Evolution of Cosmic Structure

Lecture 5: Power spectrum of initial overdensities

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Linear growth of perturbations

How did the small fluctuations visible in the CMB grow to the point where gravitationally bound structures could form?

- Characterising the fluctuations – the overdensity and its power spectrum
- Gravitational instablility in static and expanding media
- The effects of pressure – Jeans mass
- The initial power spectrum and its evolution
- The behaviour of the baryons

Gravitational instability in expanding medium

Matter-dominated

$$\delta(t) = At^{2/3} \propto a(t) \propto \frac{1}{1+z}$$

Linear growth

Radiation-dominated

$$\delta(t) = B_1 + B_2 \ln(t)$$

Logarithmic growth

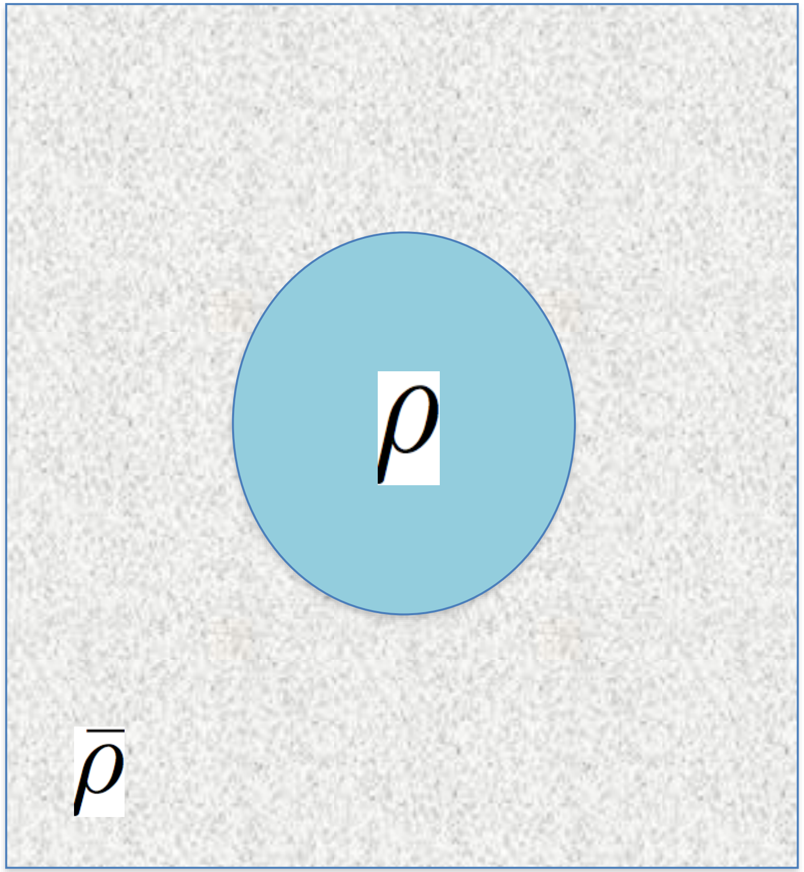
Cosmological-constant dominated

$$\delta(t) = C_1 + C_2 e^{-2H_\Lambda t}$$

$$H_\Lambda = \text{constant}$$

No growth!

