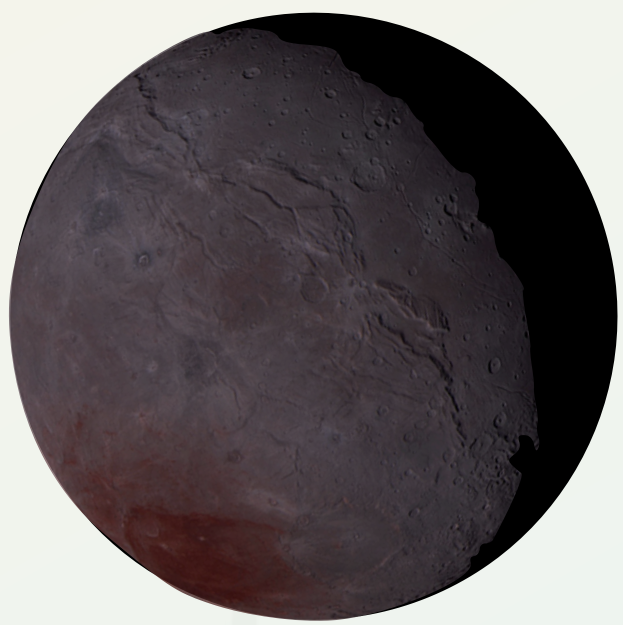
UNIVERSITY OF BIRMINGHAM

AMAURY TRIAUD
PHYSIC WEST G32
A.TRIAUD@BHAM.AC.UK
OFFICE HOURS: FRI 9-10AM

THE 2-BODY PROBLEM



Key Concepts

- The Two Body Problem
- Orbital parameters
- Deriving Kepler's laws using Newton's
- Vis-Viva Equation

The Two-Body Problem

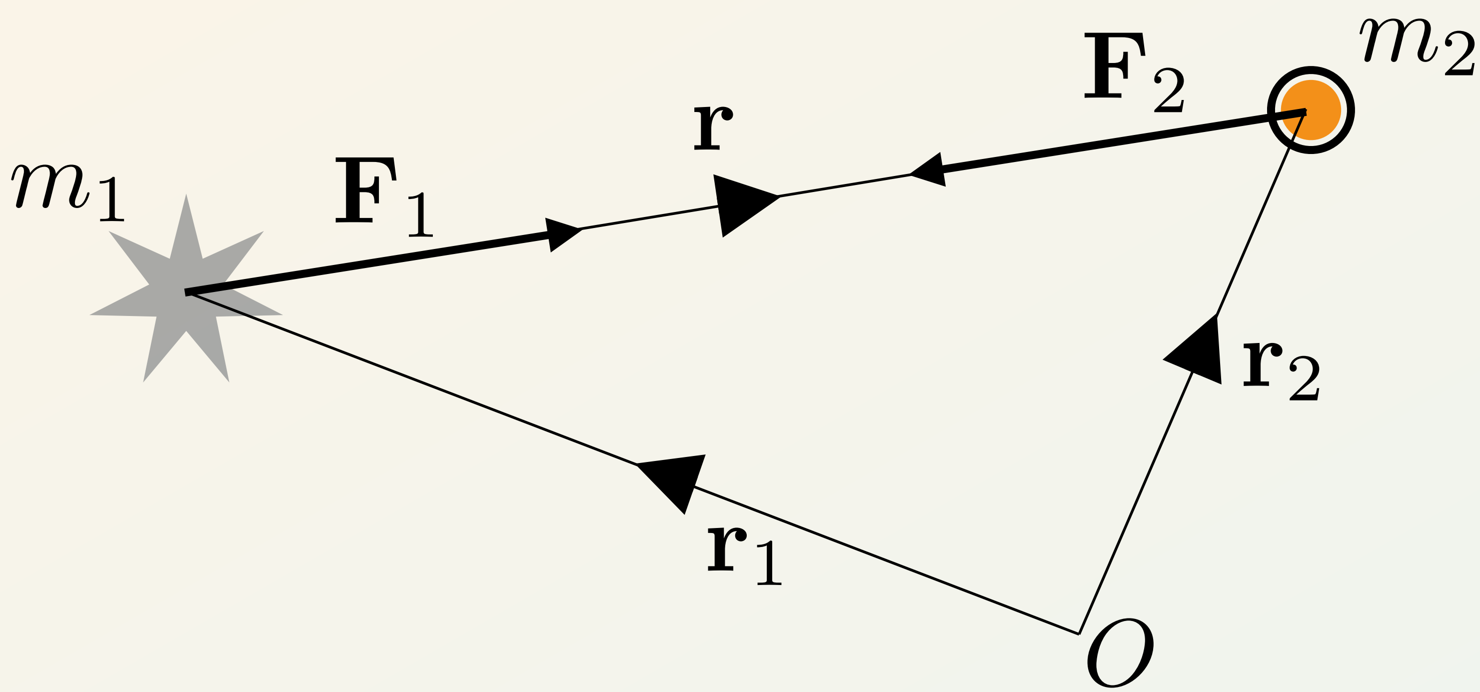
The Two Body Problem solves exactly the position and velocities of two objects that orbit one another in isolation.

