Minimal Inscribed Polyforms

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Abstract

A polyomino is a shape in the plane constructed by joining unit squares along their edges. The question of how many polyominos there are with n unit squares is a famous unsolved problem in combinatorics. This has lead many to study polyomino families, which are polyominos with additional rules that must be adhered to in construction. One such family is the minimally inscribed polyominos. This paper extends this family to other lattices, and gives novel formulae for multiple minimal inscribed polyforms.

1 Introduction

A *polyomino of area n* is a shape in $\mathbb{Z}^2$ constructed by joining n unit squares called *cells* along their edges. A domino is the case $n = 2$. Polyominos of area n are often referred to as *n-ominos*.
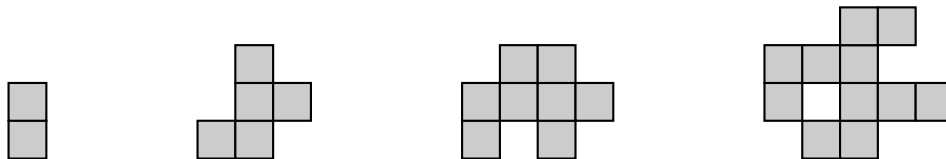
A domino (2-omino), a 5-omino, an 8-omino, and an 11-omino

Polyominos are traditionally enumerated under different equivalence classes distinguished by rotational and reflective symmetries. For this paper, we will only consider the enumeration of *fixed* polyominos.

Definition 1.1. The class of *fixed* polyominos of area n consists of all possible polyominos up to translation, regardless of other symmetries.

Whenever the enumeration of polyominos or polyomino families are mentioned within this paper, it can be assumed we are enumerating under the fixed equivalence class.

Let us call the number of fixed n-ominos t_n. The first couple terms of t_n for $n \geq 1$ are

$$1, 2, 6, 19, 63, 219, 760, 2725, 9910, 36446, \ldots (A001168 \ [\textbf{?}])$$

Example 1.1. The number of fixed 4-ominos is 19, colored by their appearance in the popular video game TETRIS.
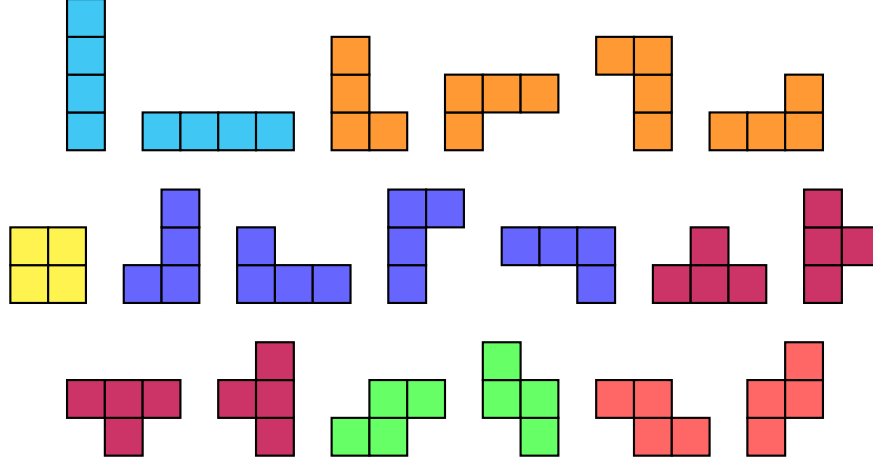
Figure 1: All 19 fixed 4-ominos

Given the simplicity of their definition, it is surprising that there is no closed formula known for t_n. It is known however that t_n is exponential.

Klarner [?] proved that $\lim_{n\to\infty}(t_n)^{\frac{1}{n}}$ exists and has a finite value λ. Madras [?] followed by proving that $\lim_{n\to\infty}\frac{t_{n+1}}{t_n}$ exists, and consequently is also equal to λ. This value is often referred to as Klarner's constant. The best current proven lower bound for λ is 3.980 [?] , and best current proven upper bound is 4.6496 [?].

The apparent difficulty of the problem has lead many to the study of polyomino families [?, ?, ?]. Polyomino families add constraints on polyomino construction to introduce additional structure. One such family was introduced by Goupil, Cloutier, and Noubound [?] called the *minimal inscribed polyominos*.

2   Minimal Inscribed Polyominos

Definition 2.1. A polyomino is *minimal inscribed* when it is contained in a $w \times \ell$ rectangular lattice, where each of the four sides of the rectangle is touched by a cell of the polyomino, and the polyomino is of minimal area $w + \ell - 1$.
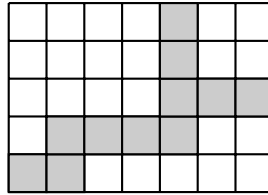

Figure 2: A minimal inscribed 11-omino in a 7×5 lattice

How many minimal inscribed polyominos are there for a given $w \times \ell$ lattice?

Theorem 1 ([?]). *Let $s_{w,\ell}$ be the number of minimal inscribed polyominos in a given $w \times \ell$ lattice. Then $s_{w,\ell} = 8\binom{w+\ell-2}{w-1} - 3w\ell + 2w + 2\ell - 8$.*

Proof. First, fix w and ℓ, and call S the set of all $(w + \ell - 1)$-ominos that can be inscribed in a $w \times \ell$ lattice. S can be split into three distinct subsets based on the number of corners of the rectangle each polyomino touches. S_2 contains polyominos that contain either 2 or 3 corners, S_1 contains polyominos that contain 1 corner, and S_0 contains polyominos that contain no corners. An example of each subset for the 4×4 lattice is shown in Figure **??**.
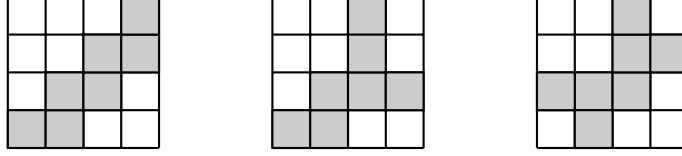

Figure 3: Examples of elements in S_2, S_1, and S_0 for a 4×4 lattice

We then have that

$$S = S_2 \uplus S_1 \uplus S_0,$$

where S_2, S_1, and S_0 are disjoint.

Case 1. We begin with S_2. Split S_2 into two subsets, $S_{T,2}$ and $S_{T,2}^c$ so that $S_{T,2} \uplus S_{T,2}^c = S_2$. Let $S_{T,2}$ represent all polyominos that are "T-shaped", while $S_{T,2}^c$ will represent all polyominos that are not. Formally, the "T-shaped" polyominos are all polyominos that contain two adjacent corner cells and a perpendicular bar, as in the left polyomino in Figure **??**.
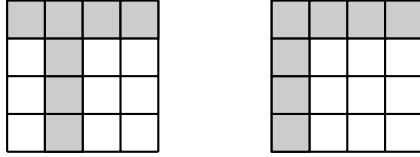

Figure 4: Examples of elements in $S_{T,2}$, and $S_{T,2}^c$ for a 4×4 lattice

For $|S_{T,2}|$ it is easy to see that $|S_{T,2}| = 2(w - 2) + 2(\ell - 2) = 2w + 2\ell - 8$, as there is a "T-shape" polyomino for every edge cell.

