New math derivation

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The purpose of this summary is to derive the math governing the way an asteroid reacts to a non-point source. It was motivated by the discovery that, under the assumption of a gravitating point source and a small asteroid, the tidal torque can be written in terms of components of the moment of inertia, which means inference is limited to the six components of the inertia matrix. By violating these assumptions and considering other effects (oblateness, etc.), we can break the degeneracy.

I. DEFINITIONS

Please see info on spherical harmonics

$$P_{lm} = \frac{1}{2^l l!}(1-t^2)^{m/2}\frac{d^{l+m}}{dl^{l+m}}(t^2-1)^l$$

$$Y_{lm}(\theta,\phi) = P_{lm}(\cos\theta)e^{im\phi}$$

$$\bar{Y}_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}Y_{lm}(\theta,\phi)$$

$$R_{lm}(\boldsymbol{r}) = (-1)^m\frac{r^l}{(l+m)!}Y_{lm}(\boldsymbol{r})$$

$$S_{lm}(\boldsymbol{r}) = (-1)^m\frac{(l-m)!}{r^{l+1}}Y_{lm}(\boldsymbol{r})$$

We write $Y(\boldsymbol{r}) = Y(\hat{\boldsymbol{r}})$.

These definitions lead to several key facts. For $r' < r$,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{lm} R_{lm}(\boldsymbol{r}')S^*_{lm}(\boldsymbol{r}). \tag{1}$$

We also have a translation rule:

$$S_{lm}(\boldsymbol{r}'-\boldsymbol{r}) = (-1)^l\sum_{l'm'}R^*_{l'm'}(\boldsymbol{r}')S_{l+l',m+m'}(\boldsymbol{r}). \tag{2}$$

Similarly, we have some gradient formulas that result in

$$\begin{aligned}\boldsymbol{r}\times\nabla R^*_{lm} = -\frac{1}{2}[&(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l-m+1)R^*_{l,m-1}\\ &+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l+m+1)R^*_{l,m+1}+2im\hat{\boldsymbol{z}}R^*_{lm}].\end{aligned} \tag{3}$$

Finally, we have the symmetry relations $R_{l,-m} = (-1)^m R^*_{lm}$, $S_{l,-m} = (-1)^m S^*_{lm}$.

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II. LINEAR LAW OF MOTION

We use Newton's law of gravity for the linear law of motion:

$$\ddot{\boldsymbol{D}} = \hat{\boldsymbol{D}}\frac{\mu}{D^2} \tag{4}$$

where $\boldsymbol{D}$ points from the asteroid to the planet, and μ is the reduced mass of the system. We assume that the asteroid mass is so much smaller than the planet's mass that the difference between μ and the mass of the planet cannot be detected. The initial conditions for $\boldsymbol{D}$ should have large magnitude of D and $\dot{D}$ greater than the escape velocity of the system.

III. ANGULAR LAW OF MOTION

Recall that angular momentum satisfies $\dot{\boldsymbol{L}} = \boldsymbol{\tau} = \frac{d}{dt}(I\boldsymbol{\omega})$, where I is the moment of inertia matrix. This equation is not easily simplified in general, but in a coordinate system fixed to the rotating body, we make use of the fact that $\dot{I} = 0$. Also,

$$\dot{\boldsymbol{L}} = \dot{\boldsymbol{L}}_{\text{rot}} + L_x\dot{\hat{\boldsymbol{x}}} + L_y\dot{\hat{\boldsymbol{y}}} + L_z\dot{\hat{\boldsymbol{z}}},$$

where $\dot{\boldsymbol{L}}_{\text{rot}} = \dot{L}_x\hat{\boldsymbol{x}} + \dot{L}_y\hat{\boldsymbol{y}} + \dot{L}_z\hat{\boldsymbol{z}}$. By considering $\boldsymbol{\omega} \parallel \hat{\boldsymbol{z}}$, we see that

$$L_x\dot{\hat{\boldsymbol{x}}} + L_y\dot{\hat{\boldsymbol{y}}} + L_z\dot{\hat{\boldsymbol{z}}} = \boldsymbol{\omega}\times\boldsymbol{L},$$

and since this pattern is replicated for all axes, it must be generally true. Furthermore, $\dot{\boldsymbol{L}}_{\text{rot}} = \dot{I}_{\text{rot}}\boldsymbol{\omega} + I\dot{\boldsymbol{\omega}}_{\text{rot}}$, and since $\dot{I}_{\text{rot}} = 0$, we have

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\boldsymbol{L}.$$

Here we have removed the rot subscript on $\dot{\boldsymbol{\omega}}$ because $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_{\text{rot}} + \omega\times\omega = \dot{\boldsymbol{\omega}}_{\text{rot}}$. Reversing to solve for $\dot{\boldsymbol{\omega}}$, we have

$$\dot{\boldsymbol{\omega}} = I^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega}\times(I\boldsymbol{\omega})\right). \tag{5}$$

Once again, this is valid only in a frame fixed to the body. The torque $\boldsymbol{\tau}$ will be calculated in the next section.

The initial conditions are that ω is initially parallel to the eigenvector of I with the largest eigenvalue, to minimize energy. The norm and direction of ω is observed.