However, the universe has more than two bodies (the Solar system has many as well). The Three-Body (or N-body) problem is unsolvable exactly, and its study led to the creation of Chaos Theory. The Solar system like most planetary systems is chaotic.

Parts of the Three-Body problem are solvable, if you assume $e=0$, or mutual inclinations $= 0$ etc.

Now what we will do is solve Newton's equations to recover Kepler's laws of planetary motion, which are at the heart of all equations we need to discover exoplanets.

Orbits occupy a plane



$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

Start with $\mathbf{F}_1 = m\,\ddot{\mathbf{r}}_1 = +\dfrac{G\,m_1\,m_2}{r^3}\,\mathbf{r}$ 2.1

$$\mathbf{F}_2 = m\,\ddot{\mathbf{r}}_2 = -\frac{G\,m_1\,m_2}{r^3}\,\mathbf{r}$$ 2.2

Find that $\ddot{\mathbf{r}} + \mu\dfrac{\mathbf{r}}{r^3} = 0$ 2.3

where $\mu = G\,(m_1 + m_2)$ 2.4

take a cross product, integrate, and reach $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ 2.5

Getting to Kepler's first Law

We start by defining a new polar coordinate system, where:

$$\mathbf{r} = r\,\hat{\mathbf{r}} \tag{2.6}$$

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\theta} \tag{2.7}$$

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\,\hat{\mathbf{r}} + \left(\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\dot{\theta}\right)\right)\,\hat{\theta} \tag{2.8}$$