For $|S_{T,2}^c|$, suppose the bottom left cell of the $w \times \ell$ lattice is called $(1, 1)$, and the top right cell (w, ℓ). The number of paths from $(1, 1)$ to (w, ℓ) is the number of ways one can make $w - 1$ unit steps right and $\ell - 1$ unit steps up among $w + \ell - 2$ total unit steps. This gives a total of $\binom{w+\ell-2}{w-1}$ paths from $(1, 1)$ to (w, ℓ). As we can make the same argument for corners $(1, \ell)$ and $(w, 1)$, we can conclude $|S_{T,2}^c| = 2\binom{w+\ell-2}{w-1}$. This gives us $|S_2| = |S_{T,2}| + |S_{T,2}^c| = 2\binom{w+\ell-2}{w-1} + 2w + 2\ell - 8$.

Case 2. For S_1, suppose we choose corner $(1, 1)$ and start a path to some cell (i, j). Extending a path from (i, j) to (i, ℓ) and from (i, j) to (w, j) creates a minimally inscribed polyomino that touches only one corner. In Figure **??**, $(i, j) = (3, 1)$. Notice that (i, j) must have $i \in [2, w - 1]$, and $j \in [2, \ell - 1]$, as otherwise the polyomino formed would have more than 1 corner. Therefore, the total number of polyominos that touch only corner $(1, 1)$ in S_1 is

$$\sum_{i=2}^{w-1} \sum_{j=2}^{\ell-1} \binom{i + j - 2}{i - 1} = \binom{w + \ell - 2}{w - 1} - w - \ell + 2.$$

As we can make the same argument starting with any corner, we have $|S_1| = 4\binom{w+\ell-2}{w-1} - 4w - 4\ell + 8$

Case 3. Finally, consider S_0. Suppose we have two cells (i, j) and (i', j'), with $i, i' \in [2, w-1]$ and $j, j' \in [2, \ell-1]$, $i \leq i'$ and $j \leq j'$.
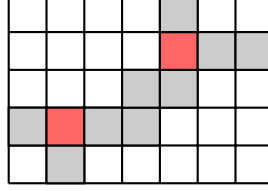

Figure 5: An example of a polyomino in S_0 with (i, j) and (i', j') in red

One can construct a path from (i, j) to (i', j'). Next, make a path from (i, j) to $(1, j)$, and a path from (i, j) to $(i, 1)$. Similarly, make a path from (i', j') to (w, j') and a path from (i', j') to (i', ℓ). This constructs a minimal inscribed polyomino that touches no corners, as shown in Figure **??**.

These polyominos can easily be enumerated by the size of the rectangle made with corners (i, j), (i, j'), (i', j), and (i', j'). If we suppose that $\Delta i = i' - i$, and $\Delta j = j' - j$, we can construct the following sum.

$$\sum_{\Delta i=1}^{w-2} \sum_{\Delta j=1}^{\ell-2} \binom{\Delta i + \Delta j - 2}{\Delta i - 1}(\ell - 1 - \Delta j)(w - 1 - \Delta i) = \binom{w + \ell - 2}{w - 1} - w\ell + w + \ell - 2.$$

As (i, j') and (i', j) define the same rectangle but different polyominos, we multiply the above result by two. However, this double counts the polyominos that are built with $i = i'$ and $j = j'$. There are $(w-2)(\ell-2)$ of these polyominos, so we subtract this quantity once. Therefore $|S_0| = 2\binom{w+\ell-2}{w-1} - 2w\ell + 2w + 2\ell - 4 - (w-2)(\ell-2) = 2\binom{w+\ell-2}{w-1} - 3w\ell + 4w + 4\ell - 8$

Combining the three subsets, we get the result.

$$|S| = |S_2| + |S_1| + |S_0| = 8\binom{w + \ell - 2}{w - 1} - 3w\ell + 2w + 2\ell - 8.$$

$\square$

Goupil et al. [**?**] proved Theorem **??** similarly, only differing on the method used to enumerate case 3. The authors also enumerated the inscribed polyominos of minimal area plus 1. In a separate paper, Goupil and Cloutier enumerated the 3-dimensional analogue of Theorem **??** in [**?**]. We chose to extend Theorem **??** instead to various other lattices, and ask how many *polyforms* of minimal area can be inscribed in different shapes.

3   Extensions to Polyforms

The purpose of this section will be to first generalize our definition of a polyomino to other shapes. Then we will generalize inscription in a general lattice. Finally, we will introduce notation to discuss lattices of certain sizes.

3.1 Polyforms

The idea behind polyforms is to construct shapes in $\mathbb{Z}^2$ with any polygons with unit side lengths.

Definition 3.1. A *polyform* of area n is a shape in the plane constructed by joining n unit side length polygons along their edges.
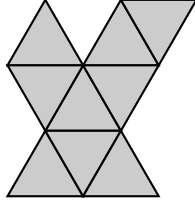

Figure 6: A polyform composed of 9 unit triangles

Polyominos are therefore polyforms made up of squares. Other polyforms are also well studied [?, ?]. *Polyiamonds* are polyforms composed of equilateral triangles. An example can be seen in Figure ??. *Polyhexes* are polyforms composed of regular hexagons. Most of the literature, including this paper, will focus solely on regular polygons when constructing polyforms.

There are other polyforms that have been studied [?, ?] and have been given special names. *Polyiamonds*, like the polyform in Figure ??, are composed of equilateral triangles. *Polyhexes*, polyforms that are composed of regular hexagons, are also well studied.

3.2 Generalization of Inscription

Polyforms can be inscribed into lattices as polyominos can. For minimal inscribed polyforms it was clear what cells were considered touching a side of the rectangular lattice. In other lattices is may be less clear. For example, consider Figure ??.
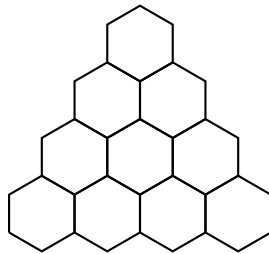

Figure 7: A triangle in the hexagonal lattice

To make it clear which cells in a given lattice belong to a side, construct the dual graph G of the lattice. Then let k be the number of sides chosen. Minimal inscribed polyominos have $k = 4$. In Figure ?? the most obvious choice is $k = 3$. After a k is decided, label each vertex of G with a subset of $[k] = \{1, 2, \ldots, k\}$, so that $\bigcup_{j \in V(G)} j = [k]$, where $V(G)$ is the vertex set of G. Let us call these graphs the *labelled dual* of G.

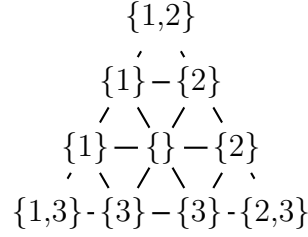**Example 3.1.** The lattice in Figure ?? has many possible labelled duals, one of which is shown in Figure ??.

$$\{1,2\}$$
$$\{1\}-\{2\}$$
$$\{1\}-\{\}-\{2\}$$
$$\{1,3\}\text{-}\{3\}-\{3\}\text{-}\{2,3\}$$

Figure 8: The labelled dual of Figure **??**

Example 3.2. We can re-frame polyomino inscription with labelled graphs. Let us call the family of $w \times \ell$ rectangles in the square lattice $\square_{w,\ell}^{S}$, where $\square$ designates the family is the shape of a rectangle and S designates we are in a square lattice. The labelled graph considered in Theorem **??** is shown below.
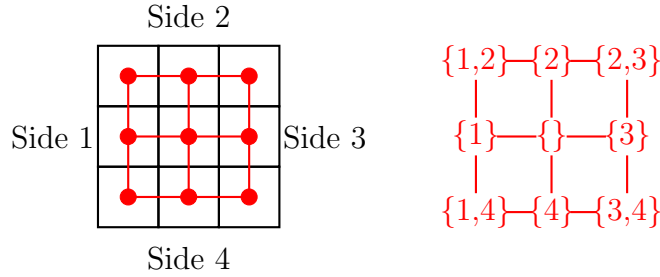
Side 2

$$\{1,2\}\text{—}\{2\}\text{—}\{2,3\}$$
$$\{1\}\text{—}\{\}\text{—}\{3\}$$
$$\{1,4\}\text{—}\{4\}\text{—}\{3,4\}$$

Side 1   Side 3

Side 4

Figure 9: $\square_{3,3}^{S}$ and the labelled dual

3.3 Minimal Inscribed Polyforms

We can now define the polyform equivalent of inscription in the labelled dual graph.

