9 Tensors (Cartesian/Euclidean)

In this section, we construct coordinate derivatives in the context of summation convention and then turn to the representation of objects, tensors, which are associated with n directions as generalisations of vectors which are associated with $n = 1$, *one* direction. We will find in the next section that the relativistic object describing electric and magnetic fields is a tensor with $n = 2$ and in general relativity, spacetime curvature is represented by an object with $n = 4$.

9.1 Derivatives

We will consider only the case where the basis vectors are independent of position (i.e. Cartesian coordinates). We will initially deal with an orthonormal basis: $\{\hat{\mathbf{e}}_i\}$, with $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$

A simple case is the the gradient of a scalar field, $\phi(\mathbf{r})$,

$$\nabla\phi = \hat{\mathbf{e}}_i \nabla_i \phi \ ,$$

using summation convention.

Turning to derivatives of vector fields, consider the vector field

$$\mathbf{A}(\mathbf{r}) = A^i(\mathbf{r})\hat{\mathbf{e}}_i \quad \text{and} \quad \nabla = \hat{\mathbf{e}}_i \partial_i \ .$$

Then, for example (suppressing the explicit dependence on $\mathbf{r}$):

$$\nabla \cdot \mathbf{A} = \hat{\mathbf{e}}_i \partial_i \cdot \hat{\mathbf{e}}_j A_j = \delta_{ij} \partial_i A_j = \partial_i A_i \ . \tag{9.1}$$

Note if had used, for example, a polar basis, then the derivative would differentiate the basis vectors as well as the components.

But summation convention allows construction of more complicated objects:

Example: $\nabla(\mathbf{A} \cdot \mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ are vector fields.

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= \hat{\mathbf{e}}_i \partial_i \left(A_j \hat{\mathbf{e}}_j \cdot B_k \hat{\mathbf{e}}_k \right) \\
&= \hat{\mathbf{e}}_i \partial_i \left(A_j \, \delta_{jk} \, B_k \right) \\
&= \hat{\mathbf{e}}_i \partial_i \left(A_j B_j \right) \\
&= \hat{\mathbf{e}}_i \left(B_j \partial_i A_j + A_j \partial_i B_j \right) && (9.2) \\
\Rightarrow \quad [\nabla(\mathbf{A} \cdot \mathbf{B})]_i &= B_j \partial_i A_j + A_j \partial_i B_j && (9.3)
\end{aligned}$$

This example shows how straightforward summation convention is here – with an answer that is difficult to write in terms of vector notation.

So with derivatives:

(i) The summation convention keeps track of vector aspects.

(ii) Just use ordinary calculus rules for respecting the order of derivatives and functions, differentiating products and the chain rule.

But what about *curls* and cross products?

9.2 The Levi-Civita Symbol

With an orthonormal basis, note the cyclic symmetric of the suffices:

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2, \quad \text{etc.} \tag{9.4}$$

This motivates the definition of the **Levi-Civita** symbol or tensor (or **alternating 3-symbol**):

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an } \textbf{even} \text{ permutation of } 123 \\ -1 & \text{if } ijk \text{ is an } \textbf{odd} \text{ permutation of } 123 \\ 0 & \text{otherwise (e.g. } 112) \end{cases} \tag{9.5}$$

Here "even" means an even number of adjacent transpositions (ATs). For example:

$$\underbrace{12}_{AT}3 \to 2\underbrace{13}_{AT} \to 231 \quad \Rightarrow \quad 2 \text{ ATs (even)}. \tag{9.6}$$

"Odd" means an odd number of adjacent transpositions. For example:

$$\underbrace{12}_{AT}3 \to 213 \quad \Rightarrow \quad 1 \text{ AT (odd)}. \tag{9.7}$$

Note that **cyclic permutations** of 123:

$$123 \to 312 \to 231 \tag{9.8}$$

are **even** for three objects. But when we come to the relativistic counterpart, a cyclic permutation of the four indices is **odd**. Also we will occasionally use the 2d version

$$\varepsilon_{ij} = \begin{cases} +1 & \text{if } ij \text{ is } 12 \\ -1 & \text{if } ij \text{ is } 21 \\ 0 & \text{otherwise (e.g. } 11) \end{cases} \tag{9.9}$$

Note this is consistent with definition of odd and even permutations: as in 4d a cyclic permutation is *odd*.