and, using 2.4, find that $\mathbf{h} = r^2\,\dot{\theta}\,\hat{\mathbf{z}}$ 2.9

then find $\dot{r}$ and $\ddot{r}$ by substituting $r = u^{-1}$

and 3.3 becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}$$

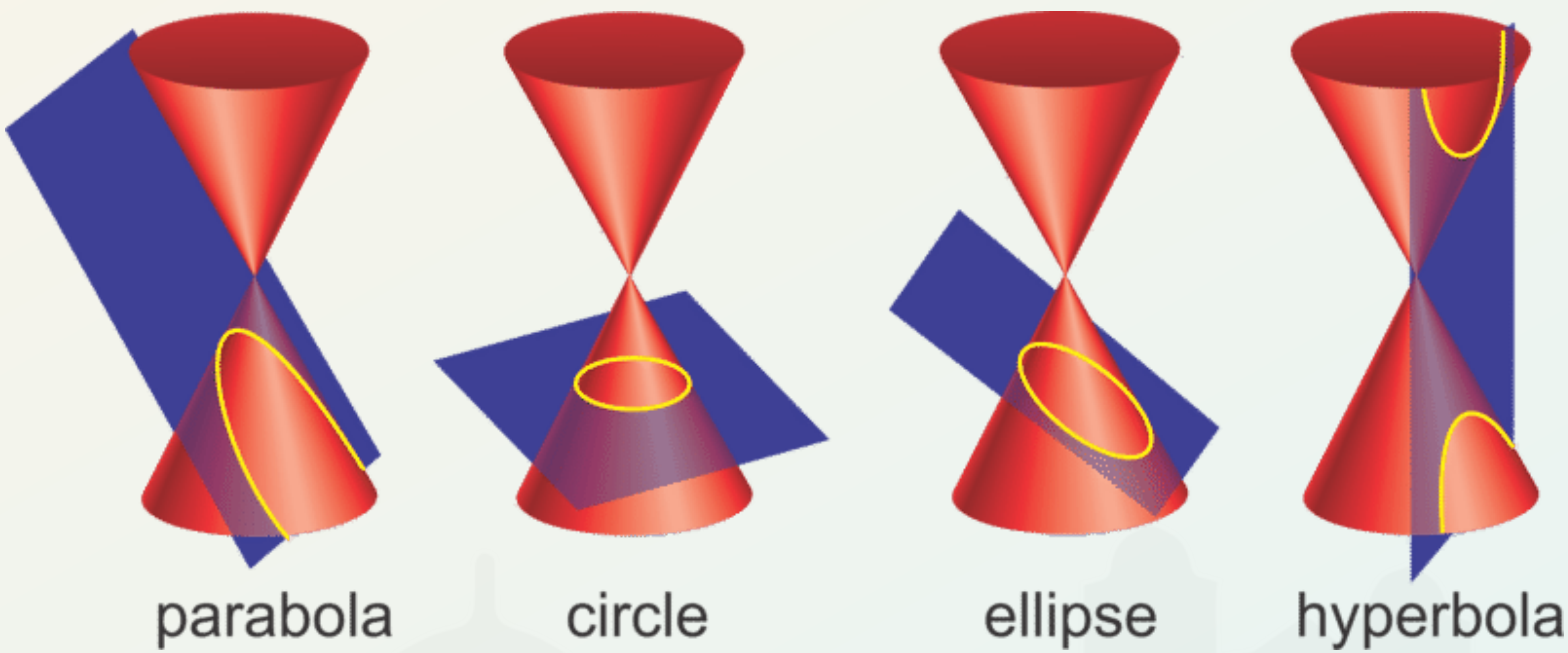