Matter phase dominates the growth of structure



The role of pressure

The typical collapse time, or dynamical time, is

$$t_{dyn} = \frac{1}{\sqrt{4\pi G\bar{\rho}}}$$

But pressure can oppose the growth of structure of perturbations

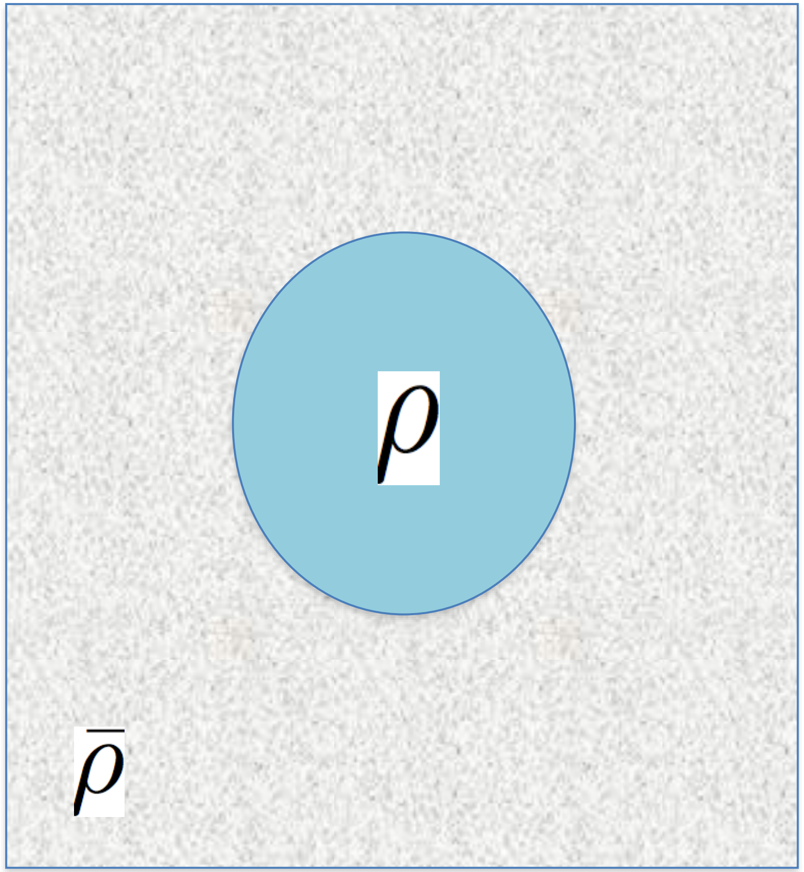
$$t_{pre} \sim \frac{R}{c_s} \qquad\qquad c_s = \sqrt{\frac{dP}{d\rho}}$$

Which leads to a Jeans length,

$$\Lambda_J = c_s t_{dyn}$$

And a Jeans Mass,

$$M_J = \frac{4\pi}{3}\Lambda_J^3 \rho = \frac{4\pi}{3} c_s^3 t_{dyn}^3 \rho$$



The role of pressure

The key point is then what the sound speed is

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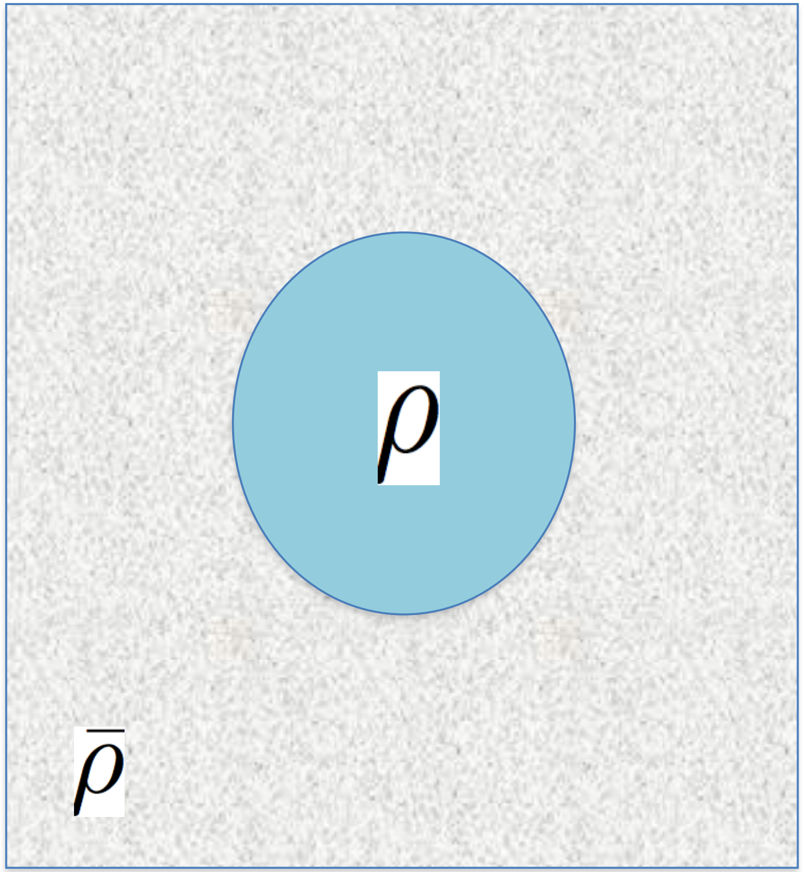

For a perfect gas,

$$P \propto \rho^\gamma$$

$$c_s = \frac{\gamma P}{\rho} = \frac{\gamma KT}{\mu m_H}$$

$$M_J \sim 10^5 M_\odot$$

For a radiation-dominated sound speed,

$$c_s = \frac{c}{\sqrt{3}}$$

$$M_J \sim 10^{18} M_\odot$$

Pressure stops all growth of baryon perturbations during the radiation dominated area