9.3 Examples and Properties of ε_{ijk}

Examples:

(i) **Examples of the values of components of ε_{ijk}**,

$$\varepsilon_{123} = 1\ , \quad \varepsilon_{213} = -1\ , \quad \varepsilon_{113} = 0 = \varepsilon_{111}\ . \tag{9.10}$$

(ii) **Orthonormal basis $\{\hat{\mathbf{e}}_i\}$**:

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk}\,\hat{\mathbf{e}}_k. \tag{9.11}$$

For example: $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \varepsilon_{12k}\,\hat{\mathbf{e}}_k = \varepsilon_{123}\,\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$ (NB: only $k = 3$ is non-zero). (This assumes a **right-handed** basis.)

(iii) **Cross product:**

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= a_i \hat{\mathbf{e}}_i \times b_j \hat{\mathbf{e}}_j \\ &= a_i b_j \, \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \\ &= a_i b_j \, \varepsilon_{ijk} \, \hat{\mathbf{e}}_k \end{aligned} \tag{9.12}$$

$$(\mathbf{a} \times \mathbf{b})_k = \varepsilon_{ijk} \, a_i b_j = \varepsilon_{kij} a_i b_j \ , \tag{9.13}$$

where we used cyclic symmetry in the last equality.

(iv) **Antisymmetry:**

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = +\varepsilon_{kij} \tag{9.14}$$

Since if ijk is an even permutation of 123, then ikj must be odd, and kij is even.

So in **3** dimensions, **cyclic permutations** leave ε_{ijk} invariant:

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}. \tag{9.15}$$

Example: Using $\varepsilon_{ijk} = \varepsilon_{kij}$:

$$(\mathbf{a} \times \mathbf{b})_k = \varepsilon_{ijk} \, a_i b_j = \varepsilon_{kij} \, a_i b_j. \tag{9.16}$$

(v) **Curl:**

$$\begin{aligned} \nabla \times \mathbf{A} &= \hat{\mathbf{e}}_i \partial_i \times A_j \hat{\mathbf{e}}_j \\ &= \partial_i A_j \, \varepsilon_{ijk} \, \hat{\mathbf{e}}_k \\ &= \varepsilon_{ijk} \, \partial_i A_j \, \hat{\mathbf{e}}_k \\ &= \varepsilon_{kij} \, \partial_i A_j \, \hat{\mathbf{e}}_k \end{aligned} \tag{9.17}$$

(Note: assumes Cartesian basis, but ε_{ijk} is constant.)

$$(\nabla \times \mathbf{A})_k = \varepsilon_{kij} \, \partial_i A_j \tag{9.18}$$

9.4 Two Important Identities

(1) Determinant identity: Denote the determinant of matrix a_{ij} as $|a|$. Then:

$$\varepsilon_{rst} \, |a| = \varepsilon_{ijk} \, a_{ri} \, a_{sj} \, a_{tk} \tag{9.19}$$

Note: ijk label **columns**, rst label **rows**.

The antisymmetry of ε_{ijk},

$$\varepsilon_{ijk} = -\varepsilon_{jik} \tag{9.20}$$

implies the change of sign of the determinant if rows/columns are interchanged. *Proof: next example sheet.*

Use of this identity:
Consider $a_{ij} = R_{ij}$, a rotation matrix. Then $|R| = 1$ as R_{ij} is orthogonal. So (with $|R| = 1$):

$$\varepsilon_{rst} = \varepsilon_{ijk} \, R_{ri} \, R_{sj} \, R_{tk}. \tag{9.21}$$

Thus ε is **unchanged** under rotations—it is "**isotropic**".

But with a **reflection** $P^-_{ij} = \delta_{ij} - 2\hat{n}_i \hat{n}_j$ (see problem sheet 1), we have $|P^-| = -1$ (why?), so:

$$-\varepsilon_{rst} = \varepsilon_{ijk} \, P^-_{ri} \, P^-_{sj} \, P^-_{tk}. \tag{9.22}$$

(2) The most useful ε identity:

$$\varepsilon_{ijk}\,\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \tag{9.23}$$

(Proof: example sheet.)

