Evolution of Cosmic Structure

Lecture 4 - Overdensity and Gravitational instability

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Learning Outcomes: This Section

How did the small fluctuations visible in the CMB grow to the point where gravitationally bound structures could form?

- Characterising the fluctuations – the overdensity and its power spectrum
- Gravitational instablility in static and expanding media
- The effects of pressure – Jeans mass
- The initial power spectrum and its evolution
- The behaviour of the baryons

§ 3 – Linear growth of perturbations

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Overdensity

We will closely follow the treatment in Ryden Chapter 12 throughout section 3 of the course, you should consult the book for further detail.

We characterize the overdensity at position **r** and time t by $\delta(\mathbf{r},t) \equiv \dfrac{\rho(\mathbf{r},t)-\bar{\rho}(t)}{\bar{\rho}(t)}$

$\delta(\mathbf{r},t) \longleftarrow$ Overdensity

$\delta(\mathbf{r},t) \longleftarrow$ Negative if underdense; minimum value of -1

$\delta(\mathbf{r},t) \longleftarrow$ Positive if overdense; no maximum

$\delta(\mathbf{r},t) \ll 1$ Linear growth This week

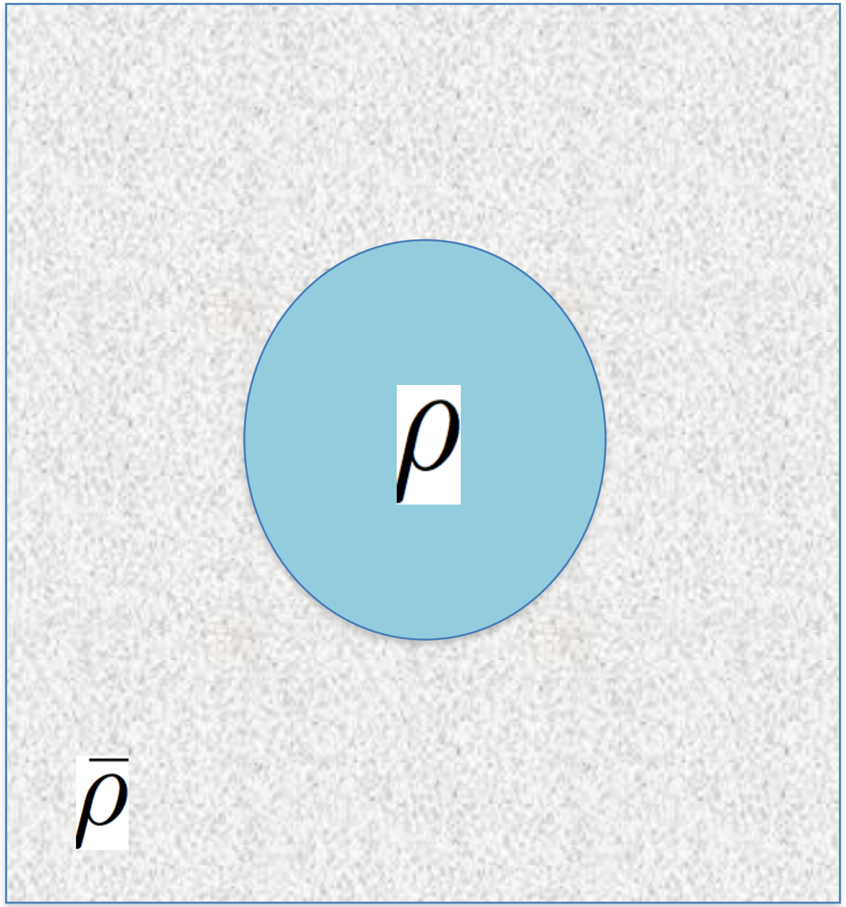
$\delta(\mathbf{r},t) \gtrsim 1$ Non-Linear growth next week

