UNIVERSITY OF BIRMINGHAM

School of Physics and Astronomy

DEGREE OF B.Sc. and M.Sci. WITH HONOURS

THIRD-YEAR FINAL EXAMINATION

03 00498

LH QUANTUM MECHANICS 3

SUMMER EXAMINATIONS 2025

Time allowed: 1 hour 30 minutes

Answer ***two*** *questions. If you answer more than two questions, credit will only be given for the* best *two answers.*

The approximate allocation of marks to each part of a question is shown in brackets **[ ]**.

All symbols have their usual meanings.

A list of formulae that may be required can be found at the end of this question paper.

No calculator is permitted in this examination.

*Answer any **two** questions. If you answer more than **two** questions, credit will only be given for the **best** two answers.*

1. Consider a qubit with the energies $-\varepsilon$ in state $|0\rangle$ and ε in state $|1\rangle$.

(a) Write down its Hamiltonian in the Dirac and matrix notations. **[2]**

Two observables, $\mathcal{A}$ and $\mathcal{B}$, have the corresponding operators $\hat{A}$ and $\hat{B}$, which are given in the energy basis $\{|0\rangle\,,\,|1\rangle\}$ by

$$\hat{A} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|\,,\quad \hat{B} = |0\rangle\langle 0| + 2|1\rangle\langle 1| + i\sqrt{2}\Big(|0\rangle\langle 1| - |1\rangle\langle 0|\Big).$$

(b) Show that the eigenvalues of $\hat{A}$ are $A_{1,2} = \pm 1$ and find the corresponding normalised eigenvectors. **[4]**

(c) Find the eigenvalues and normalised eigenvectors of $\hat{B}$. **[6]**

(d) Can $\mathcal{A}$ and $\mathcal{B}$ be simultaneously measured? **[4]**

At a time $t_0 = 0$ the result of a measurement of $\mathcal{A}$ was $A = 1$.

(e) What are possible results and their probabilities of measuring $\mathcal{A}$ and $\mathcal{B}$ immediately after this measurement? **[8]**

(f) What is the probability of measuring $A = 1$ again in a time t after its first measurement? **[6]**

2. Consider the standard quantum harmonic oscillator (QHO) (See Formula Sheet).

(a) Using the canonical commutation relation between the dimensionless position and momentum operators, $\left[\hat{X}, \hat{P}\right] = i$, prove the commutation relation between the lowering and raising ladder operators:

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1.$$ **[4]**

(b) Consider the operator function $f\left(z; \hat{a}, \hat{a}^\dagger\right) \equiv \mathrm{e}^{-z\hat{a}^\dagger} \hat{a} \mathrm{e}^{z\hat{a}^\dagger}$ where z is a complex number. Prove that it obeys the following PDE,

$$\frac{\partial f\left(z; \hat{a}, \hat{a}^\dagger\right)}{\partial z} = 1,$$

and thus show, using the value of f at $z = 0$, that $f\left(z; \hat{a}, \hat{a}^\dagger\right) = z + \hat{a}$. **[8]**

The coherent state of the QHO is defined by $|z\rangle \equiv \mathrm{e}^{z\hat{a}^\dagger} |0\rangle$, where $|0\rangle$ is the QHO ground state.

(c) Show, using Taylor's expansion of $\mathrm{e}^{z\hat{a}^\dagger}$ and the properties of the raising operator $\hat{a}^\dagger$, that the coherent state is represented by the following linear superposition of the QHO eigenstates:

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle .$$ **[4]**

(d) Use the result of part (b) to show that the coherent state $|z\rangle$ is an eigenstate of the lowering operator $\hat{a}$ with the eigenvalue z, i.e. $\hat{a} |z\rangle = z |z\rangle$. Explain why from this follows that $\langle z| \hat{a}^\dagger = \langle z| z^*$. **[8]**