It will turn out that our equation for τ will invoke the use of $\boldsymbol{D}$ in the rotating frame. To track $\boldsymbol{D}$ in that frame, we need $\dot{\boldsymbol{D}}_{\text{rot}}$. To find it,

$$\dot{\boldsymbol{D}} = \dot{\boldsymbol{D}}_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{D}$$

or

$$\dot{\boldsymbol{D}}_{\text{rot}} = \boldsymbol{D} \times \boldsymbol{\omega} + \hat{\boldsymbol{D}}. \tag{6}$$

where $\dot{D}$ is given by the evolution of equation 4

IV. TORQUE

By definition, the differential torque on a small unit of mass dm within an asteroid is $d\boldsymbol{\tau} = \boldsymbol{r} \times d\boldsymbol{F}$, where $\boldsymbol{r}$ points to dm from the center of mass of the asteroid. In the regime where $d\boldsymbol{F}$ is constant, there is no tidal torque. But if $d\boldsymbol{F}$ is allowed to vary across the asteroid, we will get tidal torque as shown.

The differential force is $d\boldsymbol{F} = -dm\nabla_{\boldsymbol{R}} V(\boldsymbol{R})$, where

$$V(\boldsymbol{R}) = -\int d^3r' \frac{\rho_M(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{R}|}$$

is the gravitational potential field of the planet, and $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{D}$ points from dm to the center of mass of the planet. Also, we have ρ_M as the mass distribution of the planet. Expanding this potential field using equation 1, we get

$$V(\boldsymbol{R}) = -\sum_{lm} S^*_{lm}(\boldsymbol{R}) \int d^3r' \rho_M(\boldsymbol{r}') R_{lm}(\boldsymbol{r}').$$

We introduce the useful symbol

$$\mathcal{J}_{lm} = \int d^3r \rho_M(\boldsymbol{r}) R_{lm}(\boldsymbol{r}) \tag{7}$$

and we have

$$V(\boldsymbol{R}) = -\sum_{lm} S^*_{lm}(\boldsymbol{R}) \mathcal{J}_{lm}. \tag{8}$$

Incidentally, since $R_{00} = 1$, we have that $\mathcal{J}_{00}$ is the total mass of the asteroid and $\mu = G\mathcal{J}_{00}$, which can be used to simplify equation 4.

The total torque on the asteroid can therefore be written as

$$\boldsymbol{\tau} = \int d^3r \rho(\boldsymbol{r}) \boldsymbol{r} \times \nabla_{\boldsymbol{R}} \sum_{lm} S^*_{lm}(\boldsymbol{R}) \mathcal{J}_{lm}. \tag{9}$$

Here, $\rho(\boldsymbol{r})$ is now the mass density of the asteroid. Since $\nabla_{\boldsymbol{R}} = \nabla_{\boldsymbol{r}}$, we will drop the subscript henceforth.

Note that the bounds of integration of equations 7 and 9 can be chosen to be the surfaces of the bodies in question, or any surface encompassing the bodies. It will be useful in the future to imagine these bounds as spherically symmetrical.

We expand equation 9 using equation 2:

$$\boldsymbol{\tau} = \sum_{lm} (-1)^l \mathcal{J}_{lm} \sum_{l'm'} S_{l+l',m+m'}(\boldsymbol{D}) \int d^3r \rho(\boldsymbol{r}) \boldsymbol{r} \times \nabla R^*_{l'm'}(\boldsymbol{r}).$$

Now we can move straight on to substitution equation 3:

$$\begin{aligned}\boldsymbol{\tau} = &-\frac{1}{2} \sum_{lm} (-1)^l \mathcal{J}_{lm} \sum_{l'm'} S_{l+l',m+m'}(\boldsymbol{D}) \\ &\int d^3r \rho(\boldsymbol{r}) [(i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})(l' - m' + 1) R^*_{l',m'-1}(\boldsymbol{r}) \\ &+ (i\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})(l' + m' + 1) R^*_{l',m'+1}(\boldsymbol{r}) + 2im'\hat{\boldsymbol{z}} R^*_{l'm'}(\boldsymbol{r})].\end{aligned}$$

This equation had better be real. To prove it, let us combine the $\pm m'$ terms, leaving out the $m' = 0$ term and using the subscript o (off-diagonal) to indicate its absence:

$$\begin{aligned}\boldsymbol{\tau}_o = &-\frac{1}{2} \sum_{lm} (-1)^l \mathcal{J}_{lm} \sum_{l'm'_+} \Big\{ S_{l+l',m+m'}(\boldsymbol{D}) \\ &\int d^3r \rho(\boldsymbol{r}) [(i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})(l' - m' + 1) R^*_{l',m'-1}(\boldsymbol{r}) \\ &+ (i\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})(l' + m' + 1) R^*_{l',m'+1}(\boldsymbol{r}) + 2im'\hat{\boldsymbol{z}} R^*_{l'm'}(\boldsymbol{r})] \\ &S_{l+l',m-m'}(\boldsymbol{D}) \\ &\int d^3r \rho(\boldsymbol{r}) [(i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})(l' + m' + 1) R^*_{l',-m'-1}(\boldsymbol{r}) \\ &+ (i\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})(l' - m' + 1) R^*_{l',-m'+1}(\boldsymbol{r}) - 2im'\hat{\boldsymbol{z}} R^*_{l',-m'}(\boldsymbol{r})] \Big\}.\end{aligned}$$

