0 Introduction and Bibliography

Aims
The aims of the module are:

1. To understand electromagnetism as a naturally relativistic subject. for example, we will find that combining electrostatics with relativity yields the Lorentz force. An illuminating thought experiment is to consider a line of, say positive, charges at rest with respect to the observer. Then there is an electric field from the charges. Now the observer walks parallel to the line of charges. In their rest frame they observe a current opposite to the direction of the observer's motion, and hence a magnetic field due to the current. Thus we see that electric and magnetic fields transform into each other under changes of reference frame.

   We know already in special relativity that time and space may be grouped together, ct and $\mathbf{r}$, and energy and momentum, E/c and $\mathbf{p}$. Both of these form *four-vectors* that we will discuss at length presently. But $\mathbf{E}$ and $\mathbf{B}$ present a problem – there are six components in total, not four. We will resolve this issue.

   Finally there is the issue of units. It seems most natural to combine objects of the same dimensions (and hence units), so we pick ct to combine with $\mathbf{r}$ and similarly E/c and $\mathbf{p}$. Since $\mathbf{E}$ and $\mathbf{B}$ transform into each other, it is natural for them to have the same dimensions and hence units. This is not consistent with the SI system, which was not designed for relativistic calculations. We will abandon SI for electromagnetic quantities.

2. The second aim is to construct the formalism which manifests the intrinsically relativistic nature of electromagnetism most clearly. As we noted already in the previous item, this is not obvious. We will along the way generate a powerful description of relativistic particle dynamics.

3. To combine the relativistic motion of charges and radiation, showing for example the overall conservation of energy and momentum. This places the charges and the radiation on an equal footing.

Bibliography The best books (despite the awkwardness of sign differences) are by Wolfgang Rindler, Fig. (0.1):
Introduction to Special Relativity, Rindler, Wolfgang, OUP (1991).
Relativity : special, general, and cosmological, Rindler, Wolfgang, Second edition, OUP (2006).
The latter reference is available as a e-book, and while much more general the first 160 pages are on special relativity. It is a very useful reference for the Y4 module on General Relativity.

Figure 0.1: Wolfgang Rindler, see the Wiki link for some comments (flattering) on his books on relativity.

1 Newtonian Mechanics and Einstein's postulates

1.1 Newton's Laws

Newton's three laws are:

(N1) First Law (Law of Inertia)
Particles under the influence of no external forces move with constant velocity, $\dot{\mathbf{r}} = \mathbf{u}$. That is, constant speed in straight lines.

(N2) Second Law
The force on a body, $\mathbf{f}$, and its acceleration, $\mathbf{a} = \ddot{\mathbf{r}}$, are related:

$$\mathbf{f} = m\mathbf{a} \tag{1.1}$$

with m the body's mass.

(N3) Third Law
Forces of action and reaction are equal in magnitude and opposite in sign.

Logically, (N1) is a special case of (N2), with $\mathbf{f} = 0$.

1.2 Velocity, Reference Frames and Galilean and Einstein Relativity

What is velocity measured with respect to?

Newton thought there was "absolute space" which allowed a definitive assignment of $\mathbf{u}$, defined by "fixed stars". In fact, Newtonian dynamics does not require absolute space. "Relativity" denotes the principle that the laws of physics should have the same form (be "covariant") for two observers moving at a *constant relative velocity*. (Generalised in Y4 for General Relativity.)

The differences between the Galilean version and Einstein's version of relativity are:

(i) **Which laws of physics?**
In Newton's time the "laws of physics" were mainly mechanics, with optics being disjoint; there was no electromagnetism or thermodynamics. It was the advent of electromagnetism and the finite speed of light which presented Galilean relativity with a problem–as we shall see.

(ii) **The relationship between the measurements of two observers.**
Let the two observers be denoted as O and O'. The most primitive measurement is the definition of an "event", for example a gun being fired. To denote the event, each observer will allocate a time, t and t', and a position, $\mathbf{r}$ and $\mathbf{r}'$. We now examine how these variables are related in both Galilean and Einsteinian relativity.

Galilean Transformation
Assume for simplicity that:
Observers agree on a common zero for t, t' and at $t = t' = 0$ their positions coincide, so $\mathbf{r}(t=0) = \mathbf{r}'(t'=0)$.

