Quantum knots and the number of knot mosaics

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Abstract Lomonaco and Kauffman developed a knot mosaic system to introduce a precise and workable definition of a quantum knot system. This definition is intended to represent an actual physical quantum system. A knot (m, n)-mosaic is an $m \times n$ matrix of mosaic tiles (T_0 through T_{10} depicted in the introduction) representing a knot or a link by adjoining properly that is called suitably connected. $D^{(m,n)}$ is the total number of all knot (m, n)-mosaics. This value indicates the dimension of the Hilbert space of these quantum knot system. $D^{(m,n)}$ is already found for $m, n \leq 6$ by the authors. In this paper, we construct an algorithm producing the precise value of $D^{(m,n)}$ for $m, n \geq 2$ that uses recurrence relations of state matrices that turn out to be remarkably efficient to count knot mosaics.

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$$D^{(m,n)} = 2 \, \|(X_{m-2} + O_{m-2})^{n-2}\|$$

where $2^{m-2} \times 2^{m-2}$ matrices X_{m-2} and O_{m-2} are defined by

$$X_{k+1} = \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} = \begin{bmatrix} O_k & X_k \\ X_k & 4 \, O_k \end{bmatrix}$$

for $k = 0, 1, \cdots, m-3$, with 1×1 matrices $X_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $O_0 = \begin{bmatrix} 1 \end{bmatrix}$. Here $\|N\|$ denotes the sum of all entries of a matrix N. For $n = 2$, $(X_{m-2} + O_{m-2})^0$ means the identity matrix of size $2^{m-2} \times 2^{m-2}$.

**Keywords** Quantum knot · Knot mosaic · Enumeration

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1 Introduction

During the last three decades, much of the theory of knots has been applied in quantum physics. One of remarkable discovery in the theory of knots is the Jones polynomial, and it turned out that the explanation of the Jones polynomial has to do with quantum theory [4,5,7,8,11,13,20]. Lomonaco and Kauffman introduced quantum knots to explain how to make quantum information versions of mathematical structures in the series of papers [12,14–16]. They build a knot mosaic system to set the foundation for a quantum knot system, which can be viewed as a blueprint for the construction of an actual physical quantum system. Their definition of quantum knots was based on the planar projections of knots and the Reidemeister moves. They model the topological information in a knot by a state vector in a Hilbert space that is directly constructed from knot mosaics.

This paper is a sequel to the research program on finding the total number of knot mosaics announced in [2,3]. This counting is very important because the total number of knot mosaics is indeed the dimension of the Hilbert space of these quantum knot mosaics. In [3], a partition matrix argument was developed by the authors to count small knot mosaics. In this sequel, we generalize this argument to give an algorithm for counting all knot mosaics that uses recurrence relations of matrices that are called state matrices. This algorithm using state matrices turns out to be remarkably efficient to count knot mosaics.

Throughout this paper, the term "knot" means either a knot or a link. We begin by explaining the basic notion of knot mosaics. Let $\mathbb{T}$ denote the set of the following eleven symbols that are called *mosaic tiles*;


$\quad T_0 \qquad T_1 \qquad T_2 \qquad T_3 \qquad T_4 \qquad T_5 \qquad T_6 \qquad T_7 \qquad T_8 \qquad T_9 \qquad T_{10}$
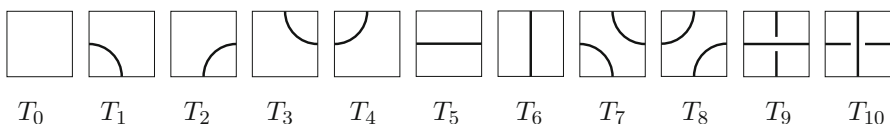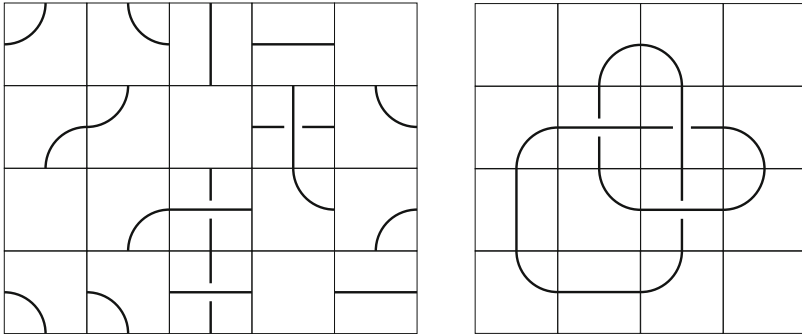
Fig. 1 Examples of mosaics

