 Y3 Radiation and Relativity 2026

2 Dynamics and Levi-Civita

Problem 2.1 **Events connected?**
Some event $\mathscr{P}$ has coordinates $x^\mu = (5, 4, 3, 2)$ in an inertial frame Σ. Could this event occur with a particle whose world line passes through the origin of Σ?

Problem 2.2 **4-velocity and rapidity**
Work in two-dimensional space-time.

(i) Re-express the 4-velocity, u^μ ($\mu = 0,\ 1$) in terms of the rapidity. ϕ, of the particle.

(ii) Write down the effects of a Lorentz transformation of rapidity ψ on the components of u^μ.

(iii) Determine an expression for the 4-acceleration in terms of the rapidity and its proper time derivative.

Problem 2.3 **Rectilinear 4-Acceleration**
4-acceleration α^μ is defined by

$$\alpha^\mu \equiv \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}$$

where u^μ is 4-velocity. Show that, in general, the acceleration invariant is:

$$\alpha^\mu \alpha_\mu = \gamma^6(v)\dot{v}^2 \ .$$

Is the form of this answer disconcerting? Consider the answer to problem (1.9).

Problem 2.4 **Relativistic fly-paper**
A body moves with relativistic velocity v through a gas containing n_0 *slowly-moving* particles per unit volume, each having rest mass m_0. Assume that collisions of the particles and the body are *completely* inelastic (i.e. the particles stick to the body).

(i) In the rest frame of the body, find the 4-force f^μ exerted by the gas on the surface of the body which is perpendicular to its velocity, with area A. [Hint consider f^0 and f^1 separately by examining the change $\mathrm{d}p^1$ and $\mathrm{d}p^0$ of the relativistic body, due to the accretion of gas particles, in time $\mathrm{d}\tau$ (in the rest frame of the body, time is proper time).]

(ii) What is the expression in the rest frame of the gas?

Problem 2.5 **Two particle decay**
A particle of rest mass M decays into two particles of rest masses m_1 and m_2 moving with velocities v_1 and v_2 in the rest frame Σ of the original particle. Prove that

$$m_1 = \frac{M}{\gamma(v_1)(1 - v_1/v_2)}$$

Problem 2.6 **Practise with summation**
Simplify using summation convention:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \qquad \text{and} \qquad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$$

Turn to differential identities. Simplify (where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{u}$ are vector fields):

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \qquad \text{and} \qquad \mathbf{u} \times (\nabla \times \mathbf{u})$$

(The latter is useful in fluid dynamics.)

Problem 2.7 **Scalar Triple product**
Show that the scalar triple product of three vectors, $\mathbf{a} = a_i\hat{\mathbf{e}}_i$, $\mathbf{b} = b_i\hat{\mathbf{e}}_i$ and $\mathbf{c} = c_i\hat{\mathbf{e}}_i$ obeys:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk}\, a_i b_j c_k$$

Problem 2.8 **Inertia tensor**
Let there be N point masses, $m^{(n)}$, $n = 1, 2, \cdots, N$ in a rigid body with positions $\mathbf{r}^{(n)}$. The rigid body is rotating steadily such that $\dot{\mathbf{r}}^{(n)} = \boldsymbol{\omega} \times \mathbf{r}^{(n)}$. The total angular momentum is

$$\mathbf{L} = \sum_{n=1}^{N} m^{(n)} \mathbf{r}^{(n)} \times \dot{\mathbf{r}}^{(n)} \ .$$

(i) Show that

$$L_i = \mathcal{I}_{i\ell}\, \omega_\ell \ ,$$

where the *inertia tensor*, $\mathcal{I}_{ij}$, is

$$\mathcal{I}_{ij} = \sum_{n=1}^{N} m^{(n)} \left(r_k^{(n)} r_k^{(n)} \delta_{ij} - r_i^{(n)} r_j^{(n)} \right) \ .$$

(ii) Determine the expression for $\mathcal{I}_{ij}$ for a continuous medium with a mass density $\rho(\mathbf{r})$.

Problem 2.9 **Non-rectilinear acceleration**
The 4-acceleration is

$$\alpha^\mu = \mathrm{c}\gamma(\dot{\gamma}, \dot{\gamma}\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}) \ .$$

Show that the acceleration invariant for acceleration in general is:

$$\alpha^\mu \alpha_\mu = \mathrm{c}^2 \left[\gamma^6 \left(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} \right)^2 + \gamma^4 \dot{\beta}^2 \right] = \mathrm{c}^2 \gamma^6 \left(\dot{\beta}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2 \right) \quad \text{with} \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{\mathrm{c}} \, , \quad \dot{\mathbf{v}} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

Problem 2.10 **Expression for determinants**
Prove the following result for the determinant, $|a|$, of a 3×3 matrix, a, by explicit evaluation of the right-hand side.

$$|a| = \epsilon_{ijk}\, a_{1i} a_{2j} a_{3k}$$

Deduce the identity

$$\epsilon_{rst} |a| = \epsilon_{ijk}\, a_{ri} a_{sj} a_{tk}$$

by examining $rst = 123$ and then using the properties of ϵ_{ijk} to demonstrate the other components.

Problem 2.11 **Products of determinants and matrix inverses**
Show that the determinant of a product of matrices a and b, $c = ab$, is the product of the determinants:

$$|c| = |ab| = |a|\,|b|$$

Show that the inverse of the matrix a_{ij} may be expressed:

$$a_{ij}^{-1} = \frac{1}{2|a|}\epsilon_{ilm}\epsilon_{jnp}a_{nl}a_{pm}$$

Problem 2.12 **Determinantal representation of Levi-Civita**
Show that the following representations of ϵ_{ijk} are faithful:

$$\begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} = \epsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix}$$

Problem 2.13 **Derivation of the most useful identity**
Show using the results of Prob. (2.10) that

$$\epsilon_{ijk}\,\epsilon_{rst} = \begin{vmatrix} \delta_{ir} & \delta_{is} & \delta_{it} \\ \delta_{jr} & \delta_{js} & \delta_{jt} \\ \delta_{kr} & \delta_{ks} & \delta_{kt} \end{vmatrix}$$

Then derive

$$\epsilon_{rjk}\epsilon_{rst} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$$

by contracting on $r = i$ and summing.