Asteroid project summary

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1 Introduction

2 Asteroid model

In an effort to make a general asteroid model, we will consider the entire effect of the tidal torque applied to an asteroid from a planet, rather than solving for the first order perturbation. To better understand the effect, we will not model perturbing effects from other bodies such as the Sun or other planets. However, nearby bodies, such as moons or rings, are considered in a later section. We will also model the asteroid as a rigid body, whose density distribution is fixed throughout the flyby.

Most asteroids are observed to have attained their minimum energy rotation state, so we will also assume that the asteroid's initial state aligns its rotational velocity parallel to the principal axis with maximal moment of inertia.

2.1 Coordinates

We will make use of two frames of reference to model this system. One is the "inertial frame," with axes denoted by $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, $\hat{\mathbf{Z}}$. As the name suggests, these axes are inertial, with $\hat{\mathbf{X}}$ pointing to the asteroid pericenter and $\hat{\mathbf{Z}}$ pointing parallel to the orbit angular momentum. The origin of this frame is set to the center of mass of the central body. We will assume that the mass distribution of the central body is known in this inertial frame.

Our second frame is the "body-fixed" frame, denoted by $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. This frame is fixed with respect to the body's principle axes and rotates with the body, with its origin at the body's center of mass. We will solve for the asteroid's mass distribution with reference to the body-fixed frame. For definiteness, we define $\hat{\mathbf{z}}$ to be the principal axis with maximal MOI, and $\hat{\mathbf{x}}$ to have minimal MOI.

We define a rotation matrix M such that $M\boldsymbol{r} = \boldsymbol{R}$, where $\boldsymbol{r}$ is in the body-fixed frame and $\boldsymbol{R}$ is in the inertial frame. It will be useful for us to represent this matrix in two different ways; one is with a normalized quaternion $\widetilde{\boldsymbol{q}}$, and the other is with $z-y-z$ Euler angles named α, β, and γ. In particular, we define

$$M = R_z(\gamma)R_y(\beta)R_z(\alpha) \tag{1}$$

where $R_i(\theta)$ is a rotation around the ith unit vector by θ (figure 1).

We use the quaternion $\widetilde{\boldsymbol{q}}$ as our dynamical variable because Euler angles suffer from gimbal lock. For the sake of this paper, we write quaternions as $\widetilde{\boldsymbol{q}} = r + i\boldsymbol{i} + j\boldsymbol{j} + k\boldsymbol{k}$, with the complex conjugate written as $\widetilde{\boldsymbol{q}}^* = r - i\boldsymbol{i} - j\boldsymbol{j} - k\boldsymbol{k}$. The inverse is then $\widetilde{\boldsymbol{q}}^{-1} = \widetilde{\boldsymbol{q}}^*/(\widetilde{\boldsymbol{q}}\widetilde{\boldsymbol{q}}^*)$, and we define real and imaginary parts as $\Re\widetilde{\boldsymbol{q}} = r$; $\Im\widetilde{\boldsymbol{q}} = i\hat{\mathbf{x}} + j\hat{\mathbf{y}} + k\hat{\mathbf{z}}$. See a description of quaternions in Ref. [1]

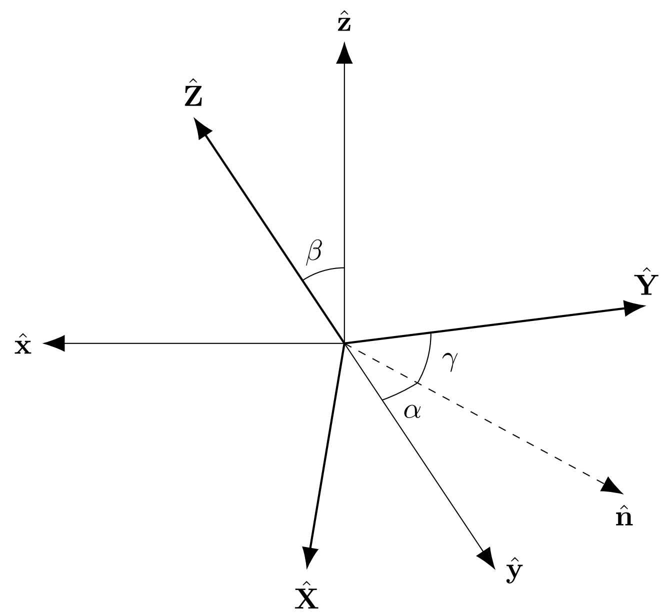

Figure 1: $z-y-z$ Euler angles used in this work to express the orientation of the asteroid. Orientation is expressed as a rotation from the body-fixed axes to the inertial axes.

