2 Lorentz transformations and the invariant interval

In this section we revise Lorentz transformations and cast them in an especially simple form in the case their relative velocities are all along on coordinate axis. We will then show that there is a geometric object, the invariant interval, which is a relativistic counterpart to distance between two points in Euclidean space.

2.1 The Lorentz Transformation Equations

Consider two inertial frames Σ, Σ' in the standard configuration, with relative velocity v along the x-direction.

Then the relation between $(\mathrm{c}t, x, y, z)$ and $(\mathrm{c}t', x', y', z')$ is:

$$\text{(L1)} \quad x' = \gamma(x - \beta \mathrm{c}t) \tag{2.1}$$

$$\text{(L2)} \quad \mathrm{c}t' = \gamma(\mathrm{c}t - \beta x) \tag{2.2}$$

$$y' = y \quad ; \quad z' = z$$

where:

$$\beta = \frac{v}{\mathrm{c}} \qquad \gamma = \frac{1}{\sqrt{1 - (v/\mathrm{c})^2}} \tag{2.3}$$

Matrix Form of the Lorentz Transformation

Disregarding y, z and y', z', we can write the Lorentz transformations (L1, L2) in 2×2 matrix form:

$$\text{(LM)} \quad \begin{pmatrix} \mathrm{c}t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \mathrm{c}t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(\mathrm{c}t - \beta x) \\ \gamma(x - \beta \mathrm{c}t) \end{pmatrix} \tag{2.4}$$

This can be more neatly represented, remember rotations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.5}$$

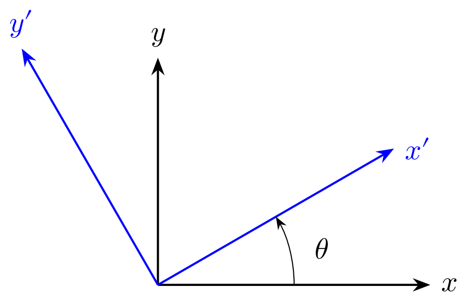

Figure 2.1: Rotation of coordinate axes by angle θ.

The rotation matrix $\mathsf{R}(\theta)$ satisfies:

$$\mathsf{R}^{-1}(\theta) = \mathsf{R}^T(\theta) = \mathsf{R}(-\theta) \tag{2.6}$$

Hyperbolic Representation and Rapidity

We can write (LM) in a more compact manner by noting:

$$\gamma^2 - \gamma^2\beta^2 = \frac{1}{1-(v/\mathrm{c})^2} - \frac{(v/\mathrm{c})^2}{1-(v/\mathrm{c})^2} = 1 \tag{2.7}$$

So try the representation:

$$\gamma = \cosh\phi \qquad \beta\gamma = \sinh\phi \tag{2.8}$$

This works since $\cosh^2\phi - \sinh^2\phi = 1$, and the ranges are consistent:

- $1 \leq \gamma \leq \infty$ is consistent with $\cosh\phi$
- $-\infty \leq \gamma\beta \leq \infty$ is consistent with $\sinh\phi$
- $-\infty \leq \phi \leq \infty$ (as v can be positive or negative)

Thus the Lorentz transformation becomes:

$$\begin{pmatrix} \mathrm{c}t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \mathrm{c}t \\ x \end{pmatrix} \tag{2.9}$$

The Rapidity Parameter
The parameter ϕ is called the "**rapidity**" or "pseudoangle" of the Lorentz transformation.

Note that:

$$\tanh\phi = \frac{\beta\gamma}{\gamma} = \frac{v}{\mathrm{c}}$$
$$\Rightarrow \phi = \tanh^{-1}(v/\mathrm{c}) \ .$$

Properties of the Lorentz Transformation Matrix
Let us denote the Lorentz transformation matrix by $\Lambda(\phi)$. We can easily check:

$$\Lambda^{-1}(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(-\phi) & -\sinh(-\phi) \\ -\sinh(-\phi) & \cosh(-\phi) \end{pmatrix}$$

$$= \Lambda(-\phi)$$

but **not** the transpose (unlike rotations).

