

## Asteroid project summary

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### ABSTRACT

Abstract.

*Keywords:* Keywords, em-dash separated

### 1. INTRODUCTION

### 2. ASTEROID MODEL

In an effort to make a general asteroid model, we will consider the entire effect of the tidal torque applied to an asteroid from a planet, rather than solving for the first order perturbation. To better understand the effect, we will not model perturbing effects from other bodies such as the Sun or other planets. However, nearby bodies, such as moons or rings, are considered in a later section. We will also model the asteroid as a rigid body, whose density distribution is fixed throughout the flyby.

Most asteroids are observed to have attained their minimum energy rotation state, so we will also assume that the asteroid's initial state aligns its rotational velocity parallel to the principal axis with maximal moment of inertia.

#### 2.1. Coordinates

We will make use of two frames of reference to model this system. One is the “inertial frame,” with axes denoted by  $\hat{\mathbf{X}}$ ,  $\hat{\mathbf{Y}}$ ,  $\hat{\mathbf{Z}}$ . As the name suggests, these axes are inertial, with  $\hat{\mathbf{X}}$  pointing to the asteroid pericenter and  $\hat{\mathbf{Z}}$  pointing parallel to the orbit angular momentum. The origin of this frame is set to the center of mass of the central body. We will assume that the mass distribution of the central body is known in this inertial frame.

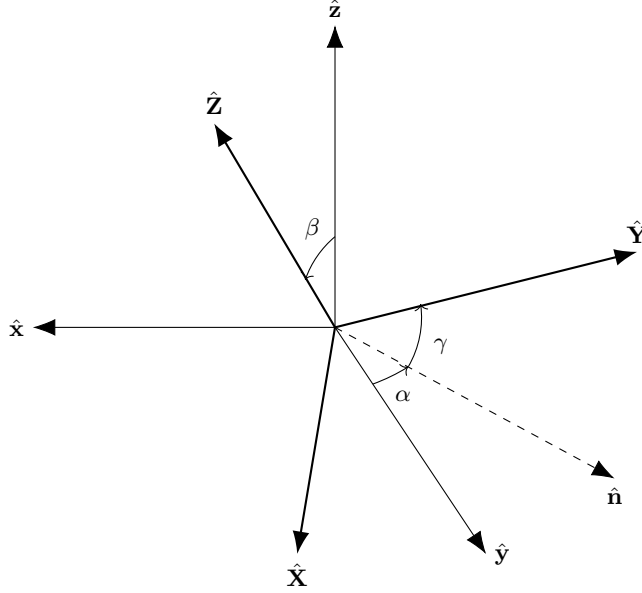
Our second frame is the “body-fixed” frame, denoted by  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ ,  $\hat{\mathbf{z}}$ . This frame is fixed with respect to the body's principle axes and rotates with the body, with its origin at the body's center of mass. We will solve for the asteroid's mass distribution with reference to the body-fixed frame. For definiteness, we define  $\hat{\mathbf{z}}$  to be the principal axis with maximal MOI, and  $\hat{\mathbf{x}}$  to have minimal MOI.

We define a rotation matrix  $M$  such that  $M\mathbf{r} = \mathbf{R}$ , where  $\mathbf{r}$  is in the body-fixed frame and  $\mathbf{R}$  is in the inertial frame. It will be useful for us to represent this matrix in two different ways; one is with a normalized quaternion  $\tilde{\mathbf{q}}$ , and the other is with  $z - y - z$  Euler angles named  $\alpha$ ,  $\beta$ , and  $\gamma$ . In particular, we define

$$M = R_z(\alpha)R_y(\beta)R_z(\gamma) \quad (1)$$

where  $R_i(\theta)$  is a rotation around the  $i$ th unit vector by  $\theta$  (figure 1).

We use the quaternion  $\tilde{\mathbf{q}}$  as our dynamical variable because Euler angles suffer from gimbal lock. For the sake of this paper, we write quaternions as  $\tilde{\mathbf{q}} = r + i\mathbf{i} + j\mathbf{j} + k\mathbf{k}$ , with the complex conjugate written as  $\tilde{\mathbf{q}}^* = r - i\mathbf{i} - j\mathbf{j} - k\mathbf{k}$ . The inverse is then  $\tilde{\mathbf{q}}^{-1} = \tilde{\mathbf{q}}^*/(\tilde{\mathbf{q}}\tilde{\mathbf{q}}^*)$ , and we define real and imaginary parts as  $\Re\tilde{\mathbf{q}} = r$ ;  $\Im\tilde{\mathbf{q}} = i\hat{\mathbf{x}} + j\hat{\mathbf{y}} + k\hat{\mathbf{z}}$ . See a description of quaternions in Ref. Graf (2008)



**Figure 1.**  $z - y - z$  Euler angles used in this work to express the orientation of the asteroid. Orientation is expressed as a rotation from the body-fixed axes to the inertial axes.

Normalized quaternions express rotations of an angle  $\theta$  by  $\tilde{\mathbf{q}} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \tilde{\mathbf{n}}$  where  $\tilde{\mathbf{n}} = 0 + n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k}$ . This rotation may be applied to any vector  $\mathbf{r}$  by

$$\mathbf{R} = \Im [\tilde{\mathbf{q}} \mathbf{r} \tilde{\mathbf{q}}^{-1}]. \quad (2)$$

Due to the real part of  $\tilde{\mathbf{q}}$  which does not vanish for  $\theta \rightarrow 0$ , quaternions do not suffer from gimbal lock. Using the above notation, we can give the full equation of quaternion multiplication:

$$(q_0 + \tilde{\mathbf{q}})(p_0 + \tilde{\mathbf{p}}) = q_0 p_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{q} \times \mathbf{p}. \quad (3)$$

Note that quaternion multiplication is not commutative.

We convert from Euler angles to quaternions by writing equation 1 in terms of quaternions:

$$\tilde{\mathbf{q}} = \begin{pmatrix} \cos \frac{\alpha}{2} \\ 0 \\ 0 \\ \sin \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos \frac{\beta}{2} \\ 0 \\ \sin \frac{\beta}{2} \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\gamma}{2} \\ 0 \\ 0 \\ \sin \frac{\gamma}{2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \cos \frac{\gamma}{2} - \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \sin \frac{\gamma}{2} \\ \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \sin \frac{\gamma}{2} - \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \cos \frac{\gamma}{2} \\ \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \cos \frac{\gamma}{2} + \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \sin \frac{\gamma}{2} \\ \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \sin \frac{\gamma}{2} + \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \cos \frac{\gamma}{2} \end{pmatrix}. \quad (4)$$

This equation can be reversed for  $\alpha, \beta, \gamma$  as follows:

$$\tan \alpha = \frac{jk - ir}{jr + ik} \quad \cos \beta = 1 - 2(i^2 + j^2) \quad \tan \gamma = \frac{jk + ir}{jr - ik}. \quad (5)$$

## 2.2. Tidal Torque

We start by expressing the gravitational potential energy field of the central body in its most general form:

$$V(\mathbf{R}') = -G \int_{\mathcal{P}} d^3 R \rho_M(\mathbf{R}) \frac{1}{|\mathbf{R} - \mathbf{R}'|}. \quad (6)$$

This integral is computed in the inertial frame and has units of energy per unit mass. We represent the central body density as  $\rho_M$  and its volume as  $\mathcal{P}$ . To reduce equation 6, we define the unnormalized spherical harmonics  $Y_{\ell m}(\theta, \phi) = P_{\ell m}(\cos \theta) e^{im\phi}$ , where  $P_{\ell m}$  are the associated Legendre Polynomials without the Condon-Shortley phase. The regular and irregular spherical harmonics are then defined as

$$S_{\ell m}(\mathbf{r}) = (-1)^m (\ell - m)! \frac{Y_{\ell m}(\hat{\mathbf{r}})}{r^{\ell+1}} \quad R_{\ell m}(\mathbf{r}) = (-1)^m \frac{r^{\ell}}{(\ell + m)!} Y_{\ell m}(\hat{\mathbf{r}}). \quad (7)$$

Assume that we can construct a sphere, centered at the center of mass of the central body, so that  $\rho_M = 0$  outside the sphere and  $\rho_m = 0$  inside the sphere. Then  $R' > R$  in equation 6, and Ref. [van Gelderen \(1998\)](#) gives the identity

$$\frac{1}{|\mathbf{R} - \mathbf{R}'|} = \sum_{\ell, m} R_{\ell m}(\mathbf{R}) S_{\ell m}^*(\mathbf{R}'), \quad (8)$$

where the sum is shorthand for  $\sum_{\ell, m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}$ .

