Evolution of Cosmic Structure

Lecture 8:
Non-linear growth and the spherical collapse model

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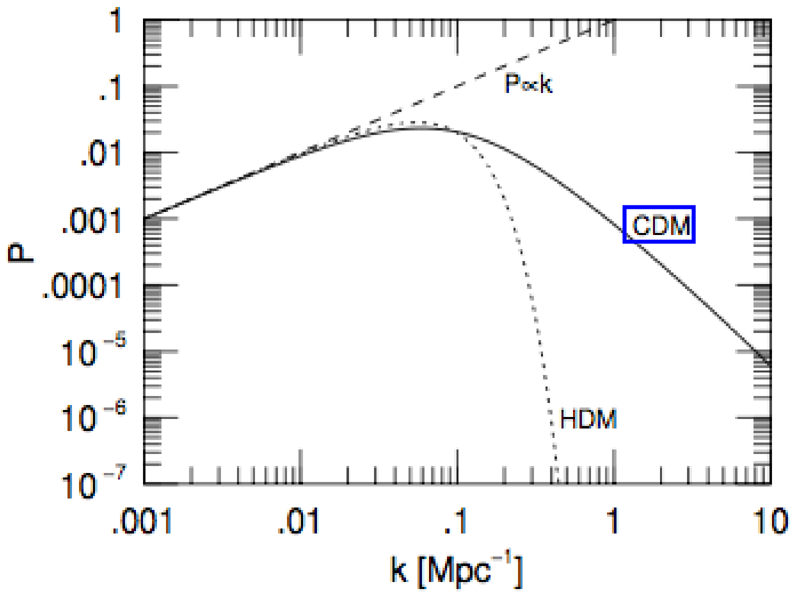
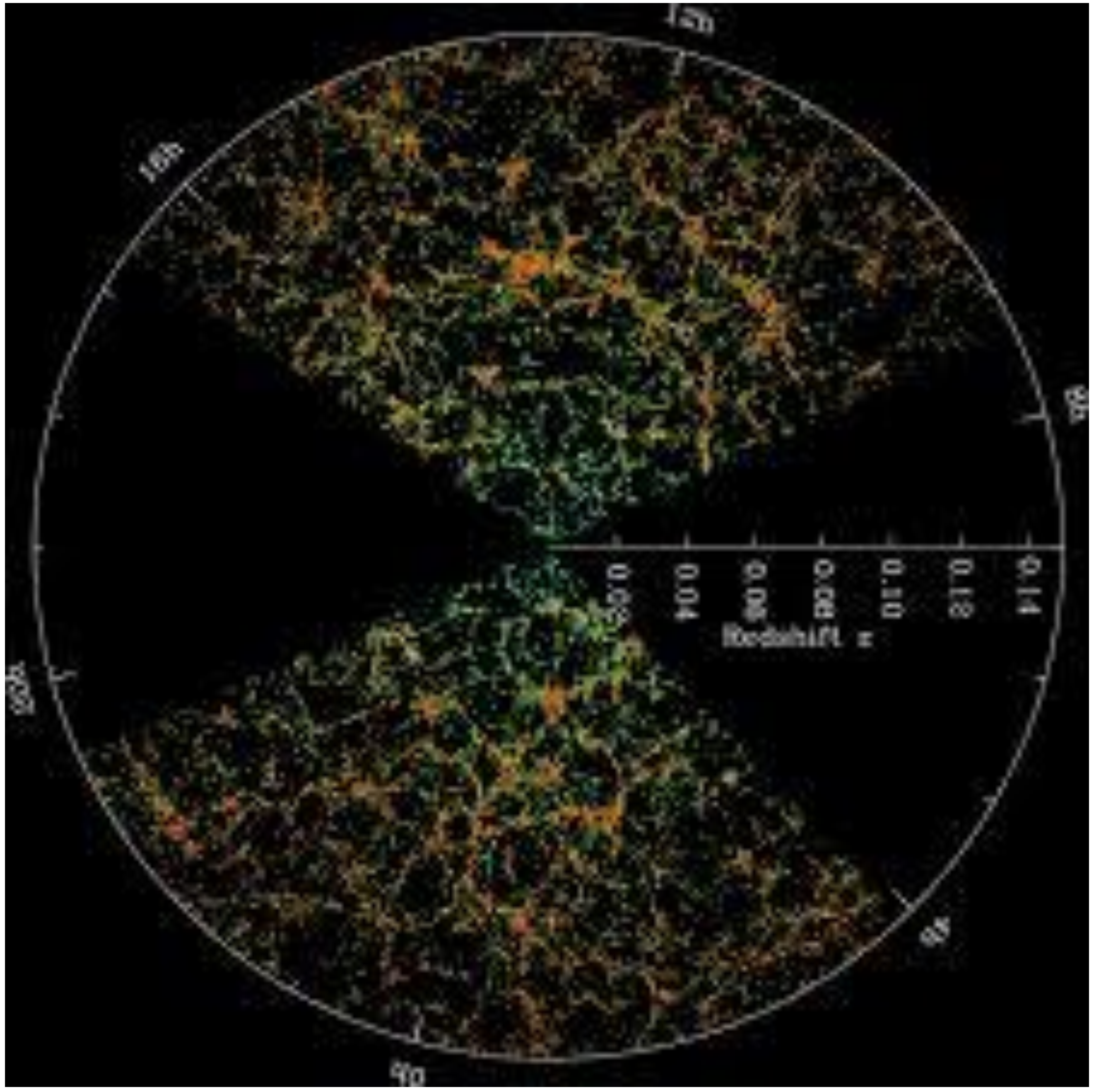
Evolution of the power spectrum



1. Perturbations in a fluid with significant pressure cannot collapse if they are inside the horizon, unless they have $L>\lambda_J$.
2. Even for a pressureless material, like dark matter, we have seen that growth of modes inside the horizon is much slower ($\delta \propto \log t$).
3. If the dark matter particles have relativistic velocities, then they will be able to stream freely on scales up to the horizon scale, and will therefore smooth out any perturbations on smaller scales.

