Mosaics

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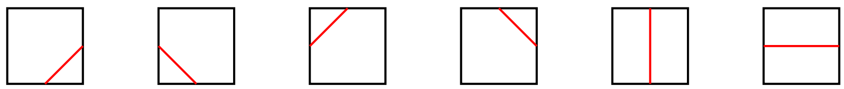
1 Introduction

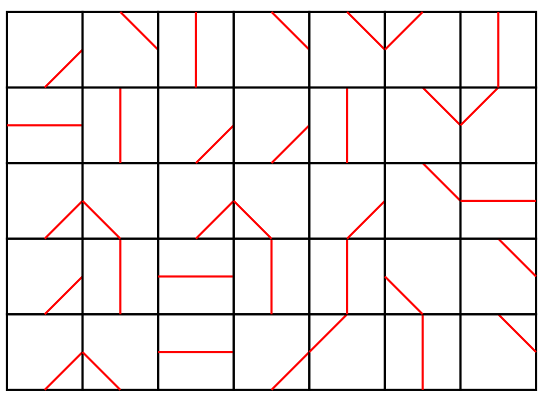
Consider the following 6 unit squares with markings on them.



Call these squares *tiles*.

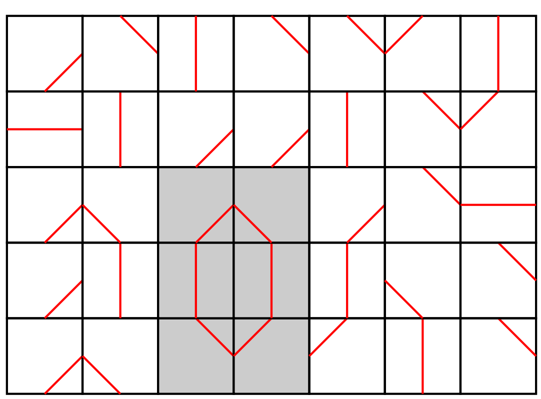
Definition 1.1. An (n, m)-*mosaic* is a rectangular grid made up of tiles.

Example 1.1. An example of a $(7, 5)$-mosaic:

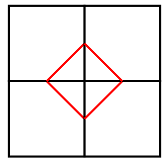


Clearly there are 6^{nm} possible mosaics. Which of these mosaics contain self-avoiding polygons?

Example 1.2. An example of a $(7, 5)$-mosaic with a self-avoiding polygon:



Let $t_{n,m}$ be the number of mosaics that have at least one self avoiding polygon (SAP). From the fact that the smallest SAP is



we have that $t_{n,1} = t_{1,m} = 0$, and $t_{2,2} = 1$. What else can be said?

Theorem 1. *Setting $m = 2$ gives*

$$T_2(x) = \sum_{n \geq 2} t_{n,2} x^n = \frac{x^2}{(1 - 36x)(1 - 37x + 37x^2)}. \tag{1}$$

This can be solved for $n \geq 2$ to give

$$t_{n,2} = 6^{2n} - \frac{1}{\beta - \alpha}((36\beta - 35)\beta^{-n+1} - (36\alpha - 35)\alpha^{-n+1}) \tag{2}$$

where $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{33}{37}}$ and $\beta = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{33}{37}}$.

Proof. We prove that $t_{n,2}$ has

$$t_{n,2} = 36 t_{n-1,2} + \sum_{i=2}^{n} (6^{2(n-i)} - t_{n-i,2}).$$

Split $t_{n,2}$ into S_n and S_n^c. S_n contains the mosaics that have just 1 SAP that contains the left-most two cells. This means S_n^c contains all mosaics that contain multiple SAPs and mosaics that contain only 1 SAP, but that does not contain the two left-most cells.

The subset S_n can be split further by the length of each SAP i.

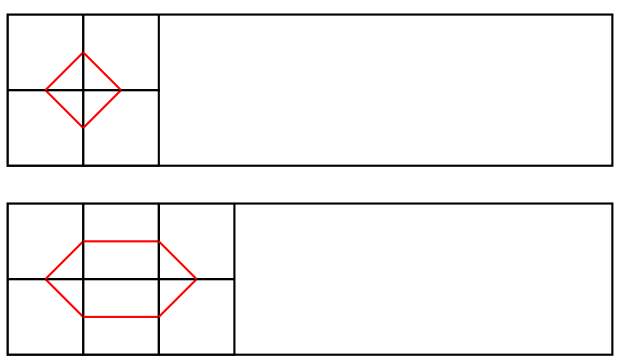


Figure 1: Members of S_n of lengths $i = 2$ and $i = 3$

As S_n counts the number of mosaics that only contain 1 SAP, the blank space in Figure 1 must have no SAPs. The number of mosaics that have no SAPs is $6^{2(n-i)} - t_{n-i,2}$. As a SAP's width can range from 2 to n, we have $|S_n| = \sum_{i=2}^{n} (6^{2(n-i)} - t_{n-i,2})$.

Now consider S_n^c. The mosaics that belong to this set can be represented by the following diagram,

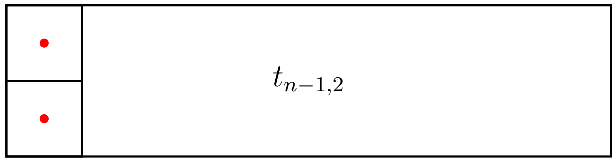


Figure 2: Representation of S_n^c

where the red dot in the left most cells indicate any marking. For this paper, we will refer to a cell that can have any marking as an *open*. We can conclude that $|S_n^c| = 6^2 t_{n-1,2}$. Combining S_n and S_n^c gives the recurrence relation. Standard techniques then give the generating function and formula. $\square$

Theorem 2. *Setting $m = 3$ gives*

$$T_3(x) = \sum_{n\geq 2} t_{n,3}x^n = \frac{(73-414x)x^2}{(1-216x)(1-228x+2699x^2-7758x^3)} \tag{3}$$

Proof. For $t_{n,3}$ we directly compute the generating function $T_3(x) = \sum_{n\geq 2} t_{n,3}x^n$ using the following recurrence relation

$$t_{3,n} = 6^3 t_{n-1,3} + \sum_{i=2}^{n}(6^{3(n-i)} - t_{n-i,3})f_i,$$

where f_i is the number of mosaics in an $i \times 3$ grid that contain just one SAP that has cells in the left-most column. We similarly split $t_{n,3}$ into S_n and S_n^c. Here let S_n be the set that contains the mosaics that have just 1 SAP that has cells in the left-most column. Therefore S_n^c contains all mosaics that contain multiple SAPs and mosaics that contain 1 SAP that does not have cells in the left-most column.

Similarly for the $n = 2$ case, S_n^c can be easily enumerated.

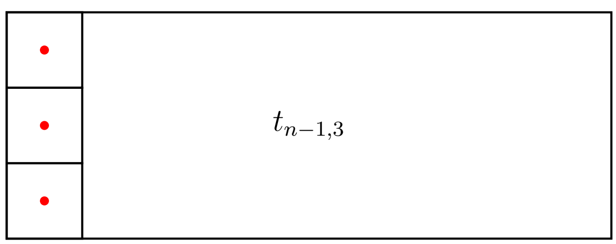
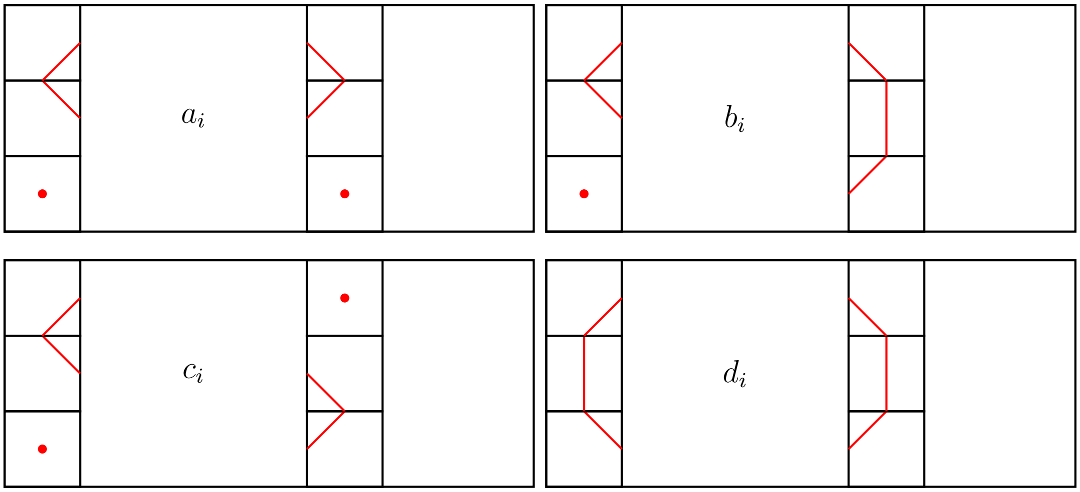


Figure 3: Representation of S_n^c

It is clear to see that $|S_n^c| = 6^3 t_{n-1,3}$, and so

$$\sum_{n\geq 2} |S_n^c|x^n = 6^3 xT(x).$$

To enumerate S_n, as in the $n = 2$ case, the SAP starts in the first column and ends at column i, after which there are no SAPs. This allows us to conclude that $|S_n| = \sum_{i=2}^{n}(6^{3(n-i)} - t_{n-i,3})f_i$,, where f_i is the number of ways the cells to the left of and including column i can contain 1 SAP that includes the left-most column. To find the identity of f_i, we study the 4 cases below.



It is easy to see that $a_2 = 36$ and $a_3 = 216$. As i increases, one can see that the enumeration of a_i is related to smaller values of a_i and b_i, more specifically

$$a_i = 6a_{i-1} + 6^2 b_{i-2}.$$

We find similar relations with the other 3 cases, namely

$$b_n = 6b_{n-1} + 6^2 d_{n-2}$$

where $b_2 = 0$ and $b_3 = 6$

$$c_n = 6c_{n-1} + 6^2 b_{n-2}$$

where $c_2 = 0$ and $c_3 = 0$

$$d_n = 6d_{n-1} + 6^2 b_{n-2}$$

where $d_2 = 1$ and $d_3 = 6$.

Combining these 4 cases, and accounting for the appropriate symmetries, we arrive at

$$f_i = 2a_i + 4b_i + 2c_i + d_i.$$

Solving this series of recurrence relations using generating functions gives

$$F(x) = \sum_{i\geq 2} f_i x^i = \frac{73 - 414x}{1 - 12x + 43x^2}$$

This allows us to write

$$\sum_{n\geq 2} |S_n| x^n = \left(\frac{1}{1 - 6^3 x} - T(x) \right) F(x).$$

Combining these two generating functions and simplifying gives the result.

$\square$

2 Full Solution

TODO writeup, credit Farstar31!

$$M(2) = \begin{bmatrix} 36 & 1 \\ -1 & 1 \end{bmatrix}$$

For $h \geq 2$, then if

$$M(h) = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

then

$$M(h+1) = \begin{bmatrix} 6M_1 & 6M_2 & \frac{1}{6}M_1 & 1M_2 \\ 6M_3 & 6M_4 & 0M_3 & 1M_4 \\ -\frac{1}{6}M_1 & 0M_2 & \frac{1}{6}M_1 & 1M_2 \\ 1M_3 & 1M_4 & -1M_3 & 6M_4 \end{bmatrix}$$

where M_i is a sub-matrix (or possible a scalar) of the block matrix M.