Suppose the vector  $\mathbf{D}$  points from the center of mass of the central body to the center of mass of the asteroid, again in the inertial frame. Then we may write  $\mathbf{R}' = \mathbf{D} + \mathbf{U}$ , and Ref. [van Gelderen \(1998\)](#) gives

$$S_{\ell m}(\mathbf{R}') = \sum_{\ell', m'} (-1)^{\ell'} R_{\ell' m'}^*(\mathbf{U}) S_{\ell+\ell', m+m'}(\mathbf{D}). \quad (9)$$

Define the complex, unitless constants

$$J_{\ell m} = \int_{\mathcal{P}} d^3 R \rho_M(\mathbf{R}) \frac{R_{\ell m}(\mathbf{R})}{\mu_M a_M^{\ell}} \quad K_{\ell m} = \int_{\mathcal{A}} d^3 r \rho_m(\mathbf{r}) \frac{R_{\ell m}(\mathbf{r})}{\mu_m a_m^{\ell}} \quad (10)$$

where the integrals are taken in the inertial and the body-fixed frames respectively. Here,  $\rho_m$  denotes the density of the asteroid and  $\mathcal{A}$  denotes its volume. The symmetry relations of  $R_{\ell m}$  guarantee  $J_{\ell m} = (-1)^m J_{\ell, -m}^*$  and similarly for  $K_{\ell m}$ . We have chosen  $\mu_M$ ,  $\mu_m$  to be the masses of the central body and asteroid respectively, and  $a_M$  and  $a_m$  to be scale lengths of both bodies (defined more precisely later). Then we can express the potential as

$$V(\mathbf{U}) = -G \left[ \sum_{\ell, m} \mu_M a_M^{\ell} J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} R_{\ell' m'}(\mathbf{U}) S_{\ell+\ell', m+m'}^*(\mathbf{D}) \right]. \quad (11)$$

To make use of  $K_{\ell m}$  in equation 10, we need to rotate  $\mathbf{U}$  from the inertial frame into  $\mathbf{u}$  in the body fixed frame via  $\mathbf{U} = M\mathbf{u}$ . The rotation of a spherical harmonic satisfies

$$Y_{\ell m}(M\mathbf{u}) = \sum_{m'=-\ell}^{\ell} (-1)^{m+m'} \sqrt{\frac{(\ell-m')!(\ell+m)!}{(\ell+m')!(\ell-m)!}} \mathcal{D}_{mm'}^{\ell}(M)^* Y_{\ell m'}(\mathbf{u}). \quad (12)$$

where  $\mathcal{D}$  are the Wigner- $D$  matrices. The corollary of this formula for the regular solid spherical harmonic is

$$R_{\ell m}(M\mathbf{u}) = \sum_{m'=-\ell}^{\ell} \sqrt{\frac{(\ell-m')!(\ell+m')!}{(\ell-m)!(\ell+m)!}} \mathcal{D}_{mm'}^{\ell}(M)^* R_{\ell m'}(\mathbf{u}) \quad (13)$$

where  $\mathcal{D}_{mm'}^{\ell}$  are the Wigner- $D$  matrix elements. In practice, these are computed from the Euler angles corresponding to the rotation  $M$ , which we specified in section 2.1 are  $z-y-z$  Euler angles. With this choice of Euler angles, the Wigner- $D$  matrix elements are real with matrix elements given by

$$d_{mm'}^{\ell}(\beta) = \sqrt{(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!} \quad (14)$$

$$\sum_{s=s_{\min}}^{s_{\max}} \left[ \frac{(-1)^{m-m'+s} \left(\cos \frac{\beta}{2}\right)^{2\ell+m'-m-2s} \left(\sin \frac{\beta}{2}\right)^{m-m'+2s}}{(\ell+m'-s)!(\ell-m-s)!(m-m'+s)!s!} \right] \quad (15)$$

where  $s_{\min} = \max(0, m' - m)$  and  $s_{\max} = \min(\ell + m', \ell - m)$ , and  $\mathcal{D}_{mm'}^{\ell}(\alpha, \beta, \gamma) = d_{mm'}^{\ell} e^{-im\alpha} e^{-im'\gamma}$ .

Combining equations 13 and 11,

$$V(\mathbf{u}) = -G \left[ \sum_{\ell, m} \mu_M a_M^{\ell} J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* R_{\ell' m''}(\mathbf{u}) \right]. \quad (16)$$

The tidal torque applied to an asteroid at a point  $\mathbf{u}$  is given by  $d\tau(\mathbf{u}) = -\rho_m(\mathbf{u}) d^3 u (\mathbf{u} \times (\nabla V_{\mathbf{u}}(\mathbf{u})))$  in the body-fixed frame, which leaves

$$\tau = G \int_{\mathcal{A}} d^3 u \rho_m(\mathbf{u}) \mathbf{u} \times \nabla \left[ \sum_{\ell, m} \mu_M a_M^{\ell} J_{\ell m} \sum_{\ell', m'} (-1)^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) \right. \quad (17)$$

$$\left. \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* R_{\ell' m''}(\mathbf{u}) \right]. \quad (18)$$

Using the identity

$$\begin{aligned} \mathbf{u} \times \nabla R_{\ell m}(\mathbf{u}) = & \frac{1}{2}[(i\hat{\mathbf{x}} - \hat{\mathbf{y}})(\ell - m + 1)R_{\ell, m-1}(\mathbf{u}) \\ & + (i\hat{\mathbf{x}} + \hat{\mathbf{y}})(\ell + m + 1)R_{\ell, m+1}(\mathbf{u}) + 2im\hat{\mathbf{z}}R_{\ell m}(\mathbf{u})], \end{aligned} \quad (19)$$

which was derived from Ref. [van Gelderen \(1998\)](#)'s analysis of the angular momentum operators applied to spherical harmonics, we write equation 18 as

$$\tau = G \frac{\mu_m \mu_M}{2} \left[ \sum_{\ell, m} a_M^\ell J_{\ell m} \sum_{\ell', m'} a_m^{\ell'} S_{\ell+\ell', m+m'}^*(\mathbf{D}) (-1)^{\ell'} \sum_{m''=-\ell'}^{\ell'} \sqrt{\frac{(\ell'-m'')!(\ell'+m'')!}{(\ell'-m')!(\ell'+m')!}} \mathcal{D}_{m'm''}^{\ell'}(M)^* \right. \quad (21)$$

$$\left. \left[ (i\hat{\mathbf{x}} - \hat{\mathbf{y}})(\ell' - m'' + 1)K_{\ell', m''-1} + (i\hat{\mathbf{x}} + \hat{\mathbf{y}})(\ell' + m'' + 1)K_{\ell', m''+1} + 2im''\hat{\mathbf{z}}K_{\ell' m''} \right] \right]. \quad (22)$$

This equation contains complex values, but it was numerically confirmed that  $\tau$  is a real-valued vector.

Note that each sum in this equation is weighted by a factor of  $a_M^\ell a_m^{\ell'}/D^{\ell+\ell'+1}$ . Since  $D \gg a_M \gg a_m$  in general, the leading order torques come from low  $\ell$  and  $\ell'$ . However,  $\ell' = 0$  contributes nothing because the inner bracketed term is zero, and  $K_{\ell'=1} = 0$  if we choose the origin of the body-fixed axes to be the center of mass of the asteroid. The leading order of equation 22 is therefore  $1/D^3$  with  $\ell = 0$  and  $\ell' = 2$ .

### 2.3. Moment of Inertia

The inertia tensor of the asteroid is expressed as a matrix  $I$ . Written in terms of spherical harmonics in the body-fixed frame, we have

$$I = \int_{\mathcal{A}} d^3r \rho_m(\mathbf{r}) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -xz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \quad (23)$$

$$= \frac{2}{3} \mu_m a_m^2 \begin{pmatrix} K_{20} - 3K_{2,-2} - 3K_{22} + 1 & 3i(K_{22} - K_{2,-2}) & \frac{3}{2}(K_{21} - K_{2,-1}) \\ 3i(K_{22} - K_{2,-2}) & K_{20} + 3K_{2,-2} + 3K_{22} + 1 - \frac{3}{2}i(K_{21} + K_{2,-1}) \\ \frac{3}{2}(K_{21} - K_{2,-1}) & -\frac{3}{2}i(K_{21} + K_{2,-1}) & -2K_{20} + 1 \end{pmatrix} \quad (24)$$

where we have defined, for neatness,

$$a_m^2 = \frac{1}{\mu_m} \int_{\mathcal{A}} d^3r \rho_m(\mathbf{r}) r^2. \quad (25)$$

With this definition of  $a_m$ , and an arbitrary definition of  $a_M$  which may be taken to be similar to equation 25 or a more convenient unit such as the central body radius, the parameters  $J_{\ell m}$  and  $K_{\ell m}$  of equation 10 are completely defined. Note that  $a_m$  is similar to the radius of the asteroid; if the asteroid is a sphere of uniform density and radius  $r$ , then  $a_m = r\sqrt{3/5}$ .

Henceforth, we define the body-fixed axis to be coincident with the principal axes of the asteroids, so that the off-diagonal components of  $I$  are zero.

### 2.4. Equation of Motion

As a simplifying assumption, we will not model the effect of the central body's non-sphericity on the asteroid orbit. This effect has been studied in much detail and is small. We therefore separate the position dynamical variables from the angular variables and derive the standard equations of motion for asteroid position in the inertial frame:

$$\dot{\mathbf{V}} = -\frac{G\mu_M}{r^3} \mathbf{D} \quad \dot{\mathbf{D}} = \mathbf{V}. \quad (26)$$

The position variables will no longer be discussed.

