Enumeration of Mosaic Systems

Jack Hanke
Northwestern University

Michael Maltenfort
Northwestern University

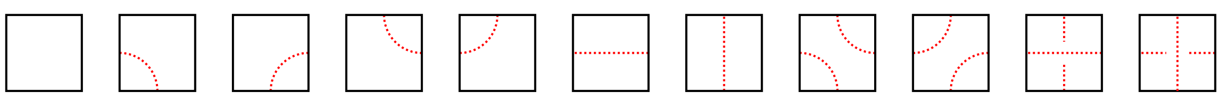
Richard Schank

Abstract

Lomonaco and Kauffman introduced a mosaic system to model quantum knots composed of an $m \times n$ grid of 11 possible tiles. Oh and colleagues introduced a state matrix recursion to exactly enumerate mosaics that have the property of being suitably connected. Hong and Oh later calculated upper and lower bounds on a modified mosaic system with 7 tiles. We introduce a general method for enumerating these mosaic systems. We also introduce a variant to mosaics we call messy mosaics that can also enumerated using our method.

1 Introduction

Lomonaco and Kauffman [3] introduced a model for quantum knots in which an $m \times n$ matrix is constructed using 11 distinct symbols called *tiles*. These tiles, diagrammed below, are composed of unit squares with dotted lines connecting 2 or 4 sides at their midpoint.

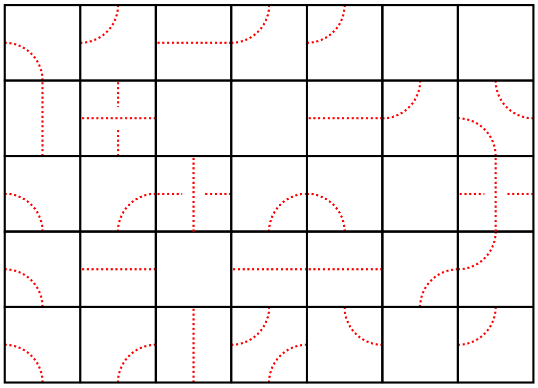


T_0 T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10}

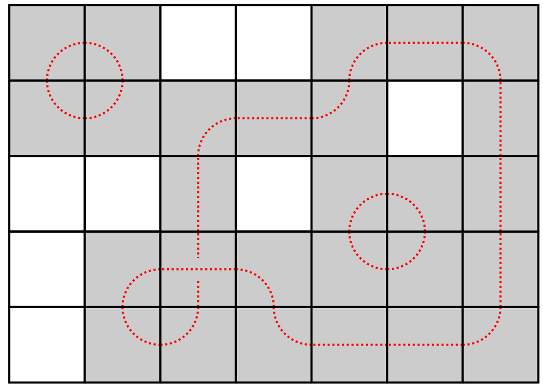
We denote this set of tiles as $\mathbb{T} = \{T_0, \ldots, T_{10}\}$. A *mosaic* of size (m, n) is an $m \times n$ matrix made up of elements in $\mathbb{T}$, and a *mosaic system* is a collection of mosaics with some property. Figure 1a shows an example mosaic of size $(5, 7)$.

Consider an edge shared between two tiles in Figure 1a. The edge has either 0, 1, or 2 dotted lines drawn from its midpoint. Also note that the edges of the tiles on the boundary of the matrix are not shared by another tile. Therefore these edges only have 0 or 1 dotted lines drawn from their midpoint. We define a *knot mosaic* to be a mosaic of tiles from $\mathbb{T}$ that has all edges having 0 or 2 dotted lines drawn from their midpoint.

Lomonaco and Kauffman [3] call these knot mosaics because, other than the mosaic consisting of all T_1 tiles, the dotted lines form *knots*. Figure 1b shows a knot mosaic of size $(5, 7)$ that contains 3 knots, with the tiles that make up the knot hilighted in gray. Note that a mosaic can contain knots isomorphic to the unknot, as well as knots that encompass other knots.



(a) A mosaic



(b) A knot mosaic, with knots hilighted

Figure 1: Examples of mosaics of size $(5,7)$ made of tiles in $\mathbb{T}$

Let $k_{m,n}$ be the number of knot mosaics of size (m,n). First notice that if either m or n is 1, one can only construct a knot mosaic wtih T_0, so $k_{m,1} = k_{1,n} = 1$. Oh et al. [9] showed the following for $m, n \geq 2$.

Theorem 1 ([9])**.** *The number of knot mosaics of size (m,n) for $m, n \geq 2$ is $k_{m,n} = 2\,\|(X_{m-2} + O_{m-2})^{n-2}\|$, where X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined as*

$$X_{k+1} = \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} = \begin{bmatrix} O_k & X_k \\ X_k & 4O_k \end{bmatrix},$$

for $k = 0, 1, \ldots, m-3$, and $\|N\|$ denotes the sum of elements of matrix N.

Oh and colleagues utilize the state matrix recursion in Theorem 1 to bound the growth rate $\delta = \lim_{n\ to\infty} k_{n,n}^{\frac{1}{n^2}}$ [6, 8, 1], and Oh further adapts the method to solve problems in monomer and dimer tilings [5, 7].

Finally, Hong and Oh [2] study the mosaic system with the tile set $\mathbb{T}^* = \{T_0, \ldots, T_7\}$. This tile set constructs shapes we call *polygons*[1]. If we let $p_{m,n}$ be the number of polygon mosaics of size (m,n), Hong and Oh showed the following results[2].

Theorem 2 ([2])**.** *The number of polygon mosaics of size (m,n) $p_{m,n}$ for $m, n \geq 2$ has*

$$2^{m+n-3}\left(\frac{17}{10}\right)^{(m-2)(n-2)} \leq p_{m,n} \leq 2^{m+n-3}\left(\frac{31}{16}\right)^{(m-2)(n-2)}.$$

The array $p_{n,m}$ is A181245 on the OEIS [4, OEIS].

We first re-prove the enumeration of $k_{n,m}$ from Theorem 1 using our general method for enumerating mosaic systems, and show it immediately gives an an analagous result for $p_{n,m}$. We state the enumeration of $p_{n,m}$ for $m, n \geq 2$ in Theorem 4. First we define

[1]Polygons are more commonly called ”self-avoiding polygons” in the literature to emphasize their relationship with self-avoiding walks.