Normalized quaternions express rotations of an angle θ by $\widetilde{\boldsymbol{q}} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\widetilde{\boldsymbol{n}}$ where $\widetilde{\boldsymbol{n}} = 0 + n_x\boldsymbol{i} + n_y\boldsymbol{j} + n_z\boldsymbol{k}$. This rotation may be applied to any vector $\boldsymbol{r}$ by

$$\boldsymbol{R} = \Im\left[\widetilde{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{r}}\widetilde{\boldsymbol{q}}\right]. \tag{2}$$

Due to the real part of $\widetilde{\boldsymbol{q}}$ which does not vanish for $\theta \to 0$, quaternions do not suffer from gimbal lock. Using the above notation, we can give the full equation of quaternion multiplication:

$$(q_0 + \widetilde{\boldsymbol{q}})(p_0 + \widetilde{\boldsymbol{p}}) = q_0 p_0 - \boldsymbol{p}\cdot\boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{q}\times\boldsymbol{p}. \tag{3}$$

Note that quaternion multiplication is not commutative.

We convert from Euler angles to quaternions by writing equation 1 in terms of quaternions:

$$\widetilde{\boldsymbol{q}} = \begin{pmatrix}\cos\frac{\gamma}{2}\\0\\0\\\sin\frac{\gamma}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2}\\0\\\sin\frac{\beta}{2}\\0\end{pmatrix}\begin{pmatrix}\cos\frac{\alpha}{2}\\0\\0\\\sin\frac{\alpha}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\\ \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\\ \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\\ \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\end{pmatrix}. \tag{4}$$

This equation can be reversed for α, β, γ as follows:

$$\tan\alpha = \frac{jk + ir}{jr - ik} \qquad \cos\beta = 1 - 2(i^2 + j^2) \qquad \tan\gamma = \frac{jk - ir}{jr + ik}. \tag{5}$$

2.2 Tidal torque

We start by expressing the gravitational potential energy of the central body in its most general form:

$$V(\boldsymbol{R}') = -G\int_{\mathcal{P}} d^3R\,\rho_M(\boldsymbol{R})\frac{1}{|\boldsymbol{R} - \boldsymbol{R}'|}. \tag{6}$$

This integral is computed in the inertial frame. We represent the central body density as ρ_M and its volume as $\mathcal{P}$. To reduce equation 6, we define the unnormalized spherical harmonics $Y_{\ell m}(\theta,\phi) = P_{\ell m}(\cos\theta)e^{im\phi}$, where $P_{\ell m}$ are the associated Legendre Polynomials without the Condon-Shortley phase. The regular and irregular spherical harmonics are then defined as

$$S_{\ell m}(\boldsymbol{r}) = (-1)^m(\ell-m)!\frac{Y_{\ell m}(\hat{\mathbf{r}})}{r^{\ell+1}} \qquad R_{\ell m}(\boldsymbol{r}) = (-1)^m\frac{r^\ell}{(\ell+m)!}Y_{\ell m}(\hat{\mathbf{r}}). \tag{7}$$

Assume that we can construct a sphere, centered at the center of mass of the central body, so that $\rho_M = 0$ outside the sphere and $\rho_m = 0$ inside the sphere. Then $r' > r$ in equation 6, and Ref. [2] gives the identity

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{\ell,m} R_{\ell m}(\boldsymbol{r})S^*_{\ell m}(\boldsymbol{r}'), \tag{8}$$

where the sum is shorthand for $\sum_{\ell,m} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$.

Suppose the vector $\boldsymbol{D}$ points from the center of mass of the central body to the center of mass of the asteroid, again in the inertial frame. Then we may write $\boldsymbol{R}' = \boldsymbol{D}+\boldsymbol{R}$, and Ref. [2] gives

$$S_{\ell m}(\boldsymbol{R}') = \sum_{\ell',m'}(-1)^{\ell'}R^*_{\ell'm'}(\boldsymbol{U})S_{\ell+\ell',m+m'}(\boldsymbol{D}). \tag{9}$$

