1 Indices and Lorentz transformations

Problem 1.1 **Practice Lorentz transformations**
Consider the following events in frame Σ and transform them into frame Σ' (standard configuration) travelling at $v = (4/5)$c.

(i) $\mathrm{c}t_1 = 1$, $x_1 = 1$.

(ii) $\mathrm{c}t_2 = 0$, $x_2 = 1$.

(iii) $\mathrm{c}t_3 = -2$, $x_3 = 1$.

Check whether they are inside, outside or on the light-cone of the event $\mathrm{c}t_0 = 0$, $x_0 = 0$, is not changed by changing frames.

Problem 1.2 **Classification of intervals**
Consider the following pairs of events and classify whether the interval, s^2_{12}, between each pair is spacelike, timelike or null (lightlike).

(i)

$$\mathrm{c}t_1 = 5 \ , \ x_1 = 4 \quad \text{and} \quad \mathrm{c}t_2 = 2 \ , \ x_2 = 1 \ .$$

(ii)

$$\mathrm{c}t_1 = -5 \ , \ x_1 = 3 \quad \text{and} \quad \mathrm{c}t_2 = 3 \ , \ x_2 = 2 \ .$$

Problem 1.3 **Causality**
Consider the following two events, S_1 and S_2

$$\mathrm{c}t_1 = 0 \ , \ x_1 = 0 \quad \text{and} \quad \mathrm{c}t_2 > 0 \ , \ x_2 \neq 0 \ .$$

(i) Consider viewing these events in a different inertial frame, moving at v. By considering the transformed coordinates, $\mathrm{c}t'_1$, x'_1 etc determine if there are frames such that $\mathrm{c}t'_2 <$

ct'_1, i.e. the apparent time order is reversed.

(ii) Is S_2 within the light cone of S_1 if the time order is reversed?

♡*Problem 1.4* **Lorentz transformation of the wave equation**
Perform the counterpart to the Galilean transformation of the wave equation with speed of the waves being $\tilde{c}$,

$$\frac{1}{\tilde{c}^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \ ,$$

performed in lectures, using Lorentz transformations. Examine the cases $c > \tilde{c}$ and $c = \tilde{c}$. Under what conditions is the wave equation covariant?

Problem 1.5 **Kronecker delta as a vector**
Choose a coordinate system such that the axes 1, 2 and 3 correspond to x, y and z. Write down the components of the following entities: δ_{i1}, δ_{1i}, δ_{i2} and δ_{i3}. What is their relation to the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$? Evaluate, using summation convention, $\delta_{1i}\delta_{2i}$ and $\delta_{i2}v_i$ where v_i are the components of a vector $\mathbf{v}$. Comment on the results in the light of the answers to the previous parts of the question.

Problem 1.6 **Relation of familiar matrix algebra to summation convention**

(i) Consider two 2×2 matrices, M ad M with

$$M_{ij} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad N_{ij} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

Calculate the product matrix $(\mathsf{M} \cdot \mathsf{N})_{ij}$ by ordinary matrix algebra and calculating $M_{ik}N_{kj}$ using summation convention to see they agree.

(ii) Consider the transformation of a matrix, M_{ij}, under rotation of the coordinate system via a rotation matrix R, in an abuse of notation,

$$\mathsf{M}' = \mathsf{R} \cdot \mathsf{M} \cdot \mathsf{R}^{\mathrm{T}} \ .$$

Show this is equivalent to

$$M'_{ij} = R_{ik}R_{jl}M_{kl} \ .$$

(iii) Consider the quantities

$$v_i M_{ij} \quad \text{and} \quad M_{ij} v_j \ .$$

what do they correspond to in matrix algebra?

Problem 1.7 **Projection matrices 1**

Let $\hat{\mathbf{n}}$ be a unit vector with components $\hat{n}_i$. Consider the matrix:

$$P^{(n)}_{ij} = \hat{n}_i \hat{n}_j$$

Determine its action on a vector whose components are v_j. What is the interpretation of the resulting vector? Note that this may be represented as

$$\mathbf{v}^{\parallel} = \mathbf{P}^{(n)} \cdot \mathbf{v}$$

where

$$\mathbf{P}^{(n)} = \hat{\mathbf{n}}\,\hat{\mathbf{n}}$$

where there is no scalar product between the two unit vectors. This is called an *outer product* of the vectors. Now consider the two matrices:

$$P^{(n)\perp}_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j \tag{1.1}$$

$$P^{(n)-}_{ij} = \delta_{ij} - 2\hat{n}_i \hat{n}_j \ . \tag{1.2}$$

Determine their actions on v_j and interpret the results.