[2]The authors did not consider the mosaic containing all T_0 tiles a polygon mosaic, and so define $p_{m,n}$ as one less than what we define.

$A(m) \in \mathbb{Z}^{2^{m-1} \times 2^{m-1}}$ for integers $m \geq 2$, which we use to enumerate $p_{m,n}$. To be clear, throughtout the paper we index the rows and columns of matrices starting at 0.

Definition 3. Let $A(2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We recursively define $A(k+1)$ given $A(k)$. Begin by writing $A(k) = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}$, where the block matrices $A_{i,j}$ are square block matrices of size $2^{k-2} \times 2^{k-2}$. We then define

$$A(k+1) = \begin{bmatrix} A_{0,0} & A_{0,0} & A_{0,1} & A_{0,1} \\ A_{0,0} & A_{0,0} & 0A_{0,1} & A_{0,1} \\ A_{1,0} & 0A_{1,0} & A_{1,1} & A_{1,1} \\ A_{1,0} & A_{1,0} & A_{1,1} & A_{1,1} \end{bmatrix},$$

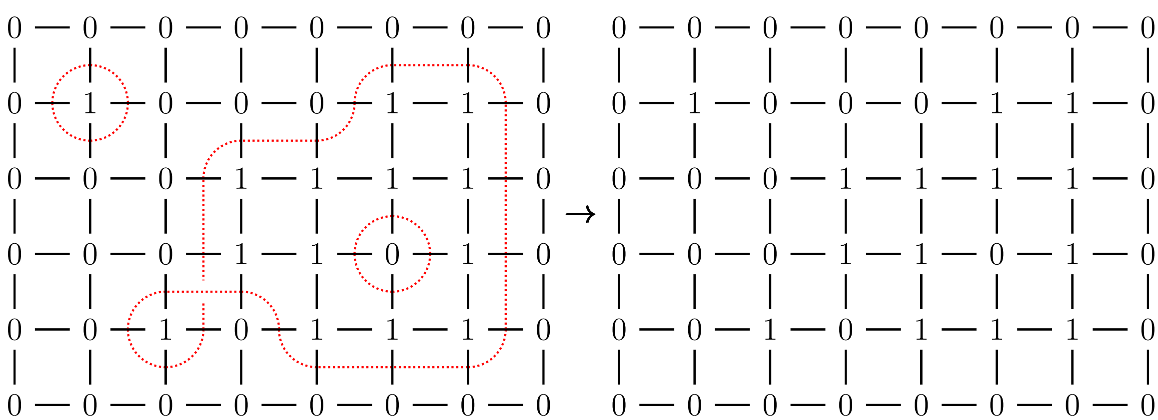
Construct $A(m)$ by starting with $k = 2$ and recursing until $k = m$.

Theorem 4. *The number of polygon mosaics $p_{m,n}$ is the $(0,0)$ entry of $A(m)^n$.*

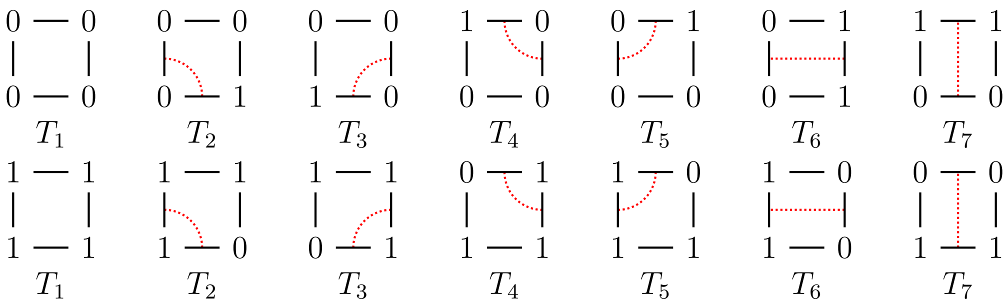
We then introduce messy knot mosaics—a variant of knot mosaics— in Section 3 and enumerate them.

2 Proof of Theorem 4

Proof. For a given mosaic, label the vertices of the tiles as follows. If the vertex is surrounded by an even number of polygons, label it 0. If the vertex is surrounded by an odd number of polygons, label it 1. To make a *vertex labeling* we also remove the dotted lines from all tiles in the mosaic. Using the mosaic from Figure 1b, we show both the labeling of the vertices, and then the removal of the dotted lines.



The critical point is this: even with the dotted lines removed, the vertex labeling uniquely identifies the polygon mosaic. This is true even if there are polygons surrounding other polygons. This is because the vertex labeling of an individual cell, which we will call a *cell labeling*, uniquely corresponds with a cell T_i. This is shown below.



We can then enumerate $p_{m,n}$ by enumerating the number of vertex labelings that correspond to a polygon mosaic.

Consider the collection of vertex labelings for a $m \times n$ grid of cells. To correspond with a polygon mosaic, each boundary vertex is necessarily labeled 0, and each interior vertex is labeled 0 or 1. Additionally, of these $2^{(m-1)(n-1)}$ labelings, the labelings that correspond with valid polygon mosaics are ones that do not contain the following two cell labelings.

$$\begin{array}{ccc} 1 & — & 0 \\ | & & | \\ 0 & — & 1 \end{array} \qquad \begin{array}{ccc} 0 & — & 1 \\ | & & | \\ 1 & — & 0 \end{array}$$

These two cell labelings aren't associated with any of the cells $T_1, \ldots, T_7$, and so cannot correspond with a polygon mosaic. We seek a recursive solution to enumerate the number of vertex labelings for an $m \times n$ grid with these conditions: the labeling has a boundary of all 0's and does not contain the above cell labelings.

