The Mosaic Problem

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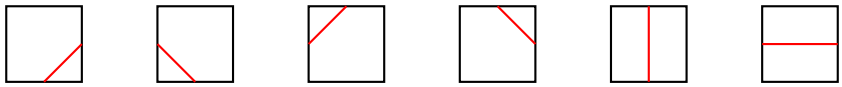
1 Introduction

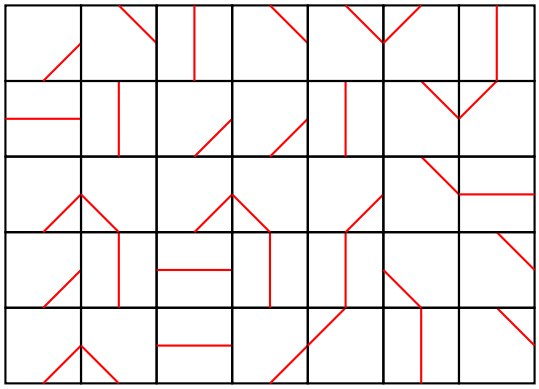
Consider the following 6 unit squares with markings on them.



Call these squares *tiles*.

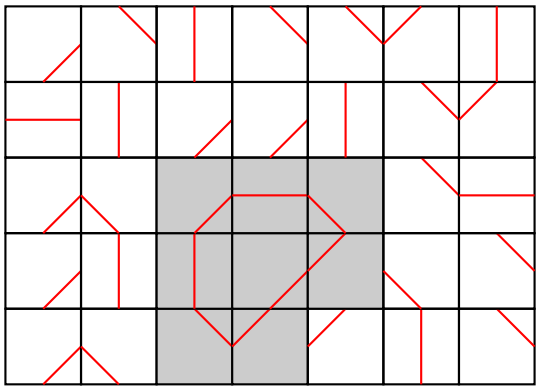
Definition 1.1. An (n, m)-*mosaic* is a rectangular lattice made up of tiles.

Example 1.1. An example of a $(7, 5)$-mosaic:

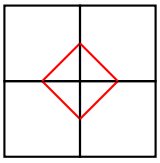


Clearly there are 6^{nm} possible mosaics. Which of these mosaics contain self-avoiding polygons?

Example 1.2. An example of a $(7, 5)$-mosaic with a self-avoiding polygon, hilighted in gray:



Let $t_{n,m}$ be the number of mosaics that have at least one self avoiding polygon (SAP). Clearly $t_{n,m} = t_{m,n}$. Also from the fact that the smallest SAP is



we have that $t_{n,1} = t_{1,m} = 0$, and $t_{2,2} = 1$. What else can be said?

Theorem 1. *Assume* $n \geq m$

$$M(m) = \begin{bmatrix} 36 & 1 \\ -1 & 1 \end{bmatrix}$$

For $h \geq 2$, *then if*

$$M(m) = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

then

$$M(m+1) = \begin{bmatrix} 6M_1 & 6M_2 & \frac{1}{6}M_1 & 1M_2 \\ 6M_3 & 6M_4 & 0M_3 & 1M_4 \\ -\frac{1}{6}M_1 & 0M_2 & \frac{1}{6}M_1 & 1M_2 \\ 1M_3 & 1M_4 & -1M_3 & 6M_4 \end{bmatrix}$$

where M_i *is a sub-matrix (or possible a scalar) of the block matrix* M.
Then for the given m, *define the rows and columns of* $M(m)$ *as*

$$M(m) = \begin{bmatrix} c_1 & c_2 & \dots & c_{2^{m-1}} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \dots \\ r_{2^{m-1}} \end{bmatrix}$$

Then let $v, L(n) \in \mathbb{R}^{2^{m-1} \times 1}$ *so that*

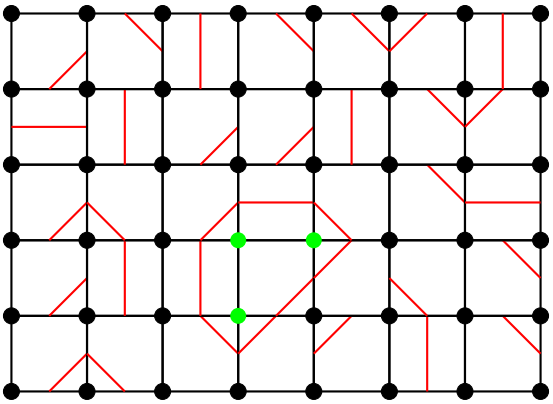
$$v_i = -1 * r_i \cdot c_1$$

excluding the first term of the dot product,
and

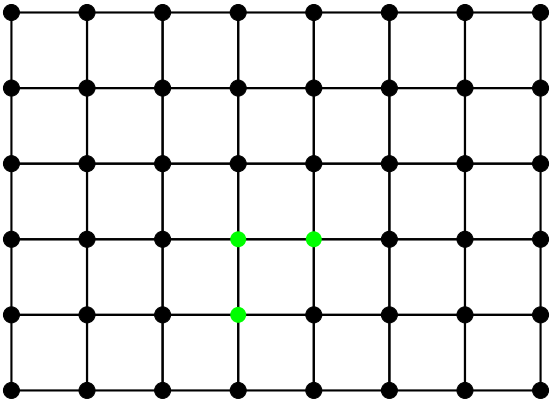
$$L(n) = M(m) \cdot L(n-1) + v$$

Then $t_{n,m} = r_1 \cdot (M(m) \cdot L(n-1) + v)$.

