

Enumeration of Messy Polygon Mosaics

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Abstract

Hong and Oh introduced a model for multiple ring polymers in physics in which an $m \times n$ rectangular lattice is constructed from a selection of 7 distinct tiles. These lattices are called *mosaics*. The authors provide bounds on a subset of these mosaics that both contain polygons and have all other tiles that are not part of a polygon set to the blank tile. We introduce and enumerate mosaics with the relaxed property of containing at least one polygon, which we call messy polygon mosaics.

1 Introduction

Imagine you are tasked with tiling a rectangular bathroom floor that is m units by n units, blindfolded. At your disposal is an unending supply of 7 distinct types of tiles. These tiles, diagrammed in Figure 1, are composed of unit squares with dotted lines connecting 2 sides at their midpoints, as well as the “blank” tile T_0 .

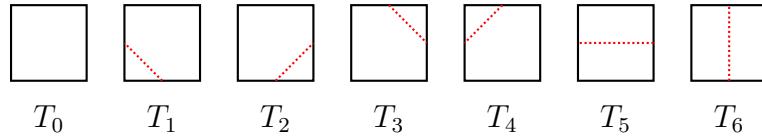


Figure 1: The tile set \mathbb{T}

We denote the set of tiles $\mathbb{T} = \{T_0, \dots, T_6\}$. The task is complete once you place mn randomly selected tiles from \mathbb{T} to cover the floor, after which you remove your blindfold. We are interested in the probability of constructing at least one polygon with the red lines with this process.

We call a fully tiled $m \times n$ floor an (m, n) *mosaic*. If $m = 5$ and $n = 7$, you may have constructed the mosaic in Figure 2a. You may have also constructed the mosaic in Figure 2b. Notice that in the mosaic in Figure 2b, the red dotted lines form multiple polygons¹, which we highlight the corresponding tiles in gray.

¹Polygons are more commonly referred to as “self-avoiding polygons” in the literature to highlight their connection with self-avoiding walks.

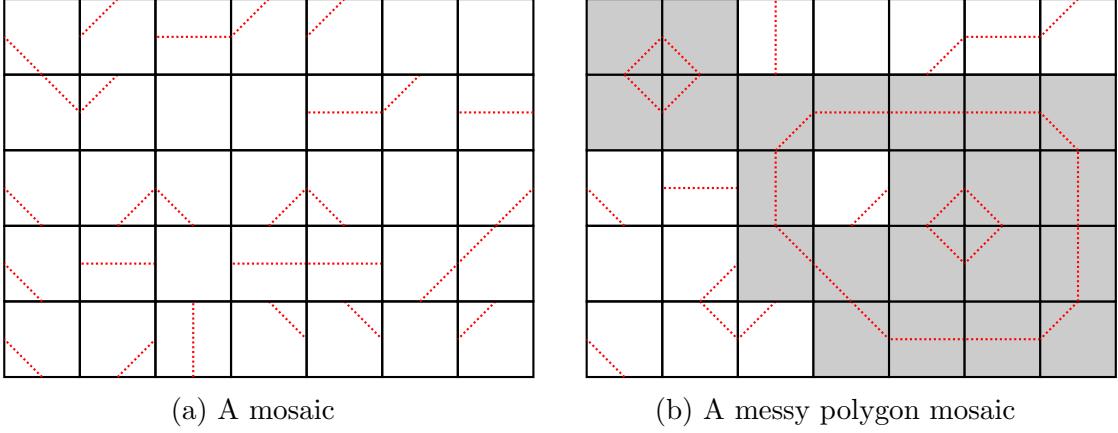


Figure 2: Examples of mosaics of size $(5, 7)$ made of tiles in \mathbb{T}

Definition 1.1. An (m, n) *messy polygon mosaic* is an (m, n) mosaic that contains at least one polygon.

As there are $|\mathbb{T}|^{mn} = 7^{mn}$ total mosaics, we focus on the number of messy polygon mosaics. In fact, it turns out to be simpler to enumerate the number of mosaics that *do not* contain a polygon. Therefore, let $\mathbb{P}^{(m,n)}$ be the subset of (m, n) mosaics that do not contain a polygon. Clearly the number of (m, n) messy polygon mosaics is then $7^{mn} - |\mathbb{P}^{(m,n)}|$.

