Enumeration of Messy Polygon Mosaics

Jack Hanke **Richard Schank**
Northwestern University

Michael Maltenfort
Northwestern University

Abstract

Hong and Oh introduced a model for multiple ring polymers in physics in which an $m \times n$ retangular lattice is constructed from a selection of 7 distinct tiles. These lattices are called *mosaics*. The authors provide bounds on a subset of these mosaics that both contain polygons and have all other tiles that are not part of a polygon set to the blank tile. We introduce and enumerate mosaics with the relaxed property of containing at least one polygon, which we call messy polygon mosaics.

1 Introduction

Imagine you are tasked with tiling a rectangular bathroom floor that is m units by n units, blindfolded. At your disposal is an unending supply of 7 distinct types of tiles. These tiles, diagrammed in Figure 1, are composed of unit squares with dotted lines connecting 2 sides at their midpoints, as well as the "blank" tile T_0.

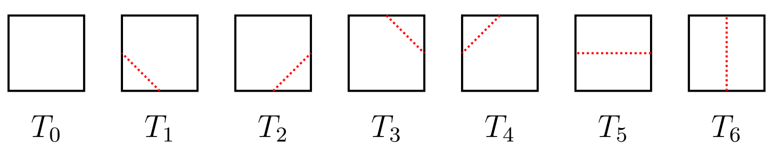


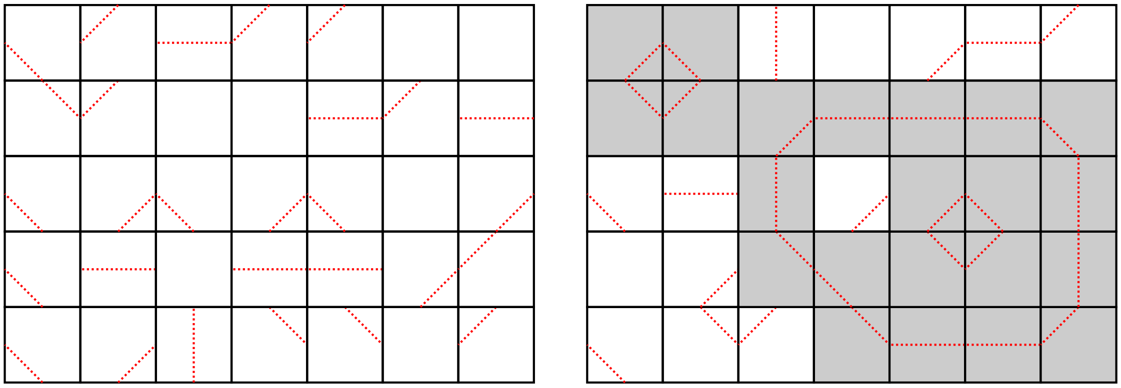
Figure 1: The tile set $\mathbb{T}$

We denote the set of tiles $\mathbb{T} = \{T_0, \ldots, T_6\}$. The task is complete once you place mn randomly selected tiles from $\mathbb{T}$ to cover the floor, after which you remove your blindfold. We call a fully tiled $m \times n$ floor an (m, n) *mosaic*.

If $m = 5$ and $n = 7$, you may have constructed the mosaic in Figure 2a. You may have also constructed the mosaic in Figure 2b. Notice that in the mosaic in Figure 2b, the tiles form multiple polygons, which we highlight in gray.

Definition 1.1. A *polygon*[1] is a nonempty subset p of edge-wise connected tiles such that if one chooses any starting tile in p, one can visit each tile in the set exactly once by doing

[1]Polygons are more commonly referred to as "self-avoiding polygons" in the literature to highlight their connection with self-avoiding walks.



(a) A mosaic (b) A messy polygon mosaic

Figure 2: Examples of mosaics of size $(5, 7)$ made of tiles in $\mathbb{T}$

the following. Identify an edge that is shared between the current tile and a tile that has not been visited and has 2 dotted lines drawn from the edge's midpoint. Set the current tile to this new tile. Repeat the procedure until the new tile is the starting tile.

Definition 1.2. An *messy polygon* (m, n) mosaic is an (m, n) mosaic that contains at least one polygon.

Example 1.1. Below is one of the polygons in the messy polygon mosaic in Figure 2b.