Example: Vector Triple Product Consider $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$:

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_i \hat{\mathbf{e}}_i \times \left(B_j \hat{\mathbf{e}}_j \times C_k \hat{\mathbf{e}}_k\right) \\
&= A_i B_j C_k\, \hat{\mathbf{e}}_i \times \left(\varepsilon_{jk\ell}\, \hat{\mathbf{e}}_\ell\right) \\
&= A_i B_j C_k\, \varepsilon_{jk\ell}\, (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_\ell) \\
&= A_i B_j C_k\, \varepsilon_{jk\ell}\, \varepsilon_{i\ell m}\, \hat{\mathbf{e}}_m .
\end{aligned} \tag{9.24}$$

So the mth component of $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is:

$$\begin{aligned}
\left[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\right]_m &= A_i B_j C_k\, \varepsilon_{jk\ell}\, \varepsilon_{i\ell m} \\
&= A_i B_j C_k\, \varepsilon_{jk\ell}\, \varepsilon_{mi\ell} \quad \text{(cyclic)} \\
&= A_i B_j C_k \left(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km}\right) \\
&= A_i B_m C_i - A_i B_i C_m \\
&= (\mathbf{A} \cdot \mathbf{C}) B_m - (\mathbf{A} \cdot \mathbf{B}) C_m .
\end{aligned} \tag{9.25}$$

Therefore, assembling these components into vectors, we see :

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{9.26}$$

Use $\varepsilon\varepsilon = \delta\delta - \delta\delta$ repeatedly for $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$, etc.

Consequences of the $\varepsilon\varepsilon$ Identity
From $\varepsilon_{ijk}\,\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ and cyclic permutation:

$$\varepsilon_{kij}\,\varepsilon_{k\ell m} = \varepsilon_{ijk}\,\varepsilon_{k\ell m} \tag{9.27}$$

Setting $\ell = i$:

$$\begin{aligned}
\varepsilon_{kij}\,\varepsilon_{kim} &= \delta_{ii}\delta_{jm} - \delta_{im}\delta_{ji} \\
&= 3\delta_{jm} - \delta_{jm} \quad \text{(in 3D)} \\
&= 2\delta_{jm}
\end{aligned} \tag{9.28}$$

Setting $j = m$:

$$\varepsilon_{kij}\,\varepsilon_{kij} = 2\delta_{jj} = 6. \quad \text{(in 3D)} \tag{9.29}$$

9.5 Definition of tensors

The components of *tensor* are a collection of numbers arranged in an array in n dimensions, with defined transformation properties under a change of basis. n is called the *rank* of the tensor. Examples we have already encountered are: $n = 1$, components of a vector, A_i; $n = 2$, δ_{ij}; $n = 3$, ε_{ijk}. Physical examples will be given presently. The subscripts run over the same range, $1 \le i \le N$, here we will be concerned with $N = 3$, non-relativistically, and $N = 4$ for relativistic applications.

For non-relativistic applications, the transformation is a rotation. for example if we consider a rank-3 tensor with components are T_{ijk}, for example then the tranformed components are

$$T'_{\ell mn} = R_{\ell i} R_{mj} R_{nk} T_{ijk} \ .$$

For $N \geq 3$ it is not natural to rewrite in a manner, familiar expression from matrix multiplication, for the components M_{ij} for $N = 2$

$$M'_{k\ell} = R_{ki} R_{\ell j} M_{ij} = R_{ki} M_{ij} R^{\mathrm{T}}_{j\ell} \ .$$

For the relativistic case Lorentz transformation replaces rotation.

One may readily check that one may add tensors and the sum also behaves like a tensor under rotations.

A "rank-2" example which **is** a matrix, M_{ij}, but **not** a tensor is that where the rows correspond to students and the columns correspond to modules, with a specific entry, M_{ij}, corresponding to the mark of student i in module j. Even if we had the accident of the number of students being equal to the number of modules, there is no sensible general transformation of this matrix (apart from a rotation of $\pi/2$ to interchange rows and columns).

We have been slack (and will continue to be slack) using the word "tensor" to mean components of a tensor. We can construct the object, of rank > 1, which does not change under a change of basis in the same manner as we constructed a vector from its components. For example if we have the objects with components T_{ij} or S_{ijk} we can form:

$$\mathbf{T} = \hat{\mathbf{e}}_i T_{ij} \hat{\mathbf{e}}_j \quad \text{and} \quad \mathbf{S} = \hat{\mathbf{e}}_i S_{ijk} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \ .$$

Note there are no scalar products between the basis vectors in these definitions; however one can perform "matrix multiplication" by

$$\begin{aligned} \mathbf{T} \cdot \mathbf{U} &= \hat{\mathbf{e}}_i T_{ij} \underbrace{\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k}_{=\delta_{jk}} U_{k\ell} \hat{\mathbf{e}}_\ell \\ &= \hat{\mathbf{e}}_i T_{ik} U_{k\ell} \hat{\mathbf{e}}_\ell \\ &= \hat{\mathbf{e}}_i (TU)_{i\ell} \hat{\mathbf{e}}_\ell \ . \end{aligned}$$