Let  $\omega$  be the rotational velocity of the asteroid in the body-fixed frame. We have two equations of motion for our system; one governs the evolution of  $\omega$  and the other governs the evolution of  $\mathbf{q}$ . Starting with the latter, consider a fixed point  $\mathbf{r}$  in the body-fixed frame on a rotating asteroid. The corresponding point  $\mathbf{R} = \mathfrak{S}[\tilde{\mathbf{q}}\mathbf{r}\tilde{\mathbf{q}}^{-1}]$  in the inertial

frame satisfies  $\dot{\mathbf{R}} = \Im[\tilde{\mathbf{q}}(\omega \times \mathbf{r})\tilde{\mathbf{q}}^{-1}]$ . Furthermore, we can demand that  $\omega \perp \mathbf{r}$  so that  $\omega \times \mathbf{r} = \Im[\tilde{\omega}\tilde{\mathbf{r}}]$ . We obtain another equation for  $\dot{\mathbf{R}}$  by differentiating equation 2 with  $\dot{\mathbf{r}} = 0$ . Setting these equal,

$$\dot{\tilde{\mathbf{q}}}\tilde{\mathbf{q}}^{-1} + \tilde{\mathbf{q}}\dot{\tilde{\mathbf{q}}}^{-1} = \tilde{\mathbf{q}}\tilde{\omega}\tilde{\mathbf{q}}^{-1}. \quad (27)$$

Substituting  $\mathbf{r} \rightarrow \tilde{\mathbf{q}}^{-1}\mathbf{R}\tilde{\mathbf{q}}$  and acknowledging that  $\Re[\dot{\tilde{\mathbf{q}}}\tilde{\mathbf{q}}^{-1}] = 0$ ,

$$\dot{\tilde{\mathbf{q}}}\tilde{\mathbf{q}}^{-1}\mathbf{R} + \mathbf{R}\tilde{\mathbf{q}}\dot{\tilde{\mathbf{q}}}^{-1} = 2\dot{\tilde{\mathbf{q}}}\tilde{\mathbf{q}}^{-1}\mathbf{R} = \tilde{\mathbf{q}}\tilde{\omega}\tilde{\mathbf{q}}^{-1}\mathbf{R}. \quad (28)$$

Since  $\mathbf{R}$  is arbitrary, this simplifies to

$$2\dot{\tilde{\mathbf{q}}}\tilde{\mathbf{q}}^{-1} = \tilde{\mathbf{q}}\tilde{\omega}\tilde{\mathbf{q}}^{-1} \Rightarrow \dot{\tilde{\mathbf{q}}} = \frac{1}{2}\tilde{\mathbf{q}}\tilde{\omega}. \quad (29)$$

The second equation of motion are Euler's equations of motion given our expression for tidal torque. Namely,

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = \tau_1 \quad (30)$$

$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = \tau_2 \quad (31)$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = \tau_3 \quad (32)$$

The components of  $\tau$  are given by equation 22, which was also in the body-fixed frame.

### 2.5. Initial Values

We assume that the asteroid begins the simulation in its minimum energy configuration; that is, with  $\omega \parallel \hat{\mathbf{z}}$ . The initial conditions are therefore that  $\omega = \omega\hat{\mathbf{z}}$ , and the quaternion is fixed to rotate  $\hat{\mathbf{z}}$  to  $\boldsymbol{\Omega}$  in the inertial frame, which fixes Euler angles  $\beta$  and  $\alpha$ . The third Euler angle  $\gamma$  represents the initial roll of the asteroid, and therefore cannot be determined from physical data. However, it is an initial condition of the simulation. Once the Euler angles are known, the quaternion is computed via equation 4.

### 2.6. Fit Parameters

By requiring that the  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$  axes coincide with principal axes, we force the diagonal components of  $I$  to be zero. Thus, we may require  $K_{21} = K_{2,-1} = 0$  and  $\Im K_{22} = \Im K_{2,-2} = 0$  with no loss of generality. By requiring that the body-fixed frame be located at the center of mass of the asteroid, we also require  $K_{1m} = 0$ . Finally,  $K_{00} = 1$  follows automatically from the definition. Thus,  $K_{20}$ ,  $\Re K_{22}$ , and  $K_{\ell>2,m}$  are the only physical parameters of the asteroid density distribution, beyond  $a_m$ .

As discussed before, the leading order of equation 22 is  $\ell' = 2$ . Therefore, the equation is proportional to  $\mu_m a_m^2$ . Factoring out this product results in an equation of motion independent of  $\mu_m$ , meaning that the asteroid dynamics are unaffected by the mass of the asteroid. This is expected, and it means that the density distribution of the asteroid can only be determined up to a scale factor.

Since the coefficient of  $J_{lm}K_{\ell'm'}$  is proportional to  $a_m^\ell a_m^{\ell'}/D^{\ell+\ell'+1}$ , to first order in the asteroid density distribution, the system is also independent of  $a_m$ . When  $K_{\ell'>2}$  terms are included,  $a_m$  scales the size of the higher order terms.

We require that the moment of inertia matrix be positive-definite, which bounds our asteroid parameters. Specifically, we have  $K_{20} \leq \frac{1}{2}$ , and  $|K_{22}| \leq (K_{20} + 1)/6$ . We can also choose without loss of generality to let the maximum moment of inertia align with  $\hat{\mathbf{z}}$ , so that  $1 - 2K_{20} \geq K_{20} + 1 + 6|K_{22}|$ . Finally, we wish to require that the density of the asteroid be everywhere positive. One consequence of this requirement is that the sum of any two distinct moments of inertia cannot be greater than the third, by the integral definition of moment of inertia in equation 24. Three inequalities result from this, but only one is not covered by the above restrictions. Namely,  $I_{xx} + I_{yy} \geq I_{zz} \Rightarrow K_{20} \geq -1/4$ . The intersection of these requirements forces  $K_{20}$  and  $K_{22}$  to lie in the triangle defined by

$$-\frac{1}{4} \leq K_{20} \leq 0 \quad |K_{22}| \leq -\frac{K_{20}}{2}. \quad (33)$$

We have one additional fit parameter, which is the only initial condition not fixed by the problem setup or observations. Namely, the initial value of the first Euler angle  $\alpha_0$ . Due to the fact that the principal axes  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are orthogonal, increments of  $\gamma_0$  by  $\pi/2$  correspond to flipping the body-fixed axes so that no physical change in the asteroid model occurs. We must therefore limit the range of  $\gamma_0$  to width  $\pi/2$ . Arbitrarily, we choose  $\gamma_0 \in [-\pi/4, \pi/4]$

### 3. EXPERIMENT DESIGN

#### 3.1. Simulation Design

Given the observed orbit of an asteroid and observational data of its resolved spin over time, we built a simulation that would integrate equations 29 and 32 to generate synthetic data. Inputs to the simulation were (1) the  $\ell$  and  $\ell'$  values at which to truncate the potential energy sum in equation 22, (2) the cadence of observations, (3) the  $J_{\ell m}$  parameters and mass and radius of the central body, (4) trial asteroid parameters  $K_{\ell m}$ ,  $\gamma_0$  and  $a_m$ , (5) the initial rotational velocity of the asteroid in the inertial frame, and (6) the asteroid's orbital parameters.

The orbital parameters we use to describe the orbit are the hyperbolic excess velocity  $v_\infty$  and the pericenter altitude  $r_p$ . The orbit orientation is fixed to lie in the  $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$  plane, with pericenter lying on  $\hat{\mathbf{X}}$ . The position of the satellite as a function of time is therefore generated by numerically integrating the Newtonian equation of motion (equation 26) from initial values  $\mathbf{r}(t=0) = r_p \hat{\mathbf{X}}$  and  $\mathbf{v}(t=0) = \dot{\mathbf{Y}} \sqrt{v_\infty^2 + 2G\mu_M/r_p}$ .

Once the inputs were specified, the moments of inertia were pre-computed from  $K_{\ell m}$ , as was the orbital path of the satellite as a function of time. Equations 29 and 32 were then integrated by the Runge-Kutta fourth order method with a time step of *describe* seconds, with a custom implementation of the spherical harmonics and Wigner-D matrices to improve runtime. At fixed time intervals of one cadence, the rotational velocity in the body-fixed frame  $\omega_{xyz}$  was converted via equation 2 to the inertial frame to produce the synthetic data set.

We begin our simulation at  $D = 10r_p$ . Since the leading order of the equations of motion is  $\ell' = 2, \ell = 0$ , this corresponds roughly to a torque of  $10^{-3}$  times the maximum torque at perigee. Unless otherwise indicated, the simulation is terminated at  $D = 10r_p$  as well.

#### 3.2. Fitting Method

To analyze the degree to which asteroid density distributions can be determined from flyby rotational velocity data, we use the forward model defined in section 3.1 to generate synthetic data, randomize the data according to one of the observational uncertainty model defined in section 3.3, then use the same forward model to fit to the data and recover parameters.

After randomizing the data, we isolate a set of parameters which are likely to provide good fits. To do this, we minimize the  $\chi^2$  value of the data given the model using the Nelder-Mead algorithm, implemented in `scipy` Gao & Han (2012). Our starting points for the minimization are spaced uniformly randomly across the allowed parameter space, which is the set of parameters with the triangle described by equation 33. The other parameter bounds are  $\gamma_0 \in [-\pi/4, \pi/4]$ ,  $|\Re K_{l>2}| \leq 1$ , and  $|\Im K_{l>2}| \leq 1$ .

If the minimization is conducted over too many parameter dimensions, convergence can be slow or can fail. However, the parameters  $K_{\ell m}$  are naturally separated into groups by order of  $\ell$ , since the torque applied by successive increment to  $\ell$  is suppressed by  $a_m/D$ . Thus, we fit the  $\ell = 2$  and  $\gamma_0$  parameters first, then lock them in place and fit the  $\ell = 3$  parameters, etc. The result is a set of points in parameter space which minimize the  $\chi^2$ .

