Minimal Inscribed Polyforms of the Extended Generalized Aztec Diamond

Jack Hanke

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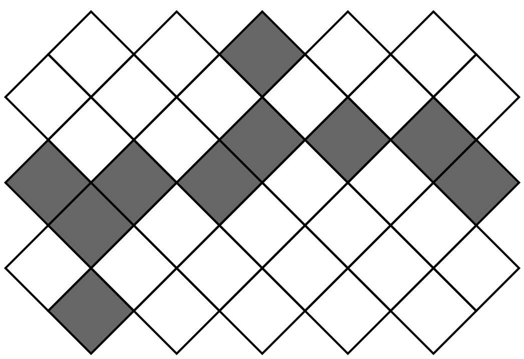
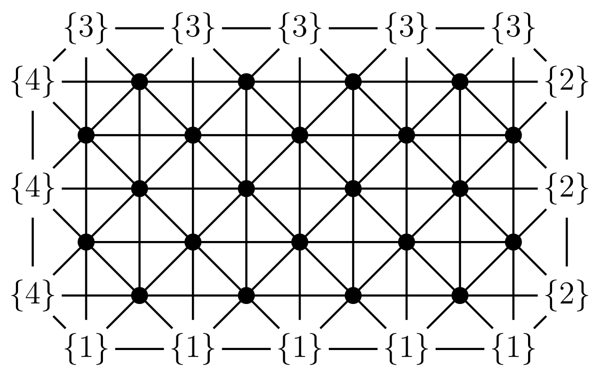
Let the family of extended generalized aztec diamonds be denoted A^{S^*}. Below is an example of a polyform in $A^{S^*}_{5,3}$ and the labelled dual.





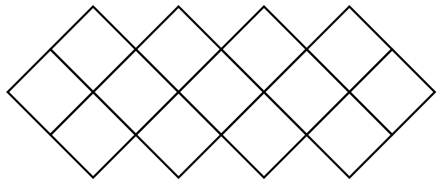
Note that $m(A^{S^*}_{n,m}) = n + m + 2$. Some values of $\rho(A^{S^*}_{n,m})$ are summarized below.

$\rho(A^{S^*}_{n,m})$	1	2	3	4	5
1	1	8	29	74	155
2	8	68	266	752	1758
3	29	266	1113	3428	8815
4	74	752	3428	11616	33036
5	155	1758	8815	33036	104097

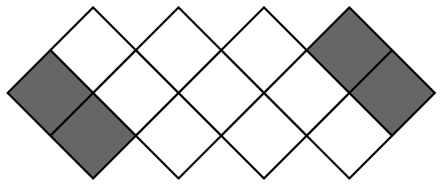
Theorem 1. *The generating function for the number of minimal inscribed polyforms in the extended generalized aztec diamond is*

$$\begin{aligned}
\sum_{n,m\geq 0} \rho(A^{S^*}_{n,m}) x^n y^m \quad = & \\
& xy(2\,x^5y - 11\,x^4y^2 + 9\,x^3y^3 - 11\,x^2y^4 + 2\,xy^5 \\
- & \; 10\,x^4y + 16\,x^3y^2 + 16\,x^2y^3 - 10\,xy^4 \\
+ & \; x^4 + 15\,x^3y - 21\,x^2y^2 + 15\,xy^3 + y^4 - 8\,x^2y - 8\,xy^2 \\
- & \; 4\,x^2 + 5\,xy - 4\,y^2 + 2\,x + 2\,y + 1) \\
/ & \; \left(1 - 2\,x - 2\,y + x^2 + xy + y^2\right)(1-x)^4(1-y)^4
\end{aligned}$$

Proof. We first handle the $m = 1$ case. $A_{4,1}^{S^*}$ is shown below as an example of what the $m = 1$ case looks like.