Using the symmetry rules for complex conjugates,

$$\begin{aligned}\boldsymbol{\tau}_o = &-\frac{1}{2} \sum_{lm} (-1)^l \mathcal{J}_{lm} \sum_{l'm'_+} \Big\{ S_{l+l',m+m'}(\boldsymbol{D}) \\ &\int d^3r \rho(\boldsymbol{r}) [(i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})(l' - m' + 1) R^*_{l',m'-1}(\boldsymbol{r}) \\ &+ (i\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})(l' + m' + 1) R^*_{l',m'+1}(\boldsymbol{r}) + 2im'\hat{\boldsymbol{z}} R^*_{l'm'}(\boldsymbol{r})] \\ &S_{l+l',m-m'}(\boldsymbol{D}) \\ &(-1)^{m'} \int d^3r \rho(\boldsymbol{r}) [-(i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})(l' + m' + 1) R_{l',m'+1}(\boldsymbol{r}) \\ &- (i\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}})(l' - m' + 1) R_{l',m'-1}(\boldsymbol{r}) - 2im'\hat{\boldsymbol{z}} R_{l',m'}(\boldsymbol{r})] \Big\}.\end{aligned}$$

Now we expand for $\pm m$ too:

$$\begin{aligned}
\boldsymbol{\tau}_o = & -\frac{1}{2}\sum_{lm_+}(-1)^l\Bigg\{\mathcal{J}_{lm}\sum_{l'm'_+}\Bigg[S_{l+l',m+m'}(\boldsymbol{D}) \\
& \int d^3r\rho(\boldsymbol{r})[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)R^*_{l',m'-1}(\boldsymbol{r}) \\
& +(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)R^*_{l',m'+1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}R^*_{l'm'}(\boldsymbol{r})] \\
& +S_{l+l',m-m'}(\boldsymbol{D}) \\
& (-1)^{m'}\int d^3r\rho(\boldsymbol{r})[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)R_{l',m'+1}(\boldsymbol{r}) \\
& -(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)R_{l',m'-1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}R_{l',m'}(\boldsymbol{r})]\Bigg] \\
& +\mathcal{J}_{l,-m}\sum_{l'm'_+}\Bigg[S_{l+l',-m+m'}(\boldsymbol{D}) \\
& +\int d^3r\rho(\boldsymbol{r})[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)R^*_{l',m'-1}(\boldsymbol{r}) \\
& +(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)R^*_{l',m'+1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}R^*_{l'm'}(\boldsymbol{r})] \\
& +S_{l+l',-m-m'}(\boldsymbol{D}) \\
& (-1)^{m'}\int d^3r\rho(\boldsymbol{r})[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)R_{l',m'+1}(\boldsymbol{r}) \\
& -(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)R_{l',m'-1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}R_{l',m'}(\boldsymbol{r})]\Bigg]\Bigg\}.
\end{aligned}$$

Let us introduce a new notation:

$$\mathcal{K}_{lm}=\int d^3r\rho(\boldsymbol{r})R_{lm}(\boldsymbol{r}) \tag{10}$$

which are the expansion coefficients of the potential of the asteroid. Then the above becomes

$$\begin{aligned}
\boldsymbol{\tau}_o = & -\frac{1}{2}\sum_{lm_+}(-1)^l\Bigg\{\mathcal{J}_{lm}\sum_{l'm'_+}\Bigg[S_{l+l',m+m'}(\boldsymbol{D}) \\
& \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}^*_{l',m'-1}(\boldsymbol{r}) \\
& +(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)K^*_{l',m'+1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}\mathcal{K}^*_{l'm'}(\boldsymbol{r})\Big] \\
& +S_{l+l',m-m'}(\boldsymbol{D}) \\
& (-1)^{m'}\Big[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}_{l',m'+1}(\boldsymbol{r}) \\
& -(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}_{l',m'-1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}\mathcal{K}_{l',m'}(\boldsymbol{r})\Big]\Bigg] \\
& +(-1)^m(-1)^{m+m'}\mathcal{J}^*_{l,m}\sum_{l'm'_+}\Bigg[S^*_{l+l',m-m'}(\boldsymbol{D}) \\
& \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}^*_{l',m'-1}(\boldsymbol{r}) \\
& +(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}^*_{l',m'+1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}\mathcal{K}^*_{l'm'}(\boldsymbol{r})\Big] \\
& +S^*_{l+l',m+m'}(\boldsymbol{D}) \\
& (-1)^{m'}\Big[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}_{l',m'+1}(\boldsymbol{r}) \\
& -(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}_{l',m'-1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}\mathcal{K}_{l',m'}(\boldsymbol{r})\Big]\Bigg]\Bigg\}.
\end{aligned}$$

which can be simplified as

$$\begin{aligned}
\boldsymbol{\tau}_o = &-\frac{1}{2}\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{\mathcal{J}_{lm}\Bigg[S_{l+l',m+m'}(\boldsymbol{D})\\
&\Bigg[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}^*_{l',m'-1}(\boldsymbol{r})\\
&+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)K^*_{l',m'+1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}\mathcal{K}^*_{l'm'}(\boldsymbol{r})\Bigg]\\
&+S_{l+l',m-m'}(\boldsymbol{D})\\
&(-1)^{m'}\Bigg[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}_{l',m'+1}(\boldsymbol{r})\\
&-(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}_{l',m'-1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}\mathcal{K}_{l',m'}(\boldsymbol{r})\Bigg]\Bigg]\\
&+\mathcal{J}^*_{lm}\Bigg[S^*_{l+l',m+m'}(\boldsymbol{D})\\
&\Bigg[-(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}_{l',m'-1}(\boldsymbol{r})\\
&-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}_{l',m'+1}(\boldsymbol{r})-2im'\hat{\boldsymbol{z}}\mathcal{K}_{l',m'}(\boldsymbol{r})\Bigg]\Bigg]\\
&+(-1)^{m'}S^*_{l+l',m-m'}(\boldsymbol{D})\\
&\Bigg[(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)\mathcal{K}^*_{l',m'+1}(\boldsymbol{r})\\
&+(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\mathcal{K}^*_{l',m'-1}(\boldsymbol{r})+2im'\hat{\boldsymbol{z}}\mathcal{K}^*_{l'm'}(\boldsymbol{r})\Bigg]\Bigg\}
\end{aligned}$$

or

$$\begin{aligned}
\boldsymbol{\tau}_o = &-\frac{1}{2}\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{\\
&(l'-m'+1)\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}_{l',m'-1}\\
&-\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l',m'-1}(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\\
&+(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\Bigg)\\
&+(l'+m'+1)\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}_{l',m'+1}\\
&-\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l',m'+1}(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\\
&+(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\Bigg)\\
&+2im'\hat{\boldsymbol{z}}\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\\
&-\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\\
&+(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\Bigg)\Bigg\}.
\end{aligned}$$