Overdensity and its power spectrum

$$\delta(\mathbf{r},t) \equiv \frac{\rho(\mathbf{r},t)-\bar{\rho}(t)}{\bar{\rho}(t)}$$

We will be interested in the way these fractional fluctuations vary with spatial scale, which can be characterised within some volume *V* of space by taking a Fourier Transform

$$\delta_{\mathbf{k}} = \frac{1}{V}\int_V \delta(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r$$

The Fourier coefficients provide a representation of δ(**r**) as a set of sine waves of vector wavenumber **k**,

$$\delta(\mathbf{r}) = \frac{V}{(2\pi)^3}\int \delta_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3k$$

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$$\delta_{\mathbf{k}} = \frac{1}{V} \int_V \delta(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r$$

The Fourier coefficients provide a representation of $\delta(\mathbf{r})$ as a set of sine waves of vector wavenumber **k**,

$$\delta(\mathbf{r}) = \frac{V}{(2\pi)^3} \int \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3 k$$

The $\delta_{\mathbf{k}}$ are complex numbers, $\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}| e^{i\phi_{\mathbf{k}}}$, and the mean power at a given wavelength can be obtained by averaging the amplitudes over all directions, giving $P(k) = \langle |\delta_{\mathbf{k}}|^2 \rangle$. The initial fluctuations emerging from inflation are usually assumed to constitute a *Gaussian Random Field*, whereby the phases (ϕ_k) of the Fourier components are random and uncorrelated. In this case, the Central Limit Theorem leads departures from the mean density on each spatial scale (e.g. mean density in randomly scattered boxes) to be Gaussian distributed.

Evolution of the power spectrum

As we discussed earlier, the relative amplitude of fluctuations on different spatial scales is typically characterised by the power spectrum, $P(k) = <|\delta_{\mathbf{k}}|^2>$. What shape does this have, and how does it evolve?

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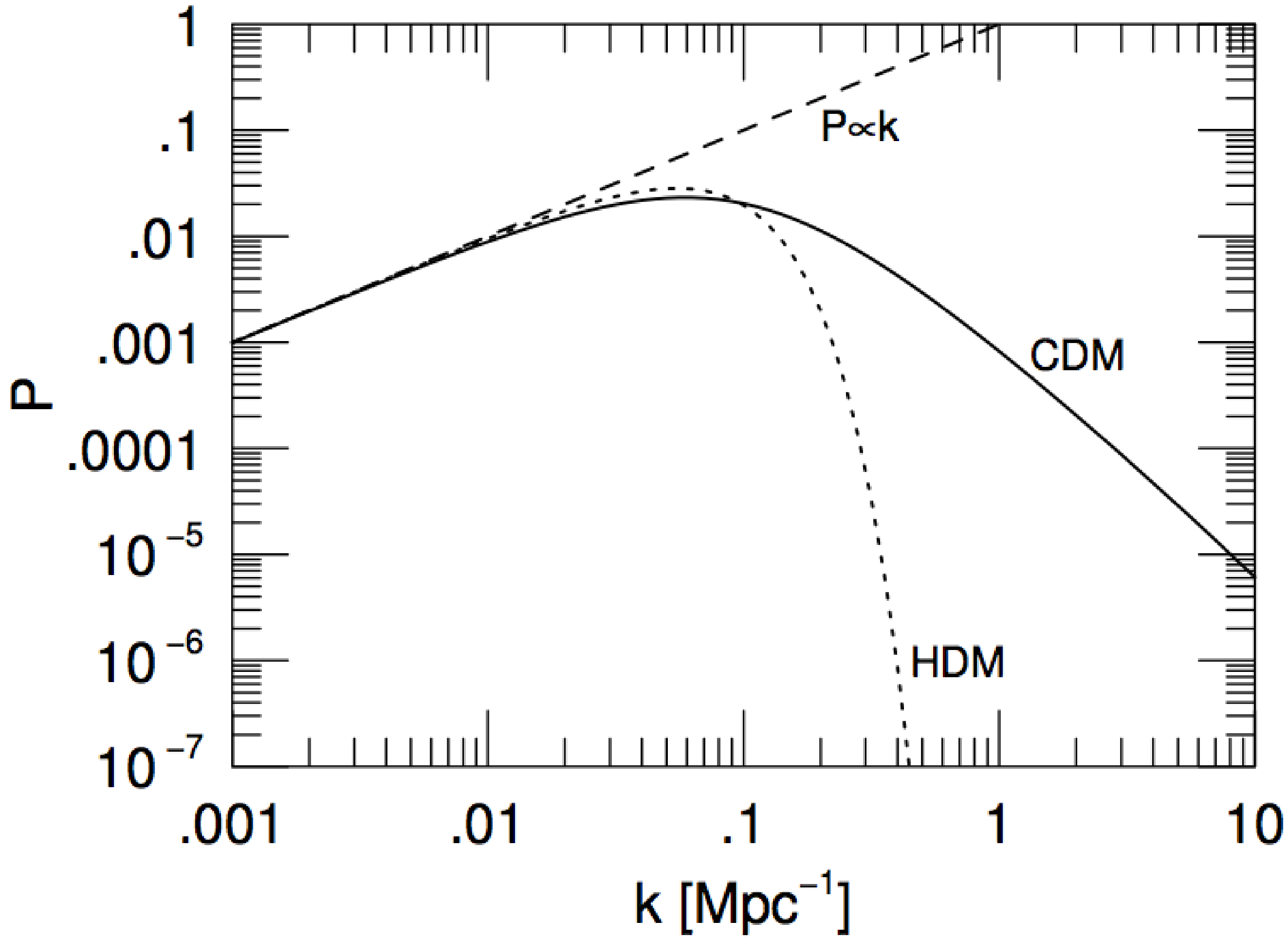