And one may define a matrix counterpart "scalar product" to that between vectors, where now both basis vectors in each tensor are involved, in a cyclic manner – the last basis vector undergoes a scalar product with the first basis vector.

$$\begin{aligned} \mathbf{T} \ : \ \mathbf{U} &= \underbrace{\hat{\mathbf{e}}_\ell \cdot \hat{\mathbf{e}}_i}_{=\delta_{\ell i}} T_{ij} \underbrace{\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k}_{=\delta_{jk}} U_{k\ell} \\ &= \delta_{\ell i} \delta_{jk} T_{ij} U_{k\ell} \\ &= T_{ij} U_{ji} \ . \end{aligned}$$

But suffix notation allows for entities which cannot readily be depicted in a linear fashion, such as $S_{ijk} a_i b_j c_k$.

9.5.1 The quotient rule

The *quotient rule* is a device for deducing some entities are tensors by the context in equations involving other objects known to be tensors. It applies to tensors of arbitrary rank but we will discuss it for rank-2 for simplicity.

We are told that for *any* rank-1 tensor (vector) b_j the following object, a_i, is a vector or rank-1 tensor

$$a_i = C_{ij} b_j \ , \tag{9.30}$$

and we wish to show that the quantities C_{ij} are components of a tensor. This is called the *quotient rule.*

Let us define the quantities measured in a different coordinate system to be

$$a'_i = C'_{ij} b'_j \ , \tag{9.31}$$

Then by multiplying both sides of Eq. (9.30) by R_{ki}, the rotation matrix linking the two coordinate systems, we see that

$$\begin{aligned} R_{ki} a_i &= R_{ki} C_{ij} b_j \\ \Rightarrow \quad R_{ki} a_i &= R_{ki} C_{ij} \delta_{j\ell} b_\ell \\ \Rightarrow \quad R_{ki} a_i &= R_{ki} C_{ij} R^{-1}_{j\ell} R_{\ell m} b_m \\ \Rightarrow \quad a'_i &= R_{ki} C_{ij} R^{-1}_{j\ell} b'_\ell \ . \end{aligned} \tag{9.32}$$

Now subtract Eq. (9.32) from Eq. (9.31), adjusting indices appropriately, to obtain

$$\begin{aligned} 0 &= \left(C'_{ij} \delta_{j\ell} - R_{ki} C_{ij} R^{-1}_{j\ell} \right) b'_\ell \\ \text{(Since } b'_\ell \text{ is arbitrary.)} \quad \Rightarrow \quad 0 &= C'_{ij} \delta_{j\ell} - R_{ki} C_{ij} R^{-1}_{j\ell} \\ \Rightarrow \quad C'_{i\ell} &= R_{ki} C_{ij} R^{-1}_{j\ell} \ . \end{aligned}$$

I.e. C_{ij} transforms as a rank-2 tensor.

9.5.2 Physical examples of Cartesian tensors − non-examinable

Here we restrict ourselves to nonrelativistic examples.

(i) **Inertia tensor**. See problem sheet 2. Let there be N point masses, $m^{(n)}$, $n = 1, 2, \cdots, N$ in a rigid body with positions $\mathbf{r}^{(n)}$. The rigid body is rotating steadily such that $\dot{\mathbf{r}}^{(n)} =$

$\boldsymbol{\omega} \times \mathbf{r}^{(n)}$. The total angular momentum of the particles is

$$
\begin{aligned}
\mathbf{L} &= \sum_{n=1}^{N} m^{(n)} \mathbf{r}^{(n)} \times \dot{\mathbf{r}}^{(n)} \\
&= \sum_{n=1}^{N} m^{(n)} \mathbf{r}^{(n)} \times \left(\boldsymbol{\omega} \times \mathbf{r}^{(n)}\right) \\
\Rightarrow \quad L_i &= \sum_{n=1}^{N} m^{(n)} \varepsilon_{ijk} r_j^{(n)} \varepsilon_{k\ell m} \omega_\ell r_m^{(n)} \\
\text{(Use identity.)} \qquad &= \sum_{n=1}^{N} m^{(n)} \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right) r_j^{(n)} \omega_\ell r_m^{(n)} \\
&= \sum_{n=1}^{N} m^{(n)} \left(r_j^{(n)} r_j^{(n)} \delta_{i\ell} - r_i^{(n)} r_\ell^{(n)}\right) \omega_\ell \\
&= \mathcal{I}_{i\ell}\, \omega_\ell \ ,
\end{aligned}
$$

where the *inertia tensor*, $\mathcal{I}_{ij}$, is

$$
\mathcal{I}_{ij} = \sum_{n=1}^{N} m^{(n)} \left(r_k^{(n)} r_k^{(n)} \delta_{ij} - r_i^{(n)} r_j^{(n)}\right) \ .
$$