Even with this ordering of parameters, the system's strong dependence on asteroid shape makes convergence rare. The data is therefore truncated at distance  $D = fr_p$ , where  $f$  is a manually-set fraction. Once minimization is conducted for the truncated data, the resulting parameters are used as the initial point for a minimization over the entire data set. Experimentation revealed that  $f = 2$  was appropriate, where the cut is performed after the perigee for the  $\ell = 2$  parameters and before the perigee for the  $\ell = 3$  parameters.

We are interested in finding not just the optimal parameters, but the probability distributions of each fit parameter. To do this, we use an Affine Invariant Markov Chain Monte Carlo Ensemble Sampler (MCMC), implemented in the Python package `emcee`, to conduct the fit to data Foreman-Mackey et al. (2013). This fit method requires the parameter space to be populated with “walkers,” indicating the a guess at the true parameters of the system. We center 32 walkers on the minimizing parameters computed by the minimization of  $\chi^2$ , and spread out the walkers such that the number density of walkers is proportional to the likelihood ( $\ln \mathcal{L} = -\chi^2/2$ ) at each point. This is done explicitly by computing the Hessian  $\Sigma$  of the  $\chi^2$  value at each minimizing point  $\theta_0$  and expressing the likelihood locally as a multi-dimensional Gaussian

$$\mathcal{L}(\theta) \propto \exp\left(-\frac{1}{2}(\theta - \theta_0)^T \Sigma (\theta - \theta_0)\right). \quad (34)$$

We may change bases into a frame in which the parameters are uncorrelated by multiplying by  $V^T$ , where  $V$  is the matrix of eigenvectors of  $\Sigma$ . Populating the walkers in this uncorrelated space according to Gaussian standard deviations given by the eigenvalues of  $\Sigma$ , we can convert these walkers back to  $\theta$ -space by multiplying by  $V$ . In practice,

we only compute the components of  $\Sigma$  for parameters that are found in the same minimization step; for example, we do not compute the entries of  $\Sigma$  corresponding to  $K_{20}$  and  $K_{30}$ , setting it equal to zero instead.

Occasionally, the likelihood is non-Gaussian enough for  $\Sigma$  to have negative or otherwise nonsensical eigenvalues (this occurs for strongly varying likelihood or due to numerical errors in the calculation of the Hessian). In these cases, we choose eigenvalues of between *limit* and *limit*.

With the initial walkers distributed, we compute the posterior probability distribution of  $\theta$  given flat priors in the range outlined above for each parameter. We run the MCMC until results converge (defined as the condition that the autocorrelation time is less than one hundredth of the current iteration number and that the fractional difference between autocorrelation times is under 1%), or for 100,000 iterations, whichever is smaller. The MCMC is observed only to run for all 100,000 iterations if the data is insufficiently precise to constrain some parameters.

### 3.3. Uncertainty Models

We model uncertainties in experimental data according to several methods, comparing them to determine the dependence of our conclusions on the sensitivity model used. In every case, the sensitivity model is used both to add randomness to the data set after it is generated, and to weight the  $\chi^2$  statistic.

#### 3.3.1. Nominal Method

The most detailed uncertainty model we use, and the one we use most often, is called the “nominal method.” It rotates each angular velocity vector by some angle  $\Theta$  drawn from a normal distribution centered on  $\Theta = 0$  with standard deviation  $\sigma_\theta$ . The period is assumed to be known to arbitrary precision at each cadence. The direction in which the velocity is rotated is chosen from a uniformly random distribution.  $\sigma_\theta$  is a general parameter which can be tweaked to model resolution of the spin axis direction.

Practically, the randomization of the data is done by creating a rotation matrix  $M$  that rotates  $\hat{\mathbf{Z}}$  to the true rotational velocity vector  $\omega$ . This same rotation matrix is then used to rotate the random vector  $\mathbf{\Omega} = \sin \Theta \cos \Phi \hat{\mathbf{X}} + \sin \Theta \sin \Phi \hat{\mathbf{Y}} + \cos \Theta \hat{\mathbf{Z}}$  to the randomized velocity vector. Here,  $\Phi$  is a uniformly random variable in the range  $[0, \pi)$ . Specifically, we write this rotation matrix as

$$M = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad (35)$$

where  $\theta$  and  $\phi$  are the spherical coordinates of the original velocity vector  $\omega$ .

Equation 35 is sufficient to randomize synthetic data, but an analytical expression for the coordinates of  $\omega$  is required to define the  $\chi^2$  value used for fitting. We approximate the distribution of each spin coordinate as normal, with covariance given by the covariance of the true distribution. Based on equation 35, the covariance matrix is

$$\text{Cov}(\omega) = \frac{1}{4} \left(1 - e^{-\sigma_\theta^2}\right) \left(1 - 3e^{-\sigma_\theta^2}\right) \begin{pmatrix} \omega_x^2 & \omega_x \omega_y & \omega_x \omega_z \\ \omega_x \omega_y & \omega_y^2 & \omega_y \omega_z \\ \omega_x \omega_z & \omega_y \omega_z & \omega_z^2 \end{pmatrix} + \frac{\omega^2}{4} \left(1 - e^{-2\sigma_\theta^2}\right) \mathbf{1}. \quad (36)$$

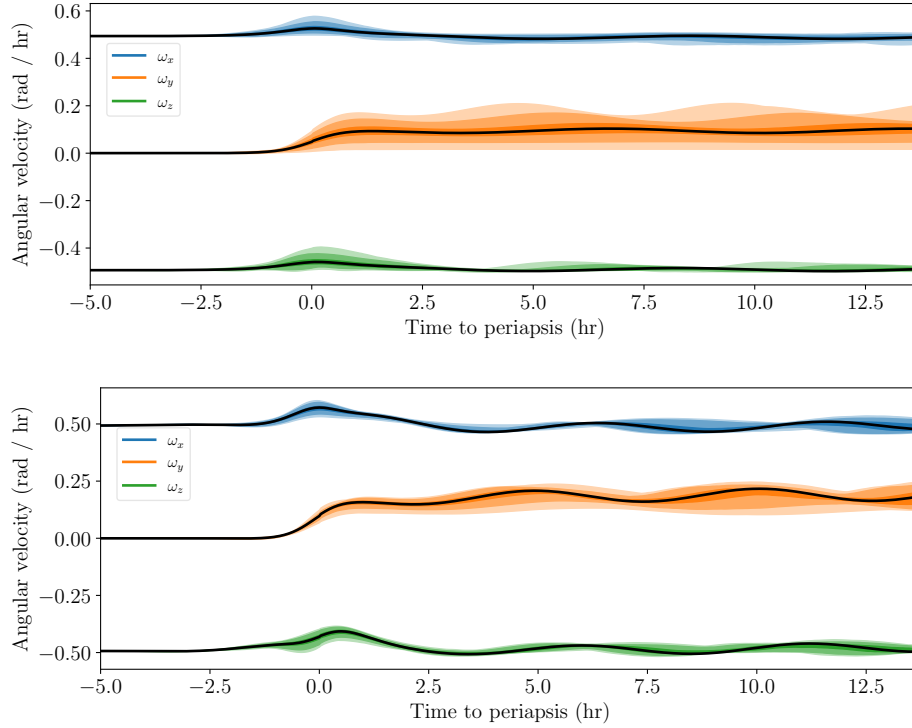
We then define the  $\chi^2$  statistic to be

$$\chi^2 = \sum_{i=0} (\omega_i^* - \omega_i)^T \text{Cov}^{-1}(\omega_i^*) (\omega_i^* - \omega_i) \quad (37)$$

where  $\omega_i$  is the  $i$ th expected spin and  $\omega_i^*$  is the data. We also refer to the reduced chi squared,  $\chi_r^2 = \chi^2/(3N)$ , where  $N$  is the number of  $\omega_i$  in the data set.

In figure 2 we present example data for this asteroid using the two reference asteroid and orbit parameters described in appendix A. Additionally, one thousand simulations are also run with asteroid parameters  $\gamma_0$ ,  $K_{20}$ , and  $K_{22}$  changed slightly, distributed according to the likelihood defined by equation ?? with this nominal uncertainty method and  $\sigma_\theta = 0.2$ . The method for distributing these shape parameters is the same as the method for distributing the MCMC initial walker positions, described in section 3.2. Bands containing 68.3%, 95.5%, and 99.7% of these additional simulations are also plotted in figure 2.





**Figure 2.** Angular velocity data simulated for the symmetric and asymmetric reference asteroid densities and the reference orbit (black lines). Also plotted is the deviation of the data for likelihood-distributed changes to the asteroid shape (bands) using the nominal uncertainty method. Bands contain 68.3%, 95.5%, and 99.7% of the 1000 simulations run.

### 3.3.2. Position dependent method

In practice, precision on the asteroid spin axis will increase with decreasing distance *Citation needed*. The nominal sensitivity model does not account for this; however, we can approximate this effect by choosing  $\sigma_\theta$  from the nominal method to be position-dependent. Specifically, we choose

$$\sigma_\theta(D) = \text{unimplemented}. \quad (38)$$

We do not expect this change to differ qualitatively from the nominal method because resolution is most important near pericenter, where  $D$  is stationary to first order.

## 3.4. System Setup

We study several environments for our asteroid, including (1) a flyby of a single, spherical planet, (2) a close flyby of a ringed planet, (3) a flyby of a planet with a moon, and (4) a close flyby of a planet with a moon. The qualitative and quantitative differences between these flyby types are discussed below.