Recap of Galaxy Clustering



Recap of Galaxy Clustering

1.Traditionally, cosmological constraints from galaxy clustering have been quoted in terms of constraints on the amplitude of mass fluctuations σ(z, R)

$$\sigma^2(R) = \int \frac{d^3k}{(2\pi)^3} P_{\rm lin}(k) \left|W(kR)\right|^2$$

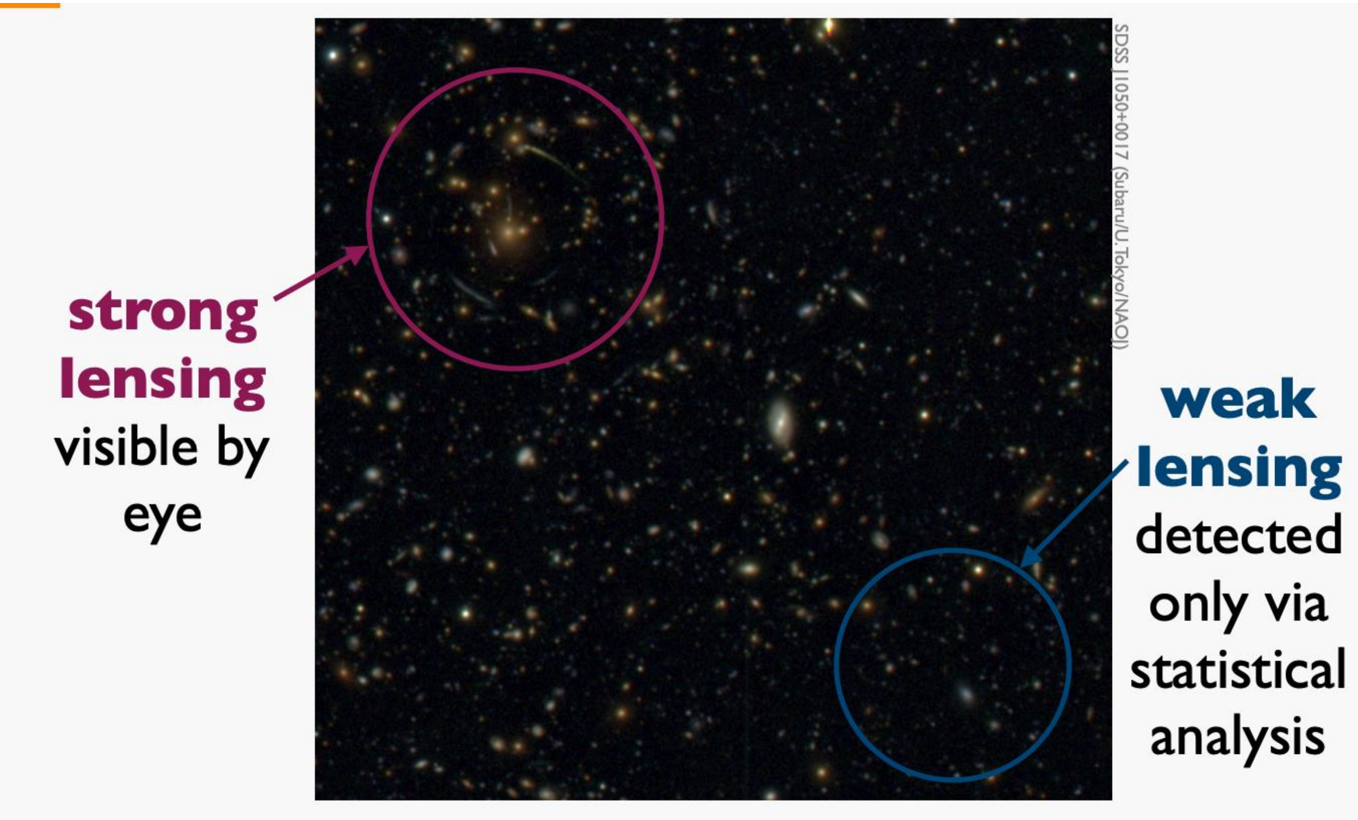
Where Δ^2 is the dimensionless power spectrum, W is the window function and k is the wavenumber.