$\delta(\mathbf{r},t) \gg 1$ Very Non-Linear growth computer simulations week 5

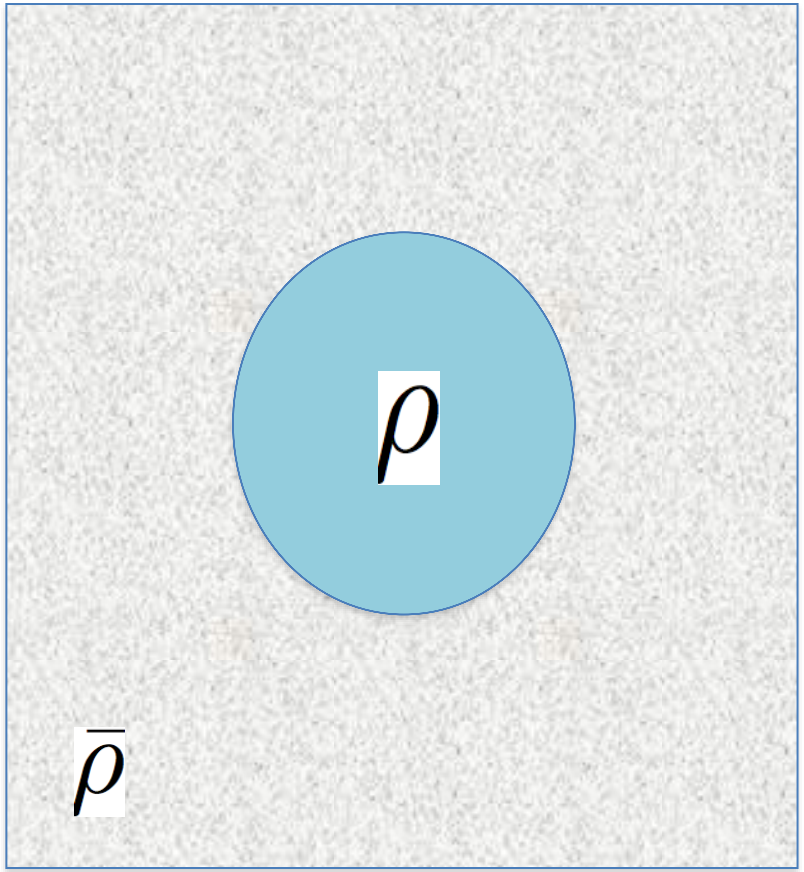
The linear growth of structure

Consider a small positive fluctuation in density in a spherical region which lies within a uniform, static medium without any pressure. Its density is $\rho(t) = \bar{\rho}(t)[1 + \delta(t)]$, where $\delta \ll 1$.

This is the regime where linear perturbation theory is relevant. And useful for the growth of superclusters and other large-scale structures. Not clusters, galaxies or people.



Gravitational instability



We will examine the gravitational instability of δ in a static medium.

Our world – but no opposing pressure

$\bar{\rho}$ Uniform density

ρ Density of the sphere

R Radius of the sphere

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} \quad \Longrightarrow \quad \rho = \bar{\rho}\,[1 + \delta]$$

$\delta(\mathbf{r}, t) \ll 1$ *Linear regime*

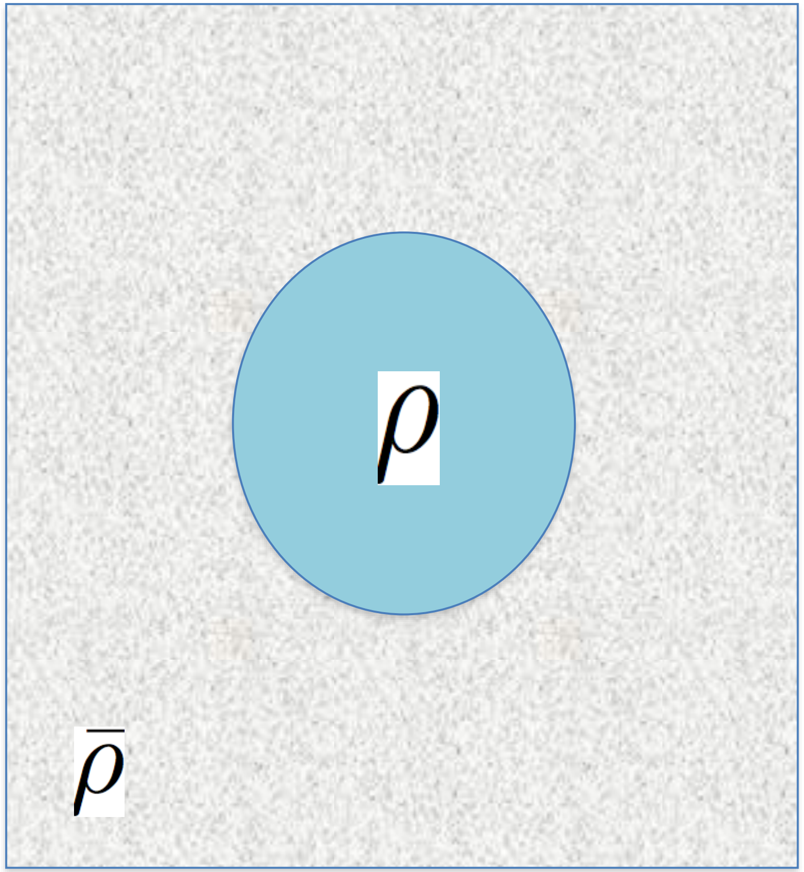
The equation of motion for this system is

$$\ddot{R} = -\frac{G\,\Delta M}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right)$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t)$$

Two unknowns, need another equation

Gravitational instability



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Our world – but no opposing pressure

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Consider the conservation of mass in the perturbation

$$M = \frac{4\pi}{3}\,\bar{\rho}[1+\delta(t)]\;R(t)^3$$

$$R(t) = R_0[1+\delta(t)]^{-1/3} \qquad R_0 \equiv \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3}$$