$\Big[$ *Hint: Act on* $|0\rangle$ *by* $\mathrm{e}^{-z\hat{a}^\dagger} \hat{a} \mathrm{e}^{z\hat{a}^\dagger} = z + \hat{a}$. $\Big]$

(e) Using the result of part (c), show that the scalar product of two coherent states is

$$\langle z_1 \,|\, z_2\rangle = \mathrm{e}^{z_1^* z_2},$$

and hence normalise $|z\rangle$. **[6]**

3. A particle of mass m is moving in the potential

$$V(x) = \begin{cases} \infty\,, & |x| > a \\ V_0 \sin \dfrac{\pi x}{2a}\,, & |x| \leqslant a \end{cases}.$$

At $V_0 = 0$, this is an infinitely deep potential well of width $2a$. Assuming that V_0 is sufficiently small, you shall consider the V_0-dependent part as a perturbation.

(a) Sketch this potential labelling all significant points. **[6]**

(b) The energy levels for a particle in this potential are quantised with energies E_k ($k = 1, 2, 3, \ldots$). The wavefunctions corresponding to the *unperturbed* k^{th} level with energy ε_k are

$$\psi_k(x) = \frac{1}{\sqrt{a}} \cos \frac{\pi k x}{2a} \text{ for odd } k\,, \quad \psi_k(x) = \frac{1}{\sqrt{a}} \sin \frac{\pi k x}{2a}, \text{ for even } k\,.$$

Calculate the unperturbed energies ε_k. **[6]**

(c) Formulate the condition under which the V_0-dependent part of $V(x)$ can be treated as a perturbation, V_{pert}, and write V_{pert} explicitly. **[4]**

(d) Explain why the first-order perturbative correction vanishes. **[3]**

(e) The second-order perturbative correction, $\varepsilon_1^{(2)}$, to the ground-state energy ε_1 is to be found using the following expression:

$$\varepsilon_1^{(2)} = \sum_{k=2}^{\infty} \frac{\left|\langle 1 \,|\, V_{\text{pert}} \,|\, k \rangle\right|^2}{\varepsilon_1 - \varepsilon_k}.$$

Calculate the matrix elements $\langle 1 \,|\, V_{\text{pert}} \,|\, k \rangle$ and use the result of part (b) to write explicitly the above series representing $\varepsilon_1^{(2)}$. **[8]**

[*Hint:*
- *You may find the relation* $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ *useful;*
- *Pay special attention to the* $k=2$ *term in the above sum.*]

(f) Show that if accuracy of 1% is sufficient, only the $k = 2$ term will contribute to the series in part (e) resulting in

$$\varepsilon_1^{(2)} = -\frac{8 m a^2 V_0^2}{3 \hbar^2 \pi^2}\,.$$ **[3]**

Formula Sheet

Resolution of unity (completeness)
$\hat{I} = \sum_n |\eta_n\rangle\langle\eta_n| = \int\limits_{-\infty}^{\infty} \mathrm{d}x |x\rangle\langle x|$

Orthonormality: $\langle \eta_n \,|\, \eta_m \rangle = \delta_{nm}$.

Hermitian conjugation: if $|\varphi\rangle = \hat{A}|\psi\rangle$, then $\langle\varphi| = \langle\psi|\hat{A}^\dagger$.

Ket and bra relations to the wavefunction:
$\langle x \,|\, \psi \rangle = \psi(x)\,; \quad \langle \psi \,|\, x \rangle = \psi^*(x)$.

Heisenberg uncertainty principle
$\Delta A \Delta B \geqslant \dfrac{1}{2} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right|$,
$\Delta A \equiv \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$.

Time evolution
$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$,
$\hat{U}(t) \equiv \mathrm{e}^{-i\hat{H}t/\hbar}$.

Evolution of operators in Heisenberg picture
$\hat{A}_{\mathrm{H}}(t) = \hat{U}^\dagger(t)\hat{A}(0)\hat{U}(t)$.

