Enumeration of Messy Polygon Mosaics

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Abstract

Hong and Oh introduced a model for multiple ring polymers in physics in which an $m \times n$ retangular lattice is constructed from a selection of 7 distinct tiles. These lattices are called *mosaics*. The authors provide bounds on a subset of these mosaics that both contain polygons and have all other tiles that are not part of a polygon set to the blank tile. We introduce and enumerate mosaics with the relaxed property of containing at least one polygon, which we call messy polygon mosaics.

1 Introduction

Imagine you are tasked with tiling a rectangular bathroom floor that is m units by n units, blindfolded. At your disposal is an unending supply of 7 distinct types of tiles. These tiles, diagrammed in Figure 1, are composed of unit squares with dotted lines connecting 2 sides at their midpoints, as well as the “blank” tile T_0.

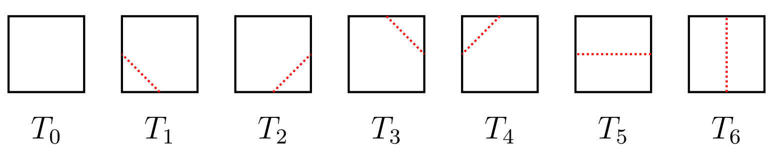
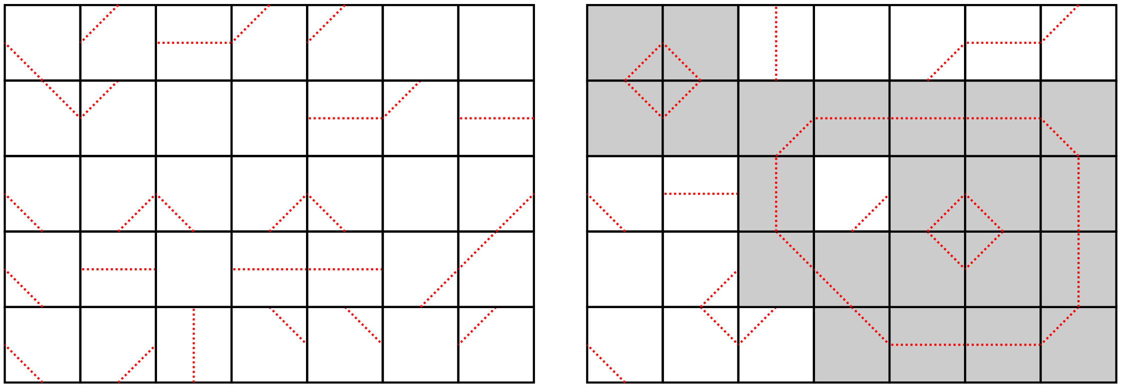


Figure 1: The tile set $\mathbb{T}$

We denote the set of tiles $\mathbb{T} = \{T_0, \ldots, T_6\}$. The task is complete once you place mn randomly selected tiles from $\mathbb{T}$ to cover the floor, after which you remove your blindfold. We call a fully tiled $m \times n$ floor an (m, n) *mosaic*.

If $m = 5$ and $n = 7$, you may have constructed the mosaic in Figure 2a. You may have also constructed the mosaic in Figure 2b. Notice that in the mosaic in Figure 2b, the red dotted lines form multiple polygons[1], which we highlight the corresponding tiles in gray.

[1]Polygons are more commonly referred to as “self-avoiding polygons” in the literature to highlight their connection with self-avoiding walks.



(a) A mosaic (b) A messy polygon mosaic

Figure 2: Examples of mosaics of size $(5,7)$ made of tiles in $\mathbb{T}$

Definition 1.1. An (m,n) *messy polygon mosaic* is an (m,n) mosaic that contains at least one polygon.

What is the probability of constructing a messy polygon mosaic, given the dimensions of the bathroom m,n? As there are $|\mathbb{T}|^{mn} = 7^{mn}$ total mosaics, we focus on the total number of messy polygon mosaics. In fact, it turns out to be simpler to enumerate the number of mosaics that *do not* contain a polygon. Therefore, let $\mathbb{P}^{(m,n)}$ be the subset of (m,n) mosaics that do not contain a polygon. Clearly the number of (m,n) messy polygon mosaics is then $7^{mn} - \left|\mathbb{P}^{(m,n)}\right|$.