But $\delta \ll 1$, so

$$R(t) = R_0\left[1 - \frac{1}{3}\delta(t)\right] \quad \Longrightarrow \quad \ddot{R} = -\frac{1}{3}R\;\ddot{\delta}(t)$$

Second equation for the system;
2 unknowns, 2 equations

Gravitational instability

Putting the two equations together, results in this differential equation

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta$$

Which has the solution

$$\delta(t) = Ae^{t/t_{dyn}} + Be^{-t/t_{dyn}}$$

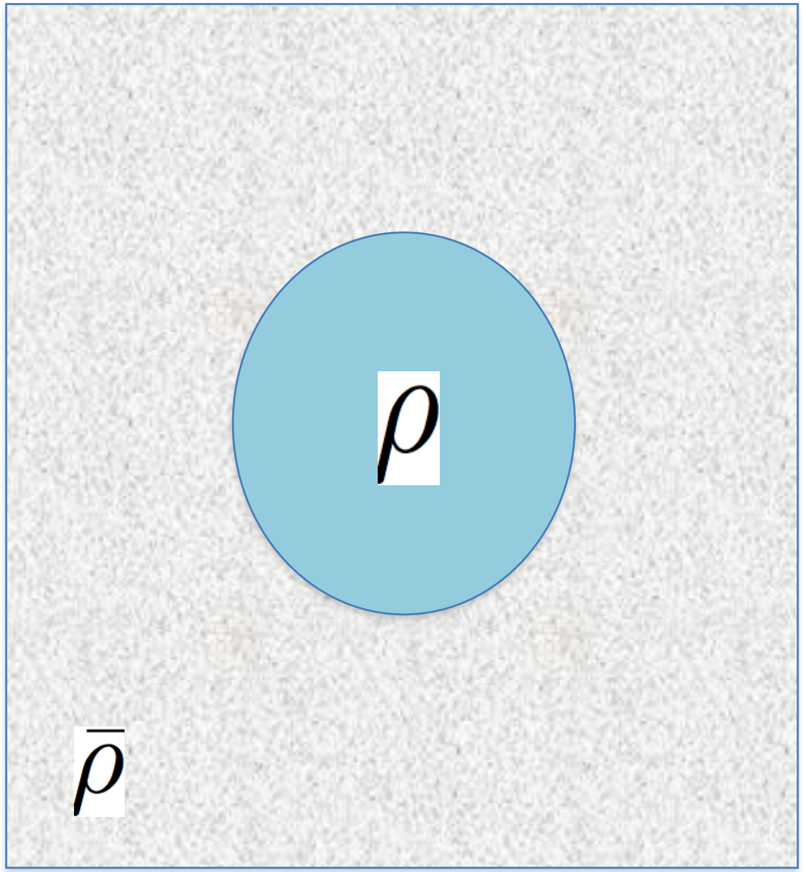
Where the dynamical time is given by

$$t_{dyn} = \frac{1}{(4\pi G\bar{\rho})^{1/2}}$$

But, with increasing time, the second term of the solution goes to zero

$$\boxed{\delta(t) = Ae^{t/t_{dyn}}}$$

Static medium solution for a perturbation in the linear regime



Gravitational instability in expanding medium

Now, let's consider the perturbation in an expanding medium instead.

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3}G\bar{\rho}R - \frac{4\pi}{3}G\bar{\rho}\delta R$$

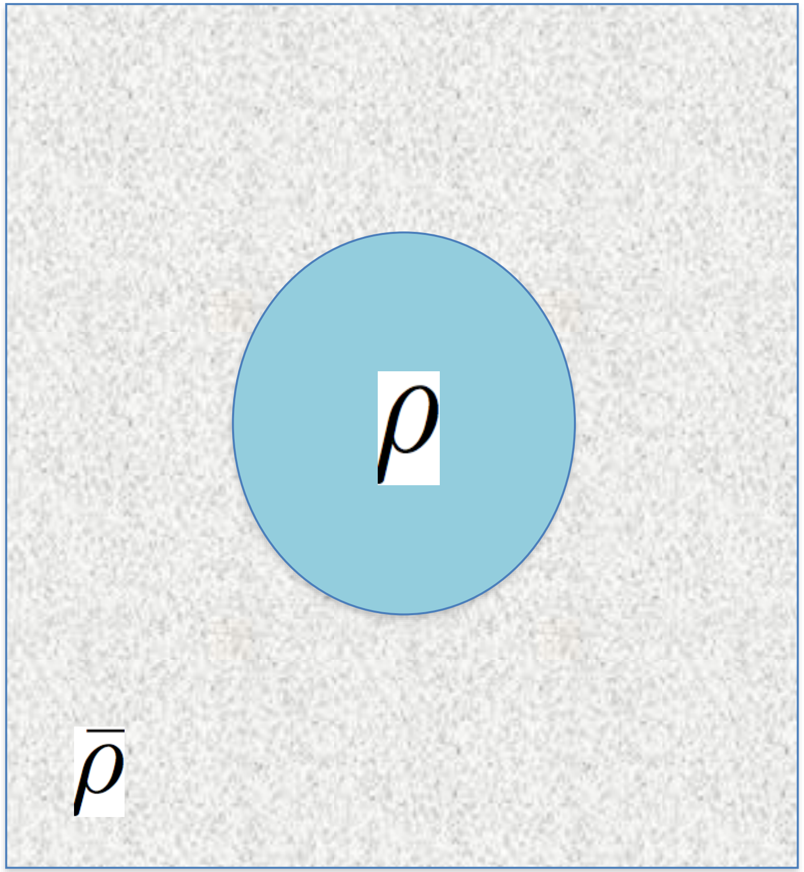
Mass conservation of the perturbation (like static case):