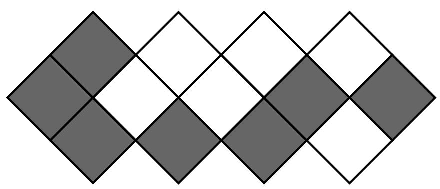


We split the polyforms into the following distinct cases. First, the polyforms that include the following cells are enumerated by



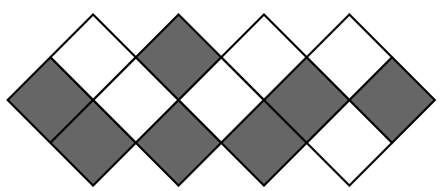
$$\binom{n}{n-2}_2$$

Next we enumerate the polyforms of the following form



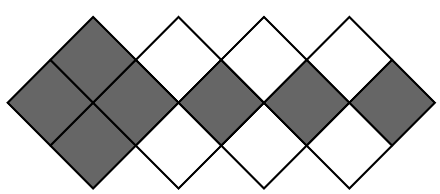
$$(n-2)$$

Next we enumerate the polyforms of the following form



$$\frac{(n-2)(n-3)}{2}$$

The single polyform below must be included.



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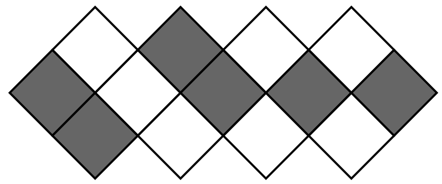
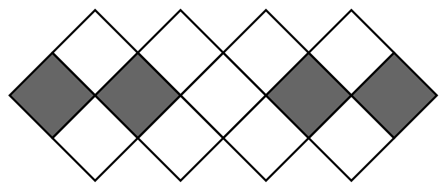
Next we enumerate the polyforms of the following form



$$\sum_{k=2}^{n-1} \binom{k}{k-2}_2$$

Finally, polyforms of this form are enumerated by



$$\rho(A^{S^*}_{n-2,1})$$

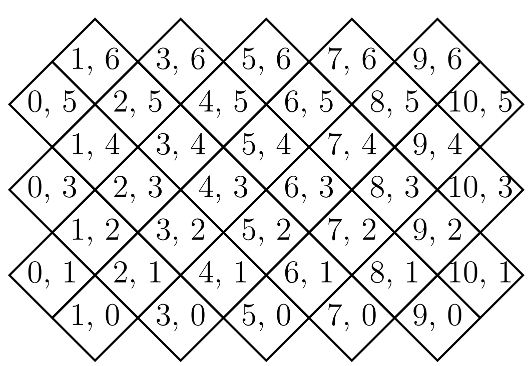
Multiplying by the appropriate constants for reflections and rotations, and re-indexing, gives

$$\rho(A^{S^*}_{n+3,1}) = 2\binom{n+3}{n+1}_2 + 2(n+1)n + 2(n+1) + 2 + 4\sum_{k=2}^{n+2} \binom{k}{k-2}_2 + \rho(A^{S^*}_{n+1,1})$$

this can be solved using common techniques to get

$$\sum_{n\geq 1} \rho(A^{S^*}_{n,1})x^n = \frac{x + 3x^2 - x^3 - x^4}{(1-x)^5}.$$

For the $n, m \geq 2$ case, we begin by indexing the cells for reference.



The path from a point (i,j) and $(\hat{i},\hat{j})$ is

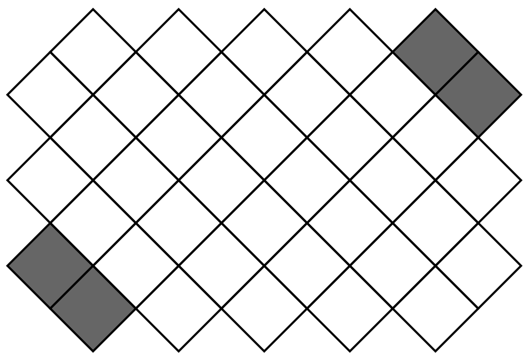
$$\binom{\frac{1}{2}((i-\hat{i})+(j-\hat{j}))}{\frac{1}{2}((i-\hat{i})-(j-\hat{j}))}_2$$

where $\binom{n}{k}$ is the n,k-th trinomial coefficient. From properties of tinomial coefficients we have

$$F(x,y) = \sum_{n,m\geq 0} \binom{n+m}{n-m}_2 x^n y^m = \frac{1-x-y}{(1-x-y)^2 - xy}$$

$$G(x,y) = \sum_{n,m\geq 0} \binom{n+m+1}{n-m}_2 x^n y^m = \frac{1}{(1-x-y)^2 - xy}.$$

We first define $cc(n,m)$ to be the number of minimal inscribed polyforms that contain the following cells.