Let us define complex, unitless constants

$$J_{\ell m} = \int_{\mathcal{P}} d^3R\rho_M(\boldsymbol{R})\frac{R_{\ell m}(\boldsymbol{R})}{\mu_M a_M^\ell} \qquad K_{\ell m} = \int_{\mathcal{P}} d^3r\rho_m(\boldsymbol{r})\frac{R_{\ell m}(\boldsymbol{r})}{\mu_m a_m^\ell} \tag{10}$$

where the integrals are taken in the inertial and the body-fixed frames respectively. The symmetry relations of $R_{\ell m}$ guarantee $J_{\ell m} = (-1)^m J_{\ell,-m}$ and similarly for $K_{\ell m}$. We have chosen μ_M, μ_m to be the masses of the central body and asteroid respectively, and a_M and a_m to be scale lengths of both bodies (defined more precisely later). Then we can express the potential as

$$V(\boldsymbol{U}) = G\left[\sum_{\ell,m}\mu_M a_M^\ell J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'}R_{\ell'm'}(\boldsymbol{U})S^*_{\ell+\ell',m+m'}(\boldsymbol{D})\right]. \tag{11}$$

To make use of $K_{\ell m}$ in equation 10, we need to rotate $\boldsymbol{U}$ from the inertial frame into $\boldsymbol{u}$ in the body fixed frame via $\boldsymbol{U} = M\boldsymbol{u}$. The rotation of a spherical harmonic satisfies

$$Y_{\ell m}(M\boldsymbol{r}) = \sum_{m'=-\ell}^{\ell}\mathcal{D}^\ell_{mm'}(M)^*Y_{\ell m'}(\boldsymbol{r}). \tag{12}$$

where $\mathcal{D}$ are the Wigner-D matrices. The corollary of this formula for the regular solid spherical harmonic is

$$R_{\ell m}(M\boldsymbol{r}) = \sum_{m'=-\ell}^{\ell}(-1)^{m+m'}\frac{(\ell+m')!}{(\ell+m)!}\mathcal{D}^\ell_{mm'}(M)^*R_{\ell m'}(\boldsymbol{r}) \tag{13}$$

where $\mathcal{D}^\ell_{mm'}$ are the Wigner-D matrix elements. In practice, these are computed from the Euler angles corresponding to the rotation M, which we specified in section 2.1 are $z-y-z$ Euler angles.

With this choice of Euler angles, the Wigner-D matrix elements are real with matrix elements given by

$$\begin{aligned}d^{\ell}_{mm'}(\beta) =&\sqrt{(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!}\\ &\sum_{s=s_{\min}}^{s_{\max}}\left[\frac{(-1)^{m-m'+s}\left(\cos\frac{\beta}{2}\right)^{2\ell+m'-m-2s}\left(\sin\frac{\beta}{2}\right)^{m-m'+2s}}{(\ell+m'-s)!(\ell-m-s)!(m-m'+s)!s!}\right]\end{aligned} \tag{14}$$

where $s_{\min}=\max(0,m'-m)$ and $s_{\max}=\min(\ell+m',\ell-m)$, and $\mathcal{D}^{\ell}_{mm'}(\alpha,\beta,\gamma)=d^{j}_{mm'}e^{-im\alpha}e^{-im'\gamma}$.

Combining equations 13 and 11,

$$V(\boldsymbol{u})=G\left[\sum_{\ell,m}\mu_M a_M^{\ell}J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'+m'+m''}S^*_{\ell+\ell',m+m'}(\boldsymbol{D})\sum_{m''=-\ell'}^{\ell'}\frac{(\ell'+m'')!}{(\ell'+m')!}\mathcal{D}^{\ell'}_{m'm''}(M)^*R_{\ell'm''}(\boldsymbol{u})\right]. \tag{15}$$

The tidal torque applied to an asteroid at a point $\boldsymbol{u}$ is given by $d\boldsymbol{\tau}(\boldsymbol{u})=-\rho_m(\boldsymbol{u})d^3u(\boldsymbol{u}\times(\nabla V_{\boldsymbol{u}}(\boldsymbol{u})))$ in the body-fixed frame, which leaves

$$\begin{aligned}\boldsymbol{\tau}=&-G\int_{\mathcal{A}}d^3u\rho_m(\boldsymbol{u})\boldsymbol{u}\times\nabla\left[\sum_{\ell,m}\mu_M a_M^{\ell}J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'+m'+m''}S^*_{\ell+\ell',m+m'}(\boldsymbol{D})\right.\\ &\left.\sum_{m''=-\ell'}^{\ell'}\frac{(\ell'+m'')!}{(\ell'+m')!}\mathcal{D}^{\ell'}_{m'm''}(M)^*R_{\ell'm''}(\boldsymbol{u})\right].\end{aligned} \tag{16}$$