$$M = \frac{4\pi}{3}\ \bar{\rho}[1+\delta(t)]\ R(t)^3$$

In a matter dominated case, this leads to:

$$\rho(t) \propto a^{-3} \propto (1+z)^3$$

$$R(t) \propto a(t)[1+\delta(t)]^{-1/3}$$



Gravitational instability in expanding medium

For the matter dominated case, the equation becomes

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta \; -2H\dot{\delta}$$

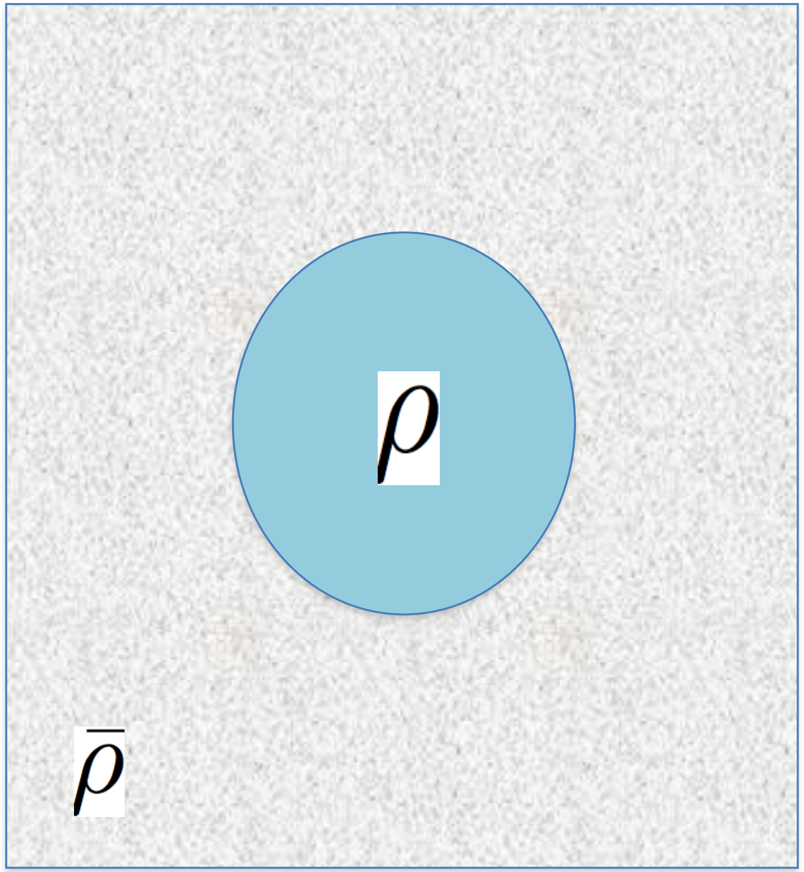
Hubble Drag

We know how H depends on the evolution of the Universe. Recall:

$$\Omega_m = \frac{\overline{\rho_m}}{\rho_c} = \rho_m \frac{8\,\pi G}{3\,H^2}$$

The equation of motion becomes

$$\ddot{\delta} = \frac{3}{2}\Omega_m H^2\delta \; - 2H\dot{\delta}$$



Gravitational instability in expanding medium

$$\ddot{\delta} = \frac{3}{2}\Omega_m H^2 \delta \; - 2H\dot{\delta}$$

We can solve this in a matter universe

$$\Omega_m = 1 \qquad\qquad H = \frac{2}{3t}$$

From the solutions to Freidmann

Substituting into the equation of motion

$$\ddot{\delta} = \frac{2}{3t^2}\delta \; - \frac{4}{3t}\dot{\delta}$$

Which has the solution (by inspection)

