Module 2: Life Insurance Draft

Jack Conway and Marcus Milazzo

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## Package:  lifecontingencies
## Authors:  Giorgio Alfredo Spedicato [cre,aut], Reinhold Kainhofer [ctb], Kevin J.
##     Owens [ctb], Christophe Dutang [ctb], Ernesto Schirmacher[ctb]
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Introduction

Life insurance is a contract between an insurance company and a policy holder where the insurer agrees to pay a sum of money contingent upon a life event (e.g. death). Death is indeed a random event, and actuaries in life insurance are tasked with developing systems that account for the probability of death and the payments to be made. This can be done in a variety of ways. In this section we aim to introduce the main types of life insurance and how these models are derived using basic principles of probability and finance. By the end of this module, the reader should have an understanding of the basics of life insurance, and some insight on how actuaries deal with these problems in real life.

2.1 Basic Insurance Computations

A major component of life insurance is the amount that the insurer pays, which is called the benefit. The benefit can of course be any positive value, but for introduction in this module we will only consider benefits of 0 or 1, as defined by the benefit function, b_t. Since insurance depends on mortality, we must also consider the future lifetime random variable, T. Lastly, recall from intro finance the discounting factor v_t for time value of money computations.

The present value of a benefit paid at time T is $b_T v_T$. Since T is a random variable, so is the present value of the payments, Z = $b_T v_T$. The expected value of the random variable Z, is what we call the **actuarial present value** (APV) of the insurance. The actuarial present value takes the time value of money concept, and adjusts it for the probability of payment at a given time.

To get a better feel for actuarial present value, we'll compute it for a simple case. Suppose a person is going to die at some point during the next four years, and there is a policy that pays 1 at the end of each year *only if* the insured dies during that year (i.e. $b_T = 1$ or 0 depending on death). Using an interest rate of .08 and a .25 probability of payment each year, we have

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.25)(v^1) + (1)(.25)(v^2) + (1)(.25)(v^3) + (1)(.25)(v^4) \approx 0.828.$$

So, the insurer expects to pay a benefit of .828.

What happens if the insured has a much higher chance of dying during the first of the next four years? For example,

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.97)(v^1) + (1)(.01)(v^2) + (1)(.01)(v^3) + (1)(.01)(v^4) \approx 0.922.$$

Or in the opposite extreme case,

$$\mathrm{E}[Z] = \mathrm{APV} = (1)(.01)(v^1) + (1)(.01)(v^2) + (1)(.01)(v^3) + (1)(.97)(v^4) \approx 0.739.$$

In the unit payment case, the actuarial present value behaves as we expect it to from time value of money principles, with the probabilities acting as "payments."

The reader may have noticed something about the nature of the probabilities that we were using in the examples above. They are exactly the probability concepts that we introduced in the life table module. For example, in the example directly above, the first probability of payment, .01, is the probability that a life dies in the first year. The second .01 is the probability that a life survives the first year, and then dies during the second year. The last probability, .97, is the probability that the life survives the first three years, and dies the fourth year.

Those probabilities are calculated using p and q for each year. With the use of this notation from the previous module, we can write the general formula for the specific examples above as

$$A^{1}_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \ {}_{k}p_{x} \ q_{x+k}$$

This can be further generalized to

$$APV = \mathrm{E}[Z] = \sum_{k=0}^{\infty} v^{t} \ {}_{t}p_{x} \ q_{x+t}.$$

2.2 More On How Insurance Works

Now that we have a basic idea of the computations involved for insurance on a single life, we can look at larger examples and their implications. Imagine we are working for a life insurance company, where 10,000 people of age 50 buy a one year term insurance policy with a benefit of \$55,000. Each of the insured pays an annual premium of \$500 at the beginning of the term so the insurance company collects (10,000)(\$500) = \$5,000,000. The life tables for this group tell us that we can expect 75 people to die in the next year, so we expect to pay out total benefits of (\$55,0000)(75) = \$4,125,000 one year from now. Discounting at a rate of .08 gives an actuarial present value of approximately \$3.8 million, so the expected profit is \$1.2 million. Of course, this is just an estimation. In reality, there will be variability, in particular to mortality. The nature of this variability will be explored in Module 5.

Let's extend this general idea over the course of four years. Imagine we started out with a group of 10,000 people, and charged them each \$500 at the beginning of each year for life insurance. Like the previous example, the insurance pays out \$55,000 at the end of the year of death, and we use an interest rate of .08. We may get a situation that looks like the table below.