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Models for inflation generally predict that the spectrum of fluctuations which result should be "scale-free" (i.e. there is no preferred spatial scale), such that $P(k) \propto k^n$, and the favoured value of n is usually n=1, known as the Harrison-Zeldovich spectrum. What would such a spectrum mean in terms of the fluctuations on different *mass* scales? If L is a comoving spatial scale, and $k=2\pi/L$ is the corresponding comoving wavenumber, then the mean mass within a sphere of comoving radius L is

$$< M >= \frac{4\pi}{3}(aL)^3\rho_m$$

but the actual mass found within a randomly chosen sphere of this size will scatter around $<M>$ due to the perturbations.

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but the actual mass found within a randomly chosen sphere of this size will scatter around <*M*> due to the perturbations. The mean square amplitude of these fluctuations is related to *P(k)* as follows

$$\left\langle \left(\frac{M - < M >}{< M >} \right)^2 \right\rangle \propto k^3 P(k)$$

so the rms fractional fluctuations scale as

$$\frac{\Delta M}{M} \propto \sqrt{k^3 P(k)} \propto L^{-(3+n)/2} \propto M^{-(3+n)/6}$$

Evolution of the power spectrum

Hence for a Harrison-Zeldovich spectrum $\Delta M/M$ scales as $M^{-2/3}$, and so fluctuations on smaller mass scales have larger amplitude.

However, there are several factors at work which can modify this primordial spectrum in the time between the end of inflation (t~10^{-34}s) and the end of the radiation-dominated era at t~5×10^{4} yr. Firstly recall (from § 2) that in the radiation-dominated era, the proper radius of the horizon expands as $d_{hor}=2ct=c/H$,reaching a size of ~0.03 Mpc by the onset of matter domination. In comoving units, this is $0.03/a_{rm}$, where $a_{rm}\approx1/3570$ is the scale factor at radiation-matter equality. The corresponding wavenumber, in comoving units, is hence $k_{rm}\approx2\pi a_{rm}/2ct\approx0.06$ Mpc^{-1}.

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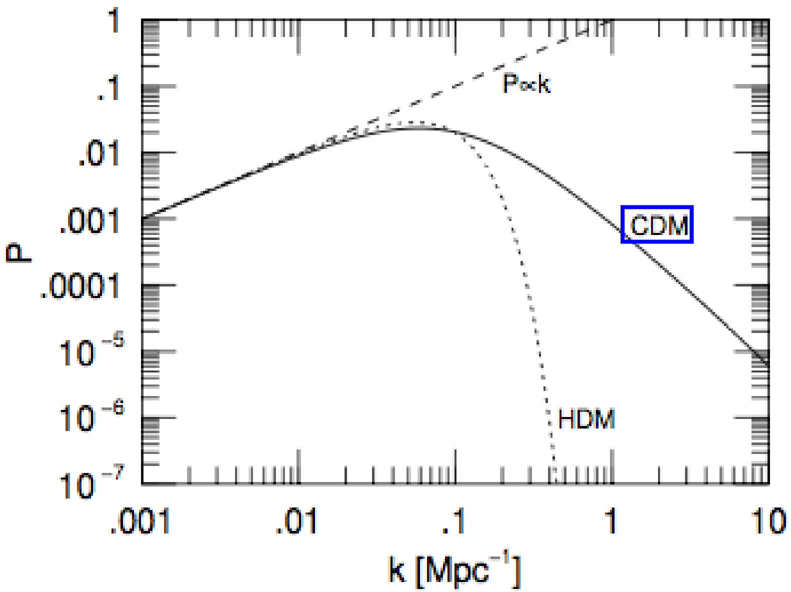
Perturbations on scales larger than the horizon can be shown to grow linearly with $a(t)$ throughout, so at comoving wavenumbers <0.06 Mpc^{-1} (where fluctuation scales exceed the horizon size throughout the radiation-dominated era) we expect to see $P(k) \propto k$ (i.e. Harrison-Zeldovich). However, modes with $k>k_{rm}$ will have spent some time inside the horizon (increasingly longer for higher values of k), and will have had their growth restricted by one of several effects:

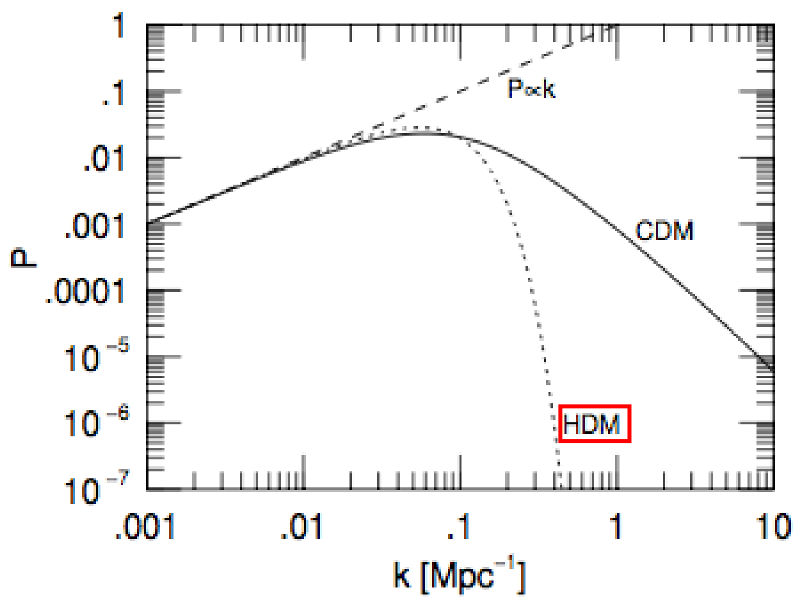
Evolution of the power spectrum



1. Perturbations in a fluid with significant pressure cannot collapse if they are inside the horizon, unless they have $L>\lambda_J$.
2. Even for a pressureless material, like dark matter, we have seen that growth of modes inside the horizon is much slower ($\delta \propto \log t$).
3. If the dark matter particles have relativistic velocities, then they will be able to stream freely on scales up to the horizon scale, and will therefore smooth out any perturbations on smaller scales.

Concentrating on the dark matter, cold dark matter is defined by the fact that its particles are sufficiently massive to already be non-relativistic when it decouples from other components of the Universe. Hence point 3 does not apply, but modes with higher *k* will still have had their growth restricted progressively more by effect 2, so the CDM spectrum *turns* down, as shown in the above plot.

