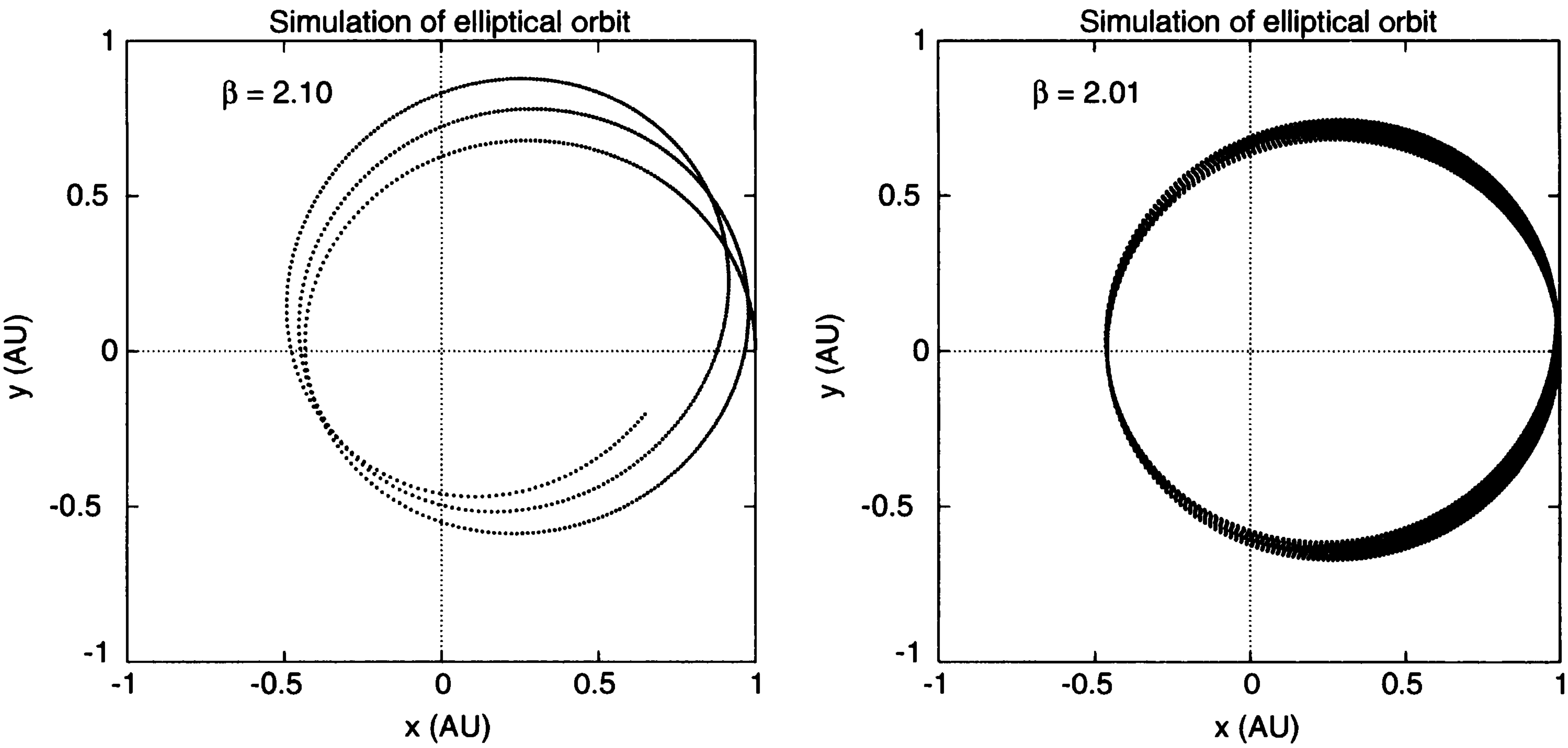

Figure 4.6: Elliptical orbits calculated for a force law (4.8) with $\beta = 2.1$ (left) and $\beta = 2.01$ (right).

an experiment aimed at determining β through deviations from Kepler's laws would have to account carefully for the effects of the other planets.

Exercises

1. Verify Kepler's third law for elliptical orbits. Run the planetary motion program with initial conditions chosen to give orbits that are noncircular. Calculate T^2/a^3 and compare with the values given in Table 4.2.