For the continuous case,

$$
\mathcal{I}_{ij} = \int_V \mathrm{d}^3\mathbf{r}\ \rho(\mathbf{r}) \left(r_k r_k \delta_{ij} - r_i r_j\right) \ .
$$

(ii) **Strain tensor**. Consider a deformable body, see Fig. (9.1), which is deformed from its initial equilibrium with the position of the material at $\mathbf{r}$ moving to:

$$
\mathbf{r} \to \mathbf{r} + \mathbf{u}(\mathbf{r})
$$

where $\mathbf{u}(\mathbf{r})$ is called the *displacement field*. A neighbouring point

$$
\mathbf{r} + \mathrm{d}\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r} + \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) \ ,
$$

and in index notation the components are approximately

$$
u_i(\mathbf{r} + \mathrm{d}\mathbf{r}) \simeq u_i(\mathbf{r}) + \mathrm{d}r_j \nabla_j u_i(\mathbf{r}) \ .
$$

Now consider how these two points move relative to each other:

$$
\begin{aligned}
(r_i + \mathrm{d}r_i) - r_i &\to \left((r_i + \mathrm{d}r_i + u_i(\mathbf{r}) + \mathrm{d}r_j \nabla_j u_i(\mathbf{r})\right) - (r_i + u_i(\mathbf{r})) \\
\Rightarrow \quad \mathrm{d}r_i &\to \mathrm{d}r_i + \mathrm{d}r_j \nabla_j u_i(\mathbf{r}) \ ,
\end{aligned}
$$

The strain tensor,

$$
e_{ij} = \nabla_j u_i = \frac{\partial u_i}{\partial x_j}
$$

characterises the relative displacement, or strain, in the material. The two “directions” associated with the tensor are the displacement field $u_i(\mathbf{r})$, and the separation of the two initial points, $\mathrm{d}r_i$.

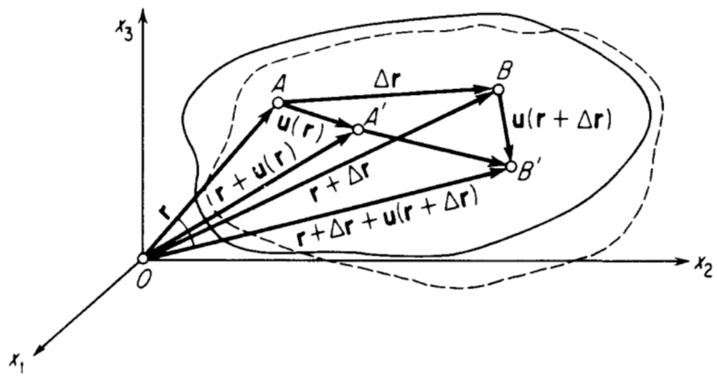

Figure 9.1: Deformation of neighbouring points, illustrating the displacement field, $\mathbf{u}(\mathbf{r})$.

(iii) **Stress tensor**. The *stress tensor*, σ_{ij}, describes the forces on the surfaces of an element of the medium. For simplicity consider a cube, with surface elements of integration being $\mathrm{d}\mathbf{S}^{(\alpha)} = \mathrm{d}S\,\hat{\mathbf{n}}^{(\alpha)}$, with $\alpha = x, y$ and z for a cube aligned with the coordinate axes. Then the force on face α is $f_i^{(\alpha)} = \sigma_{ij}\mathrm{d}S\,\hat{n}_j^{(\alpha)}$. More on this topic in *Vector and tensor analysis with applications*, A.I. Borisenko and I.E. Tarapov, Dover (1979).

(iv) The relationship between the strain and stress tensors involves quantities which generalise Young's modulus:

$$\sigma_{ij} = C_{ijk\ell}e_{k\ell}\ .$$

For an isotropic medium

$$C_{ijk\ell} = K\delta_{k\ell}\delta_{ij} + 2\mu\left(\delta_{ki}\delta_{\ell j} - \tfrac{1}{3}\delta_{ij}\delta_{k\ell}\right)$$

where K is the *bulk modulus* and μ is the *shear modulus.* A nice account of elasticity, and the use of tensors, is in chapter 39, volume II, of the Feynman Lectures in Physics.