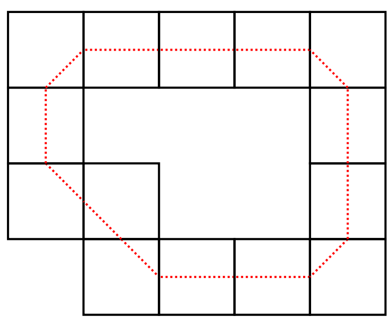


Figure 3: A polygon in Figure 2b

As in Figure 2b, a mosaic can contain polygons that surround other polygons.

Further note that when a part of a mosaic, both tiles and polygons also have an associated location within the mosaic. The mosaic in Figure 2b contains 3 polygons, although two of the polygons are the same up to location.

What is the probability of constructing a messy polygon mosaic, given the dimensions of the bathroom m, n? As there are $|\mathbb{T}|^{mn} = 7^{mn}$ total mosaics, we focus on the total number of messy polygon mosaics. In fact, it turns out to be simpler to enumerate the number of mosaics that *do not* contain a polygon. Therefore, let $\mathbb{P}^{(m,n)}$ be the subset of (m, n) mosaics that do not contain a polygon. Clearly the number of messy polygon (m, n) mosaics is then $7^{mn} - \left|\mathbb{P}^{(m,n)}\right|$.

From the fact that the smallest polygon is made of 4 tiles (appearing twice in Figure 2b), we can conclude that $\left|\mathbb{P}^{(n,1)}\right| = 7^n$, and $\left|\mathbb{P}^{(2,2)}\right| = 7^4 - 1$. For $m, n \geq 2$, we first define the matrices for messy polygon mosaics.

Definition 1.3. For integers $k \geq 1$ let A_k, B_k, C_k, D_k be $2^{k-1} \times 2^{k-1}$ matrices with integer entries, where $A_1 = \begin{pmatrix} 7 \end{pmatrix}, B_1 = \begin{pmatrix} -1 \end{pmatrix}, C_1 = \begin{pmatrix} 1 \end{pmatrix}, D_1 = \begin{pmatrix} 1 \end{pmatrix}$ and

$$
\begin{aligned}
A_{k+1} &= \begin{pmatrix} 7A_k & B_k \\ C_k & D_k \end{pmatrix} & B_{k+1} &= \begin{pmatrix} -A_k & B_k \\ 0C_k & D_k \end{pmatrix} \\
C_{k+1} &= \begin{pmatrix} A_k & 0B_k \\ C_k & D_k \end{pmatrix} & D_{k+1} &= \begin{pmatrix} A_k & -B_k \\ C_k & 7D_k \end{pmatrix}.
\end{aligned}
$$

Throughout this work, we index elements in matrices, mosaics, and later binary lattices with a pair of coordinates. The first coordinate is the row index, counted top to bottom, and the second coordinate is the column index, counted left to right, both beginning at 0.

Here is our main result.

Theorem 1.1. *The number of (m, n) mosaics that do not contain a polygon $\left|\mathbb{P}^{(m,n)}\right|$ is the $(0, 0)$ entry of A_n^m.*

2 Related Work

Hong and Oh [2] studied a similar question in which they construct mosaics from $\mathbb{T}$, but were interested in the number of polygon mosaics.

Definition 2.1. An *polygon* (m, n) mosaic is an (m, n) mosaic that contains at least one polygon and every tile that is not part of a polygon is T_0.

Clearly, all polygon mosaics are messy polygon mosaics. The sequence A181245 on the OEIS [4] is the array of 1+ the number of polygon (m, n) mosaics. The authors in [2] provide bounds for the number of polygon mosaics.

Theorem 2.1 ([2])**.** *The number of polygon (m, n) mosaics for $m, n \geq 3$ is bounded between $2^{m+n-3}\left(\frac{17}{10}\right)^{(m-2)(n-2)}$ and $2^{m+n-3}\left(\frac{31}{16}\right)^{(m-2)(n-2)}$.*

In related work, Lomonaco and Kauffman [3] introduced mosaics constructed from a tile set of 11 distinct tiles, of which $\mathbb{T}$ is a subset. The authors were interested in a subset of mosaics which they call *knot mosaics*. Oh et al. [9] enumerated the number of knot mosaics.