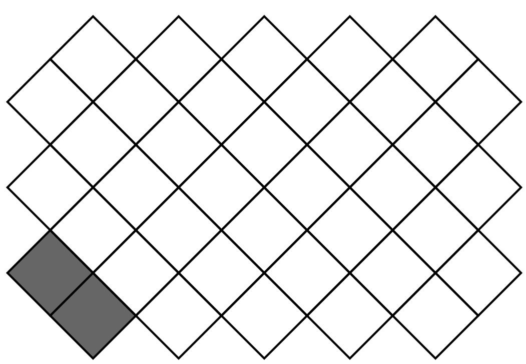
It is easy to see that

$$cc(n,m) = \binom{n+m-1}{n-m-1}_2 + \binom{n+m-1}{n-m+1}_2 - \binom{n+m-3}{n-m-1}_2 - \binom{n+m-3}{n-m+1}_2$$

for n,m and $cc(n,1) = \binom{n}{n-2}_2$ and $cc(n,0) = 1$. This gives

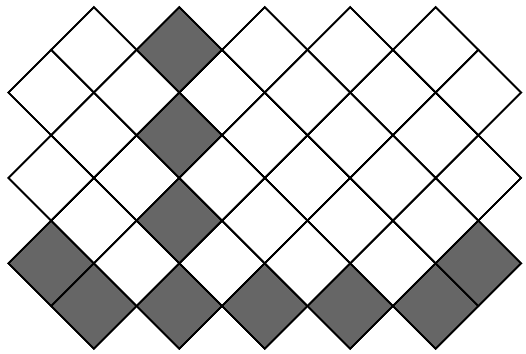
$$CC(x,y) = \sum_{n,m\geq 0} cc(n,m) x^n y^m = (1-xy)(1+(x+y)F(x,y))$$

We next define c to be the number of polyforms that contain the following cells that are not counted in cc.



To compute c we study subclasses c_0, c_1, c_2, c_3 so that $c(n,m) = c_0(n,m) + c_1(n,m) + c_2(n,m) + c_3(n,m)$.

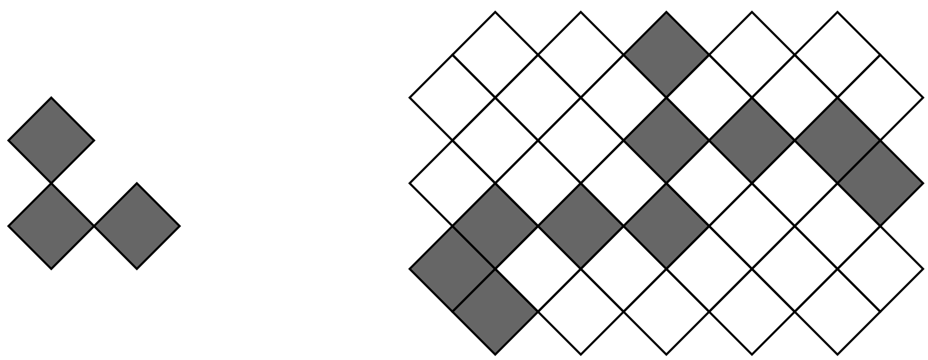
The subclass c_0 counts the number of polyforms that contain two corners and include the bottom most corner cells. An example polyform in c_0 can be found below.



We have that the generating function for c_0 is

$$C_0(x,y) = \frac{}{x}^3 y^2(1-x)^2(1-y) + \frac{}{x}^2 y^3(1-x)(1-y)^2.$$

The subclass c_1 counts the number of polyforms that contain the "thin u" substructure. The substructure and an example polyform in c_1 can be found below.