Proof. Begin by labelling the vertices of the rectangular lattice as follows. If the vertex is surrounded by an even number of SAPs, color it black. If the vertex is surrounded by an odd number of SAPs, color it green. Using the diagram from Example 1.2 above, we get the following labelling.



Notice that vertices on the boundary of the lattice will always be black. The associated *parity configuration* for the above mosaic is below.

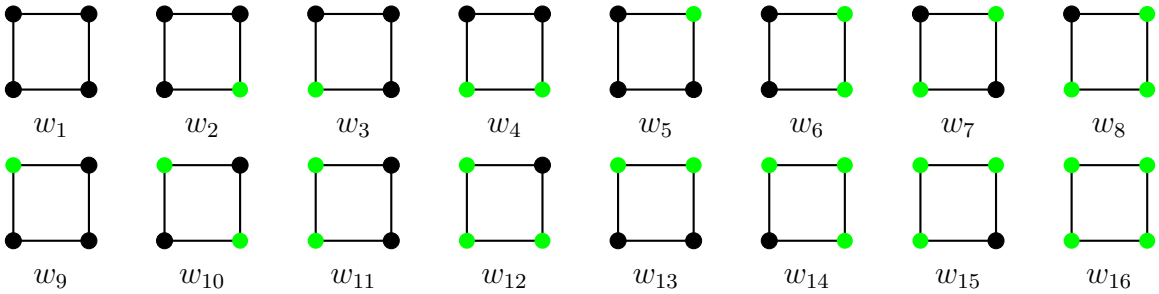


Let $\mathcal{P}(n,m)$ denote all possible parity configurations for an (n,m) rectangular lattice. Clearly $|\mathcal{P}(n,m)| = 2^{(n-1)(m-1)}$. Then let $f(p)$ be a function of a specific parity configuration that retuns the number of possible mosaics that map to p, multiplied by -1 to the number of SAPs specified by p. For example, the above parity configuration only specifies 1 SAP.

Then one can write

$$t_{n,m} = -\sum_{p\in\mathcal{P}(n,m)} f(p).$$

Amazingly, $f(p)$ can be written as a product of some choice of weights $w_1, \ldots, w_{16}$ associated with the following individual *cell-parity configurations*.



Next note the following lemma.

Lemma 2. *One can construct all larger SAPs from the smallest SAP using a finite set of transformations S.*

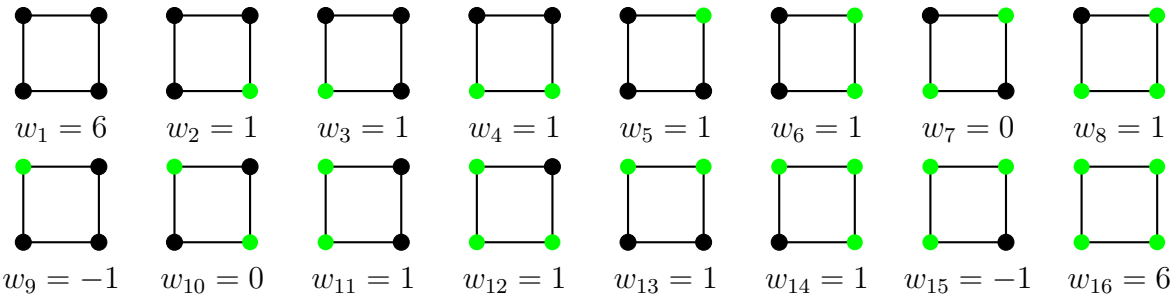
Proof. TODO □

This is because one can find $w_1, \ldots, w_{16}$ so that the following two constraints hold:

Constraint 3. *The weights associated with the smallest SAP multiply to -1, ie. $w_1 w_2 w_4 w_8 = -1$.*

Constraint 4. All *transformations in S preserve the weight product equalling -1.*

Constraint 3 and Constraint 4 amount to a series of constraints on the values of w_i. The derivation for these constraints can be found in the Appendix. Choosing a solution set from these constrains gives the following weights.



TODO

□

2 Appendix

Each constraint on $w_1, \ldots, w_{16}$ represents a change in a SAP border. For example

TODO

In total, there are 2^8 seperate constraints, one for each parity configuration of a $(2, 2)$ rectangular lattice that do not contain the parity configuratrions associated with w_7 or $w_1 0$. The constraints are as follows:

TODO

With the help of SageMath, these constraints reduce to the following simplified constraints

TODO

The solution set to these constrains are as follows.

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