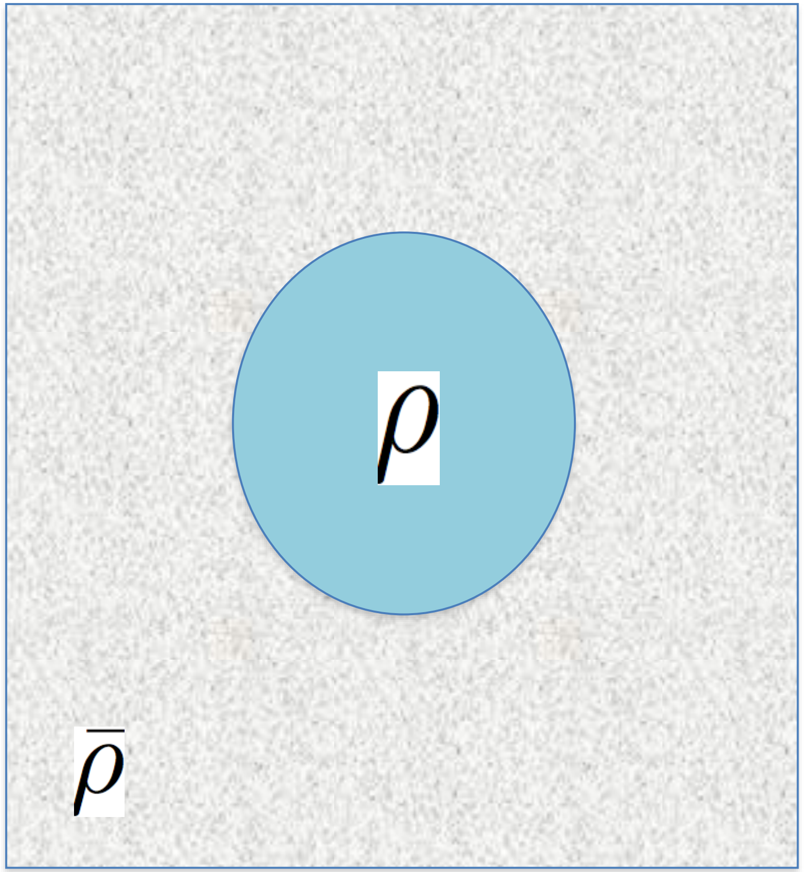
$$\delta(t) = At^{2/3} + Bt^{-1}$$

Second term again vanishes with time

$$\boxed{\delta(t) = At^{2/3} \propto a(t) \propto \frac{1}{1+z}}$$

Solution to a linear perturbation in a matter-dominate universe

Scale free!!!



Gravitational instability in expanding medium

Matter-dominated

$$\delta(t) = At^{2/3} \propto a(t) \propto \frac{1}{1+z}$$

Linear growth

Radiation-dominated

$$\delta(t) = B_1 + B_2 \ln(t)$$

Logarithmic growth

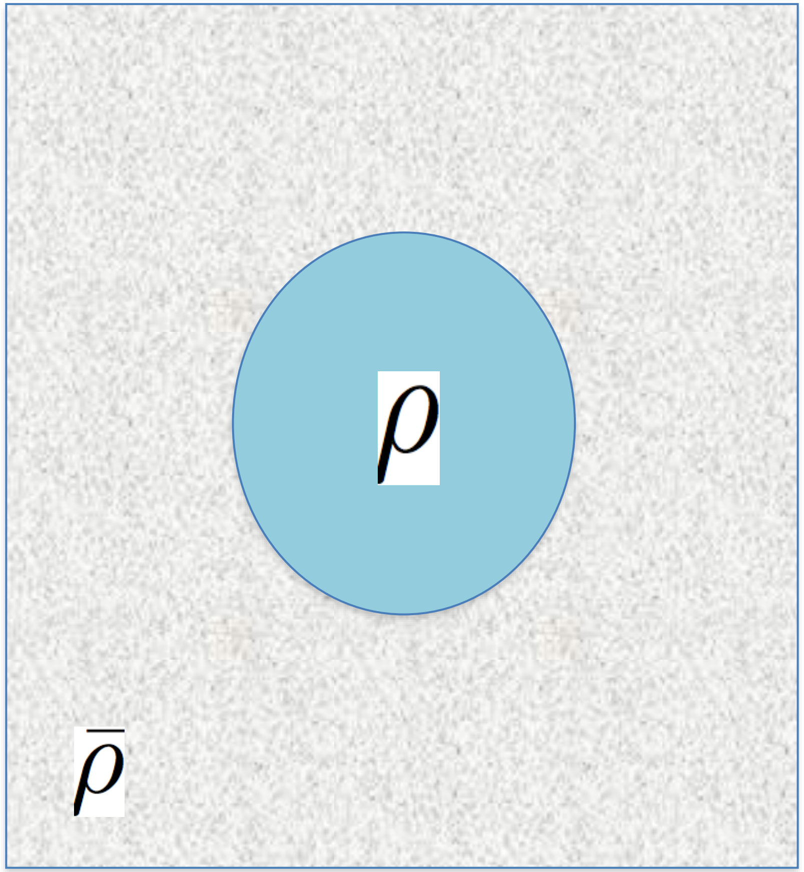
Cosmological-constant dominated

$$\delta(t) = C_1 + C_2 e^{-2H_\Lambda t}$$

$$H_\Lambda = \text{constant}$$

No growth!