Getting to Kepler's first Law

This second order differential equation has the following solution

ϖ
is curly π
or pomega

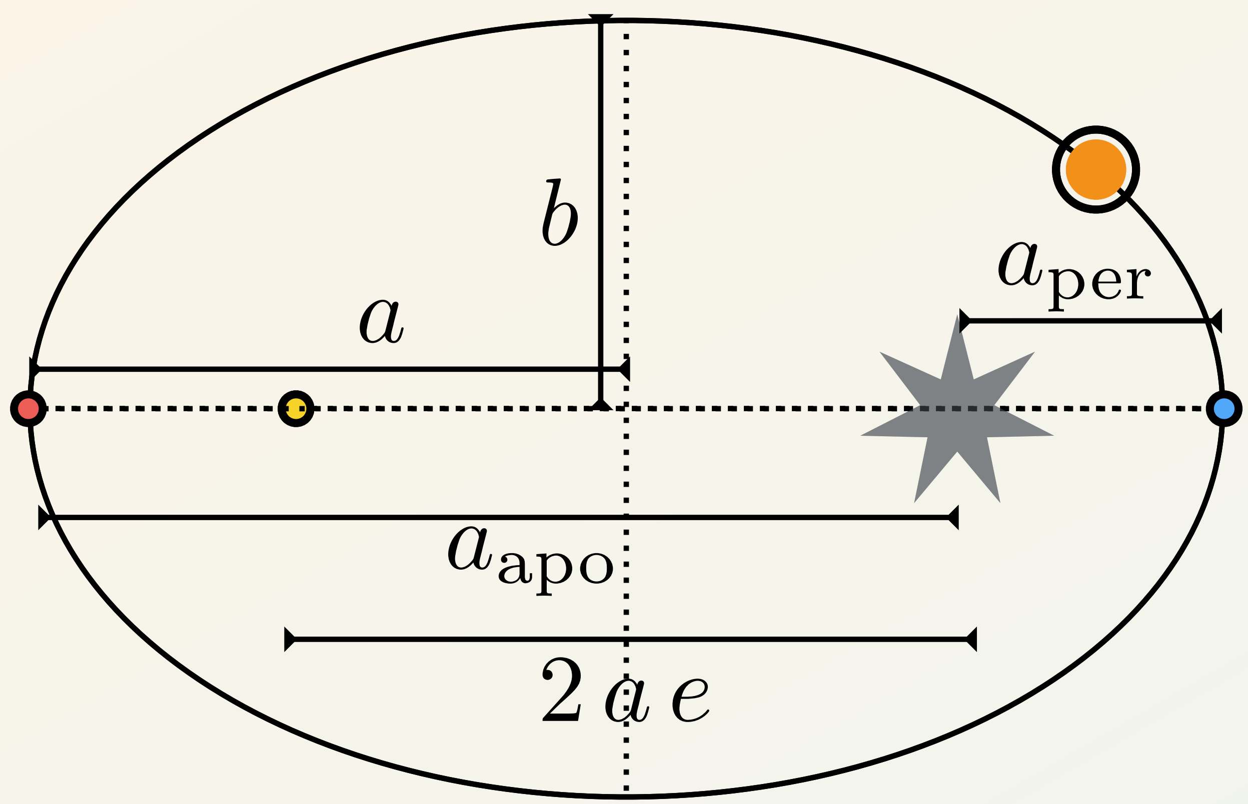
$$r = \frac{h^2/\mu}{1 + e\,\cos(\theta - \varpi)} \tag{2.10}$$