Lives	Expected Deaths	Premium Collected	Payout for Deaths	PV Premium at age 50	PV Payout at age 50
10000	25	5000000			
9975	40	4987500	1375000	4618056	1273148
9935	45	4967500	2200000	4258831	1886145
9890	50	4945000	2475000	3925500	1964735
9840		4920000	2750000	3616347	2021332
			Total	16418733	7145360
			Expected Profit		9273373

One can imagine a table like this that continues until we are left with no more lives. In this example, it's clear the actuaries have a lot to take into account when deciding how to price the insurance. The type of policyholders, length of term, benefit amount, survivorship, and interest rates are all important factors to track and predict when trying to make an accurate model. These problems quickly become more difficult as we add more detail and variation in the important factors, but the concepts remain the same. In the following section, we will introduce a few of the main types of insurance and the mathematical detail behind them.

2.3 Types of Life insurance

Recall the model we wrote at the end of Section 2.1 that pays a benefit if the insured (x) dies within n years,

$$A^{1}_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \; {}_kp_x \; q_{x+k}$$

This is the model for **n-year term life insurance**. Here we have defined b_t, v_t, and Z to be

$$b_{k+1} = \begin{cases} 1 & K = 0, 1, ..., n-1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, ..., n-1 \\ 0 & \text{elsewhere.} \end{cases}$$

For most examples in this module, we actually base the models around the *curtate future lifetime* because we focus on insurances payable at the end of the year of death. Recall that the curtate future lifetime, *k*, is the number of future years completed by (x) prior to death. We will also make the assumptions that all benefits are paid on December 31, all contracts are issued January 1, and death can occur at any point in the year . This is important to remember when defining functions that involve *k*. In the examples using curtate future lifetime, the subscripts become *k+1*.

For clarification, imagine a person that purchased life insurance on January 1st, and then dies three months later. From the perspective of the beginning of the year, this makes their future lifetime $T = \frac{3}{12}$, and their curtate future lifetime 0 since they did not survive a complete year. The assumption is the benefit is paid at the end of the year at time 1, and it is discounted from time 1, so the subscript must be k+1 = 0+1 = 1.

Another type of insurance pays a benefit regardless of the time it takes for the death of (x), called **whole-life insurance**. We make a slight adjustment to the APV formula above and write

$$A_x = \mathrm{E}[Z] = \sum_{k=0}^{\infty} v^{k+1} \; {}_kp_x \; q_{x+k}.$$

In this case, we let $b_{k+1} = 1$ and $n \to \infty$. In practice, we do not have life tables that extend to infinity. The computation of whole-life insurance closely resembles n-year term insurance but we utilize the table to the extent of the number of years that the table covers.

An **n-year endowment insurance** pays a benefit if the insured dies within n years, and also pays if the insured survives at least n years. This gives the functions

$$b_{k+1} = 1, \qquad k = 0, 1, \ldots,$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, ..., n-1 \\ v^{n} & k = n, n+1, ..., \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, ..., n-1 \\ v^{n} & k = n, n+1, ..., \end{cases}$$

In this case, it's helpful to remember that APVs are expected values of functions of random variables. So far, the only random variable is K, the (curtate) future lifetime, which means we can write the APV for *n-year endowment* as $\mathrm{E}(v_X b_X)$ where X = min(k+1, n).

The last concept we will introduce is insurance in the continuous context, with a benefit payable at the moment of death. Moving from discrete insurance to continuous insurance is exactly what you would expect from probability theory — sums become integrals, and probability functions become probability density functions. We also move away from curtate future lifetime, and return to the idea of T, the random variable representing future lifetime. This makes our present value function

$$z_t = b_t v_t$$

Similar to the discrete case, we redefine z_t, b_t, and v_t depending on what type of insurance we have. When looking at the continuous version of whole life insurance, let

$$\begin{aligned} b_t &= 1 \quad t \geq 0, \\ v_t &= v^t \quad t \geq 0, \\ Z &= v^t \quad t \geq 0. \end{aligned}$$

and then we can write the APV as

$$\bar{A}_x = E[Z] = \int_o^\infty z_t \ f_T(t) \ dt = \int_0^\infty v^t \ {}_tp_x \ \mu_x(t) \ dt$$

For example, assume the pdf of T for (x) is assumed to be

$$f_t(t) = \begin{cases} 1/85 & 0 \leq t \leq 85 \\ 0 & \text{elsewhere.} \end{cases}$$

With a force of interest, δ, we can calculate the actuarial present value as

$$\bar{A}_x = E[Z] = \int_o^\infty z_t \ f_T(t) \ dt = \int_0^{85} e^{-\delta t} \frac{1}{85} \ dt = \frac{1 - e^{-85\delta}}{85\delta}.$$