Here, ρ_m is the asteroid density and $\mathcal{A}$ is its volume. Using the identity

$$\begin{aligned}\boldsymbol{u}\times\nabla R_{\ell m}(\boldsymbol{u})=&\frac{1}{2}[(i\hat{\mathbf{x}}-\hat{\mathbf{y}})(\ell-m+1)R_{\ell,m-1}(\boldsymbol{u})\\ &+(i\hat{\mathbf{x}}+\hat{\mathbf{y}})(\ell+m+1)R_{\ell,m+1}(\boldsymbol{u})+2im\hat{\mathbf{z}}R_{\ell m}(\boldsymbol{u})],\end{aligned} \tag{17}$$

which was derived from Ref. [2]'s analysis of the angular momentum operators applied to spherical harmonics, we write equation 20 as

$$\begin{aligned}\boldsymbol{\tau}=&-G\int_{\mathcal{A}}d^3u\rho_m(\boldsymbol{u})\left[\sum_{\ell,m}\mu_M a_M^{\ell}J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'+m'+m''}S^*_{\ell+\ell',m+m'}(\boldsymbol{D})\right.\\ &\sum_{m''=-\ell'}^{\ell'}\frac{(\ell'+m'')!}{(\ell'+m')!}\mathcal{D}^{\ell'}_{m'm''}(M)^*\frac{1}{2}\left[(i\hat{\mathbf{x}}-\hat{\mathbf{y}})(\ell'-m''+1)R_{\ell',m''-1}(\boldsymbol{u})\right.\\ &\left.\left.+(i\hat{\mathbf{x}}+\hat{\mathbf{y}})(\ell'+m''+1)R_{\ell',m''+1}(\boldsymbol{u})+2im''\hat{\mathbf{z}}R_{\ell'm''}(\boldsymbol{u})\right]\right].\end{aligned} \tag{18}$$

This equation reduces to

$$\begin{aligned}\boldsymbol{\tau}=&-G\frac{\mu_m\mu_M}{2}\left[\sum_{\ell,m}a_M^{\ell}J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'+m'+m''}a_m^{\ell'}S^*_{\ell+\ell',m+m'}(\boldsymbol{D})\sum_{m''=-\ell'}^{\ell'}\frac{(\ell'+m'')!}{(\ell'+m')!}\mathcal{D}^{\ell'}_{m'm''}(M)^*\right.\\ &\left.\left[(i\hat{\mathbf{x}}-\hat{\mathbf{y}})(\ell'-m''+1)K_{\ell',m''-1}+(i\hat{\mathbf{x}}+\hat{\mathbf{y}})(\ell'+m''+1)K_{\ell',m''+1}+2im''\hat{\mathbf{z}}K_{\ell'm''}\right]\right].\end{aligned} \tag{19}$$

Thus,

$$\begin{aligned}\boldsymbol{\tau} =&G\sum_{\ell,m}\mu_m a_M^\ell J_{\ell m}\sum_{\ell',m'}(-1)^{\ell'}S^*_{\ell+\ell',m+m'}(\boldsymbol{D})+\int_{\mathcal{A}}d^3U\frac{1}{2}\Big[(i\hat{\mathbf{X}}-\hat{\mathbf{Y}})(l'-m'+1)R_{\ell',m'-1}(\boldsymbol{U})\\&(i\hat{\mathbf{X}}+\hat{\mathbf{Y}})(\ell'+m'+1)R_{\ell',m'+1}(\boldsymbol{U})+2im'\hat{\mathbf{Z}}R_{\ell'm'}(\boldsymbol{U})\Big].\end{aligned}\tag{20}$$

To convert the remaining volume integral into the form of, we need to rotate $\boldsymbol{U}$ to $\boldsymbol{u}$ in the body-fixed frame.

We wish Then, by rotating $K_{\ell m}$ from the body-fixed axis orientation to the inertial axis orientation, we may use $J_{\ell m}$ and $K_{\ell m}$ to simplify the potential energy. Rotations of the spherical harmonics are given by the Wigner-D matrices, via the formula

$$Y_{\ell m}(M\boldsymbol{r})=\sum_{m'=-\ell}^{\ell}\mathcal{D}^{\ell}_{mm'}(M)^*Y_{\ell m'}(\boldsymbol{r}).\tag{21}$$