### 3.4.1. One Spherical Planet

The simplest setup for an asteroid flyby is a close flyby of a single, spherical body. We call this central body a “planet” because it is likely that a body massive enough to observably torque an asteroid would be a planet. For such a body, the only relevant physical parameter is its mass,  $\mu_M$ , since  $J_{00} = 1$  and the other  $J_{\ell m}$  parameters are zero.

### 3.4.2. Ringed Planet

Equation 22 demonstrates that parameters controlling non-sphericity in the central body ( $J_{2m}$ ,  $J_{3m}$ , etc.) are damped by a factor of  $(a_M/D)^\ell$ , where  $a_M$  can be thought to be the radius of the central body. Furthermore, the  $J_{1m}$  terms are all zero by choosing the center of mass of the central body to be the origin of the coordinate system. Thus, we are interested in the extreme values of  $J_{\ell \geq 2}$  and whether they have a significant effect on the observability of asteroid density distributions. An example of such an extreme value is the presence of a planetary ring.



For a ring of inner radius  $a$ , outer radius  $b$ , and mass  $\mu_r$  oriented in the  $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$  plane, the only changed  $J_{\ell m}$ s have  $m = 0$  by rotational symmetry. Furthermore,  $Y_{\ell, m} = 0$  on the equator for odd  $m$ , and  $(-1)^{\ell/2} \ell! / (2^\ell (\ell/2)!)^2$  otherwise. Thus, we have

$$J_{\ell m} = \begin{cases} \frac{2\pi\sigma}{\mu_M + \pi(b^2 - a^2)\sigma} \frac{(-1)^{\ell/2}}{a_M^\ell} \frac{b^{\ell+2} - a^{\ell+2}}{2^\ell (\ell/2)!^2 (\ell+2)} & \text{for } m = 0, \ell \text{ even}, \ell \neq 0 \\ 1 & \ell = 1 \\ 0 & \text{else} \end{cases} \quad (39)$$

where  $\mu_M$  is the mass of the planet without the ring and  $\sigma_s$  is the surface density of the ring.

The oblateness  $\epsilon$ , also called  $J_2$  of a body is defined as  $\epsilon = (I_z - I_x) / (\mu_M a_M^2)$ , where  $I_x = I_y$  (that is,  $J_{22} = 0$ ) and  $a_M$  is the true radius of the body, not the value defined in equation 25. Expressed in terms of  $J_{\ell m}$ , this is  $\epsilon = -2J_{20}$ . We can compare typical oblateness to the  $J_{20}$  value from equation 39 to see which effect is more dramatic.

Using data from the NASA Saturn Fact Sheet, we arrive at  $J_{20} \sim 10^{-8}$  from rings, but  $10^{-4}$  from oblateness. So oblateness is in fact the stronger effect.

*Don't actually use this section in the paper. I'm keeping it here as a math repository. Change the section to an oblate planet.*

### 3.5. Planet with a Moon: Distant Flyby

Consider a system with a moon of mass  $\mu_o$  at constant displacement  $\mathbf{R}$  from the central body. We assume that both the planet and the moon are spherical. As long as the distance from the barycenter of the planet-moon system to the asteroid  $D$  always satisfies  $D > r$  we are free to use the formalism developed above, absorbing the moon into  $J_{\ell m}$ . Making use of the identity van Gelderen (1998)

$$R_{\ell m}(\mathbf{R} + \mathbf{R}') = \sum_{\ell'=0}^{\ell} \sum_{m'=-\ell'}^{\ell'} R_{\ell' m'}(\mathbf{R}') R_{\ell-\ell', m-m'}(\mathbf{R}) \quad (40)$$

*Is this true?* for  $R > R'$ , we have

$$J_{\ell m} = \frac{1}{(\mu_M + \mu_o) a_M^\ell} \left[ \delta_{\ell 0} \delta_{m 0} \mu_M + \int_{\mathcal{P}} d^3 r' \rho_o(\mathbf{r}') R_{\ell m}(\mathbf{R} + \mathbf{r}') \right] \quad (41)$$

$$J_{\ell m} = \frac{1}{(\mu_M + \mu_o)} \left[ \mu_M \delta_{\ell 0} \delta_{m 0} + \mu_o \frac{R_{\ell m}(\mathbf{R})}{a_M^\ell} \right] \quad (42)$$

where the  $\delta$ s are Kronecker deltas. It is apparent that if  $a_M^\ell$  remains the radius of the planet,  $|J_{\ell m}| \rightarrow \infty$  as  $\ell \rightarrow \infty$ . However, since each  $J_{\ell m}$  is damped by  $a_M^\ell / D^\ell$  and since  $R < D$  by construction, the series in equation 22 remains roughly geometric and does not blow up. For the sake of this calculation, we neglect motion of the moon during the flyby.

For our tests, we use parameters similar to our Moon: we set  $\mu_o = 0.01\mu_M$  and  $R = 60a_M$ .

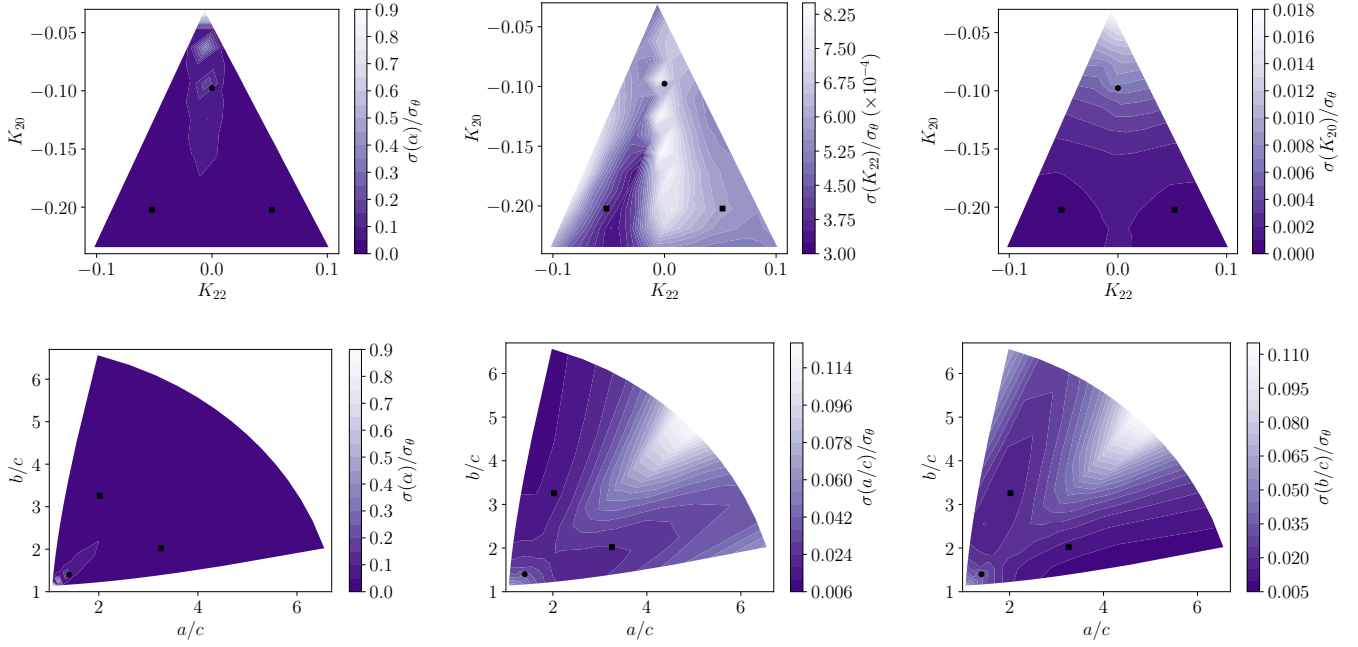
### 3.6. Planet with a Moon: Close Flyby

If, contrary to the previous section, we want to fly by both the moon and the planet of a binary system, we must consider the tidal torque contributions from both bodies separately. The formalism above cannot be extended to write one set of  $J_{\ell m}$  for both the moon and planet. To do this, we generate two separate paths of  $\mathbf{D}(t)$ , one pointing from the central body to the asteroid and one pointing from the moon to the asteroid. Both are defined in the inertial reference frames. Then the torque from both can be computed by equation 22 and added to get the net torque on the asteroid. Again, we neglect motion of the moon during the flyby.

## 4. RESULTS

To interpret the capacity of this tidal torque technique to constrain the density distribution of asteroids, we generate synthetic data in multiple simulated settings and fit to this data using an MCMC, reporting the standard deviation of the posterior parameter distributions as a measure of the effectiveness of the fit. Fits with narrow posterior distributions are considered more effective than those with wide posterior distributions, and the correlations between parameters are also of interest when computing uncertainties on physical models for the asteroid shape.

The distributions are nearly Gaussian in most cases (see appendix C) and all fits are run until convergence.



**Figure 3.** Ratios of fit uncertainties (the standard deviations of parameter posterior distributions) to experimental uncertainty ( $\sigma_\theta$ ) for fit parameters  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top row) and  $\gamma_0$ ,  $a/c$ , and  $b/c$  (bottom row). Also shown are the reference asteroid shapes used elsewhere in this paper (appendix A); the symmetric case is marked with a circle and the asymmetric with a square.

