Module 2: Life Insurance Draft

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## Package:  lifecontingencies
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Introduction

Life insurance is a contract between an insurance company and a policy holder where the insurer agrees to pay a sum of money contingent upon a life event (e.g. death). Death is indeed a random event, and actuaries in life insurance are tasked with developing systems that account for the probability of death and the payments to be made. This can be done in a variety of ways. In this section we aim to introduce the main types of life insurance and how these models are derived using basic principles of probability and finance. By the end of this module, the reader should have an understanding of the basics of life insurance, and some insight on how actuaries deal with these problems in real life.

2.1 Basic Insurance Computations

A major component of life insurance is the amount that the insurer pays, which we call the benefit. The benefit can of course be any positive value, but for our introduction we will only consider benefits of 0 or 1. Since insurance depends on mortality, we must also consider the future lifetime random variable, T.

For most examples this module, we actually base the models around the *curtate future lifetime* because we focus on insurances payable at the end of the year of death. Recall that the curtate future lifetime, *k*, is the number of future years completed by (x) prior to death. We will also make the assumptions that all benefits are paid on December 31, all contracts are issued January 1, and death can occur at any point in the year . This is important to remember when defining functions that involve *k*.

We rely on slight variations of the benefit function, b_t, and the discount function, v_t, where v_t is the discount factor from the time of the payment to the time of policy issue. In the examples using curtate future lifetime, the subscripts become *k+1*. Imagine a person that purchased life insurance on January 1st, and then dies three months later. From the perspective of the beginning of the year, this makes their future lifetime $T = \frac{3}{12}$, and their curtate future lifetime 0 since they did not survive a complete year. The assumption is the benefit is paid at the end of the year at time 1, and it is discounted from time 1, so the subscripts must be k+1 = 0+1 = 1.

The present value of a benefit paid at time T is $b_T v_t$. Since T is a random variable, so is the present value of the payments, Z = $b_T v_T$. The expected value of the random variable Z, is what we call the **actuarial present value** (APV) of the insurance. The actuarial present value takes the time value of money concept, and adjusts it for the probability of payment at a given time. To get a better feel for actuarial present value, we'll compute it for a simple case.

Suppose a person is going to die at some point during the next four years, and there is a policy that pays 1 at the end of each year *only if* the insured dies during that year (i.e. $b_T = 1$ or 0 depending on death). Using an interest rate of .08 and a .25 probability of payment each year, we have

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.25)(v^1) + (1)(.25)(v^2) + (1)(.25)(v^3) + (1)(.25)(v^4) \approx 0.828.$$

So, the insurer expects to pay a benefit of .828. What happens if the insured has a much higher chance of dying during the first of the next four years? For example,

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.97)(v^1) + (1)(.01)(v^2) + (1)(.01)(v^3) + (1)(.01)(v^4) \approx 0.922.$$

Or in the opposite extreme case,

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.01)(v^1) + (1)(.01)(v^2) + (1)(.01)(v^3) + (1)(.97)(v^4) \approx 0.739.$$

In the unit payment case, the actuarial present value behaves as we expect it to from time value of money principles, with the probabilities acting as "payments." The reader may have noticed something about the nature of the probabilities that we were using in the examples above. They are exactly the probability concepts that we introduced in the life tables modules. For example, in the example directly above, the first probability of payment, .01, is the probability that a life dies in the first year. The second .01 is the probability that a life survives the first year, and then dies during the second year. The last probability, .97, is the probability that the life survives the first three years, and dies the fourth year.

Those probabilities are calculated using p and q for each year. With the use of this notation from the last module, we can write the general formula for the specific examples above as

$$A^1_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \ {}_kp_x \ q_{x+k}$$

This can be further generalized to

$$APV = \mathrm{E}[Z] = \sum_{k=0}^{\infty} v^t \ {}_tp_x \ q_{x+t}.$$

2.2 More On How Insurance Works

Now that we have a basic idea of the computations involved for insurance on a single life, we can look at a larger example and its implications. Imagine we are working for a life insurance company, where 10,000 people of age 50 buy a one year term insurance policy with a benefit of \$55,000. Each of the insured pays an annual premium of \$500 at the beginning of the term so the insurance company collects (10,000)(\$500) = \$5,000,000. The life tables for this group tell us that we can expect 75 people to die in the next year, so we expect to pay out total benefits of (\$55,0000)(75) = \$4,125,000 one year from now. Discounting at a rate of .08 gives an actuarial present value of approximately \$3.8 million, so the expected profit is \$1.2 million. Of course, this is just an estimation. In reality, there will be variability, in particular to mortality. The nature of this variability will be explored in Module 5.

Let's extend this general idea over the course of four years. Imagine we started out with a group of 10,000 people, and charged them each \$500 at the beginning of each year for life insurance. Like the previous example, the insurance pays out \$55,000 at the end of the year of death, and we use a interest rate of .08. We may get a situation that looks like the table below.