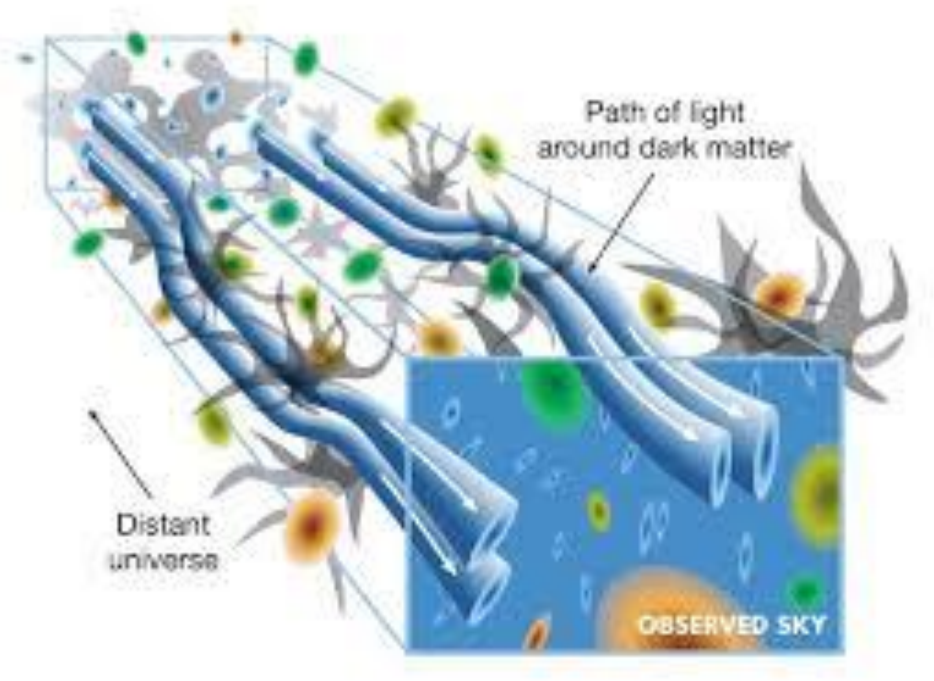
Conventionally, σ is measured at z = 0 and R = 8 h-1Mpc, hence, the parameter of interest is σ8.

- Galaxy clustering traces the galaxy overdensity and not directly the matter density. The two are related as δm = bg δ_g. The relationship between the galaxy power spectrum
and the matter power spectrum is consequently

$$P_{gg}(k,z) = b^2(k,z)P(k,z),$$

Gravitational Lensing and cosmology

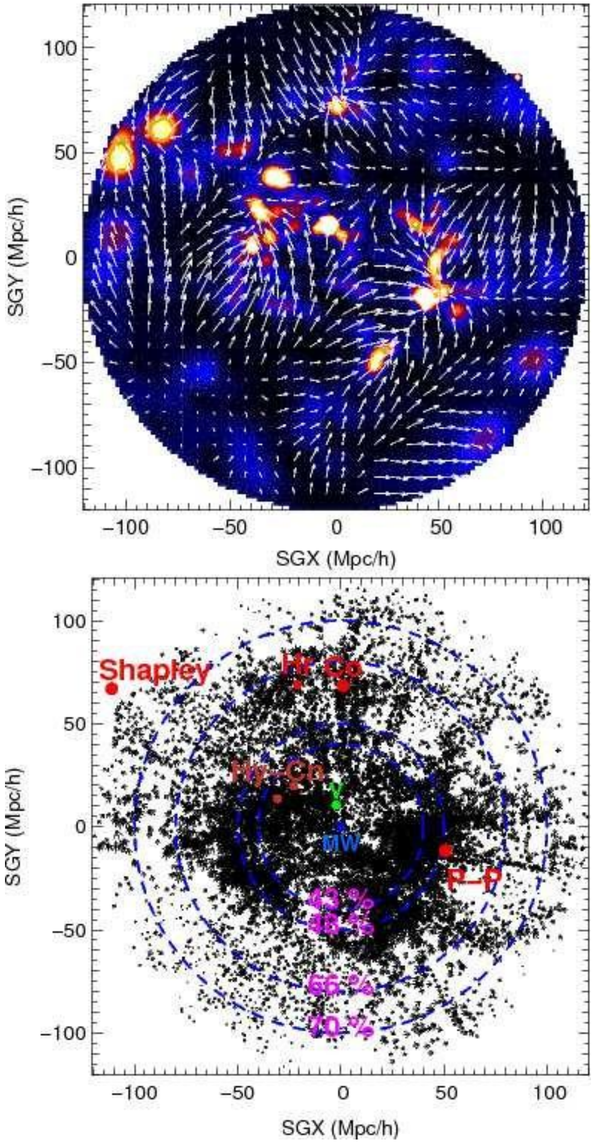




Lensing occurs in different regimes. Here strong lensing is a rare phenomenon, seen as the arcs of the lensed background source. Weak Lensing is an imprint of the dark matter on the shapes of galaxies

What are peculiar velocities?

Deviations from the Hubble flow due to the gravitational attraction of the matter density field



From: Guillhelm Lavaux

Relating Velocity and Matter Power Spectra

The matter power spectrum is defined by

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k}')\rangle = (2\pi)^3 \delta_D(\mathbf{k}-\mathbf{k}')P_{\delta\delta}(k)$$

The velocity power spectrum is defined analogously:

$$\langle v_i(\mathbf{k})v_j^*(\mathbf{k}')\rangle = (2\pi)^3 \delta_D(\mathbf{k}-\mathbf{k}')P_{v_i v_j}(k)$$

Substituting the velocity–density relation:

$$v_i(\mathbf{k}) = iHf\frac{k_i}{k^2}\delta(\mathbf{k})$$

we obtain

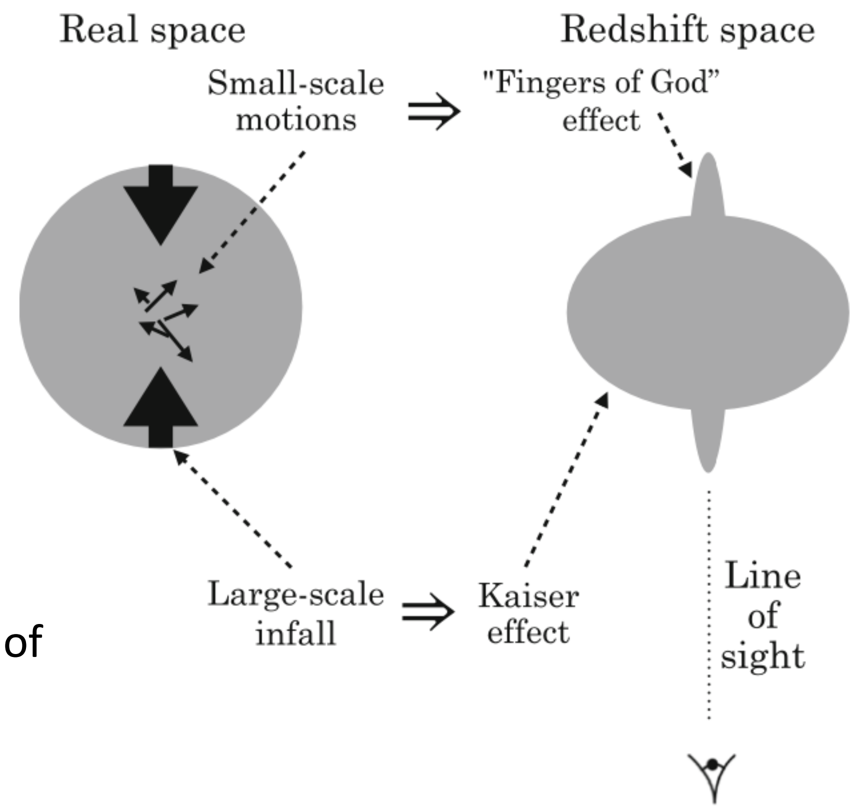
$$P_{v_i v_j}(k) = (Hf)^2 \frac{k_i k_j}{k^4} P_{\delta\delta}(k)$$

$$P_{vv}(k) \equiv \langle |\mathbf{v}(\mathbf{k})|^2\rangle = (Hf)^2 \frac{1}{k^2} P_{\delta\delta}(k)$$

$$\boxed{P_{vv}(k) = (Hf)^2 \frac{P_{\delta\delta}(k)}{k^2}}$$

Redshift space distortions

A complementary approach to measure peculiar velocities and hence, fσ8 is redshift space distortions. This is a more "statistical" approach compared to using individual distances, e.g. with Type Ia supernovae



There are two principal effects:

(1) On large scales, velocity flows into large overdensities, "squishing" the appearance of the object along the line of sight; this is the Kaiser effect;

(2) On smaller scales, random motions introduce apparent elongation along the line of sight; this is the somewhat hyperbolically called "fingers of god" effect.

Non-linear growth of structure

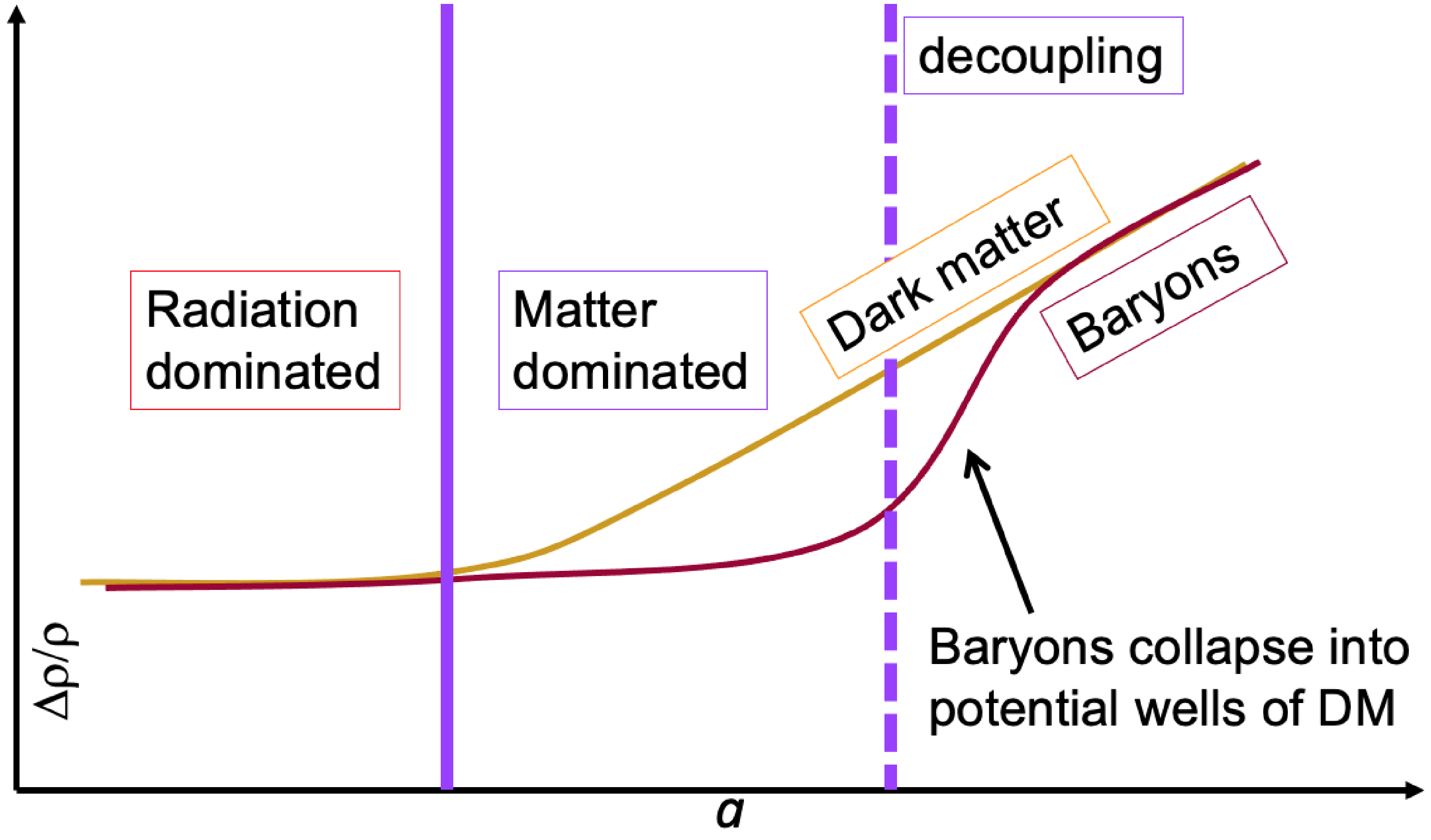
How does the structure develop once the fractional density perturbations become large, and subject to gravitational collapse?

How do the baryons respond to the collapse of dark halos?

- Non-linear evolution and collapse
- The spherical collapse model
- Non-spherical collapse
- The halo mass function
- Hierarchical merging
- Baryon cooling and galaxy formation

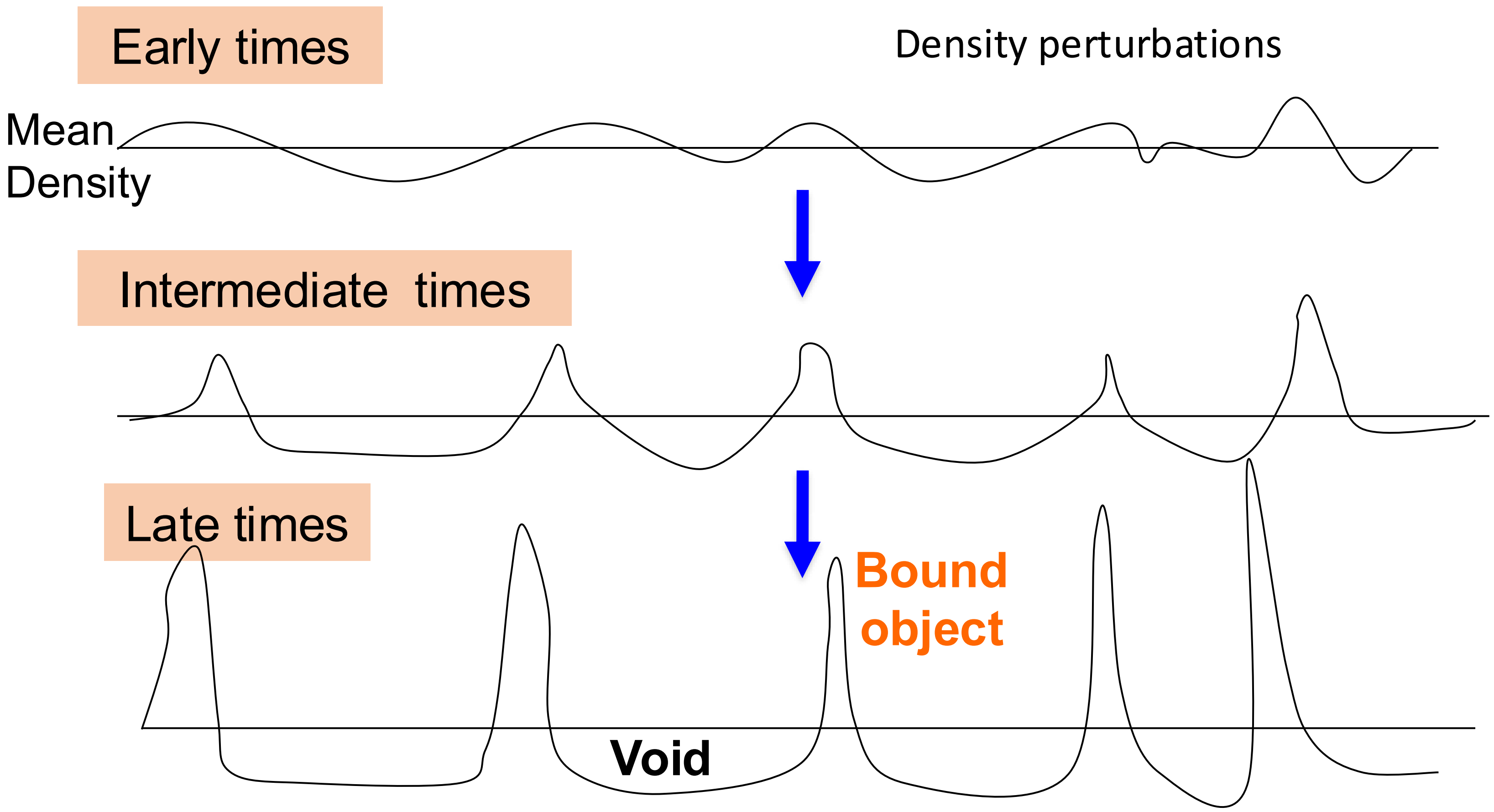
Recall – linear growth

Baryons fall into growing dark matter structures after decoupling.



Non-linear evolution and collapse

In the matter dominated era, if $\Omega_m \approx 1$, then perturbations on all wavelengths grow according to $\delta \propto a$, until $\delta \sim 1$. Once the local density in a perturbation exceeds the critical density of the Universe, its expansion will slow down and stall, and it can collapse.

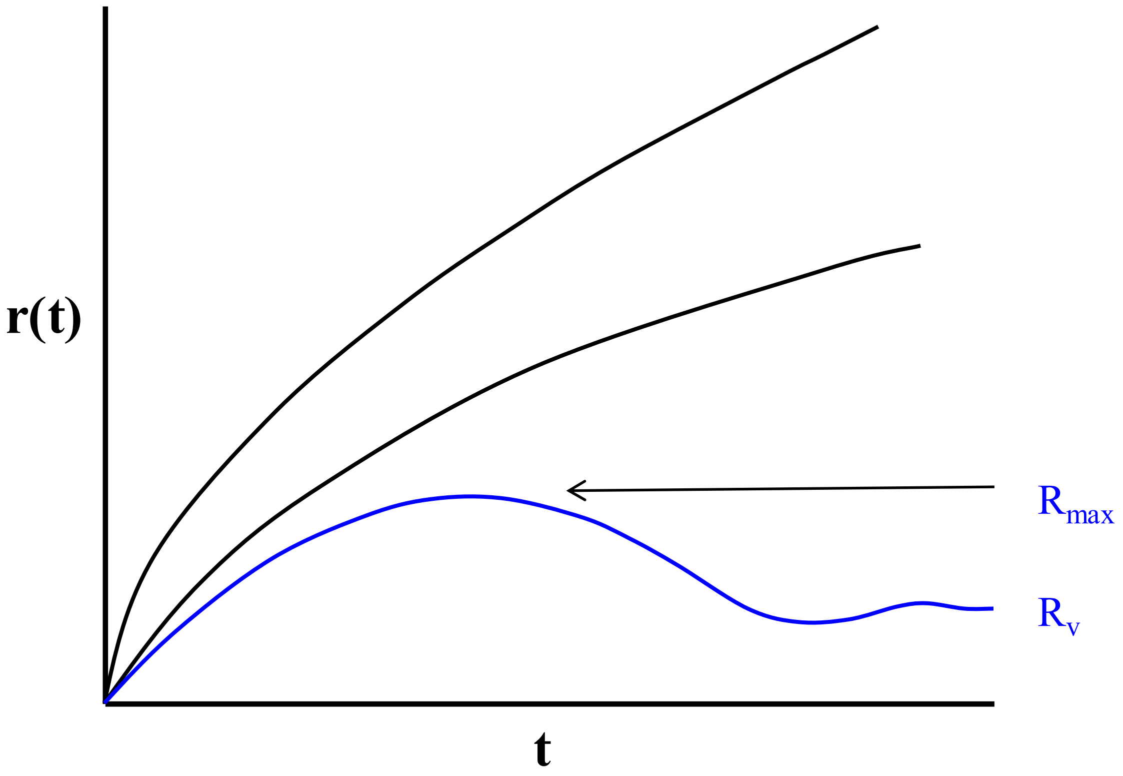


Non-linear evolution and collapse

Within a single overdense perturbation the densest part will separate out from the Hubble flow and then collapse and virialise.

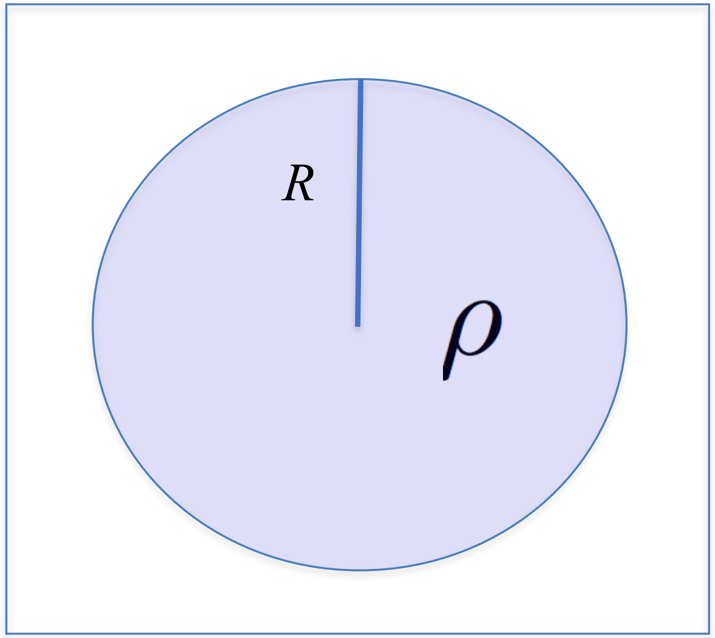
In the simple case of a density perturbation with spherical symmetry we can use Birkhoff's Theorem (which says that the gravitational force due to a spherical shell is zero inside it in GR, as it is in the case of Newtonian gravity) to treat the spherical overdensity as effectively a "mini-universe" of its own.

Evolution of radii of concentric shells near a spherical overdensity.



The spherical top-hat model

Take a simple model of a sphere of a uniform density of mass M and radius R



The total energy of a shell of mass m at the outer edge of the sphere is

$$E = V + T = -\frac{GMm}{R} + \frac{1}{2}m\dot{R}^2.$$

Dividing through by m,

$$\mathcal{E} \equiv E/m = -\frac{GM}{R} + \frac{1}{2}\dot{R}^2,$$

and so

$$\left(\frac{dR}{dt}\right)^2 = \frac{2GM}{R} + 2\mathcal{E}\ ,$$

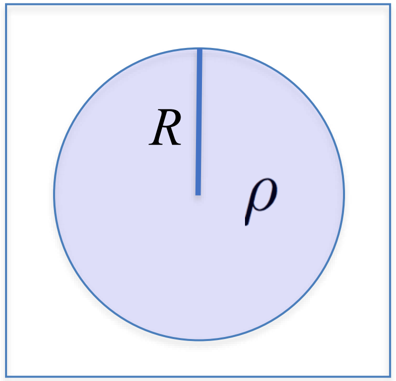
where the total specific energy of the shell, $\mathcal{E}$, will be negative for a bound shell, and will be conserved as the balance between kinetic and potential energy shifts.

The spherical top-hat model

If the spherical overdensity has uniform density – known as a "top-hat" overdensity – then its evolution is governed by the equation:

$$\left(\frac{dR}{dt}\right)^2 = \frac{2GM}{R} + 2\mathcal{E} \ ,$$

where $\mathcal{E}$ is the total specific energy of material at the outer edge of the sphere, and M is the mass within it.



This is more easily solved by introducing the *conformal time* η, related to universal time, *t*, by $dt = \alpha R d\eta$.

We introduce the quantities

$$\alpha = \left(-\frac{1}{2\mathcal{E}}\right)^{1/2} \qquad R_* = -\frac{GM}{2\mathcal{E}}$$

which depend on the mass and specific energy. R_* is a characteristic size scale for the system, related to the maximum radius it can reach, whilst *1/α* is a characteristic escape velocity.

The spherical top-hat model

$$\left(\frac{dR}{dt}\right)^2 = \frac{2GM}{R} + 2\mathcal{E} \ ,$$

We now change variables, so as to convert the last equation into the form $(dy/d\eta)^2 = 2y - y^2$, which can be solved. To achieve this, we transform the variables from R and t to y and η, where $R = R_* y$ and $dt = \alpha R d\eta$.

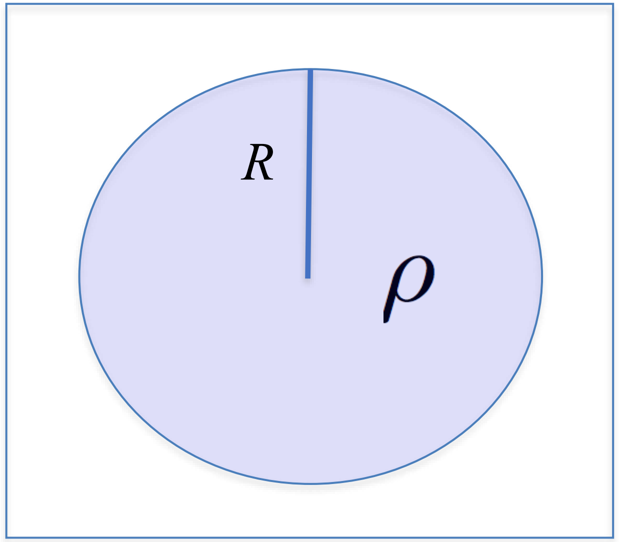
$$\left(\frac{dy}{d\eta}\right)^2 = \frac{2GM\alpha^2}{R_*} y + 2\mathcal{E}\alpha^2 y^2 \ ,$$

and this reduces to the desired form, $(dy/d\eta)^2 = 2y - y^2$, if we choose α and R_* to be:

$$\alpha = \left(-\frac{1}{2\mathcal{E}}\right)^{1/2} \qquad R_* = -\frac{GM}{2\mathcal{E}} \ .$$

The solution is $y = 1 - \cos\eta$, or in terms of R,

$$R = R_*(1 - \cos\eta) = -\frac{GM}{2\mathcal{E}}(1 - \cos\eta) \ .$$

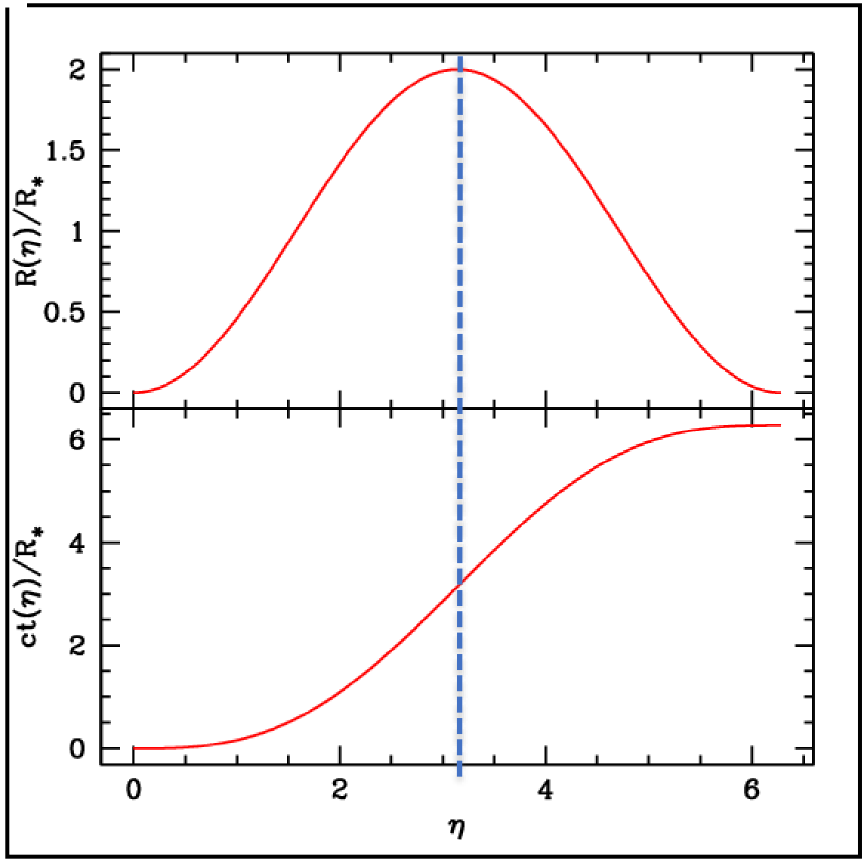


The spherical top-hat model

The equation for the evolution of R can then be cast into the form

$$R = R_*(1 - \cos\eta) = -\frac{GM}{2\mathcal{E}}(1 - \cos\eta) \; .$$

So the radius reaches a maximum of R=2R_* when η =π, i.e. at time t_{max}=παR_*, and then collapses.



We need to relate this behaviour back to the initial linear perturbations. To do this, we note that at early times η≈0, and so $\cos\eta \approx 1 - \eta^2/2! + \eta^4/4!$ and $\sin\eta \approx \eta - \eta^3/3! + \eta^5/5!$

The spherical top-hat model

Substituting into $R = R_*(1 - \cos\eta)$ and $t = \alpha R_*(\eta - \sin\eta)$, we obtain, at early times

$$R(\eta) \approx R_* \frac{\eta^2}{2}(1 - \eta^2/12) \ , \qquad t(\eta) \approx \alpha R_* \frac{\eta^3}{6}(1 - \eta^2/20) \ .$$

It follows that R can be expressed as a function of t thus:

$$R(t) \approx \frac{R_*}{2}\left(\frac{6t}{\alpha R_*}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{\alpha R_*}\right)^{2/3}\right] \ .$$

Hence $R \propto t^{2/3}$ at the outset – the well-known evolution for an Ω_m=1 universe.

The higher order term shows us how the overdensity evolves as the radius lags the Hubble expansion, i.e.

$$\delta \equiv \frac{\delta\rho}{\rho} = -3\frac{\delta R}{R} \approx \frac{3}{20}\left(\frac{6t}{\alpha R_*}\right)^{2/3} .$$

This is the linear regime behaviour whereby δ increases in proportion to *a*.

The spherical top-hat model

We can use now calculate the overdensity of the perturbations at different times throughout its evolution. We can compare to the $R(\mathrm{t})$ equation to understand the radius of the system in the absence of any perturbation.

$$R_{un} = \frac{R_*}{2}\left(\frac{6t}{\alpha R_*}\right)^{2/3}$$

As we have seen, the turnaround of the perturbation occurs at time $t_{max} = \pi\alpha R_*$ when the perturbation has radius $2R_*$.

So, we can calculate the overdensity at that time (assuming mass conservation):

$$\frac{\rho}{\bar{\rho}} = 1 + \delta = \frac{M_{turn}}{M_{un}}\left(\frac{R_{un}}{2\,R_*}\right)^3 = \left(\frac{R_{un}}{2\,R_*}\right)^3 = \left(\frac{\frac{R_*}{2}\left(\frac{6t_{max}}{\alpha R_*}\right)^{2/3}}{2\,R_*}\right)^3 = \frac{9\pi^2}{16} \approx 5.5$$

If we assume that the universe is flat, matter-only (e.g., $\Omega_m = 1$) then this is 5.5 times the critical density.

The spherical top-hat model

What happens at the final stage? The analytical top hat model gives system collapses to a point at $t_{coll} = 2\pi\alpha R_*$. In practice, however, we expect the dark matter particles to stream through the centre and *virialise*. The virial theorem requires a stable self-gravitating system to have a balance between gravitational potential energy (*V*) and kinetic energy (*T*), such that $V = -2T$.

If the energy in the collapse is conserved, then

$$E_{max} = E_{vir}$$

$$U_{max} = U_{vir} + T_{vir} = \frac{1}{2} U_{vir}$$

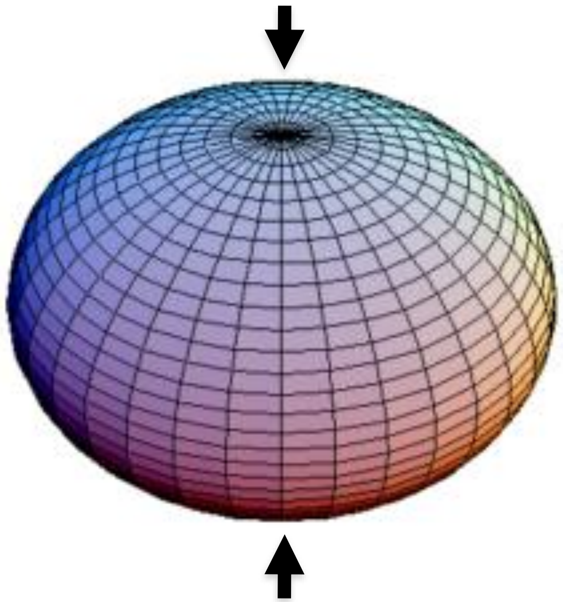
$$-\frac{GM^2}{r_{max}} = -\frac{GM^2}{2r_{vir}}$$

$$r_{vir} = \frac{r_{max}}{2}$$

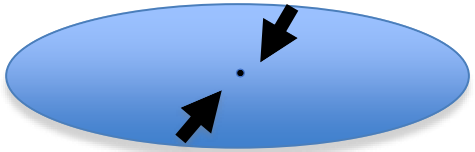
A similar calculation to the previous one, with $r_{vir}$and t_{coll} relative to the unperturbed radius give $\frac{\rho}{\bar{\rho}} = \frac{(6\,\pi)^2}{2} = 178$. This will be a key definition for a collapsed halo!

Non-spherical collapse

In practice, perturbations are most unlikely to be spherical. An *ellipsoidal* perturbation would collapse first along its short axis…



then along its next shortest axis…



and finally along its longest

producing a compact virialised halo

Hence unvirialised structures in the Universe tend to be flattened or filamentary in shape, as seen in galaxy redshift surveys and numerical simulations.