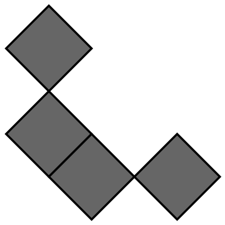
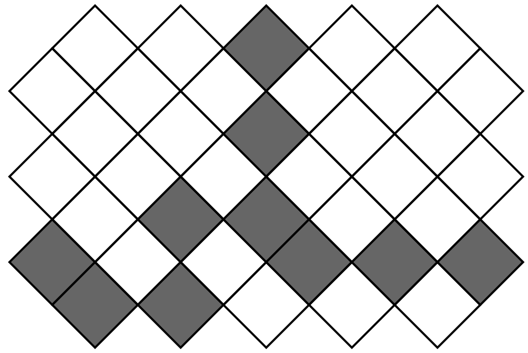
This subclass can be counted with the following sums.

$$\begin{aligned} c_1(n,m) &= 2\sum_{\substack{0\le i\le n-1\\ 1\le j\le m-1}} \binom{i+j}{i-j}_2 (n-1-i) + 2\sum_{\substack{0\le i\le n-1\\ 0\le j\le m-3}} \binom{i+j+1}{i-j}_2 (n-1-i) \\ &+ 2\sum_{\substack{1\le i\le n-1\\ 0\le j\le m-1}} \binom{i+j}{i-j}_2 (m-1-j) + 2\sum_{\substack{0\le i\le n-3\\ 0\le j\le m-1}} \binom{i+j+1}{i-j}_2 (m-1-j)), \end{aligned}$$

and has the following generating function

$$\begin{aligned} C_1(x,y) &= \frac{2xy}{(1-x)(1-y)}\left(F(x,y)\left(\frac{x}{1-x}+\frac{y}{1-y}\right) + G(x,y)\left(\frac{xy^2}{1-x}+\frac{x^2y}{1-y}\right)\right) \\ &- \frac{2x^2y}{(1-x)^3(1-y)} - \frac{2xy^2}{(1-x)(1-y)^3}. \end{aligned}$$

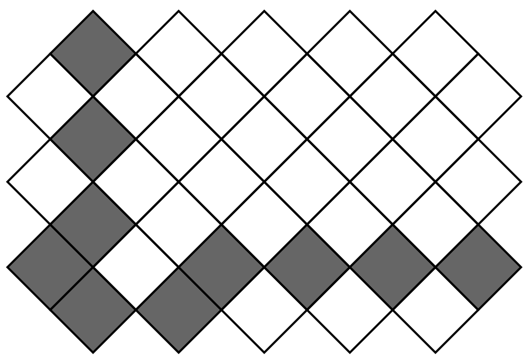
The subclass c_2 counts the number of polyforms that contain the "thick u" substructure. The substructure and an example polyform in c_1 can be found below.

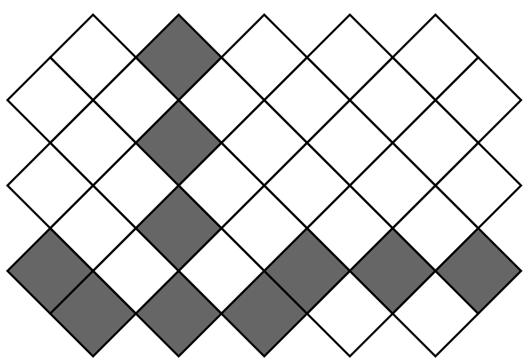
The subclass can be counted by the following sum $c_2(n,m) = \sum_{\substack{1\le i\le n\\ 1\le j\le m\\ i,j\neq(1,1)\\ i,j\neq(n,m)}} cc(i,j)$, and has the following generating function

$$C_2(x,y) = \left(\frac{1}{(1-x)(1-y)} - 1\right)\left(CC(x,y) - \frac{x}{1-x} - \frac{y}{1-y} - 2\right) + \frac{x}{1-x} + \frac{y}{1-y} + xy.$$