Evolution of the power spectrum



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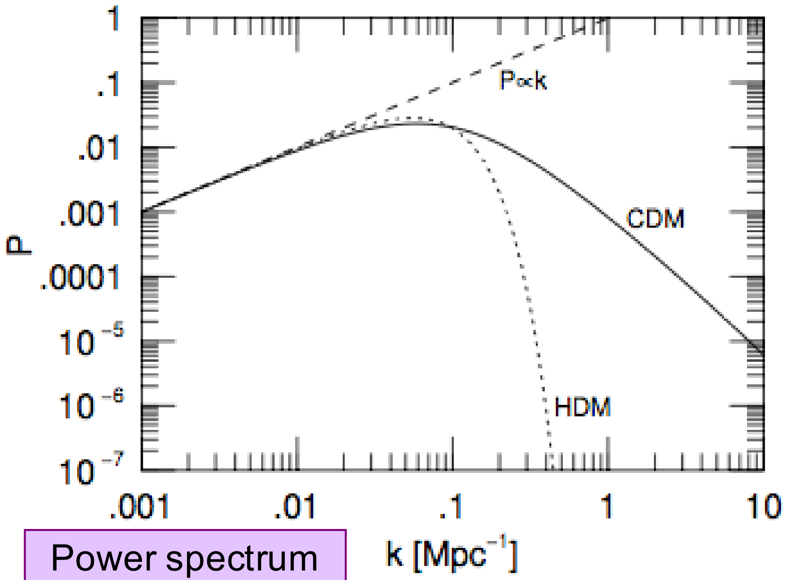
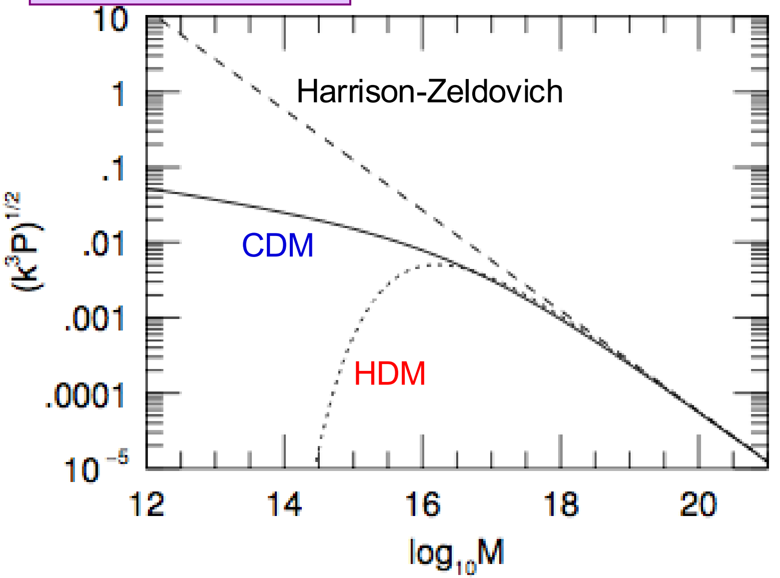
Hot dark matter is relativistic when it decouples – neutrinos are an example – and it will therefore free stream and eliminate short wavelength perturbations. For example, neutrinos decouple at t~1 s, when kT~1 