Theorem 2.2 ([9])**.** *The number of knot (m, n) mosaics for $m, n \geq 2$ is $2\left\|(X_{m-2} + O_{m-2})^{n-2}\right\|$, where $X_0 = O_0 = \begin{bmatrix} 1 \end{bmatrix}$ and X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined as*

$$
X_{k+1} = \begin{pmatrix} X_k & O_k \\ O_k & X_k \end{pmatrix} \text{ and } O_{k+1} = \begin{pmatrix} O_k & X_k \\ X_k & 4O_k \end{pmatrix},
$$

for $k = 0, 1, \ldots, m-3$. Here $\|N\|$ denotes the sum of elements of matrix N.

Oh and colleagues go beyond enumeration by bounding the growth rate of knot mosaics [6, 8, 1], and Oh further adapts the matrix recursion method to solve problems in monomer and dimer tilings [5, 7]. Related ideas were independently used to enumerate the number of rectangular partitions of a rectangle in [10] for the sequence A182275 on the OEIS [4].

Our work can be seen as an extension to Hong and Oh [2] and further generalizing the techniques in Oh et al. [9] to explore a new direction in mosaic enumeration.

3 Preliminaries

We begin by defining a map that takes an (m, n) mosaic and gives an (m, n) binary lattice. An (m, n) binary lattice is a rectangular lattice of $m + 1$ by $n + 1$ vertices, with each vertex labeled 0 or 1. We also define a *framed* binary lattice to be a binary lattice in which the boundary vertices are labeled 0. An example of a $(5, 7)$ framed binary lattice is shown on the right of Figure 4. Also let $\mathbb{L}^{(m,n)}$ be the set of all (m, n) binary lattices and $\mathbb{F}^{(m,n)}$ be the set of all (m, n) framed binary lattices. We immediately have $\left|\mathbb{L}^{(m,n)}\right| = 2^{(m+1)(n+1)}$, $\left|\mathbb{F}^{(m,n)}\right| = 2^{(m-1)(n-1)}$.

Definition 3.1. Let f map an (m, n) mosaic and labels each vertex with the following rule. If the vertex is surrounded by the red dotted lines of an even number of polygons (including 0 polygons), label it 0. If the vertex is surrounded by the red dotted lines of an odd number of polygons, label it 1. Removing the red dotted lines from the tiles gives the framed binary lattice.

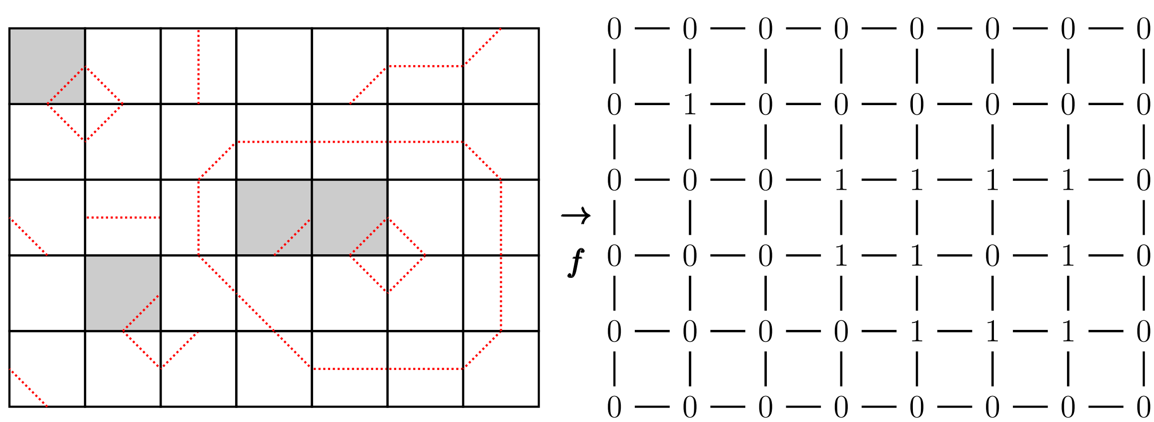


Figure 4: f applied to the left mosaic in Figure 2b, resulting in a binary lattice. We highlight each possible way a T_2 tile can map to a cell by shading a representative tile in gray.