Definition 3.2. Suppose we have a labelled dual G. A given subgraph G' is an *inscribed polyform* in G if the following condition holds.

$$\bigcup_{j \in V(G')} j = [k] \tag{1}$$

Informally, condition **??** says that the subgraph G' touches all k sides. Notice that the number of vertices in an inscribed polyform of G can vary. If A is an inscribed polyform, then $|V(A)| \in [m(G), |V(G)|]$. Here $m(G)$ represents the minimum number of vertices for which Condition **??** holds, and can range from 1 to $|V(G)|$, depending on the structure and labelling of G.

Example 3.3. The labelled dual in Example **??** is $\square_{3,3}^{S}$, and so $m(\square_{3,3}^{S}) = 5$, and so at least 5 cells are required to touch all sides in $\square_{3,3}^{S}$. The $w \times \ell$ rectangular lattice has minimal area $m(\square_{w,\ell}^{S}) = w + \ell - 1$.

Our focus is on the minimal inscribed polyforms, which are defined as follows.

Definition 3.3. An inscribed polyform A is *minimal* if $|V(A)| = m(G)$.

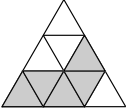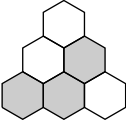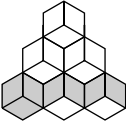Finally, we are concerned with the number of polyforms that are minimal.
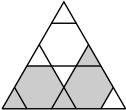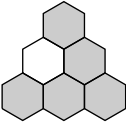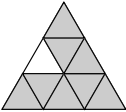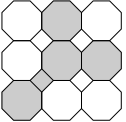
Definition 3.4. For a labelled graph G, let $\rho(G)$ denote the number of minimal inscribed polyforms.

Example 3.4. Theorem **??** gives us that $\rho(\square_{w,\ell}^S) = 8\binom{w+\ell-2}{w-1} - 3w\ell + 2w + 2\ell - 8$.

The results of this paper will be finding expressions for $\rho(G)$ for various labelled graphs G.

4 New Results on Minimal Inscribed Polyforms

We catalogue the known solved cases, which constitute the main new results of the paper. A summary of the results in this section can be found in the table below.

Example Polyform	Notation	ρ(lattice)
	$\triangle_n^T$	$((n-1)^2 + 2)2^{n-2}$
	$\triangle_n^H$	$\left(\binom{n}{2} + 2\right)2^{n-2}$
	$\triangle_n^R$	$n(n+1)2^{n-2}$
	$\triangle_n^K$	$\left(n^2 + 3n + \frac{10}{3}\right)4^{n-3} - \frac{1}{3}$
	$\triangle_n^{H*}$	$\binom{2(n-1)}{n-1} - \sum_{k=0}^{n-2}\binom{2k}{k}$
	$\triangle_n^{T*}$	$\sum_{k=0}^{n-1}\binom{2k}{k}$
	$R_{w,\ell}^\theta$	See Theorem **??**

Each result includes visuals for both the traditional representation and the labelled dual representation. The empty set in the dual graph representation will be replaced with a black node for simplicity.

4.1 Minimal Inscribed Polyforms in $\triangle_n^T$

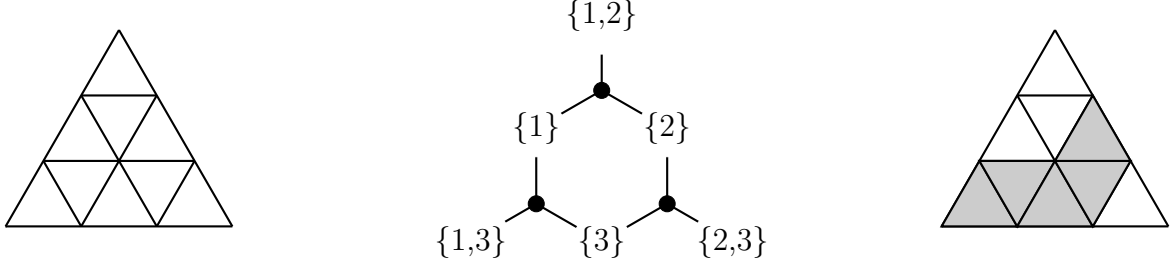The first solved case we will examine is the triangular analogue to Theorem **??**.

Figure 10: The tiling and dual representation of $\triangle_3^T$ and an example polyform

Let us call the family of triangles one can make $\triangle_n^T$, where n designates the number of triangles touching a side, and the superscript T designates that it is in a triangular lattice. Here $k = 3$, and $m(\triangle_n^T) = 2n - 1$.

Theorem 2. *The number of minimal inscribed polyforms $\rho(\triangle_n^T)$ for $n \geq 1$ is given by the following formula.*

$$\rho(\triangle_n^T) = ((n-1)^2 + 2)2^{n-2}. \tag{2}$$

Proof. Here we partition $\triangle_n^T$ into 3 distinct subsets. Let us call the subset that contains the polyforms that contain 1 or 2 corner cells S_1. Then let us call the subset of polyforms not contained in S_1 that have cells with maximum degree of 2 to be S_2. Finally let us call the subset of polyforms that have a maximum degree of 3 to be S_3. We then write $\rho(\triangle_n^T) = |S_1| + |S_2| + |S_3|$.
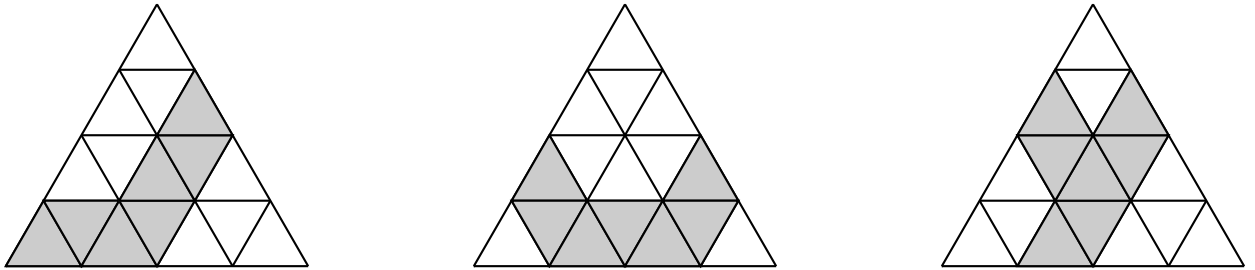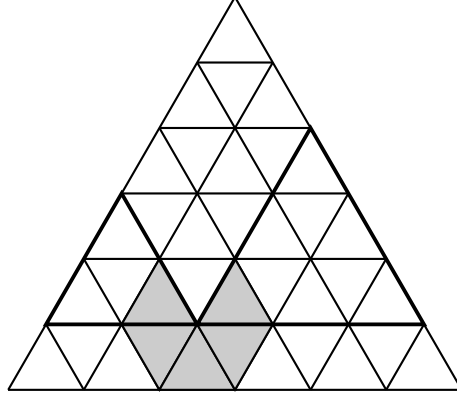

Figure 11: Polyforms in S_1, S_2, and S_3

Case 1. Let $C(n)$ be the number of polyforms that contain the bottom left cell. The binomial theorem immediately gives that $C(n) = 2^{n-1}$. Applying this argument to each corner will double count the 3 minimal polyominos that contain 2 corners, and so $|S_1| = 3(2^{n-1} - 1)$.

