Calculus

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Contents

Part I

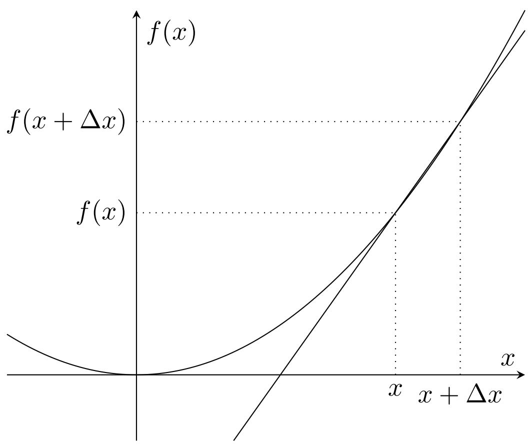
Single Variable Calculus

Chapter 1

Differentiation

1.1 The Derivative

The derivative measures change. More specifically, if we have a function, f(x), then the derivative of this function (often denoted f'(x)) tells us how quickly the function is increasing or decreasing at any given point. If we consider a general function and think how we might measure how it changes, we may look at the difference in the function between two values of x.



Here we have two points, one at $x = x_1$ and $x = x_2 = x_1 + \Delta x$. Here Δ means "change in" and so Δx denotes the difference between our two x values. The corresponding function values are $f(x_1)$ and $f(x_1+\Delta x)$ where the difference between these two values is Δf. We can say a reasonable estimate of the rate of change would be the gradient of the line that passes through these two points. We can write this as

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1.1}$$

We can see that this estimate for the rate of change gets better the closer the two points are, or the smaller Δx gets. If we take the limit of $\Delta x \to 0$, then our two points become infinitesimally close and the line through them becomes the tangent line to the curve. Thus we arrive at the definition of the derivative and the idea of "instantaneous" rate of change. The derivative is defined as a function which gives us the gradient of the tangent line to a curve at a point x. Mathematically we can write this as

$$f'(x) \equiv \frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Using the expression for the gradient above we then get

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1.2}$$

Which is our formal definition of the derivative. The process of finding a functions derivative is called differentiation. We can differentiate from first principles using the expression above.

Example: Differentiate the function x^2 using first principles.

Using equation 1.2, we have that

$$\begin{aligned}
\frac{d}{dx}(x^2) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + \Delta x^2) - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x \\
&= 2x
\end{aligned}$$

We can generalise the above example by differentiating x^n instead. With which we get

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}\Delta x^2 + \ldots) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}\Delta x^2}{\Delta x} + \ldots \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \binom{n}{2}x^{n-2}\Delta x + \ldots \\
&= nx^{n-1}
\end{aligned}$$

We know this is true for all n as any subsequent terms will contain a factor of Δx which follows from the binomial theorem. The expression we have just derived is known as the power rule which we can use to differentiate any power term. There are further rules of differentiation which will allow us to differentiate any function thrown at us.

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

1.2 The Chain Rule

The first rule is the chain rule which we use to differentiate composite functions, or "functions of functions". If we have a composite function, $f[g(x)]$, then the chain rule says the derivative of f with respect to x can be written as

$$\frac{d}{dx}f[g(x)] = \frac{df}{dg}\frac{dg}{dx}$$

1.3 The Product Rule

The second rule is the product rule which tells us how to derivative products of functions. Formally, we ay derive this using equation 1.2. If we define the product of two functions $f(x) = u(x)v(x)$, then we can write

$$\begin{aligned}
f(x+\Delta x) - f(x) &= u(x+\Delta x)v(x+\Delta x) - u(x)v(x) \\
&= u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x) + u(x+\Delta x)v(x) - u(x)v(x) \\
&= u(x+\Delta x)\left[v(x+\Delta x) - v(x)\right] + \left[u(x+\Delta x) - u(x)\right]v(x)
\end{aligned}$$

and so using our definition of the derivative,

$$\begin{aligned}
\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ u(x+\Delta x)\left[\frac{v(x+\Delta x) - v(x)}{\Delta x}\right] + \left[\frac{u(x+\Delta x) - u(x)}{\Delta x}\right] v(x) \right\} \\
&= u(x)\frac{dv}{dx} + \frac{du}{dx}v(x)
\end{aligned}$$

This can be written much more neatly using prime notation which gives us,

$$f' = (uv)' = u'v + uv'$$

This also works for any numbers of products. Namely, if we had $f(x) = u(x)v(x)w(x)$ then,

$$f' = u'vw + uv'w + uvw'$$

1.4 The Quotient Rule

Like the product rule with products of functions, the quotient rule tells how to differentiate quotients of functions. Consider the function $f(x) = u(x)/v(x)$, the product rule tells us that

$$f' = \left(\frac{u}{v}\right)' = u\left(\frac{1}{v}\right)' + u'\left(\frac{1}{v}\right) = u\left(-\frac{v'}{v^2}\right) + \frac{u'}{v}$$

Combining our two fractions and we get

$$f' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

Which is our quotient rule.