Our solution accomplishes this by building $m \times n$ vertex labelings for fixed m using vertex labelings of $m \times 1$ columns of cells. These columns have vertex labelings such that the top and bottom edge all labeled 0. One of these columns for $m = 3$ is below.

$$\begin{array}{ccc} 0 & — & 0 \\ | & & | \\ 0 & — & 0 \\ | & & | \\ 0 & — & 1 \\ | & & | \\ 0 & — & 0 \end{array}$$

It is useful to define an index for vertex labelings of these $m \times 1$ columns. To do this, consider reading a column of vertex labelings from bottom to top, ignoring the first and last 0's. If we interpret these sequences for the left and right column as binary numbers b_{left} and b_{right}, the index in base ten is the pair (b_{left}, b_{right}). For example, for the above $m = 3$ column, we have sequences 00 for the left column and 10 for the right column, so the index is $(0, 2)$.

Next consider a column with index $(0, b_1)$ for some $b_1 \in [0, 2^{m-1} - 1]$. For reasons we will see later, let's call this the *starting column*. Now consider appending a column with index (b_1, b_2) for some $b_2 \in [0, 2^{m-1} - 1]$. For example, below is a diagram for appending our starting column $(0, 2)$ with column $(2, 3)$.

$$\begin{array}{ccc} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{array} \quad + \quad \begin{array}{ccc} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 1 & - & 1 \\ | & & | \\ 0 & - & 0 \end{array} \quad \rightarrow \quad \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 1 \\ | & & | & & | \\ 0 & - & 1 & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

Notice that in the above example we have created a vertex labeling for an $m \times 2$ grid of cells, in which the left-most vertex column labels are all 0 (left index of 0). Also note that we did not create either illegal cell labelings with column $(2, 3)$. However, we could append the column $(2, 1)$, which would create an illegal cell labeling.

Finally, if we further appended a column with label $(b_2, 0)$, we would have created a vertex labeling with a boundary of all 0's. For this reason, let's call a column with index $(b, 0)$ for $b \in [0, 2^{m-1} - 1]$ an *ending column*. This vertex labeling would correspond with 1 polygon mosaic if the middle column had index $(2, 3)$, but not if the middle column had index $(2, 1)$.

This motivates the creation of a matrix $A(m)$ to every column index (i, j), where $A(m)_{i,j} = 1$ if the column labeling corresponds with a legal polygon mosaic, and 0 otherwise. For example, the $A(3)$ matrix corresponds with the following labeled columns.

$$\begin{array}{cccc}
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} \\
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} \\
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 1\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} \\
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix} &
\begin{matrix}0 & \text{—} & 0\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 1 & \text{—} & 1\\ | & & |\\ 0 & \text{—} & 0\end{matrix}
\end{array}
\quad \rightarrow \quad
\underset{A(3)}{\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}}$$

$A(m)$ has the property that the 0-th row represets all starting columns, and the 0-th column represents all ending columns. Even more importantly, notice that $A(m)^2_{i,j}$ represents the number of $m \times 2$ grids with left-most index i and right-most index j that correspond with a legal polygon mosaic. In general, $(A(m)^n)_{i,j}$ represents this quantity for an $m \times n$ grid of cells, and so if we know $A(m)$ for some m, then $(A(m)^n)_{0,0} = p_{m,n}$.

The final component of the proof is constructing $A(m)$ for any m. Begin by calculating $A(2)$ by identifying the number of legal polygon mosaics that correspond with each vertex coloring index $\{(0,0),(0,1),(1,0),(1,1)\}$, like so.

$$
\begin{matrix}
\begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \\ | & & | \\ 0 & - & 0 \end{matrix} & \begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix} \\
\\
\begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 0 \\ | & & | \\ 0 & - & 0 \end{matrix} & \begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix}
\end{matrix}
\quad \rightarrow \quad
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

Next consider an arbitrary value $A(k)_{i,j}$ for any $k \geq 2$. This value is 1 if the $k \times 1$ column with index (i, j) can be part of a polygon mosaic, and 0 otherwise. We can determine that specific values of $A(k + 1)$ are multiples of $A(k)_{i,j}$ by considering the following operation on an arbitrary column with index (i, j). Copy the column four times and replace the top two 0's of each column with the bottom row of labels from one of the four cell labelings below.

$$
\begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \end{matrix} \qquad
\begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \end{matrix} \qquad
\begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 0 \end{matrix} \qquad
\begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 1 \end{matrix}
$$

For example, for the $m = 2$ column with index $(0, 1)$, this operation looks like the following.

$$
\begin{matrix} \begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix} \\ \\ A(2)_{0,1} \end{matrix}
\quad \rightarrow \quad
\begin{matrix}
\begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix} & \begin{matrix} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix} \\
\\
\begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 0 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix} & \begin{matrix} 0 & - & 0 \\ | & & | \\ 1 & - & 1 \\ | & & | \\ 0 & - & 1 \\ | & & | \\ 0 & - & 0 \end{matrix}
\end{matrix}
\quad \rightarrow \quad
\begin{bmatrix} A(3)_{0,2} & A(3)_{0,3} \\ A(3)_{1,2} & A(3)_{1,3} \end{bmatrix}
$$

This operation results in 4 new columns that are represented in $A(k+1)$. In our example, specifically we get the following.

$$
\begin{bmatrix} A(3)_{0,2} & A(3)_{0,3} \\ A(3)_{1,2} & A(3)_{1,3} \end{bmatrix} = \begin{bmatrix} 1A(2)_{i,j} & 1A(2)_{i,j} \\ 0A(2)_{i,j} & 1A(2)_{i,j} \end{bmatrix},
$$