Case 2. All polyforms in S_2 are of the form

where the triangles in bold can be seen as smaller versions of case 1. Suppose we consider the polyforms with this "U" shape on the bottom edge. If we index the cells that touch the bottom edge 1 to n from left to right, then there exist minimal inscribed polyforms that contain this "U" shape on cells 2 through $n-1$. These "U" shapes can also be longer, shown below.
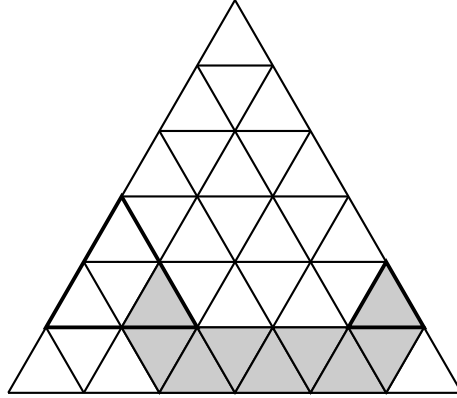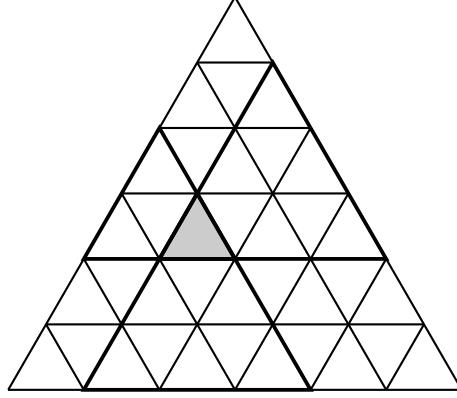

If we let k represent the index of the left most edge cell that the "U" shape contains, and p the total number of edge cells the "U" shape contains, we can write the following sum

$$E(n) = \sum_{k=2}^{n-1} \sum_{p=1}^{n-k} (2^{k-2})(2^{n-k-p}),$$

which after simplifying and applying the same argument for each edge, gives

$$|S_2| = 3E(n) = 3(n-3)2^{n-2} + 3.$$

Case 3. All polyforms in S_3 are of the form

again where the triangles in bold can be seen as smaller versions of case 1. This leads us to write the sum

$$|S_3| = \sum_{i=1}^{n-3}\sum_{j=1}^{i}(2^j)(2^{n-i-2})(2^{i-j+1}) = (n-2)(n-3)2^{n-2},$$

where i and j count over the cells where the sides of the cell are parallel to the side of the larger triangle in the interior $(n-3)$-triangle. This gives $\rho(\triangle_n^T) = |S_1| + |S_2| + |S_3| = ((n-1)^2 + 2)2^{n-2}$.

$\square$

The first terms of $\rho(\triangle_n^T)$ for $n \geq 1$ are $1, 3, 12, 44, 144, 432, 1216, \ldots$ (A356888 [**?**]). The next solved family is similar.

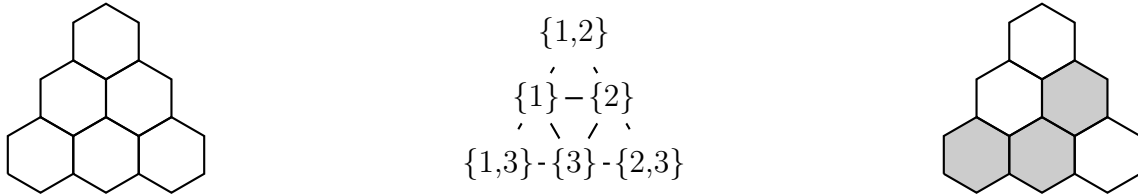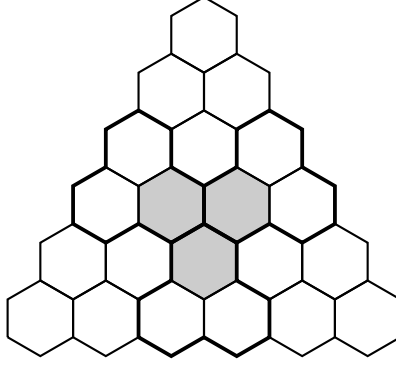## 4.2   Minimal Inscribed Polyforms in $\triangle_n^{\mathbf{H}}$

Figure 12: The tiling and dual representation of $\triangle_3^H$ and an example polyform

Let us call the family of triangles you can make in the hexagonal lattice $\triangle_n^H$, where n designates the number of hexagons touching a triangular side, and the superscript H designates that it is in a hexagonal lattice. Here $k = 3$, and $m(\triangle_n^H) = n$.

Theorem 3. *The number of minimal inscribed polyforms $\rho(\triangle_n^H)$ for $n \geq 1$ is given by the following formula.*

$$\rho(\triangle_n^H) = \left(\binom{n}{2} + 2\right)2^{n-2}. \tag{3}$$

Proof. We partition $\triangle_n^H$ in the same way as $\triangle_n^T$. S_1 and S_2 are enumerated the same way. However, the sum for S_3 given in the proof of Theorem **??** triple counts shapes of the form

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where the bold triangles can be seen as smaller cases of S_1. The number of polyominos that are of this form are $(n-1)(n-2)2^{n-4}$. As these are triple counted by the sum for S_3, we get that

$$\rho(\triangle_n^H) = \rho(\triangle_n^T) - (n-1)(n-2)2^{n-3} = \left(\binom{n}{2} + 2 \right) 2^{n-2}.$$

□

The first terms in this sequence, for $n \geq 1$, are $1, 3, 10, 32, 96, 272, 736, \ldots$ (A104270 [?]).
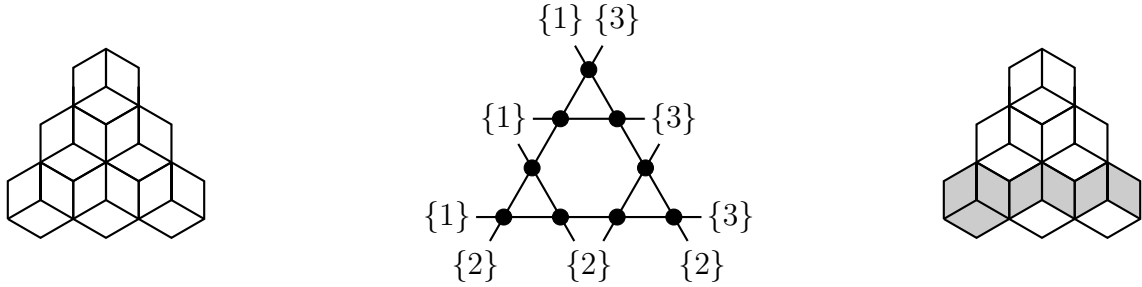
4.3 Minimal Inscribed Polyforms in $\triangle_n^R$

Figure 13: The tiling and dual representation of $\triangle_3^R$ and an example polyform

Let us call the family of triangles one can make $\triangle_n^R$, where n designates the number of rhombuses touching a side, and the superscript R designates that it is in a rhombic lattice. Here $k = 3$, and $m(\triangle_n^R) = 2n + 1$.

