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New Results of Energy-Based Swing-Up Control for Rotational Pendulum ⋆

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Abstract: In this paper, we revisit the energy-based swing-up control problem for a rotational pendulum. Different from the existing energy-based control solution, first, we present a necessary and sufficient condition for avoiding singular points in the control law and carry out a global motion analysis of the pendulum. Next, we remove the previous required constraint on the initial state of the pendulum and the control parameters for preventing the pendulum getting stuck at the downward equilibrium point by revealing that the point is saddle (hyperbolic and unstable). Specifically, we show that the Jacobian matrix evaluated at the point has two and two eigenvalues in the open left- and right-half planes, respectively. We prove that the pendulum will eventually be swung up into the basin of attraction of any locally stabilizing controller for all initial conditions with the exception of a set of Lebesgue measure zero. Finally, we validate the presented theoretical results via numerical simulation of two rotational pendulums. Our simulation results show that the swing-up control can be achieved quickly under the improved conditions on the control parameters.

Keywords: Rotational pendulum, energy-based control, swing-up control, underactuated system, Lyapunov stability theory.

1. INTRODUCTION

The rotational pendulum, also known as the Furuta pendulum, consists of a driven arm which rotates in the horizontal plane and a pendulum attached to that arm which is free to rotate in the vertical plane Furuta et al. (1992). It has been widely used in control experiments and in verification the effectiveness of various control design methods, see, e.g., Yamakita et al. (1999). It, together with the cart-pole system in Åström and Furuta (2000), the Pendubot in Spong and Block (1995); Fantoni et al. (2000), the Acrobot in Spong (1995); Lai et al. (1999), has been studied as one of typical examples of underactuated mechanical systems which have fewer actuators than degrees of freedom.

In recent years, many researchers studied the swing-up and stabilizing control of pendulum-type systems, see, e.g., Åström and Furuta (2000); Fantoni et al. (2000); Xin and Kaneda (2007); Izutsu et al. (2008). The swing-up control for the rotational pendulum is to swing the pendulum up to its unstable upright equilibrium point (the pendulum is still at the upright position) so that the pendulum can be balanced about that point by a locally stabilizing controller.

Fantoni and Lozano (2002) proposed an energy-based swing-up control law for a rotational pendulum. They reduced the swing-up control problem for the pendulum to a tracking problem for a homoclinic orbit. They gave a sufficient condition for avoiding the singular points in the control law. Moreover, to guarantee the tracking of the homoclinic orbit and to prevent the pendulum getting stuck at the downward equilibrium point (the pendulum is still at the downward position), they presented a constraint on the initial state of the pendulum and the control parameters.

In this paper, we revisit the energy-based control solution in Fantoni and Lozano (2002). We improve the solution further and obtain the following results. First, we present a necessary and sufficient condition rather than the existing sufficient condition for avoiding singular points in the control law. We carry out a global motion analysis of the pendulum. Next, we remove the required constraint on the initial state of the pendulum and the control parameters for preventing the pendulum getting stuck at the downward equilibrium point by revealing that the point is unstable saddle (hyperbolic and unstable). Specifically, we show by using the Routh-Hurwitz criterion that the Jacobian matrix evaluated at the point has two and two eigenvalues in the open left- and right-half planes, respectively. In this way, we prove that the pendulum will

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eventually be swung up into the basin of attraction of any (locally) stabilizing controller for all initial conditions with the exception of a set of Lebesgue measure zero. In this way, we provide a bigger control parameter region for achieving the control objective and require almost no constraint on the initial state of the pendulum. Finally, we validate the presented theoretical results via numerical simulation of two rotational pendulums. Our simulation results show that the swing-up control can be achieved quickly under the improved conditions on the control parameters.

2. PRELIMINARY KNOWLEDGE

Consider the rotational pendulum shown in Fig. 1. I_0 is the moment of inertia of the arm, L_0 is the total length of the arm, m_1 is the mass of the pendulum, l_1 is the distance to the center of gravity of the pendulum, J_1 is the moment of inertia of the pendulum around its center of gravity, θ_0 is the rotational angle of the arm, θ_1 is rotational angle of the pendulum, and τ is the control input torque applied on the arm.