As another example of the continuous case, the n**-year pure endowment** provides a payment at the end of the nth year if the insured survives at least n years from the time of the policy issue. Now we let

$$b_t = \begin{cases} 0 & t \leq n \\ 1 & t > n, \end{cases}$$

$$v_t = v^t \quad t \geq 0,$$

$$Z = \begin{cases} 0 & T \leq n \\ v^n & T > n. \end{cases}$$

Then we add that $Z = v^n Y$, where Y is the indicator with the value 1 if if the insured survives to $(x+n)$. With this endowment, the size and time of the payment is predetermined so the only uncertainty in this problem is whether or not the payment occurs. Noting that Y is a Bernoulli random variable, it's clear that

$$Ax : n = E[Z] = v^n \ E[Y] = v^n \ {}_np_x.$$

As one can imagine, there are many types of insurance that can be created with slight variations to the functions z_t, b_t, and v_t. For example, insurances that pay with increasing payments, decreasing payments, or payments at different intervals. Over time, one can memorize the details of each type of insurance and endowment, but there is value in understanding where the formulas come from and the ability to recreate them if needed. For reference, summary tables have been provided in the appendix that include the insurances covered in this section and more.

2.4 Life Insurance in R

The lifecontingencies package introduced in the first module has some functions that are useful for the calculations that we saw in these lessons. For example, the package has a present value function that calculates the present value as learned in finance class, but it also has the option to input probabilities to compute APV.

For example, observe the use of the function to compute the actuarial present value of the first example from this module, n-year term life insurance with an interest rate of .08 and .25 probability of payment each year.

```
presentValue(cashFlows = c(1, 1, 1, 1), timeIds = c(1, 2, 3, 4), interest = .08,
             probabilities = c(.25, .25, .25, .25))
```

```
## [1] 0.8280317
```

This agrees with the calculation that we did by hand earlier.

With curtate future lifetime being relevant to many life insurance computations, actuaries may want to get estimates of the expected future lifetime of alife. The *lifecontingencies* package also has a quick function for evaluating the expected future lifetime from a life table object. To find the expected lifetime for (10), enter

```
data(soa08Act)
exn(object = soa08Act, x = 10)
```

```
## [1] 63.4282
```

So, we expect a 10 year old to live approimately 63 more years based off of this life table.

Problems

There are currently 93,131 people of age 40 that want to purchase a whole life insurance policy that pays \$150,000 at the end of the year of death. Your goal is to use the life table, mod2prob1.csv, to evaluate the possibility of collecting a \$2,400 yearly premium paid at the beginning of the year. Specifically, compute the actuarial present value of the profit using an interest rate of .04 using Excel.

Hint: See the examples in Section 2.2

Solution

Row 40 of the given life table has 93,131 people, so it can be used to model what we think will happen with this population of 40 year olds that want life insurance.

x	Lives	Epected Deaths	Premium Collected	Payout for Deaths	PV Premiums at age 40	PV Payouts at age 40
40	93131.64	259.02	223515936	38853000	223515936	38853000
41	92872.62	276.92	222894288	41538000	214321430.8	39940384.62
42	92595.7	296.47	222229680	44470500	205463831.4	41115477.07
43	92299.23	317.76	221518152	47664000	196928830.5	42373122.44
44	91981.47	340.97	220755528	51145500	188702750.5	43719387.75
45	91640.5	366.25	219937200	54937500	180772346.5	45154620.43
46	91274.25	393.77	219058200	59065500	173124877.4	46680322.62
47	90880.48	423.7	218113152	63555000	165748069.5	48296576.62
48	90456.78	456.23	217096272	68434500	158630119.5	50004418.83
49	90000.55	491.55	216001320	73732500	151759662.3	51803476.48
50	89509	529.89	214821600	79483500	145125775.6	53696204.61
51	88979.11	571.43	213549864	85714500	138717919.6	55678504.76
52	88407.68	616.42	212178432	92463000	132526022.6	57752117
53	87791.26	665.06	210699024	99759000	126540373.8	59912670.26
54	87126.2	717.6	209102880	107640000	120751702.9	62159417.91
55	86408.6	774.27	207380640	116140500	115151107.9	64488696.98
56	85634.33	835.26	205522392	125289000	109730085.4	66892821.42
57	84799.07	900.82	203517768	135123000	104480577.1	69368533.1
58	83898.25	971.14	201355800	145671000	99394885.21	71907302.02
59	82927.11	1046.38	199025064	156957000	94465738.82	74498450.95
60	81880.73	1126.72	196513752	169008000	89686311.16	77133045
61	80754.01	1212.23	193809624	181834500	85050175.42	79795088.62
62	79541.78	1303	190900272	195450000	80551398.08	82471180.32
63	78238.78	1399	187773072	209850000	76184480	85141671.05
64	76839.78	1500.15	184415472	225022500	71944435.83	87786109.46
65	75339.63	1606.26	180815112	240939000	67826786.61	90380267.22
66	73733.37	1717.04	176960088	257556000	63827598.4	92897676.08
67	72016.33	1832.02	172839192	274803000	59943495.75	95306233.92
68	70184.31	1950.65	168442344	292597500	56171726.93	97574674.4
69	68233.66	2072.12	163760784	310818000	52510127.06	99664231.41
70	66161.54	2195.46	158787696	329319000	48957210.92	101535195.4
71	63966.08	2319.46	153518592	347919000	45512161.34	103144156.4
72	61646.62	2442.69	147951888	366403500	42174860.42	104446226.9
73	59203.93	2563.43	142089432	384514500	38945885.38	105393183.9
74	56640.5	2679.7	135937200	401955000	35826533.11	105936080.2
75	53960.8	2789.29	129505920	418393500	32818803.68	106027385.8
76	51171.51	2889.7	122811624	433455000	29925351.45	105619425.8
77	48281.81	2978.21	115876344	446731500	27149462.15	104667782.3
78	45303.6	3051.98	108728640	457797000	24494978.49	103134994.3
79	42251.62	3107.98	101403888	466197000	21966171.71	100987876.9
80	39143.64	3143.27	93944736	471490500	19567659.31	98206305.81