Problem 1.8 **Projection matrices 2: retroreflectors**

Consider three mutually orthogonal vectors $\{\hat{\boldsymbol{n}}^a\}$, where $a = 1$, 2 or 3. Let $P^a_{ij} = \hat{n}^a_i\, \hat{n}^a_j$ be the projection matrix for $\hat{\boldsymbol{n}}^a$.

(i) What is the effect of the matrix

$$P^{\text{tot}}_{ij} = P^1_{ij} + P^2_{ij} + P^3_{ij}$$

on a vector, v_j?

(ii) What is the product of two Ps equal to, e.g. a square, $P^1_{ij} P^1_{jk}$, or two different P's, $P^1_{ij} P^2_{jk}$?

(iii) Define the counterpart to Eq. (1.1) for each of $a = 1$, 2 and 3, $P^{a\perp}_{ij}$. Determine the product $P^{1\perp}_{ij} P^{2\perp}_{jk} P^{3\perp}_{kl}$.

(iv) Perform the same for the counterparts of $P_{ij}^{(n)-}$ from Eq. (1.2).

(v) Can you see the connection with retroreflectors (objects which reflect light coming from any directionback in the same direction)?

Problem 1.9 **Complex and non-orthogonal vectors**
Consider complex vectors, $\mathbf{u} = u^i \mathbf{e}_i$, with the basis vectors real ($\mathrm{Im}\,\mathbf{e}_i = 0$), but the coefficients may be complex (i.e. $u^{i*} \neq u^i$, in general). Define the scalar product is $\mathbf{u}^* \cdot \mathbf{v}$. Using the real orthogonal basis $\{\hat{\mathbf{e}}_i\}$, $i = 1,\ 2,\ 3$, deduce whether the set $\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm \mathrm{i}\hat{\mathbf{e}}_2)$ and $\hat{\mathbf{e}}_3$ form an orthogonal set. If one now has the *non*-orthogonal set $\{\mathbf{e}_i\}$, $i = 1,\ 2,\ 3$, construct a set corresponding to the complex set above, analyse the orthogonality properties and construct the dual set. Consider the quantity:

$$\mathbf{E} = \mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^3\mathbf{e}_3 = \mathbf{e}^i\mathbf{e}_i$$

(**Note** there is no scalar product in this expression.) Determine the action of this quantity on a vector, $\mathbf{v}$, i.e. $\mathbf{E}\cdot\mathbf{v}$. You may find it useful to express the vector in terms of its covariant components. What is the answer if we replace $\mathbf{E}$ by $\mathbf{E}'$ where

$$\mathbf{E}' = \mathbf{e}_1\mathbf{e}^1 + \mathbf{e}_2\mathbf{e}^2 + \mathbf{e}_3\mathbf{e}^3 = \mathbf{e}_i\mathbf{e}^i$$

Problem 1.10 **Some practice with LT's**
A 4-vector G^μ has components $(1, 2, 3, 4)$ in an inertial frame Σ in Minkowski space-time. What are the components of G_μ? Find $G^\mu G_\mu$. Find G'^μ in Σ' that moves at speed $0.75\,c$ along x axis of Σ. Verify directly that $G^\mu G_\mu = G'^\mu G'_\mu$.

Problem 1.11 **Products of Lorentz transformations**
The Lorentz transformation between an inertial frame Σ' moving with a velocity v_1 (and associated rapidity ϕ_1) with respect to a laboratory frame Σ is $x'^\mu = \overline{\Lambda}^\mu{}_\nu(v_1)x^\nu$ where

$$\overline{\Lambda}^\mu{}_\nu(v) = \begin{pmatrix} \gamma(v) & -\beta(v)\gamma(v) & 0 & 0 \\ -\beta(v)\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \overline{\Lambda}^\mu{}_\nu(\phi)$$

Another inertial frame Σ'' has a velocity v_2 (and rapidity ϕ_2) with respect to Σ'. All the frames are in standard configuration (the x axes are parallel and the origins coincide at $t = 0$). Show by direct matrix multiplication that coordinates x''^μ in Σ'' and x^μ in Σ are related by the single Lorentz transformation

$$x''^\mu = \overline{\Lambda}^\mu{}_\nu(\Phi)x^\nu$$

with rapidity $\Phi = \phi_1 + \phi_2$. Show that the associated velocity, V, obeys $V = (v_1 + v_2)/(1 + v_1 v_2/c^2)$. Comment on whether rapidity or velocity is the more useful variable.

Problem 1.12 **Combination of non-collinear LT's**

Can you find an explicit expression for the matrix of the Lorentz transformation $\Lambda^\mu{}_\nu$, if in the problem as above Σ'' is moving with a velocity v_2 along y' axis in Σ'. Does it matter if the order of the Lorentz transformations is reversed?