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ENM 502-001

2021-03-08

Assignment 3 Newton method and Arc-length continuation

Introduction

In this assignment, we would solve a non-linear boundary-value problem defined on the unit-square domain

$$
\begin{aligned}
&D = (0 \leq x \leq 1) \cup (0 \leq y \leq 1) \\
&\nabla^2 u + \lambda u(1+u) = 0 \\
&u(x,y) = 0 \text{ on all boundaries,}
\end{aligned}
$$

with **newton method**. After we guess the initial solution on some typical `lambda`, we track track and iterate `lambda` from

$$0 \leq \lambda \leq 60$$

with **analytic continuation(AyC)** and **arc-length continuation(ARCLC)**.

Problem Setup and Formulation

Discretization and centered finite difference applied to this problem.

We discretize the `U` with finite difference method, in specifically a `30 X 30` uniform grid.

Newton's Method

As a fixed-point iteration method, given a initial approximation at `U_0`, newton's method would approximate the solution in the derivative direction.

The iteration goes like

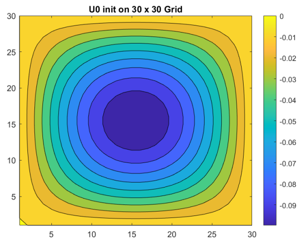
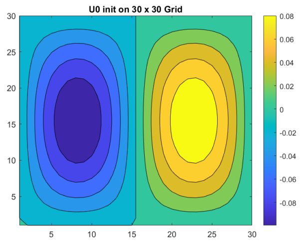
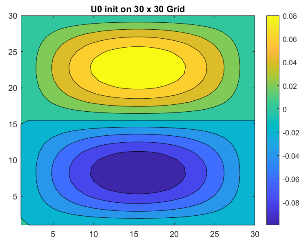
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta \mathbf{u}^k$$

where $ \delta \mathbf{u}^{k} $ is calculated by

$$\mathbf{J}|_k (\, \partial \mathbf{u}\,)_k = -(\partial \mathbf{R})_k$$

where `J` is the Jacobin matrix.

For the initial value, when L2norm of λ is near to zero, it could be visualized as a eigen value problem. The initial value near is like

lambda = 2 * pi^2	lambda = 5 * pi^2	lambda = 5 * pi^2
		

Methods to generate first two non-trivial solutions (analytical continuation)

We would use newton's method to solve for `U_0` from first initial guess(see image above). To calculate `U_1` , we use **AyC** to calculate the initial guess `U_1_guess`, then use newton's method to solve the solution at `lambda_1`.

Arc-Length Continuation

During iterating the `lambda`, the `J` would become nearly singular during turning point. To prevent this, we introduce a new independent variable `s`.

$$(\delta S)^2 = (\delta\lambda)^2 + \|\delta\mathbf{u}\|_2^2$$

and start to iterate on `s` for the rest steps.

For each iteration, let's say we have `U_cur`, `lmbd_cur`, `U_prv` `lmbd_prv`, which is the solution of `k - 1`, `k - 2`iteration(it's `kth iteration`)

during every iteration, we first use **arc-length initialization** to get initial guess , let's say it be `U_cur_init`, `lmbd_cur_init`.

$$\mathbf{u}_2^0 = \mathbf{u}_1 + (\delta s)\left(\frac{\partial \mathbf{u}}{\partial s}\right)_1$$
$$\lambda_2^0 = \lambda_1 + (\delta s)\left(\frac{\partial \lambda}{\partial s}\right)_1$$

And during **arc-length initialization** **, we would use `U_cur`, `lmbd_cur`to create the Jacobin matrix.

The we would use `U_cur_init`, `lmbd_cur_init` together with `U_prv` `lmbd_prv` to do the full newton iteration. The detailed iteration could be viewed as below.