This is the conic's equation.



for e < 1, we arrived at **Kepler's first law**.

Orbits




occupied focus

unoccupied focus

periapse

apoapse

a semimajor axis

b semiminor axis

e eccentricity

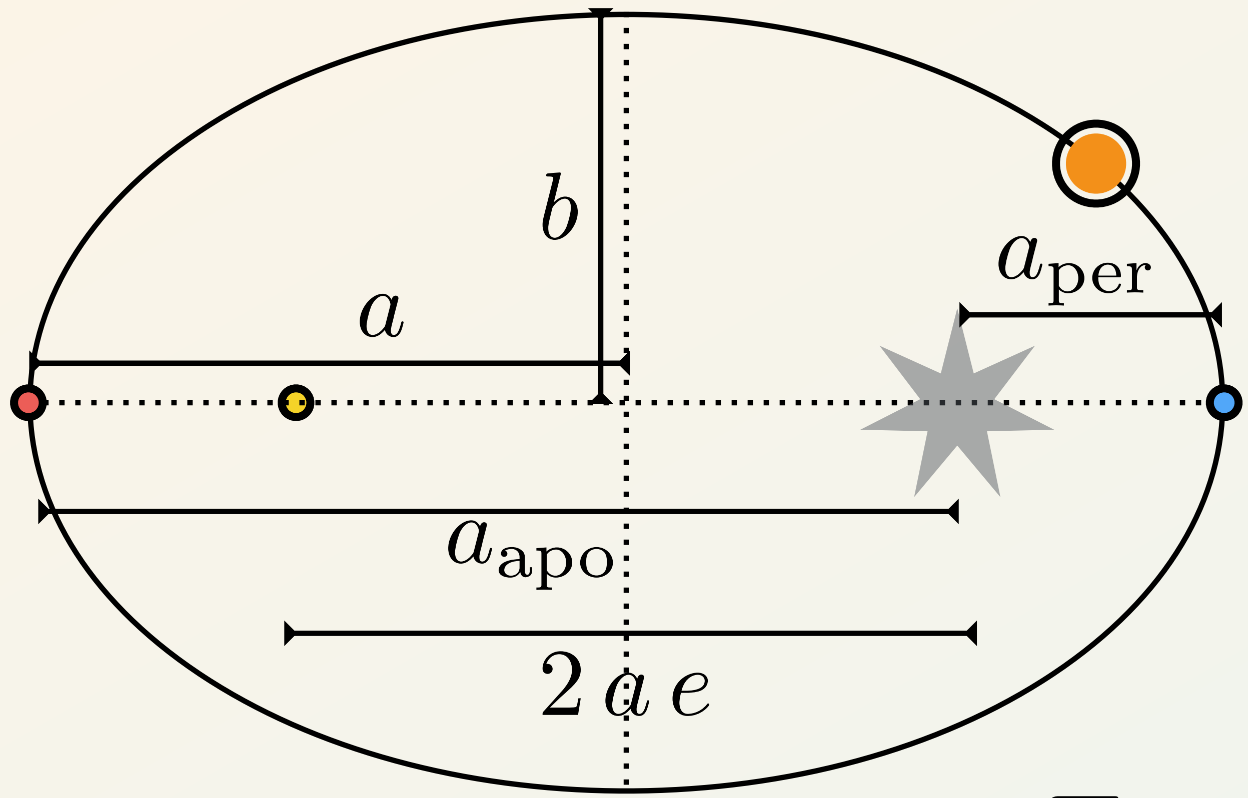
$$a_{\mathrm{apo}} = a\,(1+e)$$

$$a_{\mathrm{per}} = a\,(1-e)$$

$$\frac{b}{a} = \sqrt{1-e^2}$$

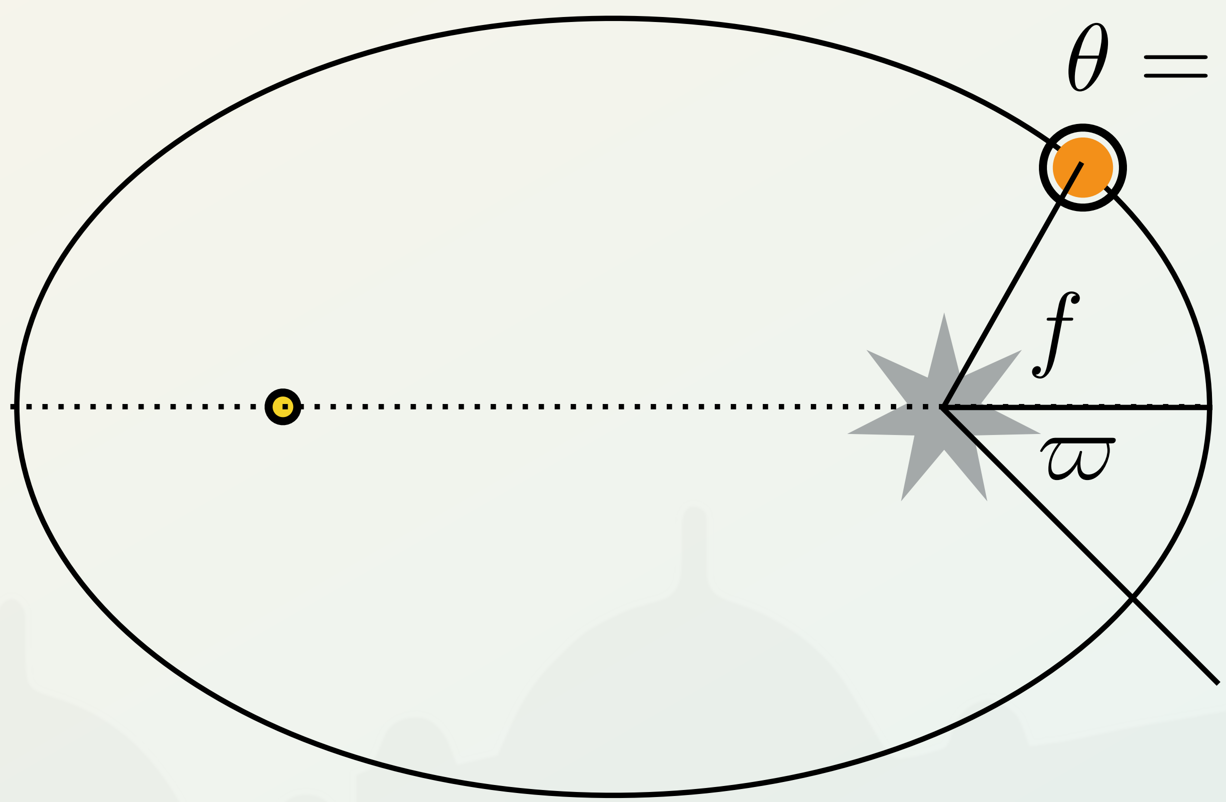
$$A = \pi\,a\,b = \pi\,a^2\,\sqrt{1-e^2}$$