Matter phase dominates the growth of structure



The role of pressure

The typical collapse time, or dynamical time, is

$$t_{dyn} = \frac{1}{\sqrt{4\pi G\bar{\rho}}}$$

But pressure can oppose the growth of structure of perturbations

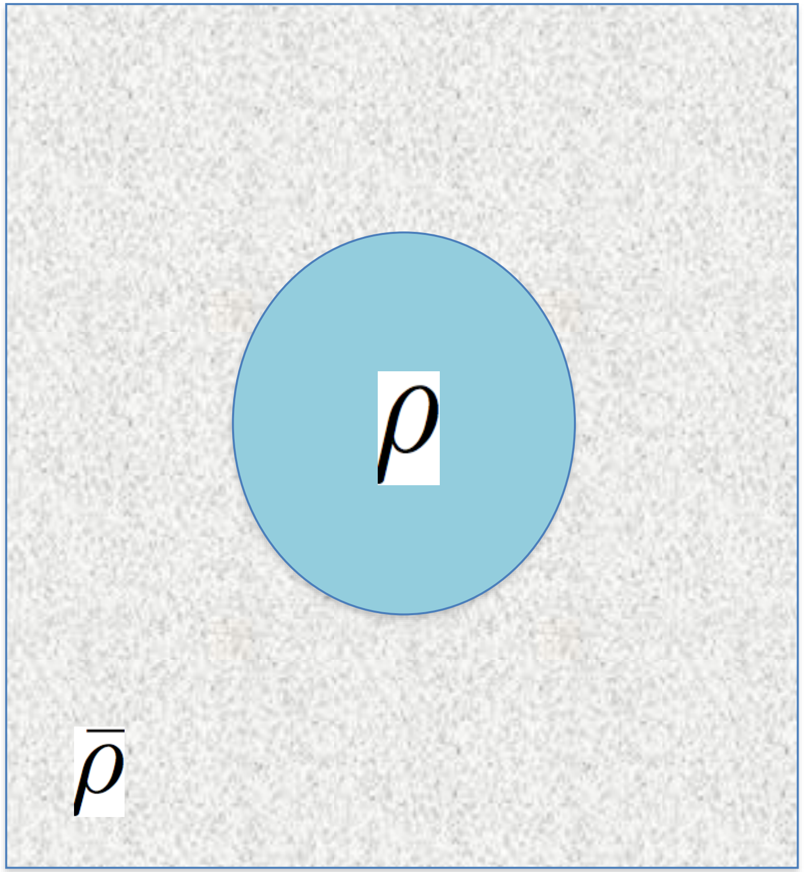
$$t_{pre} \sim \frac{R}{c_s} \qquad\qquad c_s = \sqrt{\frac{dP}{d\rho}}$$

Which leads to a Jeans length,

$$\Lambda_J = c_s t_{dyn}$$

And a Jeans Mass,

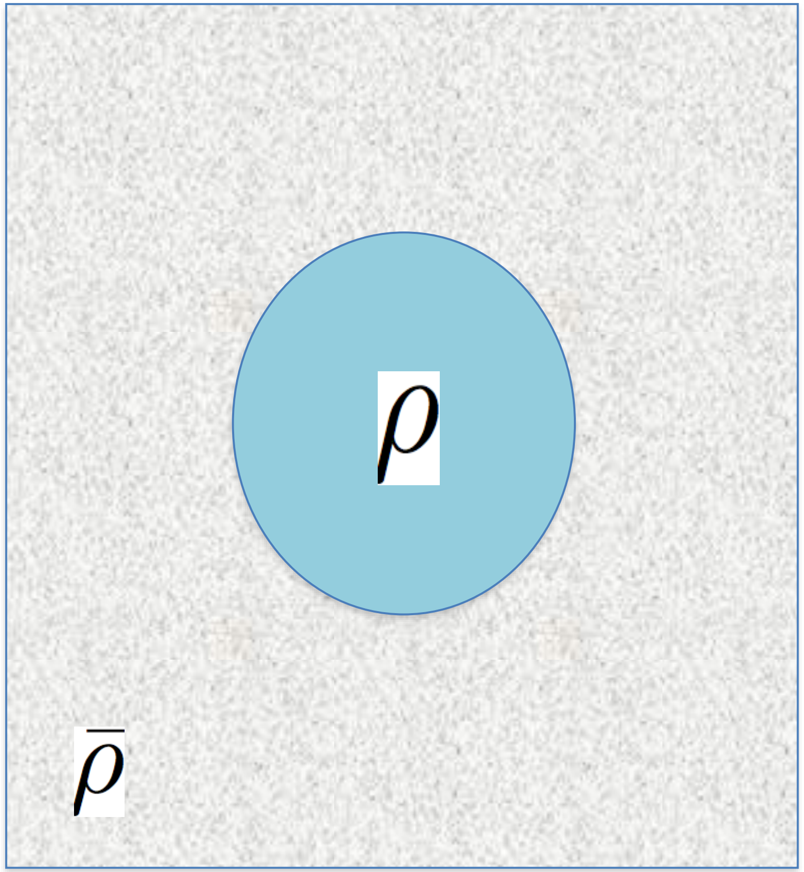
$$M_J = \frac{4\pi}{3}\Lambda_J^3\rho = \frac{4\pi}{3}c_s^3 t_{dyn}^3\rho$$