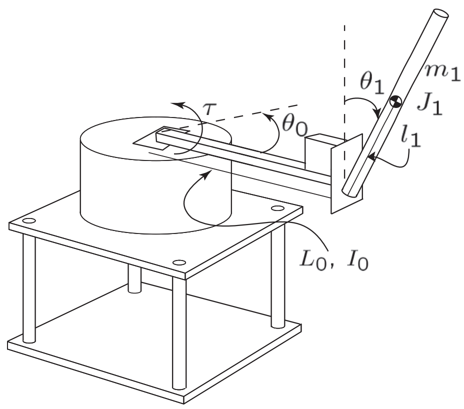

Fig. 1. The rotational pendulum system.

The motion equation of this system is

$$D(q)\ddot{q} + H(q,\dot{q}) + G(q) = B\tau, \tag{1}$$

where $q = [\,\theta_0,\ \theta_1\,]^{\mathrm{T}}$ is the generalized coordinate, and

$$D(q) = \begin{bmatrix} I_0 + m_1\left(L_0^2 + l_1^2 \sin^2\theta_1\right) & m_1 l_1 L_0 \cos\theta_1 \\ m_1 l_1 L_0 \cos\theta_1 & J_1 + m_1 l_1^2 \end{bmatrix}, \tag{2}$$

$$H(q,\dot{q}) = \begin{bmatrix} m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0\dot{\theta}_1 - m_1 L_0 l_1 \sin(\theta_1)\dot{\theta}_1^2 \\ -\dfrac{1}{2} m_1 l_1^2 \sin(2\theta_1)\dot{\theta}_0^2 \end{bmatrix}, \tag{3}$$

$$G(q) = [0,\ \ -m_1 g l_1 \sin\theta_1]^{\mathrm{T}}, \tag{4}$$

$$B = [1,\ 0]^{\mathrm{T}}, \tag{5}$$

where g is the acceleration of gravity.

We express the total mechanical energy of the system as

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} D(q)\dot{q} + P(q), \tag{6}$$

where $P(q)$ is the potential energy and is defined as

$$P(q) = m_1 g l_1(\cos\theta_1 - 1) \tag{7}$$

with the potential energy at the upright equilibrium point being zero.

3. SWING-UP CONTROL LAW

We examine the energy-based swing-up control law in Fantoni and Lozano (2002). The new result of this section is the presence of a necessary and sufficient condition for avoiding any singular point in the control law.

Consider the following upright equilibrium point of the pendulum.

$$\theta_0 = 0,\quad \theta_1 = 0,\quad \dot{\theta}_0 = 0,\quad \dot{\theta}_1 = 0. \tag{8}$$

For $E(q,\dot{q})$, θ_0, and $\dot{\theta}_0$, Fantoni and Lozano (2002) studied how to design τ such that

$$\lim_{t\to\infty} E(q,\dot{q}) = 0,\quad \lim_{t\to\infty} \theta_0 = 0,\quad \lim_{t\to\infty} \dot{\theta}_0 = 0. \tag{9}$$

We use the following Lyapunov function candidate proposed in Fantoni and Lozano (2002).

$$V(q,\dot{q}) = \frac{k_E}{2}E(q,\dot{q})^2 + \frac{k_\omega}{2}\dot{\theta}_0^2 + \frac{k_\theta}{2}\theta_0^2, \tag{10}$$

where scalars $k_E > 0$, $k_\omega > 0$, and $k_\theta > 0$ are control parameters. Taking the time-derivative of V along the trajectories of (1), and using

$$\dot{E} = \dot{q}^T B\tau = \dot{\theta}_0\tau, \tag{11}$$

we obtain

$$\dot{V} = \dot{\theta}_0(k_E E\tau + k_\omega\ddot{\theta}_0 + k_\theta\theta_0). \tag{12}$$

If we can choose τ such that

$$k_E E\tau + k_\omega\ddot{\theta}_0 + k_\theta\theta_0 = -k_\delta\dot{\theta}_0, \tag{13}$$

for some constant $k_\delta > 0$, then we have

$$\dot{V} = -k_\delta\dot{\theta}_0^2 \le 0. \tag{14}$$

We discuss under what condition (13) is solvable with respect to τ for any $(q,\dot{q})$. From (1), we obtain

$$\ddot{\theta}_0 = \ddot{q}_1 = B^{\mathrm{T}}\ddot{q} = B^{\mathrm{T}}D^{-1}(B\tau - H - G). \tag{15}$$