The corrolary of this formula for the regular solid spherical harmonic is

$$R_{\ell m}(M\boldsymbol{r})=\sum_{m'=-\ell}^{\ell}(-1)^{m+m'}\frac{(\ell+m')!}{(\ell+m)!}\mathcal{D}^{\ell}_{mm'}(M)^*R_{\ell m'}(\boldsymbol{r})\tag{22}$$

where $\mathcal{D}^{\ell}_{mm'}$ are the Wigner-D matrix elements. In practice, these are computed from the Euler angles corresponding to the rotation M, which we specified in section 2.1 are $z-y-z$ Euler angles. With this choice of axes, the Wigner-D matrix elements are real with matrix elements given by

$$\begin{aligned}d^{\ell}_{mm'}(\beta)=&\sqrt{(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!}\\&\sum_{s=s_{\min}}^{s_{\max}}\left[\frac{(-1)^{m-m'+s}\left(\cos\frac{\beta}{2}\right)^{2\ell+m'-m-2s}\left(\sin\frac{\beta}{2}\right)^{m-m'+2s}}{(\ell+m'-s)!(\ell-m-s)!(m-m'+s)!s!}\right]\end{aligned}\tag{23}$$

where $s_{\min}=\max(0,m'-m)$ and $s_{\max}=\min(\ell+m',\ell-m)$, and $\mathcal{D}^{\ell}_{mm'}(\alpha,\beta,\gamma)=d^{j}_{mm'}e^{-im\alpha}e^{-im'\gamma}$. For neatness, we define the unitless constants

$$C_{\ell\ell'mm'm''}(\alpha,\beta,\gamma,t)=D(t)(-1)^{\ell'+m'+m''}S^*_{\ell+\ell',m+m'}(\boldsymbol{D}(t))a_M^\ell a_m^{\ell'}\frac{(\ell'+m'')!}{(\ell'+m')!}\mathcal{D}(\boldsymbol{q})^{\ell}_{m'm''}(M)^*\tag{24}$$

where $\boldsymbol{D}(t)$ is purely dependent on time because the orbital path of the asteroid is assumed to be Keplerian. Then, combining equations 8 through 23, we get a potential energy of

$$V=-\frac{G\mu_M\mu_m}{D(t)}\sum_{\ell,m}\sum_{\ell',m'}\sum_{m''=-\ell'}^{\ell'}C_{\ell\ell'mm'm''}(\alpha,\beta,\gamma,t)J_{\ell,m}K_{\ell'm''}.\tag{25}$$

2.3 Moment of inertia

The inertia tensor of the asteroid is expressed as a matrix I. Written in terms of spherical harmonics in the body-fixed frame, we have

$$\begin{aligned}I=&\int_{\mathcal{A}}d^3r\rho_m(\boldsymbol{r})\begin{pmatrix}y^2+z^2&-xy&-xz\\-xy&x^2+z^2&-xz\\-xz&-yz&x^2+y^2\end{pmatrix}\\=&\frac{2}{3}\mu_m a_m^2\begin{pmatrix}K_{20}-3K_{2,-2}-3K_{22}+1&3i\left(K_{22}-K_{2,-2}\right)&\frac{3}{2}\left(K_{21}-K_{2,-1}\right)\\3i\left(K_{22}-K_{2,-2}\right)&K_{20}+3K_{2,-2}+3K_{22}+1&-\frac{3}{2}i\left(K_{21}-K_{2,-1}\right)\\\frac{3}{2}\left(K_{21}-K_{2,-1}\right)&-\frac{3}{2}i\left(K_{21}-K_{2,-1}\right)&-2K_{20}+1\end{pmatrix}\end{aligned}\tag{26}$$

where we have defined, for neatness,

$$a_m^2 = \frac{1}{\mu_m} \int_{\mathcal{A}} d^3 r \rho_m(\boldsymbol{r}) r^2. \tag{27}$$

With this definition of a_m, and an arbitrary definition of a_M which may be taken to be similar to equation 27 or a more convenient unit such as the central body radius, the parameters $J_{\ell m}$ and $K_{\ell m}$ of equation 10 are completely defined. Note that a_m is similar to the radius of the asteroid; if the asteroid is a sphere of uniform density and radius r, then $a_m = r\sqrt{3/5}$.

Henceforth, we define the body-fixed axis to be coincident with the principal axes of the asteroids, so that the off-diagonal components of I are zero.

2.4 Equation of motion

As a simplifying assumption, we will not model the effect of the central body's non-sphericity on the asteroid orbit. This effect has been studied in much detail and is small. We therefore separate the position dynamical variables from the angular variables and derive the standard equations of motion for asteroid position:

$$\dot{\boldsymbol{v}} = -\frac{G\mu_M}{r^3}\boldsymbol{r} \qquad \dot{\boldsymbol{r}} = \boldsymbol{v}. \tag{28}$$

The position variables will no longer be discussed.

Let $\boldsymbol{\omega}$ be the rotational velocity of the asteroid in the body-fixed frame. We have two equations of motion for our system; one governs the evolution of $\boldsymbol{\omega}$ and the other governs the evolution of $\boldsymbol{q}$. Starting with the latter, consider a fixed point $\boldsymbol{r}$ in the body-fixed frame on a rotating asteroid. The corresponding point $\boldsymbol{R}$ in the inertial frame satisfies $\dot{\boldsymbol{R}} = \Im[\widetilde{\boldsymbol{q}}^{-1}(\boldsymbol{\omega} \times \boldsymbol{r})\widetilde{\boldsymbol{q}}]$. Furthermore, we can demand that $\boldsymbol{\omega} \perp \boldsymbol{r}$ such that $\boldsymbol{\omega} \times \boldsymbol{r} = \Im[\widetilde{\boldsymbol{\omega}}\widetilde{\boldsymbol{r}}]$ Plugging this into equation 2, we have

$$\dot{\widetilde{\boldsymbol{q}}}^{-1}\widetilde{\boldsymbol{r}}\widetilde{\boldsymbol{q}} + \widetilde{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{r}}\dot{\widetilde{\boldsymbol{q}}} = \widetilde{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{\omega}}\widetilde{\boldsymbol{r}}\widetilde{\boldsymbol{q}}.$$

Substituting $\boldsymbol{r} \to \boldsymbol{R}$ and acknowledging that $\Re[\dot{\widetilde{\boldsymbol{q}}}\widetilde{\boldsymbol{q}}^{-1}]$,

$$2\dot{\widetilde{\boldsymbol{q}}}^{-1}\widetilde{\boldsymbol{q}}\widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{\omega}}\widetilde{\boldsymbol{q}}\widetilde{\boldsymbol{R}}.$$

Since $\boldsymbol{R}$ is arbitrary, this simplifies to

$$\dot{\widetilde{\boldsymbol{q}}} = \frac{1}{2}\widetilde{\boldsymbol{\omega}}\widetilde{\boldsymbol{q}}. \tag{29}$$

The second equation of motion are Euler's equations of motion given our expression for tidal torque. Namely,

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= \tau_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= \tau_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= \tau_3 \end{aligned} \tag{30}$$

2.5 Initial values

We assume that the asteroid begins the simulation in its minimum energy configuration; that is, with $\boldsymbol{\omega} \parallel \hat{\mathbf{z}}$. The initial conditions are therefore that $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, and the quaternion is fixed to rotate $\hat{z}$ to $\boldsymbol{\Omega}$ in the inertial frame, which fixes Euler angles β and γ. The third Euler angle α represents the initial roll of the asteroid, and therefore cannot be determined from physical data. However, it is an initial condition of the simulation. Once the Euler angles are known, the quaternion is computed via equation 4.

2.6 Fit parameters

By requiring that the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes coincide with principal axes, we force the diagonal components of I to be zero. Thus, we may require $K_{21} = K_{2,-1} = 0$ and $\Im K_{22} = \Im K_{2,-2} = 0$ with no loss of generality. By requiring that the body-fixed frame be located at the center of mass of the asteroid, we also require $K_{1m} = 0$. Finally, $K_{00} = 1$ follows automatically from the definition. Thus, K_{20}, $\Re K_{22}$, and $K_{\ell>2,m}$ are the only physical parameters of the asteroid density distribution, beyond a_m.

Note from equation **??** that the $K_{00}J_{00}$ term vanishes. Since $J_{\ell=1} = K_{\ell=1} = 0$ as well, equation **??** is proportional to $\mu_m a_m^2$. Factoring out this product results in an equation of motion independent of μ_m, meaning that the asteroid dynamics are unaffected by the mass of the asteroid. This is expected, and it means that the density distribution of the asteroid can only be determined up to a scale factor.

Also note that the coefficient of $J_{lm}K_{l'm'}$ is proportional to $a_M^\ell a_m^{\ell'}/D^{\ell+\ell'}$. Usually, $D \gg a_M \gg a_m$, so we expect this fraction to be small. The nonzero term first order in this fraction in the potential energy has $\ell = 0$ and $\ell' = 2$. To first order in the asteroid density distribution, therefore, the system is also independent of a_m. When $K_{\ell'>2}$ terms are included, a_m scales the size of the higher order terms.

We require that the moment of inertia matrix be positive-definite, which bounds our asteroid parameters. Specifically, we have $K_{20} \leq \frac{1}{2}$, and $|K_{22}| \leq (K_{20}+1)/6$. We can also choose without loss of generality to let the maximum moment of inertia align with $\hat{\mathbf{z}}$, so that $1-2K_{20} \geq K_{20}+1+6|K_{22}|$. Finally, we wish to require that the density of the asteroid be everywhere positive. One consequence of this requirement is that the sum of any two distinct moments of inertia cannot be greater than the third, by the integral definition of moment of inertia in equation 26. Three inequalities result from this, but only one is not covered by the above restrictions. Namely, $I_{xx}+I_{yy} \geq I_{zz} \implies K_{20} \geq -1/4$. The intersection of these requirements forces K_{20} and K_{22} to lie in the triangle defined by

$$-\frac{1}{4} \leq K_{20} \leq 0 \qquad |K_{22}| \leq -\frac{K_{20}}{2}. \tag{31}$$

We have one additional fit parameter, which is the only initial condition not fixed by the problem setup or observations. Namely, the initial value of the first Euler angle α_0. Due to the fact that the principal axes $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthogonal, Increments of α_0 by $\pi/2$ correspond to flipping the body-fixed axes so that no physical change in the asteroid model occurs. We must therefore limit the range of α_0 to width $\pi/2$. Arbitrarily, we choose $\alpha_0 \in [-\pi/4, \pi/4)$

3 Experiment design

3.1 Simulation design

Given the observed orbit of an asteroid and observational data of its resolved spin over time, we built a simulation that would integrate equations 29 and 30 to generate synthetic data. Inputs to the simulation were (1) the ℓ and ℓ' values at which to truncate the potential energy sum in equation **??**, (2) the cadence of observations, (3) the $J_{\ell m}$ parameters and mass and radius of the central body, (4) a guess at the asteroid parameters $K_{\ell m}$, α_0 and a_m, (5) the initial rotational velocity of the asteroid in the inertial frame, and (6) the asteroid's orbital parameters.

The orbital parameters we use to describe the orbit are the hyperbolic excess velocity v_∞ and the pericenter altitude r_p. The orbit orientation is fixed to lie in the $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$ plane, with pericenter lying on $\hat{X}$. The position of the satellite as a function of time is therefore generated by numerically integrating the Newtonian equation of motion (equation 28) from initial values $\boldsymbol{r}(t=0) = r_p\hat{\mathbf{X}}$ and $\boldsymbol{v}(t=0) = \hat{\mathbf{Y}}\sqrt{v_\infty^2 + 2G\mu_M/r_p}$

Once the inputs were specified, the moments of inertia were pre-computed from $K_{\ell m}$, as was the orbital path of the satellite as a function of time. Equations 29 and 30 were then integrated by the fourth order Runge-Kutta fourth order method with a time step of *Insert time step*, with a custom implementation of the spherical harmonics and Wigner-D matrices to improve runtime. At fixed time intervals, the rotational velocity in the body-fixed frame $\boldsymbol{\omega}_{xyz}$ was converted via equation 2 to the inertial frame to produce synthetic data.

Since the leading order of the equations of motion is $\ell' = 2, \ell = 0$, we begin our simulation at $D = Df^{-1/3}$, where f is a manually set parameter to control the length of the simulation. For all simulations done here, $f = 10^{-3}$ We end the simulation also at $Df^{-1/3}$ on the other side of the orbit.

3.2 Fitting method

To analyze the degree to which asteroid density distributions can be determined from flyby rotational velocity data, we use the forward model defined in section 3.1 to generate synthetic data, randomize the data according to one of the observational uncertainty model defined in section 3.3, then use the same forward model to fit to the data and recover parameters.

After randomizing the data, we isolate a set of parameters which are likely to provide good fits. To do this, we minimize the χ^2 value of the data given the model using the BFGS algorithm. Our starting points for the minimization are spaced uniformly randomly across the allowed parameter space, which is the set of parameters with the triangle described by equation 31. The other parameter bounds are $\alpha_0 \in [-\pi/4, \pi/4)$, and $|\Re K_{l>2}| \leq 1$, $|\Im K_{l>2}| \leq 1$.