Theorem 4. *The number of minimal inscribed polyforms $\rho(\triangle_n^R)$ for $n \geq 1$ is given by the following formula.*

$$\rho(\triangle_n^R) = n(n+1)2^{n-2}. \tag{4}$$

Proof. We partition $\triangle_n^R$ into two subsets. Let S_1 be the polyforms that contain the three corner most cells, and let $S_2 = S_1^c$.
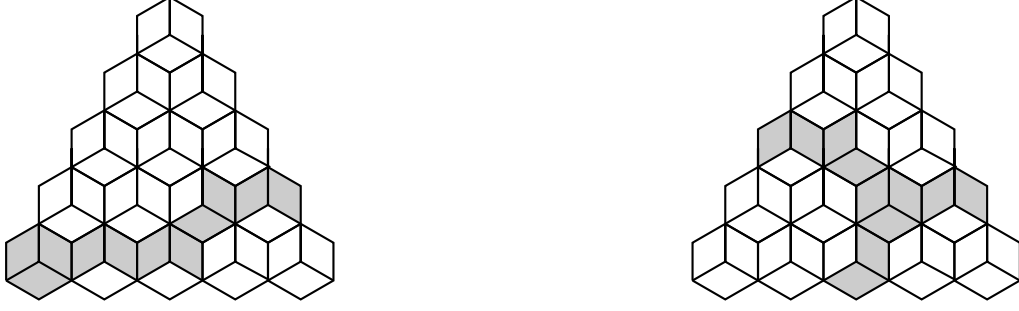
Figure 14: Examples of Polyforms in S_1 and S_2 for $\rho(\triangle_5^R)$

Similarly to Theorems **??** and Theorem **??**, S_1 can easily be enumerated using the binomial theorem to get $|S_1| = 2^{n-1}$. Also similarly, enumerating S_2 can be seen as the product of smaller cases of S_1. So each of the $\frac{n(n+1)}{2}$ locations where the following three cells

can be found has 2^{n-1} polyforms, which gives $\rho(\triangle_n^R) = n(n+1)2^{n-2}$.

$\square$

The first terms of this sequence, for $n \geq 1$, are $1, 6, 24, 80, 240, 672, 1792, \ldots$ (A001788 [**?**]).
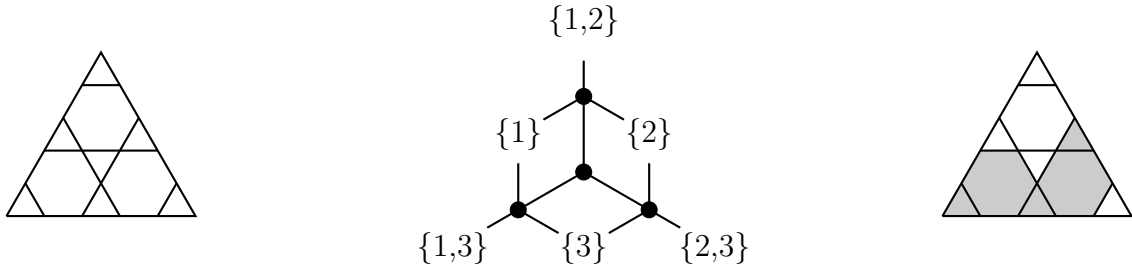
4.4 Minimal Inscribed Polyforms in $\triangle_n^K$

Figure 15: The tiling and dual representation of $\triangle_n^K$ and an example polyform

Let us call the family of triangles one can make $\triangle_n^K$, where n designates the number of triangles touching a side, and the superscript K designates that it is the hexagonal and triangular lattice shown above, which is often called a *kagome* lattice.

Again $k = 3$, and $m(\triangle_n^K) = 2n - 1$.

Theorem 5. *The number of minimal inscribed polyforms $\rho(\triangle_n^K)$ for $n \geq 2$ is given by the following formula.*

$$\rho(\triangle_n^K) = \left(n^2 + 3n + \frac{10}{3}\right)4^{n-3} - \frac{1}{3}. \tag{5}$$

12

Proof. We again partition $\triangle_n^K$ into S_1, S_2, and S_3. This proof is closely related to the proof of Theorem **??**, as we choose analogous definitions for S_1, S_2, and S_3.
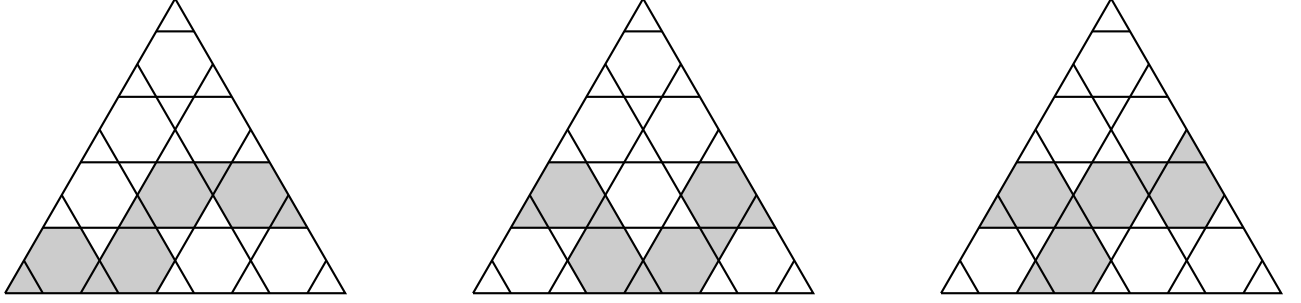

Figure 16: Polyforms in S_1, S_2, and S_3

Case 1 Similar to the proof of Theorem **??**, we use the binomial theorem to get $C(1) = 1$ and $C(n) = 2^{2n-3}$ for $n \geq 2$, and consequently $|S_1| = 3(2^{2n-3} - 2^{n-2})$. We introduce a new function $H(n) = C(n) - 1 - \sum_{k=2}^{n-1} C(k)$. $H(n)$ counts all of the polyforms that are counted in $C(n)$ that do not contain any other cells on one of the shared sides. For example, in Figure **??**, $H(3)$ is the number of polyforms that contains the node labelled $\{1,3\}$, but not the node labelled $\{3\}$ or $\{2,3\}$. H is used to build the sums in case 2.

Case 2 Similar to the proof of Theorem **??**, we next consider the "U"-shapes that appear in S_2 and introduce $E(n)$ to count the number of these "U" shapes on a given side. In this lattice we need to split $E(n)$ into two parts E_1 and E_2. E_1 counts the polyforms with "U"-shapes that only contain 1 of the triangles that make up the bottom edge, while E_2 counts the number of polyforms that contain 2 or more triangles.
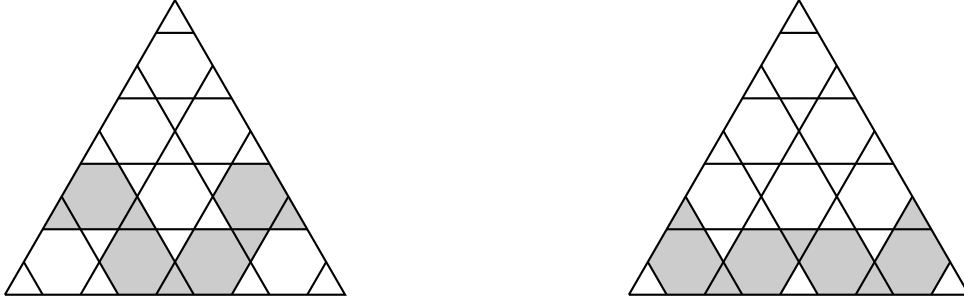

Figure 17: Polyforms counted by E_1 and E_2

We number the triangles that belong to the bottom edge 1 to n from left to right. Polyforms that have the "U" shape with one triangle can be found at cells 2 through $n-1$. This gives

$$E_1(n) = \sum_{i=2}^{n-1} H(i)H(n+1-i)$$

For E_2, call the leftmost triangle on the bottom row i and the rightmost j. There are $i - j - 1$ triangles between i and j, and so there are 2^{i-j-1} different paths between the cells i and j. This gives

$$E_2(n) = \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} 2^{j-i-1} H(i)H(n+1-j).$$

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Therefore $|S_2| = 3E(n) = 3E_1(n) + 3E_2(n)$.