Substituting (15) into (13) yields

$$\Lambda(q,\dot{q})\tau = k_\omega B^{\mathrm{T}}D^{-1}(H+G) - k_\theta\theta_0 - k_\delta\dot{\theta}_0, \tag{16}$$

where

$$\Lambda(q,\dot{q}) = k_E E + k_\omega B^{\mathrm{T}}D^{-1}B. \tag{17}$$

Therefore, when

$$\Lambda(q,\dot{q}) \neq 0,\quad \text{for}\ \ \forall q,\ \forall\dot{q}, \tag{18}$$

we obtain

$$\tau = \frac{k_\omega B^{\mathrm{T}}D^{-1}(H+G) - k_\theta\theta_0 - k_\delta\dot{\theta}_0}{\Lambda}. \tag{19}$$

Below we provide a necessary and sufficient condition for avoiding any singular point in the control law. Since

$E(q, \dot{q}) \geq P(q)$, the controller (19) has no singular points for any $(q, \dot{q})$ if and only if

$$\frac{k_\omega}{k_E} > \max_q \frac{-P(q)}{B^{\mathrm{T}} D^{-1}(q) B}, \tag{20}$$

which is equivalent to

$$\frac{k_\omega}{k_E} > \frac{m_1 g l_1}{J_1 + m_1 l_1^2} \max_{\theta_1} \{(1 - \cos\theta_1)\det(D)\} \tag{21}$$

by computing the right-hand side of (20).

Then, we apply LaSalle's theorem Khalil (2002) to the closed-loop system consisting of (1) and (19) to determine the largest invariant set that the close-loop solution approaches. We obtain the following lemma.

Lemma 1. Consider the closed-loop system consisted of (1) and (19) with positive parameters k_E, k_ω, k_θ, and k_δ. Then the controller (19) has no singular points for any $(q, \dot{q})$ if and only if k_ω and k_E satisfies (21). In this case,

$$\lim_{t\to\infty} E(q, \dot{q}) = E^*, \quad \lim_{t\to\infty} \theta_0 = \theta_0^*, \quad \lim_{t\to\infty} \dot{\theta}_0 = 0, \tag{22}$$

where E^* and θ_0^* are constants. Moreover, every closed-loop solution approaches the following invariant set as $t \to \infty$:

$$W = \left\{ (q, \dot{q}) \,\middle|\, \theta_0 \equiv \theta_0^*,\ \dot{\theta}_1^2 \equiv \frac{2E^* + 2m_1 g l_1 (1 - \cos\theta_1)}{J_1 + m_1 l_1^2} \right\}, \tag{23}$$

where "$\equiv$" denotes the equation holds for all time t.

Remark 1. Note that Fantoni and Lozano (2002) proposed the following sufficient condition for avoiding singular point in the control law (19).

$$\frac{k_\omega}{k_E} > 2m_1 g l_1 \left(I_0 + m_1 l_1^2 + m_1 L_0^2\right). \tag{24}$$

The condition in (24) can be derived from (21) by using

$$\max_{\theta_1}(1 - \cos\theta_1)\det(D) < 2(J_1 + m_1 l_1^2)\left(I_0 + m_1 l_1^2 + m_1 L_0^2\right).$$

4. MOTION ANALYSIS OF THE PENDULUM

To investigate whether the pendulum can be swung up under the energy-based control law (19), we analyze the invariant set W defined in (23). We study the relationship between the convergent value of the energy and motion of the pendulum, and we analyze the stability of the downward equilibrium point.

The main result of this paper is described in the following theorem.