*2. In this section we saw that orbits are unstable for any value of β that is not precisely 2 in (4.8). A related question, which we did not address (until now), is *how* unstable an orbit might be. That is, how long will it take for an unstable orbit to become obvious. The answer to this question depends on the nature of the orbit. If the initial velocity is chosen so as to make the orbit precisely circular, then the value of β in (4.8) will make absolutely no difference. Of course, in practice it is impossible to construct an orbit that is exactly circular, so the instabilities when $\beta \neq 2$ will always be apparent given enough time. Even so, orbits that start out as nearly circular will remain almost stable for a longer period than those that are highly elliptical. Investigate this by studying orbits with the same value of β (say, $\beta = 2.05$) and comparing the behavior with different values of the ellipticity of the orbit. You should find that the orientation of orbits that are more nearly circular will rotate more slowly than those that are highly elliptical.

4.3 Precession of the Perihelion of Mercury

In Section 4.2 we mentioned the possibility of using the solar system in an experiment to test the accuracy of the inverse-square law. In fact, such an experiment essentially has already been performed.

Astronomy is a precise science. We mentioned earlier that Kepler inferred his laws from observations made by Tycho Brahe. What we didn't mention was that Brahe's observations were all made *by eye*; the telescope had not yet been invented! With the availability of telescopes and improvements in timekeeping, astronomical observations became even more precise.[12] This prompted impressive advances in celestial mechanics, the branch of physics concerned with the motion of planets and other objects in the universe. You might think that Kepler's laws provide all of the information a celestial mechanic would ever require, but this is far from the case. In a solar system with more than one planet there will be deviations from Kepler's laws. For our solar system these deviations are small, so Kepler's laws form an extremely useful starting point for discussing the planets we know. Nevertheless, there are deviations from these laws. These deviations come from a number of sources, including the effects of the planets on each other. It turns out that this is a problem for which very few exact results are known (even today). One of the jobs of a celestial mechanic is to calculate such effects.

We have noted several times that most of the planets have orbits that are very nearly circular. The planets whose orbits deviate the most from circular are Mercury and Pluto, and this leads to interesting consequences in both cases. For Mercury it was known by the early part of the 19th century that the orientation of the axes of the ellipse that describes its orbit rotate with time. This is known as the precession of the perihelion of Mercury (the perihelion is the point in an orbit when a planet is nearest the Sun). The magnitude of this precession is approximately 566 arcseconds per century (an arcsecond is 1/3600 of a degree). That is, Mercury's perihelion makes one complete rotation every $\approx$ 230,000 years. That such a small effect can be measured so accurately is a striking demonstration of the precision of astronomical measurements. This deviation from Kepler's law was of great interest to celestial mechanics, and by the middle of the 19th century it had been calculated that the gravitational forces of the other known planets[13] lead to a precession of 523 arcseconds/century. Jupiter, which is by far the largest planet, is responsible for most of this. Perhaps most amazing is the fact that this calculation of the precession was performed "by hand" in a numerical computation involving log tables and the like, long before computers were available!

While the precision of both the experimental measurement and the theoretical calculation of the precession of Mercury's perihelion are very impressive, the fact remained that they did not agree. It was realized, of course, that this disagreement might be evidence for some interesting new physics, and various solutions to this puzzle were proposed. One of the most popular (for a while at least) was the suggestion that there might be another planet whose orbit was inside that of Mercury. However, such a planet was never found. It was also suggested that there might be a large amount of dust orbiting near the Sun, whose gravitational attraction was affecting Mercury, but again this proposal was never confirmed.

This troubling discrepancy was not explained until 1917 when Einstein developed the theory of general relativity. That theory deals with the geometry of space and views gravity in a much more complicated manner than our simple picture of Euclidean space and the inverse-square law (4.1). Nevertheless, general relativity leads to a similar prediction for the force due to gravity, provided that the two objects are not too close together (or equivalently, not too massive). However, if the separation between the two objects is made small enough, general relativity predicts deviations from

[12] Astronomers have to measure not only *where* an object is located, but also *when* it is there.

[13] At that time only eight planets were known. Pluto was not discovered until 1930, but it is too small and too far away to have a significant effect on Mercury.

the inverse-square law. It turns out that the Sun and Mercury are close enough for these deviations to just be significant, and Einstein showed that they precisely account for the previously unexplained 43 arcseconds of precession. This was one of the first triumphs of the theory of general relativity.

The precession due to general relativity can be calculated analytically, although to do so is fairly complicated. However, it is actually very straightforward to deal with this problem computationally (which is why we have devoted a section to it!). All we have to do is simulate the orbital motion using the force law predicted by general relativity and measure the rate of precession of the orbit, much as we did in the previous section. However, the precession rate is fairly small, so we have to design our simulation with that in mind (we don't want to have to compute the motion for 230,000 years).

The force law predicted by general relativity is

$$F_G \approx \frac{G\, M_S\, M_M}{r^2}\left(1 + \frac{\alpha}{r^2}\right), \tag{4.9}$$

where M_M is the mass of Mercury and[14] $\alpha \approx 1.1 \times 10^{-8}$ AU^2. The force is seen to be an inverse-square law with a very small additional piece[15] that is proportional to $1/r^4$. The effects of this tiny deviation from an inverse-square law are too small to measure easily in a computer simulation. The approach we take here is to calculate the rate of precession as a function of α, with values of α that are much larger than the actual value for Mercury. It will turn out that the rate of precession is given by $C\alpha$, where C is a constant that we will calculate. After we have obtained the value of C, we can then estimate the rate of precession for $\alpha = 1.1 \times 10^{-8}$, which is the case that we are really interested in.

To carry out this approach we first need to modify our planetary motion program to employ the force law (4.9), with α as an adjustable parameter. We will leave this job for the exercises. Next we need to determine the initial conditions required to obtain the orbit of Mercury. This is important, since the behavior depends on both the size of the orbit and also its eccentricity. The length of the semimajor axis (see Figure 4.7) for Mercury's orbit is $a = 0.39$ AU. The corresponding velocity, v_1, which we require as an initial condition for the simulation, can be estimated in either of two ways. One is to calculate the motion for different trial values of v_1 and adjust its value until the eccentricity of the resulting orbit agrees with the known value, $e = 0.206$. A second approach is to make use of the conservation of both energy and angular momentum over the course of an orbit to calculate v_1. This approach can be carried out with the help of Figure 4.7.

Conservation of total energy (kinetic plus potential) implies that the energies at points 1 and 2 in Figure 4.7 are the same. Thus

$$-\frac{G\, M_S\, M_M}{r_1} + \frac{1}{2}\, M_M\, v_1^2 = -\frac{G\, M_S\, M_M}{r_2} + \frac{1}{2}\, M_M\, v_2^2\,. \tag{4.10}$$

The terms on the left-hand side of this equation are just the potential and kinetic energies at point 1 in Figure 4.7, and the terms on the right are the corresponding energies at point 2 (here we don't need to worry about the extremely small contribution of the general relativistic term in the potential). Since the

[14] α can be expressed in terms of the speed of light, the mass of the Sun, the eccentricity of the orbit, and other similar parameters (see Goldstein [1990], Chapter 11). This general relativistic effect is most noticeable for Mercury because it is the planet closest to the Sun.

[15] The term involving α in (4.9) is actually just the first of a series of such terms, which can be written as an expansion in powers of r^{-1}. For Mercury it is accurate to stop with the first correction term, as we have done here.

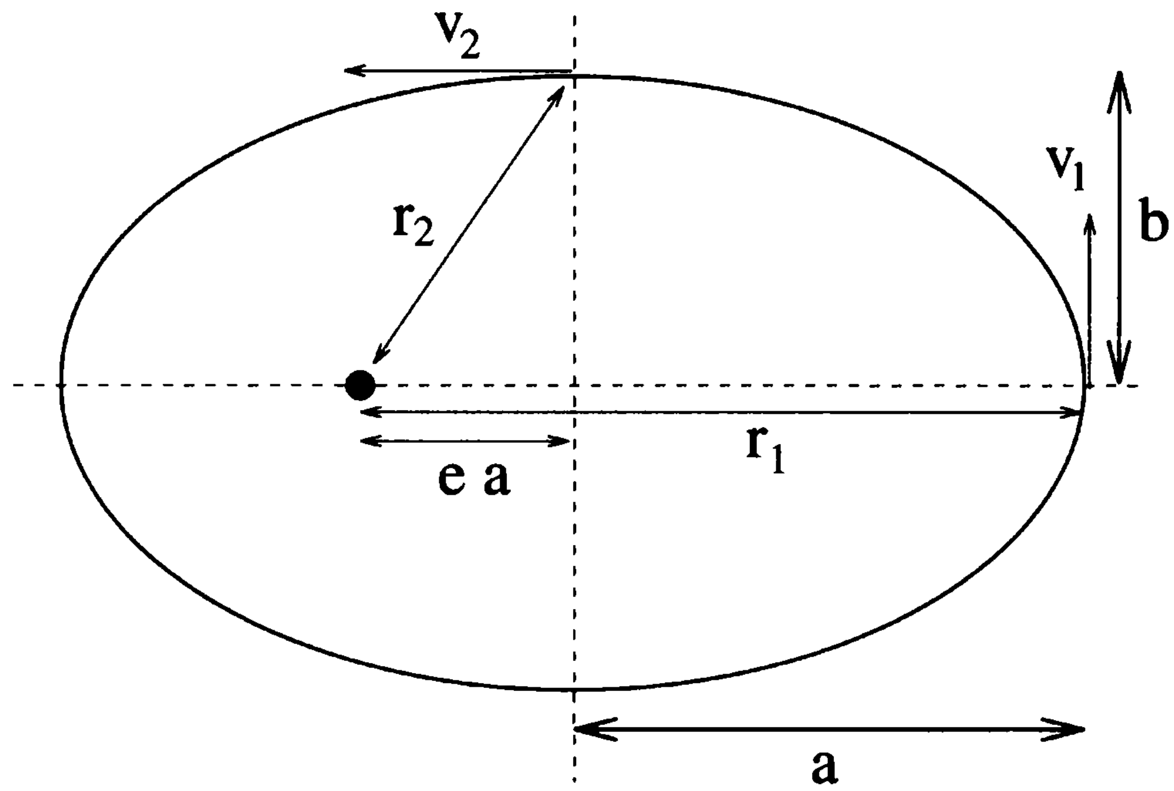

Figure 4.7: Definition of the parameters necessary for a calculation of the initial conditions needed for a simulation of Mercury's elliptical orbit. Points 1 and 2 are the places where the orbit crosses the x and y axes, respectively.

force of gravity is a central force, it exerts no torque on the planet. Hence, the angular momentum at point 1 is equal to that at point 2, which yields

$$r_1 \, v_1 \; = \; b \, v_2 \, . \tag{4.11}$$

We thus have two equations involving the unknowns v_1 and v_2. After several lines of algebra (we'll leave that to you) we find

$$v_1 \; = \; \sqrt{2 \, G \, M_S \left[\frac{b^2}{a^2(1+e)^2 \; - \; b^2} \right] \left[\frac{1}{\sqrt{e^2 a^2 \; + \; b^2}} \; - \; \frac{1}{a \; + \; ea} \right]} \; = \; \sqrt{\frac{G \, M_S \, (1 \; - \; e)}{a \, (1 \; + \; e)}} \, , \tag{4.12}$$

where in the last step we have used the fact that $b = a\sqrt{1 - e^2}$. Inserting the values of a and e given above yields $v_1 = 8.2$ AU/yr, and this is one of our initial conditions. The other is the distance from Mercury to the Sun, which is $r_1 = (1 + e)a = 0.47$ AU.

Now that the initial conditions are known we can simulate the motion of Mercury. The result obtained using the force law (4.9) with $\alpha = 0.01$ is shown in Figure 4.8. This value of α is *much* larger than the true value for Mercury, and it is seen that the ellipse precesses very noticeably in this case. The lines drawn from the Sun to the orbit show the orientation of the long axis of the ellipse. They are drawn from the origin (which is where the Sun is located) to the points on each orbit that are farthest from the Sun. These points are found by monitoring the distance of Mercury from the Sun and noting when its time derivative changes from positive to negative. These lines are very useful for our calculation, since the angles they make with the x axis are the amounts the orbit has precessed.

The next step is to calculate the angle of precession as a function of time for a particular value of α. A plot of this angle, which is the angle the radial lines in Figure 4.8 make with the x axis as a function of time is shown in Figure 4.9. We see that the precession angle θ varies linearly with time. This means that the precession rate is a constant, which is what we have assumed implicitly all along.

The rate of precession, $d\theta/dt$, is the slope of the line in Figure 4.9. To obtain this slope we have two choices. One is to simply draw in a line by hand through the points and estimate its slope. A

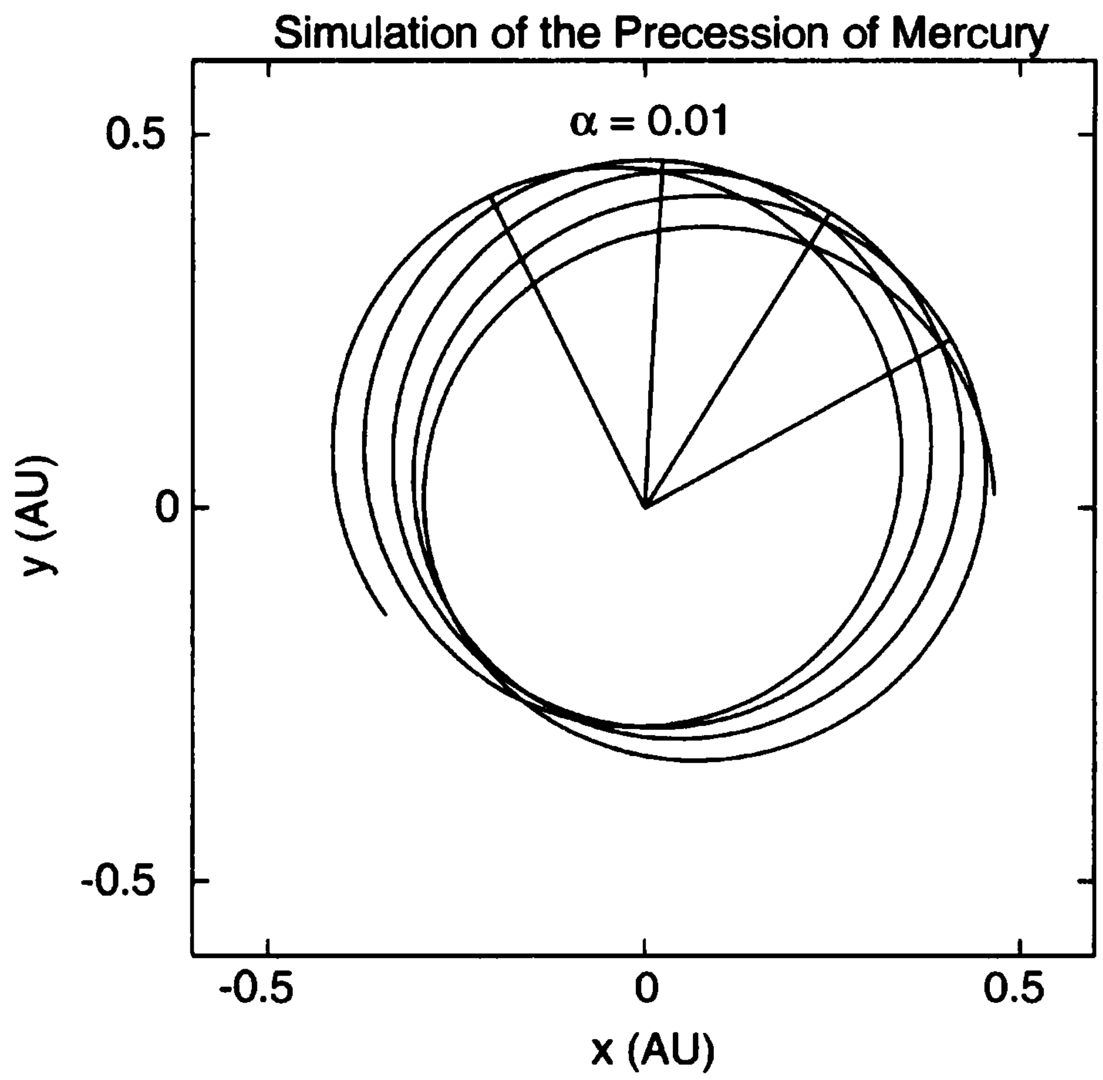

Figure 4.8: Simulated orbit for Mercury orbiting the Sun. The force law (4.9) was used, with $\alpha = 0.01$. The time step was 0.0001 yr. The program was stopped after several orbits. The solid lines emanating from the Sun (i.e., the origin) are drawn to the points on the orbit that are farthest from the Sun, so as to show the precession of the orientation of the orbit.

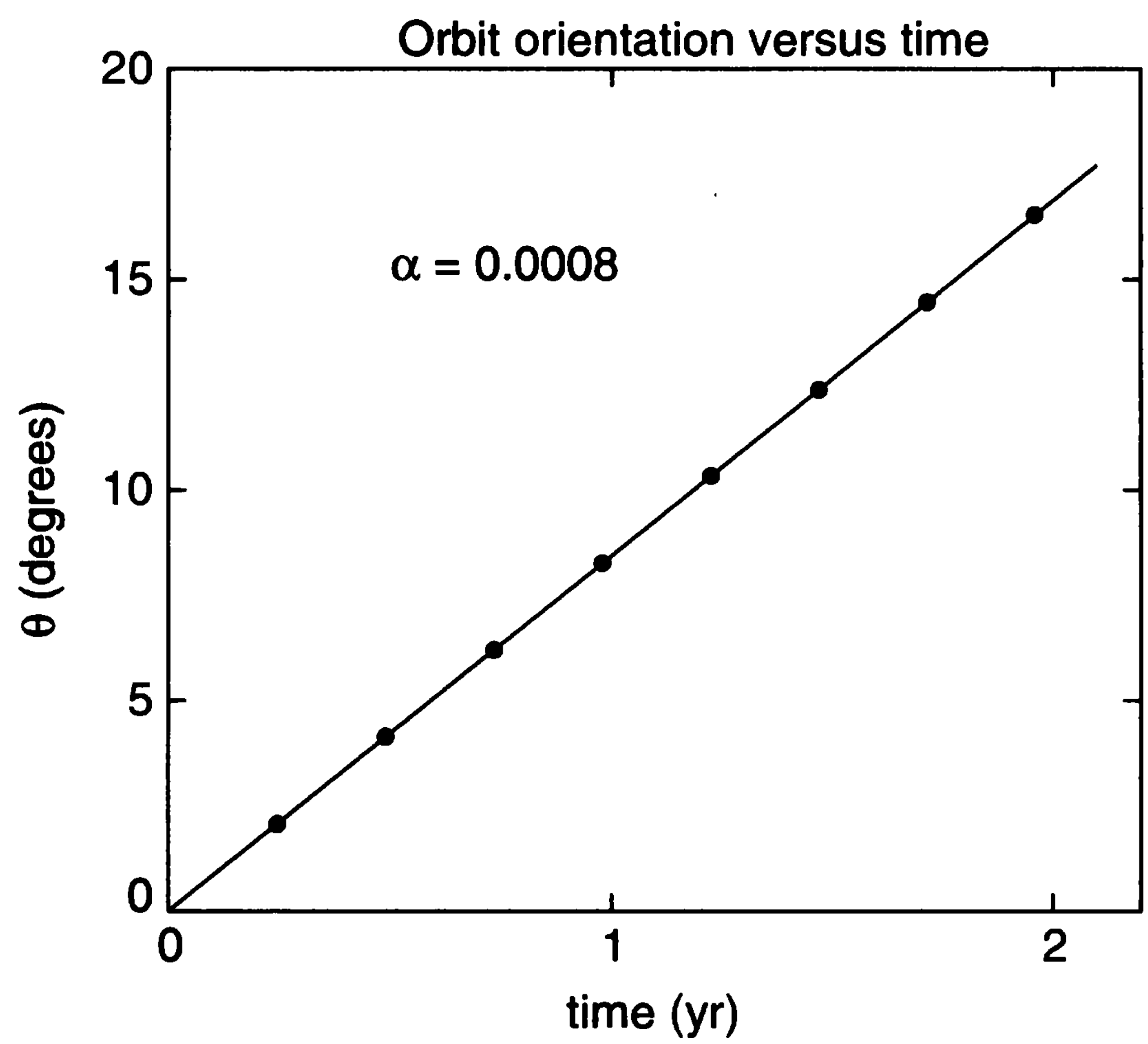

Figure 4.9: Precession of the axis of Mercury's orbit as a function of time, calculated for $\alpha = 0.0008$. The time step was 0.0001 yr. The solid line is a least-squares fit.

much better approach is to calculate the equation of the best-fit line. By this we mean the line that comes closest to passing through all of the points. If we had somehow been able to perform a perfect calculation of θ as a function of t, all of our points would lie exactly on a line. Alas, things are not this simple. Numerical errors (due to the nonzero value of the time step, etc.) will cause the points to deviate from such an ideal result. We would, therefore, like to somehow choose a line that comes as close as possible to all of the points (sort of a grand compromise). The conventional way to choose this line is with the method of least squares, which is discussed in Appendix 3. This is a very powerful method of "fitting" straight lines (or, more generally, smooth curves) to noisy data; here the data are our results for θ. This least-squares line is determined in the following way. Since, as we have already mentioned, the data do not all fall precisely on a single line, any line we imagine will deviate somewhat from at least some of the points. In our specific case, the values of θ defined by any line will in general deviate somewhat from the calculated values in Figure 4.9. With the method of least squares we choose a line so as to minimize the sum of the squares of these deviations. This approach is discussed at greater length in Appendix 3, where we show how to calculate the slope and intercept of the least-squares line. It turns out that there is a unique solution for this problem. That is, there is one and only one least-squares line. Moreover, it is not hard to calculate.

Let us now return to the precession of the perihelion of Mercury. The solid line in Figure 4.9 is the least-squares line for these calculated values of $\theta(t)$. The equation of this line was obtained using the least-squares subroutine that is part of the *True Basic* scientific graphics library, so it took us very little effort to modify our program to include it. If you are using a different language package, or just prefer to do things for yourself, Appendix 4 contains the listing of a least-squares program. We see from Figure 4.9 that the least-squares line does indeed provide an excellent fit to the results for $\theta(t)$. The slope of this best-fit line then gives the precession rate for this particular value of α. We can now repeat the calculation with different values of α and obtain the best-fit precession rate in each case. The results of such a calculation are shown in Figure 4.10 where we plot the precession rate as a function of α. We see that the precession rate itself varies linearly with α. We can again use the method of least squares to describe these results, and the best-fit line is also shown in Figure 4.10. The slope of this line is 1.11×10^4 degrees per year per unit α. Now we are finally ready to finish off our calculation. Since we have found that the precession rate varies linearly with α and we have calculated the coefficient of proportionality, we can extrapolate to the case $\alpha = 1.1 \times 10^{-8}$ predicted by the theory of general relativity. This yields a precession rate of $1.1 \times 10^{-8} \times 1.11 \times 10^4 \approx 1.2 \times 10^{-4}$ degrees/year, which is also equal to ≈ 43 arcseconds/century, in agreement with the experimental result mentioned at the beginning of this section.

Our study of the precession of the perihelion of Mercury has illustrated several useful techniques. One is the use of extrapolation to deal with situations in which the effects of interest are too small to conveniently estimate directly with a numerical approach. Of course, you must have some confidence in how the extrapolation should be made. In the present case we were able to show that a linear extrapolation as a function of α is appropriate. The second useful technique we have introduced is the method of least squares. We will use it again in later chapters.

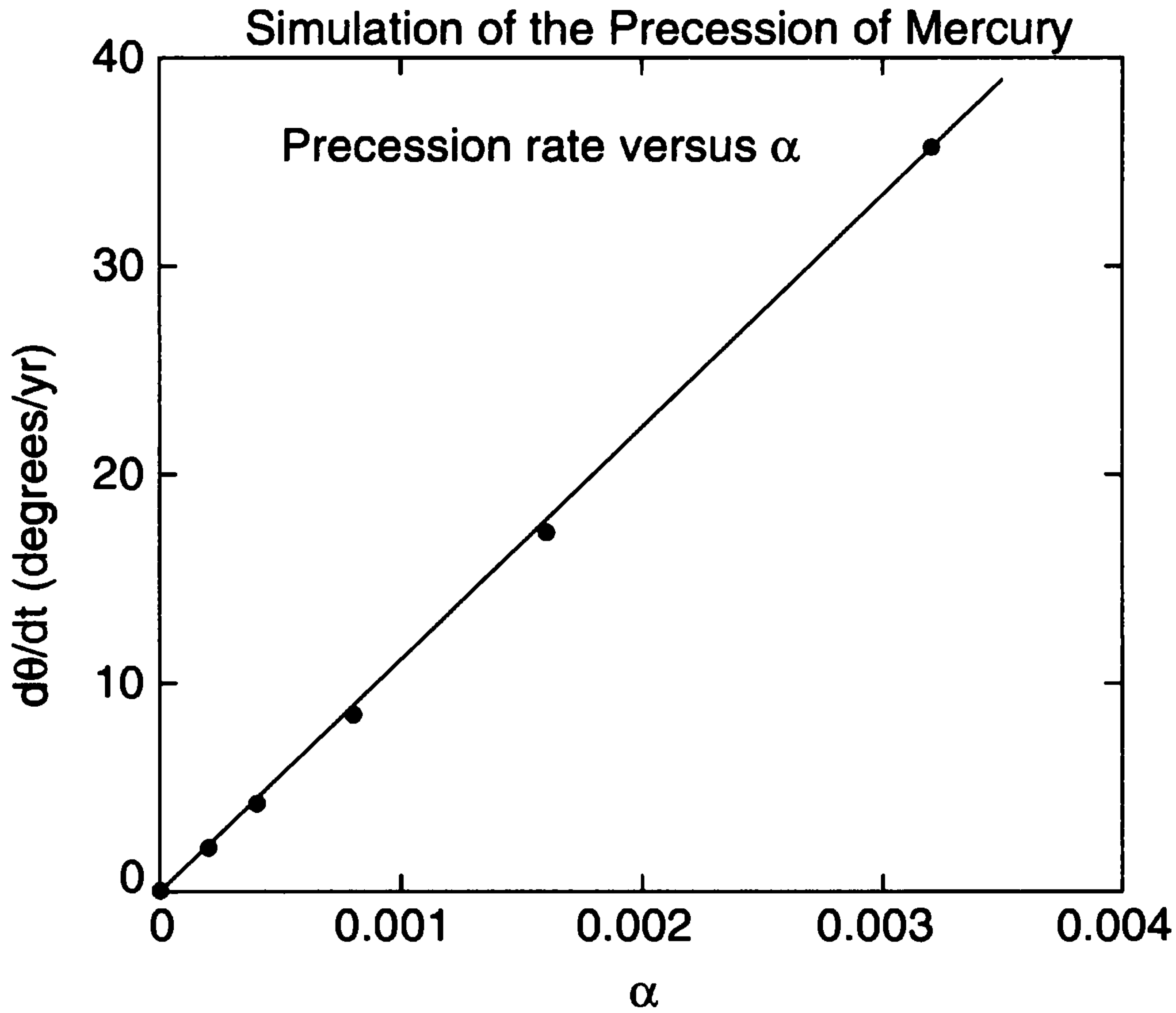

Figure 4.10: Precession rate of Mercury as a function of α. The solid line is a least-squares fit, which yielded a slope of 1.11×10^4 degrees per year per unit α.

Exercises

1. Calculate the precession of the perihelion of Mercury, following the approach described in this section.

*2. Investigate how the precession of the perihelion of a planet's orbit due to general relativity varies as a function of the eccentricity of the orbit. Study the precession of different elliptical orbits with different eccentricities, but with the same value of the perihelion. Let the perihelion have the same value as for Mercury, so that you can compare it with the results shown in this section.

4.4 The Three-body Problem and the Effect of Jupiter on Earth

To this point all of our planetary simulations have involved two-body solar systems. It is now time to consider some of the things that can happen when there are three or more objects in the solar system. The problem of two objects interacting through the inverse-square law (4.1) can be solved exactly (as we have already mentioned), leading to Kepler's laws. However, if we add just one more planet to give what is known as the three-body problem, an analytic theory becomes *much* more difficult. In fact, there are very few exact results in this case, even though it has been studied extensively for several centuries. Indeed, the three-body, or more generally the n-body problem, is *the* problem of celetial mechanics.

In this section we consider one of the simplest three-body problems, the Sun and two planets, which we will take to be Earth and Jupiter. We know that without Jupiter, Earth's orbit is stable and