Minimal Inscribed Polyforms Summary

Jack Hanke

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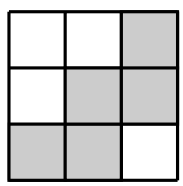
This document serves as a summary of known enumerations of minimal inscribed polyforms. The rows in each entry correspond with the notation for the family followed by an example polyform. Unless otherwise stated, the size of the example is $n = 3$ or $n, m = 3, 3$. Following this are the first couple terms of the sequence starting at $n = 1$ and, if available, the OEIS index. If the sequence has two parameters, the first couple terms are the terms of the main diagonal $n = m$. Finally, the generating function, the ρ function and its domain, and finally any additional notes complete the entry.

$\square^S_{n,m}$



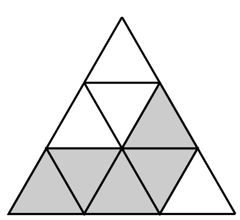
$1, 4, 25, 120, 497, 1924, 7265, \ldots (A334551)$

$$\sum_{n,m\geq 1} \rho(\square^S_{n,m})x^n y^m = \left(1 + \frac{xy}{(1-x)(1-y)}\right)^2 \frac{2xy}{1-x-y} - \frac{xy}{(1-x)^2(1-y)^2} + \frac{x^2y}{(1-x)^2} + \frac{xy^2}{(1-y)^2}$$

$$\rho(\square^S_{n,m}) = 8\binom{n+m-2}{n-1} - 3nm + 2n + 2m - 8 \text{ for } n, m \geq 2$$

Found by Goupil, Cloutier, and Noubound.

$\triangle^T_n$

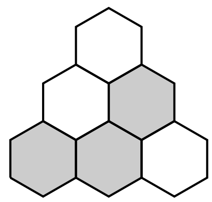


$$1, 3, 12, 44, 144, 432, 1216, \ldots (A356888)$$

$$\sum_{n\geq 1} \rho(\triangle_n^T) x^n = \frac{x - 3x^2 + 6x^3}{(1-2x)^3}$$

$$\rho(\triangle_n^T) = ((n-1)^2 + 2)2^{n-2} \text{ for } n \geq 1$$

$\triangle_n^H$

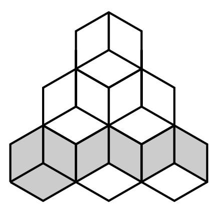


$$1, 3, 10, 32, 96, 272, 736, \ldots (A104270)$$

$$\sum_{n\geq 1} \rho(\triangle_n^H) x^n = \frac{x - 3x^2 + 4x^3}{(1-2x)^3}$$

$$\rho(\triangle_n^H) = \left(\binom{n}{2} + 2\right) 2^{n-2} \text{ for } n \geq 1$$

$\triangle_n^R$

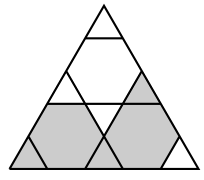


$$1, 6, 24, 80, 240, 672, 1792, \ldots (A001788)$$

$$\sum_{n\geq 1} \rho(\triangle_n^R) x^n = \frac{x}{(1-2x)^3}$$

$$\rho(\triangle_n^R) = n(n+1)2^{n-2} \text{ for } n \geq 1$$

$\triangle_n^K$



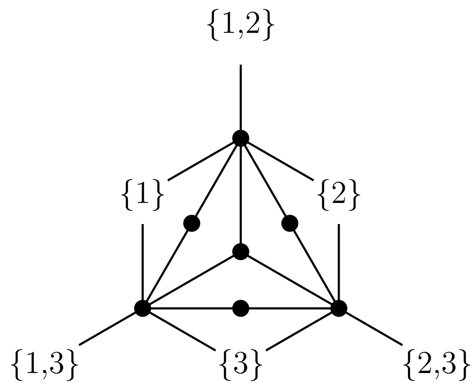
$$1, 3, 21, 125, 693, 3669, 18733, \ldots (A356889)$$

$$\sum_{n \geq 1} \rho(\triangle_n^K) x^n = \frac{x - 10x^2 + 42x^3 - 80x^4 + 56x^5}{(1-x)(1-4x)^3}$$

$$\rho(\triangle_n^K) = \left(n^2 + 3n + \frac{10}{3}\right) 4^{n-3} - \frac{1}{3} \text{ for } n \geq 2$$

The above results can be extended by adding $a-2$ additional edges in a symmetric manner, shown below in the labelled dual graph.

