

1 Indices and Lorentz Transformations: Solutions

Problem 1.1 Practice Lorentz transformations

Note that (Pythagorean triples are useful!)

$$\beta = \frac{4}{5} \Rightarrow \gamma = \frac{1}{\sqrt{1 - \left(\frac{4}{5}\right)^2}} = \frac{5}{\sqrt{5^2 - 4^2}} = \frac{5}{3}.$$

Label the three events S_1 , S_2 and S_3 ,

(i) $ct_1 = 1$, $x_1 = 1$.

$$\begin{aligned} ct'_1 &= \gamma \left(1 - \frac{4}{5} \times 1 \right) = \frac{5}{3} \times \frac{1}{5} = \frac{1}{3} \\ x'_1 &= \gamma \left(1 - \frac{4}{5} \times 1 \right) = \frac{5}{3} \times \frac{1}{5} = \frac{1}{3}. \end{aligned}$$

(ii) $ct_2 = 0$, $x_2 = 1$.

$$\begin{aligned} ct'_2 &= \gamma \left(0 - \frac{4}{5} \times 1 \right) = -\frac{5}{3} \times \frac{4}{5} = -\frac{4}{3} \\ x'_2 &= \gamma (1 - 0) = \frac{5}{3} \times 1 = \frac{5}{3}. \end{aligned}$$

(iii) $ct_3 = -2$, $x_3 = 1$.

$$\begin{aligned} ct'_3 &= \gamma \left(-2 - \frac{4}{5} \times 1 \right) = -\frac{5}{3} \times \frac{14}{5} = -\frac{14}{3} \\ x'_3 &= \gamma \left(1 + \frac{4}{5} \times 2 \right) = \frac{5}{3} \times \frac{13}{5} = \frac{13}{3}. \end{aligned}$$

S_1 and S_3 are inside the light-cone of the origin, $ct = 0$, $x = 0$, S_2 is outside.

Problem 1.2 Classification of intervals

(i)

$$ct_1 = 5, \quad x_1 = 4 \quad \text{and} \quad ct_2 = 2, \quad x_2 = 1.$$

Then:

$$s_{12}^2 = -(5-2)^2 + (4-1)^2 = -9 + 9 = 0$$

so null or lightlike.

(ii)

$$ct_3 = -5, \quad x_3 = 3 \quad \text{and} \quad ct_4 = 3, \quad x_4 = 2.$$

Then:

$$s_{34}^2 = -(-5-3)^2 + (3-2)^2 = -4 + 1 = -3$$

so timelike.

♡ Problem 1.3 Causality

Consider the following two events, S_1 and S_2 in frame Σ be

$$ct_1 = 0, \quad x_1 = 0 \quad \text{and} \quad ct_2 > 0, \quad x_2 \neq 0.$$

Let the new frame, Σ' , be in the standard configuration with Σ with relative velocity v and associated β and γ .

(i) Then the transformed coordinates of S_1 and S_2 are:

$$\begin{aligned} ct'_1 &= 0 & x'_1 &= 0 \\ ct'_2 &= \gamma(ct_2 - \beta x_2) & x'_2 &= \gamma(x_2 - ct_2). \end{aligned}$$

Wish that $ct'_2 < 0$, so require $\beta x_2 > ct_2$, so for sufficiently large x_2 the order of the temporal coordinates are inverted.

(ii) To be inside the light cone of the origin, we inspect the invariant. We may use the original frame, so:

$$s_{12}^2 = -(ct_2)^2 + x_2^2$$

thus need $ct_2 > x_2$ to be inside the light cone. But from the first part we require $\beta x_2 > ct_2$, implying $\beta > 1$ to be inside the light cone, but then $v > c$, so impossible. Thus causality is preserved for events which may influence each other.

♡ Problem 1.4 Lorentz transformation of the wave equation

As usual

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct). \end{aligned}$$

Then using change of variables for partial derivatives, firstly for t :

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} \\
&= \gamma \frac{\partial}{\partial t'} - \gamma \beta c \frac{\partial}{\partial x'} \\
\Rightarrow \frac{1}{\tilde{c}^2} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\tilde{c}^2} \left(\gamma \frac{\partial}{\partial t'} - \gamma \beta c \frac{\partial}{\partial x'} \right)^2 u \\
&= \frac{\gamma^2}{\tilde{c}^2} \left(\frac{\partial^2}{\partial t'^2} + (\beta c)^2 \frac{\partial^2}{\partial x'^2} - 2\beta c \frac{\partial^2}{\partial x' \partial t'} \right) u
\end{aligned} \tag{1.1}$$

Secondly for x

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} \\
&= -\gamma \frac{\beta}{c} \frac{\partial}{\partial t'} + \gamma \frac{\partial}{\partial x'} \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= \gamma^2 \left(-\frac{\beta}{c} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \right)^2 u \\
&= \gamma^2 \left(\frac{\beta^2}{c^2} \frac{\partial^2}{\partial t'^2} + \frac{\partial^2}{\partial x'^2} - 2\frac{\beta}{c} \frac{\partial^2}{\partial x' \partial t'} \right) u
\end{aligned} \tag{1.2}$$