1.5 Implicit Differentiation

So far we have differentiated functions in the form $y = f(x)$ which is its explicit form. However, not every function can be expressed explicitly, for example $x^3 - 3xy + y^3 = 2$. Such functions are called implicit functions. We can differentiate implicit functions term by term using the chain rule. In the case of function above we have,

$$\begin{aligned} \frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(2) \\ 3x^2 - \left(3x\frac{dy}{dx} + 3y\right) + 3y^2\frac{dy}{dx} &= 0 \end{aligned}$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

1.6 Maxima and Minima

Chapter 2

Integration

2.1 The Antiderivative

$$I = \int_a^b f(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c, \qquad \int \frac{1}{x} dx = ln(x) + c$$

2.2 Integration by Substitution

2.3 Integration by Parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

$$uv = \int u\frac{dv}{dx}dx + \int \frac{du}{dx}vdx$$

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx$$

2.4 Surfaces and Volumes of Revolution

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

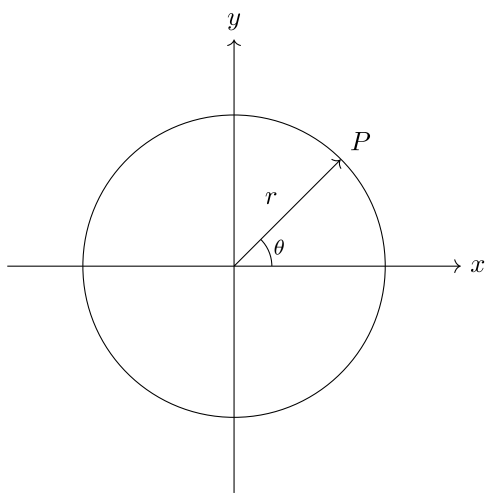
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$S = \int_a^b 2\pi y ds$$

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$V = \int_a^b \pi y^2 dx$$

2.5 Plane Polar Coordinates



Chapter 3

Series

3.1 Power Series

3.2 Taylor Series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_n^\infty a_n x^n$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_n^\infty n a_n x^{n-1}$$

$$f''(x) = \sum_n^\infty n(n-1) a_n x^{n-2}$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) +$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

3.3 L'Hopital's Rule

Chapter 4

Complex Numbers

4.1 Imaginary Numbers

$$i = \sqrt{-1}$$

$$z = x + iy$$

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + i(x_1 y_2 + y_1 x_2) + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 x_2 - y_1 y_2) \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= z_2 z_1 \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3) \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \arctan\left(\frac{y}{x}\right)$$

4.2 Polar Representation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \dots\right)
\end{aligned}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

$$z = re^{i\theta}$$

$$re^{i\theta} = re^{i(\theta+2n\pi)}$$

4.3 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$$z + \frac{1}{z} = 2\cos\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

4.4 Hyperbolic Functions

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cos(ix) = \frac{1}{2}(e^x + e^{-x})$$
$$\sin(ix) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \cos(ix)$$
$$i\sinh(x) = \sin(ix)$$
$$\cos(x) = \cosh(ix)$$
$$i\sin(x) = \sinh(ix)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

Part II

Multi Variable Calculus

Chapter 5

Partial Differentiation

5.1 The Partial Derivative

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Delta f \approx \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

$$df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

$$df = \sum_i^n \frac{\partial f}{\partial x_i} dx_i$$

$$\frac{df}{dx_1} = \sum_i^n \left(\frac{\partial f}{\partial x_i}\right) \frac{dx_i}{dx_1}$$

Chapter 6

Multiple Integrals

6.1 Double Integrals

$$I = \int f(x, y) dA$$

$$I = \int \int f(x, y) dx dy$$

6.2 Triple Integrals

Chapter 7

Vector Calculus

7.1 Grad, Div, and Curl

$$\nabla \cdot \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\nabla \times \boldsymbol{a} =$$

7.2 Line Integrals

7.3 Surface Integrals

7.4 Volume Integrals

7.5 Divergence Theorem

7.6 Stoke's Theorem