The fact that $\Lambda^{-1}(\phi) = \Lambda(-\phi)$ is sensible as $-\phi(v) = \phi(-v)$. So the inverse Lorentz transformation corresponds to $-v$, as expected.

2.2 Coordinate Independence and invariants

- Coordinate systems are chosen to make calculations or observations simple.
- They are not intrinsic, so any physical quantity must be independent of choice of coordinates. This distinguishes a vector such as force, $\mathbf{f}$, from its components, f_i, where $i = x, y, z$: the vector, $\mathbf{f}$ is invariant under changes of the coordinate system, while the components depend on the coordinate system, i.e its orientation.

- A particularly simple example is provided by two vectors $\mathbf{u}$ and $\mathbf{v}$: the scalar product $\mathbf{u} \cdot \mathbf{v}$ is independent of coordinates.
- A special case is $\mathbf{u} = \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{u}$ is invariant, the magnitude of the vector, u, squared.

Distance as an Invariant
One of the simplest invariants under changes of coordinates is the distance between two points, r_{12}^2:

$$r_{12}^2 = (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \tag{2.10}$$

This is invariant under both rotations and changes of origin of coordinates (including Galilean transformations).

We will now construct an analogous entity for *events* in relativity.

2.3 The Invariant Interval

There are several important physical quantities invariant under Lorentz transformations. The simplest is the **invariant interval**.

Consider two events, $(\mathrm{c}t_2, x_2)$ and $(\mathrm{c}t_1, x_1)$.

Then form:

$$\varDelta \mathrm{c}t = \mathrm{c}(t_2 - t_1) \tag{2.11}$$
$$\varDelta x = x_2 - x_1 \tag{2.12}$$

$$\boxed{s_{12}^2 = -(\varDelta \mathrm{c}t)^2 + (\varDelta x)^2} \tag{2.13}$$

We will now see that s_{12}^2 is invariant under Lorentz transformations.

Proof of Invariance
Use the Lorentz transformation:

$$(\varDelta \mathrm{c}t)' = \cosh\phi\,(\varDelta \mathrm{c}t) - \sinh\phi\,(\varDelta x) \tag{2.14}$$
$$(\varDelta x)' = \cosh\phi\,(\varDelta x) - \sinh\phi\,(\varDelta \mathrm{c}t) \tag{2.15}$$

So:

$$\begin{aligned}
(s'_{12})^2 &= -(\varDelta \mathrm{c}t)'^2 + (\varDelta x)'^2 \\
&= -\cosh^2\phi\,(\varDelta \mathrm{c}t)^2 + 2\cosh\phi\sinh\phi\,(\varDelta \mathrm{c}t)(\varDelta x) - \sinh^2\phi\,(\varDelta x)^2 \\
&\qquad + \cosh^2\phi\,(\varDelta x)^2 - 2\cosh\phi\sinh\phi\,(\varDelta \mathrm{c}t)(\varDelta x) + \sinh^2\phi\,(\varDelta \mathrm{c}t)^2 \\
&= (\cosh^2\phi - \sinh^2\phi)\Big(-(\varDelta \mathrm{c}t)^2 + (\varDelta x)^2\Big) \\
&= -(\varDelta \mathrm{c}t)^2 + (\varDelta x)^2 \\
&= (s_{12})^2
\end{aligned} \tag{2.16}$$

Thus s_{12}^2 is indeed invariant under Lorentz transformations.

2.4 Spacetime

We will now see that the invariant interval allows a classification of the relations between two events, which indicates whether they can be causally related.

Returning to the four dimensional version of the invariant interval, choosing the first event to be the origin, i.e. $ct_1 = 0$ and $(x_1, y_1, z_1) = (0, 0, 0)$, then denoting $r_2^2 = x_1^2 + y_1^2 + z_1^2$ we see

$$s_2^2 = -\mathrm{c}t_2^2 + r_2^2 \tag{2.17}$$

is the invariant interval from the origin to (t_2, r_2).