For positive integers m and n, an (m, n)-*mosaic* is an $m \times n$ matrix $M = (M_{ij})$ of mosaic tiles. We denote the set of all (m, n)-mosaics by $\mathbb{M}^{(m,n)}$. Note that $\mathbb{M}^{(m,n)}$ has 11^{mn} elements. This definition is an extended version of the definition of an *n-mosaic* as an $n \times n$ matrix of mosaic tiles in [14].

A *connection point* of a mosaic tile is defined as the midpoint of a mosaic tile edge that is also the endpoint of a curve drawn on the tile. Then each tile has zero, two or four connection points as follows;
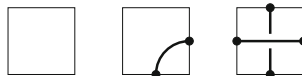
We say that two tiles in a mosaic are *contiguous* if they lie immediately next to each other in either the same row or the same column. A mosaic is said to be *suitably connected* if any pair of contiguous mosaic tiles have or do not have connection points simultaneously on their common edge. A *knot* (m, n)-*mosaic* is a suitably connected (m, n)-mosaic whose boundary edges do not have connection points. Then this knot (m, n)-mosaic represents a specific knot. $\mathbb{K}^{(m,n)}$ denotes the subset of $\mathbb{M}^{(m,n)}$ of all knot (m, n)-mosaics. A knot (n, n)-mosaic is simply specified by a *knot n-mosaic*. The examples of mosaics in Fig. 1 are a non-knot $(4, 5)$-mosaic and the trefoil knot 4-mosaic. Also the reader finds a complete list of all 22 knot 3-mosaics in Appendix A in [14].

As an analog to the planar isotopy moves and the Reidemeister moves for standard knot diagrams, Lomonaco and Kauffman [14] created the 11 mosaic planar isotopy moves and the mosaic Reidemeister moves on knot mosaics. They conjectured that for any two tame knots K_1 and K_2, and their arbitrary chosen mosaic representatives M_1 and M_2, respectively, K_1 and K_2 are of the same knot type if and only if M_1 and M_2 are of the same knot mosaic type, which is defined in [14]. This means that knot mosaic type is a complete invariant of tame knots. Kuriya and Shehab [9] verified Lomonaco–Kauffman conjecture.

Lomonaco and Kauffman also proposed several questions related to knot mosaics. $D^{(m,n)}$ denotes the total number of elements of $\mathbb{K}^{(m,n)}$. In the series of recent papers [2,3], the authors found some results about $D^{(m,n)}$. They showed that $D^{(1,n)} = 1$, $D^{(2,n)} = 2^{n-1}$ and $D^{(3,n)} = \frac{2}{5}(9 \times 6^{n-2} + 1)$ for a positive integer n and found a

table of the precise values of $D^{(m,n)}$ for $m, n = 4, 5, 6$ (note that $D^{(m,n)} = D^{(n,m)}$);

$D^{(m,n)}$	$n = 4$	$n = 5$	$n = 6$
$m = 4$	2,594	54,226	1,144,526
$m = 5$		4,183,954	331,745,962
$m = 6$			101,393,411,126

Furthermore a lower and an upper bounds on $D^{(m,n)}$ for $m, n \geq 3$ were established;

$$2^{(m-3)(n-3)} \leq \frac{275}{2(9 \times 6^{m-2} + 1)(9 \times 6^{n-2} + 1)} \cdot D^{(m,n)} \leq 4.4^{(m-3)(n-3)}.$$

In this paper, we construct an algorithm producing the precise value of $D^{(m,n)}$ in general.

Theorem 1 *For integers $m, n \geq 2$, the total number $D^{(m,n)}$ of all knot (m, n)-mosaics is the following;*

$$D^{(m,n)} = 2 \, \|(X_{m-2} + O_{m-2})^{n-2}\|$$

where $2^{m-2} \times 2^{m-2}$ matrices X_{m-2} and O_{m-2} are defined by

$$X_{k+1} = \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} = \begin{bmatrix} O_k & X_k \\ X_k & 4 O_k \end{bmatrix}$$

for $k = 0, 1, \ldots, m - 3$, with 1×1 matrices $X_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $O_0 = \begin{bmatrix} 1 \end{bmatrix}$.