This can be written as follows

$$\begin{aligned}
\boldsymbol{\tau}_o = &-\frac{1}{2}\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{\\
&(l'-m'+1)\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\\
&+\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l',m'-1}(-i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}_{l',m'-1}\\
&-(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\Bigg)\\
&+(l'+m'+1)\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\\
&+\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l',m'+1}(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}_{l',m'+1}\\
&-(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})(-i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\Bigg)\\
&+2im'\hat{\boldsymbol{z}}\Bigg(\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\\
&-\mathcal{J}^*_{lm}S^*_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\\
&-(-1)^{m'}\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\\
&+(-1)^{m'}\mathcal{J}^*_{lm}S^*_{l+l',m-m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\Bigg)\Bigg\},
\end{aligned}$$

or

$$\boldsymbol{\tau}_o = -\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{ + (l'-m'+1)\Big(\Re\left[\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(-i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\right] - (-1)^{m'}\Re\left[\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}_{l',m'-1}\right]\Big) + (l'+m'+1)\Big(\Re\left[\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\right] - (-1)^{m'}\Re\left[\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}_{l',m'+1}\right]\Big) - 2m'\hat{\boldsymbol{z}}\Big(\Im\left[\mathcal{J}_{lm}S_{l+l',m+m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\right] - (-1)^{m'}\Im\left[\mathcal{J}_{lm}S_{l+l',m-m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\right]\Big)\Bigg\}.$$

Define

$$\mathcal{J}^{\pm}_{lml'm'} = \mathcal{J}_{lm}S_{l+l',m\pm m'}(\boldsymbol{D})(\pm 1)^{m'+1}.$$

Then

$$\boldsymbol{\tau}_o = -\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{ + (l'-m'+1)\Big(\Re\left[\mathcal{J}^{+}_{lml'm'}(-i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'-1}\right] + \Re\left[\mathcal{J}^{-}_{lml'm'}(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}_{l',m'-1}\right]\Big) + (l'+m'+1)\Big(\Re\left[\mathcal{J}^{+}_{lml'm'}(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',m'+1}\right] + \Re\left[\mathcal{J}^{-}_{lml'm'}(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}_{l',m'+1}\right]\Big) - 2m'\hat{\boldsymbol{z}}\Big(\Im\left[\mathcal{J}^{+}_{lml'm'}\mathcal{K}^*_{l'm'}\right] + \Im\left[\mathcal{J}^{-}_{lml'm'}\mathcal{K}_{l'm'}\right]\Big)\Bigg\}.$$

Further defining

$$\begin{aligned}\chi^{\pm,s}_{lml'm'} &= \mathcal{J}^{\pm}_{lml'm'}\mathcal{K}^*_{l',m'+s} \\ &= (\pm 1)^{m'+1}S_{l+l',m\pm m'}(\boldsymbol{D})\mathcal{J}_{lm}\mathcal{K}^*_{l',m'+s},\end{aligned} \tag{11}$$

we get

$$\boldsymbol{\tau}_o = -\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{ + (l'-m'+1)\Big(\Re\left[\chi^{+-}_{lml'm'}(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\right] + \Re\left[\chi^{--}_{lml'm'}(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\right]\Big) + (l'+m'+1)\Big(\Re\left[\chi^{++}_{lml'm'}(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\right] + \Re\left[\chi^{-+}_{lml'm'}(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\right]\Big) - 2m'\hat{\boldsymbol{z}}\Big(\Im\left[\chi^{+0}_{lml'm'}\right] + \Im\left[\chi^{-0}_{lml'm'}\right]\Big)\Bigg\}$$

(Here I realized I made a mistake earlier; I took the complex conjugate of the top line and forgot to conjugate the vector term, so I do it here) , or

$$\boldsymbol{\tau}_o = -\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{ + (l'-m'+1)\Big(\Re\chi^{+-}_{lml'm'}\hat{\boldsymbol{y}} - \Im\chi^{+-}_{lml'm'}\hat{\boldsymbol{x}} - \Re\chi^{--}_{lml'm'}\hat{\boldsymbol{y}} - \Im\chi^{--}_{lml'm'}\hat{\boldsymbol{x}}\Big) + (l'+m'+1)\Big(- \Re\chi^{++}_{lml'm'}\hat{\boldsymbol{y}} - \Im\chi^{++}_{lml'm'}\hat{\boldsymbol{x}} + \Re\chi^{-+}_{lml'm'}\hat{\boldsymbol{y}} - \Im\chi^{-+}_{lml'm'}\hat{\boldsymbol{x}}\Big) - 2m'\hat{\boldsymbol{z}}\Big(\Im\chi^{+0}_{lml'm'} + \Im\chi^{-0}_{lml'm'}\Big)\Bigg\}.$$

