5 4d Vectors

We can now construct 4-vectors and their components, using the ideas we have developed in the non-orthogonal basis case. The counterpart to g_{ij} is $\eta_{\mu\nu}$, which we introduced in sec 2.

5.1 4d Basis Set

A basis set is:

$$\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{or} \quad \{\mathbf{e}_\mu\} \tag{5.1}$$

Convention:

- **Greek** indices run from $0 \to 3$ (0 is the time component)
- **Latin** indices run from $1 \to 3$ (spatial components only)

Then, as in the non-orthogonal case:

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu} \tag{5.2}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.3}$$

The Inverse Metric
Then construct $\eta^{\mu\nu}$:

$$\eta^{\mu\nu}\eta_{\nu\kappa} = \delta^\mu{}_\kappa \tag{5.4}$$

As is easily checked, entries of $\eta^{\mu\nu}$ are the same:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

and $\delta^\mu{}_\nu$ is just the **unit matrix**. (Note: in **general** $g^{\mu\nu} \neq g_{\mu\nu}$.)

5.2 4-Vectors

Then we may form vectors:

$$\mathbf{A} = A^\mu \mathbf{e}_\mu \tag{5.6}$$

The first example is the **position 4-vector** of an event:

$$\begin{aligned} x^\mu &= (\mathrm{c}t, x, y, z) \\ &= (\mathrm{c}t, \mathbf{x}) \\ &= (\mathrm{c}t, x^i) \\ &= (x^0, x^i) \end{aligned} \tag{5.7}$$

5.3 Contravariant and Covariant Components

x^μ are **contravariant** components and x_μ are **covariant** components.

$$x_\mu = \eta_{\mu\nu} x^\nu = \eta_{\mu 0} x^0 + \eta_{\mu i} x^i \tag{5.8}$$

Note: here we sum over Latin index $i = 1, 2, 3$.

Then pick $\mu = 0$:

$$\begin{aligned} x_0 &= \eta_{00} x^0 + \eta_{0i} x^i \\ &= (-1) x^0 + 0 \cdot x^i \\ &= -\mathrm{c}t \end{aligned} \tag{5.9}$$

Pick $\mu = j$:

$$\begin{aligned} x_j &= \eta_{j0} x^0 + \eta_{ji} x^i \\ &= 0 \cdot x^0 + \delta_{ji} x^i \\ &= x^j \end{aligned} \tag{5.10}$$

Note there is no difference between the **spatial** covariant and contravariant components of a four vector as the spatial part of the metric, η_{ij}, is just the unit matrix.

So in terms of four (or five) different but equivalent, notations (all are useful in some circumstances):

$$\begin{aligned} x_\mu &= (-\mathrm{c}t, x, y, z) \\ &= (-\mathrm{c}t, \mathbf{x}) \\ &= (-\mathrm{c}t, x_i) \\ &= (x_0, x_i) \end{aligned} \tag{5.11}$$

5.4 Note on Spatial Indices

Note: On spatial (Latin) indices, it does not matter if covariant or contravariant, as $\eta_{ij} = \delta_{ij}$, so: $A_i = A^i$.

Not true for:

- Greek indices, or
- Non-Cartesian coordinates.

Invariant Interval in Index Notation
Pick $\mathrm{c}t_1 = 0$, $\mathbf{x}_1 = \mathbf{0}$; $\mathrm{c}t_2 = \mathrm{c}t$, $\mathbf{x}_2 = \mathbf{x}$.

Propose:

$$
\begin{aligned}
s^2 &= x^\mu \eta_{\mu\nu} x^\nu \\
&= x^\mu x_\mu = x_\nu x^\nu \quad \text{[Note dummy indices]} \qquad (5.12) \\
&= x^0 x_0 + \mathbf{x}\cdot\mathbf{x} \\
&= x^0 x_0 + x^i x_i \\
&= \mathrm{c}t(-\mathrm{c}t) + x^2 + y^2 + z^2 \\
&= -(\mathrm{c}t)^2 + x^2 + y^2 + z^2 \qquad (5.13)
\end{aligned}
$$

as before.

Important: a very useful device is illustrated in Eq. (5.12) of interchanging, "flip-flopping", which index is covariant and which is contravariant. We will use this repeatedly during the module. Perhaps it is useful to demonstrate it another way. Although we have only introduced the components of one four-vector so far, x^μ, we will meet many others, so here we will be more generic with components a^μ and b^μ

$$
\begin{aligned}
\text{(Insert a unit matrix)} \quad a^\mu b_\mu &= a^\mu \delta_\mu{}^\nu b_\nu \\
\text{(Write the unit matrix as } \eta \text{ times its inverse)} \quad &= a^\mu \eta_{\mu\kappa} \eta^{\kappa\nu} b_\nu \\
&= a_\kappa b^\kappa \ , \qquad (5.14)
\end{aligned}
$$

6 Lorentz Transformations (Index Notation)

6.1 Definition of Lorentz Transformations

We now describe Lorentz transformations using index notation, deriving expressions for the transformation of both covariant and contravariant components.

We represent Lorentz transformations for covariant components via:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{6.1}$$

which transforms from reference frame Σ to reference frame Σ'.

We may deduce the corresponding representation for *contravariant* components, using a “flip-flop”

$$x'_{\lambda} = \eta_{\lambda\mu} x'^{\mu} = \eta_{\lambda\mu} \Lambda^{\mu}{}_{\nu} x^{\nu} = \Lambda_{\lambda\nu} x^{\nu} = \Lambda_{\lambda}{}^{\nu} x_{\nu} \tag{6.2}$$

Thus:

$$x'^{0} = \Lambda^{0}{}_{0} x^{0} + \Lambda^{0}{}_{i} x^{i} \tag{6.3}$$

$$x'^{i} = \Lambda^{i}{}_{0} x^{0} + \Lambda^{i}{}_{j} x^{j} \tag{6.4}$$

6.2 Standard Configuration

Convenient to use notation for the standard configuration:

$$\begin{gathered} x'^{\mu} = \bar{\Lambda}^{\mu}{}_{\nu} x^{\nu} \quad \text{(note bar)} \\ \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \end{gathered} \tag{6.5}$$

So:

$$\begin{aligned} \bar{\Lambda}^{0}{}_{0} &= \cosh\phi = \gamma \\ \bar{\Lambda}^{0}{}_{1} &= -\sinh\phi = -\beta\gamma \\ \bar{\Lambda}^{1}{}_{0} &= -\sinh\phi = -\beta\gamma \\ \bar{\Lambda}^{1}{}_{1} &= \cosh\phi = \gamma \end{aligned}$$

To be more evocative, sometimes write $\Lambda^{t}{}_{t} = \bar{\Lambda}^{0}{}_{0}$, $\Lambda^{t}{}_{x} = \bar{\Lambda}^{0}{}_{1}$, etc., and $x^{t} = \mathrm{c}t$, $x^{x} = x$, $x^{y} = y$, etc.

6.3 Matrix Forms

So:

$$\bar{\Lambda}^{\mu}{}_{\nu}(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{aligned} \bar{\Lambda}_{\nu}{}^{\mu}(\phi) = \eta_{\nu\kappa} \bar{\Lambda}^{\kappa}{}_{\lambda}\, \eta^{\lambda\mu} &= \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \Lambda^{\mu}{}_{\nu}(-\phi) \\ &= (\Lambda^{-1})^{\mu}{}_{\nu}(\phi)\ . \end{aligned}$$

So $\bar{\Lambda}^{\mu}{}_{\nu}(\phi)$ "=" $\bar{\Lambda}_{\nu}{}^{\mu}(-\phi)$. This statement is rather special – see later. In general it is meaningless to compare expressions with different covariant/contravariant indices.