Critically, this transformation *only* changes the identity of the top two tiles. This implies that the same value coefficients computed by comparing $A(2)$ and $A(3)$ can be used for any $m \times 1$ column, as long as both column indices (i, j) are congruent mod 2. Furthermore, if one writes $A(k)$ as the block matrix

$$A(k) = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix},$$

where $A_{\hat{i},\hat{j}} \in \mathbb{R}^{2^{k-2} \times 2^{k-2}}$, then all column's represented in $A_{\hat{i},\hat{j}}$ have indices $(i, j) \equiv (\hat{i}, \hat{j}) \bmod 2$. This allows us to write that in general, if $A(k) = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}$, then

$$\begin{bmatrix} V_{0,0}A_{0,0} & V_{0,1}A_{0,0} & V_{0,2}A_{0,1} & V_{0,3}A_{0,1} \\ V_{1,0}A_{0,0} & V_{1,1}A_{0,0} & V_{1,2}A_{0,1} & V_{1,3}A_{0,1} \\ V_{2,0}A_{1,0} & V_{2,1}A_{1,0} & V_{2,2}A_{1,1} & V_{2,3}A_{1,1} \\ V_{3,0}A_{1,0} & V_{3,1}A_{1,0} & V_{3,2}A_{1,1} & V_{3,3}A_{1,1} \end{bmatrix}. \tag{1}$$

where $V \in \mathbb{R}^{4\times 4}$ can be found after directly computing $A(2)$ and $A(3)$, then solving the following equation.

$$\begin{bmatrix} A(3)_{0,0} & A(3)_{0,1} & A(3)_{0,2} & A(3)_{0,3} \\ A(3)_{1,0} & A(3)_{1,1} & A(3)_{1,2} & A(3)_{1,3} \\ A(3)_{2,0} & A(3)_{2,1} & A(3)_{2,2} & A(3)_{2,3} \\ A(3)_{3,0} & A(3)_{3,1} & A(3)_{3,2} & A(3)_{3,3} \end{bmatrix} = \begin{bmatrix} V_{0,0}A(2)_{0,0} & V_{0,1}A(2)_{0,0} & V_{0,2}A(2)_{0,1} & V_{0,3}A(2)_{0,1} \\ V_{1,0}A(2)_{0,0} & V_{1,1}A(2)_{0,0} & V_{1,2}A(2)_{0,1} & V_{1,3}A(2)_{0,1} \\ V_{2,0}A(2)_{1,0} & V_{2,1}A(2)_{1,0} & V_{2,2}A(2)_{1,1} & V_{2,3}A(2)_{1,1} \\ V_{3,0}A(2)_{1,0} & V_{3,1}A(2)_{1,0} & V_{3,2}A(2)_{1,1} & V_{3,3}A(2)_{1,1} \end{bmatrix}$$

This is solved by

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which completes the proof.

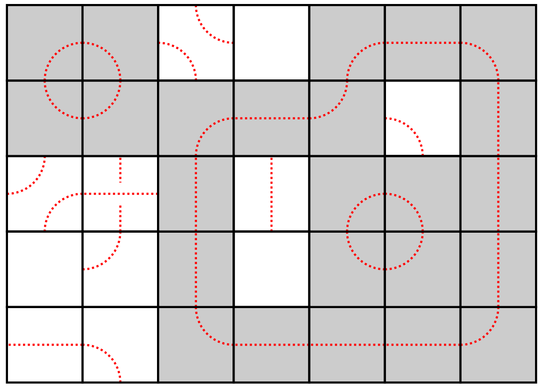
□

The method detailed in Theorem 4 generalizes to other tile sets, which we tabularize in Section 5 without proof. Interestingly, the method not only generalizes to other tile sets, but can also be augmented to enumerate the more complicated "messy" polygon mosaics.

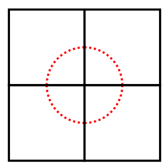
3 Messy Polygon Mosaics

A *messy polygon mosaic* is a mosaic that contains at least one polygon, with no restriction on other shared edges.

Example 3.1. Below is a messy polygon mosaic of size $(5, 7)$ that contains 1 polygon, with the squares that make it up hilighted in gray.



It turns out that it is simpler to enumerate the number of mosaics that *do not* contain a polygon. Therefore, let $t_{m,n}$ be the number of mosaics that do not contain a polygon. Also from the fact that the smallest polygon is



we have that $t_{n,1} = 7^n$, and $t_{2,2} = 7^4 - 1$. We can use similar techniques to Theorem 4 to enumerate messy polygon mosaics. We define $A(m) \in \mathbb{Z}^{2^{m-1} \times 2^{m-1}}$, which we use to enumerate $t_{m,n}$.

Definition 5. First define $A(2) = \begin{bmatrix} 7^2 & 1 \\ -1 & 1 \end{bmatrix}$. We recursively define $A(k+1) \in \mathbb{Z}^{2^{m-1} \times 2^{m-1}}$ given $A(k)$. Begin by writing $A(k) = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}$, where the block matrices $A_{i,j}$ are square block matrices of size $2^{k-2} \times 2^{k-2}$. We then have

$$A(k+1) = \begin{bmatrix} 7A_{0,0} & \frac{1}{7}A_{0,0} & 7A_{0,1} & A_{0,1} \\ -\frac{1}{7}A_{0,0} & A_{0,0} & 0A_{0,1} & A_{0,1} \\ 7A_{1,0} & 0A_{1,0} & 7A_{1,1} & A_{1,1} \\ A_{1,0} & -A_{1,0} & A_{1,1} & 7A_{1,1} \end{bmatrix}.$$

Construct $A(m)$ by starting with $k = 2$ and recursing until $k = m$.

Theorem 6. *The number of mosaics that do not contain a polygon is the* $(0,0)$*-th entry of* $A(m)^n$.

4 Proof of Theorem 6

Proof. We consider the same vertex labeling in the proof of Theorem 4, namely labeling a vertex by the number of polygons that contain it mod 2. Though avoided in the previous proof for clarity, it is important to think of each individual cell labeling as having its own weight, which corresponds with how many distinct tiles can map to the cell labeling.