From the fact that the smallest polygon is made of 4 tiles (appearing twice in Figure 2b), we can conclude that $|\mathbb{P}^{(n,1)}| = 7^n$, and $|\mathbb{P}^{(2,2)}| = 7^4 - 1$. For $m, n \geq 2$, we first define the following matrices.

Definition 1.2. For integers $k \geq 1$ let A_k, B_k, C_k, D_k be $2^{k-1} \times 2^{k-1}$ matrices with integer entries, where $A_1 = (7), B_1 = (-1), C_1 = (1), D_1 = (1)$ and

$$\begin{aligned} A_{k+1} &= \begin{pmatrix} 7A_k & B_k \\ C_k & D_k \end{pmatrix} & B_{k+1} &= \begin{pmatrix} -A_k & B_k \\ 0C_k & D_k \end{pmatrix} \\ C_{k+1} &= \begin{pmatrix} A_k & 0B_k \\ C_k & D_k \end{pmatrix} & D_{k+1} &= \begin{pmatrix} A_k & -B_k \\ C_k & 7D_k \end{pmatrix}. \end{aligned}$$

For matrices, mosaics, and later binary lattices, we number the rows $0, 1, \dots, m-1$ top to bottom, and columns $0, 1, \dots, n-1$ left to right. Here is our main result.

Theorem 1.1. *The number of (m, n) mosaics that do not contain a polygon $|\mathbb{P}^{(m,n)}|$ is the $(0, 0)$ entry of A_n^m .*

2 Related Work

Hong and Oh [2] studied a similar question in which they construct mosaics from \mathbb{T} , but were interested in the number of polygon mosaics.

Definition 2.1. An (m, n) polygon mosaic is an (m, n) mosaic that contains at least one polygon and every tile that is not part of a polygon is T_0 .

Clearly, all polygon mosaics are messy polygon mosaics. The sequence A181245 on the OEIS [4] is the array of 1+ the number of (m, n) polygon mosaics. The authors in [2] provide bounds for the number of polygon mosaics.

Theorem 2.1 ([2]). *The number of (m, n) polygon mosaics for $m, n \geq 3$ is bounded between $2^{m+n-3} \left(\frac{17}{10}\right)^{(m-2)(n-2)}$ and $2^{m+n-3} \left(\frac{31}{16}\right)^{(m-2)(n-2)}$.*

In related work, Lomonaco and Kauffman [3] introduced mosaics constructed from a tile set of 11 distinct tiles, of which \mathbb{T} is a subset. The authors were interested in a subset of mosaics which they call *knot mosaics*. Oh et al. [9] enumerated the number of knot mosaics.

Theorem 2.2 ([9]). *The number of (m, n) knot mosaics for $m, n \geq 2$ is $2 \| (X_{m-2} + O_{m-2})^{n-2} \|$, where $X_0 = O_0 = [1]$ and X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined as*

$$X_{k+1} = \begin{pmatrix} X_k & O_k \\ O_k & X_k \end{pmatrix} \text{ and } O_{k+1} = \begin{pmatrix} O_k & X_k \\ X_k & 4O_k \end{pmatrix},$$

for $k = 0, 1, \dots, m-3$. Here $\|N\|$ denotes the sum of elements of matrix N .

Oh and colleagues go beyond enumeration by bounding the growth rate of knot mosaics [6, 8, 1], and Oh further adapts the matrix recursion method to solve problems in monomer and dimer tilings [5, 7]. Related ideas were independently used to enumerate the number of rectangular partitions of a rectangle in [10] for the sequence A182275 on the OEIS [4].

Our work can be seen as an extension to Hong and Oh [2] and further generalizing the techniques in Oh et al. [9] to explore a new direction in mosaic enumeration.

3 Preliminaries

We begin by defining a map that takes an (m, n) mosaic and gives an (m, n) binary lattice. An (m, n) binary lattice is a rectangular lattice of $m+1$ by $n+1$ vertices, with each vertex labeled 0 or 1. We also define a *framed binary lattice* to be a binary lattice in which the boundary vertices are labeled 0. An example of a $(5, 7)$ framed binary lattice is shown on the right of Figure 3.