Notice that by the definition of f, in a binary lattice the red dotted lines of a polygon draw out the boundary between edge-connected vertices with the same label, excluding the vertices that are edge-connected to the boundary 0's. For example in Figure 4 the three red-dotted lines of the polygons correspond with the boundary of the three edge-connected regions of vertices that are not edge-connected to the boundary 0's.

To enumerate $|\mathbb{P}^{(m,n)}|$, it will be useful to consider how f maps individual tiles in $\mathbb{T}$ to individual cells in a binary lattice.

Definition 3.2. Let a *cell* be a $(1,1)$ binary lattice.

Example 3.1. Applying f to the mosaic in Figure 4 results in the T_2 tile at position $(0,0)$ mapping to the cell at position $(0,0)$, diagrammed below.

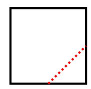
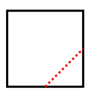
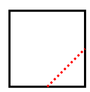
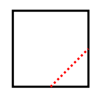
 $\rightarrow \begin{array}{ccc} 0 & — & 0 \\ | & & | \\ 0 & — & 1 \end{array}$

Figure 4 also illustrates the three other cells T_2 can map to. We diagram applying f to the T_2 cells at positions $(2,3),(2,4)$, and $(3,1)$ below.

 $\rightarrow \begin{array}{ccc} 1 & — & 1 \\ | & & | \\ 1 & — & 1 \end{array}$  $\rightarrow \begin{array}{ccc} 1 & — & 1 \\ | & & | \\ 1 & — & 0 \end{array}$  $\rightarrow \begin{array}{ccc} 0 & — & 0 \\ | & & | \\ 0 & — & 0 \end{array}$

For convenience, we denote a cell by the 2×2 matrix of its vertex labels. For example, we denote the cell in Example 3.1 as $\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$. There are sixteen cells:

$$\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}, \text{ and } \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}.$$

It would be convenient to compute $f^{-1}(\{\ell\})$ for any framed binary lattice ℓ "cell-by-cell". For a binary lattice ℓ, let $u(\ell)$ be the product over all cells in ℓ, with each term being

$$\begin{cases} 7 & \text{for cells } \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \\ 0 & \text{for cells } \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ 1 & \text{otherwise} \end{cases}$$

where these terms come from the number of tiles in $\mathbb{T}$ that can map to a specific cell under f. For a given binary lattice ℓ that does not contain $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ cells, the function $u(\ell)$ counts the number of mosaics of a specific form, in which at each square a tile is either uniquely specified, or can be any tile in $\mathbb{T}$. We can depict this set with a *mosaic diagram*, where we introduce a notational tile with a red dot at the center to indicate that any tile in $\mathbb{T}$ can be in that location (to avoid confusion with the T_0 tile). For example, the mosaic diagram for the binary lattice in Figure 4 is depicted in Figure **??**. Consequently, the mosaic in Figure 4 is included in the set represented by the mosaic diagram in Figure **??**.

Definition 3.3. For a given (m,n), let ℓ^* be the framed binary lattice made of only $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ cells.

Therefore, as all mosaics that do not contain a polygon map to ℓ^* under f, we have

$$\mathbb{P}^{(m,n)} = f^{-1}(\{\ell^*\}), \tag{1}$$

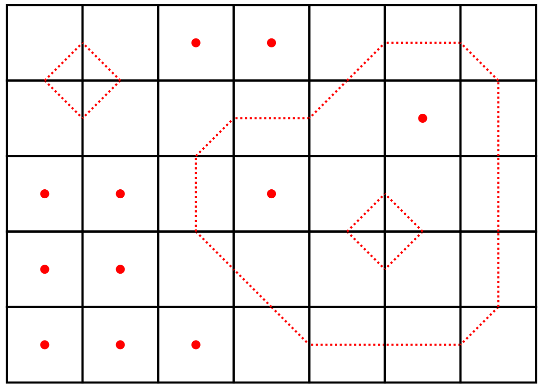


Figure 5: Mosaic Diagram of Binary Lattice in Figure 4

and so it suffices to compute $|f^{-1}(\{\ell^*\})|$. One would hope that the function u could be used to compute $|\mathbb{P}^{(m,n)}|$ in some way. However for the (m,n) binary lattice ℓ^*, we have