$$
\begin{array}{cccccccc}
\begin{array}{c}0 - 0\\ | \quad\ | \\ 0 - 0\\ w_1\end{array} &
\begin{array}{c}0 - 0\\ | \quad\ | \\ 0 - 1\\ w_2\end{array} &
\begin{array}{c}0 - 0\\ | \quad\ | \\ 1 - 0\\ w_3\end{array} &
\begin{array}{c}0 - 0\\ | \quad\ | \\ 1 - 1\\ w_4\end{array} &
\begin{array}{c}0 - 1\\ | \quad\ | \\ 0 - 0\\ w_5\end{array} &
\begin{array}{c}0 - 1\\ | \quad\ | \\ 0 - 1\\ w_6\end{array} &
\begin{array}{c}0 - 1\\ | \quad\ | \\ 1 - 0\\ w_7\end{array} &
\begin{array}{c}0 - 1\\ | \quad\ | \\ 1 - 1\\ w_8\end{array} \\
\begin{array}{c}1 - 0\\ | \quad\ | \\ 0 - 0\\ w_9\end{array} &
\begin{array}{c}1 - 0\\ | \quad\ | \\ 0 - 1\\ w_{10}\end{array} &
\begin{array}{c}1 - 0\\ | \quad\ | \\ 1 - 0\\ w_{11}\end{array} &
\begin{array}{c}1 - 0\\ | \quad\ | \\ 1 - 1\\ w_{12}\end{array} &
\begin{array}{c}1 - 1\\ | \quad\ | \\ 0 - 0\\ w_{13}\end{array} &
\begin{array}{c}1 - 1\\ | \quad\ | \\ 0 - 1\\ w_{14}\end{array} &
\begin{array}{c}1 - 1\\ | \quad\ | \\ 1 - 0\\ w_{15}\end{array} &
\begin{array}{c}1 - 1\\ | \quad\ | \\ 1 - 1\\ w_{16}\end{array}
\end{array}
$$

The proof of Theorem 4 can be seen as assigning $w_7 = w_{10} = 0$, and all other weights to 1. The usefulness of this view can be seen when considering the operation for creating the recursive definition for $A(k)$ in the previous proof. Previously, we defined the coefficient matrix V by computing $A(2)$ and $A(3)$ directly and comparing. Now with a weight assigned to individual cell labelings, we can define the values of $A(2)$ and V directly in terms of these weights.

$$
A(2) = \begin{bmatrix} w_1 w_1 & w_2 w_5 \\ w_3 w_9 & w_4 w_{13} \end{bmatrix}, V = \begin{bmatrix} \frac{w_1 w_1}{w_1} & \frac{w_2 w_5}{w_1} & \frac{w_1 w_2}{w_2} & \frac{w_2 w_6}{w_2} \\ \frac{w_3 w_9}{w_1} & \frac{w_4 w_{13}}{w_1} & \frac{w_3 w_{10}}{w_2} & \frac{w_4 w_{14}}{w_2} \\ \frac{w_1 w_3}{w_3} & \frac{w_2 w_7}{w_3} & \frac{w_1 w_4}{w_4} & \frac{w_2 w_8}{w_4} \\ \frac{w_3 w_{11}}{w_3} & \frac{w_4 w_{15}}{w_3} & \frac{w_3 w_{12}}{w_4} & \frac{w_4 w_{16}}{w_4} \end{bmatrix} = \begin{bmatrix} w_1 & \frac{w_2 w_5}{w_1} & w_1 & w_6 \\ \frac{w_3 w_9}{w_1} & \frac{w_4 w_{13}}{w_1} & \frac{w_3 w_{10}}{w_2} & \frac{w_4 w_{14}}{w_2} \\ w_1 & \frac{w_2 w_7}{w_3} & w_1 & \frac{w_2 w_8}{w_4} \\ w_{11} & \frac{w_4 w_{15}}{w_3} & \frac{w_3 w_{12}}{w_4} & w_{16} \end{bmatrix}.
$$

With this identity, we can enumerate $t_{m,n}$ once we have proper assignments for the 16 weights. As in the previous proof, we have $w_7 = w_{10} = 0$, as again these are impossible vertex labelings for our tile set.

Next consider the cell labelings for w_1 and w_{16}. When enumerating polygon mosaics (and their messy variant), these cell labelings do not contribute to the cells of a polygon. For polygon mosaics, only the T_1 tile are permitted to not contribute to the shape of a polygon. However, in messy polygon mosaics, all 7 tiles are permitted to not contribute to the shape of the polygon, so $w_1 = w_{16} = 7$.

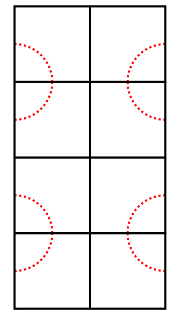
However, this means we now lose the uniqueness of the map from vertex labeling to messy polygon mosaics. For instance, the sub-grid vertex labelings below are now ambiguous as to whether or not they represent a polygon.

$$
\begin{array}{cc}
\begin{array}{c}0 - 0 - 0\\ | \quad\ | \quad\ | \\ 0 - 0 - 0\\ | \quad\ | \quad\ | \\ 0 - 0 - 0\end{array} &
\begin{array}{c}1 - 1 - 1\\ | \quad\ | \quad\ | \\ 1 - 1 - 1\\ | \quad\ | \quad\ | \\ 1 - 1 - 1\end{array}
\end{array}
$$

This ambiguity is explored in the following example.

Example 4.1. Consider the four vertex labelings below, along with the messy polygon mosaic on the right. Write the product of the weights of all cell labelings below each, assuming all weights not defined above are 1.

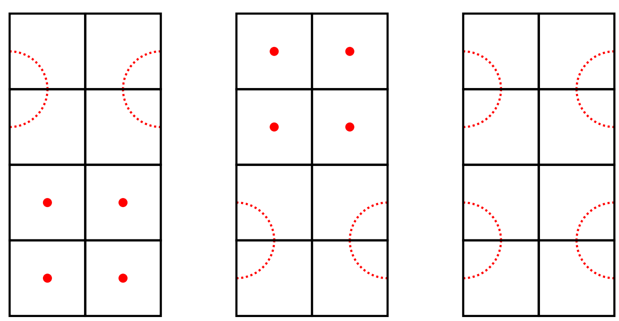
$$\begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$



$7^8 \qquad 7^4 \qquad 7^4 \qquad 7^0$

Notice that each vertex labeling could include the right-most messy polygon mosaic. In fact, the left-most vertex labeling contains all possible mosaics! If we were to add these weight products together, we would count the right messy polygon mosaic 4 times.