Definition 3.1. Let f be the map that takes an (m, n) mosaic and labels each vertex with the following rule. If the vertex is surrounded by the red dotted lines of an even number of polygons (including 0 polygons), label it 0. If the vertex is surrounded by the red dotted lines of an odd number of polygons, label it 1. Removing the red dotted lines from the tiles gives the framed binary lattice.

To enumerate $|\mathbb{P}^{(m,n)}|$, it will be useful to consider how f maps the tiles in \mathbb{T} to the cells in a binary lattice.

Definition 3.2. Let a *cell* be a $(1, 1)$ binary lattice.

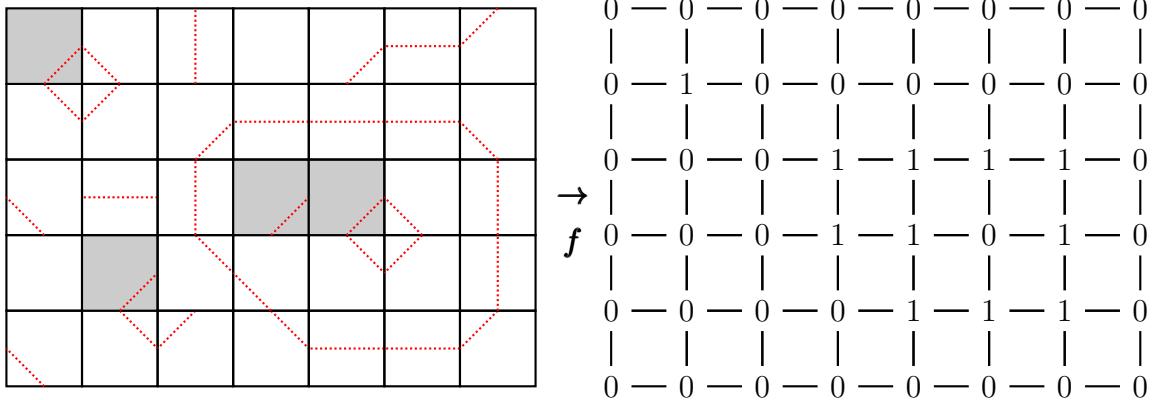


Figure 3: f applied to the mosaic in Figure 2b, resulting in a binary lattice. We highlight each possible way a T_2 tile can map to a cell by shading a representative tile in gray.

Example 3.1. Applying f to the mosaic in Figure 3 results in the T_2 tile at position $(0, 0)$ mapping to the cell at position $(0, 0)$, diagrammed below.

$$\begin{array}{c} \text{[Gray } T_2 \text{ tile]} \\ \rightarrow \end{array} \begin{array}{c} 0 - 0 \\ | \\ 0 - 1 \end{array}$$

Figure 3 also illustrates the three other cells T_2 can map to. We diagram applying f to the T_2 cells at positions $(2, 3), (2, 4)$, and $(3, 1)$ below.

$$\begin{array}{ccc} \text{[Gray } T_2 \text{ tile]} & \rightarrow & \begin{array}{c} 1 - 1 \\ | \\ 1 - 1 \end{array} \\ \text{[Gray } T_2 \text{ tile]} & \rightarrow & \begin{array}{c} 1 - 1 \\ | \\ 1 - 0 \end{array} \\ \text{[Gray } T_2 \text{ tile]} & \rightarrow & \begin{array}{c} 0 - 0 \\ | \\ 0 - 0 \end{array} \end{array}$$

For convenience, we denote a cell by the 2×2 matrix of its vertex labels. For example, we denote the cell in Example 3.1 as $\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$. There are sixteen cells:

$$\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}, \text{ and } \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}.$$

Under f , no tile can map to $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$. Furthermore, tiles that do not form a polygon must map to cells $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$, and any tile that is part of a polygon must map to one of the twelve remaining possible cells.

We can then define a general function v that maps a cell to some integer, then extend v to (m, n) binary lattices by taking the product over v of the mn individual cells in the binary lattice. We choose the following definition for v .