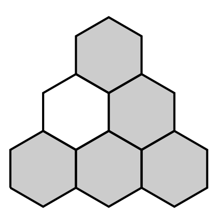
$\triangle_n^{K,a}$



$$\rho(\triangle_n^{K,a}) = \left(n^2 + \frac{(6a^2 - 5a - 5)n}{a+1} + \frac{6(a^4 - 2a^3 + 2a + 1)}{(a+1)^2}\right) 2^{n-2} a^{n-4} - \frac{3(a-1)^n}{(a+1)^2} \text{ for } n \geq 2$$

Interestingly, though initially defined for $a \geq 2$, $\rho(\triangle_n^{K,1}) = ((n-1)^2 + 2)2^{n-2} = \rho(\triangle_n^T)$. So in the case of minimal inscribed polyforms $\triangle_n^T$ can be seen as the $a = 1$ case of $\triangle_n^{K,a}$.

$\triangle_n^{H*}$

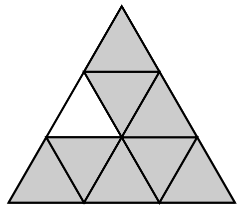


$$1, 1, 3, 11, 41, 153, 573, \ldots (A281593)$$

$$\sum_{n \geq 1} \rho(\triangle_n^{H*}) x^n = \frac{x - 2x^2}{(1-x)\sqrt{1-4x}}$$

$$\rho(\triangle_n^{H*}) = \binom{2(n-1)}{n-1} - \sum_{k=0}^{n-2} \binom{2k}{k} \text{ for } n \geq 1$$

$\triangle_n^{T*}$



$$1, 3, 9, 29, 99, 351, 1275 \ldots (A006134)$$

$$\sum_{n\geq 1} \rho(\triangle_n^{T*}) x^n = \frac{x}{(1-x)\sqrt{1-4x}}$$

$$\rho(\triangle_n^{T*}) = \text{ for } n \geq 1$$

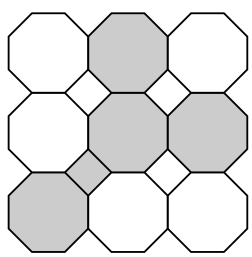
Both $\rho(\triangle_n^{T*})$ and $\rho(\triangle_n^{H*})$ rely on the following identity.

Theorem 1.

$$\sum_{k_1+k_2+k_3=n} \binom{k_1+k_2}{k_1}\binom{k_2+k_3}{k_2}\binom{k_3+k_1}{k_3} = \sum_{k=0}^{n}\binom{2k}{k}$$

Proof. TODO □

$R_{n,m}^{\theta}$

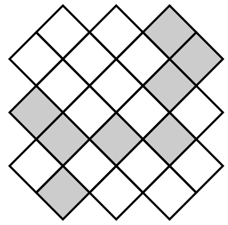


$$1, 6, 43, 256, 1201, 7510, 40363, \ldots$$

$$\sum_{n,m\geq 1} \rho(R_{n,m}^{\theta}) x^n y^m = \left(1 + \frac{xy}{(1-x)(1-y)}\right)^2 \frac{2xy}{1-x-y-xy} - \frac{xy}{(1-x)^2(1-y)^2} + \frac{x^2y}{(1-x)^2} + \frac{xy^2}{(1-y)^2}$$

$$\rho(R_{n,m}^{\theta}) = 2D(n-1, m-1) - nm + 2\sum_{i=0}^{n-2}\sum_{j=0}^{m-2} D(i,j)(2 + (n-2-i)(m-2-j)) \text{ for } n, m \geq 2$$

$A_{n,m}^{S^*}$



$$1, 68, 1113, 11616, 104097, \ldots$$

$$\begin{aligned}
\sum_{n,m\geq 0} \rho(A^{S^*}_{n,m}) x^n y^m &= \\
& \quad xy(2\,x^5y - 11\,x^4y^2 + 9\,x^3y^3 - 11\,x^2y^4 + 2\,xy^5 \\
&- \quad 10\,x^4y + 16\,x^3y^2 + 16\,x^2y^3 - 10\,xy^4 \\
&+ \quad x^4 + 15\,x^3y - 21\,x^2y^2 + 15\,xy^3 + y^4 - 8\,x^2y - 8\,xy^2 \\
&- \quad 4\,x^2 + 5\,xy - 4\,y^2 + 2\,x + 2\,y + 1) \\
&/ \quad \left(1 - 2\,x - 2\,y + x^2 + xy + y^2\right)(1-x)^4(1-y)^4
\end{aligned}$$