Finally, the c_3 subclass counts the number of polyforms that contain "u" and "t" polyforms that don't include an adjacent corner. An example polyform can be found below.



or



This class has the following generating function.

$$C_3(x,y) = \frac{xy}{(1-x)^2(1-y)^2} + \frac{x^4y^2}{(1-x)^3(1-y)} + \frac{x^2y^4}{(1-x)(1-y)^3}.$$

$$C_3(x,y) = \frac{xy}{(1-x)^2(1-y)^2} + \frac{x^4y^2}{(1-x)^3(1-y)} + \frac{x^2y^4}{(1-x)(1-y)^3} - \frac{xy}{(1-x)^2} - \frac{xy}{(1-y)^2} + xy.$$

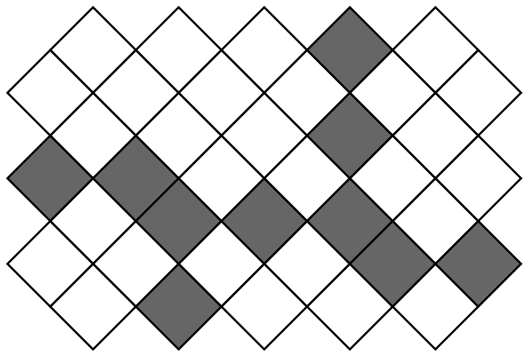
Note that we must add additional corrective terms the generating function of c, namely

$$C(x,y) = (C_0 + C_1 + C_2 + C_3)(x,y) - \frac{x^2y}{(1-x)^4} - \frac{xy^2}{(1-y)^4} + \frac{x^2+y}{1-x} + \frac{x+y^2}{1-y} - xy.$$

All polyforms not counted by cc or c are counted in i. i can also be split into subclasses, namely $i_0, i_1, i_{1,1}, r_1, i_2, i_3, i_4, i_5$, as well as the corrective term r_3 so that

$$i(n,m) = (2i_0 + 4i_1 + 4i_{1,1} + r_1 + 2i_2 + 2i_3 - r_3 + 4i_4 + i_5)(n,m).$$

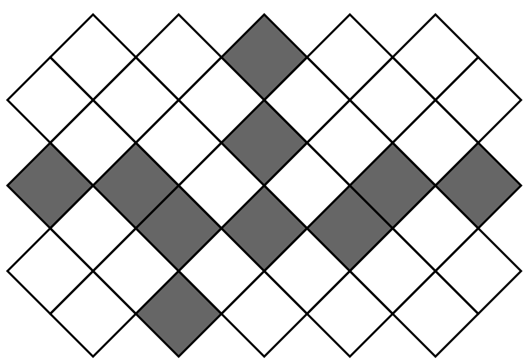
The subclass i_0 counts the number of polyforms of the following type



$$i_0(n,m) = 2(n-1)(m-1) + \sum_{i=0}^{n-2}\sum_{j=0}^{m-2}(i+j)$$

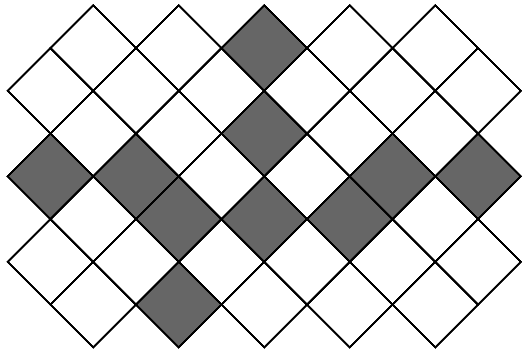
$$I_0(x,y) = \sum_{n,m\geq 0} i_0(n,m)x^ny^m = \frac{2x^2y^2}{(1-x)^2(1-y)^2} + \frac{x^3y^2}{(1-x)^3(1-y)^2} + \frac{x^2y^3}{(1-x)^2(1-y)^3}$$