Here $\|N\|$ denotes the sum of all entries of a matrix N. For $n = 2$, $(X_{m-2}+O_{m-2})^0$ means the identity matrix of size $2^{m-2} \times 2^{m-2}$. We have calculated $D^{(n,n)}$ for $m = n = 1, 2, \ldots, 13$ as given in the following table.

n	$D^{(n,n)}$
1	1
2	2
3	22
4	2594
5	4183954
6	101393411126
7	38572794946976688
8	2348550528709544480828416
9	2305409936220039965604617545 3184
10	3656462755944109221731040977 7161751756800
11	9372731425713264236416769564 68995920021677311787008
12	3882160215193708062213464345 1310239313398559084431296175 9051776
13	2597619491722287317211028202 2623847240168723042091634469 5982604770638561 2800

Indeed $D^{(n,n)}$ grows in a quadratic exponential rate. The growth constant $\lim_{n\to\infty}$ $(D^{(n,n)})^{\frac{1}{n^2}}$ exists and lies between 4 and $\frac{5+\sqrt{13}}{2}$ (≈ 4.303). This result was proved recently by Oh [17]. Other issues for knot mosaics involve considering mosaic representations on the torus rather than in the plane. A knot toroidal (m, n)-mosaic is a suitably connected (m, n)-mosaic constructed on a torus by identifying their boundaries properly. Recently the authors and Yeon [19] improved this state matrix algorithm to find the total number of knot toroidal (m, n)-mosaics for positive co-prime integers m and n. Another result about knot toroidal mosaics is found in [1]. Also a result about mosaic representations of graphs with at most 4 valencies is considered in [18]. Mirror-curve representations of knots that are similar to mosaic representations were treated in [6].

Another interesting question related to knot mosaics is the mosaic number of knots. Define the *mosaic number* $m(K)$ of a knot K as the smallest integer n for which K is representable as a knot n-mosaic. For example, the mosaic number of the trefoil is 4 as illustrated in Fig. 1. One question is the following: *Is this mosaic number related to the crossing number of a knot?* The authors [10] found an upper bound on the mosaic number as follows; If K is a nontrivial knot or a non-split link except the Hopf link, then $m(K) \leq c(K) + 1$. Moreover if K is prime and non-alternating except the 6_3^3 link, then $m(K) \leq c(K) - 1$. Note that the mosaic numbers of the Hopf link and the 6_3^3 link are 4 and 6, respectively.

2 Suitably connected mosaics and state matrices

Let p and q be positive integers. $\mathbb{S}^{(p,q)}$ denotes the set of all suitably connected (p, q)-mosaics that possibly have connection points on their boundary edges. A suitably connected (5,3)-mosaic is depicted in Fig. 2 as an example. This is a submosaic of a knot mosaic in Lomonaco and Kauffman's definition.
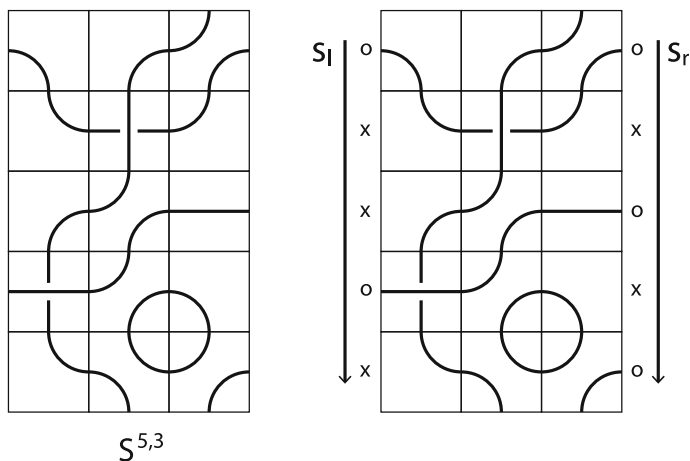
Fig. 2 Suitably connected (5,3)-mosaic $S^{5,3}$

For simplicity of exposition, a mosaic tile is called l-, r-, t- and b-cp if it has a connection point on its left, right, top and bottom, respectively. Sometimes we use two or more letters such as lt-cp for the case of both l-cp and t-cp. Also we use the sign $\hat{}$ for negation such as $\hat{l}$-cp means not l-cp, $\hat{l}\hat{t}$-cp means both $\hat{l}$-cp and $\hat{t}$-cp, and $\widehat{lt}$-cp (which is differ from $\hat{l}\hat{t}$-cp) means the negation of lt-cp, i.e., $\hat{l}t$-, $l\hat{t}$- or $\hat{l}\hat{t}$-cp.

Choice rule. Each M_{ij} in a suitably connected mosaic has four choices of mosaic tiles as T_7, T_8, T_9 and T_{10} if it is lrb-cp (so automatically t-cp), and unique choice if it is $\widehat{lrb}$-cp.

For a suitably connected (p, q)-mosaic $S^{p,q} = (M_{ij})$ where $i = 1, \ldots, p$ and $j = 1, \ldots, q$, an l-*state* of $S^{p,q}$ indicates the presence of connection points of p mosaic tile edges on the leftmost boundary, and we denote that $s_l(S^{p,q}) = s_l(M_{11})s_l(M_{21})\ldots s_l(M_{p1})$ where each $s_l(M_{i1})$ denotes "x" if M_{i1} is $\hat{l}$-cp and "o" if M_{i1} is l-cp. Similarly we define an r-*state* of $S^{p,q}$, which indicates the presence of connection points of p mosaic tile edges on the rightmost boundary. For $(5, 3)$-mosaic $S^{5,3}$ drawn in the figure,

$$s_l(S^{5,3}) = \text{oxxox and } s_r(S^{5,3}) = \text{oxoxo.}$$

Note that $\mathbb{S}^{(p,q)}$ has possibly 2^p kinds of l-states and also 2^p kinds of r-states. We arrange the elements of the set of all states in the backward of lexicographical order such as xxx, oxx, xox, oox, xxo, oxo, xoo and ooo for $p = 3$.

Now we are ready to define a state matrix that turns out to be remarkably efficient to count the number of suitably connected mosaics. A *state matrix* for $\mathbb{S}^{(p,q)}$ is a $2^p \times 2^p$ matrix $N^{(p,q)} = (N_{ij})$ where N_{ij} is the number of all suitably connected (p, q)-mosaics that have the i-th l-state and the j-th r-state in the set of 2^p states of the order arranged above.

Furthermore, we split the state matrix $N^{(p,1)}$, only when $q = 1$, into two $2^p \times 2^p$ matrices, namely X_p and O_p as follows. Each (i, j)-entry of X_p (or O_p) indicates the number of all suitably connected $(p, 1)$-mosaics that have the i-th l-state and the j-th r-state, and additionally whose bottom mosaic tiles are $\hat{b}$-cp (or b-cp, respectively). Obviously $N^{(p,1)} = X_p + O_p$.

Direct construction of the state matrix $N^{(1,1)}$. From eleven mosaic tiles in Fig. 3, we get the following state matrices;

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad O_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } N^{(1,1)} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Each of four suitably connected $(1, 1)$-mosaics on the first line in the figure represents each entry 1 of X_1, each of left three mosaics on the second line represents each entry 1 of O_1, and the remaining four mosaics represent $(2, 2)$-entry 4 of O_1. Note that the sum of all entries of $N^{(1,1)}$ is the total number of elements of $\mathbb{S}^{(1,1)}$, which is obviously 11.

Note that each element of $\mathbb{S}^{(1,1)}$ can be extended to exactly two knot 3-mosaics because we have two choices of adjoining eight mosaic tiles surrounding it, satisfying that all mosaic tiles are suitably connected. This implies $D^{(3,3)} = 22$.
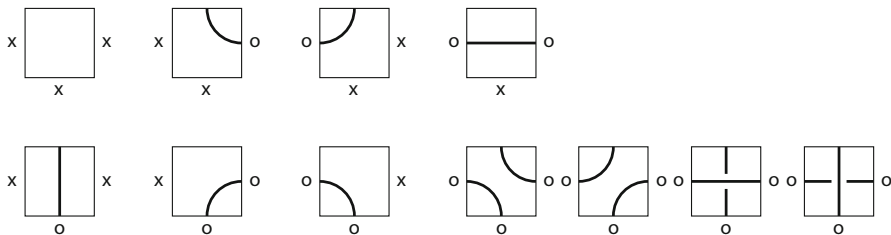
Fig. 3 Finding the state matrix $N^{(1,1)}$
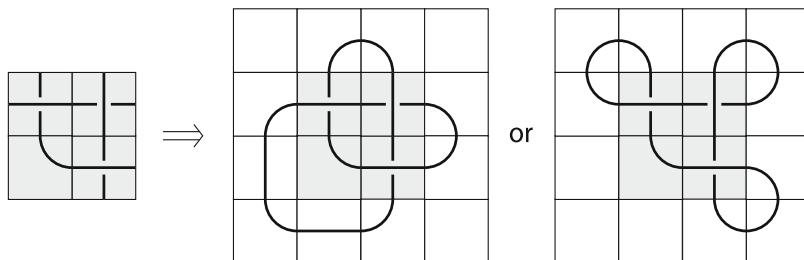
Fig. 4 The twofold rule

We can easily extend this argument to each element of $\mathbb{S}^{(m-2,n-2)}$ by adjoining $2m + 2n - 4$ proper mosaic tiles surrounding it. Since each mosaic tile has even number of connection points, a suitably connected $(m - 2, n - 2)$-mosaic has exactly even number of connection points on its boundary. To make a knot (m, n)-mosaic, all these connection points must be connected pairwise via mutually disjoint arcs when we adjoin new mosaic tiles. There are exactly two ways to do as illustrated in Fig. 4. Note that if it has no connection point on the boundary, then we may add empty tiles or encircle the mosaic with a new circle.

Twofold rule. A suitably connected $(m - 2, n - 2)$-mosaic can be extended to exactly two different knot (m, n)-mosaics by augmenting the four sides with a row/column of tiles.

3 State matrix $N^{(p,1)}$

In this section, we establish the state matrix $N^{(p,1)}$ for $\mathbb{S}^{(p,1)}$. For a $2^{k+1} \times 2^{k+1}$ matrix $N = (N_{ij})$, the 11-quadrant (similarly 12-, 21- or 22-quadrant) of N denotes the $2^k \times 2^k$ submatrix (N_{ij}) where $1 \leq i, j \leq 2^k$ ($1 \leq i \leq 2^k$ and $2^k + 1 \leq j \leq 2^{k+1}$, $2^k + 1 \leq i \leq 2^{k+1}$ and $1 \leq j \leq 2^k$, or $2^k + 1 \leq i, j \leq 2^{k+1}$, respectively).

Proposition 2 *For the set $\mathbb{S}^{(p,1)}$ of all suitably connected $(p, 1)$-mosaics, the associated state matrix $N^{(p,1)}$ can be obtained as follows;*

$$N^{(p,1)} = X_p + O_p$$
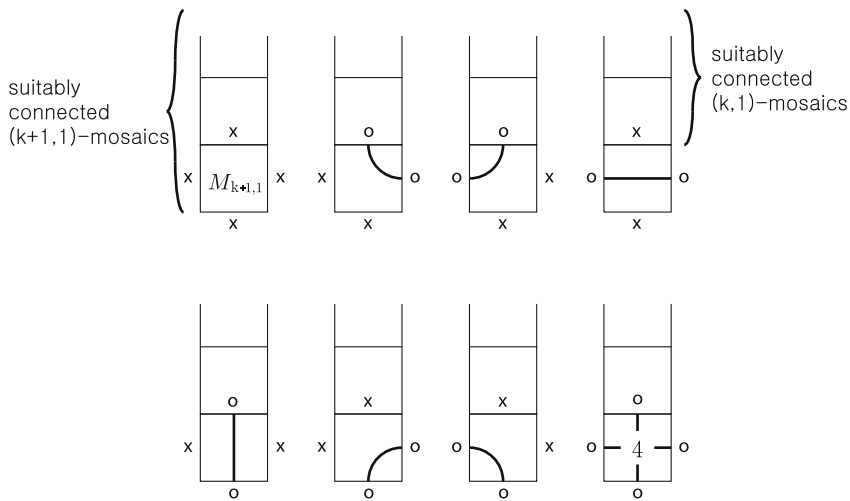
Fig. 5 Suitably connected $(k + 1, 1)$-mosaics

where matrices X_p and O_p are defined by

$$X_{k+1} = \begin{bmatrix} X_k & O_k \\ O_k & X_k \end{bmatrix} \text{ and } O_{k+1} = \begin{bmatrix} O_k & X_k \\ X_k & 4\,O_k \end{bmatrix}$$

for $k = 1, \ldots, p - 1$, starting with $X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $O_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$.

Proof The identity $N^{(p,1)} = X_p + O_p$ follows immediately from the definition of X_p and O_p. We will use the induction on p. Matrices X_1 and O_1 are already found in the previous section.

Assume that matrices X_k and O_k satisfy the statement. Let $S^{k+1,1} = (M_{i,1})$ be a suitably connected $(k + 1, 1)$-mosaic of $\mathbb{S}^{(k+1,1)}$. Consider the bottom mosaic tile $M_{k+1,1}$. If it is $\hat{l}\hat{r}\hat{b}$-cp, for example, then $S^{k+1,1}$ should be counted in an entry of the 11-quadrant of X_{k+1}. This is because of the backwardness of lexicographical order of 2^{k+1} states. In this case, $M_{k+1,1}$ has unique choice T_0 of mosaic tiles because of Choice rule. Let $S^{k,1}$ be the associated suitably connected $(k, 1)$-mosaic obtained from $S^{k+1,1}$ by ignoring $M_{k+1,1}$. Then the bottom mosaic tile of $S^{k,1}$ must be $\hat{b}$-cp, so the associated state matrix for all possible $S^{k,1}$ is X_k. The eight figures in Fig. 5 and the table below explain all eight cases according to the presence of connection points of $M_{k+1,1}$. Notice that only when $M_{k+1,1}$ is lrb-cp, it has four choices of mosaic tiles T_7, T_8, T_9 and T_{10}. Thus the associated submatrix must be $4\,O_k$ instead of O_k. Now we complete the proof. □

Fig. 6 Finding $(10, 11)$-entry of O_4
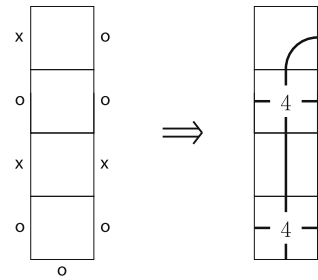
	Quadrants	Associated $M_{k+1,1}$		Submatrix
X_{k+1}	11-quadrant	$\hat{l}\hat{r}\hat{b}$-cp	T_0	X_k
	12-quadrant	$\hat{l}r\hat{b}$-cp	T_3	O_k
	21-quadrant	$l\hat{r}\hat{b}$-cp	T_4	O_k
	22-quadrant	$lr\hat{b}$-cp	T_5	X_k
O_{k+1}	11-quadrant	$\hat{l}\hat{r}b$-cp	T_6	O_k
	12-quadrant	$\hat{l}rb$-cp	T_2	X_k
	21-quadrant	$l\hat{r}b$-cp	T_1	X_k
	22-quadrant	lrb-cp	T_7, T_8, T_9, T_{10}	$4\,O_k$

For example, let us try to find $(10, 11)$-entry of O_4. This entry can be written as $(2^3 + 0 \times 2^2 + 2^1 + 0 \times 2^0, 2^3 + 0 \times 2^2 + 2^1 + 2^0)$-entry, and so counts the total number of all suitably connected $(4, 1)$-mosaics with $xoxo$ l-state and $ooxo$ r-state, and additionally whose bottom mosaic tiles are b-cp as shown in Fig. 6. In this case, M_{41} is lrb-cp, so has 4 choices of mosaic tiles. Thus M_{31} is $\hat{l}\hat{r}b$-cp, so it must be unique choice T_6. Similarly M_{21} and M_{11} have 4 choices and unique choice, respectively. Thus the entry is 4^2.

4 State matrix $N^{(p,q)}$ and the proof of Theorem 1

In this section, we find the state matrix $N^{(p,q)}$ for $\mathbb{S}^{(p,q)}$ and prove Theorem 1.