Case 3 In a similar fashion to Theorem **??** we see that $|S_3| = I(n) = (n-2)(n-3)2^{2n-4}$. Solving the sums in case 2 and combining $|S_1|$, $|S_2|$, and $|S_3|$ gives the result.

$\square$

The first terms of this sequence, for $n \geq 2$, starts $3, 21, 125, 693, 3669, 18733, \ldots$ (A356889 [?])

Theorem **??** can be extended by adding $a - 2$ additional edges in a symmetric manner, shown in Figure **??**. We will call the family of these graphs $\triangle_n^{K,a}$, where K, a represents the dual with a connections, and n represents the size.

Figure 18: Duals of $\triangle_3^{K,3}$ and $\triangle_3^{K,4}$

Notice that we recover $\triangle_n^K$ for $a = 2$.

Theorem 6. *For a given $a \geq 2$, the number of minimal inscribed polyforms $\rho(\triangle_n^{K,a})$ for $n \geq 2$ is given by the following formula.*

$$\rho(\triangle_n^{K,a}) = \left(n^2 + \frac{(6a^2 - 5a - 5)n}{a+1} + \frac{6(a^4 - 2a^3 + 2a + 1)}{(a+1)^2} \right) 2^{n-2} a^{n-4} - \frac{3(a-1)^n}{(a+1)^2} \qquad (6)$$

Proof. This proof extends the sums of Theorem **??** for a given a.

Case 1 We parameterize the method in Theorem **??**. We have $C_a(1) = 1$, and $C_a(n) = 2^{n-1}a^{n-2}$ for $n \geq 2$. This gives $|S_1| = 3(C_a(n) - a^{n-2})$. We $H_a(n) = C_a(n) - (a-1)^{n-2} - \sum_{k=2}^{n-1}(a-1)^{n-1-k}C_a(k)$, which represents all the polyforms counted in $C_a(n)$ that do not contain any other cells on one of the shared sides. For example in Figure **??** $H_a(3)$ is the number of polyforms that contains the cell labelled $\{1,3\}$, but not $\{3\}$ or $\{2,3\}$.

Case 2 We form the sums in a similar way to Theorem **??**. $|S_2| = 3E_1(n) + 3E_2(n)$, where

$$E_{1,a}(n) = \sum_{i=2}^{n-1} H_a(i) H_a(n+1-i)$$

and

$$E_{2,a}(n) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} a^{j-i-1} H_a(i) H_a(n+1-j).$$

Case 3 Similar to Theorem **??** we have $|S_3| = I_a(n) = (n-2)(n-3)2^{n-2}a^{n-4}$. Solving the sums in case 2 and combining $|S_1|, |S_2|$, and $|S_3|$ gives the result. $\qquad \square$

Interestingly, though initially defined for $a \geq 2$, $\rho(\triangle_n^{K,1}) = ((n-1)^2 + 2)2^{n-2} = \rho(\triangle_n^T)$. So in the case of minimal inscribed polyforms $\triangle_n^T$ can be seen as the $a = 1$ case of $\triangle_n^{K,a}$.

4.5 Minimal Inscribed Polyforms in $\triangle_n^{H*}$

Different labellings of the same lattice can also give interesting results.
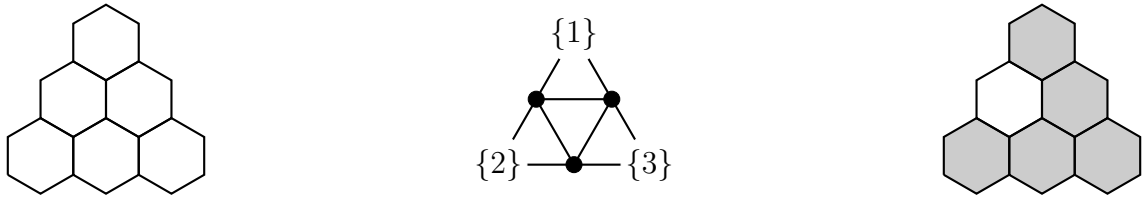
Figure 19: The tiling and dual representation of $\triangle_3^{H*}$ and an example polyform

Let us call the family shown in Figure **??** $\triangle_n^{H*}$, where $H*$ indicates an alternative labelling of the hexagonal lattice. Here $k = 3$ and $m(\triangle_n^{H*}) = 2n - 1$.

Theorem 7. *The number of minimal inscribed polyforms $\rho(\triangle_n^{H*})$ for $n \geq 2$ is given by the following formula.*

$$\rho(\triangle_n^{H*}) = \binom{2(n-1)}{n-1} - \sum_{k=0}^{n-2} \binom{2k}{k}. \tag{7}$$

Proof. Each unit hexagon in the triangle corresponds with a unique weak integer 3-composition of $n - 1$. Figure **??** shows the visual interpretation of the integer composition for the dark grey hexagon. The lengths of the 3 light grey "spokes" are the components of the composition. Suppose the components of the composition are labelled k_1, k_2, and k_3. Then Figure **??** represents the cell for $k_1 = 2$, $k_2 = 1$, and $k_3 = 1$, read counter-clockwise around the dark grey hexagon.
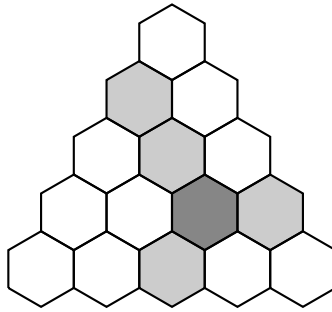

Figure 20: A cell with spokes

Notice that k_i can be 0, as there are cells in which one cannot extend a spoke in a certain direction (the dark grey hexagon being located on a side or corner).

Using the spokes as guides, we can group paths from each corner to the dark grey hexagon. An example of this grouping is shown in Figure ??.
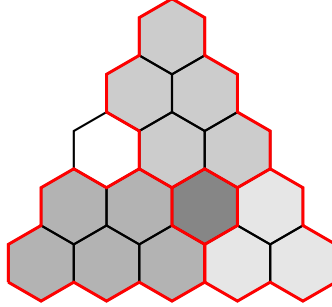

Figure 21: A cell in the $n = 5$ triangle with groupings

Using this grouping we can construct the following sum for the number of possible sets of paths from each corner to a given vertex.

$$s(n) = \sum_{k_1+k_2+k_3=n-1} \binom{k_1 + k_2}{k_1}\binom{k_2 + k_3}{k_2}\binom{k_3 + k_1}{k_3} \tag{8}$$

Notice that this grouping, and consequently $s(n)$, will count certain polyforms multiple times. An example of a polyform that is counted multiple times is shown in Figure ??.
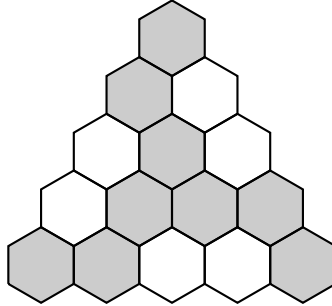
Figure 22: A minimal inscribed polyform that is counted multiple times