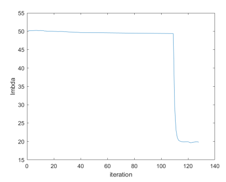
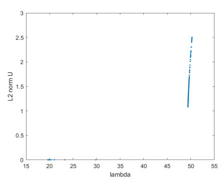
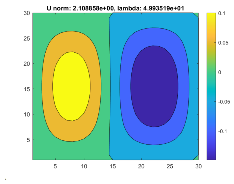
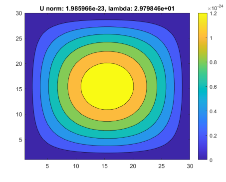
$$\left(\hat{\mathbf{J}}^k\right)\Big|_{s2}\begin{pmatrix}\delta\mathbf{u}^k\\ \delta\lambda^k\end{pmatrix}\Bigg|_{s2} = -\left(\hat{\mathbf{R}}^k\right)\Big|_{s2}$$
$$\lambda^{k+1} = \lambda^k + \delta\lambda^k$$
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}^k$$

Results and Discussion

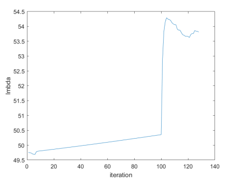
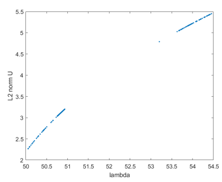
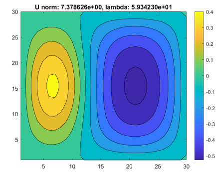
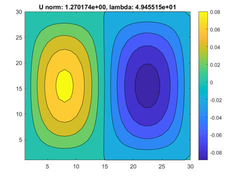
From my circumstances, I notice that with different mixture of parameters(**ARCLC** step size, initial point, etc), the results vary rapidly.

Failed cases

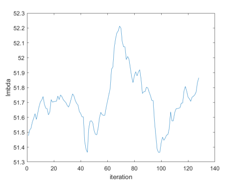
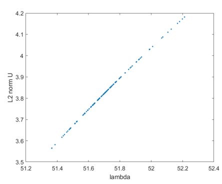
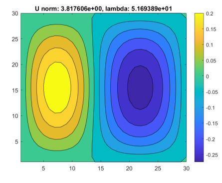
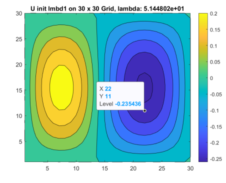
Rush to bottom

λ iteration	L2norm of U $lambda$	U	U
			

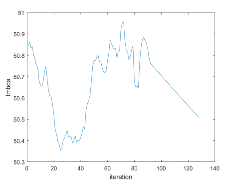
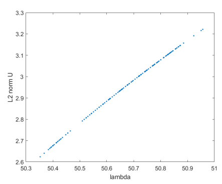
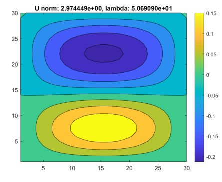
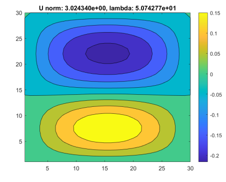
Rush to top

λ iteration	L2norm of U $lambda$	U	U
			

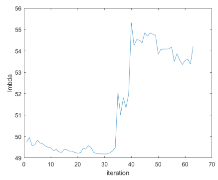
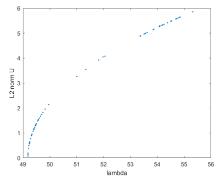
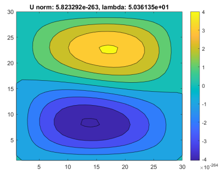
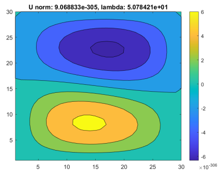
Rush to top

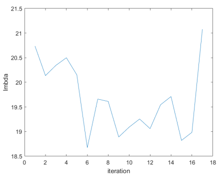
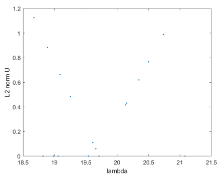
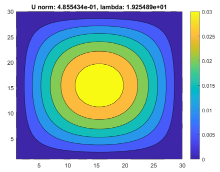
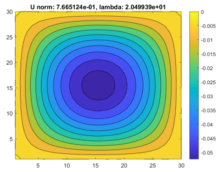
λ iteration	L2norm of U $lambda$	U	U
			