The subclass i_1 counts the number of polyforms of the following type



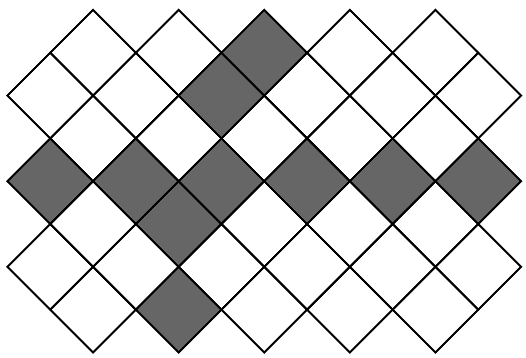
$$I_1(x,y) = \sum_{n,m\geq 0} i_1(n,m)x^ny^m = C_1(x,y)\left(\frac{1}{(1-x)(1-y)} - 1\right)$$

The subclass $i_{1,1}$ counts the number of polyforms of the following type



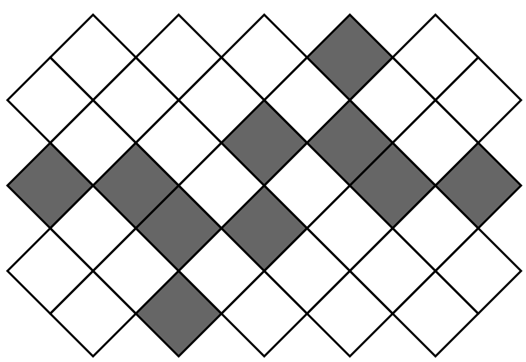
$$I_{1,1}(x,y) = \sum_{n,m\geq 0} i_{1,1}(n,m)x^n y^m = \frac{2x^2y^3}{(1-x)^2(1-y)^3} + \frac{2x^3y^2}{(1-x)^3(1-y)^2} + \frac{2x^2y^4}{(1-x)^2(1-y)^4} + \frac{2x^4y^2}{(1-x)^4(1-y)^2}$$

The corrective term r_1 counts all the polyforms in i_1 that are double counted when reflecting along both axes. An example polyform is below.



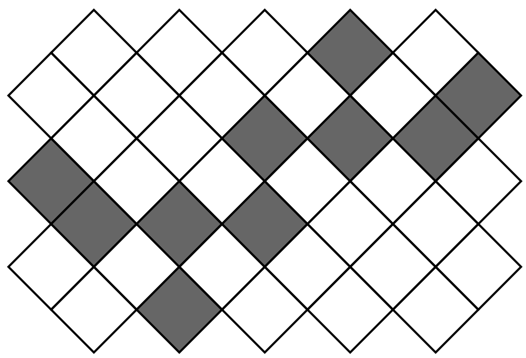
$$R_1(x,y) = \sum_{n,m\geq 0} r_1(n,m)x^n y^m = \frac{2x^2y^4}{(1-x)^4(1-y)} + \frac{2x^4y^2}{(1-x)(1-y)^4}$$

The subclass i_2 counts the number of polyforms of the following type



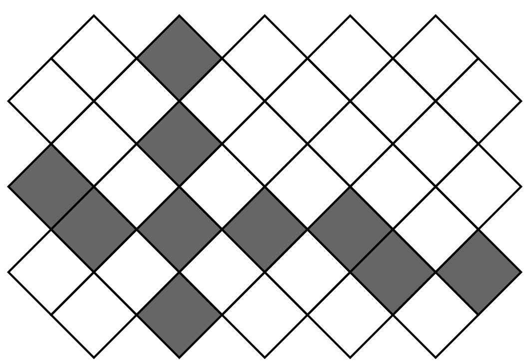
$$
\begin{aligned}
I_2(x,y) = \sum_{n,m\geq 0} i_2(n,m)x^n y^m \;&=\; C_2(x,y)\left(\frac{1}{(1-x)(1-y)} - 1\right) - \frac{x^3y}{(1-x)^5} + \frac{x^2y}{(1-x)^2} - \frac{x^2y}{(1-x)} \\
&- \frac{xy^3}{(1-y)^5} + \frac{xy^2}{(1-y)^2} - \frac{xy^2}{(1-y)} - \frac{2x^2y^2}{(1-x)^2(1-y)^2} + \frac{2}{(1-x)(1-y)} \\
&- \frac{2xy+2}{1-x} - \frac{2xy+2}{1-y} + 2xy + 2
\end{aligned}
$$