Definition 3.3. Define a map v from cells to integers as follows.

$$v = \begin{cases} 7 & \text{for cells } \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \\ 0 & \text{for cells } \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ -1 & \text{for cells } \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \\ 1 & \text{otherwise} \end{cases}$$

More generally, we define v from binary lattices to integers by taking the product of v applied to each cell in the binary lattice. By definition, v applied to a binary lattice with no cells equals 1.

Example 3.2. If we let ℓ be the framed binary lattice on the right of Figure 3, we have $v(\ell) = -7^{11}$, as there are $2 \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ cells, $1 \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ cells, $9 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ cells, and $2 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ cells.

We show our choice of v has useful properties by first considering the magnitude, then the sign. For the magnitude of v , for fixed m, n give an index $1 \leq i \leq N$ to every way one polygon can be formed in an (m, n) mosaic. We remind the reader that a polygon is determined not only by its shape and orientation, but also its location within the mosaic. For example, Figure 4 depicts three distinct polygons, which for reference we call polygons 1, 2, and 3.

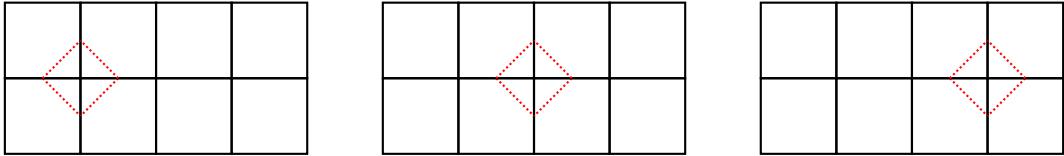


Figure 4: Polygons 1, 2, and 3

Let M_i be the set of (m, n) mosaics that contain polygon i . We immediately have the number of messy polygon mosaics is

$$7^{mn} - |\mathbb{P}^{(m,n)}| = \left| \bigcup_{i=1}^N M_i \right|. \quad (1)$$

Notice that not every subset of polygons $1 \leq i \leq N$ has a mosaic that contains them. For example, for the polygons in Figure 4 the second-from-left column contains different tiles in polygon 1 and polygon 2, so no mosaic exists that contain these two polygons, and so $M_1 \cap M_2 = \emptyset$.

Proposition 3.1. *Let \mathcal{M} be a mosaic and let S be the indices of the polygons in \mathcal{M} . We then have*

$$|v(f(\mathcal{M}))| = \left| \bigcap_{i \in S} M_i \right|.$$

Proof. From the definition of v , if a cell comes from a tile that is part of a polygon, the absolute value of v of the cell is 1. Therefore, $|v(f(\mathcal{M}))|$ is just 7 to the number of $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ cells in $f(\mathcal{M})$. As all tiles in \mathbb{T} can map to $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$, $|v(f(\mathcal{M}))|$ counts all mosaics that contain the polygons in \mathcal{M} . This is the definition of $\bigcap_{i \in S} M_i$, and so the equality holds. \square

Definition 3.4. If ℓ is a framed binary lattice that does not contain $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, let $P(\ell)$ to be the number of polygons in the polygon mosaic formed by replacing all $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ cells with the blank tile T_0 , and replacing all other cells with the unique tile that maps to that cell under f .

For the sign of v , we show that though $v(\ell)$ is computed “cell-by-cell”, we recover global information about $P(\ell)$.

Proposition 3.2. *If ℓ is a framed binary lattice that does not contain the cells $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$, then*

$$\text{sign}(v(\ell)) = (-1)^{P(\ell)}. \quad (2)$$

Proof. For this proof, we use LHS and RHS to abbreviate the left and right hand side of Equation 2. We prove the result by induction. For the base case, construct the $(0, n)$ framed binary lattice ℓ for some $n \geq 0$. As there are no cells in ℓ , from the definition of v the LHS is $\text{sign}(v(\ell)) = 1$. As there are no polygons in $g(\ell)$, we have the RHS is 1.

For the induction step, consider any $(m+1, n)$ framed binary strip ℓ' for $m \geq 1$ such that there are no $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ cells. Similarly, define ℓ to be the framed binary lattice that shares the top m rows of vertices with ℓ' , and the bottom most row all being 0. We show that we can construct ℓ' from ℓ with a procedure that preserves Equation 2 with each intermediate step.

Procedure:

Step 1. Add a bottom row to ℓ of $n+1$ vertices, all labeled 0. This results in a new $(m+1, n)$ framed binary lattice we denote ℓ_1 .