We can write this as

$$\boldsymbol{\tau}_o = \sum_{lm_+}\sum_{l'm'_+}(-1)^l\Bigg\{ + (l'-m'+1)\Big(\Im\chi^{+-}_{lml'm'}\hat{\boldsymbol{x}} - \Re\chi^{+-}_{lml'm'}\hat{\boldsymbol{y}} + \Re\chi^{--}_{lml'm'}\hat{\boldsymbol{y}} + \Im\chi^{--}_{lml'm'}\hat{\boldsymbol{x}}\Big) + (l'+m'+1)\Big(\Re\chi^{++}_{lml'm'}\hat{\boldsymbol{y}} + \Im\chi^{++}_{lml'm'}\hat{\boldsymbol{x}} \; \Im\chi^{-+}_{lml'm'}\hat{\boldsymbol{x}} - \Re\chi^{-+}_{lml'm'}\hat{\boldsymbol{y}}\Big) + 2m'\hat{\boldsymbol{z}}\Big(\Im\chi^{+0}_{lml'm'} + \Im\chi^{-0}_{lml'm'}\Big)\Bigg\},$$

or, joining vectors,

$$\begin{aligned}\boldsymbol{\tau}_o = &\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Big((l'-m'+1)\Im\chi^{+-}_{lml'm'}+(l'-m'+1)\Im\chi^{--}_{lml'm'}\\ &+(l'+m'+1)\Im\chi^{++}_{lml'm'}+(l'+m'+1)\Im\chi^{-+}_{lml'm'}\Big)\\ &+\hat{\boldsymbol{y}}\Big(-(l'-m'+1)\Re\chi^{+-}_{lml'm'}+(l'-m'+1)\Re\chi^{--}_{lml'm'}\\ &+(l'+m'+1)\Re\chi^{++}_{lml'm'}-(l'+m'+1)\Re\chi^{-+}_{lml'm'}\Big)\\ &+2m'\hat{\boldsymbol{z}}\Big(\Im\chi^{+0}_{lml'm'}+\Im\chi^{-0}_{lml'm'}\Big)\Big\}.\end{aligned}$$

Finally,

$$\begin{aligned}\boldsymbol{\tau}_o = &\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\\ &+(l'-m'+1)\chi^{+-}_{lml'm'}+(l'+m'+1)\chi^{-+}_{lml'm'}\Big)\\ &+\hat{\boldsymbol{y}}\Re\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\\ &-(l'-m'+1)\chi^{+-}_{lml'm'}-(l'+m'+1)\chi^{-+}_{lml'm'}\Big)\\ &+2m'\hat{\boldsymbol{z}}\Im\Big(\chi^{+0}_{lml'm'}+\chi^{-0}_{lml'm'}\Big)\Big\}.\end{aligned}$$

Note that

$$\begin{aligned}\chi^{\pm,s}_{l,-m,l',m'} =&(\pm1)^{m'+1}S_{l+l',-m\pm m'}(\boldsymbol{D})\mathcal{J}_{l,-m}\mathcal{K}^*_{l',m'+s}\\ =&(-1)^s(\pm1)^{m'+1}S^*_{l+l',m\mp m'}(\boldsymbol{D})\mathcal{J}^*_{lm}\mathcal{K}_{l',-m'-s}\\ =&(-1)^s{\chi^{\pm,-s}_{l,m,l',-m'}}^*.\end{aligned} \tag{12}$$

and

$$\begin{aligned}\chi^{\pm,s}_{l,-m,l',-m'} =&(\pm1)^{-m'+1}S_{l+l',-m\mp m'}(\boldsymbol{D})\mathcal{J}_{l,-m}\mathcal{K}^*_{l',-m'+s}\\ =&(-1)^s(\pm1)^{m'+1}S^*_{l+l',m\pm m'}(\boldsymbol{D})\mathcal{J}^*_{lm}\mathcal{K}_{l',m'-s}\\ =&(-1)^s{\chi^{\pm,-s}_{lml'm'}}^*\end{aligned} \tag{13}$$

so

$$\begin{aligned}\boldsymbol{\tau}_o = &\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\\ &+(l'-m'+1)\chi^{++}_{l,-m,l',-m'}+(l'+m'+1)\chi^{--}_{l,-m,l',-m'}\Big)\\ &+\hat{\boldsymbol{y}}\Re\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\\ &+(l'-m'+1)\chi^{++}_{l,-m,l',-m'}+(l'+m'+1)\chi^{--}_{l,-m,l',-m'}\Big)\\ &+2m'\hat{\boldsymbol{z}}\Im\Big(\chi^{+0}_{lml'm'}+\chi^{-0}_{lml'm'}\Big)\Big\}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\tau}_o = &\sum_{lm_+}\sum_{l'm'_+}(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+\hat{\boldsymbol{y}}\Re\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+m'\hat{\boldsymbol{z}}\Im\Big(\chi^{+0}_{lml'm'}+\chi^{-0}_{lml'm'}\Big)\Big\}\\ &+\sum_{lm_-}\sum_{l'm'_-}(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+\hat{\boldsymbol{y}}\Re\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+m'\hat{\boldsymbol{z}}\Im\Big(\chi^{+0}_{lml'm'}+\chi^{-0}_{lml'm'}\Big)\Big\}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\tau}_o = &\left(\sum_{lm_+}\sum_{l'm'_+}+\sum_{lm_-}\sum_{l'm'_-}\right)(-1)^l\Big\{\\ &\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+\hat{\boldsymbol{y}}\Re\Big((l'+m'+1)\chi^{++}_{lml'm'}+(l'-m'+1)\chi^{--}_{lml'm'}\Big)\\ &+m'\hat{\boldsymbol{z}}\Im\Big(\chi^{+0}_{lml'm'}+\chi^{-0}_{lml'm'}\Big)\Big\}\end{aligned}$$