The role of pressure

The key point is then what the sound speed is

$$M_J = \frac{4\pi}{3}\Lambda_J^3\rho = \frac{4\pi}{3}c_s^3 t_{dyn}^3\rho$$



For a perfect gas,

$$P \propto \rho^\gamma$$

$$c_s = \frac{\gamma P}{\rho} = \frac{\gamma KT}{\mu m_H}$$

$$M_J \sim 10^5 M_\odot$$

For a radiation-dominated sound speed,

$$c_s = \frac{c}{\sqrt{3}}$$

$$M_J \sim 10^{18} M_\odot$$

Pressure stops all growth of baryon perturbations during the radiation dominated area

Overdensity and its power spectrum

$$\delta(\mathbf{r}, t) \equiv \frac{\rho(\mathbf{r}, t) - \bar{\rho}(t)}{\bar{\rho}(t)}$$

We will be interested in the way these fractional fluctuations vary with spatial scale, which can be characterised within some volume *V* of space by taking a Fourier Transform

$$\delta_{\mathbf{k}} = \frac{1}{V} \int_V \delta(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r$$

The Fourier coefficients provide a representation of δ(**r**) as a set of sine waves of vector wavenumber **k**,

$$\delta(\mathbf{r}) = \frac{V}{(2\pi)^3} \int \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3 k$$

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$$\delta_{\mathbf{k}} = \frac{1}{V}\int_V \delta(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

The Fourier coefficients provide a representation of $\delta(\mathbf{r})$ as a set of sine waves of vector wavenumber **k**,

$$\delta(\mathbf{r}) = \frac{V}{(2\pi)^3}\int \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3k$$

The $\delta_{\mathbf{k}}$ are complex numbers, $\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}|e^{i\phi_{\mathbf{k}}}$, and the mean power at a given wavelength can be obtained by averaging the amplitudes over all directions, giving $P(k) = <|\delta_{\mathbf{k}}|^2>$. The initial fluctuations emerging from inflation are usually assumed to constitute a *Gaussian Random Field*, whereby the phases (ϕ_k) of the Fourier components are random and uncorrelated. In this case, the Central Limit Theorem leads departures from the mean density on each spatial scale (e.g. mean density in randomly scattered boxes) to be Gaussian distributed.

Evolution of the power spectrum

As we discussed earlier, the relative amplitude of fluctuations on different spatial scales is typically characterised by the power spectrum, $P(k) = < |\delta_{\mathbf{k}}|^2 >$. What shape does this have, and how does it evolve?

