7 Relativistic Velocity and Acceleration

Non-relativistic mechanics uses $\mathbf{x}(t) = x_i(t)$ to describe trajectory. This is not a candidate for a relativistic treatment as the dependent variable, $\mathbf{x}$, is a function of the independent variable t, but these should be on a more even footing in relativity. x^μ is more natural – but how may a worldline be labeled, i.e. a sequence of "events" $x_1^\mu, x_2^\mu, \ldots$ describing where/when a particle was?

Proper Time as a Parameter and its use in defining the relativistic velocity
The time as measured in the particle's rest frame is invariant under Lorentz transformations: the **proper time** τ. Since τ is a Lorentz invariant it allows an observer-independent label for the worldline, $x^\mu(\tau)$.

To obtain the velocity, note the **infinitesimal** invariant interval:

$$
\begin{aligned}
(\mathrm{d}s)^2 = -\mathrm{c}^2\mathrm{d}\tau^2 &= -\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 \\
\Rightarrow \quad \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 &= 1 - \frac{1}{\mathrm{c}^2}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2 \\
&= 1 - (v/\mathrm{c})^2 \\
\Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \frac{1}{\sqrt{1-(v/\mathrm{c})^2}} \\
&= \gamma(v) \ .
\end{aligned}
$$

which is the expression of **time dilation** ($\mathrm{d}t \gg \mathrm{d}\tau$ as $v \to \mathrm{c}$).

The 4-Velocity
Now note:

$$
\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(\mathrm{c}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\right) \tag{7.1}
$$

This transforms the same way as x^μ under Lorentz transformations (as τ is invariant), i.e.:

$$
\begin{aligned}
\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} &= \Lambda^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \\
\Rightarrow \quad \dot{x}'^\mu &= \Lambda^\mu{}_\nu \dot{x}^\nu
\end{aligned}
$$

i.e. $\dot{x}^\mu$ are the components of a **four-vector**. $\dot{x}^\mu$ measures "rate of change" of particle along worldline. This is the natural generalisation of velocity – called the **4-velocity**.

What is the relation to 3-velocity, $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{v}$?

$$
\begin{aligned}
u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} &= \left(\mathrm{c}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\right) = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right) \\
&= \left(\mathrm{c}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) \\
&= \mathrm{c}\frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1, \frac{\mathbf{v}}{\mathrm{c}}\right) \\
&= \mathrm{c}\gamma(v)\left(1, \frac{\mathbf{v}}{\mathrm{c}}\right) && (7.2) \\
\Rightarrow \quad u^\mu &= \gamma(\mathrm{c}, \mathbf{v}) = (\gamma\mathrm{c}, \gamma\mathbf{v}) \ . && (7.3)
\end{aligned}
$$

Notice the very important distinction between the spatial components of u^μ, $\mathbf{u} = \gamma(v)\mathbf{v}$, and the three-velocity $\mathbf{v}$.

7.1 The 4-Acceleration

(Note: $\dot{}$ on non-invariant or non-4-vector objects denotes $\mathrm{d}/\mathrm{d}t$; $\dot{}$ on invariant or 4-vector quantities denotes $\mathrm{d}/\mathrm{d}\tau$.)

The **4-acceleration** is naturally defined as:

$$\begin{aligned}\alpha^\mu &= \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{d}{\mathrm{d}t}\left(\mathrm{c}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)\\ &= \gamma\frac{d}{\mathrm{d}t}\gamma(\mathrm{c},\mathbf{v}) \quad \text{(using } \gamma = \mathrm{d}t/\mathrm{d}\tau\text{)}\\ &= \gamma\left(\mathrm{c}\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v} + \gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\\ \Rightarrow \quad \alpha^\mu &= \gamma(\mathrm{c}\dot{\gamma}, \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}) \ . \end{aligned} \tag{7.4}$$

Properties of the 4-Acceleration
Note:

$$\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}(1-(v/\mathrm{c})^2)^{-1/2} \tag{7.5}$$

$$= -\frac{1}{2}(1-(v/\mathrm{c})^2)^{-3/2}\cdot(-2)\cdot\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{\mathrm{c}^2} \tag{7.6}$$

$$= \gamma(v)^3\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{\mathrm{c}^2} \ . \tag{7.7}$$

So in the rest frame of the particle, Σ^0:

$$\begin{aligned}&\dot{\gamma} \propto \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \quad \text{as} \quad \mathbf{v} = \mathbf{0}\\ &\Rightarrow \quad \alpha^\mu = (0, \mathbf{a}) \ .\end{aligned}$$

Observations:

(i) No paradox: we can have $\mathbf{v} = \mathbf{0}$ and $\mathbf{a} \neq \mathbf{0}$.