Theorem 1. Consider the closed-loop system consisted of (1) and (19) with positive parameters k_E, k_ω, k_θ, and k_δ. Suppose that k_E and k_ω satisfy (21). Then the following statements hold:

1 The closed-loop solution $(q(t), \dot{q}(t))$ approaches

$$W = W_r \cup \{(0, \pi, 0, 0)\}, \tag{25}$$

as $t \to \infty$, where

$$W_r = \left\{ (q, \dot{q}) \,\middle|\, \theta_0 \equiv 0,\ \dot{\theta}_1^2 \equiv \frac{2m_1 g l_1 (1 - \cos\theta_1)}{J_1 + m_1 l_1^2} \right\}. \tag{26}$$

2 The Jacobian matrix evaluated at the downward equilibrium point $(\theta_0, \theta_1, \dot{\theta}_0, \dot{\theta}_1) = (0, \pi, 0, 0)$ for the closed-loop system has two and two eigenvalues in the open left- and right-half planes; it is a saddle.

Proof. First, we prove Statement 1, since $E \equiv E^*$ and $\theta_0 \equiv \theta^*$ hold in the invariant set W in (23), putting these equations into (13) yields

$$k_E E^* \tau + k_\theta \theta_0^* \equiv 0. \tag{27}$$

We address two cases of $E^* = 0$ and $E^* \neq 0$, separately.

First, if $E^* = 0$, then $\theta_0^* = 0$ follows directly from (27). Putting $E^* = 0$ and $\theta_0^* = 0$ into (23) shows that the closed-loop solution $(q(t), \dot{q}(t))$ approaches the invariant set W_r defined in (26) as $t \to \infty$.

Next, if $E^* \neq 0$, we prove that the closed-loop solution $(q(t), \dot{q}(t))$ approaches the downward equilibrium point $(\theta_0, \theta_1, \dot{\theta}_0, \dot{\theta}_1) = (0, \pi, 0, 0)$. To this end, using $E^* \neq 0$ yields that τ in (27) is a constant, which is denoted as τ^*. From (27), we obtain

$$k_E E^* \tau^* + k_\theta \theta_0^* \equiv 0. \tag{28}$$

Substituting $\theta_0 \equiv \theta_0^*$ and $\tau \equiv \tau^*$ into (1), we obtain

$$m_1 l_1 L_0 \left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) \equiv \tau^*. \tag{29}$$

Integrating (29) with respect to time t yields

$$\dot{\theta}_1 \cos\theta_1 \equiv \frac{\tau^*}{m_1 l_1 L_0} t + \lambda_1, \quad \forall t, \tag{30}$$

where λ_1 is a constant. Since it follows from (23) that $\dot{\theta}_1$ is bounded, we obtain $\tau^* = 0$. Otherwise, the right-hand side of (30) becomes infinity as $t \to \infty$.

Now integrating (30) with respect to time t yields

$$\sin\theta_1 = \lambda_1 t + \lambda_2, \quad \forall t, \tag{31}$$

where λ_2 is a constant. Since $\sin\theta_1$ is bounded, $\lambda_1 = 0$ holds; otherwise, the right-hand side of (31) is unbounded as $t \to \infty$. We have $\sin\theta_1 = \lambda_2$. This implies that θ_1 is a constant denoted as θ_1^*.

From (28) and $\tau^* = 0$, we have $\theta_0^* = 0$. Putting $\theta_0 \equiv \theta_0^* = 0$ and $\theta_1 \equiv \theta_1^*$ into (1) and (6) yields

$$-m_1 g l_1 \sin\theta_1^* = 0 \tag{32}$$

$$E^* = m_1 g l_1 (\cos\theta_1^* - 1), \tag{33}$$

respectively. From (32), we have $\sin\theta_1^* = 0$. Note that $\theta_1^* = 0$ is impossible since it yields $E^* = 0$ which contradicts the assumption $E^* \neq 0$. Therefore, we obtain $\theta_1^* = \pi \pmod{2\pi}$ and $E^* = -2m_1 g l_1$. Thus, the closed-loop solution approaches the downward equilibrium point $(\theta_0, \theta_1, \dot{\theta}_0, \dot{\theta}_1) = (0, \pi, 0, 0)$.