Orbital Elements



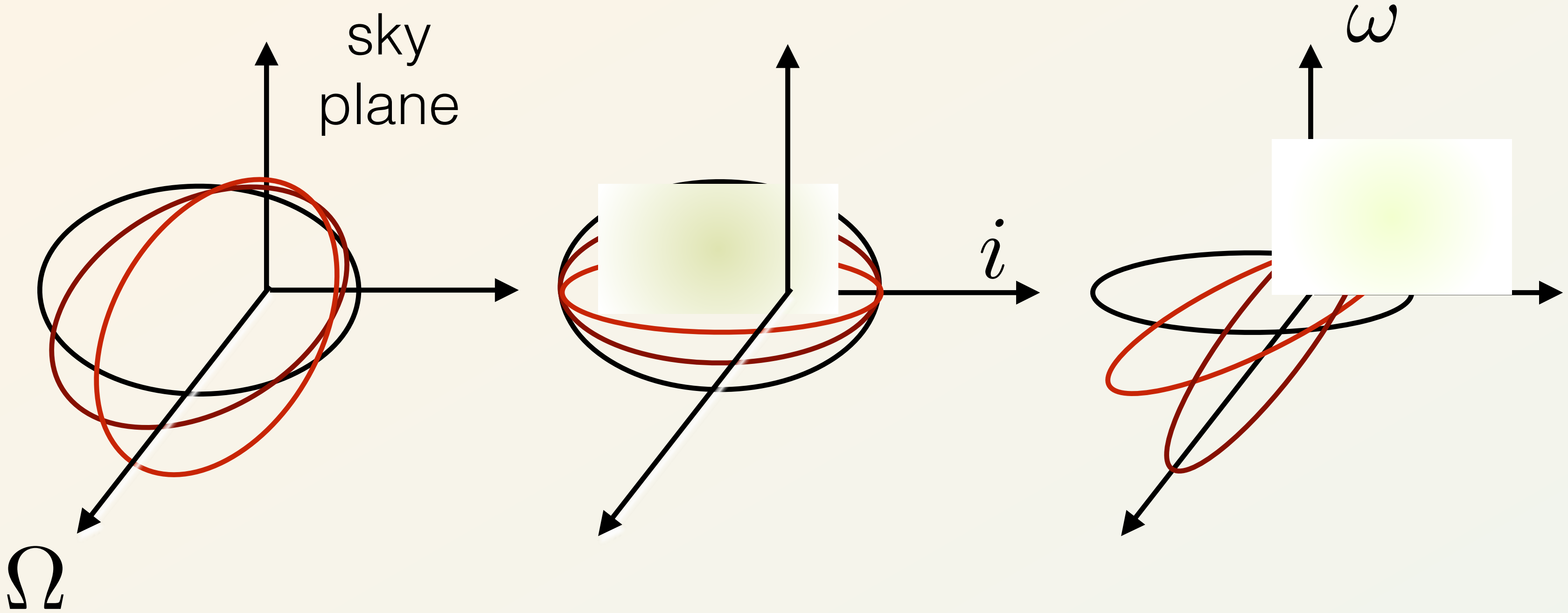
$$\varpi = \Omega + \omega$$

$$\theta = f + \varpi$$



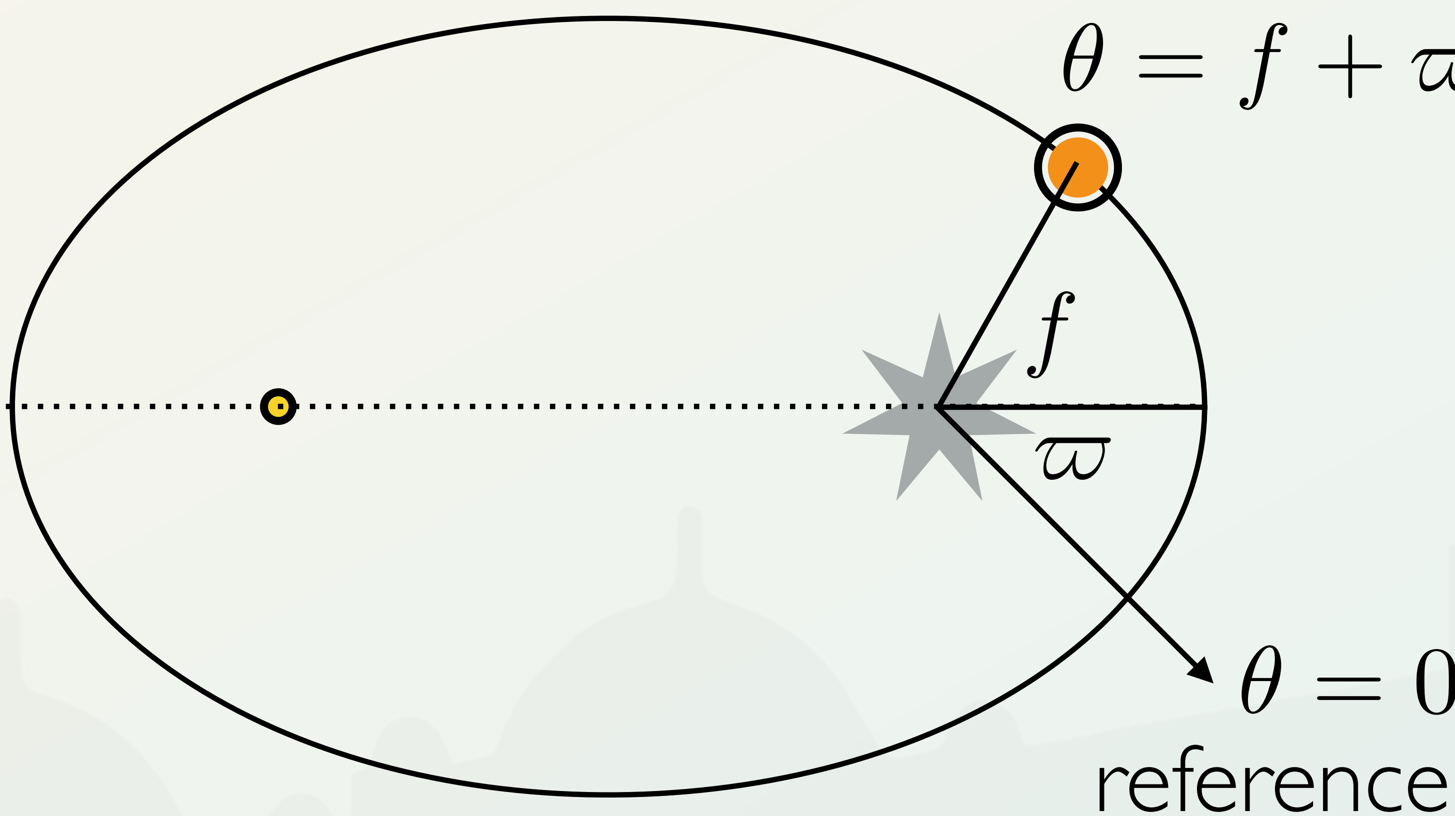
	occupied focus
	unoccupied focus
a	semimajor axis
e	eccentricity
f	true anomaly
i	inclination
Ω	longitude of the ascending node
ω	argument of periaspse
θ	true longitude
ϖ	longitude of periapse