The double counting demonstrated in Example 4.1 motivates the following idea. If vertex labelings with an odd number of polygons are negative, then the addition of these weight products would incorporate the *inclusion-exclusion principle*, mitigating the double counting. In Example 4.1, the sum $7^8 - 7^4 - 7^4 + 7^0$ would then represent the number of mosaics that *do not* contain the following three classes of messy polygon mosaics, where cells that can be any tile are marked with a dot.



Therefore, the sum over the products for all vertex labelings, where the product is negative if the vertex labeling represents an odd number of polygons in the mosaic, would be the number of mosaics that *do not* include messy polygon mosaics.

We can accomplish this by finding a weight assignment such that the product over the cell labeling weights of *any* single polygon equals -1. It is not obvious that such an assignment can even be found!

Luckily such assignments exist. The proof of this fact can be found in the Appendix, and choosing an assignment gives us the following weight assignments.

$$
\begin{array}{cccccccc}
\begin{matrix}0 & — & 0\\ | & & |\\ 0 & — & 0\end{matrix} &
\begin{matrix}0 & — & 0\\ | & & |\\ 0 & — & 1\end{matrix} &
\begin{matrix}0 & — & 0\\ | & & |\\ 1 & — & 0\end{matrix} &
\begin{matrix}0 & — & 0\\ | & & |\\ 1 & — & 1\end{matrix} &
\begin{matrix}0 & — & 1\\ | & & |\\ 0 & — & 0\end{matrix} &
\begin{matrix}0 & — & 1\\ | & & |\\ 0 & — & 1\end{matrix} &
\begin{matrix}0 & — & 1\\ | & & |\\ 1 & — & 0\end{matrix} &
\begin{matrix}0 & — & 1\\ | & & |\\ 1 & — & 1\end{matrix} \\
w_1 = 7 & w_2 = 1 & w_3 = 1 & w_4 = 1 & w_5 = 1 & w_6 = 1 & w_7 = 0 & w_8 = 1 \\
\begin{matrix}1 & — & 0\\ | & & |\\ 0 & — & 0\end{matrix} &
\begin{matrix}1 & — & 0\\ | & & |\\ 0 & — & 1\end{matrix} &
\begin{matrix}1 & — & 0\\ | & & |\\ 1 & — & 0\end{matrix} &
\begin{matrix}1 & — & 0\\ | & & |\\ 1 & — & 1\end{matrix} &
\begin{matrix}1 & — & 1\\ | & & |\\ 0 & — & 0\end{matrix} &
\begin{matrix}1 & — & 1\\ | & & |\\ 0 & — & 1\end{matrix} &
\begin{matrix}1 & — & 1\\ | & & |\\ 1 & — & 0\end{matrix} &
\begin{matrix}1 & — & 1\\ | & & |\\ 1 & — & 1\end{matrix} \\
w_9 = -1 & w_{10} = 0 & w_{11} = 1 & w_{12} = 1 & w_{13} = 1 & w_{14} = 1 & w_{15} = -1 & w_{16} = 7
\end{array}
$$

This immediately gives us a way to construct an analagous definition for $A(k+1)$ given $A(k)$. Once we write $A(k) = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}$, we have

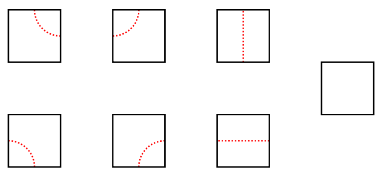
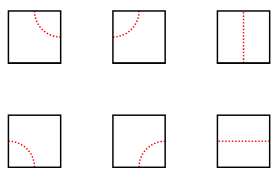
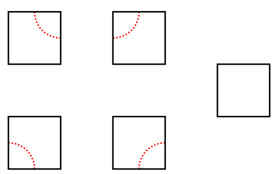
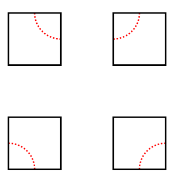
$$
A(2) = \begin{bmatrix} 7^2 & 1 \\ -1 & 1 \end{bmatrix}, V = \begin{bmatrix} 7 & \frac{1}{7} & 7 & 1 \\ -\frac{1}{7} & 1 & 0 & 1 \\ 7 & 0 & 7 & 1 \\ 1 & -1 & 1 & 7 \end{bmatrix}
$$

Subsituting V into Equation 1 gives the result.

$\square$

5 Summary of Results

As demonstrated in the proof of Theorem 6, the enumeration of both polygon mosaics and mosaics that do not contain a polygon share the same structure, and only differ by the identity of the matrices $A(2), V$. We summarize these matrices for various collections of tiles below.

Tile Set	Polygon Mosaics	Messy Polygon Mosaics
	$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$	$\begin{bmatrix}7^2 & 1\\ -1 & 1\end{bmatrix}, \begin{bmatrix}7 & \frac{1}{7} & 7 & 1\\ -\frac{1}{7} & 1 & 0 & 1\\ 7 & 0 & 7 & 1\\ 1 & -1 & 1 & 7\end{bmatrix}$
	TODO	$\begin{bmatrix}6^2 & 1\\ -1 & 1\end{bmatrix}, \begin{bmatrix}6 & \frac{1}{6} & 6 & 1\\ -\frac{1}{6} & 1 & 0 & 1\\ 6 & 0 & 6 & 1\\ 1 & -1 & 1 & 6\end{bmatrix}$
	$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$, TODO	TODO
	TODO	TODO

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6 Appendix

We demonstrate that weight assignments exist such that the product over the cell labeling weights of *all* single polygons equals -1. We do this by first asserting that the product of the weights associated with the smallest polygon multiply to -1, ie. $w_2 w_3 w_5 w_9 = -1$.

Lemma 7. *One can construct all larger polygons from the smallest polygon using a finite set of transformations S.*

Proof. TODO Something about chaging vertex values □

This is because one can find $w_1, \ldots, w_{16}$ so that the following two constraints hold:

Constraint 8. *The weights associated with the smallest polygon multiply to -1, ie. $w_2 w_3 w_5 w_9 = -1$.*

Constraint 9. *All transformations in S preserve the weight product of a changed polygon.*

Constraint 8 and Constraint 9 amount to a series of constraints on the values of w_i. Choosing a solution set from these constraints gives the following weights.