To prove Statement 2, first, straightforward calculation of the characteristic equation of the Jacobian matrix evaluated at the downward equilibrium point $(\theta_0, \theta_1, \dot{\theta}_0, \dot{\theta}_1) = (0, \pi, 0, 0)$ yields

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0, \tag{34}$$

where

$$a_1 = \frac{(J_1 + m_1 l_1^2)k_\delta}{\gamma},$$
$$a_2 = \frac{J_1 k_\theta + m_1 l_1 (g k_\omega + l_1 k_\theta - 2 m_1 g^2 l_1 (I_0 + m_1 L_0^2) k_E)}{\gamma},$$
$$a_3 = \frac{g m_1 l_1 k_\delta}{\gamma}, \ a_4 = \frac{g m_1 l_1 k_\theta}{\gamma},$$

where

$$\gamma = (J_1 + m_1 l_1^2) k_\omega - 2 m_1 g l_1 (J_1 (I_0 + m_1 L_0^2) + I_0 m_1 l_1^2) k_E.$$

Next, note that the condition (21) on k_ω and k_E implies $\gamma > 0$. Indeed, taking $\theta_1 = \pi$ in $(1 - \cos\theta_1)\det(D)$ contained in (21) shows $\gamma > 0$. This gives

$$a_1 > 0, \ a_3 > 0, \ a_4 > 0.$$

The Hurwitz determinants (see, for example, Kuo (1991)) of (34) are given by

$$D_1 = a_1, \quad D_2 = a_1 a_2 - a_3, \tag{35}$$
$$D_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4, \quad D_4 = a_4 D_3. \tag{36}$$

Direct computation yields the following important inequality:

$$D_3 = -\frac{2 m_1^5 l_1^5 L_0^2 g^3 k_E k_\delta^2}{\gamma^3} < 0. \tag{37}$$

Thanks to (37), we use Routh's tabulation Kuo (1991) to study the root location of (34) in the complex plane. If D_2 defined in (35) is not zero, then the coefficients in the first column of Routh's tabulation for (34) are $\{1, a_1, D_2/a_1, D_3/D_2, a_4\}$. Using $D_3 < 0$ yields that the signs in the coefficients of the first column are $\{+,+,-,+,+\}$ and $\{+,+,+,-,+\}$ for $D_2 < 0$ and $D_2 > 0$, respectively. This shows that the changes of the signs are both 2. Hence, if $D_2 \neq 0$, then J has *two* and *two* eigenvalues in the open left- and right-half planes, respectively.

If $D_2 = 0$, then we replace the zero in the s^2 row in Routh's tabulation by a small positive number ϵ, and proceed with the tabulation. In this way, we can show that there are two sign changes in the first column of the tabulation. Thus, J has *two* and *two* eigenvalues in the left- and right-half planes, respectively. This shows that the downward equilibrium point is saddle. ■

5. DISCUSSION

Fantoni and Lozano (2002) studied how to guarantee that the closed-loop solution does not approach the downward equilibrium point. Since *the potential energy at the downward equilibrium point is* $-2m_1 g l_1$, they proposed the following constraint on the initial state of the pendulum and the control parameters:

$$V(0) < 2 k_E m_1^2 g^2 l_1^2 \tag{38}$$

which is equivalent to

$$E(q, \dot{q})^2 + \frac{k_\omega}{k_E}\dot{\theta}_0^2 + \frac{k_\theta}{k_E}\theta_0^2 < (2 m_1 g l_1)^2 \tag{39}$$

for $t = 0$. Since the Lyapunov function V is non-increasing under the controller (19), the constraint (38) guarantees that $|E| < 2m_1 g l_1$ holds for all $t \geq 0$ and the closed-loop solution does not approach the downward equilibrium point.

Note that the condition on the initial state of the pendulum and the control parameters in (38) (or equivalent (39)) for guaranteeing the convergence to the invariant set W_r is rather strict. Obviously, such a condition fails for an initial state $(\theta_0(0), \theta_1(0), \dot{\theta}_0(0), \dot{\theta}_1(0))$ with its energy E satisfying $|E| \geq 2m_1 g l_1$. Even if $|E| < 2m_1 g l_1$ holds at $t = 0$, taking control parameters $(k_E, k_\delta, k_\omega)$ satisfying (39) may not yield a good swing-up control performance.