Then consider

$$\sum_{l,m\neq 0}\sum_{l',m'\neq 0'}(-1)^l\Big\{ \hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}-(l'-m'+1)\chi^{+-}_{lml'm'}\Big)$$

$$=\sum_{l,m_+}\sum_{l',m'\neq 0'}(-1)^l\Big\{ \hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}-(l'-m'+1)\chi^{+-}_{lml'm'}\Big) \hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{+-}_{l,m,l',-m'}-(l'-m'+1)\chi^{++}_{l,m,l',-m'}\Big)$$

$$=\sum_{l,m_+}\sum_{l',m'_+}(-1)^l\Big\{ \hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}-(l'-m'+1)\chi^{+-}_{lml'm'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{+-}_{l,m,l',-m'}-(l'-m'+1)\chi^{++}_{l,m,l',-m'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'-m'+1)\chi^{++}_{l,m,l',-m'}-(l'+m'+1)\chi^{+-}_{l,m,l',-m'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'-m'+1)\chi^{+-}_{lml'm'}-(l'+m'+1)\chi^{++}_{lml'm'}\Big)$$

$$=\sum_{l,m_+}\sum_{l',m'_+}(-1)^l\Big\{ \hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{++}_{lml'm'}-(l'-m'+1)\chi^{+-}_{lml'm'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'+m'+1)\chi^{+-}_{l,m,l',-m'}-(l'-m'+1)\chi^{++}_{l,m,l',-m'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'-m'+1)\chi^{++}_{l,m,l',-m'}-(l'+m'+1)\chi^{+-}_{l,m,l',-m'}\Big) +\hat{\boldsymbol{x}}\Im\Big((l'-m'+1)\chi^{+-}_{lml'm'}-(l'+m'+1)\chi^{++}_{lml'm'}\Big)$$

Now we want to compute the on-diagonal component of torque, which is where $m'=0$, and $m\in\mathbb{Z}$. That's

$$\boldsymbol{\tau}_d=-\frac{1}{2}\sum_{lm}(-1)^l\mathcal{J}_{lm}\sum_{l'}(l'+1)S_{l+l',m}(\boldsymbol{D}) [(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',-1}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',1}].$$

Now we combine $\pm m$,

$$\boldsymbol{\tau}_d=-\frac{1}{2}\sum_{lm_+}(-1)^l\Big\{\mathcal{J}_{lm}\sum_{l'}(l'+1)S_{l+l',m}(\boldsymbol{D}) [(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',-1}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',1}] +\mathcal{J}_{l,-m}\sum_{l'}(l'+1)S_{l+l',-m}(\boldsymbol{D}) [(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',-1}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\mathcal{K}^*_{l',1}]\Big\} -\frac{1}{2}\sum_{l}(-1)^l\mathcal{J}_{l0}\sum_{l'}(l'+1)S_{l+l',0}(\boldsymbol{D}) [(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})\mathcal{K}^*_{l',-1}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}}\mathcal{K}^*_{l',1})].$$

Note that

$$\chi^{\pm,s}_{lml'0}=\pm S_{l+l',m}(\boldsymbol{D})\mathcal{J}_{lm}\mathcal{K}^*_{l',s},$$

so

$$\boldsymbol{\tau}_d=-\frac{1}{2}\sum_{lm_+}(-1)^l\sum_{l'}(l'+1)\Big\{ [(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{--}_{lml'0}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{++}_{lml'0}] +[(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}}){\chi^{++}_{lml'0}}^*+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}}){\chi^{--}_{lml'0}}^*]\Big\} -\frac{1}{2}\sum_{l}(-1)^l\sum_{l'}(l'+1) [(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{--}_{l0l'0}+(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{++}_{l0l'0}].$$

Since $\chi^{--}_{l0l'0}={\chi^{++}_{l0l'0}}^*$, this simplifies as follows:

$$\boldsymbol{\tau}_d=-\sum_{lm_+}(-1)^l\sum_{l'}(l'+1)\Big(\Re[(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{--}_{lml'0}]+\Re[(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{++}_{lml'0}]\Big) -\sum_{l}(-1)^l\sum_{l'}(l'+1)\Re[(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})\chi^{++}_{l0l'0}]$$

or

$$\boldsymbol{\tau}_d=-\sum_{lm_+}(-1)^l\sum_{l'}(l'+1)\Big\{ \left(-\hat{\boldsymbol{y}}\Re\chi^{--}_{lml'0}+\hat{\boldsymbol{x}}\Im\chi^{--}_{lml'0}\right) +\left(-\hat{\boldsymbol{y}}\Re\chi^{++}_{lml'0}-\hat{\boldsymbol{x}}\Im\chi^{++}_{lml'0}\right)\Big\} -\sum_{l}(-1)^l\sum_{l'}(l'+1)\left(-\hat{\boldsymbol{y}}\Re\chi^{++}_{l0l'0}-\hat{\boldsymbol{x}}\Im\chi^{++}_{l0l'0}]\right)$$

which is

$$\boldsymbol{\tau}_d=-\sum_{lm_+}(-1)^l\sum_{l'}(l'+1)\Big\{ \hat{\boldsymbol{x}}\Im\left(\chi^{--}_{lml'0}-\chi^{++}_{lml'0}\right)-\hat{\boldsymbol{y}}\Re\left(\chi^{++}_{lml'0}+\chi^{--}_{lml'0}\right)\Big\} -\sum_{l}(-1)^l\sum_{l'}(l'+1)\left(-\hat{\boldsymbol{y}}\Re\chi^{++}_{l0l'0}-\hat{\boldsymbol{x}}\Im\chi^{++}_{l0l'0}\right).$$