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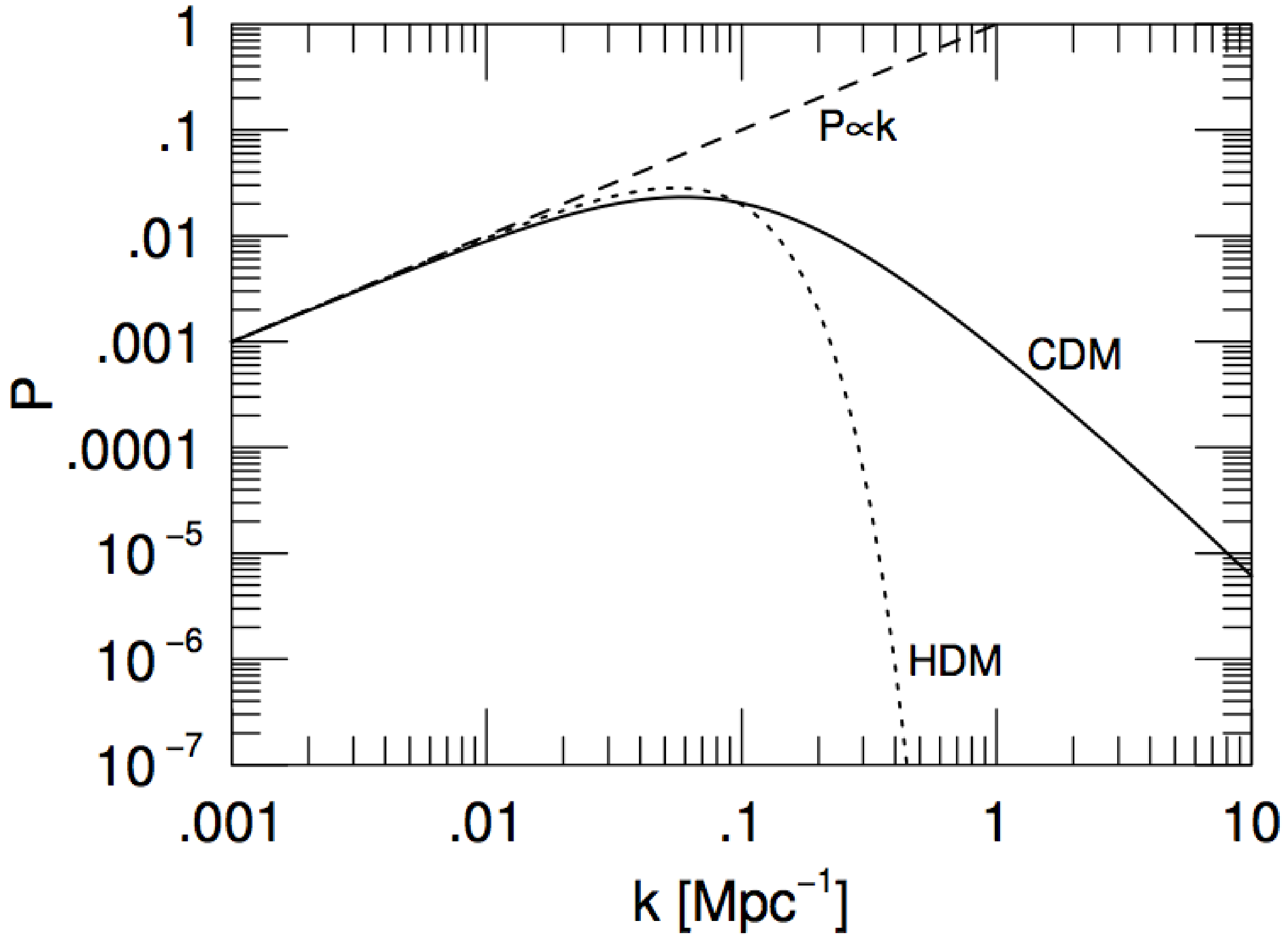

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Models for inflation generally predict that the spectrum of fluctuations which result should be "scale-free" (i.e. there is no preferred spatial scale), such that $P(k) \propto k^n$, and the favoured value of *n* is usually *n*=1, known as the Harrison-Zeldovich spectrum. What would such a spectrum mean in terms of the fluctuations on different *mass* scales? If *L* is a comoving spatial scale, and *k*=2π/*L* is the corresponding comoving wavenumber, then the mean mass within a sphere of comoving radius *L* is

$$< M >= \frac{4\pi}{3}(aL)^3\rho_m$$

but the actual mass found within a randomly chosen sphere of this size will scatter around <*M*> due to the perturbations.

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but the actual mass found within a randomly chosen sphere of this size will scatter around <*M*> due to the perturbations. The mean square amplitude of these fluctuations is related to *P(k)* as follows

$$\left\langle \left(\frac{M- < M >}{< M >} \right)^2 \right\rangle \propto k^3 P(k)$$

so the rms fractional fluctuations scale as

$$\frac{\Delta M}{M} \propto \sqrt{k^3 P(k)} \propto L^{-(3+n)/2} \propto M^{-(3+n)/6}$$

Evolution of the power spectrum

Hence for a Harrison-Zeldovich spectrum $\Delta M/M$ scales as $M^{-2/3}$, and so fluctuations on smaller mass scales have larger amplitude.

However, there are several factors at work which can modify this primordial spectrum in the time between the end of inflation (t~10^{-34}s) and the end of the radiation-dominated era at t~5×10^4 yr. Firstly recall (from § 2) that in the radiation-dominated era, the proper radius of the horizon expands as $d_{hor}=2ct=c/H$,reaching a size of ~0.03 Mpc by the onset of matter domination. In comoving units, this is $0.03/a_{rm}$, where $a_{rm}\approx 1/3570$ is the scale factor at radiation-matter equality. The corresponding wavenumber, in comoving units, is hence $k_{rm}\approx 2\pi a_{rm}/2ct\approx 0.06$ Mpc^{-1}.

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Perturbations on scales larger than the horizon can be shown to grow linearly with *a(t)* throughout, so at comoving wavenumbers <0.06 Mpc^{-1} (where fluctuation scales exceed the horizon size throughout the radiation-dominated era) we expect to see $P(k) \propto k$ (i.e. Harrison-Zeldovich). However, modes with $k>k_{rm}$ will have spent some time inside the horizon (increasingly longer for higher values of k), and will have had their growth restricted by one of several effects: