Writeup of numerical integration algorithm

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Abstract

In this project, a luminosity function was extracted from the image of a plot in another paper. This extracted luminosity function, consisting of a finite set of points, then had to be integrated. The methods for doing so accurately are found below.

1 Introduction

A luminosity function was extracted from a plot in Ploeg et al.[1] See the Jan-2021 summary for the analysis of this luminosity function. The extracted function is shown in figure 1. The task is to compute the following two integrals:

$$\int_{L_{min}}^{\infty} \frac{dN}{dL} dL \qquad\qquad \int_{L_{min}}^{\infty} L\frac{dN}{dL} dL \tag{1}$$

which are known as the number integral and luminosity integral respectively. Here, L_{min} is some arbitrary value, which in practice was either a value near the peak of the luminosity function, $L_{min} = L_{th}$ (see figure 1) or $L_{min} = 0$.

The methods for performing these integrals are given below, followed by a comparison between the values attained by the numerical method, and those attained by analytically integrating over the log normal fit.

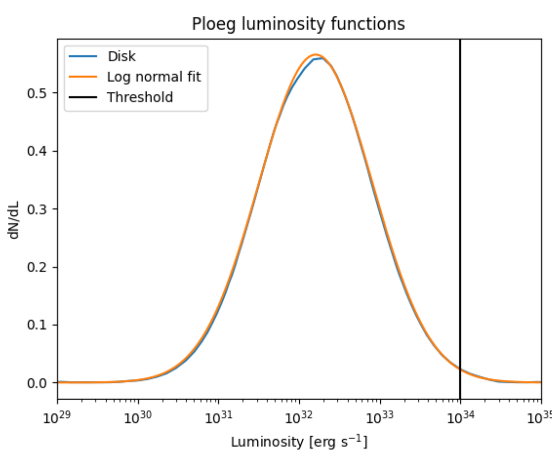


Figure 1: Luminosity function to be integrated over. The blue plot is the extracted function, while the orange is a log-normal fit to it. The black line is the threshold luminosity L_{th}, sometimes used as an integration bound.

2 Methods

Methods employed for both integrals approximate the luminosity function as piecewise-linear, that is:

$$\frac{dN}{dL}(x) \approx \begin{cases} \frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1 & x_1 \le x < x_2 \\ \frac{y_3-y_2}{x_3-x_2}(x-x_2)+y_2 & x_2 \le x < x_3 \\ \vdots & \vdots \\ \frac{y_N-y_{N-1}}{x_N-x_{N-1}}(x-x_{N-1})+y_{N-1} & x_{N-1} \le x \le x_N \end{cases} \tag{2}$$

where $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are the points extracted from the luminosity function found in the literature, structured so that $x_1 < x_2 < \cdots < x_N$ and $y_i = \frac{dN}{dL}(x_i)$.

The bounds of the integral are then restricted to the domain $[x_1, x_N]$. For example, the integral over $[0, \infty)$ becomes an integral over $[x_1, x_N]$.

2.1 Number integral

We will compute the first integral given in equation 1 by integrating the components of equation 2. The result is

$$\int_{L_{min}}^{\infty} \frac{dN}{dL} dL = \int_{L_{min}}^{x_n} \frac{y_{min}-y_n}{L_{min}-x_n}(x-L_{min})+y_{n-1}dx + \sum_{i=n}^{N-1} \int_{x_i}^{x_{i+1}} \frac{y_{i+1}-y_i}{x_{i+1}-x_i}(x-x_i)+y_i dx$$

where we have chosen n such that $x_{n-1} \le L_{min} < x_n$, and set $y_{min} = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}(L_{min}-x_{n-1})+y_{n-1}$. Performing the integral, we get

$$\int_{L_{min}}^{\infty} \frac{dN}{dL} dL = \frac{1}{2}(x_n-L_{min})(y_n+y_{min}) + \sum_{i=n}^{N-1} \frac{1}{2}(x_{i+1}-x_i)(y_{i+1}+y_i) \tag{3}$$

which is equivalent to trapezoidal integration. This series can be simplified, but we will not do so.

2.2 Luminosity integral

The same process can be applied to integrate the second integral given in equation 1. We will compute

$$\int_{L_{min}}^{\infty} \frac{dN}{dL} dL = \int_{L_{min}}^{x_n} \frac{y_{min}-y_n}{L_{min}-x_n}(x^2-xL_{min})+xy_{n-1}dx + \sum_{i=n}^{N-1} \int_{x_i}^{x_{i+1}} \frac{y_{i+1}-y_i}{x_{i+1}-x_i}(x^2-xx_i)+xy_i dx.$$

The result is

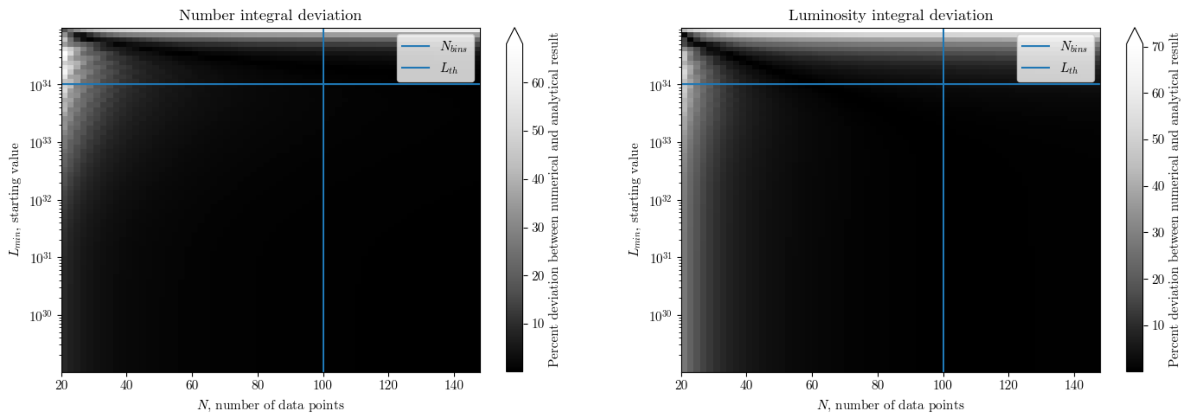
$$\begin{aligned} \int_{L_{min}}^{\infty} \frac{dN}{dL} dL = & \frac{1}{6}(x_n-L_{min})\left[y_{min}(2L_{min}+x_n)+y_n(L_{min}+2x_n)\right] \\ & + \frac{1}{6}\sum_{i=n}^{N-1}(x_{i+1}-x_i)\left[y_i(2x_i+x_{i+1})+y_{i+1}(x_i+2x_{i+1})\right]. \end{aligned} \tag{4}$$

3 Results

We may compare the accuracy of this numerical method by fitting a log-normal distribution to the luminosity function and comparing the analytical and numerical results of the integrals in equation 1. The log normal distribution used is

$$\frac{dN}{dL} = \frac{\log_{10} e}{\sigma\sqrt{2\pi}L} \exp\left(-\frac{(\log_{10} L - \log_{10} L_0)^2}{2\sigma^2}\right), \tag{5}$$

and the fit achieved values of $L_0 = 1.6084 \times 10^{32} \frac{\text{erg}}{\text{s}}$ and $\sigma = 0.7003$. The integrals were computed analytically, and the function was sampled with N points in $[10^{29} \frac{\text{erg}}{\text{s}}, 10^{35} \frac{\text{erg}}{\text{s}}]$ spaced evenly in log space. The percent deviation of the numerical integrals from the analytical integrals is shown in figure 2. The percent deviations as calculated for four points commonly used in the jan-2021 summary of this UROP are shown in table 1.



(a) Number integral

(b) Luminosity integral

Figure 2: Percent deviation between the numerical integral, calculated by sampling N points, equally-spaced in log space, from a log normal distribution and using the described methods to integrate over the range from L_{min} to the edge of the domain, which was $10^{35} \frac{\text{erg}}{\text{s}}$ here.

	Number integral	Luminosity integral
$L_{min} = 10^{29} \frac{\text{erg}}{\text{s}}$	0.321%	0.0796%
$L_{min} = L_{th} = 10^{34} \frac{\text{erg}}{\text{s}}$	0.934%	3.65%

Table 1: Percent deviations between the numerical and analytical values of the integrals shown in equation 1, for the log-normal fit to Ploeg et al.'s data.

4 Conclusion

We developed a method to numerically integrate a luminosity function and its expectation value, and affirmed that it was accurate to within four percent at relevant points.

The position-dependent sensitivity model cited in the jan-2021 summary may be more inaccurate than table 1 suggests, because for pixels near the galactic plane, the threshold luminosity can rise to as much as five times the value of L_{th} used here.

The luminosity function extracted from Ploeg et al. does not match the log-normal fit perfectly, particularly at high luminosities. The deviations present between the "custom" luminosity function and the log normal fit in the jan-2021 summary are most likely caused by this disagreement, not numerical error.

References

[1] H. Ploeg, C. Gordon, R. Crocker, O. Macias. "Comparing the Galactic Bulge and Galactic Disk Millisecond Pulsars." arXiv:2008.10821v2 (2020).