Step 2. Scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$, change the associated $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ column in ℓ_1 to $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$. Completing this scan results in a new framed binary lattice denoted ℓ_2 .

Step 3. Again scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, change the associated $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ column in ℓ_2 to $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$. Completing this scan results in the framed binary lattice ℓ' .

For step 1, only $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ cells are added. As $\text{sign}(v(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix})) = 1$, the LHS is unchanged. For the RHS no new polygons are created in $g(\ell_1)$, so Equation 2 is preserved by step 1.

For steps 2 and 3, we prove the result for each intermediate vertex change as we scan left to right. Furthermore, as each intermediate change only flips a single vertex label from 0 to 1, we only need to consider the 4 cells that share this vertex. We diagram all possible cases for both steps in Figure 5 and Figure 7, where the # symbol indicates the vertex flipping from a 0 to a 1. We know that the bottom most row of vertices must all be 0 by construction.

For step 2, the vertex above the # symbol must be 1 by definition. Also, the vertex right of the # symbol must be 0, as the procedure moves left to right on ℓ_1 . Additionally, the $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ cell cannot be created or destroyed, and so Figure 5 depicts all 6 possible cases.

As no case in Figure 5 creates or destroys any $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ cells, the LHS sign does not change for any intermediate vertex flip. Similarly, one can check that both before and after the vertex is changed from 0 to 1 that the following is true. Replacing each $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ and $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ cell with T_0 and replacing all other cells with their unique corresponding tile only forms tiles that must all be part of the same polygon. For example, the corresponding tiles for Case 5e is shown in Figure 6.

Therefore the number of polygons remains the same, and so the RHS is preserved.

$\begin{array}{c c c} 0 & 1 & 0 \\ \hline & & \\ 0 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 1 & 0 \\ \hline & & \\ 0 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 1 & 0 \\ \hline & & \\ 1 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$
(a)	(b)	(c)
$\begin{array}{c c c} 0 & 1 & 1 \\ \hline & & \\ 0 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 1 & 1 \\ \hline & & \\ 0 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 1 & 1 \\ \hline & & \\ 1 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$
(d)	(e)	(f)

Figure 5: Step 2 Cases



Figure 6: Tile configurations before and after flipping the vertex in Case 5e

$\begin{array}{c c c} 0 & 0 & 0 \\ \hline & & \\ 0 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 0 & 0 & 0 \\ \hline & & \\ 1 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 0 & 0 \\ \hline & & \\ 1 & \# & 0 \\ \hline & & \\ 0 & 0 & 0 \end{array}$
(a)	(b)	(c)
$\begin{array}{c c c} 0 & 0 & 1 \\ \hline & & \\ 0 & \# & 1 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 0 & 0 & 1 \\ \hline & & \\ 1 & \# & 1 \\ \hline & & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{c c c} 1 & 0 & 1 \\ \hline & & \\ 1 & \# & 1 \\ \hline & & \\ 0 & 0 & 0 \end{array}$
(d)	(e)	(f)

Figure 7: Step 3 Cases

For step 3, the vertex above the $\#$ symbol must be 0 by definition. As we cannot create or destroy $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ cells, Figure 7 depicts all 6 possible cases.

We show Equation 2 holds case by case. In Case 7a, the flip add one $\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$ cell, so the sign of the LHS changes. As the flip creates a new polygon the sign of the RHS changes. In Case

7b, the flip removes and adds a $\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 7c, the flip adds both a $\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$ cell and a $\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 7d, the flip does not add a $\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$ cell or $\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new polygon, the RHS stays the same. In Case 7e, the flip removes a $\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$ cell, so the sign of the LHS changes. Before the flip, replacing the cells with the appropriate tiles results in a configuration that can either represent portions of one or two polygons, as shown in Figure 8.