MeV. Hence, so long as they have mass <1 MeV, they will be "hot". It now seems likely that neutrino masses are ~1 eV, in which case they would remain relativistic all the way to t_{rm}, when kT~1 eV. The result is a much sharper cut-off in *P(k)*, as shown above (HDM).

Evolution of the power spectrum



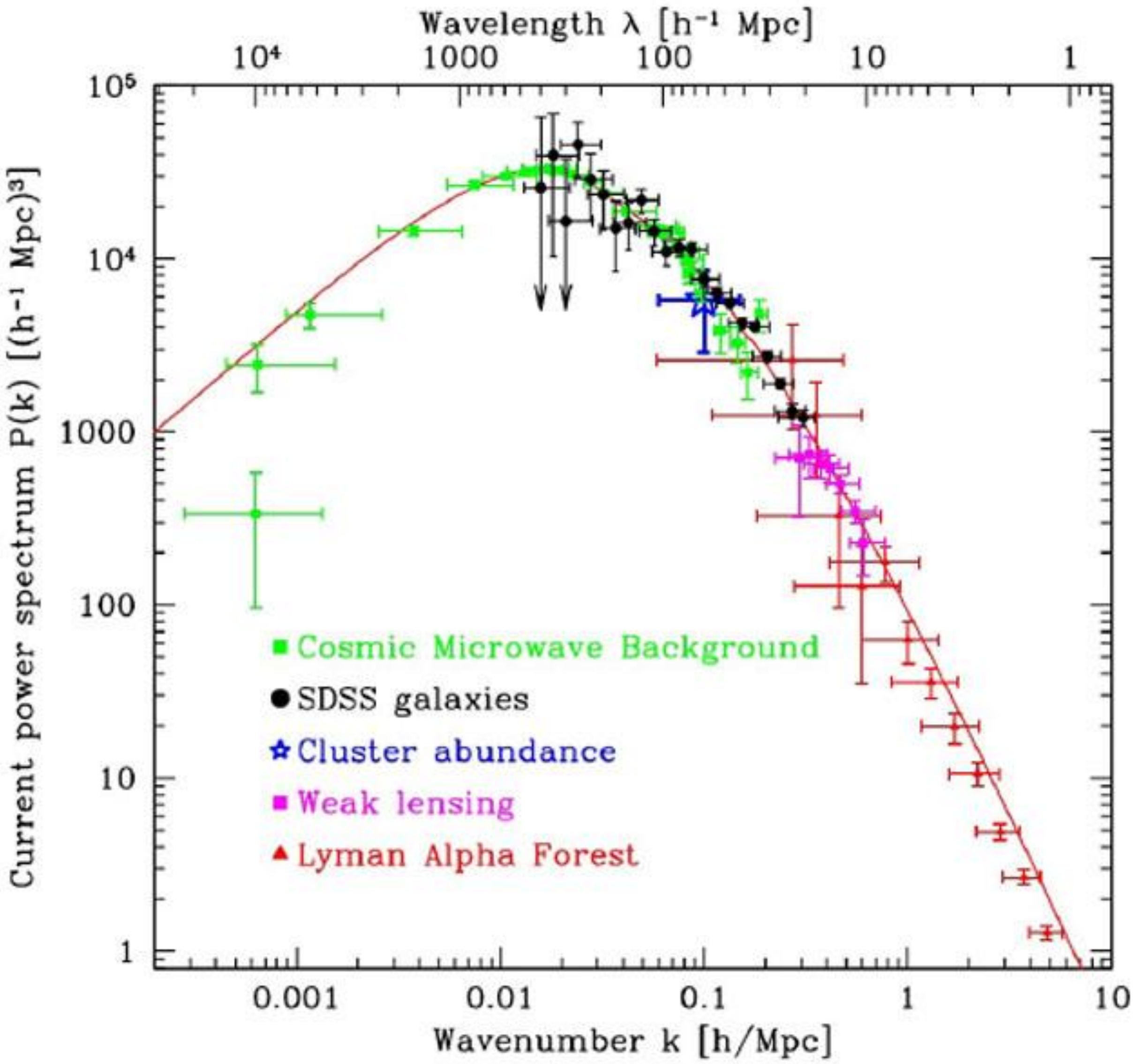
Power spectrum and $\Delta M/M$ at t_{rm}