Different from Fantoni and Lozano (2002), as shown in Theorem 1, we do not need such a constraint on the initial state of the pendulum and the control parameters. Since we proved that the downward equilibrium point is a saddle (hyperbolic and unstable), based on the statement of Ortega et al. (2002) (p. 1225), for all initial conditions with the exception of a set of Lebesgue measure zero, every closed-loop solution, $(\theta_0(t), \theta_1(t), \dot{\theta}_0(t), \dot{\theta}_1(t))$ approaches the invariant set W_r as $t \to \infty$. Therefore, we have more freedom to tune the control gains for achieving a better control performance such as swinging up the pendulum more quickly into a small neighborhood of the upright equilibrium point.

Next, since the equation about $(\theta_1(t), \dot{\theta}_1(t))$ in (26) describes a homoclinic orbit with $(\theta_1(t), \dot{\theta}_1(t)) = (0, 0)$ being its equilibrium point, $(\theta_1(t), \dot{\theta}_1(t))$ will have $(0, 0)$ as an ω-limit point Sastry (1999); that is, there exists a sequence of times t_m $(m = 1, \ldots, \infty)$ such that $t_m \to \infty$ as $m \to \infty$ for which $\lim_{m\to\infty}(\theta_1(t), \dot{\theta}_1(t)) = (0, 0)$. Thus, there exists a sequence of times that the pendulum can enter any arbitrarily small neighborhood of the upright equilibrium point.

6. SIMULATION

We validated the theoretical results obtained in this paper via numerical simulation of two rotational pendulums in Fantoni and Lozano (2002) and Quanser Inc. (2008). We took the gravity acceleration $g = 9.80$ m/s^2.

6.1 Pendulum 1

Pendulum 1 has the following mechanical parameters Fantoni and Lozano (2002):

$$\begin{cases} L_0 = 0.215 \text{ m}, \ l_1 = 0.113 \text{ m}, \ m_1 = 5.38 \times 10^{-2} \text{ kg}, \\ I_0 = 1.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2, \ J_1 = 1.98 \times 10^{-4} \text{ kg} \cdot \text{m}^2. \end{cases}$$

To avoid singular points in (19) for any initial condition, the sufficient condition (24) proposed in Fantoni and Lozano (2002) is $k_\omega/k_E > 2.46 \times 10^{-3}$, and our necessary and sufficient condition (21) is $k_\omega/k_E > 2.23 \times 10^{-3}$.

Consider the same initial state of the pendulum as that in Fantoni and Lozano (2002):

$$\theta_0(0) = -\pi/2 \text{ rad}, \ \theta_1(0) = 2.5\pi/3 \text{ rad},$$
$$\dot\theta_0(0) = 0 \text{ rad/s}, \ \dot\theta_1(0) = 0 \text{ rad/s}. \tag{40}$$

The following control parameters were chosen in Fantoni and Lozano (2002):

$$k_E = 480, \quad k_\omega = 1, \quad k_\delta = 1, \quad k_\theta = 1. \tag{41}$$

Note that the four parameters k_E, k_ω, k_δ, and k_θ in the controller (19) are not independent. Indeed, from (19), we can see that $(k_E, k_\omega, k_\delta, k_\theta)$ and $\frac{1}{k_E}(k_E, k_\omega, k_\delta, k_\theta)$ give the same controller. Therefore, (41) and

$$k_E = 1, \ k_\omega = 1/480, \ k_\delta = 1/480, \ k_\theta = 1/480 \tag{42}$$

produce the same controller.

Clearly $k_\omega/k_E = 1/480 = 2.08 \times 10^{-3}$ in (41) violates the sufficient condition $k_\omega/k_E > 2.46 \times 10^{-3}$ proposed in Fantoni and Lozano (2002) and violates also our necessary and sufficient condition $k_\omega/k_E > 2.23 \times 10^{-3}$. This indicates that the controller (19) with the parameters in (41) contains singular points.

Using our necessary and sufficient condition $k_\omega/k_E > 2.23 \times 10^{-3}$, we chose

$$\begin{cases} k_E = 1, \ k_\omega = 2.30 \times 10^{-3}, \\ k_\delta = 6.70 \times 10^{-3}, \ k_\theta = 6.53 \times 10^{-2}. \end{cases} \tag{43}$$

Note that k_ω/k_E does not satisfy the sufficient condition $k_\omega/k_E > 2.46 \times 10^{-3}$ in Fantoni and Lozano (2002). Moreover, the constraint condition $V(0) < 2k_E m_1^2 g^2 l_1^2$ in (38) proposed in Fantoni and Lozano (2002) is not satisfied due to $V(0) = 0.0867$ and $2k_E m_1^2 g^2 l_1^2 = 0.0071$.