(ii) In general, $\alpha^i \neq \lambda a^i$, i.e. the spatial components of α are **not** parallel to $\mathbf{a}$.

Invariants of 4-Velocity and 4-Acceleration
We may form the invariant of the 4-velocity

$$u^\mu u_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\eta_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \ . \tag{7.8}$$

But:

$$-\mathrm{c}^2\mathrm{d}\tau^2 = \mathrm{d}x^\mu\eta_{\mu\nu}\mathrm{d}x^\nu \tag{7.9}$$

So by comparison (i.e. divide by $\mathrm{d}\tau^2$):

$$u^\mu u_\mu = -\mathrm{c}^2 \tag{7.10}$$

i.e. $u^m u$ is **timelike**, appropriately for a massive particle.

Notice with Lorentz invariants we may always chose the frame to evaluate them. So for the acceleration, evaluate in the instantaneous rest frame of the particle:

$$\alpha^\mu\alpha_\mu = -0^2 + \mathbf{a}\cdot\mathbf{a} = a^2 \geq 0 \ ,$$

so 4-acceleration is spacelike.

Orthogonality of 4-Velocity and 4-Acceleration
Note (as c^2 is constant):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}(u^\mu u_\mu) = 0 \\ \Rightarrow \quad &\dot{u}^\mu u_\mu + u^\mu \dot{u}_\mu = 0 \ . \end{aligned}$$

But we know in general $A_\mu B^\mu = A^\mu B_\mu$. So:

$$\dot{u}^\mu u_\mu + u_\mu \dot{u}^\mu = 0 \Rightarrow \quad \alpha^\mu u_\mu = 0 \tag{7.11}$$

So α and u are "$\perp$" (orthogonal in the Minkowski sense). **Check** in Σ^0: $u^\mu = (\mathrm{c}, \mathbf{0})$, $\alpha^\mu = (0, \mathbf{a})$.
So: $u^\mu \alpha_\mu = \mathrm{c} \cdot 0 + \mathbf{0} \cdot \mathbf{a} = 0$ (verified)

8 Equations of Motion

We need to seek a generalisation of Newton's 2nd law:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} \tag{8.1}$$

Remember this was covariant under Galilean transformations (i.e. its **form** was the same).

The previous section suggests a generalisation of Eq. (8.1), where m_0 being the rest mass of the particle:

$$\begin{aligned}
\mathbf{p} &\to m_0 u^\mu = p^\mu \\
\mathbf{F} &\to f^\mu && (8.2)\\
t &\to \tau \, . && (8.3)
\end{aligned}$$

I.e. $\mathbf{p}$, $\mathbf{F}$ become 4-vectors. We will take the non-relativistic limit to aid in understanding these four-vectors.

The 4-Momentum
Examine p^μ:

$$p^\mu = m_0 u^\mu = m_0 \gamma(v)(\mathrm{c}, \mathbf{v}) \tag{8.4}$$

Does it reduce to sensible quantities as $(v/\mathrm{c}) \to 0$?

Firstly the spatial components, p^i:

$$\begin{aligned}
p^i &= m_0 u^i = m_0 \gamma(v) v^i \\
&\approx m_0 \left(1 + \frac{1}{2}\left(\frac{v}{\mathrm{c}}\right)^2\right) v^i \\
&\approx m_0 v^i + m_0 v^i \cdot \frac{1}{2}\left(\frac{v}{\mathrm{c}}\right)^2 && (8.5)
\end{aligned}$$

So we find non-relativistic momentum as the limiting expression, with the first correction being cubic in the velocity. We interpret m_0 as '**rest mass**' or '**proper mass**', i.e. mass in rest frame of particle.