We can also write this as

$$\boldsymbol{\tau}_d = -\sum_{lm_+}(-1)^l \sum_{l'}(l'+1)\Bigg\{ \hat{\boldsymbol{x}}\Im\left(\chi^{--}_{lml'0}+\chi^{-+}_{lml'0}\right) - \hat{\boldsymbol{y}}\Re\left(\chi^{++}_{lml'0}-\chi^{+-}_{lml'0}\right)\Bigg\} - \sum_{l}(-1)^l \sum_{l'}(l'+1)\left(-\hat{\boldsymbol{y}}\Re\chi^{++}_{l0l'0}-\hat{\boldsymbol{x}}\Im\chi^{++}_{l0l'0}\right)$$

which is

$$\boldsymbol{\tau}_d = -\sum_{lm_+}(-1)^l \sum_{l'}(l'+1)\Bigg\{ \hat{\boldsymbol{x}}\Im\left(\chi^{--}_{lml'0}+\chi^{--}_{l,-m,l',0}\right) - \hat{\boldsymbol{y}}\Re\left(\chi^{++}_{lml'0}+\chi^{++}_{l,-m,l',0}\right)\Bigg\} + \sum_{l}(-1)^l \sum_{l'}(l'+1)\left(\hat{\boldsymbol{y}}\Re\chi^{++}_{l0l'0}+\hat{\boldsymbol{x}}\Im\chi^{++}_{l0l'0}\right)$$

or

$$\boldsymbol{\tau}_d = -\sum_{lm}(-1)^l \sum_{l'}(l'+1)\Bigg\{\hat{\boldsymbol{x}}\Im\chi^{--}_{lml'0}-\hat{\boldsymbol{y}}\Re\chi^{++}_{lml'0}\Bigg\}.$$

Note that the sum now extends over all m.

I forgot to add the case where $m=0$, $m'\neq 0$. So add this part:

$$\boldsymbol{\tau}_d = -\frac{1}{2}\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'\neq 0} S_{l+l',m'}(\boldsymbol{D}) \int d^3r\rho(\boldsymbol{r})[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)R^*_{l',m'-1}(\boldsymbol{r}) + (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)R^*_{l',m'+1}(\boldsymbol{r}) + 2im'\hat{\boldsymbol{z}}R^*_{l'm'}(\boldsymbol{r})]$$

$$\boldsymbol{\tau}_d = -\frac{1}{2}\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'_+}\Bigg\{ \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l',m'-1} + (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l',m'+1} + 2im'\hat{\boldsymbol{z}}S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\Big] + \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)S_{l+l',-m'}(\boldsymbol{D})\mathcal{K}^*_{l',-m'-1} + (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)S_{l+l',-m'}(\boldsymbol{D})\mathcal{K}^*_{l',-m'+1} - 2im'\hat{\boldsymbol{z}}S_{l+l',-m'}(\boldsymbol{D})\mathcal{K}^*_{l'-m'}\Big]\Bigg\}$$

$$\boldsymbol{\tau}_d = -\frac{1}{2}\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'_+}\Bigg\{ \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l',m'-1} + (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l',m'+1} + 2im'\hat{\boldsymbol{z}}S_{l+l',m'}(\boldsymbol{D})\mathcal{K}^*_{l'm'}\Big] + \Big[-(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'+m'+1)S^*_{l+l',m'}(\boldsymbol{D})\mathcal{K}_{l',m'+1} - (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'-m'+1)S^*_{l+l',m'}(\boldsymbol{D})\mathcal{K}_{l',m'-1} - 2im'\hat{\boldsymbol{z}}S^*_{l+l',m'}(\boldsymbol{D})\mathcal{K}_{l'm'}\Big]\Bigg\}$$

$$\chi^{\pm,s}_{l0l'm'} = (\pm 1)^{m'+1}S_{l+l',\pm m'}(\boldsymbol{D})\mathcal{J}_{l0}\mathcal{K}^*_{l',m'+s}, \tag{14}$$

$$\boldsymbol{\tau}_d = -\frac{1}{2}\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'_+}\Bigg\{ \Big[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\chi^{+-}_{l0l'm'} + (i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)\chi^{++}_{l0l'm'} + 2im'\hat{\boldsymbol{z}}\chi^{+0}_{l0l'm'}\Big] + \Big[(-i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1){\chi^{++}_{l0l'm'}}^* + (-i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1){\chi^{+-}_{l0l'm'}}^* - 2im'\hat{\boldsymbol{z}}{\chi^{+0}_{l0l'm'}}^*\Big]\Bigg\}$$

$$\boldsymbol{\tau}_d = -\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'_+}\Bigg\{ \Re[(i\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})(l'-m'+1)\chi^{+-}_{l0l'm'}] + \Re[(i\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}})(l'+m'+1)\chi^{++}_{l0l'm'}] - 2m'\hat{\boldsymbol{z}}\Im\chi^{+0}_{l0l'm'}\Bigg\}$$