If the minimization is conducted over too many parameter dimensions, convergence can be slow and potentially inaugurate. However, the parameters $K_{\ell m}$ are naturally separated into groups by order of ℓ, since the torque applied by successive increment to ℓ is suppressed by a_m/D. Thus, we fit the $\ell = 2$ and α_0 parameters first, then lock them in place and fit the $\ell = 3$ parameters, etc. The result is a set of points in parameter space which minimize the χ^2.

We are interested in finding not just the optimal parameters, but the probability distributions of each fit parameter. To do this, we use an Affine Invariant Markov Chain Monte Carlo Ensemble Sampler (MCMC), implemented in the Python package `emcee`, to conduct the fit to data [3]. This fit method requires the parameter space to be populated with "walkers," indicating the a guess at the true parameters of the system. We center these walkers on the minimizing parameters computed by the minimization of χ^2, and spread out the walkers such that the number density of walkers is proportional to the likelihood ($\ln \mathcal{L} = -\chi^2$) at each point. This is done explicitly by computing the Hessian Σ of the likelihood at each minimizing point θ_0 and expressing the likelihood locally as a multi-dimensional Gaussian

$$\mathcal{L}(\boldsymbol{\theta}) \propto \exp\left(\frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)^T \Sigma (\boldsymbol{\theta} - \boldsymbol{\theta}_0)\right). \tag{32}$$

We may change bases into a frame in which the parameters are uncorrelated by multiplying by D^T, which is the matrix of eigenvectors of Σ. Populating the walkers in this uncorrelated space according to Gaussian standard deviations given by the eigenvalues of Σ, we can convert these walkers back to $\boldsymbol{\theta}$-space by multiplying by D.

With the initial walkers distributed, we use the χ^2 value as the log-likelihood to compute the posterior probability distribution of $\boldsymbol{\theta}$ given flat priors in the range outlined above for each parameter.

3.3 Uncertainty models

We model uncertainties in experimental data according to several methods, comparing them to determine the dependence of our conclusions on the sensitivity model used. In every case, the sensitivity model is used both to add randomness to the data set after it is generated, and to weight the χ^2 statistic.

3.3.1 Nominal method

The most detailed uncertainty model we use, and the one we use most often, is called the "nominal method." It rotates each angular velocity vector by some angle Θ drawn from a normal distribution centered on $\Theta = 0$ with standard deviation σ_θ. The period is assumed to be known to arbitrary precision at each cadence. The direction in which the velocity is rotated is chosen from a uniformly random distribution. σ_θ is a general parameter which can be tweaked to model resolution of the spin axis direction.

Practically, the randomization of the data is done by creating a rotation matrix M that rotates $\hat{\mathbf{Z}}$ to the true rotational velocity vector $\boldsymbol{\omega}$. This same rotation matrix is then used to rotate the random vector $\boldsymbol{\Omega} = \sin\Theta\cos\Phi\hat{\mathbf{X}} + \sin\Theta\sin\Phi\hat{\mathbf{Y}} + \cos\Theta\hat{\mathbf{Z}}$ to the randomized velocity vector. Here, Φ is a uniformly random variable in the range $[0, \pi)$. Specifically, we write this rotation matrix as

$$M = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{33}$$

where θ and ϕ are the spherical coordinates of the original velocity vector $\boldsymbol{\omega}$.

Equation 33 is sufficient to randomize synthetic data, but an analytical expression for the coordinates of $\boldsymbol{\omega}$ is required to define the χ^2 value used for fitting. We approximate the distribution of each spin coordinate as normal, with covariance given by the covariance of the true distribution. Based on equation 33, the covariance matrix is

$$\mathrm{Cov}(\omega_i^*, \omega_j^*) = \frac{1}{4\omega^2}\left(1 - e^{-\sigma_\theta^2}\right)\left(1 - 3e^{-\sigma_\theta^2}\right)\begin{pmatrix} \omega_x^2 & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & \omega_y^2 & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2 \end{pmatrix} + \frac{1}{4}\left(1 - e^{-2\sigma_\theta^2}\right)\mathbb{1}. \tag{34}$$

We then define the χ^2 statistic to be

$$\chi^2 = \sum_{i=0}(\boldsymbol{\omega}_i^* - \boldsymbol{\omega}_i)^T \mathrm{Cov}^{-1}(\omega_i^*, \omega_j^*)(\boldsymbol{\omega}_i^* - \boldsymbol{\omega}_i) \tag{35}$$

where $\boldsymbol{\omega}_i$ is the ith expected spin and $\boldsymbol{\omega}_i^*$ is the data.

4 Results

5 Uncertainty testing

6 Conclusion

References

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