113	0.001410556	0.001161733	3.3853344	174.25995	0.1932714948	9.948642301
114	0.000248823	2.12E-04	0.5971752	3.18E+01	0.03278194034	1.74E+00
115	3.71E-05	3.25E-05	8.90E-02	4.87E+00	4.70E-03	2.57E-01
116	4.61E-06	4.14E-06	1.11E-02	6.21E-01	5.62E-04	3.15E-02
117	4.68E-07	4.30E-07	1.12E-03	6.45E-02	5.48E-05	3.15E-03
118	3.82E-08	3.58E-08	9.17E-05	5.36E-03	4.30E-06	2.52E-04
119	2.44E-09	2.32E-09	5.86E-06	3.48E-04	2.64E-07	1.57E-05
120	1.20E-10	1.16E-10	2.88E-07	1.73E-05	1.25E-08	7.52E-07
121	4.39E-12	4.27E-12	1.05E-08	6.41E-07	4.40E-10	2.67E-08
122	1.17E-13	1.15E-13	2.81E-10	1.72E-08	1.13E-11	6.91E-10
123	2.20E-15	2.17E-15	5.28E-12	3.26E-10	2.04E-13	1.26E-11
124	2.82E-17	2.80E-17	6.77E-14	4.19E-12	2.51E-15	1.56E-13
125	2.38E-19	2.37E-19	5.71E-16	3.55E-14	2.04E-17	1.27E-15
126	1.26E-21	1.26E-21	3.02E-18	1.88E-16	1.04E-19	6.46E-18
127	4.04E-24	4.03E-24	9.70E-21	6.05E-19	3.20E-22	1.99E-20
128	7.45E-27	7.44E-27	1.79E-23	1.12E-21	5.67E-25	3.54E-23
129	7.47E-30	7.47E-30	1.79E-26	1.12E-24	5.46E-28	3.41E-26
130	3.85E-33	3.85E-33	9.24E-30	5.77E-28	2.71E-31	1.69E-29
131	9.54E-37	9.54E-37	2.29E-33	1.43E-31	6.45E-35	4.03E-33
132	1.06E-40	1.06E-40	2.54E-37	1.59E-35	6.89E-39	4.31E-37
133	4.92E-45	4.92E-45	1.18E-41	7.38E-40	3.08E-43	1.92E-41
134	8.70E-50	8.70E-50	2.09E-46	1.30E-44	5.23E-48	3.27E-46
135	5.35E-55	5.35E-55	1.28E-51	8.02E-50	3.09E-53	1.93E-51
136	1.03E-60	1.03E-60	2.47E-57	1.54E-55	5.73E-59	3.58E-57
137	5.61E-67	5.61E-67	1.35E-63	8.41E-62	3.00E-65	1.87E-63
138	7.58E-74	7.58E-74	1.82E-70	1.14E-68	3.90E-72	2.43E-70
139	2.23E-81	2.23E-81	5.35E-78	3.35E-76	1.10E-79	6.89E-78
				Total	4222283791	3972826363
				Profit		249457428.48