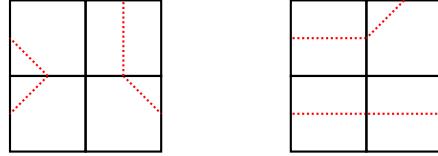


Figure 8: Tile configurations before and after flipping the vertex in Case 7e

If the tiles were part of one polygon before the flip, the tiles after the flip must be part of two distinct polygons. Similarly, if the tiles were part of two polygons before the flip, the tiles after the flip must be of one polygon. Either way the total number of polygons changes by 1, and so the RHS changes sign. In Case 7f, the flip adds $\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}$, so the sign of the LHS changes. The same logic from Case 7e applies for the RHS, as the flip either removes or adds a polygon.

As all cases preserve Equation 2, the $(m+1, n)$ framed binary lattice ℓ' also follows Equation 2. \square

Theorem 3.3. *The number of (m, n) mosaics that do not contain a polygon has*

$$|\mathbb{P}^{(m,n)}| = \sum_{\ell} v(\ell), \quad (3)$$

where the sum is over all (m, n) framed binary lattices.

Proof. We begin by rewriting Equation 1 to

$$|\mathbb{P}^{(m,n)}| = 7^{mn} - \left| \bigcup_{i=1}^N M_i \right|.$$

By the inclusion-exclusion principle, we have

$$-\left| \bigcup_{i=1}^N M_i \right| = \sum_{\emptyset \neq J \subseteq \{1, 2, \dots, N\}} (-1)^{|J|} \left| \bigcap_{j \in J} M_j \right|.$$

Consider the collection of terms in the above sum such that $|J| = k \geq 1$. From the definition of M , this collection of terms counts the number of mosaics that have at least k polygons. Propositions 3.1 and 3.2 give the magnitude and sign of the terms is

$$\sum_{J \subseteq \{1, 2, \dots, N\}, |J|=k} (-1)^{|J|} \left| \bigcap_{j \in J} M_j \right| = \sum_{\{\ell | P(\ell)=k\}} v(\ell).$$

TODO something about how prop 3.1 extends to the sum above

TODO something about overlapping polygons

Additionally notice that the binary lattice ℓ^* made up of all $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ cells has $v(\ell^*) = 7^{mn}$, and is the only framed binary lattice such that $P(\ell) = 0$. Therefore, we can write

$$\left| \mathbb{P}^{(m,n)} \right| = \sum_{k \geq 0} \sum_{\{\ell | P(\ell)=k\}} v(\ell).$$

Finally, notice that the sum $\sum_{\ell} v(\ell) = 0$ for the framed binary lattices that contains $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$, as $v(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = v(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = 0$. Therefore, summing over all framed binary lattices in Equation 3 gives the desired result. \square

We point out here that the function v is defined for all binary lattices, while Proposition 3.2 only holds for framed binary lattices that do not contain cells $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$.

As in Theorem 2.2 from [9], we can compute the sum in Equation 3 efficiently using the matrix recursion method, which we describe next.

4 Proof of Theorem 1.1

We begin by recognizing the matrices A_k, B_k, C_k, D_k in Definition 1.2 can be rewritten using any function v from cells to integers that is extended to (m, n) binary lattices as the product over individual cells. We write $A_1 = (v(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}))$, $B_1 = (v(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}))$, $C_1 = (v(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}))$, $D_1 = (v(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}))$, and for integers $k \geq 1$,

$$A_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix})D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix})D_k \end{pmatrix}$$

$$C_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} v(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix})A_k & v(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix})B_k \\ v(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})C_k & v(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})D_k \end{pmatrix}.$$

We work with general v in this section, and substitute the specific v from Definition 3.3 to give Theorem 1.1, as well as exactly enumerating the number of polygon mosaics that were bounded in [2].

Definition 4.1. Let the n digit binary representation of the number k be written as $\beta_n(k)$. If n is 0, $\beta_n(k)$ returns the empty string.

We extend our shorthand for cells to $(1, n)$ binary lattice using analogous matrix notation and the definition for $\beta_n(k)$. For example, we write

$$v\left(\begin{smallmatrix} 0 & \beta_2(1) & 0 \\ 0 & \beta_2(0) & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)v\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)v\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

Finally, we remind the reader matrix elements are indexed starting at 0.