To provide some physical insight into the information contained in the invariant interval, let us consider two special cases, where the separation is purely temporal or spatial.

1. If $r_2 = 0$, i.e. two events at the same spatial point, then $s_2^2 < 0$. For example this describes the interval associated with an object stationary at the origin at two times.
2. If $t_2 = 0$, i.e. two events at same time, but different spatial locations, then $s_2^2 > 0$

Since s^2 is **invariant**, this property (the sign of s^2) is invariant.

2.5 Timelike, Spacelike, and Lightlike Intervals

Motivated by the observations above, It is useful to call those events obeying (in general, no longer insisting one is at origin):

$$\boxed{\begin{array}{lcl} s_{12}^2 < 0 & \Rightarrow & \textbf{timelike} \\ s_{12}^2 > 0 & \Rightarrow & \textbf{spacelike} \end{array}} \tag{2.18}$$

Briefly returning

$$|r_2| = \mathrm{c}t_2 \tag{2.19}$$

this is the equation obeyed by events reached by a flash of light from the origin ($r_1 = 0$, $t_1 = 0$). So:

$$s_{12}^2 = -\mathrm{c}t_{12}^2 + r_{12}^2 = 0 \tag{2.20}$$

This case of zero invariant is called *null* or *light-like*.

Example – Proper time
Time elapsed in the frame where a particle is at rest ($r_{12} = 0$): $\mathrm{c}t_{12}$ is associated with a timelike invariant:

$$s_{12}^2 = -\mathrm{c}^2\tau_{12}^2 \tag{2.21}$$

(NB minus sign.) τ_{12} is called the **proper time**. Given it is an invariant, it is surprising that the meaning is simplest in one particular frame.

Example – Proper Length
Separation of two particles (or length of a rod) measured when $t_{12} = 0$:

$$s_{12}^2 = \ell_{12}^2 \tag{2.22}$$

ℓ_{12} is called the **proper length**.

2.6 Minkowski Spacetime Diagrams

It is useful to collect these ideas into a graphical representation: **Minkowski spacetime** or a **spacetime diagram**, see Fig. (2.2).

