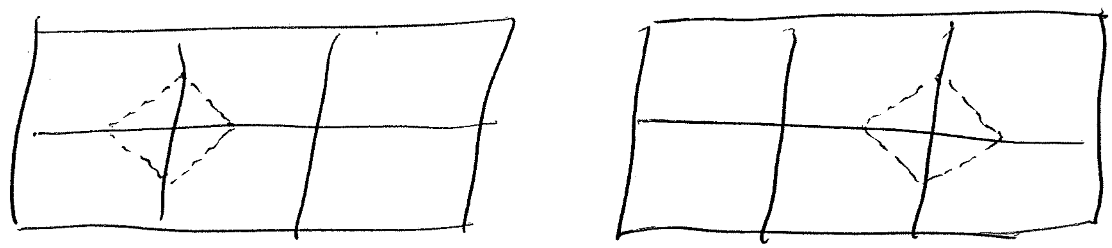
4-11-1

Given positive integers $m$ and $n$, an <u>$m \times n$ grid</u> is a rectangle, ~~[illegible]~~ $m$ units high and $n$ units across, that is subdivided into $mn$ squares of unit length that have disjoint interiors. The $mn$ squares are called <u>cells</u>. A <u>connection</u> is a line segment ~~[illegible]~~ whose endpoints are midpoints of two sides of the same cell. We draw connections as dashed red lines. An <u>$m \times n$ mozaic</u> is an $m \times n$ grid plus at most one connection in each cell.

11-11-2

Given an $m \times n$ grid, a $m \times n$ gridded polygon is a mozaic in which all connections together form exactly one polygon.

Note that gridded polygons are determined not only by their shape and orientation but also by their placement. For example these are distinct $2 \times 3$ gridded polygons. ~~that are both 2×3 mozaics.~~



If $P$ is an $m \times n$ gridded polygon and $M$ an $m \times n$ mozaic we say $M$ includes $P$ if $M = P$ or $M$ can be formed from $P$ by adding connections.

11-11-3

If $P_1, P_2, \ldots, P_r$ are $m \times n$ gridded polygons, $r \geq 1$, we say they are <u>compatible</u> if there is a mozaic that includes $P_i$ for all $i$. In that case the <u>simple mozaic including</u> $P_1, P_2, \ldots, P_r$ is the mozaic that contains $P_i$ for all $i$ and in which every connection of $M$ is a connection of some $P_i$.

Note that distinct $m \times n$ gridded polygons $P_1, P_2, \ldots, P_r$ are compatible if and only if every cell of an $m \times n$ grid has a connection in at most one $P_i$, and in this case the simple mozaic including $P_1, P_2, \ldots, P_r$ contains exactly the connections of all the connetions in every $P_i$.

11-11-4

Any cell in a mozaic can have no connections or one of six connections.

~~Fix integers~~ Therefore, for ~~Fix~~ positive integers $m$ and $n$, there are $7^{mn}$ mozaics. The goal of this paper is to show how to calculate the number of these mosaics that contain at least one gridded polygon.

Let us restate this goal in the following way. Enumerate all $m \times n$ gridded polygons for fixed $m$ and $n$, as $P_1, P_2, \ldots, P_N$. For all $1 \le i \le N$, let $A_i$ be the set of all $m \times n$ mozaics that include $P_i$. Our goal then is to find the number of elements in $\bigcup_{i=1}^{N} A_i$.

11-11-5

By the inclusion-exclusion principle,

$$\left|\bigcup_{i=1}^{N} A_i\right| = \sum_{i=1}^{N} |A_i| - \sum_{1\le i<j\le N} |A_i \cap A_j| + \sum_{1\le i<j<k\le N} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{N+1} |A_1 \cap A_2 \cap \cdots \cap A_N|$$

$$= \sum_{\emptyset \ne J \subseteq \{1,2,\dots,N\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|.$$

It turns out to be easier to instead find the number of elements in the complement of this set. Since there are $7^{mn}$ mozaics, this equals

$$7^{mn} + \sum_{\emptyset \ne J \subseteq \{1,2,\dots,N\}} (-1)^{|J|} \left|\bigcap_{j\in J} A_j\right|.$$

11-11-6

Note that for any $\emptyset \neq J \subseteq \{1, 2, \ldots, N\}$ $\bigcap_{j \in J} A_j$ is nonempty if and only if the $P_j$, $j \in J$, are compatible.

Therefore our goal is to find

$$7^{mn} + \sum (-1)^{|J|} \left| \bigcap_{j \in J} A_j \right|,$$

where the sum is over all nonempty $J \subseteq \{1, 2, \ldots, N\}$ such that the gridded polygons $P_j$, $j \in J$, are compatible.

My suggestion: do not state the main theorem now but give a forward reference to it.

11-11-7

Begin new section:
Binary Grids.

Let us broaden our earlier definition of grids by defining, for a positive integer $n$, a $\underline{0 \times n \text{ grid}}$ to be a horizontal line segment of length $n$ units, subdivided into $n$ line segments of unit length that have disjoint interiors. It has zero cells.

Given integers $m \geq 0$ and $n \geq 1$, the $\underline{\text{vertices}}$ of an $m \times n$ grid are the corners of each cell if $m \geq 1$ or the endpoints of the unit line segments if $m = 0$. Note that an $m \times n$ grid has $(m+1)(n+1)$ vertices.

We define an $\underline{m \times n \text{ binary grid}}$ to be an $m \times n$ grid in which each vertex has been replaced by 0 or 1.

11-11-8

~~Next steps:~~

~~define~~

In a binary grid, each cell has a *type*, given by the numbers at each corner. For example, a cell can have type $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ or $\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$.

~~Given an mxn binary grid~~

We define the valuation $V$ of a cell as $7$ for cells of type $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ or $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$, $0$ for cells of type $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, $-1$ for cells of type $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, and otherwise $\underline{1}$.

11-11-~~0~~9

Given an ~~[illegible]~~ $m \times n$ binary grid $G$, we define $V(G)$ to be the product of the valuations of the type of each of its $mn$ cells. In the case $n = 0$ we define $V(G) = 1$.

11-11-10

## Next steps

Rather than the maps $\mathcal{G}$ (formerly $f$) and $\mathcal{S}$, instead show a one-to-one correspondence, based on the same idea as the old functions, between, for pos integers $m, n$

the simple $m\times n$ mozaics that contain $P_j$, $j \in J$ for nonempty $J$ such that the $P_j$ are compatible

and

$m \times n$ binary grids in which the top row, bottom row, left column, and right column, are all zeros and which have no cells of type $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$.
— and there is at least one 1.

# 11-11-11

Then prove that the sum in the middle of 11-11-6 above equals the sum $\sum_G V(G)$ where the sum is over all $m \times n$ binary grids $G$ such that the top and bottom rows are all zeros and the left and right columns are all zeros.

Next:

Define matrices $A(k)$, $B(k)$, $C(k)$, $D(k)$ as before and show that $\sum_G V(G)$ as above is the $(0,0)$ entry of the appropriate matrix (probably $[A(n)]^m$ but perhaps my indices are off?)