For comparison, the simulation results (for $0 \le t \le 10$ s) with the parameters in (42) and (43) are depicted in Figs. 2–4, where the *solid lines* denote our results and the *dotted lines* denote the results of Fantoni and Lozano (2002). Our simulation results show that the swing-up control objective was achieved quickly under the improved conditions on the control parameters. Figure 2 shows that V was non-increasing and converged to 0, and that E converged to 0. From Fig. 3, we know that θ_0 converged to 0, and the controller with parameters in (43) swung up the pendulum much more quickly close to the upright position than that with parameters in (42). From the phase portrait of $(\theta_1, \dot\theta_1)$ and the time response of the control input torque τ in Fig. 4, we can see that $(\theta_1, \dot\theta_1)$ approached the homoclinic orbit described in W_r in (26). Since there existed a sequence of time at which the pendulum was swung up very close to the upright position, a locally stabilizing controller could be switched to balance the pendulum about the vertical.

6.2 Pendulum 2

The mechanical parameters of Pendulum 2 Quanser Inc. (2008) are

$$\begin{cases} L_0 = 0.2159 \text{ m}, \ l_1 = 0.1556 \text{ m}, \ m_1 = 12.7 \times 10^{-2} \text{ kg}, \\ I_0 = 2 \times 10^{-3} \text{ kg} \cdot \text{m}^2, \ J_1 = 1.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2. \end{cases}$$

From the constraint on k_ω and k_E in (21), we obtained $k_\omega/k_E > 2.13 \times 10^{-3}$. We took the initial state in (40), and chose

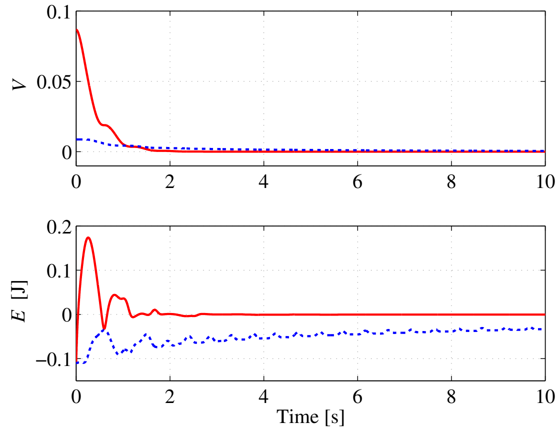

Fig. 2. Time responses of V and E for Pendulum 1.

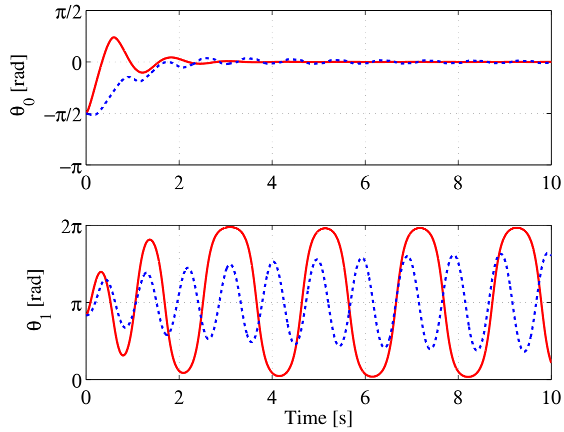

Fig. 3. Time responses of θ_0 and θ_1 for Pendulum 1.