Recalling that the rms fractional mass fluctuations scale as $\frac{\Delta M}{M} \propto \sqrt{k^3 P(k)}$ we see in the plot below that the difference between CDM and HDM is fundamental. In CDM fluctuations on smaller mass scales have progressively larger fractional amplitudes, whilst for HDM there is a maximum in $\Delta M/M$ at a mass ~10^{16} $M_{\odot}$.

Note that in the matter-dominated era, which starts at t_{rm}, fluctuations both smaller and larger than the horizon grow with $\delta \propto a$, so that the power simply rises across the whole spectrum, and there is no change in shape. This means that the first structures to reach δ~1, after which they start to turn around and collapse, will be small for CDM, but ~10^{16} M for HDM. Small galaxies seen at high z therefore show that most of the DM must be cold.

Observations of the power spectrum



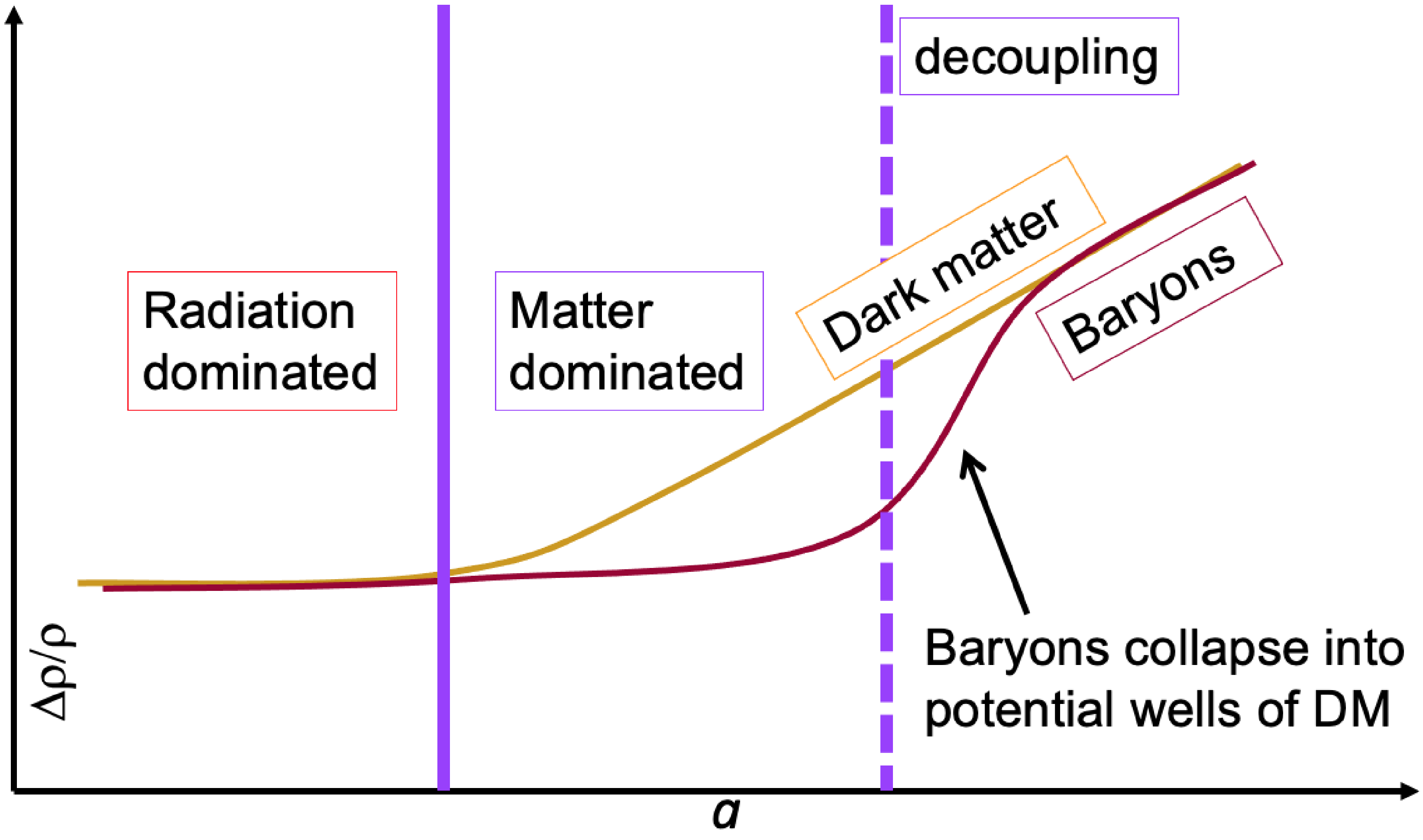
Behaviour of the baryons

Since the growth of perturbations in the matter-dominated era proceeds according to $\delta \propto a$, we can easily calculate the growth expected since decoupling in the matter fluctuations which are seen in the CMB. Since these are imprinted at z~1100, we know that a has since increased by a factor of $(1+z)=1101$, bringing the radiation temperature down from 3000K to ~3K. Hence the fluctuations should have grown from $\delta \sim 10^{-5}$ at decoupling, to $1101 \times 10^{-5} \sim 10^{-2}$ now. Clearly this is not right!

The reason is that the Jeans mass before decoupling, when the radiation pressure supports the baryons, is very large ($\sim 10^{20}\ M_\odot$), so the baryons cannot start to clump. However, the dark matter does not feel this pressure, and so perturbations in the d.m. can grow, and then when radiation decouples, the baryons fall into these developing dark matter overdensities. These must already have had amplitude $\delta > 10^{-3}$ at decoupling, in order to have led to collapsed structures by the present day.

Behaviour of the baryons

Baryons fall into growing dark matter structures after decoupling.



Summary of linear growth of perturbations

We defined an overdensity: $\delta(\mathbf{r},t) \equiv \frac{\rho(\mathbf{r},t) - \bar{\rho}(t)}{\bar{\rho}(t)}$

We discussed the the scenario where $\delta(\mathbf{r},t) \ll 1$ which is the regime where linear growth of perturbations is relevant. This is the scale of initial perturbations, (CMB) and very large-scale structures (superclusters).

We showed that in a static medium these perturbations grow exponentially.

In an expanding medium, the perturbation growth will depend on the type of material in the universe. In the absence of restoring pressure and inside the horizon:

Radiation dominated - perturbations grow logarithmically

Matter dominated – perturbations grow linearly with the scale factor

Cosmological constant dominated – perturbations do not grow

Summary of linear growth of perturbations

When considering internal pressure, the photon pressure will cause radiation dominated perturbations grow even slower than logarithmically.

We introduced a way to characterize the perturbations – the power spectrum of density fluctuations.

We showed that the horizon and the type of dark matter can both imprint features on the growth of this power spectrum.

We showed the power spectrum has been measured and agrees with a cold dark matter scenario

We showed that baryons must be impeded in their growth until the radiation pressure is released, and then they catch up to the dark matter growth.

Next stage: Show what happens to those perturbations when they reach $\delta(\mathbf{r}, t) \gtrsim 1$