Flipping the parity of a single vertex in a vertex labeling changes the 4 surrounding cells. This creates a constraint on a subset of $w_1, \ldots, w_{16}$.

The flipping of parity of a single vertex results in 2 distinct types of constraints. Let a constraint of *Type 1* be a parity flip that does not change the number of polygons represented in the vertex labeling. For example, consider the following flip of the center vertex in the following sub vertex labeling.

$$\begin{array}{ccccc} 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 1 \\ | & & | & & | \\ 0 & — & 1 & — & 1 \end{array} \rightarrow \begin{array}{ccccc} 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 1 & — & 1 \\ | & & | & & | \\ 0 & — & 1 & — & 1 \end{array}$$

As this does not change the associated number of polygons in the larger vertex labeling, we want this to preserve the sign of the weight product. This gives the following associated constraint.

$$\text{sign}(w_1 w_2 w_5 w_9) = \text{sign}(w_2 w_4 w_6 w_{16}).$$

Now let a constraint of *Type 2* be a parity flip that does change the number of polygons. For example, consider flipping the center vertex of the following portion of a vertex labeling.

$$\begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & 0 & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \rightarrow \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & 1 & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

The above transformation corresponds with *either* two distinct polygons joining into one polygon *or* one polygon splitting into two distinct polygons. In either case, we want the sign of the product to switch. This corresponds with the following constraint.

$$\text{sign}(w_3 w_2 w_9 w_5) = -\text{sign}(w_4 w_4 w_{13} w_{13}).$$

All Type 1 constraints are as follows. For the following set of equations, assume the equals sign (=) means *only* equal in sign.

$$\begin{aligned}
w_1w_1w_2w_3 &= w_2w_3w_6w_{11} & w_1w_1w_2w_4 &= w_2w_3w_6w_{12} & w_1w_1w_4w_3 &= w_2w_3w_8w_{11} \\
w_1w_1w_4w_4 &= w_2w_3w_8w_{12} & w_1w_2w_1w_5 &= w_2w_4w_5w_{13} & w_1w_2w_1w_6 &= w_2w_4w_5w_{14} \\
w_1w_2w_2w_8 &= w_2w_4w_6w_{16} & w_1w_2w_4w_8 &= w_2w_4w_8w_{16} & w_3w_1w_9w_1 &= w_4w_3w_{13}w_9 \\
w_3w_1w_{11}w_1 &= w_4w_3w_{15}w_9 & w_3w_1w_{12}w_3 &= w_4w_3w_{16}w_{11} & w_3w_1w_{12}w_4 &= w_4w_3w_{16}w_{12} \\
w_3w_2w_{12}w_8 &= w_4w_4w_{16}w_{16} & w_1w_6w_1w_5 &= w_2w_8w_5w_{13} & w_1w_6w_1w_6 &= w_2w_8w_5w_{14} \\
w_1w_6w_2w_8 &= w_2w_8w_6w_{16} & w_1w_6w_4w_8 &= w_2w_8w_8w_{16} & w_3w_6w_{12}w_8 &= w_4w_8w_{16}w_{16} \\
w_5w_9w_1w_1 &= w_6w_{11}w_5w_9 & w_5w_{13}w_1w_1 &= w_6w_{15}w_5w_9 & w_5w_{14}w_1w_5 &= w_6w_{16}w_5w_{13} \\
w_5w_{14}w_1w_6 &= w_6w_{16}w_5w_{14} & w_5w_{14}w_2w_8 &= w_6w_{16}w_6w_{16} & w_5w_{14}w_4w_8 &= w_6w_{16}w_8w_{16} \\
w_{11}w_1w_9w_1 &= w_{12}w_3w_{13}w_9 & w_{11}w_1w_{11}w_1 &= w_{12}w_3w_{15}w_9 & w_{11}w_1w_{12}w_3 &= w_{12}w_3w_{16}w_{11} \\
w_{11}w_1w_{12}w_4 &= w_{12}w_3w_{16}w_{12} & w_{11}w_2w_{12}w_8 &= w_{12}w_4w_{16}w_{16} & w_{11}w_6w_{12}w_8 &= w_{12}w_8w_{16}w_{16} \\
w_{13}w_9w_1w_1 &= w_{14}w_{11}w_5w_9 & w_{15}w_9w_9w_1 &= w_{16}w_{11}w_{13}w_9 & w_{15}w_9w_{11}w_1 &= w_{16}w_{11}w_{15}w_9 \\
w_{15}w_9w_{12}w_3 &= w_{16}w_{11}w_{16}w_{11} & w_{15}w_9w_{12}w_4 &= w_{16}w_{11}w_{16}w_{12} & w_{13}w_{13}w_1w_1 &= w_{14}w_{15}w_5w_9 \\
w_{13}w_{14}w_1w_5 &= w_{14}w_{16}w_5w_{13} & w_{13}w_{14}w_1w_6 &= w_{14}w_{16}w_5w_{14} & w_{13}w_{14}w_2w_8 &= w_{14}w_{16}w_6w_{16} \\
w_{13}w_{14}w_4w_8 &= w_{14}w_{16}w_8w_{16} & w_{15}w_{13}w_9w_1 &= w_{16}w_{15}w_{13}w_9 & w_{15}w_{13}w_{11}w_1 &= w_{16}w_{15}w_{15}w_9 \\
w_{15}w_{13}w_{12}w_3 &= w_{16}w_{15}w_{16}w_{11} & w_{15}w_{13}w_{12}w_4 &= w_{16}w_{15}w_{16}w_{12} & w_{15}w_{14}w_9w_5 &= w_{16}w_{16}w_{13}w_{13} \\
w_{15}w_{14}w_9w_6 &= w_{16}w_{16}w_{13}w_{14} & w_{15}w_{14}w_{11}w_5 &= w_{16}w_{16}w_{15}w_{13} & w_{15}w_{14}w_{11}w_6 &= w_{16}w_{16}w_{15}w_{14}
\end{aligned}$$

Similarly, all Type 2 constraints are as follows. Again, for the following set of equations, assume the equals sign (=) means *only* equal in sign.