Ladder operators for QHO, $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$:

$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\ell} + \frac{i\ell}{\hbar}\hat{p}\right) \equiv \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right)$, $\ell \equiv \sqrt{\frac{\hbar}{m\omega}}$,

$\hat{X} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$, $\hat{P} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right)$.

Commutation relations for $\hat{a}$, $\hat{a}^\dagger$, and $\hat{N} \equiv \hat{a}^\dagger\hat{a}$:
$[\hat{a}, \hat{a}^\dagger] = 1\,, \quad [\hat{a}, \hat{N}] = \hat{a}\,, \quad [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger$.

Action on eigenstates of $\hat{N}$:
$\hat{a}|n\rangle = \sqrt{n}|n{-}1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n{+}1}|n{+}1\rangle$.

Angular momentum commutation relations
$[\hat{L}_x, \hat{L}_y] = i\hbar L_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar L_x$,
$[\hat{L}_z, \hat{L}_x] = i\hbar L_y; \quad [\hat{L}_k, \hat{L}^2] = 0,\ k = x, y, z$.

Orbital angular momentum eigenstates
$\hat{L}^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle$,
$\hat{L}_z|\ell, m\rangle = \hbar m|\ell, m\rangle$
$\ell = 0, 1, 2, \ldots;\ m = 0, \pm1, \ldots, \pm\ell$.

Ladder operators for the angular momentum:
$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y\,;$
$[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$, $[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm\,;$
$\hat{L}_\pm|\ell, m\rangle = \hbar c^\pm_{\ell,m}|\ell, m \pm 1\rangle$ where
$c^\pm_{\ell,m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}$.

The spin operators $\hat{S}_k$ have the same properties as $\hat{L}_k$ when replacing $\hat{L}_k$ by $\hat{S}_k$, ℓ by s, m by σ, and allowing s and σ to be either integer or half-integer.

For spin $\frac{1}{2}$, $\hat{S}_k = \frac{1}{2}\hbar\hat{\sigma}_k$ where $k = x, y$, or z and σ_k are the Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac notations for spin $\frac{1}{2}$:
$|\frac{1}{2}, \frac{1}{2}\rangle \equiv |+\rangle \equiv |\!\uparrow\rangle; \quad |\frac{1}{2}, -\frac{1}{2}\rangle \equiv |-\rangle \equiv |\!\downarrow\rangle$.

Density matrix $\hat{\rho} = \sum\limits_n p_n|\psi_n\rangle\langle\psi_n|$
where p_n is the classical probability of a system being in a pure state $|\psi_n\rangle$.

Von Neumann entropy
$S = -\operatorname{Tr}[\hat{\rho}\log_2\hat{\rho}]$.

Time-independent perturbation theory:
$(\hat{H}_0 + \hat{V})|\psi_n\rangle = E_n|\psi_n\rangle$, $\hat{H}_0|n\rangle = \varepsilon_n|n\rangle$;
$E_n \simeq \varepsilon_n + \varepsilon_n^{(1)} + \varepsilon_n^{(2)} + \cdots;$
$\varepsilon_n^{(1)} = \langle n|\hat{V}|n\rangle$, $\varepsilon_n^{(2)} = \sum\limits_{m \neq n} \dfrac{\left|\langle m|\,\hat{V}\,|n\rangle\right|^2}{\varepsilon_n - \varepsilon_m}$;

$|\psi_n\rangle \simeq |n\rangle + \sum\limits_{m \neq n} \dfrac{\langle m|\,\hat{V}\,|n\rangle}{\varepsilon_n - \varepsilon_m}|m\rangle$.

Time-dependent perturbation theory:
Transition probability $|i\rangle \to |f\rangle$
$P_{i\to f} = |d_{fi}(t)|^2$,
$d_{fi}(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^t \mathrm{d}t' \langle f|\hat{V}(t')|i\rangle \mathrm{e}^{it'\omega_{fi}}$,
$\omega_{fi} = \frac{E_f - E_i}{\hbar}$.

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so.

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- Coats/outwear should be placed in the designated area.
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- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
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