Lives	Expected Deaths	Premium Collected	Payout for Deaths	PV Premium at age 50	PV Payout at age 50
10000		5000000			
9975	25	4987500	1375000	4618056	1273148
9935	40	4967500	2200000	4258831	1886145
9890	45	4945000	2475000	3925500	1964735
9840	50	4920000	2750000	3616347	2021332
			Total	16418733	7145360
			Expected Profit		9273373

One can imagine a table like this that continues until we are left with no more lives. In this example, it's clear the actuaries have a lot to take into account when deciding how to price the insurance. The type of policyholders, length of term, benefit amount, survivorship, and interest rates are all important factors to track and predict when trying to make an accurate model. These problems quickly become more difficult as we add more detail and variation in the important factors, but the concepts remain the same. In the following section, we will introduce a few of the main types of insurance and the mathematical detail behind them.

2.3 Types of Life insurance

Recall the model we wrote at the end of Section 2.1 that pays a benefit if the insured (x) dies within n years,

$$A^1_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \; {}_kp_x \; q_{x+k}$$

This is the model for **n-year term life insurance**. Here we have defined b_t, v_t, and Z to be

$$b_{k+1} = \begin{cases} 1 & K = 0, 1, ..., n-1 \\ 0 & \text{elsewhere}, \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, ..., n-1 \\ 0 & \text{elsewhere}. \end{cases}$$

Another type of insurance pays a benefit regardless of the time it takes for the death of (x), called **whole-life insurance**. We make a slight adjustment to the APV formula above and write

$$A_x = \mathrm{E}[Z] = \sum_{k=0}^{\infty} v^{k+1} \; {}_kp_x \; q_{x+k}.$$

In this case, we let $b_{k+1} = 1$ and $n \to \infty$.

An **n-year endowment insurance** pays a benefit if the insured dies within n years, and also pays if the insured survives at least n years. This gives the functions

$$b_{k+1} = 1, \qquad k = 0, 1, \ldots,$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, ..., n-1 \\ v^n & k = n, n+1, ..., \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, ..., n-1 \\ v^n & k = n, n+1, ..., \end{cases}$$

In this case, it's helpful to remember that APVs are expected values of functions of random variables. So far, the only random variable is K, the (curtate) future lifetime, which means we can write the APV for *n-year endowment* as E($v_X b_X$) where X = min(k+1, n).

The last concept we will introduce before the exercises is the concept of insurance in the continuous context, with a benefit payable at the moment of death. Moving from discrete insurance to continuous insurance is exactly what you would expect from probability theory — sums become integrals, and probability functions become probability density functions. We also move away from curtate future lifetime, and return to the idea of T, the random variable representing future lifetime. This makes our present value function

$$z_t = b_t v_t$$

Similar to the discrete case, we redefine z_t, b_t, and v_t depending on what type of insurance we have. When looking at the continuous version of whole life insurance, let

$$\begin{aligned} b_t &= 1 \quad t \geq 0, \\ v_t &= v^t \quad t \geq 0, \\ Z &= v^t \quad t \geq 0. \end{aligned}$$

and then we can write the APV as

$$\bar{A}_x = E[Z] = \int_o^\infty z_t \ f_T(t) \ dt = \int_0^\infty v^t \ {}_tp_x \ \mu_x(t) \ dt$$

For example, assume the pdf of T for (x) is assumed to be

$$f_t(t) = \begin{cases} 1/85 & 0 \leq t \leq 85 \\ 0 & \text{elsewhere.} \end{cases}$$

With a force of interest, δ, we can calculate the actuarial present value as

$$\bar{A}_x = E[Z] = \int_o^\infty z_t \ f_T(t) \ dt = \int_0^{85} e^{-\delta t} \frac{1}{85} \ dt = \frac{1 - e^{-85\delta}}{85\delta}.$$

As another example of the continuous case, the n**-year pure endowment** provides a payment at the end of the nth year if the insured survives at least n years from the time of the policy issue. Now we let

$$b_t = \begin{cases} 0 & t \leq n \\ 1 & t > n, \end{cases}$$

$$v_t = v^t \quad t \geq 0,$$

$$Z = \begin{cases} 0 & T \leq n \\ v^n & T > n. \end{cases}$$

Then we add that $Z = v^n Y$, where Y is the indicator with the value 1 if if the insured survives to $(x+n)$. With this endowment, the size and time of the payment is predetermined so the only uncertainty in this problem is whether or not the payment occurs. Noting that Y is a Bernoulli random variable, it's clear that

$$Ax : n = E[Z] = v^n \ E[Y] = v^n \ {}_np_x.$$

As one can imagine, there are many types of insurance that can be created with slight variations to the functions z_t, b_t, and v_t. For example, insurances that pay with increasing payments, decreasing payments, or payments at different intervals. Over time, one can memorize the details of each type of insurance and endowment, but there is value in understanding where the formulas come from and the ability to recreate them if needed. For reference, summary tables have been provided in the appendix that include the insurances covered in this section and more.

2.4 Life Insurance in R

Present value function

Present value with probabilities - APV

Expected curtate lifetime

The lifecontingencies package introduced in the first module has some functions that are useful in the calculations that we saw in these lessons. For example, the package has a present value function that calculates the present value as learned in finance class, but it also has the option to input probabilities as an argument.

For example, observe the use of the function to compute the actuarial present value of the first example from this module, n-year term life insurance with an interest rate of .08 and .25 probability of payment each year.

```r
presentValue(cashFlows = c(1, 1, 1, 1), timeIds = c(1, 2, 3, 4), interest = .08,
             probabilities = c(.25, .25, .25, .25))
```

```
## [1] 0.8280317
```

This agrees with the calculation that we did by hand earlier.

Problems