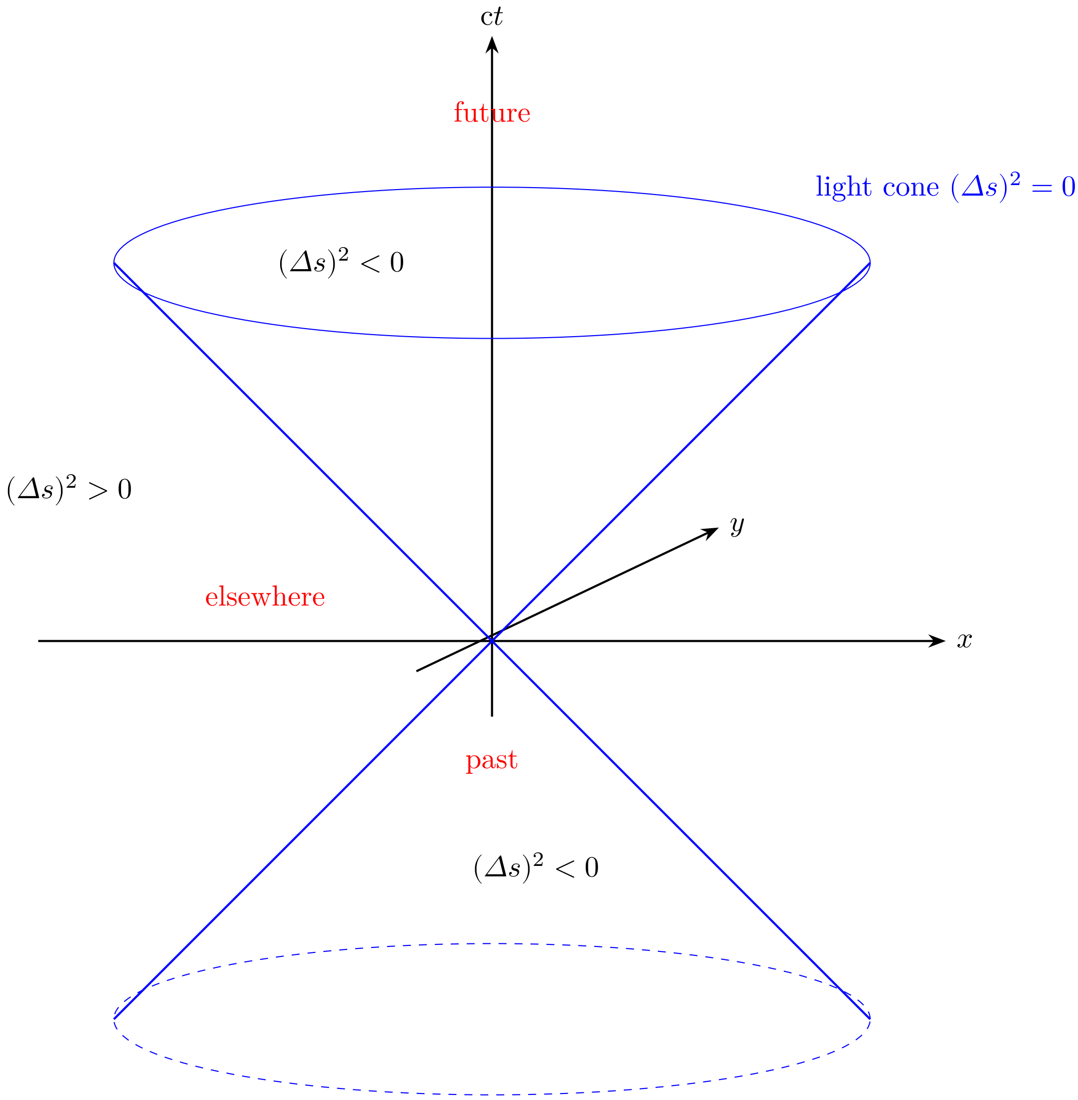

Figure 2.2: Minkowski spacetime diagram showing the light cone structure.

Key observations:

1. *Timelike* intervals reside **inside** the light-cone. They may be:
   - "future-pointing" ($ct > 0$)
   - "past-pointing" ($ct < 0$)
2. *Spacelike* intervals lie **outside** the light-cone.
3. Light from O resides **on** the light-cone.
4. "Proper" *orthochronous* Lorentz transformations preserve the future/past property. We will almost exclusively deal with these.
5. Note to make contact with non-relativistic descriptions, use t along for the vertical axis (*not* ct), the section of the light cone is at $t = x/\mathrm{c}$. If we now imagine $\mathrm{c} \to \infty$letting the

speed of light c $\to \infty$, then "elsewhere" becomes compressed to a plane–"the present"–with which Newton would have been happy. So Minkowski space is the product of having a speed limit.

