Learning Objectives

> In this tutorial you will work on rephrasing problems with mathematical language—this is an essential skill if you ever plan on applying mathematical techniques to the world!
>
> These problems relate to the following course learning objective: *work independently to understand concepts and procedures that have not been previously explained to you.*

Problems

1. Use vectors, sets, set operations, and set-builder notation to describe the following as subsets of $\mathbb{R}^2$.

   (a) The x-axis.

   (b) The corners of a square S, which is centered at the origin and whose sides have length 3 and are aligned with the axes.

   (c) The diagonal of S (from before) starting from the lower-left to the upper-right.

   (d) Both diagonals of S.

   (e) The line segment from $(2,3)$ to $(4,1)$, including the endpoints.

   (f) The line segment from $(2,3)$ to $(4,1)$, not including the endpoints.

2. Let's make a smiley face![1]

   (a) Describe the lower half of a circle of radius 1 centered at the origin. Call this set M (for mouth!).

   (b) Pick a location for the eyes and describe them as small, filled in circles. Call the left eye L and the right eye R.

   (c) Describe the whole face using M, L, and R. Call the face F.

   (d) When we draw a set, we usual draw black for points in the set and leave points not in the set white. Let's draw a reverse-face. Come up with a set F_R for a face where the "skin" of the face is included in F_R, but the eyes and mouth are not.

3. *Interpolation* is the process of filling in points that might not exist already. It's commonly used when zooming-in or rotating a picture on your computer. The picture P consists of four colored pixels, *red* at $(0,0)$, *green* at $(1,0)$, and *blue* at $(2,0)$ and $(3,0)$. To your brain, what is important about this picture is the *relative spacing* between the colors, not their absolute positions. We will interpolate the color positions for a transformed P.

   (a) Give the coordinates of each color if P were translated, going from $(1,4)$ to $(4,4)$.

   (b) Give the coordinates of each color if P were twice as big, going from $(0,0)$ to $(6,0)$.

   (c) Give the coordinates of each color if P were rotated and zoomed, going from $(-1,-1)$ to $(7,0)$.

[1] This question is not a joke, and a version of it may show up on your midterm.

1. (a) $\left\{\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2 : y=0\right\}$.

   (b) $\left\{\begin{bmatrix}-3/2\\3/2\end{bmatrix},\begin{bmatrix}3/2\\3/2\end{bmatrix},\begin{bmatrix}3/2\\-3/2\end{bmatrix},\begin{bmatrix}-3/2\\-3/2\end{bmatrix}\right\}$.

   (c) $\left\{\begin{bmatrix}t\\t\end{bmatrix} : t\in[-3/2,3/2]\right\}$.

   (d) $\left\{\begin{bmatrix}t\\t\end{bmatrix} : t\in[-3/2,3/2]\right\}\cup\left\{\begin{bmatrix}t\\-t\end{bmatrix} : t\in[-3/2,3/2]\right\}$.

   (e) $\left\{\vec{v} : \vec{v}=\alpha\begin{bmatrix}2\\3\end{bmatrix}+(1-\alpha)\begin{bmatrix}4\\1\end{bmatrix} \text{ for some } \alpha\in[0,1]\right\}$.

   (f) $\left\{\vec{v} : \vec{v}=\alpha\begin{bmatrix}2\\3\end{bmatrix}+(1-\alpha)\begin{bmatrix}4\\1\end{bmatrix} \text{ for some } \alpha\in(0,1)\right\}$.

2. (a) $M=\left\{\begin{bmatrix}x\\y\end{bmatrix} : x^2+y^2=1 \text{ and } y\leq 0\right\}$.

   (b) Define $\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|=\sqrt{x^2+y^2}$ to be the length of a vector in $\mathbb{R}^2$. Let $\vec{l}=\begin{bmatrix}-1/2\\1\end{bmatrix}$ and $\vec{r}=\begin{bmatrix}1/2\\1\end{bmatrix}$. Then

   $$L=\{\vec{v} : \|\vec{v}-\vec{l}\|\leq 1/4\} \qquad \text{and} \qquad R=\{\vec{v} : \|\vec{v}-\vec{r}\|\leq 1/4\}.$$

   (c) $F=M\cup L\cup R$.