$$-w_3w_2w_9w_5 = w_4w_4w_{13}w_{13} \quad -w_3w_2w_9w_6 = w_4w_4w_{13}w_{14} \quad -w_3w_2w_{11}w_5 = w_4w_4w_{15}w_{13}$$
$$-w_3w_2w_{11}w_6 = w_4w_4w_{15}w_{14} \quad -w_3w_6w_9w_5 = w_4w_8w_{13}w_{13} \quad -w_3w_6w_9w_6 = w_4w_8w_{13}w_{14}$$
$$-w_3w_6w_{11}w_5 = w_4w_8w_{15}w_{13} \quad -w_3w_6w_{11}w_6 = w_4w_8w_{15}w_{14} \quad -w_5w_9w_2w_3 = w_6w_{11}w_6w_{11}$$
$$-w_5w_9w_2w_4 = w_6w_{11}w_6w_{12} \quad -w_5w_9w_4w_3 = w_6w_{11}w_8w_{11} \quad -w_5w_9w_4w_4 = w_6w_{11}w_8w_{12}$$
$$-w_5w_{13}w_2w_3 = w_6w_{15}w_6w_{11} \quad -w_5w_{13}w_2w_4 = w_6w_{15}w_6w_{12} \quad -w_5w_{13}w_4w_3 = w_6w_{15}w_8w_{11}$$
$$-w_5w_{13}w_4w_4 = w_6w_{15}w_8w_{12} \quad -w_{11}w_2w_9w_5 = w_{12}w_4w_{13}w_{13} \quad -w_{11}w_2w_9w_6 = w_{12}w_4w_{13}w_{14}$$
$$-w_{11}w_2w_{11}w_5 = w_{12}w_4w_{15}w_{13} \quad -w_{11}w_2w_{11}w_6 = w_{12}w_4w_{15}w_{14} \quad -w_{11}w_6w_9w_5 = w_{12}w_8w_{13}w_{13}$$
$$-w_{11}w_6w_9w_6 = w_{12}w_8w_{13}w_{14} \quad -w_{11}w_6w_{11}w_5 = w_{12}w_8w_{15}w_{13} \quad -w_{11}w_6w_{11}w_6 = w_{12}w_8w_{15}w_{14}$$
$$-w_{13}w_9w_2w_3 = w_{14}w_{11}w_6w_{11} \quad -w_{13}w_9w_2w_4 = w_{14}w_{11}w_6w_{12} \quad -w_{13}w_9w_4w_3 = w_{14}w_{11}w_8w_{11}$$
$$-w_{13}w_9w_4w_4 = w_{14}w_{11}w_8w_{12} \quad -w_{13}w_{13}w_2w_3 = w_{14}w_{15}w_6w_{11} \quad -w_{13}w_{13}w_2w_4 = w_{14}w_{15}w_6w_{12}$$
$$-w_{13}w_{13}w_4w_3 = w_{14}w_{15}w_8w_{11} \quad -w_{13}w_{13}w_4w_4 = w_{14}w_{15}w_8w_{12}$$

Solving all Type 1 and Type 2 constraints gives the following solution set.

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	w_{16}
7	-1	-1	-1	-1	-1	0	-1	1	0	-1	-1	-1	-1	1	7
7	-1	-1	-1	-1	1	0	1	1	0	1	1	-1	1	-1	7
7	-1	-1	-1	1	-1	0	-1	-1	0	-1	-1	-1	1	-1	7
7	-1	-1	-1	1	1	0	1	-1	0	1	1	-1	-1	1	7
7	-1	-1	1	-1	-1	0	1	1	0	-1	1	1	1	-1	7
7	-1	-1	1	-1	1	0	-1	1	0	1	-1	1	-1	1	7
7	-1	-1	1	1	-1	0	1	-1	0	-1	1	1	-1	1	7
7	-1	-1	1	1	1	0	-1	-1	0	1	-1	1	1	-1	7
7	-1	1	-1	-1	-1	0	-1	-1	0	-1	1	-1	-1	-1	7
7	-1	1	-1	-1	1	0	1	-1	0	1	-1	-1	1	1	7
7	-1	1	-1	1	-1	0	-1	1	0	-1	1	-1	1	1	7
7	-1	1	-1	1	1	0	1	1	0	1	-1	-1	-1	-1	7
7	-1	1	1	-1	-1	0	1	-1	0	-1	-1	1	1	1	7
7	-1	1	1	-1	1	0	-1	-1	0	1	1	1	-1	-1	7
7	-1	1	1	1	-1	0	1	1	0	-1	-1	1	-1	-1	7
7	-1	1	1	1	1	0	-1	1	0	1	1	1	1	1	7
7	1	-1	-1	-1	-1	0	1	-1	0	-1	-1	-1	-1	-1	7
7	1	-1	-1	-1	1	0	-1	-1	0	1	1	-1	1	1	7
7	1	-1	-1	1	-1	0	1	1	0	-1	-1	-1	1	1	7
7	1	-1	-1	1	1	0	-1	1	0	1	1	-1	-1	-1	7
7	1	-1	1	-1	-1	0	-1	-1	0	-1	1	1	1	1	7
7	1	-1	1	-1	1	0	1	-1	0	1	-1	1	-1	-1	7
7	1	-1	1	1	-1	0	-1	1	0	-1	1	1	-1	-1	7
7	1	-1	1	1	1	0	1	1	0	1	-1	1	1	1	7
7	1	1	-1	-1	-1	0	1	1	0	-1	1	-1	-1	1	7
7	1	1	-1	-1	1	0	-1	1	0	1	-1	-1	1	-1	7
7	1	1	-1	1	-1	0	1	-1	0	-1	1	-1	1	-1	7
7	1	1	-1	1	1	0	-1	-1	0	1	-1	-1	-1	1	7
7	1	1	1	-1	-1	0	-1	1	0	-1	-1	1	1	-1	7
7	1	1	1	-1	1	0	1	1	0	1	1	1	-1	1	7
7	1	1	1	1	-1	0	-1	-1	0	-1	-1	1	-1	1	7
7	1	1	1	1	1	0	1	-1	0	1	1	1	1	-1	7

Any of these assignments are sufficient for calculating $t_{m,n}$.