Worldlines

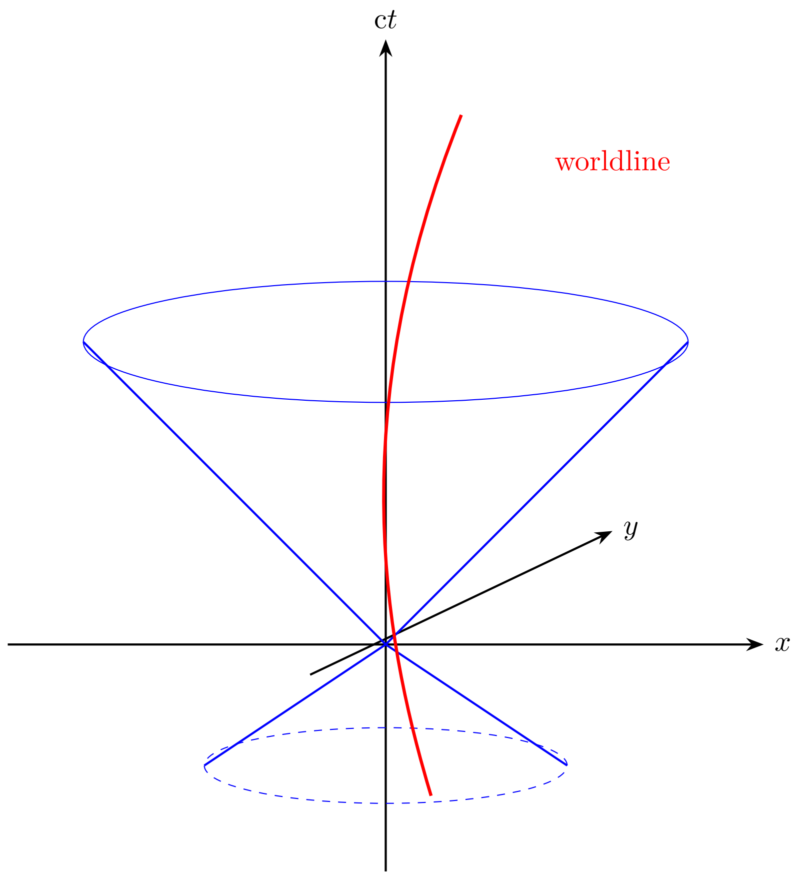

Figure 2.3: A worldline representing the trajectory of a particle through spacetime.

The trajectory of a particle through spacetime is denoted by a "**worldline**", see see Fig. (2.3)..

It must always have an angle $\alpha < \frac{\pi}{4}$ with respect to the ct-axis–i.e. "speed" < c.

Comparison with Euclidean Distance

The distance between points, r_{12}^2, was expressible as a scalar product (Galilean invariant):

$$r_{12}^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \begin{pmatrix} \Delta x & \Delta y & \Delta z \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{2.23}$$

For the invariant interval:

$$s_{12}^2 = -(\Delta \mathrm{c}t)^2 + (\Delta x)^2 = \begin{pmatrix} -\Delta \mathrm{c}t & \Delta x \end{pmatrix} \begin{pmatrix} \Delta \mathrm{c}t \\ \Delta x \end{pmatrix} \tag{2.24}$$

Note the minus sign–so this is not a standard scalar product of a vector with itself.

2.7 The Minkowski Metric

However, this may be written as:

$$s_{12}^2 = \begin{pmatrix} \Delta \mathrm{c}t & \Delta x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta \mathrm{c}t \\ \Delta x \end{pmatrix} \tag{2.25}$$

where the matrix

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.26}$$

is the *Minkowski metric* in 2 dimensions.

The generalisation to 4 dimensions is:

$$\boxed{\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \tag{2.27}$$

So we can write:

$$s_{12}^2 = \Delta x \cdot \eta \cdot \Delta x \tag{2.28}$$

(No bold in the relativistic case denoting scalar products.) This notation (using η) is cumbersome and inconvenient.

Why Not Use Imaginary Time?
Note that

$$\begin{pmatrix} -\mathrm{c}t & x \end{pmatrix} \begin{pmatrix} \mathrm{c}t \\ x \end{pmatrix} \tag{2.29}$$

could be rewritten as:

$$\begin{pmatrix} \mathrm{ic}t & x \end{pmatrix} \begin{pmatrix} \mathrm{ic}t \\ x \end{pmatrix} \tag{2.30}$$

restoring symmetry without η. (So no need for distinction between (see later) x^μ and x_μ.)

Several older books use this device, **but** almost never in General Relativity, so we will not use it here.