$$u(\ell^*) = 7^{mn} > |\mathbb{P}^{(m,n)}|.$$

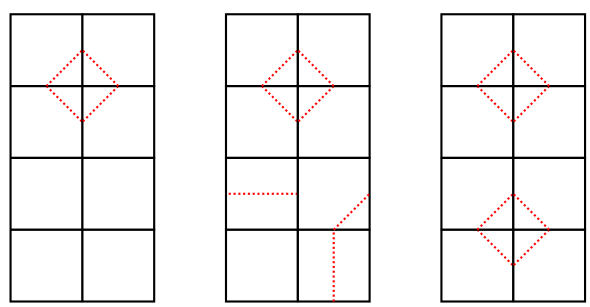
Clearly, this approach overcounts $|\mathbb{P}^{(m,n)}|$. We can examine this phenomena more closely by first definining g.

Definition 3.4. $g : \mathbb{F}^{(m,n)} \to \mathbb{M}^{(m,n)}$ takes a framed binary lattice that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, and constructs a mosaic by replacing all $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells with T_0, and replacing all other cells with the unique tile that maps to that cell under f.

Notice that for some binary lattice ℓ, $g(\ell)$ is not necessarily equal to $f^{-1}(\ell)$, as g only returns polygon mosaics and not messy polygon mosaics. We can now examine the over-counting more closely.

Example 3.2. Consider the following binary lattices and mosaics for $(m,n) = (4,2)$.

$$\begin{array}{ccccc} 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 1 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array} \qquad \begin{array}{ccccc} 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 1 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array} \qquad \begin{array}{ccccc} 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 1 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \\ | & & | & & | \\ 0 & — & 1 & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array}$$



ℓ_1 ℓ_2 ℓ_3 Mosaic 1 Mosaic 2 Mosaic 3

We have $u(\ell_1) = 7^4, u(\ell_2) = 7^4$, and $u(\ell_3) = 1$. We also have $f(\text{Mosaic 3}) = \ell_3$. Each cell in the bottom two rows of ℓ_1 map to 7 possible tiles. However, 1 of the 7^4 combinations forms a new polygon in these bottom two rows. This is also true for the top two rows of ℓ_2. Therefore, $u(\ell_1)$, $u(\ell_2)$, and $u(\ell_3)$ all count Mosaic 3.

Example 3.2 leads us to conclude the following.

Proposition 3.1. *For a framed binary lattice ℓ that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, $u(\ell)$ counts the number of mosaics that have at least the polygons in $g(\ell)$.*

Proof. Choose an framed binary lattice ℓ that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. By the definition of g, the function $u(\ell)$ counts the number of mosaics that have exactly the polygons in $g(\ell)$. Additionally, as u counts each possible combination of tiles for $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells, this includes all polygons that do not conflict with the polygons in $g(\ell)$. $\square$

Surprisingly, this overcounting phenomena can be remedied by a small modificiation to u, which we call v.

Definition 3.5. Let $v : \mathbb{L}^{(m,n)} \to \mathbb{Z}$ map a binary lattice ℓ to the product over all cells in ℓ with each term being

$$\begin{cases} 7 & \text{for cells } \begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \\ 0 & \text{for cells } \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \\ -1 & \text{for cells } \begin{smallmatrix}0&0\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\1&1\end{smallmatrix} \\ 1 & \text{otherwise} \end{cases}$$

and the empty product being 1.

Example 3.3. If we let ℓ be the framed binary lattice on the right of Figure 4, we have $v(\ell) = -7^{11}$. Notice that there are 2 $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cells, 1 $\begin{smallmatrix}1&1\\0&1\end{smallmatrix}$ cells, 9 $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells, and 2 $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells.

We will first show that $v(\ell)$, which is computed "cell-by-cell" from ℓ, recovers global information about the number of polygons in $g(\ell)$ if ℓ is framed.

Definition 3.6. For a framed binary lattice ℓ, let $P(\ell)$ to be the number of polygons in the polygon mosaic $g(\ell)$.

Proposition 3.2. *If ℓ is a framed binary lattice that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, then*

$$sign(v(\ell)) = (-1)^{P(\ell)}. \tag{2}$$

Proof. For this proof, we use LHS and RHS to abbreviate the left and right hand side of Equation 2. We prove the result by induction. For the base case, construct the $(1, n)$ framed binary lattice ℓ for some $n \geq 1$. As there are no cells in ℓ, from the definition of v the LHS is $\text{sign}(v(\ell)) = 1$. As there no polygons in $g(\ell)$, we have the RHS is 1.

For the induction step, fix an (m, n) framed binary strip ℓ for $m \geq 1$. Then consider an $(m+1, n)$ framed binary strip ℓ' that shares the top $m-1$ rows of vertices with ℓ and an arbitrarily-labeled m-th row, such that there are no $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells. We show that we can construct ℓ' from ℓ with a procedure that preserves Equation 2 with each intermediate step.
Procedure:

Step 1. Add a bottom row to ℓ of $n+1$ vertices, all labeled 0. This results in a new $(m+1, n)$ framed binary lattice we denote ℓ_a.

Step 2. Scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}1\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}1\\0\end{smallmatrix}$ column in ℓ_a to $\begin{smallmatrix}1\\1\end{smallmatrix}$. Completing this scan results in a new framed binary lattice denoted ℓ_b.

Step 3. Againk scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}0\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}0\\0\end{smallmatrix}$ column in ℓ_b to $\begin{smallmatrix}0\\1\end{smallmatrix}$. Completing this scan results in the framed binary lattice ℓ'.

For step 1, only $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells are added. As $\operatorname{sign}\left(v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)=1$, the LHS is unchanged. For the RHS, no new edge-connected regions are created, no new polygons are created in $g(\ell_a)$ so Equation 2 is preserved by step a.

For steps 2 and 3, we encounter two distinct cases which we diagram in Figure 6. These diagrams have $a, b, c, d \in [0,1]$, and use the # symbol to indicate the vertex being changed from a 0 to a 1.

$$\begin{array}{ccccc} a & — & 1 & — & b \\ | & & | & & | \\ c & — & \# & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array}$$

(a) Step 2 Diagram

$$\begin{array}{ccccc} a & — & 0 & — & b \\ | & & | & & | \\ c & — & \# & — & d \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array}$$

(b) Step 3 Diagram

Figure 6: Step Diagrams

For step 2, Figure 6a depicts all possible cases one can encounter scanning left to right on ℓ_a. Note that the vertex right of the # symbol must be 0, as the procedure moving left to right on ℓ_a gives that this vertex must be 0. Notice that no assignment of a, b, and c can create cells $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$. Additionally, it cannot be the case that $a=0$ and $c=1$, as that would imply that ℓ' contains the cell $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$. Therefore we have $\operatorname{sign}(v(\ell_b)) = \operatorname{sign}(v(\ell_a))$, so the LHS is preserved. Also note that because a polygon in $g(\ell)$ is an edge-connected set of vertices with the same label (not including the boundary 0's), that for all assignments of a, b, c, step b only involves one edge-connected set of vertices. Therefore, we have $P(\ell_a) = P(\ell_b)$, and so the RHS is preserved.

For step 3, Figure 6b depicts all possible cases one can encounter scanning left to right on ℓ_b. It cannot be the case that $a=1$ and $c=0$, as this would imply ℓ' contains $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. Similarly, it cannot be the cases that $b=1$ and $d=0$, as this would imply ℓ_b contains $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$. Additionally, it cannot be the case that $b=0$ and $d=1$, as the procedure moves left-to-right. This results in the following 6 cases, which we show each preserve Equation 2.

In Case 7a, the flip add one $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. As the flip creates a new edge-connected region, a new polygon is created, and so the sign of the RHS changes.

In Case 7b, the flip does not add a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 7c, the flip removes and adds a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 7d, the flip removes a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. Before the flip,

$$\begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(a)

$$\begin{array}{ccccc} 0 & - & 0 & - & 1 \\ | & & | & & | \\ 0 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(b)

$$\begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(c)

$$\begin{array}{ccccc} 0 & - & 0 & - & 1 \\ | & & | & & | \\ 1 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(d)

$$\begin{array}{ccccc} 1 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(e)

$$\begin{array}{ccccc} 1 & - & 0 & - & 1 \\ | & & | & & | \\ 1 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(f)

Figure 7: Step 3. Cases

either the two edge-connected regions of 1's are distinct in the full framed binary lattice, or they are joined. After the flip, if the regions were distinct, they are now joined, and so 1 polygon is removed. If the regions were joined, a new edge-connected region of 0's (that is not connected to the boundary) is created, and so one polygon is created. Either way, the RHS changes sign.

In Case 7e, the flip adds both a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell and a $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 7f, the flip adds $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, so the sign of the LHS changes. The same logic from Case 7d applies, as the flip either removes or adds a polygon, so the RHS changes.

As all cases preserve Equation 2, the $(m+1, n)$ framed binary lattice ℓ' also follows Equation 2. □

We point out here that the function v if defined for all binary lattices, while Proposition 3.2 only holds for framed binary lattices that do not contain cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$.

Proposition 3.3. *The number of (m, n) mosaics that do not contain a polygon has*

$$\left|\mathbb{P}^{(m,n)}\right| = \sum_{\ell \in \mathbb{F}^{(m,n)}} v(\ell).$$

Proof. Choose a mosaic $\mathcal{M}$ that has $P(\ell')$ polygons, where $\ell' = f(\mathcal{M})$. We begin by determining how many times $\mathcal{M}$ is counted in the related sum

$$\sum_{\ell \in \mathbb{F}^{(m,n)}} u(\ell).$$

We point out here that as $u(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}) = u(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}) = v(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}) = v(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}) = 0$, framed binary lattices with these cells don't contribute to either summation, and so can be ignored for the remainder of this argument.

Proposition 3.1 shows that the function $u(\ell)$ counts the mosaics with at least the polygons in $g(\ell)$. Consequently, for a framed binary lattice ℓ, $u(\ell)$ counts $\mathcal{M}$ if the polygons in $g(\ell)$ are a subset of the polygons in $\mathcal{M}$. Therefore, as the number of ways to choose a size p subset of $P(\ell)$ polygons is $\binom{P(\ell)}{p}$, $\mathcal{M}$ is counted $\sum_{p=0}^{P(\ell)}\binom{P(\ell)}{p}$ times. This then gives

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} u(\ell) = \sum_{\mathcal{M}\in\mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} \binom{P(f(\mathcal{M}))}{p}.$$

Following the same logic for $\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell)$, Proposition 3.2 gives that size p subsets where p is odd are subtracted, which gives

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\mathcal{M}\in\mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} (-1)^p \binom{P(f(\mathcal{M}))}{p}.$$

Finally, by the binomial theorem, for $P(\ell) > 0$ we have

$$\sum_{p=0}^{P(\ell)} (-1)^p \binom{P(\ell)}{p} = 0,$$

and as the only binary lattice with $P(\ell) = 0$ is ℓ^*, we have

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\{\mathcal{M}\in\mathbb{M}^{(m,n)} | P(f(\mathcal{M}))=0\}} \binom{0}{0} = |f^{-1}(\ell^*)| = |\mathbb{P}^{(m,n)}|.$$

□

As in Theorem 2.2 from [9], we can compute $\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell)$ efficiently using the matrix recursion method.

4 Proof of Theorem 1.1

We begin by defining the matrices.

Definition 4.1. Let A_k, B_k, C_k, D_k be $2^{k-1} \times 2^{k-1}$ matrices with integer entries, where $A_1 = \left(v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right), B_1 = \left(v\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right), C_1 = \left(v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right), D_1 = \left(v\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\right)$, and for integers $k \geq 1$,

$$A_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)D_k \end{pmatrix}$$
$$C_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)D_k \end{pmatrix}.$$

Definition 4.2. Let the n digit binary representation of the number k be written as $\beta_n(k)$. If k is 0, $\beta_k(n)$ returns the empty string.

Proposition 4.1. *The (i,j)-th entry of A_n is $v(\ell)$, where ℓ is the $(1,n)$ binary lattice where the top row of vertices has labels $0\beta_{n-1}(i)0$ and the bottom row of vertices has labels $0\beta_{n-1}(j)0$, both read left to right. Furthermore, the same statement is true for B_n with top row $1\beta_{n-1}(i)0$ and bottom row $0\beta_{n-1}(j)0$, for C_n with top row $0\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$, and for D_n with top row $1\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$.*

Proof. We prove by induction. For $n = 1$, the definition of the matrices gives A_1 is $v(\ell)$ where ℓ is the $(1,1)$ binary lattice with top row $0\beta_0(0)0 = 00$ and bottom row $0\beta_0(0)0 = 00$, as $\beta_0(0)$ is the empty string. The same is true for B_1 with top row $0\beta_0(0)0 = 10$ and bottom row $1\beta_0(0)0 = 00$, for C_1 with top row $1\beta_0(0)0 = 00$ and bottom row $0\beta_0(0)0 = 10$, and for D_1 with top row $1\beta_0(0)0 = 10$ and bottom row $1\beta_0(0)0 = 10$.