#### 4.1. Dependence on asteroid parameters

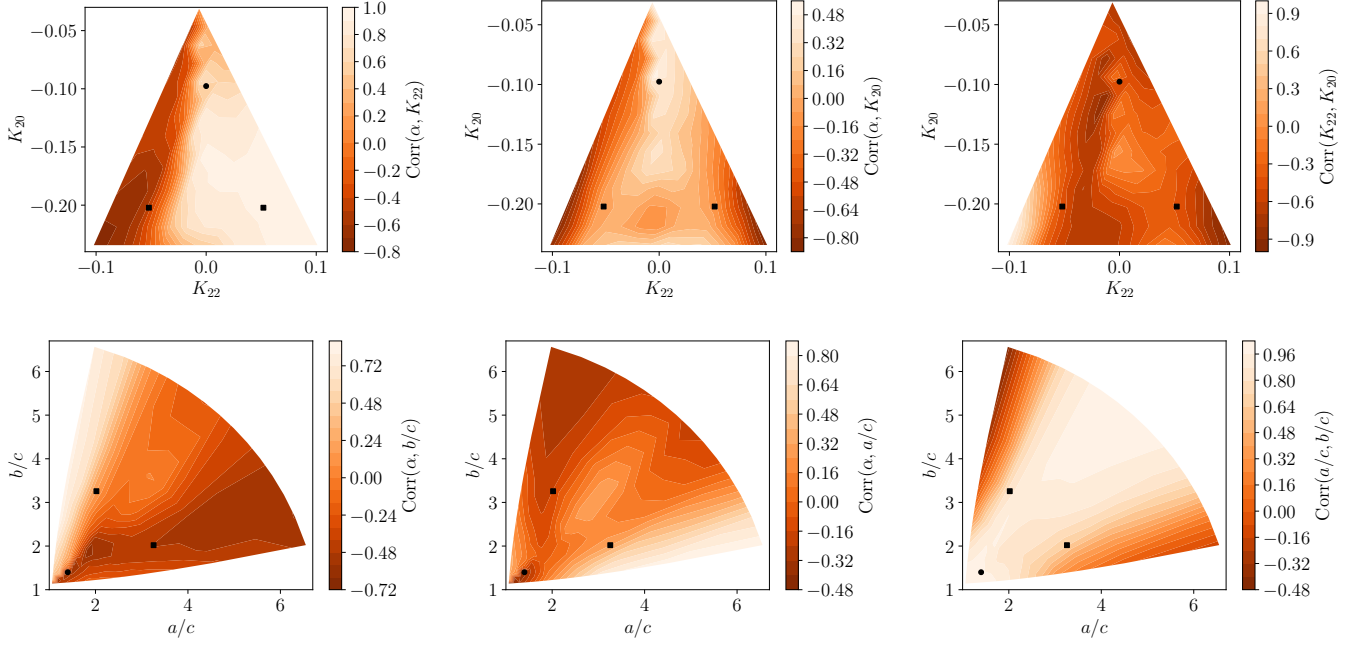
Of first importance is the dependence of parameter uncertainties on the true parameter values. To measure this, we run 120 fits, all for the two  $K_{\ell=2}$  parameters and  $\gamma_0$  with the standard orbit (appendix ??) and experimental uncertainty  $\sigma_\theta = 0.01$  with the nominal uncertainty model (section 3.3). Each fit is run for  $\gamma_0 = \pi/8$  and  $K_{20}, K_{22}$  distributed in a triangular lattice with 17 layers, spanning the parameter space. The 33 cases with extremal values of  $K_{22}$  are not fitted as being too close to the edge of the parameter space.

The parameters  $K_{22}$  and  $K_{20}$  are then converted to the more physical parameters  $a/c$  and  $b/c$  under the assumption of an elliptical asteroid of uniform density with minor axes  $a, b, c$  (section 5.1.1). Uncertainties on  $\gamma_0$  and the two shape parameters are given in figure 3, and correlations between pairs of parameters ( $\text{Corr}(X, Y) = \text{Var}(X, Y)/(\sigma_X \sigma_Y)$ ) are given in figure 4.

These figures demonstrate several key properties. Figure 3 demonstrates that fit uncertainties for all three parameters are generally for asteroids with an axis of rotational symmetry — i.e.  $a = b$  or  $K_{22} = 0$ . This is likely due to the fact that the local axes, which are required to define the parameters, are set such that they align with the asteroid principal axes, and therefore are degenerate in the case of a symmetrical asteroid. For similar reasons, uncertainty in  $\gamma_0$  especially is large for  $a \approx b \approx c$ , which is the spherical case in which no tidal torque is generated. Beyond these two trends, uncertainty in  $a/c$  and  $b/c$  appear to scale linearly with these ratios.

Figure 4 demonstrates that the parameters are generally correlated. Specifically,  $\gamma_0$  and  $|K_{22}|$  are strongly correlated for large  $|K_{22}|$  which corresponds to highly asymmetric asteroids. Likewise, for the correlation of  $a/c$  and  $b/c$  with  $\gamma_0$ .  $\text{Corr}(K_{20})$  shows slightly different behavior, since they are still correlated for asteroids with  $a = b$ . However, the correlation still increases for highly asymmetric asteroids. The correlation between shape parameters  $K_{20}$  and  $K_{22}$  is often near zero except in highly asymmetric asteroids, but the correlation between  $a/c$  and  $b/c$  is rarely zero; these correlate for  $a \approx b$  and anti-correlate for  $a \not\approx b$ . The  $K_{\ell m}$  coordinates may therefore be a better lens through which to view asteroid shape, due to their low correlation.

Also shown in figures 3 and 4 are the reference asteroid configurations which we study when fixing the asteroid shape and varying other parameters. We note that these reference configurations capture the above trends in variance and correlation, so that they can be viewed as representative of all but the most extreme asteroid configurations. The measured values of variance and covariance are displayed in table 1.



**Figure 4.** Correlations between fit uncertainties and experimental uncertainty ( $\sigma_\theta$ ) for fit parameters  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top row) and  $\gamma_0$ ,  $a/c$ , and  $b/c$  (bottom row). Also shown are the reference asteroid shapes used elsewhere in this paper (appendix A); the symmetric case is marked with a circle and the asymmetric with a square.

Reference point	$\sigma(\gamma_0) / \sigma_\theta$	$\sigma(K_{22} [a/c]) / \sigma_\theta$	$\sigma(K_{20} [b/c]) / \sigma_\theta$	$\text{Corr}(\gamma_0, K_{22})$	$\text{Corr}(\gamma_0, K_{20})$	$\text{Corr}(K_{22}, K_{20})$
Symmetric: $a = b$	0.190	0.000763	0.00726	0.707	0.508	-0.205
Asymmetric: $a < b$	0.0148	0.0412	0.0390	-0.400	-0.614	0.966
Asymmetric: $a > b$	0.0110	0.000410	0.000505	-0.553	-0.0716	-0.356
		0.0142	0.0153	-0.190	0.213	0.791
		0.000582	0.000651	0.913	-0.138	-0.432
		0.0238	0.0128	0.328	-0.485	0.636

**Table 1.** Values for the standard deviation and correlation of  $\gamma_0$ ,  $K_{22}$ , and  $K_{20}$  (top entry of each row) or  $\gamma_0$ ,  $a/c$ , and  $b/c$  respectively (bottom entry of each row) for the symmetric and asymmetric reference points.

#### 4.2. Dependence on observational precision

#### 4.3. Dependence on orbital parameters

#### 4.4. Dependence on spin pole

#### 4.5. Dependence on cadence

#### 4.6. Dependence on blocking out perigee

#### 4.7. Exotic scenarios

##### 4.7.1. Oblate planet

##### 4.7.2. Distant moon

##### 4.7.3. Close moon

### 5. INTERPRETATION OF THE FIT PARAMETERS

The physical parameters  $K_{\ell m}$  resulting from a fit to flyby data, while physical, do not completely describe an asteroid. Since the density distribution of the asteroid has many more degrees of freedom in general than the number of known  $K_{\ell m}$ , there are often many degenerate density distributions associated with one set of  $K_{\ell m}$ . As an example,

consider a spherical asteroid of radius  $A$  and density distribution of the form  $\rho(\mathbf{r}) = \sum_{\ell m} f_{\ell m}(r) Y_{\ell m}^*(\hat{\mathbf{r}})$ . Then we have

$$K_{\ell m} = N_{\ell m} \frac{(-1)^m}{(\ell + m)!(\mu_m a_m^\ell)} \int_0^A r^{2+\ell} dr f_{\ell m}(r) \quad (43)$$

where  $N_{\ell m}$  is given by equation ???. Thus,  $K_{\ell m}$  constrains only the integrals of  $f_{\ell m}$ , not the functions themselves. By relaxing the assumption that the asteroid is a sphere, even more degeneracy is introduced.

To avoid this degeneracy, we either (1) assume that the density is constant and solve for the asteroid surface (section ???), (2) assume that the surface of the asteroid is fixed and solve for  $\rho(\mathbf{r})$ . In the latter case, we can avoid the above degeneracy either by assuming  $\nabla^2 \rho = 0$  (section ???) or by using a numerical approach to find a more characteristic density distribution (section ???).

*Not all of these models will be in the paper. This is a list of all the models I've thought of. I'll give some examples of applying these models to "realistic" asteroid density profiles (uniform, regolith over rock, etc) and see what they get, using that filter to describe the best.*

### 5.1. Constant Density Asteroid Model

Suppose our asteroid has constant density  $\rho_0$  and known parameters  $K_{\ell m}$  and  $a_m$ . We do not expect to solve for  $\rho_0$  or equivalently  $\mu_m$  since the mass of the asteroid cannot be extracted from tidal torque. However, we can solve for the asteroid shape required to match the known  $K_{\ell m}$ . In this process, we ignore  $a_m$  until the last step, when we rescale the asteroid shape model until the correct  $a_m$  is yielded.