$$\boldsymbol{\tau}_d = -\sum_{l}(-1)^l \mathcal{J}_{l0}\sum_{l',m'_+}\Bigg\{ \hat{\boldsymbol{y}}\Re\chi^{+-}_{l0l'm'}(l'-m'+1) - \hat{\boldsymbol{x}}\Im\chi^{+-}_{l0l'm'}(l'-m'+1) - \hat{\boldsymbol{y}}\Re\chi^{++}_{l0l'm'}(l'+m'+1) - \hat{\boldsymbol{x}}\Re\chi^{++}_{l0l'm'}(l'+m'+1) - 2m'\hat{\boldsymbol{z}}\Im\chi^{+0}_{l0l'm'}\Bigg\}$$

$$
\begin{aligned}
\boldsymbol{\tau}_d =& \sum_l (-1)^l \mathcal{J}_{l0} \sum_{l',m'_+} \Big\{ \\
& \hat{\boldsymbol{x}} \Im \left(\chi^{+-}_{l0l'm'}(l'-m'+1) + \chi^{++}_{l0l'm'}(l'+m'+1) \right) \\
& + \hat{\boldsymbol{y}} \Re \left(\chi^{++}_{l0l'm'}(l'+m'+1) - \chi^{+-}_{l0l'm'}(l'-m'+1) \right) \\
& + 2m' \hat{\boldsymbol{z}} \Im \chi^{+0}_{l0l'm'} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\tau}_d =& \sum_l (-1)^l \mathcal{J}_{l0} \sum_{l',m'_+} \Big\{ \\
& \hat{\boldsymbol{x}} \Im \left(\chi^{++}_{l,0,l',-m'}(l'-m'+1) + \chi^{++}_{l0l'm'}(l'+m'+1) \right) \\
& + \hat{\boldsymbol{y}} \Re \left(\chi^{++}_{l0l'm'}(l'+m'+1) + \chi^{++}_{l,0,l',-m'}(l'-m'+1) \right) \\
& + m' \hat{\boldsymbol{z}} \Im \chi^{+0}_{l0l'm'} - m' \hat{\boldsymbol{z}} \Im \chi^{+0}_{l,0,l',-m'} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\tau}_d =& \sum_l (-1)^l \sum_{l',m'\neq 0} \Big\{ \\
& \hat{\boldsymbol{x}} \Im \chi^{++}_{l0l'm'}(l'+m'+1) + \hat{\boldsymbol{y}} \Re \chi^{++}_{l0l'm'}(l'+m'+1) \\
& + m' \hat{\boldsymbol{z}} \Im \chi^{+0}_{l0l'm'} \Big\}
\end{aligned}
$$

So in all,

$$
\begin{aligned}
\boldsymbol{\tau}_d =& -\sum_{lm} (-1)^l \sum_{l'} (l'+1) \Big\{ \hat{\boldsymbol{x}} \Im \chi^{--}_{lml'0} - \hat{\boldsymbol{y}} \Re \chi^{++}_{lml'0} \Big\} + \\
& \sum_l (-1)^l \sum_{l',m'\neq 0} \Big\{ \\
& \hat{\boldsymbol{x}} \Im \chi^{++}_{l0l'm'}(l'+m'+1) + \hat{\boldsymbol{y}} \Re \chi^{++}_{l0l'm'}(l'+m'+1) \\
& + m' \hat{\boldsymbol{z}} \Im \chi^{+0}_{l0l'm'} \Big\}.
\end{aligned}
$$

When the limits are extended to $m_-, m'_- \cup m \geq 0, m' \geq 0$, the off diagonal formula reduces to

$$
\begin{aligned}
\boldsymbol{\tau}_o =& \sum_{l'} \sum_{l,m\geq 0} (-1)^l (l'+1) \Bigg\{ \\
& \hat{\boldsymbol{x}} \Im \Big(\chi^{++}_{lml'0} + \chi^{--}_{lml'0} \Big) + \hat{\boldsymbol{y}} \Re \Big(\chi^{++}_{lml'0} + \chi^{--}_{lml'0} \Big) \Bigg\} \\
& + \sum_l (-1)^l \sum_{l',m'>0} \Bigg\{ \\
& \hat{\boldsymbol{x}} \Im \Big((l'+m'+1) \chi^{++}_{l0l'm'} + (l'-m'+1) \chi^{--}_{l0l'm'} \Big) \\
& + \hat{\boldsymbol{y}} \Re \Big((l'+m'+1) \chi^{++}_{l0l'm'} + (l'-m'+1) \chi^{--}_{l0l'm'} \Big) \\
& + m' \hat{\boldsymbol{z}} \Im \Big(\chi^{+0}_{l0l'm'} + \chi^{-0}_{l0l'm'} \Big) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\tau}_o =& \sum_{l'} \sum_{l,m\geq 0} (-1)^l (l'+1) \Bigg\{ \\
& \hat{\boldsymbol{x}} \Im \Big(-\chi^{--}_{l,-m,l',0} + \chi^{--}_{lml'0} \Big) + \hat{\boldsymbol{y}} \Re \Big(\chi^{++}_{lml'0} + \chi^{++}_{l,-m,l',0} \Big) \Bigg\} \\
& + \sum_l (-1)^l \sum_{l',m'>0} \Bigg\{ \\
& \hat{\boldsymbol{x}} \Im \Big((l'+m'+1) \chi^{++}_{l0l'm'} + (l'+m'+1) \chi^{-+}_{l,0,l',-m'} \Big) \\
& + \hat{\boldsymbol{y}} \Re \Big((l'+m'+1) \chi^{++}_{l0l'm'} - (l'+m'+1) \chi^{-+}_{l,0,l',-m'} \Big) \\
& + m' \hat{\boldsymbol{z}} \Im \Big(\chi^{+0}_{l0l'm'} + \chi^{-0}_{l0l'm'} \Big) \Bigg\}
\end{aligned}
$$