Up and down

λ iteration	L2norm of U $lambda$	U	U
			

Relatively make sense cases

λ iteration	L2norm of U λ	U	U
			

λ iteration	L2norm of U λ	U	U
			

Performance with sparse matrix

Here we test the running time during solving the

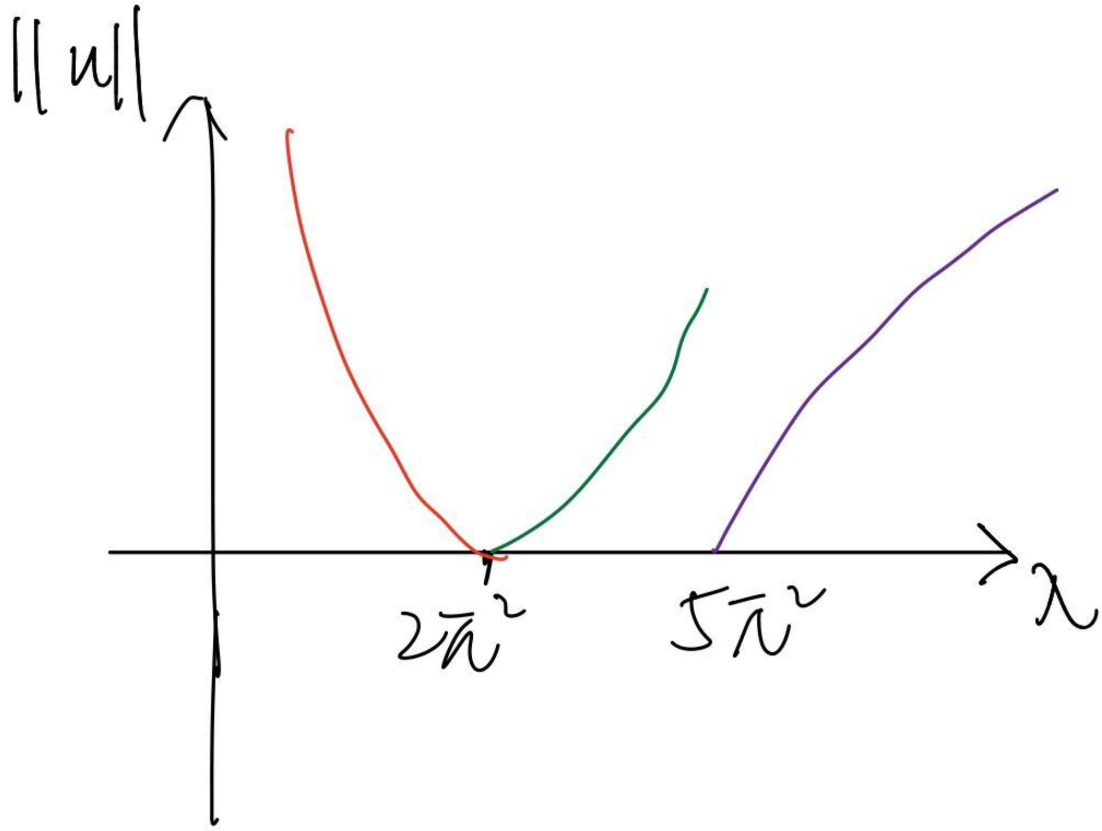
$$\mathbf{J}|_k(\partial \mathbf{u})_k = -(\partial \mathbf{R})_k$$

	Sparse	Full
time(seconds) spent in 64 iterations	2.158603	2.75824

Here `J` is a banded matrix with bandwidth `2n` (the whole matrix J is `(n+1)^2`. As expected, matrix solving under this setting could benefit greatly from sparse matrix.

Conclusion

This is the expected whole graph of `L2 norm of U` vs `iteration`.



In my current version or ARCLC, the full newton's method(with `J_hat`) could hardly converge. I think that's the reason why the `lambda` vs iteration is not stable and smooth.

Despite, we still manage to see that the `U` could jump on another branch. As in last two examples, the hill-like and bowl-like solutions change signs when we are moving along the curve.

In some circumstances(the first results), the `U` could sometimes move to `2pi^2` branch.

Code

Code repository is here.