Suppose the asteroid is "star shaped" in that every point on its surface maps to a unique set of spherical coordinates  $\theta, \phi$ , and  $r(\theta, \phi)$ . We use the helpful identity

$$K_{\ell m} = \frac{\rho_m}{\mu_m a_m^\ell} \int_{\mathcal{A}} d^3 r R_{\ell m}(\mathbf{r}) = \frac{\rho_m}{\mu_m a_m^\ell} \int_{\mathcal{A}} d^3 r \nabla \cdot \mathbf{v}(\mathbf{r}) = \frac{\rho_m}{\mu_m a_m^\ell} \oint_{\partial \mathcal{A}} d^2 \mathbf{r} \cdot \mathbf{v}(\mathbf{r}) \quad (44)$$

where  $\mathbf{v}(\mathbf{r}) = \hat{\mathbf{r}} R_{\ell m}(\mathbf{r}) r / (3 + \ell)$ . We know the area element satisfies  $d^2 \mathbf{r} = \partial \mathbf{r} / \partial \theta \times \partial \mathbf{r} / \partial \phi$  in our coordinates, and when dotted with  $\mathbf{v} \propto \hat{\mathbf{r}}$ , this gives

$$K_{\ell m} = \frac{\rho_m}{\mu_m a_m^\ell (3 + \ell)} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi r(\theta, \phi)^3 R_{\ell m}(r(\theta, \phi), \theta, \phi). \quad (45)$$

The integrand of each  $K_{\ell m}$  therefore is proportional to  $r(\theta, \phi)^{(3 + \ell)}$  times constants and a spherical harmonic. We can write  $r(\theta, \phi) = \sum_{\ell m} Y_{\ell m}^* C_{\ell m}$  without loss of generality, in which case equation 45 becomes a polynomial of degree  $3 + \ell$  in terms of  $C_{\ell' m'}$  and integrals of products of spherical harmonics. These integrals can be precomputed, since they are taken over the whole sphere  $\theta \in (0, \pi)$  and  $\phi \in (0, 2\pi)$ . Equation 45 can then be numerically solved to get  $C_{\ell m}$ . Specifically, the ratios of  $K_{\ell m} / K_{\ell 0}$  can be solved for to eliminate the prefactor, then  $\rho_m / \mu_m$  chosen such that  $K_{00} = 1$ . If  $n$  values of  $K_{\ell m}$  are known, then this method solves exactly for  $n$  values of  $C_{\ell m}$ . *Implement this as a check.*

#### 5.1.1. Special Case: Constant Density Ellipsoid

Suppose that only  $K_{\ell=2}$  are known. Then the above method is not necessary to compute the uniform density shape; we can assume the asteroid is an ellipsoid and solve for the axis lengths, which are

$$a = \sqrt{\frac{5}{3}} a_m \sqrt{1 - 2K_{20} + 12K_{22}} \quad b = \sqrt{\frac{5}{3}} a_m \sqrt{1 - 2K_{20} - 12K_{22}} \quad c = \sqrt{\frac{5}{3}} a_m \sqrt{1 + 4K_{20}}. \quad (46)$$

These arise out of computing the moments of inertia for a constant density ellipsoid and using equation 24 to relate the result to  $K_{\ell m}$  and  $a_m$ . The values  $a$ ,  $b$ , and  $c$ , represent semi-axes aligned with  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$  respectively.

Again, the density of this asteroid cannot be extracted because, as is a general principle of gravitational interactions, the mass of the asteroid does not affect its dynamics. If  $a_m$  is unknown, the ratios  $a/c$  and  $b/c$  can be computed instead. Since  $I_{zz}$ , is the greatest of the moments of inertia, we expect  $a/c > 1$  and  $b/c > 1$ .

### 5.2. Fixed Surface $\nabla^2 \rho = 0$ model

Now suppose the shape of the asteroid is known, perhaps by a simplifying guess or perhaps by direct observation during a flyby. Then a density distribution  $\rho(\mathbf{r})$  can be solved for which is consistent with observed values of  $K_{\ell m}$ . Supposing  $\nabla^2 \rho = 0$  and removing terms which are undefined at  $r = 0$ , we write  $\rho(\mathbf{r}) = \sum_{\ell m} C_{\ell m} R_{\ell m}^* a_m^\ell \mu_m$ . Thus,  $K_{\ell m} = \sum_{\ell' m'} C_{\ell' m'} B_{\ell \ell' m m'}$  where

$$B_{\ell \ell' m m'} = \int_{\mathcal{A}} d^3 r R_{\ell m}(\mathbf{r}) R_{\ell' m'}(\mathbf{r}) \quad (47)$$

can be precomputed due to the shape of  $\mathcal{A}$  being known. If we write  $K_{\ell m}$  as a vector  $\mathbf{K}$  and similarly for  $\mathbf{C}$ , we can write  $B_{\ell \ell' m m'}$  as a matrix  $\hat{B}$ , obtaining  $\mathbf{K} = \hat{B} \mathbf{C}$ . It follows that the density distribution coefficients are given by

$$\mathbf{C} = \hat{B}^{-1} \mathbf{K}. \quad (48)$$

The scale parameter  $a_m$  can be computed from the resulting values of  $\mathbf{C}$  and the distance scale can be adjusted until  $a_m$  matches the true value. As before,  $\mu_m$  is a free parameter representing the impossibility of fitting for the mass of the asteroid. Since only a finite number of  $K_{\ell m}$  are known, we can only solve for a finite number of  $C_{\ell m}$  and neglect the higher order terms.

Physically, there is no reason to expect  $\nabla^2 \rho = 0$ . However, the space of density distributions with  $\nabla^2 \rho = 0$  is a useful proxy for the true density distribution, since this section demonstrates how the density distribution can be solved for within this subspace. Conceptually, the density distribution derived from the  $C_{\ell m}$ s can be viewed as a projection from the true density distribution into the closest density distribution with  $\nabla^2 \rho = 0$  everywhere.

Note that if  $\mathcal{A}$  is assumed to be a sphere,  $B_{\ell \ell' m m'} \propto \delta_{\ell \ell'} \delta_{m m'}$  and  $B_{\ell \ell m m}$  can be computed analytically. Thus, the matrix inversion step is not required in this case.

### 5.3. Fixed Surface Numerical Model

Now we approach the problem of solving for the density distribution of an asteroid from the  $K_{\ell m}$  parameters with more general assumptions. Suppose the shape of the asteroid is known, where “known” in this case means that an indicator function  $\mathbf{1}(\mathbf{r})$  exists such that  $\mathbf{1}(\mathbf{r}) = 1$  inside the asteroid and 0 elsewhere. Then we suggest a numerical algorithm to compute a representative distribution of the asteroid density distribution. Given  $n$  known values of  $K_{\ell m}$ , the algorithm proceeds as follows.

1. Fill the asteroids with  $n$  uniformly random “seed points” by distributing them inside a bounding box containing the asteroid and rejecting those with positions  $\mathbf{r}_i$  such that  $\mathbf{1}(\mathbf{r}_i) = 0$ .
2. Construct  $n$  regions  $\mathcal{R}_i$  around each point such that the  $i$ th region is the set of all points inside the asteroid closest to the  $i$ th point. Suppose  $\mathcal{R}_i$  has uniform density  $\rho_i$ .
3. For each region, compute  $\bar{K}_{\ell m}^i = \frac{1}{\mu_m a_m^\ell} \int_{\mathcal{R}_i} d^3 r R_{\ell m}(\mathbf{r})$ . Once again,  $\mu_m$  is a free parameter and units can be chosen such that  $a_m = 1$ . Then
4. Write  $\bar{K}_{\ell m}^i$  as a square matrix  $\hat{\bar{K}}$  with columns indexed by  $i$  and rows indexed by  $\ell$  and  $m$ . Invert it to get  $\rho = \hat{\bar{K}}^{-1} \mathbf{K}$ .
5. Go back to step 1 and compute new density maps with new draws of seed points. Since  $K_{\ell m}$  is linear in  $\rho$ , the average density distribution also satisfies the fitted  $K_{\ell m}$ . Use the average as the asteroid density distribution.

*What does this give mathematically?*

## 6. UNCERTAINTY TESTING

## 7. CONCLUSION

*Talk about the walker spacing: the clamps, the scale with sigma, and the block diagonal hermitian matrices*

*Talk about first order torque*

*Talk about how numerical accuracy was chosen*

*Show plot of likelihood? (May not be necessary, since the corner plots are also shown)*

*Move up error bar section so that plots can be explained?*  
*Compare the results to gaussians (they are not gaussian near the edges). Could do moment testing, or the Shapiro-Wilk test, which is frequentist so I don't want to do it.*  
*Question: How to express non-gaussianity of distro?*

*Facilities:* MIT Supercloud

*Software:* emcee, scipy.optimize.minimize, numdifftools.Hessian

## APPENDIX

### A. REFERENCE FLYBY CONFIGURATIONS

Except when otherwise mentioned, we use the following fit parameters when running the simulation. Many of the parameter choices are made to maximize the quality of observations (a close orbit, large asteroid, etc.) This is so that if it is desired to test the dependence of fit results on perigee distance, for instance, the other parameters will not be fixed at values that are so unfavorable to observations that good fits are already excluded. Otherwise, the fit results for all perigee distances would always be poor and the dependence on perigee would be difficult to measure.