Proposition 3 *For the set $\mathbb{S}^{(p,q)}$ of all suitably connected (p, q)-mosaics, the associated state matrix $N^{(p,q)}$ is the following;*

$$N^{(p,q)} = (N^{(p,1)})^q.$$

Proof We use the induction on q. Assume that $N^{(p,k)} = (N^{(p,1)})^k$. Let $S^{p,k+1}$ be a suitably connected $(p, k + 1)$-mosaic in $\mathbb{S}^{(p,k+1)}$. Also let $S^{p,k}$ and $S^{p,1}$ be the suitably connected (p, k)-mosaic obtained by ignoring the rightmost column of $S^{p,k+1}$ and the suitably connected $(p, 1)$-mosaic which is just the rightmost column of $S^{p,k+1}$, respectively. Then r-state of $S^{p,k}$ is the same as l-state of $S^{p,1}$ as shown in Fig. 7. Remark that $N^{(p,k+1)} = (N_{ij}^{(k+1)})$ is the state matrix for $\mathbb{S}^{(p,k+1)}$ where each entry
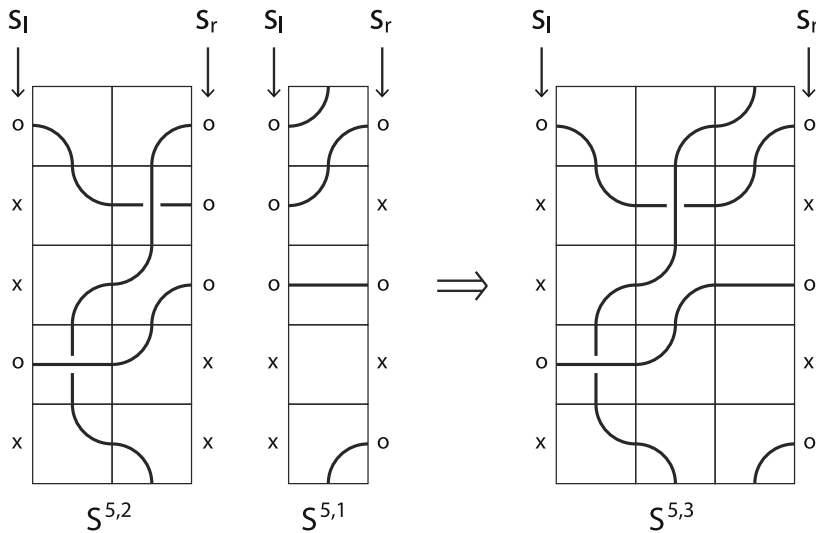
Fig. 7 Adjoining two suitably connected mosaics

$N_{ij}^{(k+1)}$ counts the number of all suitably connected $(p, k+1)$-mosaics that have the i-th l-state and the j-th r-state in the set of 2^p states. Also consider the state matrices $N^{(p,k)} = (N_{is}^{(k)})$ and $N^{(p,1)} = (N_{sj}^{(1)})$ defined similarly. Among these suitably connected $(p, k+1)$-mosaics counted in each entry $N_{ij}^{(k+1)}$, the number of all mosaics whose r-state of the k-th column (or equally l-state of the $(k+1)$-th column) is the s-th state in the set of 2^p states is the product of $N_{is}^{(k)}$ and $N_{sj}^{(1)}$. Since all 2^p states can be appeared as states of connection points where $S^{p,k}$ and $S^{p,1}$ meet, we get

$$N_{ij}^{(k+1)} = \sum_{s=1}^{2^p} N_{is}^{(k)} N_{sj}^{(1)}.$$

This implies that

$$N^{(p,k+1)} = N^{(p,k)} N^{(p,1)} = (N^{(p,1)})^{k+1}.$$

$\square$

Now we are ready to prove the main theorem.

Proof of Theorem 1 First assume that m and n are any integers at least 3. Consider the set $\mathbb{S}^{(m-2,n-2)}$ of all suitably connected $(m-2, n-2)$-mosaics and the associated state matrix $N^{(m-2,n-2)}$. By the definition of the state matrix, all rows represent all 2^{m-2} l-states and all columns represent all 2^{m-2} r-states of mosaics of $\mathbb{S}^{(m-2,n-2)}$. And each entry of the matrix counts the number of all suitably connected $(m-2, n-2)$-mosaics having specific l-state and r-state. Thus the total number of elements of $\mathbb{S}^{(m-2,n-2)}$ is the sum of all entries of the state matrix, which is $\|N^{(m-2,n-2)}\|$.

Each suitably connected mosaic in $\mathbb{S}^{(m-2,n-2)}$ can be extended to exactly two knot (m,n)-mosaics by Twofold rule. Thus the total number of all knot (m,n)-mosaics $D^{(m,n)}$ is twice of $\|N^{(m-2,n-2)}\|$. This fact combined with Proposition 2 and 3 completes the proof except for the case that m or n is 2.

For the case of $m = 2$, we denote two 1×1 matrices $X_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $O_0 = \begin{bmatrix} 1 \end{bmatrix}$. Then the same matrices X_1 and O_1 are obtained from the recurrence relations in Proposition 2, and also $D^{(2,n)} = 2\,\|(X_0 + O_0)^{n-2}\| = 2^{n-1}$, which is already known.

For the case of $n = 2$, $D^{(m,2)} = 2\,\|(X_{m-2} + O_{m-2})^0\| = 2^{m-1}$ because $(X_{m-2} + O_{m-2})^0$ is the $2^{m-2} \times 2^{m-2}$ identity matrix. $\qquad\square$

References

1. Carlisle, M.J., Laufer, M.S.: On upper bounds for toroidal mosaic numbers. Quantum Inf. Process. **12**, 2935–2945 (2013)
2. Hong, K., Lee, H., Lee, H.J., Oh, S.: Upper bound on the total number of knot n- mosaics. J. Knot Theory Ramif. (in the press)
3. Hong, K., Lee, H., Lee, H.J., Oh, S.: Small knot mosaics and partition matrices. J. Phys. A: Math. Theor. **47**, 435201 (2014)
4. Jones, V.: A polynomial invariant for links via von Neumann algebras. Bull. Am. Math. Soc. **129**, 103–112 (1985)
5. Jones, V.: Hecke algebra representations of braid groups and link polynomials. Ann. Math. **126**, 335–338 (1987)
6. Jablan, S., Radovic, L., Sazdanovic, R., Zekovic, A.: Mirror-curves and knot mosaics. Comput. Math. Appl. **64**, 527–543 (2012)
7. Kauffman, L.: Knots and Physics, 3rd edn. World Scientific, Singapore (2001)
8. Kauffman, L.: Quantum computing and the Jones polynomial, in quantum computation and information. AMS CONM. **305**, 101–137 (2002)
9. Kuriya, T., Shehab, O.: The Lomonaco–Kauffman conjecture. J. Knot Theory Ramif. **23**, 1450003 (2014)
10. Lee, H.J., Hong, K., Lee, H., Oh, S.: Mosaic number of knots. J. Knot Theory Ramif. (in the press)
11. Lomonaco, S.: Quantum computation. Proc. Symp. Appl. Math. **58**, 358 (2002)
12. Lomonaco, S., Kauffman, L.: Quantum knots, in quantum information and computation II. Proc. SPIE. **5436**, 268–284 (2004)
13. Lomonaco, S., Kauffman, L.: A 3-stranded quantum algorithm for the Jones polynomial. Proc. SPIE. **6573**, 1–13 (2007)
14. Lomonaco, S., Kauffman, L.: Quantum knots and mosaics. Quantum Inf. Process. **7**, 85–115 (2008)
15. Lomonaco, S., Kauffman, L.: Quantum knots and lattices, or a blueprint for quantum systems that do rope tricks. Proc. Symp. Appl. Math. **68**, 209–276 (2010)
16. Lomonaco, S., Kauffman, L.: Quantizing knots and beyond, in quantum information and computation IX. Proc. SPIE. **8057**, 1–14 (2011)
17. Oh, S.: Quantum knot mosaics and the growth constant (Preprint)
18. Oh, S., Hong, K.: Enumeration of graph mosaics (Preprint)
19. Oh, S., Hong, K., Lee, H., Lee, H.J., Yeon, M.J.: Quantum knots and toroidal mosaics (Preprint)
20. Shor, P., Jordan, S.: Estimating Jones polynomials is a complete problem for one clean qubit. Quantum Inf. Comput. **8**, 681–714 (2008)