From the fact that the smallest polygon is made of 4 tiles (appearing twice in Figure 2b), we can conclude that $\left|\mathbb{P}^{(n,1)}\right| = 7^n$, and $\left|\mathbb{P}^{(2,2)}\right| = 7^4 - 1$. For $m,n \geq 2$, we first define the following matrices.

Definition 1.2. For integers $k \geq 1$ let A_k, B_k, C_k, D_k be $2^{k-1} \times 2^{k-1}$ matrices with integer entries, where $A_1 = \begin{pmatrix}7\end{pmatrix}, B_1 = \begin{pmatrix}-1\end{pmatrix}, C_1 = \begin{pmatrix}1\end{pmatrix}, D_1 = \begin{pmatrix}1\end{pmatrix}$ and

$$A_{k+1} = \begin{pmatrix} 7A_k & B_k \\ C_k & D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} -A_k & B_k \\ 0C_k & D_k \end{pmatrix}$$
$$C_{k+1} = \begin{pmatrix} A_k & 0B_k \\ C_k & D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} A_k & -B_k \\ C_k & 7D_k \end{pmatrix}.$$

Throughout this work, we index elements in matrices, mosaics, and later binary lattices with a pair of coordinates. The first coordinate is the row index, counted top to bottom, and the second coordinate is the column index, counted left to right, both beginning at 0.

Here is our main result.

Theorem 1.1. *The number of (m,n) mosaics that do not contain a polygon $\left|\mathbb{P}^{(m,n)}\right|$ is the $(0,0)$ entry of A_n^m.*

2 Related Work

Hong and Oh [**Hong2018**] studied a similar question in which they construct mosaics from $\mathbb{T}$, but were interested in the number of polygon mosaics.

Definition 2.1. An (m, n) *polygon mosaic* is an (m, n) mosaic that contains at least one polygon and every tile that is not part of a polygon is T_0.

Clearly, all polygon mosaics are messy polygon mosaics. The sequence A181245 on the OEIS [**oeis**] is the array of 1+ the number of (m, n) polygon mosaics. The authors in [**Hong2018**] provide bounds for the number of polygon mosaics.

Theorem 2.1 ([**Hong2018**])**.** *The number of (m, n) polygon mosaics for $m, n \geq 3$ is bounded between $2^{m+n-3}\left(\frac{17}{10}\right)^{(m-2)(n-2)}$ and $2^{m+n-3}\left(\frac{31}{16}\right)^{(m-2)(n-2)}$.*

In related work, Lomonaco and Kauffman [**Lomonaco08**] introduced mosaics constructed from a tile set of 11 distinct tiles, of which $\mathbb{T}$ is a subset. The authors were interested in a subset of mosaics which they call *knot mosaics*. Oh et al. [**Oh2014**] enumerated the number of knot mosaics.

Theorem 2.2 ([**Oh2014**])**.** *The number of (m, n) knot mosaics for $m, n \geq 2$ is $2\,\|(X_{m-2} + O_{m-2})^{n-2}\|$, where $X_0 = O_0 = \begin{bmatrix}1\end{bmatrix}$ and X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined as*

$$X_{k+1} = \begin{pmatrix} X_k & O_k \\ O_k & X_k \end{pmatrix} \text{ and } O_{k+1} = \begin{pmatrix} O_k & X_k \\ X_k & 4O_k \end{pmatrix},$$

for $k = 0, 1, \ldots, m-3$. Here $\|N\|$ denotes the sum of elements of matrix N.

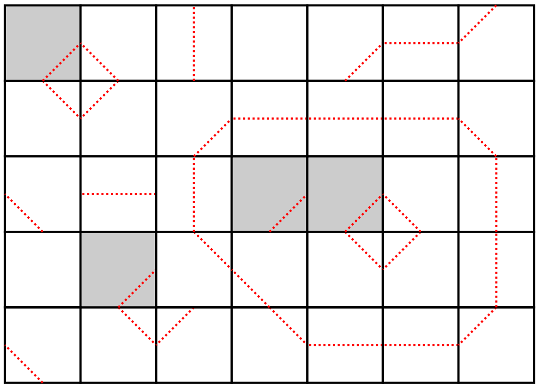
Oh and colleagues go beyond enumeration by bounding the growth rate of knot mosaics [**Oh2016**, **Oh2019**, **Choi2024**], and Oh further adapts the matrix recursion method to solve problems in monomer and dimer tilings [**Oh2018Aztec**, **Oh2019tiling**]. Related ideas were independently used to enumerate the number of rectangular partitions of a rectangle in [**oeispaper**] for the sequence A182275 on the OEIS [**oeis**].