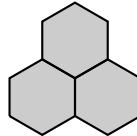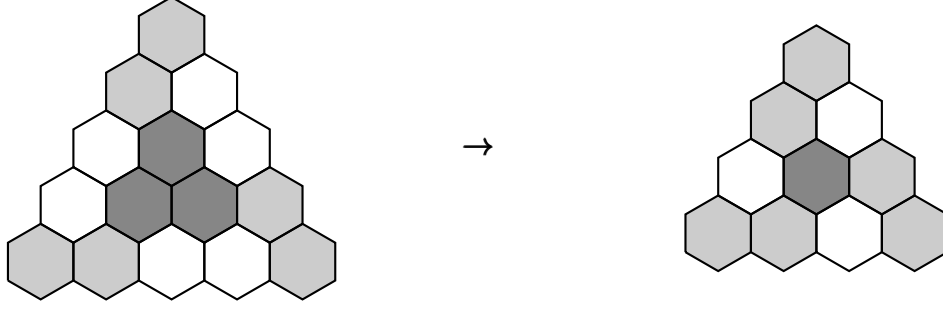All polyforms that are grouped multiple ways contains the following substructure.

Figure 23: The substructure that allows for multiple groupings.

Polyforms with these substructures are counted 3 separate times. Fortunately, the number of polyforms that contain this substructure are easy to count, after the following transformation is made.

Therefore the number of polyforms that contain this substructure is $s(n-1)$. This, and the following identity

$$\sum_{k_1+k_2+k_3=n} \binom{k_1+k_2}{k_1}\binom{k_2+k_3}{k_2}\binom{k_3+k_1}{k_3} = \sum_{k=0}^{n}\binom{2k}{k} \tag{9}$$

completes the proof, as

$$\rho(\triangle_n^{H*}) = s(n) - 2s(n-1) = \sum_{k=0}^{n-1}\binom{2k}{k} - 2\sum_{k=0}^{n-2}\binom{2k}{k} = \binom{2(n-1)}{n-1} - \sum_{k=0}^{n-2}\binom{2k}{k}$$

$\square$

The first terms of $\rho(\triangle_n^{H*})$ for $n \geq 1$ are $1, 1, 3, 11, 41, 153, 573, \ldots$ (A281593 [?]).

4.6 Minimal Inscribed Polyforms in $\triangle_n^{T*}$

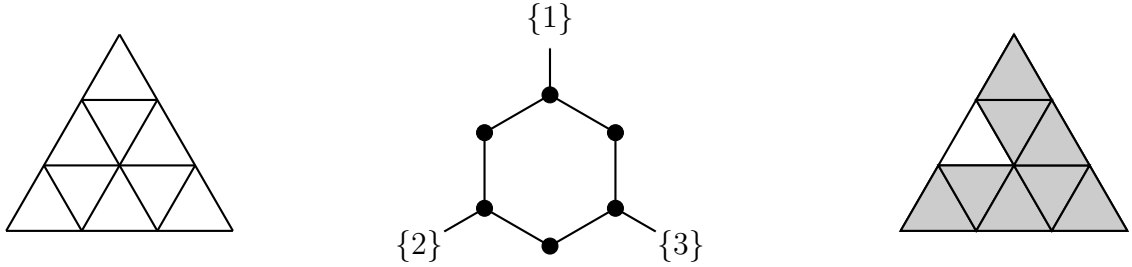We can ask a similar question for the triangular lattice.

Figure 24: The tiling and dual representation of $\triangle_3^{T*}$ and an example polyform

Let us call the family in Figure ?? $\triangle_n^{T*}$. Here $k = 3$, and $m(\triangle_n^T) = 4(n-1)$.

Theorem 8. *The number of minimal inscribed polyforms $\rho(\triangle_n^{T*})$ for $n \geq 2$ is given by the following formula.*

$$\rho(\triangle_n^{T*}) = \sum_{k=0}^{n-2}\binom{2k}{k}. \tag{10}$$

The first terms of this sequence, for $n \geq 1$, are $1, 1, 3, 9, 29, 99, 351, \ldots$ (A006134).

Proof. The proof for Theorem **??** is similar to the proof for Theorem **??**, except that there is no double-counting. The structure of the lattice allows for each polyform to be counted once by a similar grouping method, and $s(n)$ can again be used to describe the number of polyforms in this lattice, this time without correction. $\square$
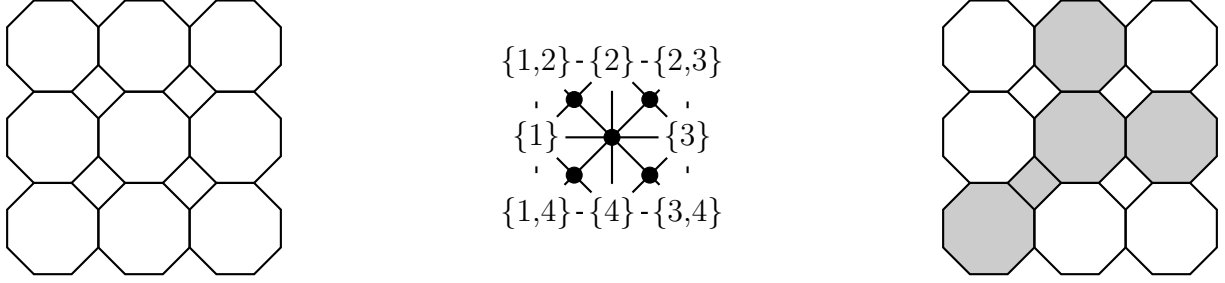
4.7 Minimal Inscribed Polyforms in $\mathbf{R}_{w,\ell}^{\theta}$

Figure 25: The tiling and dual representation of $R_{3,3}^{\theta}$ and an example polyform

Let us call the family of rectangles one can make $R_{w,\ell}^{\theta}$, where w designates how many octagons the lattice is wide, and ℓ designates how many octagons the lattice is long. R will indicate that we are forming a rectangle, and the superscript θ will indicate we are in the octagonal tiling. Here $k = 4$, and $m(R_{w,\ell}^{\theta}) = w + \ell - 1$.

Theorem 9. *The number of minimal inscribed polyforms $\rho(R_{w,\ell}^{\theta})$ for $w, \ell \geq 1$ is given by the formula*

$$\rho(R_{w,\ell}^{\theta}) = 2D(w-1, \ell-1) - w\ell + 2\sum_{i=0}^{w-2}\sum_{j=0}^{\ell-2} D(i,j)(2 + (w-2-i)(\ell-2-j)). \quad (11)$$

Here $D(i,j)$ designates the i,j-th Delannoy number, which is given by

$$D(i,j) = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i}\binom{i}{k}.$$

Proof. The proof has the same structure as the proof to Theorem **??**, with the only difference being the number of minimal paths between two cells (i,j) and (i', j') is counted by $D(|i' - i| + 1, |j' - j| + 1)$ instead of $\binom{|i'-i|+|j'-j|}{|i'-i|}$.
$\square$

The first terms of $\rho(R_{n,n}^{\theta})$, for $n \geq 1$, are $1, 6, 43, 256, 1401, 7510, \ldots$.

The lattices examined in this section all grow exponentially. This is not the case for all lattices. As it turns out, some lattices don't grow.