Then equation (8.4) implies the inertial mass, m, is velocity-dependent:

$$m = m_0 \gamma(v) \tag{8.6}$$

What about the time component, p^0? Let us identify its role by taking the non-relativistic limit

$$\begin{aligned}
p^0 &= m_0 \gamma(v) \mathrm{c} \\
&\approx m_0 \left(\mathrm{c} + \frac{1}{2}\frac{v^2}{\mathrm{c}}\right) \\
&= \frac{1}{\mathrm{c}}\left(m_0 \mathrm{c}^2 + \frac{1}{2} m_0 v^2\right) \, , && (8.7)
\end{aligned}$$

i.e. the sum of the rest mass energy and the non-relativistic kinetic energy. Thus it is natural to interpret:

$$p^0 = \frac{E}{\mathrm{c}} \quad \text{with} \quad E = m_0 \gamma \mathrm{c}^2 = m\mathrm{c}^2 \, , \tag{8.8}$$

Einstein's famous relation. We have not shown that E contains potential energy, see later discussion.

Relativistic form of Newton's Second Law
Newton's second law has become:

$$\dot{p}^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = f^\mu \tag{8.9}$$

In the non-relativistic case, if $\mathbf{F} = \mathbf{0}$ then $\dot{\mathbf{p}} = \mathbf{0}$, so momentum is conserved.

Here:

$$\dot{p}^\mu = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{E}{\mathrm{c}}, m_0\gamma\mathbf{v}\right) = 0 \tag{8.10}$$

expressing conservation of **both** E and "$\mathbf{p}$".

Lorentz Invariant of 4-Momentum
What is the Lorentz invariant $p^\mu p_\mu$?

$$p^\mu p_\mu = m_0^2 u^\mu u_\mu = -m_0^2 \mathrm{c}^2 \ , \tag{8.11}$$

i.e. p^μ is timelike with a magnitude 'momentum at speed of light' squared. The use of Lorentz invariants in collisions between fast particles is powerful, especially combined with energy/momentum conservation.

Example: Linearly Accelerated Motion
Consider the case of **rectilinear** motion, constant 3-acceleration $\mathbf{a}$, where $\mathbf{a} = a\hat{\mathbf{x}}$. We will use the standard configuration in changing reference frames.

In the instantaneous rest frame of the particle:

$$\alpha'^\mu = (0, a, 0, 0) \tag{8.12}$$

This is transformed from the lab frame, α^μ:

$$\alpha'^\mu = \bar{\Lambda}^\mu{}_\nu \alpha^\nu \tag{8.13}$$

In matrix form (omitting the untransformed y- and z-components):

$$\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \alpha^0 \\ \alpha^1 \end{pmatrix} \quad \text{(particle rest frame} \leftarrow \bar{\Lambda} \leftarrow \text{lab frame)} \ . \tag{8.14}$$

Applying the inverse Lorentz transformation to both sides of Eq. (8.14):

$$\begin{aligned} \begin{pmatrix} \gamma & +\beta\gamma \\ +\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ a \end{pmatrix} &= \begin{pmatrix} \alpha^0 \\ \alpha^1 \end{pmatrix} \quad \text{(lab frame)} \\ \begin{pmatrix} \beta\gamma a \\ \gamma a \end{pmatrix} &= \begin{pmatrix} \alpha^0 \\ \alpha^1 \end{pmatrix} \ . \end{aligned}$$

So $\alpha^1 = \gamma a$.

$$\begin{aligned} \frac{\mathrm{d}u^1}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}u^1}{\mathrm{d}t} &= \gamma a = \alpha^1 \\ \Rightarrow \quad \frac{\mathrm{d}u^1}{\mathrm{d}t} &= a \\ \Rightarrow \quad u^1 &= at \ . \end{aligned}$$

(Note: a is constant. Assuming from rest in lab frame, $t = 0$.)