Our work can be seen as an extension to Hong and Oh [**Hong2018**] and further generalizing the techniques in Oh et al. [**Oh2014**] to explore a new direction in mosaic enumeration.

3 Preliminaries

We begin by defining a map that takes an (m, n) mosaic and gives an (m, n) binary lattice. An (m, n) binary lattice is a rectangular lattice of $m + 1$ by $n + 1$ vertices, with each vertex labeled 0 or 1. We also define a *framed binary lattice* to be a binary lattice in which the boundary vertices are labeled 0. An example of a $(5, 7)$ framed binary lattice is shown on the right of Figure 3.

Definition 3.1. Let f be the map that takes an (m, n) mosaic and labels each vertex with the following rule. If the vertex is surrounded by the red dotted lines of an even number of polygons (including 0 polygons), label it 0. If the vertex is surrounded by the red dotted lines of an odd number of polygons, label it 1. Removing the red dotted lines from the tiles gives the framed binary lattice.



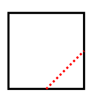
$$\xrightarrow{f} \begin{array}{ccccccccccccccc} 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & | \\ 0 & - & 1 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & | \\ 0 & - & 0 & - & 0 & - & 1 & - & 1 & - & 1 & - & 1 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & | \\ 0 & - & 0 & - & 0 & - & 1 & - & 1 & - & 0 & - & 1 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & | \\ 0 & - & 0 & - & 0 & - & 0 & - & 1 & - & 1 & - & 1 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & | \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \end{array}$$

Figure 3: f applied to the mosaic in Figure 2b, resulting in a binary lattice. We highlight each possible way a T_2 tile can map to a cell by shading a representative tile in gray.

To enumerate $|\mathbb{P}^{(m,n)}|$, it will be useful to consider how f maps the tiles in $\mathbb{T}$ to the cells in a binary lattice.

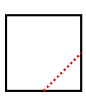
Definition 3.2. Let a *cell* be a $(1,1)$ binary lattice.

Example 3.1. Applying f to the mosaic in Figure 3 results in the T_2 tile at position $(0,0)$ mapping to the cell at position $(0,0)$, diagrammed below.

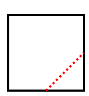


$$\rightarrow \begin{array}{ccc} 0 & - & 0 \\ | & & | \\ 0 & - & 1 \end{array}$$

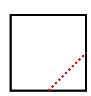
Figure 3 also illustrates the three other cells T_2 can map to. We diagram applying f to the T_2 cells at positions $(2,3), (2,4)$, and $(3,1)$ below.



$$\rightarrow \begin{array}{ccc} 1 & - & 1 \\ | & & | \\ 1 & - & 1 \end{array}$$



$$\rightarrow \begin{array}{ccc} 1 & - & 1 \\ | & & | \\ 1 & - & 0 \end{array}$$



$$\rightarrow \begin{array}{ccc} 0 & - & 0 \\ | & & | \\ 0 & - & 0 \end{array}$$

For convenience, we denote a cell by the 2×2 matrix of its vertex labels. For example, we denote the cell in Example 3.1 as $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$. There are sixteen cells:

$$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}0&0\\0&1\end{smallmatrix}, \begin{smallmatrix}0&0\\1&0\end{smallmatrix}, \begin{smallmatrix}0&0\\1&1\end{smallmatrix}, \begin{smallmatrix}0&1\\0&0\end{smallmatrix}, \begin{smallmatrix}0&1\\0&1\end{smallmatrix}, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, \begin{smallmatrix}0&1\\1&1\end{smallmatrix}, \begin{smallmatrix}1&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, \begin{smallmatrix}1&0\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\1&1\end{smallmatrix}, \begin{smallmatrix}1&1\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\0&1\end{smallmatrix}, \begin{smallmatrix}1&1\\1&0\end{smallmatrix}, \text{ and } \begin{smallmatrix}1&1\\1&1\end{smallmatrix}.$$

Under f, no tile can map to $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. Furthermore, tiles that do not form a polygon must map to cells $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ or $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$, and any tile that is part of a polygon must map to one of the twelve remaining possible cells.

We can then define a general function v that maps a cell to some integer, then extend v to (m,n) binary lattices by taking the product over v of the mn individual cells in the binary lattice. We choose the following definition for v.

Definition 3.3. Define a map v from cells to integers as follows.

$$\begin{cases} 7 & \text{for cells } \begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \\ 0 & \text{for cells } \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \\ -1 & \text{for cells } \begin{smallmatrix}0&0\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\1&1\end{smallmatrix} \\ 1 & \text{otherwise} \end{cases}$$

More generally, we define v from binary lattices to integers by taking the product of v applied to each cell in the binary lattice. By definition, v applied to a binary lattice with no cells equals 1.

Example 3.2. If we let ℓ be the framed binary lattice on the right of Figure 3, we have $v(\ell) = -7^{11}$, as there are 2 $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cells, 1 $\begin{smallmatrix}1&1\\0&1\end{smallmatrix}$ cells, 9 $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells, and 2 $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells.

We show our choice of v has useful properties. To begin, for fixed m, n give an index $1 \leq i \leq N$ to every way one polygon can be formed in an (m, n) mosaic. Let A_i be the set of (m, n) mosaics that contain polygon i. We immediately have that the number of mosaics that contain a polygon is

$$7^{mn} - \left|\mathbb{P}^{(m,n)}\right| = \left|\bigcup_{i=1}^{N} A_i\right|. \tag{1}$$

Proposition 3.1. *Let $\mathcal{M}$ be a mosaic and let S be the indices of the polygons $\mathcal{M}$. We then have*

$$\left|v(f(\mathcal{M}))\right| = \left|\bigcap_{i\in S} A_i\right|.$$

Proof. If a cell is part of a polygon, the absolute value of v of the cell is 1. Therefore, $|v(f(\mathcal{M}))|$ is just 7 to the number of $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells in $f(\mathcal{M})$. As all tiles in $\mathbb{T}$ can map to $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$, **TODO** □

Definition 3.4. If ℓ is a framed binary lattice that does not contain $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, let $P(\ell)$ to be the number of polygons in the polygon mosaic formed by replacing all $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells with the blank tile T_0, and replacing all other cells with the unique tile that maps to that cell under f.

Proposition 3.2. *If ℓ is a framed binary lattice that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, then*

$$sign(v(\ell)) = (-1)^{P(\ell)}. \tag{2}$$

Proof. For this proof, we use LHS and RHS to abbreviate the left and right hand side of Equation 2. We prove the result by induction. For the base case, construct the $(0, n)$ framed binary lattice ℓ for some $n \geq 0$. As there are no cells in ℓ, from the definition of v the LHS is $\text{sign}(v(\ell)) = 1$. As there no polygons in $g(\ell)$, we have the RHS is 1.

For the induction step, consider any $(m+1, n)$ framed binary strip ℓ' for $m \geq 1$ such that there are no $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells. Similarly, define ℓ to be the framed binary lattice that shares the top m rows of vertices with ℓ', and the bottom most row all being 0. We show that we can construct ℓ' from ℓ with a procedure that preserves Equation 2 with each intermediate step.

Procedure:

Step 1. Add a bottom row to ℓ of $n+1$ vertices, all labeled 0. This results in a new $(m+1, n)$ framed binary lattice we denote ℓ_1.

Step 2. Scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}1\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}1\\0\end{smallmatrix}$ column in ℓ_1 to $\begin{smallmatrix}1\\1\end{smallmatrix}$. Completing this scan results in a new framed binary lattice denoted ℓ_2.

Step 3. Again scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}0\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}0\\0\end{smallmatrix}$ column in ℓ_2 to $\begin{smallmatrix}0\\1\end{smallmatrix}$. Completing this scan results in the framed binary lattice ℓ'.

For step 1, only $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells are added. As $\operatorname{sign}\left(v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) = 1$, the LHS is unchanged. For the RHS no new polygons are created in $g(\ell_1)$, so Equation 2 is preserved by step 1.

For steps 2 and 3, we prove the result for each intermediate vertex change as we scan left to right. Furthermore, as each intermediate change only flips a single vertex label from 0 to 1, we only need to consider the 4 cells that share this vertex. We diagram all possible cases for both steps in Figure 4 and Figure 6, where the # symbol indicates the vertex flipping from a 0 to a 1. We know that the bottom most row of vertices must all be 0 by construction.

For step 2, the vertex above the # symbol must be 1 by definition. Also, the vertex right of the # symbol must be 0, as the procedure moves left to right on ℓ_1. Additionally, the $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ cell cannot be created or destroyed, and so Figure 4 depicts all 6 possible cases.

$$\begin{array}{ccc}
\begin{array}{ccccc} 0 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} &
\begin{array}{ccccc} 1 & - & 1 & - & 0 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} &
\begin{array}{ccccc} 1 & - & 1 & - & 0 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \\
\text{(a)} & \text{(b)} & \text{(c)} \\
\begin{array}{ccccc} 0 & - & 1 & - & 1 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} &
\begin{array}{ccccc} 1 & - & 1 & - & 1 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} &
\begin{array}{ccccc} 1 & - & 1 & - & 1 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \\
\text{(d)} & \text{(e)} & \text{(f)}
\end{array}$$

Figure 4: Step 2 Cases

As no case in Figure 4 creates or destroys any $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cells, the LHS sign does not change for any intermediate vertex flip. Similarly, one can check that both before and after the vertex is changed from 0 to 1 that the following is true. Replacing each $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cell with T_0 and replacing all other cells with their unique corresponding tile only forms tiles that must all be part of the same polygon. For example, the corresponding tiles for Case 4e is shown in Figure 5.

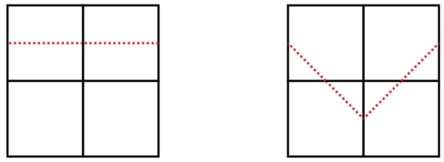


Figure 5: Tile configurations for before and after flipping the vertex in Case 4e

Therefore the number of polygons remains the same, and so the RHS is preserved.

For step 3, the vertex above the # symbol must be 0 by definition. As we cannot create or destroy $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells, Figure 6 depicts all 6 possible cases.

$$\begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 0 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 0 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 1 & - & 0 & - & 0 \\ | & & | & & | \\ 1 & - & \# & - & 0 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(a) (b) (c)

$$\begin{array}{ccccc} 0 & - & 0 & - & 1 \\ | & & | & & | \\ 0 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 0 & - & 0 & - & 1 \\ | & & | & & | \\ 1 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array} \qquad \begin{array}{ccccc} 1 & - & 0 & - & 1 \\ | & & | & & | \\ 1 & - & \# & - & 1 \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

(d) (e) (f)

Figure 6: Step 3 Cases

We show Equation 2 holds case by case. In Case 6a, the flip add one $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. As the flip creates a new polygon the sign of the RHS changes. In Case 6b, the flip removes and adds a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 6c, the flip adds both a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell and a $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 6d, the flip does not add a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 6e, the flip removes a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. Before the flip, replacing the cells with the appropriate tiles results in a configuration that can either represent portions of one or two polygons, as shown in Figure 7.
If the tiles were part of one polygon before the flip, the tiles after the flip must be part of two distinct polygons. Similarly, if the tiles are were part of two polygons before the flip, the tiles after the flip must be of one polygon. Either way the total number of polygons changes by 1, and so the RHS changes sign. In Case 6f, the flip adds $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, so the sign of the

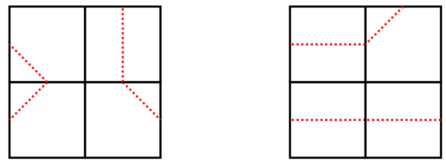


Figure 7: Tile configurations for before and after flipping the vertex in Case 6e

LHS changes. The same logic from Case 6e applies for the RHS, as the flip either removes or adds a polygon.

As all cases preserve Equation 2, the $(m+1, n)$ framed binary lattice ℓ' also follows Equation 2. $\square$

Theorem 3.3. *The number of (m, n) mosaics that do not contain a polygon has*

$$\left|\mathbb{P}^{(m,n)}\right| = \sum_{\ell} v(\ell), \tag{3}$$

where the sum is over all (m, n) framed binary lattices.

Proof. First notice for the sum in Equation 3 that v of any framed binary lattice that contains $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ is 0. Therefore we simplify to only summing over all framed binary lattices that do not contain these cells.

Consider the set of

For all $1 \leq i \leq N$, let A_i be the set of all (m, n) mosaics that include P_i.

$\bigcup_{i=1}^{N} A_i$

By the inclusion-exclusion principle,

$$\left|\bigcup_{i=1}^{N} A_i\right| = \sum_{i=1}^{N} |A_i| - \sum_{1 \leq i \leq j \leq N} |A_i \cap A_j| + \cdots +$$

Choose a mosaic $\mathcal{M}$ that has $P(\ell')$ polygons, where $\ell' = f(\mathcal{M})$. We begin by determining how many times $\mathcal{M}$ is counted in the related sum

$$\sum_{\ell \in \mathbb{F}^{(m,n)}} u(\ell).$$

We point out here that as $u\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = u\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = v\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = v\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = 0$, framed binary lattices with these cells don't contribute to either summation, and so can be ignored for the remainder of this argument.

Proposition **??** shows that the function $u(\ell)$ counts the mosaics with at least the polygons in $g(\ell)$. Consequently, for a framed binary lattice ℓ, $u(\ell)$ counts $\mathcal{M}$ if the polygons in $g(\ell)$ are a subset of the polygons in $\mathcal{M}$. Therefore, as the number of ways to choose a size p subset of $P(\ell)$ polygons is $\binom{P(\ell)}{p}$, $\mathcal{M}$ is counted $\sum_{p=0}^{P(\ell)} \binom{P(\ell)}{p}$ times. This then gives

$$\sum_{\ell \in \mathbb{F}^{(m,n)}} u(\ell) = \sum_{\mathcal{M} \in \mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} \binom{P(f(\mathcal{M}))}{p}.$$

Following the same logic for $\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell)$, Proposition 3.2 gives that size p subsets where p is odd are subtracted, which gives

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\mathcal{M}\in\mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} (-1)^p \binom{P(f(\mathcal{M}))}{p}.$$

Finally, by the binomial theorem, for $P(\ell) > 0$ we have

$$\sum_{p=0}^{P(\ell)} (-1)^p \binom{P(\ell)}{p} = 0,$$

and as the only binary lattice with $P(\ell) = 0$ is ℓ^*, we have

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\{\mathcal{M}\in\mathbb{M}^{(m,n)} | P(f(\mathcal{M}))=0\}} \binom{0}{0} = |f^{-1}(\ell^*)| = |\mathbb{P}^{(m,n)}|.$$

$\square$

We point out here that the function v if defined for all binary lattices, while Proposition 3.2 only holds for framed binary lattices that do not contain cells $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$.

As in Theorem 2.2 from [**Oh2014**], we can compute the sum in Equation 3 efficiently using the matrix recursion method, which we describe next.

4 Proof of Theorem 1.1

We begin by recognizing the matrices A_k, B_k, C_k, D_k in Definition 1.2 can be rewritten using any function v from cells to integers that is extended to (m, n) binary lattices as the product over individual cells. We write $A_1 = \left(v(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})\right), B_1 = \left(v(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})\right), C_1 = \left(v(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})\right), D_1 = \left(v(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix})\right)$, and for integers $k \geq 1$,

$$A_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix})D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix})D_k \end{pmatrix}$$

$$C_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})D_k \end{pmatrix}.$$

We work with general v in this section, and substitute the specific v from Definition 3.3 to give Theorem 1.1 and a related result at the end.

Definition 4.1. Let the n digit binary representation of the number k be written as $\beta_n(k)$. If n is 0, $\beta_n(k)$ returns the empty string.

We extend our shorthand for cells to $(1, n)$ binary lattice using analagous matrix notation and the definition for $\beta_n(k)$. For example, we write

$$v\begin{pmatrix} 0 & \beta_2(1) & 0 \\ 0 & \beta_2(0) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = v\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, we remind the reader matrix elements are indexed starting at 0.

Proposition 4.1. *For all integers $n \geq 1$, the (i,j)-th entry of*

$$A_n \text{ is } v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix}, \quad B_n \text{ is } v\begin{pmatrix} 1 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix},$$
$$C_n \text{ is } v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix}, \quad D_n \text{ is } v\begin{pmatrix} 1 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix}.$$

Proof. We prove by induction. The $n = 1$ case is trivial, as $\beta_0(0)$ is the empty string. We next assume our result for some fixed $n \geq 1$. Since A_n has size $2^{n-1} \times 2^{n-1}$, the $(i, j + 2^{n-1})$ element of A_{n+1} equals $v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ times the (i,j) element of B_n. By our induction hypothesis, this equals

$$v\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & 0 & \beta_{n-1}(i) & 0 \\ 0 & 1 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & \beta_n(i) & 0 \\ 0 & \beta_n(j+2^{n-1}) & 0 \end{pmatrix},$$

as desired. Analagous arguments show that the other blocks in A_{n+1} have the desired values. Similarly, since C_n has size $2^{n-1} \times 2^{n-1}$, the $(i + 2^{n-1}, j)$ element of D_{n+1} equals $v\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ times the (i,j) element of C_n. By our induction hypothesis, this equals

$$v\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} v\begin{pmatrix} 1 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 1 & 1 & \beta_{n-1}(i) & 0 \\ 1 & 0 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 1 & \beta_n(i+2^{n-1}) & 0 \\ 1 & \beta_n(j) & 0 \end{pmatrix},$$

again as desired. These arguments must be adapted to prove the desired results for all blocks of B_{n+1}, C_{n+1}, and D_{n+1}, but all the work is similar. □

We further extend our shorthand for binary lattices. If a binary lattices ℓ has the top row of vertices $0\beta_{n-1}(i)0$ and the bottom row of vertices $0\beta_{n-1}(j)0$, and left and right-most column of vertices being labeled 0, we write ℓ is like $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$.

Proposition 4.2. *For all positive integers m, n, the (i,j)-th entry of A_n^m is $\sum v(\ell)$, where the sum is over all (m,n) binary lattices like $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$.*

Proof. We prove by induction. The base case $m = 1$ is Proposition 4.1, as the sum is only over $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$. We next assume our result for fixed positive m, n. From the induction hypothesis, for any integer $k \in \{0, 1, \ldots, 2^{n-1} - 1\}$ the (i,k)-th entry of A_n is $v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(k) & 0 \end{pmatrix}$ and the (k,j)-th entry of A_n^m is

$$\sum_{\ell \text{ like } \begin{smallmatrix} 0 & \beta_{n-1}(k) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}} v(\ell).$$

The (i,j)-th element of $A_n \cdot A_n^m$ is the dot product of the i-th row of A_n and the j-th column of A_n^m. By construction, we have

$$\sum_{k=0}^{2^{n-1}-1} v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(k) & 0 \end{pmatrix} \sum_{\ell \text{ like } \begin{smallmatrix} 0 & \beta_{n-1}(k) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}} v(\ell) = \sum_{\ell \text{ like } \begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}} v(\ell),$$

which gives the desired result. □

Theorem 1.1. The number of (m,n) mosaics that do not contain a polygon is the $(0,0)$ entry of A_n^m.

Proof. By Proposition 4.2, the $(0,0)$ entry of A_n^m is $\sum v(\ell)$, where the sum is over all (m,n) binary lattices of the form

$$\begin{smallmatrix} 0 & \beta_{n-1}(0) & 0 \\ & \vdots & \\ 0 & \beta_{n-1}(0) & 0 \end{smallmatrix},$$

which is simply all framed binary lattices. Substituting the values for v from Definition 3.3 gives the sum from Theorem 3.3, which completes the proof. $\square$

TODO immediate extension to polygon mosaics