$$

$$\frac{\partial L}{\partial a} = -\frac{\sqrt{(n-1)s^2}}{2\sqrt{a^3}} + \lambda g(a) = 0$$

$$a^{\frac{3}{2}} g(a) = \frac{\sqrt{(n-1)s^2}}{2\lambda} \quad \ldots(1)$$

$$\frac{\partial L}{\partial b} = \frac{\sqrt{(n-1)s^2}}{2\sqrt{b^3}} - \lambda g(b) = 0$$

$$b^{\frac{3}{2}} g(b) = \frac{\sqrt{(n-1)s^2}}{2\lambda} \quad \ldots(2)$$

$$\frac{\partial L}{\partial \lambda} = -[G(b)-G(a)-(1-\alpha)] = 0$$

$$G(b) - G(a) = 1-\alpha$$

with (1) and (2), $a^{\frac{3}{2}} g(a) = b^{\frac{3}{2}} g(b)$

$$a^{\frac{3}{2}} \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} a^{\frac{n-1}{2}-1} e^{-\frac{a}{2}} \quad b^{\frac{3}{2}} \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}} b^{\frac{n-1}{2}-1} e^{-\frac{b}{2}}$$

$$a^{\frac{n}{2}} e^{-\frac{a}{2}} - b^{\frac{n}{2}} e^{-\frac{b}{2}} = 0$$

Question 2

a) $\hat{p}_1 = \frac{650}{1325}$, $\hat{p}_2 = \frac{425}{675}$

95% CI for $p_1 - p_2$

$$\left(\frac{650}{1325} - \frac{425}{675}\right) \pm 1.96\sqrt{\frac{1}{1325}\left(\frac{650}{1325}\right)\left(\frac{675}{1325}\right) + \frac{1}{675}\left(\frac{425}{675}\right)\left(\frac{250}{675}\right)}$$

$$\approx [-0.1844, -0.0938]$$

b) $\hat{p} = \frac{1075}{2000}$

95% CI for p

$$\frac{1075}{2000} \pm 1.96\sqrt{\frac{1}{2000}\left(\frac{1075}{2000}\right)\left(\frac{925}{2000}\right)}$$

$$\approx [0.5156, 0.5594]$$

Question 3

a)
$$\alpha = P(Y \leq 7 \mid p = 0.08) = \sum_{y=0}^{7} \binom{50}{y} (0.08)^y (0.92)^{50-y} \approx 0.9562$$

b)
$$1-\beta = 1 - P(Y > 7 \mid p = 0.05) = 1 - \left[1 - \sum_{y=0}^{7} \binom{50}{y} (0.05)^y (0.95)^{50-y}\right] \approx 0.9968$$

Question 4

a) test statistic $= \frac{(516)^2(80)}{(580)^2} \approx 63.3189$

critical value $= 60.39$

Since $63.3189 > 60.39$, we do not reject H_0 at $\alpha = 0.05$

b) Given $\chi^2_{0.95}(80) = 60.39$ and $\chi^2_{0.9}(80) = 64.28$, p-value is ranged in between $(0.05, 0.1)$

Question 5

a) test statistic $= \frac{114-110}{\sqrt{\frac{100}{16}}} = 1.6$

critical value $= 2.33$

Since $z_0 = 1.6 < 2.33$, we do not reject H_0 at $\alpha = 0.01$

b) critical value $= 1.645$

Since $z_0 = 1.6 < 1.645$, we do not reject H_0 at $\alpha = 0.05$

c) p-value $= 1 - \Phi(1.6)$
$= 0.0548$

Question 6

a) Let Y be #. of student who find 6F's
by CLT, $\frac{Y}{n} \sim N(p, \frac{p(1-p)}{n})$

critical region is $P(Z \leq -z_\alpha) H_0$ is true) $\approx \alpha$
$\Rightarrow \frac{\frac{Y}{230} - 0.5}{\sqrt{\frac{0.5(0.5)}{230}}} \leq -1.645$

$Y \leq 102.5262$
$\therefore$ the critical region is $Y \leq 102$

b) Since $y = 110 > 102$, we do not reject H_0 at $\alpha = 0.05$

c) p-value $= P(Z \leq z_0)$
$= P(Z \leq \frac{\frac{110}{230} - 0.5}{\sqrt{\frac{0.5(0.5)}{230}}})$
$\approx P(Z \leq -0.66)$
$= 0.2546$

Question 7

Let Y be the #. of head
if it is a fair coin, the expected #. of head is 500, but $Y = 560$
$H_0: p = 0.5$ vs $H_1: p > 0.5$
by CLT, $\frac{Y}{n} \sim N(p, \frac{p(1-p)}{n})$
Critical region is $P(Z \geq z_\alpha | p = 0.5) \approx \alpha$
$\Rightarrow \frac{\frac{Y}{1000} - 0.5}{\sqrt{\frac{0.5(0.5)}{1000}}} \geq 1.645$
$Y \geq 526.0097$
Since $y = 560 \geq 526$, we reject H_0 at $\alpha = 0.05$.
We have sufficient evidence to conclude that the coin is not fair.

Question 8

Let $Y = \sum X_i$, such that $Y \sim b(n, p)$
$\alpha = P(Y \geq y | p = 0.49) = 0.01$
by CLT, $P(Z \geq \frac{y - np}{\sqrt{np(1-p)}} | p = 0.49) \approx 0.01$
$\Rightarrow \frac{y - n(0.49)}{\sqrt{n(0.49)(0.51)}} = 2.33$
$y = 2.33\sqrt{0.2499n} + 0.49n \quad \ldots (1)$

$\beta = P(Y \leq y | p = 0.51) = 0.99$
by CLT, $P(Z \leq \frac{y - np}{np(1-p)} | p = 0.51) \approx 0.99$
$\Rightarrow \frac{y - n(0.51)}{\sqrt{n(0.51)(0.49)}} = -2.33$
$y = -2.33\sqrt{0.2499n} + 0.51n \quad \ldots (2)$

with (1) and (2), $2.33\sqrt{0.2499n} + 0.49n = -2.33\sqrt{0.2499n} + 0.51n$
$n = 13566.8211$
≈ 13567
with (1), $y = 2.33\sqrt{0.2499(13567)} + 0.49(13567)$
≈ 6783.4991

Question 9

critical region : $P(Y_n \geq 1 \cup Y_1 \geq k | \theta = 0) = \alpha$
under H_0, $P(Y_n \geq 1 | \theta = 0) = 0$
$\Rightarrow P(\max(X_i) \geq k | \theta = 0) = \alpha$
$[\int_k^1 dx]^n =$
$(1-k)^n =$
$k = 1 - \sqrt[n]{\alpha}$

Question 10

Let $Y \sim Pois(15)$, by CLT $\frac{Y - \lambda}{\sqrt{\lambda}} \sim N(0,1)$
$P(Y \leq 10 | \lambda = 15) = P(Y < 10.5 | \lambda = 15)$
$= P(Z \leq \frac{10.5 - 15}{\sqrt{15}})$
$= 1 - \Phi(1.16)$
$= 0.123$
Since $\alpha = 0.123$ is quite large in some sense (>0.1), we do not have sufficient evidence to conclude that the accident rate has dropped.