Orbital Elements



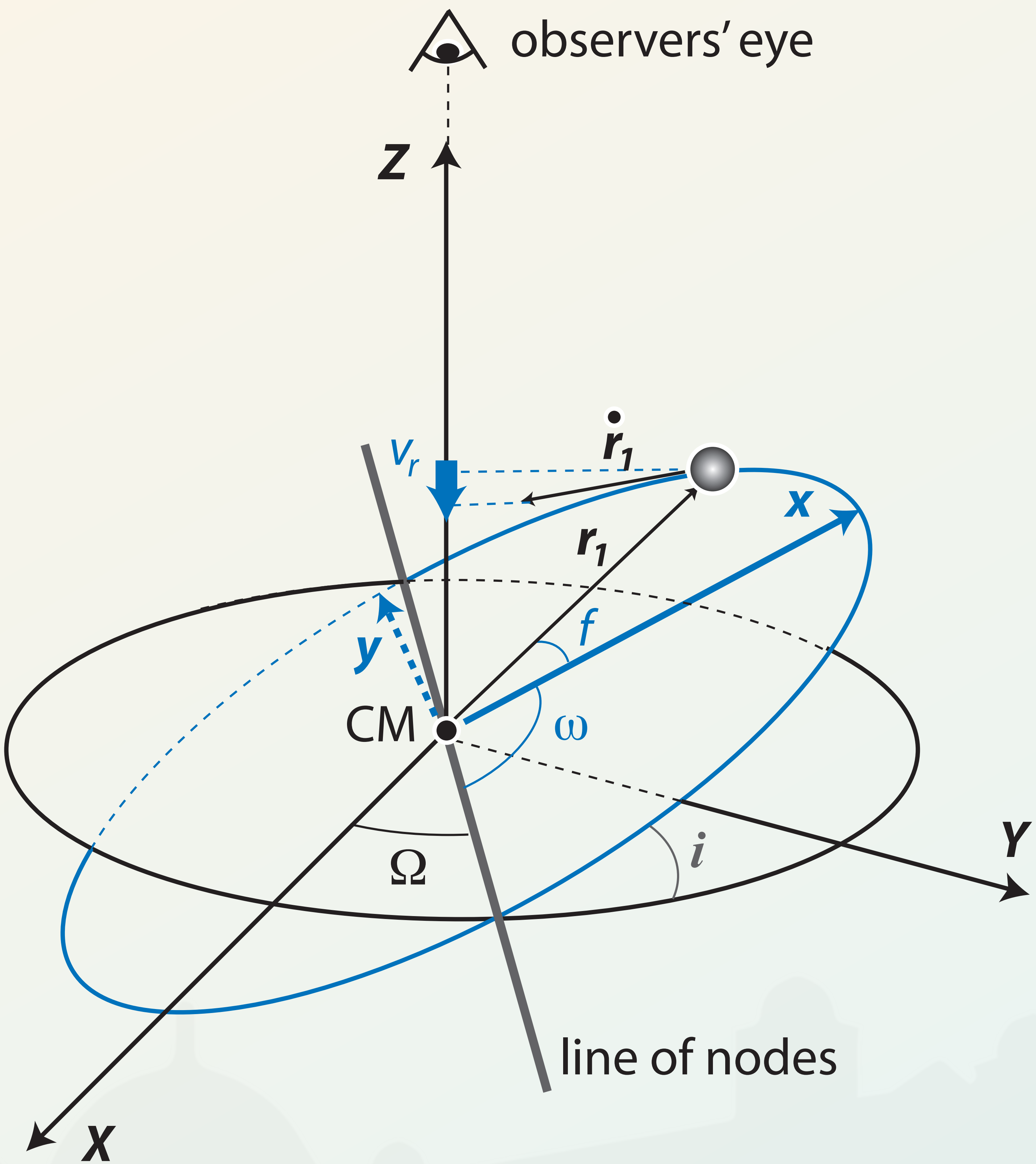
$$\varpi = \Omega + \omega$$

$$\theta = f + \varpi$$



✶	occupied focus
•	unoccupied focus
a	semimajor axis
e	eccentricity
f	true anomaly
i	inclination
Ω	longitude of the ascending node
ω	argument of periaspse
θ	true longitude
ϖ	longitude of periapse

Orbits in 3D



Kepler's Second & Third Laws

solve 2.10 at periastron, and find $$h = \sqrt{\mu\, a\,(1 - e^2)}$$ 2.11

rewrite 2.10 as $$r = \frac{a\,(1 - e^2)}{1 + e\, \cos f}$$ 2.12

the swept area is $$\mathrm{d}A = \int_0^r r\,\mathrm{d}r\,\mathrm{d}\theta$$

using 2.9, we finally find $$\dot{A} = \frac{1}{2}\sqrt{\mu\, a\,(1 - e^2)}$$ 2.13

this is **Kepler's second law**
area is swept constantly

in one period we sweep an ellipse area, getting us to

$T = \dfrac{2\,A}{h}$, then rearrange to $T^2 = \dfrac{4\pi^2}{\mu}\, a^3$ 2.14

Kepler's third law

Vis-Viva equation

With Eq. 1.17 we calculated the Keplerian velocity, however this assumed a circular orbit. Let's rederive it for any position on an orbit

Using conservation of energy (since the orbital energy is constant for a given orbit).

$$\frac{1}{2} m_{\mathrm{p}} v_{\mathrm{peri}}^2 - \frac{G M_\star m_{\mathrm{p}}}{a_{\mathrm{peri}}} = -\frac{G M_\star m_{\mathrm{p}}}{2a}$$

We obtain

$$v_{\mathrm{peri}}^2 = \frac{G M_\star}{a} \frac{(1+e)}{(1-e)} \qquad 2.15$$

And to the vis-viva equation

$$v_{\mathrm{pl}}^2 = G M_\star \left(\frac{2}{r} - \frac{1}{a}\right) \qquad 2.16$$