   (d) $F_R=\{\vec{v} : \|\vec{v}\|\leq 3/2 \text{ and } \vec{v}\notin F\}$.

3. (a) red at $(1,4)$, green at $(2,4)$, and blue at $(3,4)$ and $(4,4)$.

   (b) red at $(0,0)$, green at $(2,0)$, and blue at $(4,0)$ and $(6,0)$.

   (c) red at $(-1,-1)$, green at $(5/3,-2/3)$, and blue at $(13/3,-1/3)$ and $(7,0)$.

Learning Objectives

Students need to be able to...

- Turn geometric descriptions and pictures into equations/formulas/sets suitable for manipulation with mathematics.
- Be comfortable enough with set notation and operations to combine the operations in new ways.

Context

Students in class have gone over sets, set operations, vectors, linear combinations, vector form of lines and planes, and have just started span. Some sections may also have covered *restricted* linear combinations, for example convex combinations. Sections have *not* covered norm notation (i.e., $\|\vec{x}\|$) or lengths of vectors in general. However, they all know the Pythagorean theorem from high school.

What to Do

This is the first tutorial of the term, and it is your chance to win the students over! This is a groupwork tutorial, but students may not be used to working in groups.

- Arranged for group work. Reorganize the desks and chairs (if possible) to facilitate groups of 3 or 4. Ask students to form groups of 3 or 4 with other students nearby. Don't allow larger groups.
- Begin the tutorial by introducing yourself (your name, your program of study, and your year). You might also want to give them some more personal information, such as where you are from or when you first started liking math.
- Introduce the structure and purpose of tutorials: students will be working to (1) better understand concepts from lecture, (2) practice tackling concepts that have not been explained in lecture, and (3) effectively communicate. They can expect to spend most of the tutorial working in small groups.
- Emphasize the importance of working with others when learning mathematics—they should be working with others in this tutorial *and* outside of class.

This introduction should take no more than 5 minutes.

Next, introduce the learning objectives for the day's tutorial. Explain that the goal of this tutorial. Their worksheet has the "formal" objectives stated and these instructions have the "hidden" objectives. Feel free to share with them the hidden objectives as well.

Ask the students to pair up and start working on the problem list. Circulate around the room during this time and ask groups what they're thinking. They will be tempted to move quickly through the list without thoroughly checking their new answers—encourage them to think deeply.

Problem 1 is a straightforward question, but students will struggle starting with part (d) and especially with (e) and (f). They may have forgotten about unions! Ask them to review the set operations that they know and be creative. When most people are on parts (e) and (f), go over parts (a)–(d). Then, let them continue working through number 2. If most of the class gets stuck at any point, draw the class's attention to the front of the room and work on the difficult part together.

There are too many problems to finish in 50 minutes and *you should not be going over the solution to every problem*. Solutions will be posted for the students. The goal of tutorial is for students to spend time *doing* mathematics with an expert around to help them if they get stuck. Don't feel any time pressure, even if you only get through 1.5 questions, that's okay!

During the last 6 minutes of class, pick one problem (perhaps a few parts of one problem) that most groups have at least started, and do this problem as a wrapup. Seeing an expert do the problem is the student's reward for working so hard.

Notes:

- Students won't have a good conceptualization of convex combinations which make 1(f) and 3(c). These problems can also be done by describing a line in vector form (i.e., $\vec{x} = t\vec{d} + \vec{p}$) and restricting the scalar to get points on the line segment.
- For 1(e), some students might write $\{\vec{x} : \vec{x}$ is a convex linear combination of $\vec{p}$ and $\vec{q}\}$. Other students might think that this description is "mathy" enough. This description is mathy enough, but we can also expand it by inserting the definition of *convex linear combination* into the set.
- Problem 2 is more open-ended than they're used to. Some will get excited about this, and others will be turned off because it's not a "plug and chug" question. Emphasize to them that they will be hired for their creativity and problem-solving, and not their ability to answer precisely laid out questions, and this is what we're practicing!
- Students will be confused what part 2(d) means. If so, take some time at the front of the room to make a chalk drawing of a face and a reverse face. Remember, if you're writing on a chalkboard, black and white are already reversed!