So:

$$u^1 = \gamma \frac{\mathrm{d}x^1}{\mathrm{d}t} = at$$
$$\Rightarrow \quad \frac{v_x}{\sqrt{1-(v_x/\mathrm{c})^2}} = at$$

Solve for v_x:

$$v_x = \frac{at}{\sqrt{1+(at/\mathrm{c})^2}} \approx \begin{cases} at & \text{for } t \ll \frac{\mathrm{c}}{a} \quad \text{(Newtonian)} \\ \mathrm{c} & \text{for } t \gg \frac{\mathrm{c}}{a} \quad \text{(Relativistic limit)} \end{cases} . \tag{8.15}$$

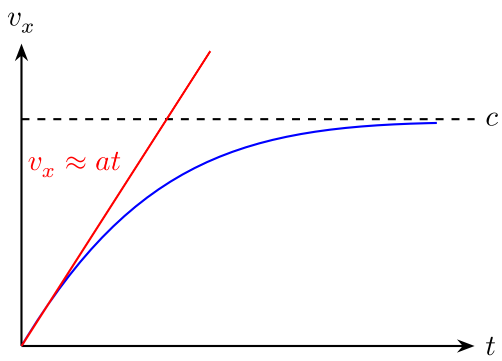

Figure 8.1: Velocity as a function of time for constant proper acceleration, showing the Newtonian limit at small t and asymptotic approach to c at large t.

We may also deduce:

$$\begin{aligned} x(T) &= \int_0^T \mathrm{d}t\, v_x(t) \\ &= \int_0^T \mathrm{d}t\, \frac{at}{\sqrt{1+(at/\mathrm{c})^2}} \\ &= \frac{\mathrm{c}^2}{a}\left[\sqrt{1+(at/\mathrm{c})^2}\right]_0^T \\ &= \frac{\mathrm{c}^2}{a}\left(\sqrt{1+(aT/\mathrm{c})^2}-1\right) && (8.16) \\ &\approx \begin{cases} \frac{1}{2}aT^2 & \text{for } T \ll \frac{\mathrm{c}}{a} \quad \text{(Newtonian)} \\ \mathrm{c}T & \text{for } T \gg \frac{\mathrm{c}}{a} \quad \text{(Relativistic limit)} \end{cases} . && (8.17) \end{aligned}$$

Exercise: Calculation using rapidity.

8.1 4-Force

We postulated that:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = f^\mu \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 u^\mu) = f^\mu \tag{8.18}$$

Note: $\mathrm{d}m_0/\mathrm{d}\tau$ may not be zero. E.g.: (i) rockets (obviously); (ii) particles (e.g. "balls") during collision, deformation implies potential energy and that adds to mass; (iii) More subtly, a body being heated in its rest frame.

Let us now examine the time and spatial components of f^μ

$$f^\mu = (f^0, f^i) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{E}{\mathrm{c}}, \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{p}\right) = \gamma\left(\frac{\mathrm{d}E}{\mathrm{d}t}\frac{1}{\mathrm{c}}, \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) \tag{8.19}$$

where $\mathbf{p} = m_0\gamma\mathbf{v}$.

So **power**, $\mathrm{d}E/\mathrm{d}t$, seems a natural ingredient in the time component of force. Note that:

$$f^i = \gamma\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \gamma\mathbf{F} \tag{8.20}$$

So unlike Newtonian mechanics, "$\mathbf{F}$" is not invariant under Lorentz transformations.

Rate of Doing Work
Note that $\mathbf{u}\cdot\mathbf{F}$ is power or rate of doing work (Newtonian).

So consider:

$$\begin{aligned} u_\mu f^\mu &= u_\mu\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} \\ &= \frac{1}{2m_0}\frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu p^\mu) \\ &= \frac{-1}{2m_0}\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0\mathrm{c})^2 \\ &= -\frac{dm_0}{\mathrm{d}\tau}\mathrm{c}^2 \end{aligned} \tag{8.21}$$

i.e. "4-version" is only non-zero if rest mass is changing with time. So:

$$u_\mu f^\mu = -\gamma^2\mathrm{c}\frac{\mathrm{d}E}{\mathrm{d}t}\cdot\frac{1}{\mathrm{c}} + \gamma\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\mathbf{u} \tag{8.22}$$

But $\mathbf{u} = \gamma\mathbf{v}$:

$$\Rightarrow \quad u_\mu f^\mu = \gamma^2\left\{-\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\mathbf{v}\right\} \tag{8.23}$$

So if $\mathrm{d}m_0/\mathrm{d}\tau = 0$ then $u_\mu f^\mu = 0$, and:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{v}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{v}\cdot\mathbf{F} \tag{8.24}$$

So apparently the Newtonian relation—BUT remember that $\mathbf{p} = m_0\gamma\mathbf{v}$, so the relation between $\mathbf{F}$ and $\mathbf{a}$ is still Einsteinian.