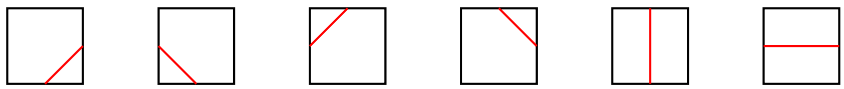
# Mosaics

Jack Hanke

January 29, 2025

## 1 Introduction

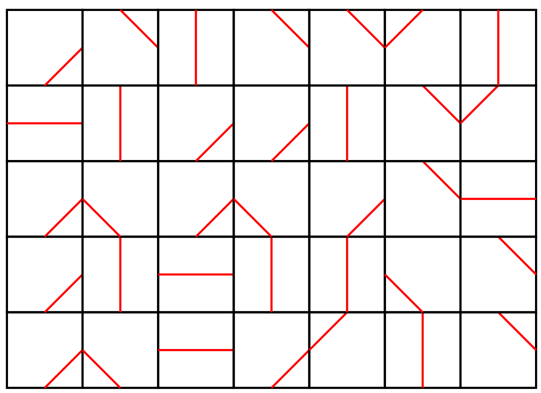
Consider the following 6 unit squares with markings on them.



Call these squares *tiles*.

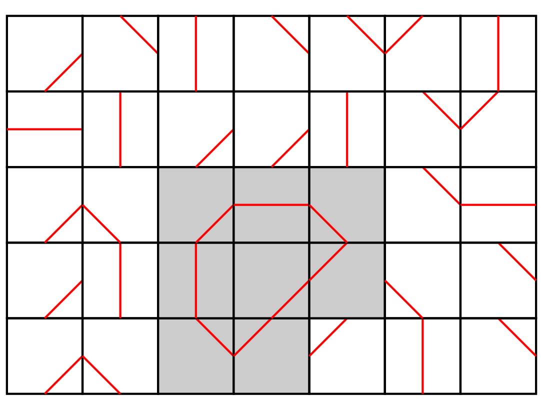
**Definition 1.1.** An $(n, m)$-*mosaic* is a rectangular grid made up of tiles.

**Example 1.1.** An example of a $(7, 5)$-mosaic:

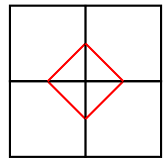


Clearly there are $6^{nm}$ possible mosaics. Which of these mosaics contain self-avoiding polygons?

**Example 1.2.** An example of a $(7, 5)$-mosaic with a self-avoiding polygon, hilighted in gray:



Let $t_{n,m}$ be the number of mosaics that have at least one self avoiding polygon (SAP). From the fact that the smallest SAP is



we have that $t_{n,1} = t_{1,m} = 0$, and $t_{2,2} = 1$. What else can be said?

**Theorem 1.** *TODO finish theorem writeup*

$$M(2) = \begin{bmatrix} 36 & 1 \\ -1 & 1 \end{bmatrix}$$

*For $h \geq 2$, then if*

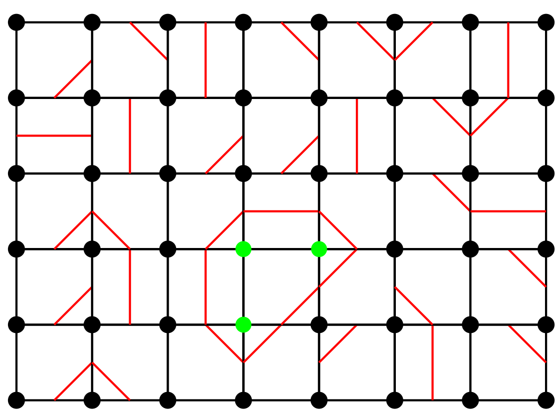
$$M(h) = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

*then*

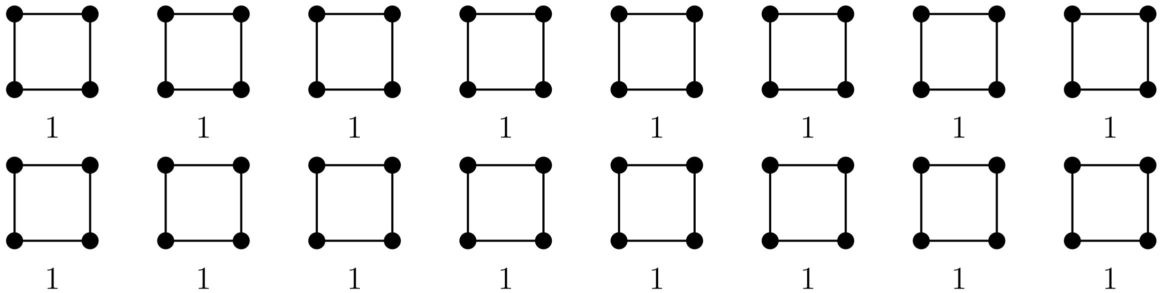
$$M(h+1) = \begin{bmatrix} 6M_1 & 6M_2 & \frac{1}{6}M_1 & 1M_2 \\ 6M_3 & 6M_4 & 0M_3 & 1M_4 \\ -\frac{1}{6}M_1 & 0M_2 & \frac{1}{6}M_1 & 1M_2 \\ 1M_3 & 1M_4 & -1M_3 & 6M_4 \end{bmatrix}$$

*where $M_i$ is a sub-matrix (or possible a scalar) of the block matrix $M$.*
*Then $t_{n,m} =$?*

*Proof.* Begin by labelling the vertices of the rectangular lattice as follows. If the vertex is surrounded by an even number of SAPs, color it black. If the vertex is surrounded by an odd number of SAPs, color it green. Using the diagram from Example 1.2 above, we get the following labelling. Notice that vertices on the boundary of the lattice will always be black.



Considering an individual cell, with the four associated labels. We call the surrounding vertice labels the cell's *parity*. The 16 possible parities, along with their associated weight, are shown in the following diagram.




TODO complete proof writeup

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