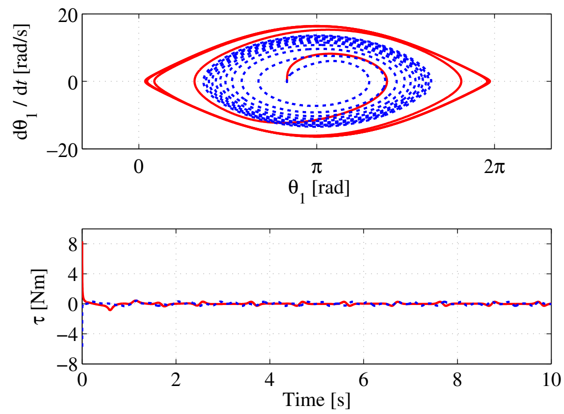

Fig. 4. Phase portrait of $(\theta_1, \dot\theta_1)$ and time response of τ for Pendulum 1.

$$\begin{cases} k_E = 1, \ k_\omega = 2.70 \times 10^{-3}, \\ k_\delta = 1.20 \times 10^{-2}, \ k_\theta = 1.20 \times 10^{-1}. \end{cases} \tag{44}$$

Note that k_ω/k_E does not satisfy the sufficient condition $k_\omega/k_E > 4.30 \times 10^{-3}$ obtained from (24). Moreover, $V(0) = 0.213$ does not satisfy the constraint condition $V(0) < 0.075$ obtained from (38).

The simulation results with above parameters are depicted in Figs. 5 and 6. From Fig. 5, we know that θ_0 and

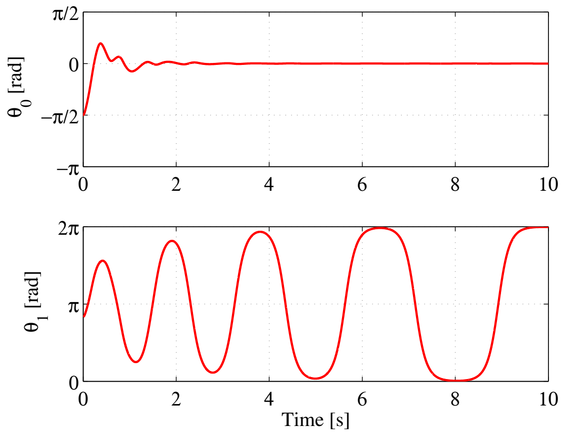

Fig. 5. Time responses of θ_0 and θ_1 for Pendulum 2.

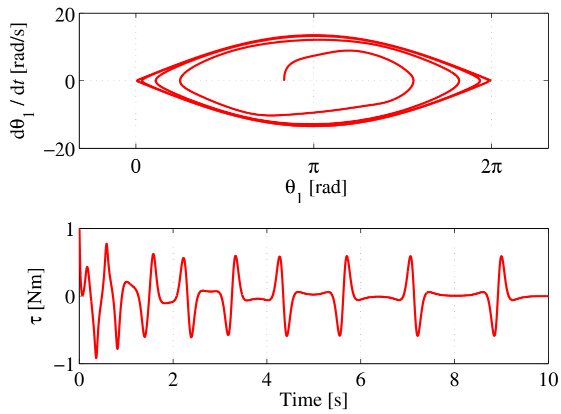

Fig. 6. Phase portrait of $(\theta_1, \dot{\theta}_1)$ and time responses of τ for Pendulum 2.

converged to 0, and the pendulum was swung up quickly to the upright position. From Fig. 6, we can see that $(\theta_1, \dot{\theta}_1)$ approached the homoclinic orbit described in W_r in (26).

7. CONCLUSION

In this paper, we revisited the energy-based swing-up control problem for a rotational pendulum. Different from the existing energy-based control solution, first, we presented the necessary and sufficient condition for avoiding singular points in the control law and carried out a global motion analysis of the pendulum. Next, we removed the previous required constraint on the initial state of the pendulum and the control parameters for preventing the pendulum getting stuck at the downward equilibrium point by revealing that the point is saddle. Specifically, we showed by using the Routh-Hurwitz criterion that the Jacobian matrix evaluated at the downward equilibrium point has two and two eigenvalues in the open left- and right-half planes, respectively. We proved that the pendulum will eventually be swung up into the basin of attraction of any stabilizing controller for all initial conditions with the exception of a set of Lebesgue measure zero. In this way, we provided a bigger control parameter region for achieving the control objective and needed almost no constraint on the initial state of the pendulum. Finally, we validated the presented theoretical results via numerical simulation of two rotational pendulums. Our simulation results showed that the swing-up control can be achieved quickly under the improved conditions on the control parameters.

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