Now form the wave equation by subtracting Eq. (??) from Eq. (??):

$$\begin{aligned}
\frac{1}{\tilde{c}^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= \frac{\gamma^2}{\tilde{c}^2} \left(\frac{\partial^2}{\partial t'^2} + (\beta c)^2 \frac{\partial^2}{\partial x'^2} - 2\beta c \frac{\partial^2}{\partial x' \partial t'} \right) u + \\
&\quad - \gamma^2 \left(\frac{\beta^2}{c^2} \frac{\partial^2}{\partial t'^2} + \frac{\partial^2}{\partial x'^2} - 2\frac{\beta}{c} \frac{\partial^2}{\partial x' \partial t'} \right) u \\
&= \frac{\gamma^2}{\tilde{c}^2} \left[1 - \beta^2 \frac{\tilde{c}^2}{c^2} \right] \frac{\partial^2 u}{\partial t'^2} - \gamma^2 \left[1 - \beta^2 \frac{c^2}{\tilde{c}^2} \right] \frac{\partial^2 u}{\partial x'^2} + \\
&\quad + 2\gamma^2 \beta \left[\frac{1}{c} - \frac{c}{\tilde{c}^2} \right] \frac{\partial^2 u}{\partial x' \partial t'}
\end{aligned}$$

This is **not** covariant unless $\tilde{c} = c$, then the extraneous factor of the mixed partial derivative has a zero coefficient, and noting that $\gamma^2(1 - \beta^2) = 1$ we see

$$\frac{1}{\tilde{c}^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{\tilde{c}^2} \frac{\partial^2 u}{\partial t'^2} - \frac{\partial^2 u}{\partial x'^2},$$

i.e. covariance.

As an example, if the wave equation represented elastic waves in a solid, then

$$\tilde{c} = \sqrt{\frac{E(1 - \nu)}{\rho(1 + \nu)(1 - 2\nu)}}$$

where E is Young's modulus and $-1 \leq \nu \leq 1/2$ is Poisson's ratio. If, for simplicity, we assume $\nu = 0$, the value for cork (which is not an accident), Then $\tilde{c} = \sqrt{E/\rho}$.

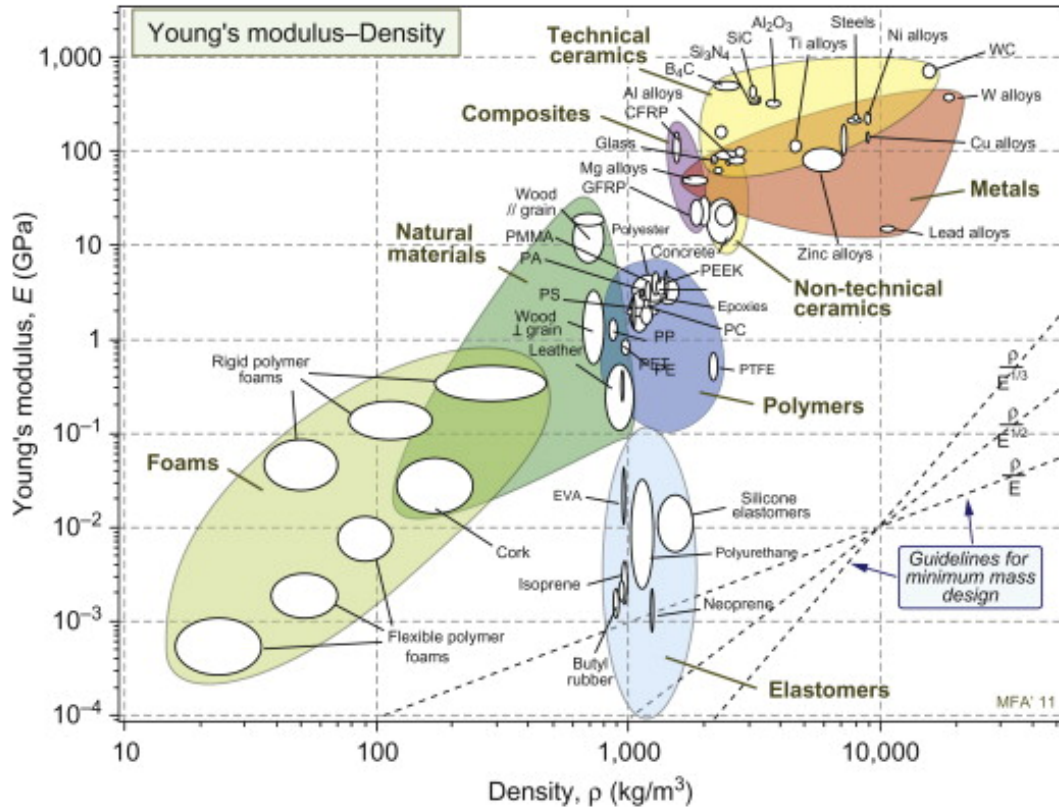


Figure 1.1: A plot (note log scales on both axes) of $E(\rho)$. Credit: *Materials and Sustainable Development*, Ashby, M.F. Second edition, Butterworth (2024).

It is perhaps interesting to ask how close \tilde{c} can be to c . The values of ρ and E for a variety of materials are shown in Fig. (??). Diamond is the material with the highest known speed of sound, with $E \simeq 1.2 \text{ TPa}$, $\nu = 0.2$ and $\rho \simeq 3.5 \times 10^3 \text{ kgm}^{-3}$, yielding $\tilde{c} \simeq 18,000 \text{ ms}^{-1}$ (*Properties of diamond*.) So $\tilde{c}/c \sim 10^{-4}$. In neutron stars the speed of sound is thought to approach $\tilde{c}/c = 1/\sqrt{3}$ (*Altiparmak, S. et al, ApJL 939 L34, (2022)*), reflecting their current position as having the greatest compressive strength to avoid collapse into a black hole.