For $t > 0$ we assume (*Galilean transformation*):

$$\text{(G1)} \quad t = t' \quad \text{(time is absolute)} \tag{1.2}$$

$$\text{(G2)} \quad \mathbf{r} - \mathbf{v}t = \mathbf{r}' \tag{1.3}$$

where $\mathbf{v}$ is the velocity of O' with respect to O.

Standard Configuration
It is convenient to introduce the "standard configuration" for the coordinate axes associated with the observers, that they are oriented in the same manner, and the relative motion is along the x-axis: where $\mathbf{v} = v\hat{\mathbf{x}}$, and we denote the coordinate systems S and S' which have the observers O and O' at their origins. The two coordinate systems are called "**inertial frames**

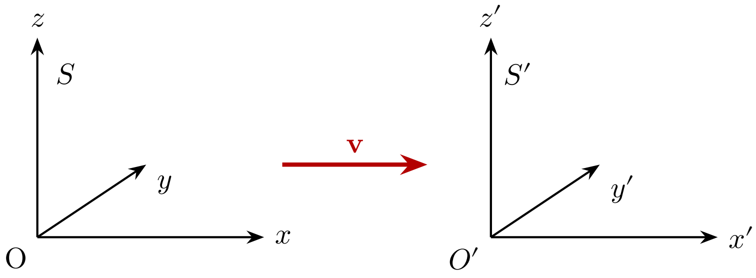

Figure 1.1: Standard configuration: Frame S has observer O at origin; Frame S' has observer O' at origin, moving with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ relative to S.

of reference" (IF's).

To derive the Galilean relationship, Eq. 1.3, consider a point P, which is stationary in frame O' with position vector

$$\mathbf{r}'(t') = \overrightarrow{O'P}$$

The relation of the two origins is

$$\overrightarrow{OO'} = \mathbf{v}t \ ,$$

So we may deduce the relation between the position vectors of the point P in the two frames. In O $\mathbf{r}(t)$ and in O', $\mathbf{r}'$:

$$\begin{aligned} \overrightarrow{OP} &= \overrightarrow{OO'} + \overrightarrow{O'P} \\ \Rightarrow \quad \mathbf{r}(t) &= \mathbf{v}t + \mathbf{r}' \\ \Rightarrow \quad \mathbf{r}(t) - \mathbf{v}t &= \mathbf{r}' \end{aligned}$$

Newtonian Mechanics in Different Inertial Frames
To see that Newtonian mechanics has the same formulation in different IF's (as observed by O'):

$$\begin{aligned} &\textbf{(N1):} \quad \dot{\mathbf{r}} = \mathbf{u} \quad \text{(for all } t\text{)} \Rightarrow \dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{v} \\ &\text{(velocity addition rule)} \quad \Rightarrow \dot{\mathbf{r}}' = \mathbf{u} - \mathbf{v} \ . \end{aligned}$$

So O' merely sees a *different* constant velocity, $\mathbf{u} - \mathbf{v}$.

$$\begin{aligned} &\textbf{(N2):} \quad \dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{v} \Rightarrow \ddot{\mathbf{r}}' = \ddot{\mathbf{r}} \ \text{(as } \mathbf{v} \text{ is constant)} \\ &\textit{Empirically} \quad \mathbf{f} = \mathbf{f}' \ ; \ \text{and } m = m' \\ &\qquad \Rightarrow \mathbf{f}' = m'\mathbf{a}' = m'\ddot{\mathbf{r}}' \end{aligned}$$

is *unchanged* both in form (i.e. 'covariant') and in numerical value ('invariant').

(N3) remains the same through observation that $\mathbf{f} = \mathbf{f}'$.

For *velocity-dependent* forces, the treatment is more difficult. (E.g. friction or Lorentz force.)

Waves Behave Differently – the form of the wave equation
Consider waves (on a string along the x-axis, say) where $u(x,t)$ is the displacement of the string at x at time t obeying:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

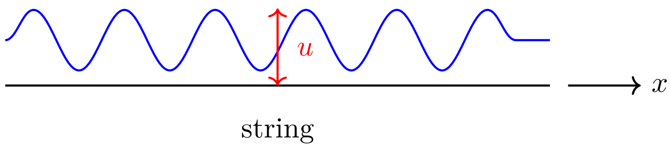

Figure 1.2: Wave displacement u on a string along the x-axis.

Now make a 'Galilean transformation':

$$x' = x - vt \tag{1.5}$$
$$t' = t \tag{1.6}$$

and rewrite the wave equation:

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'}$$

$$\text{and:} \quad \frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

$$\text{The wave equation becomes:} \quad \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)^2 u - \frac{\partial^2 u}{\partial x'^2} = 0\ . \tag{1.7}$$

Now look for wave-like solutions, with dispersion equation in S as $\omega = \pm ck$:

$$u(x', t') = U e^{i(kx' - \omega t')}$$
$$\Rightarrow -\frac{1}{c^2}(\omega + vk)^2 + k^2 = 0 \tag{1.8}$$
$$\Rightarrow \omega = \pm(c \mp v)k \tag{1.9}$$
$$c \to c'_{\pm} = c \mp v \tag{1.10}$$

is the modified speed of the waves, where $c'_{\pm}$ depends on direction of v relative to direction of wave. There are a number of consequences.

1. So if observer travels parallel to string at $v = c$, the wave travelling in the same direction as v will appear stationary. On the other hand if v and c are antiparallel, the waves will appear to travel at twice the "normal" velocity..
2. Thus addition of velocities occurs as in velocity addition rule in mechanics (Newtonian).
3. **Does this happen for light?**

   NO (Michelson–Morley Experiment).

1.3 The Michelson–Morley Experiment (1887)

Electromagnetic waves were thought to occur in a medium which surrounded the Earth, the"aether" (The University's first Vice-chancellor, Oliver Lodge, still believed in this theory until the twenties.) Since Earth must move with respect to aether at some point in orbit, the speed of light was expected to vary to vary.

Historical Note

The Michelson–Morley experiment has been called *"perhaps the most significant negative experiment in the history of science"* (Encyclopaedia Britannica).

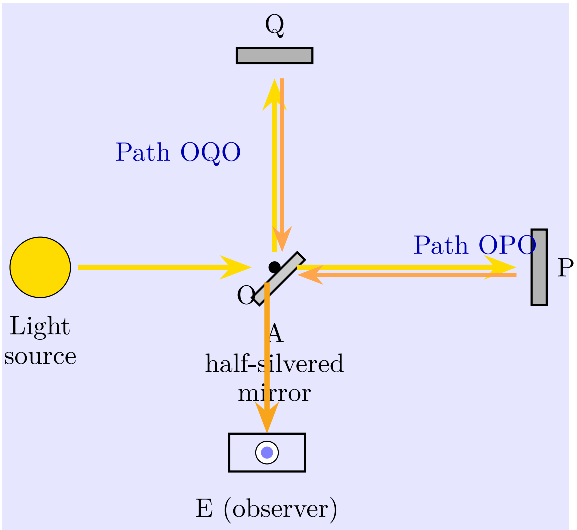

Figure 1.3: The Michelson–Morley interferometer.

The experiment was performed with an interferometer as shown in Figure 1.3. Light from the source was split into two beams at a half-silvered mirror A. One beam then took the path OPO and the other OQO and then they were superposed together and observed at E (eye). If the two paths were of different lengths, or took different times to travel, there would be interference fringes.

Result: No fringe shift was observed when the apparatus was rotated, indicating that the speed of light is the same in all directions—contrary to what would be expected if Earth were moving through an aether.

1.4 Einstein's Postulates

Einstein was lead to the postulates of Special Relativity to agree with the result of the Michelson-Morley experiment and by a thought experiment. He considered the consequence of the speed of light not being independent of reference frame. Say the speed of light in the observer's initial (stationary) frame was c_0, then the observer could go to a frame which was travelling at the same speed ie $v = c_0$. In the latter frame the electromagnetic wave would be a standing wave, in contradiction to the solutions of Maxwell's equations.

Einstein's resulting postulates are:

1. **Laws of physics are the same in all inertial frames** (if no medium selects a special frame).
2. **Speed of light is constant in all inertial frames**

One could regard (2) as an example of (1). Note that "laws of physics" may need modification, as we will see.