5 Trivial Growth

Some families do not have a strictly increasing number of minimal inscribed polyforms in n. A family of labelled duals with this property has growth rate 1 and is referred to as a *trivial* family.

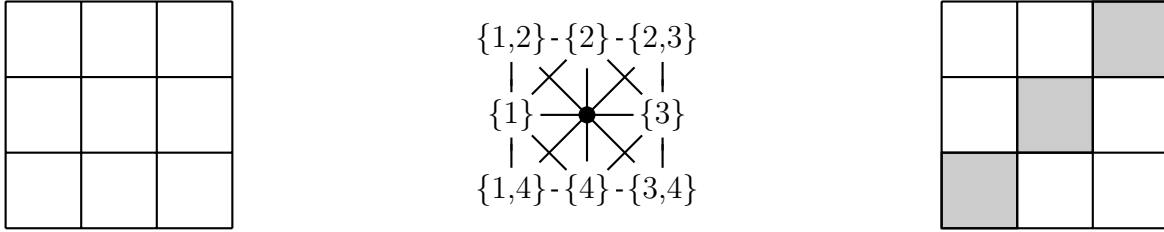**Example 5.1.** An example of a trivial labelled graph family is below.

$\{1,2\}$-$\{2\}$-$\{2,3\}$

$\{1\}$———$\{3\}$

$\{1,4\}$-$\{4\}$-$\{3,4\}$

Figure 26: The tiling and dual representation of $\square_{3,3}^{S*}$ and an example polyform

We will call this family $\square_{n,n}^{S*}$ to match the notation following Theorem **??**.

For $n \geq 2$ the minimal path between the vertices labelled $\{1,2\}$ and $\{3,4\}$ and the minimal path between the vertices labelled $\{1,4\}$ and $\{2,3\}$ are the only 2 minimal inscribed polyforms in $\square_{n,n}^{S*}$. So $\rho(\square_{n,n}^{S*}) = 2$ for $n \geq 2$.

Example 5.2. Interestingly, $\square_n^S$ (a square formed in the square lattice) and $\triangle_n^T$ (a triangle formed in the triangular lattice) both show non-trivial growth (Theorem **??** and Theorem **??**), but the family $\bigcirc_n^H$ (a hexagon formed in the hexagonal lattice) is trivial.

$\{5,6\}$ $\{4,5\}$
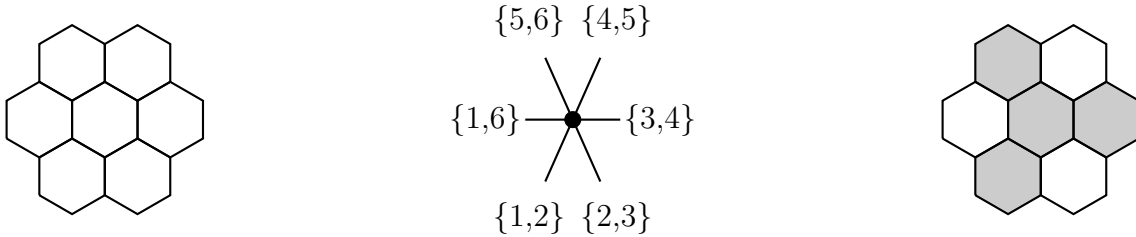
$\{1,6\}$———$\{3,4\}$

$\{1,2\}$ $\{2,3\}$

Figure 27: The tiling and dual graph representation of a hexagon in the hexagonal grid and an example polyform

The family $\bigcirc_n^H$ has $k = 6$, and $m(\bigcirc_n^H) = 3n - 2$. The polyform shown in Figure **??** and its $60°$ degree rotation are the only 2 minimal inscribed polyforms, and so $\rho(\bigcirc_n^H) = 2$ for $n \geq 2$.

Example 5.3. An interesting shape in combinatorics is the Aztec diamond. Let us call the family of Aztec diamonds of width $2n$ A_n^S.
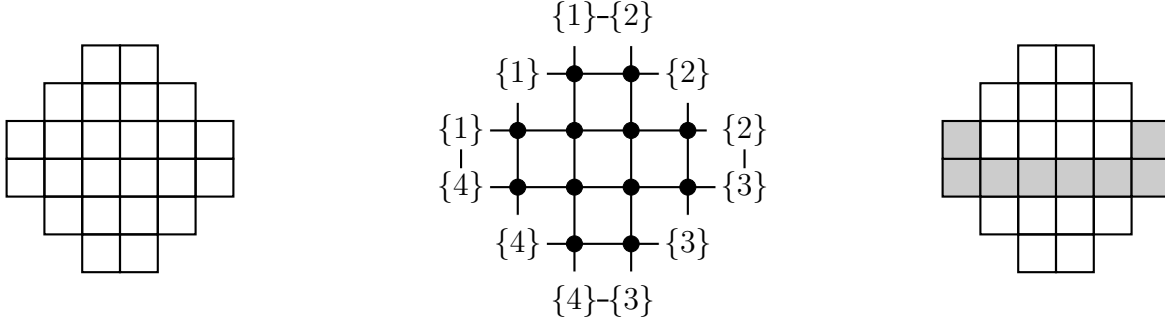
Figure 28: The tiling and dual representation of A_3^S and an example polyform

This family is trivial. Here $k = 4$, and $m(A_n^S) = 2n + 2$. The 4 minimal inscribed polyforms are the polyform shown in Figure **??**, and so $\rho(A_n^S) = 4$.

However, we can extend the duals for A_n^S. If instead cells diagonal from one another are considered adjacent, this results in a new labelled graph. We will call this family $A_n^{S^*}$.
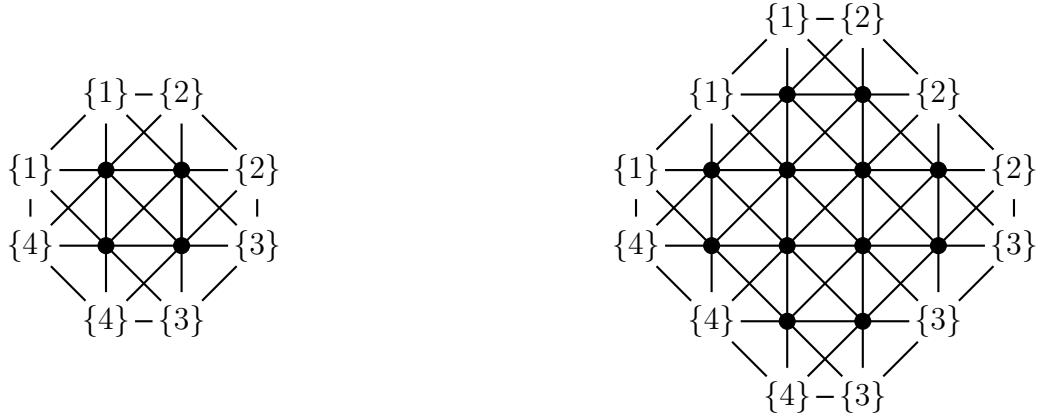

Figure 29: $A_2^{S^*}$ and $A_3^{S^*}$

$A_n^{S^*}$ appears to be non-trivial. $\rho(A_n^{S^*})$ is known up to $n = 5$, with the first terms being $1, 68, 1113, 11616, 104097$. This appears exponential, but an exact formula for $\rho(A_n^{S^*})$ is unknown.

6   Acknowledgements

7   Student Biographies

Jack Hanke: graduated from the University of Connecticut in 2022 with a B.A. in Mathematics. He is currently working as a data analyst at Lockheed Martin, and plans to pursue a Master's in Data Science for the Fall 2024 semester.