The subclass i_3 counts the number of polyforms of the following type



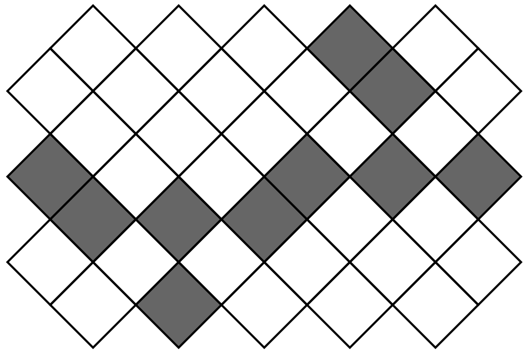
$$
I_3(x,y) = \sum_{n,m\geq 0} i_3(n,m)x^n y^m \;=\; F(x,y)\left(\frac{4x^2y^3}{(1-x)^2(1-y)^4} + \frac{4x^3y^2}{(1-x)^4(1-y)^2}\right)
$$

The corrective term r_3 counts the number of polyforms in i_3 that are double-counted when reflecting along bothe axes. An example polyform is below.



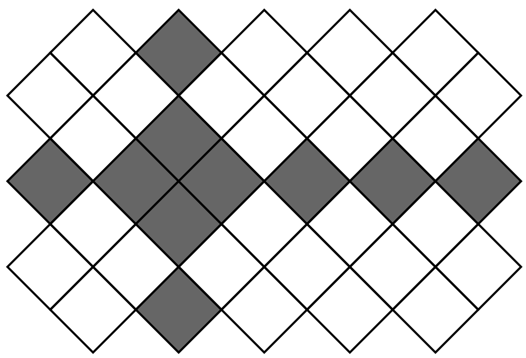
$$
R_3(x,y) = \sum_{n,m\geq 0} r_3(n,m)x^n y^m = \left(\frac{4x^2y^3}{(1-x)^2(1-y)^4} + \frac{4x^3y^2}{(1-x)^4(1-y)^2}\right)
$$

The subclass i_4 counts the number of polyforms of the following type



$$I_4(x,y) = \sum_{n,m\geq 0} i_4(n,m)x^n y^m \quad = \quad G(x,y)\left(\frac{4x^3y^3}{(1-x)^3(1-y)^3}\right)$$

The subclass i_5 counts the number of polyforms of the following type



$$I_5(x,y) = \sum_{n,m\geq 0} i_5(n,m)x^n y^m = \frac{xy}{(1-x)^2(1-y)^2} - \frac{4x^2y^2}{(1-x)(1-y)} - \frac{xy}{(1-x)^2} - \frac{xy}{(1-y)^2} + xy$$

which gives

$$I(x,y) = (2I_0 + 4I_1 + 4I_{1,1} + R_1 + 2I_2 + 2I_3 - R_3 + 4I_4 + I_5)(x,y).$$

Introducing the following corrective term

$$T_r(x,y) = \frac{6x^2y^2}{(1-x)^2(1-y)} + \frac{6x^2y^2}{(1-x)(1-y)^2}$$

allows us to write a full expression for the generating function, namely

$$\begin{aligned}\sum_{n,m\geq 0} \rho(A^{S^*}_{n,m})x^n y^m \quad &= \quad (2CC + 4C + I - T_r)(x,y) + 2 + xy - \frac{4x}{(1-x)} - \frac{4y}{(1-y)} - \frac{2xy}{(1-x)^3} - \frac{2xy}{(1-y)^3} \\ &+ \quad \frac{(x+3x^2-x^3-x^4)y}{(1-x)^5} + \frac{(y+3y^2-y^3-y^4)x}{(1-y)^5}.\end{aligned}$$

This gives the result.

$\square$