Proposition 4.1. *For all integers $n \geq 1$, the (i, j) -th entry of*

$$A_n \text{ is } v\left(\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right), \quad B_n \text{ is } v\left(\begin{smallmatrix} 1 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right),$$

$$C_n \text{ is } v\left(\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right), \quad D_n \text{ is } v\left(\begin{smallmatrix} 1 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right).$$

Proof. We prove by induction. The $n = 1$ case is trivial, as $\beta_0(0)$ is the empty string. We next assume our result for some fixed $n \geq 1$. Since A_n has size $2^{n-1} \times 2^{n-1}$, the $(i, j + 2^{n-1})$ element of A_{n+1} equals $v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ times the (i, j) element of B_n . By our induction hypothesis, this equals

$$v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)v\left(\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 0 & 0 & \beta_{n-1}(i) & 0 \\ 0 & 1 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 0 & \beta_n(i) & 0 \\ 0 & \beta_n(j+2^{n-1}) & 0 \end{smallmatrix}\right),$$

as desired. Analogous arguments show that the other blocks in A_{n+1} have the desired values. Similarly, since C_n has size $2^{n-1} \times 2^{n-1}$, the $(i + 2^{n-1}, j)$ element of D_{n+1} equals $v\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ times the (i, j) element of C_n . By our induction hypothesis, this equals

$$v\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)v\left(\begin{smallmatrix} 1 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 1 & 1 & \beta_{n-1}(i) & 0 \\ 1 & 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}\right) = v\left(\begin{smallmatrix} 1 & \beta_n(i+2^{n-1}) & 0 \\ 1 & \beta_n(j) & 0 \end{smallmatrix}\right),$$

again as desired. These arguments must be adapted to prove the desired results for all blocks of B_{n+1} , C_{n+1} , and D_{n+1} , but all the work is similar. \square

We further extend our shorthand for binary lattices. If a binary lattice ℓ has the top row of vertices $0\beta_{n-1}(i)0$ and the bottom row of vertices $0\beta_{n-1}(j)0$, and left and right-most column of vertices being labeled 0, we write ℓ is like $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ \dots & & \dots \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$.

Proposition 4.2. *For all positive integers m, n , the (i, j) -th entry of A_n^m is $\sum v(\ell)$, where the sum is over all (m, n) binary lattices like $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ \dots & & \dots \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$.*

Proof. We prove by induction. The base case $m = 1$ is Proposition 4.1, as the sum is only over $\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{smallmatrix}$. We next assume our result for fixed positive m, n . From the induction hypothesis, for any integer $k \in \{0, 1, \dots, 2^{n-1} - 1\}$ the (i, k) -th entry of A_n is $v\left(\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(k) & 0 \end{smallmatrix}\right)$ and the (k, j) -th entry of A_n^m is

$$\sum_{\substack{\ell \text{ like } 0 & \beta_{n-1}(k) & 0 \\ 0 & \beta_{n-1}(j) & 0}} v(\ell).$$

The (i, j) -th element of $A_n \cdot A_n^m$ is the dot product of the i -th row of A_n and the j -th column of A_n^m . By construction, that is

$$\sum_{k=0}^{2^{n-1}-1} v\left(\begin{smallmatrix} 0 & \beta_{n-1}(i) & 0 \\ 0 & \beta_{n-1}(k) & 0 \end{smallmatrix}\right) \sum_{\substack{\ell \text{ like } 0 & \beta_{n-1}(k) & 0 \\ \dots & & \dots \\ 0 & \beta_{n-1}(j) & 0}} v(\ell) = \sum_{\substack{\ell \text{ like } 0 & \beta_{n-1}(i) & 0 \\ \dots & & \dots \\ 0 & \beta_{n-1}(j) & 0}} v(\ell),$$

which gives the desired result. \square

Theorem 1.1. The number of (m, n) mosaics that do not contain a polygon is the $(0, 0)$ entry of A_n^m .

Proof. By Proposition 4.2, the $(0, 0)$ entry of A_n^m is $\sum v(\ell)$, where the sum is over all (m, n) binary lattices of the form

$$\begin{matrix} 0 & \beta_{n-1}(0) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(0) & 0 \end{matrix},$$

which is simply all framed binary lattices. Substituting the values for v from Definition 3.3 gives the sum from Theorem 3.3, which completes the proof. \square

TODO immediate extension to polygon mosaics. We point out that this result is likely known to the authors of [2] and [9], though to our knowledge this result has not appeared in print.

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