We next assume matrices A_n, B_n, C_n, and D_n follow their associated statements in Proposition 4.1. Therefore, the entry (i,j) in, say, B_n is $v(\ell)$ where ℓ is the binary lattice with top row $0\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$. The argument is analagous for any choice of A_n, B_n, C_n, D_n. For $n > 1$ we can depict the $(1,n)$ binary lattice using similar notation to cells, namely

$$v(\ell) = v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix}.$$

We show that A_{n+1} also follows Proposition 4.1. From the definition, we have

$$A_{n+1} = \begin{pmatrix} v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) A_n & v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) B_n \\ v\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) C_n & v\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) D_n \end{pmatrix}.$$

By construction, the (i,j)-th entry in B_n is located in the $(i, j+2^{n-1})$-th entry of A_{n+1}, as each block matrix A_n, B_n, C_n, D_n are $2^{n-1} \times 2^{n-1}$ matrices. Also by construction, the value in the (i,j)-th entry in B_n which we call $v(\ell)$ is multiplied by $v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The definition of v gives

$$v\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & 0 & \beta_n(i) & 0 \\ 0 & 1 & \beta_n(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & \beta_n(i) & 0 \\ 0 & \beta_n(j+2^{n-1}) & 0 \end{pmatrix},$$

as desired.

Therefore, the $(i, j+2^{n-1})$-th entry of A_{n+1} is $v(\ell)$ where ℓ is the $(1, n+1)$ binary lattice where the top row of vertices is $0\beta_n(i)0$ and the bottom row of vertices is $0\beta_n(j+2^{n-1})0$, which completes the induction step for A_{n+1}. Similar arguments hold for matrices B_{n+1}, C_{n+1}, D_{n+1}.

□

Proposition 4.2. *The (i,j)-th entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m,n) binary lattices with the top row of vertices having labels $0\beta_{n-1}(i)0$ and the bottom row of vertices having labels $0\beta_{n-1}(j)0$, both read left to right.*

Proof. We prove by induction. The base case $m = 1$ is Proposition 4.1, as the set L only has the unique $(1,n)$ binary lattice.

We next assume that the (i,j)-th entry of A_n^m satisfies the statement in Proposition 4.2, and show the statement also holds for A_n^{m+1}. To begin, consider the product $A_n^m \cdot A_n$, and choose an integer $k \in [0, 2^{n-1} - 1]$. From the induction hypothesis the value at the (i,k)-th entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m, n) binary lattices with the top row of vertices having labels $0\beta_{n-1}(i)0$ and the bottom row of vertices having labels $0\beta_{n-1}(k)0$. Similarly, the (k, j)-th entry of A_n is the $(1, n)$ binary lattice $\begin{smallmatrix} 0 & \beta_{n-1}(k) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$.

Therefore, the dot product of the i-th row and the j-th column is (i, j)-th value of A_n^{m+1}, which is

$$\sum_{k=0}^{2^{n-1}-1} v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(k) & 0 \end{pmatrix} v\begin{pmatrix} 0 & \beta_{n-1}(k) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix},$$

which gives the desired result for A_n^{m+1}. □

Proposition 4.3. *The number of* (m, n) *mosaics that do not contain a polygon is the* $(0, 0)$ *entry of* A_n^m*.*

Proof. By Proposition 4.2, the $(0, 0)$ entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m, n) binary lattices with the top and bottom rows of vertices having labels $0\beta_{n-1}(0)0$. As A has left-most and right-most columns all labeled 0, all binary lattices counted at $(0, 0)$ are framed, so $L = \mathbb{F}^{(m,n)}$. Substituting the values for v from Definition 3.5 gives the sum from Proposition 3.3, which completes the proof. □

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