1. An orbit around Earth with  $4 \text{ km s}^{-1}$  excess velocity and perigee at 5 Earth radii.
2. An initial roll of  $\gamma_0 = \pi/8$ .
3. A cadence of 2 minutes and observational uncertainty of  $\sigma_\theta = 0.01$ .
4. A rotational period of 5 hours, with spin axis pointing  $45^\circ$  below the orbital plane, toward the pericenter.
5. An asteroid with radius  $a_m = 1 \text{ km}$  and  $K_{3m} = 0$ . For  $K_{22}$  and  $K_{20}$ , we use two standard values. One with  $(K_{22}, K_{20}) = (0, -0.097)$  and one with  $(0.052, -0.202)$ . Including the third point obtained by reflection  $K_{22} \rightarrow -K_{22}$ , these are the three points that minimize the mean distance between an arbitrary point in the allowed parameter space (equation 33) and these reference values. The first point is called the symmetric case because the corresponding uniform-density-ellipsoid model is rotationally symmetric around  $\hat{\mathbf{z}}$ . The second case and its reflection are called the asymmetric cases. Values of  $(0.052, -0.202)$  have  $a < b$  in the ellipsoid model, and the reflected value has  $a > b$ . If not specified, we use the  $a < b$  case.

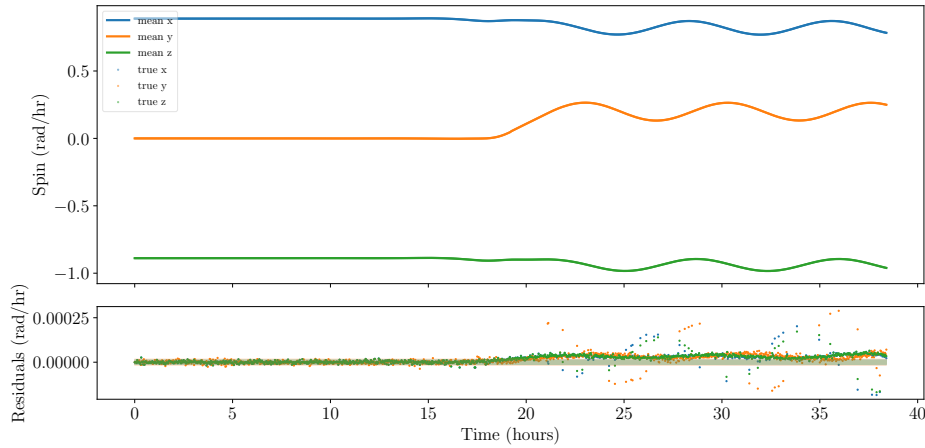
### B. EXAMPLE FIT RESULTS

In the interest of demonstrating the result of a fit to synthetic data, we present the results of a fit to synthetic data generated with the nominal sensitivity model and  $\sigma_\theta = 10^{-5}$ . The symmetric reference asteroid is used. The reduced  $\chi^2$  value of this fit (defined as  $\chi^2/N$ , where  $N$  is the number of degrees of freedom) was 0.83, and fit results were in good agreement with the true parameters. Nearly 12,000 MCMC iterations were required before convergence.

Figure 5 shows the data with the best-fit overlaid and the residuals. Uncertainties are plotted on the residuals corresponding to the square root of the diagonal entries of the covariance matrix — i.e.,  $1\text{-}\sigma$  uncertainties. However, the data often exceeds these uncertainty bands. This is because uncertainties are correlated in truth, and points that fall outside of the band are not necessary in conflict with the best fit.

Figure 6 shows a corner plot of the ten fitted parameters' posterior distributions, both marginalized to be a function of one variable and of two. The true parameters are shown as lines.

Figure 5 demonstrates that a good fit to data was achieved, despite the small error bars and strong dependence of the model on initial conditions. Figure 6 shows that the posterior distributions are largely Gaussian-distributed, with several of the parameters strongly correlated. The true values lie within  $2\sigma$  contours in most cases, as expected. Parameters are computed to very high accuracy —  $O(10^{-7})$  for  $\gamma_0$  and  $O(10^{-9})$  for  $K_{2m}$  (the first tier of parameters as ordered by the sensitivity of the asteroid system to their values), and  $O(10^{-4})$  —  $O(10^{-6})$  for  $K_{3m}$ . These ranges should be compared to the width of the prior distributions for these parameters, which is  $O(1)$  for all. The success of this fit demonstrates that these higher order parameters can be fitted to data as long as good enough precision on the data is achieved.



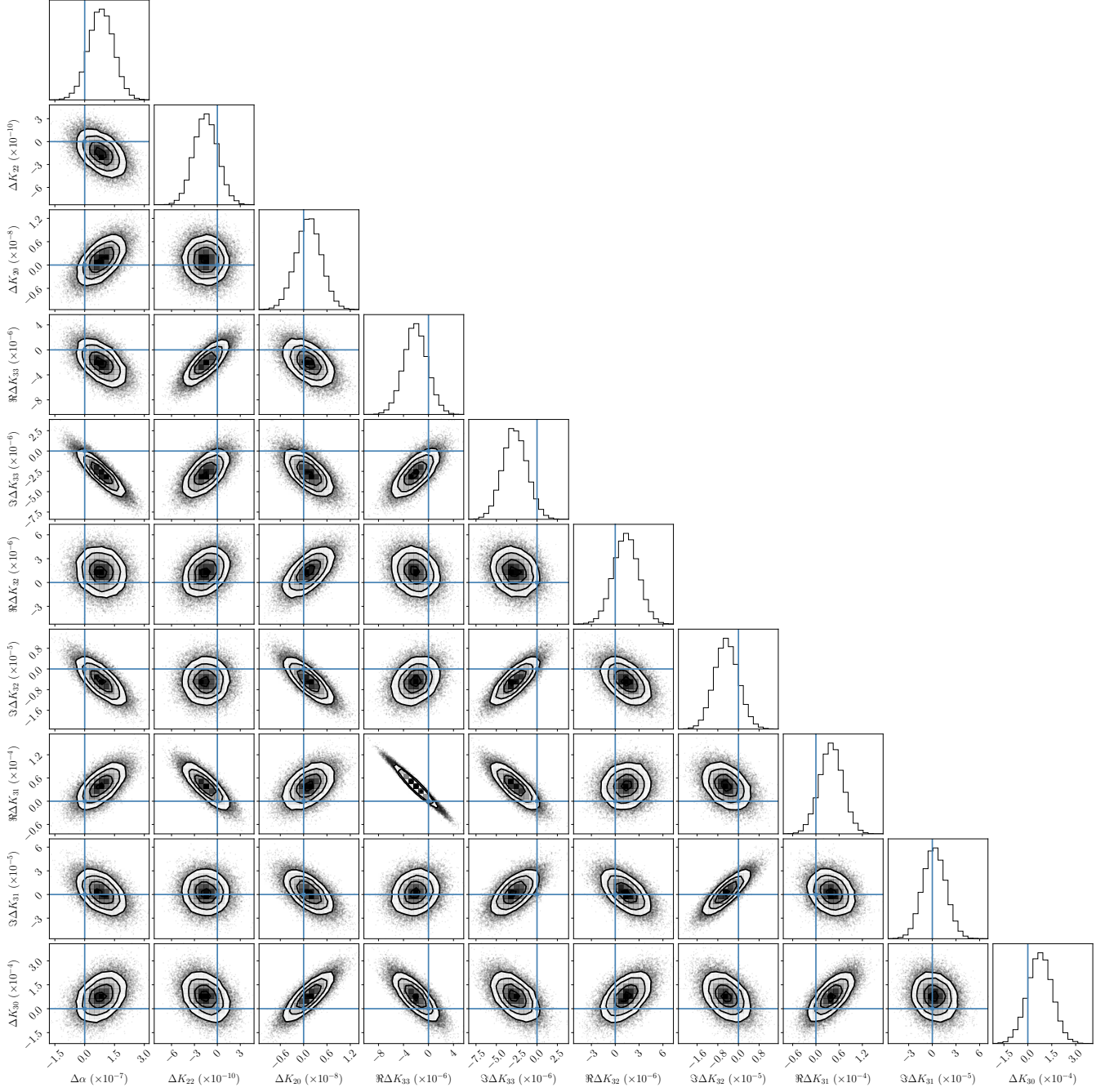
**Figure 5.** Data, best fit results, and residuals for a fit to synthetic data simulated for a symmetric reference asteroid with  $\sigma_\theta = 10^{-5}$ . The standard deviation of the data is plotted as an uncertainty band in the residuals plot, but these do not capture covariance between data points, so that the fact that many residuals lie outside the band does not disqualify the fit, which has a reduced  $\chi^2$  value of 0.83.

### C. GAUSSIANTY OF PARAMETER POSTERIOR DISTRIBUTIONS

#### REFERENCES

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**Figure 6.** A corner plot for the fit results to ten parameters to synthetic data simulated for a symmetric reference asteroid with  $\sigma_\theta = 10^{-5}$ . Marginal posterior PDFs for each parameter are shown (histograms) along with two-dimensional PDFs (contours). Individual points in the contours are samples from the MCMC fit, and the contours enclose 1-, 2-, and 3- $\sigma$  of them. True values (which are zero here due to the parameters being displayed relative to the true values) are also shown (blue lines).