Mosaics

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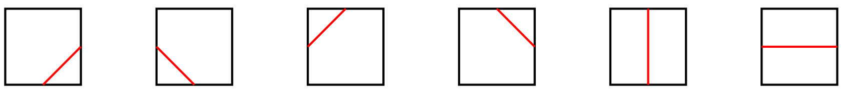
1 Introduction

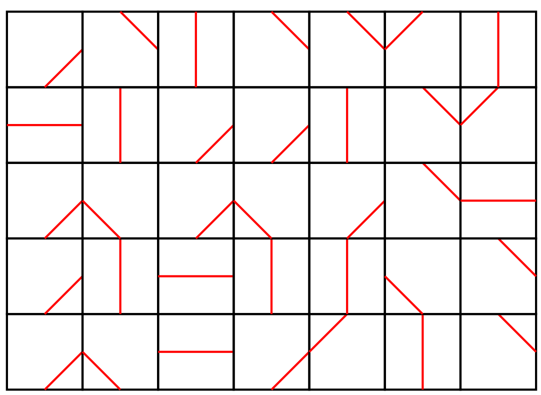
Consider [**oeis**] [**Hong˙2018**] [**10.5555/1717824**] [**Bna2006AWT**] the following 6 unit squares with markings on them.



Call these squares *tiles*.

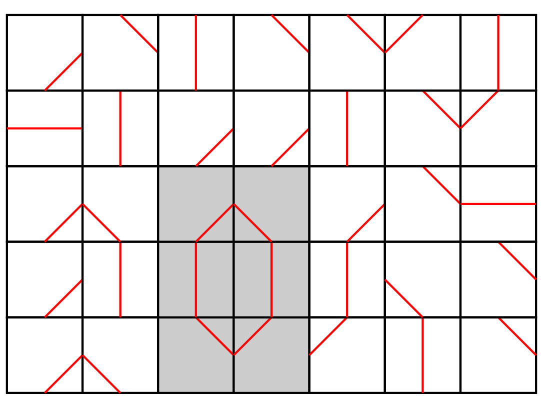
Definition 1.1. An (n, m)-*mosaic* is a rectangular grid made up of tiles.

Example 1.1. An example of a $(7, 5)$-mosaic:

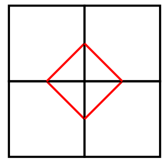


Clearly there are 6^{nm} possible mosaics. Which of these mosaics contain self-avoiding polygons?

Example 1.2. An example of a $(7, 5)$-mosaic with a self-avoiding polygon:



Let $t_{n,m}$ be the number of mosaics that have at least one self avoiding polygon or SAP. From the fact that the smallest SAP is



we have that $t_{n,1} = t_{1,m} = 0$, and $t_{2,2} = 1$. What else can be said?

Theorem 1. *Setting $m = 2$ gives*

$$\sum_{n\geq 2} t_{n,2}x^n = \frac{x^2}{(1-36x)(1-37x+37x^2)}. \tag{1}$$

This can be solved for $n \geq 2$ to give

$$t_{n,2} = 6^{2n} - \frac{1}{\beta-\alpha}((36\beta-35)\beta^{-n+1} - (36\alpha-35)\alpha^{-n+1}) \tag{2}$$

where $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{33}{37}}$ and $\beta = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{33}{37}}$.

Proof. We prove that $t_{n,2}$ has

$$t_{n,2} = 36t_{n-1,2} + \sum_{i=2}^{n}(6^{2(n-i)} - t_{n-i,2}).$$

Split $t_{n,2}$ into S and S^c. S contains the mosaics that have 1 SAP only, and the SAP contains the left-most two cells. This means S^c contains all mosaics that contain multiple SAPs and mosaics that contain only 1 SAP, but that does not contain the two left-most cells.

The subset S can be split further by the length of each SAP i.

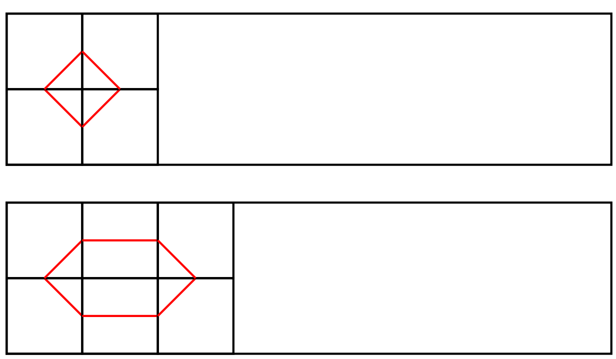


Figure 1: Members of S of lengths $i = 2$ and $i = 3$

As S counts the number of mosaics that only contain 1 SAP, the blank space in Figure **??** must have no SAPs. The number of mosaics that have no SAPs is $6^{2(n-i)} - t_{n-i,2}$. As a SAP's width can range from 2 to n, we have $|S| = \sum_{i=2}^{n}(6^{2(n-i)} - t_{n-i,2}) = \sum_{i=0}^{n-2}(6^{2(n-i)} - t_{n-i,2})$

Now consider S^c. The mosaics that belong to this set can be represented by the following diagram,

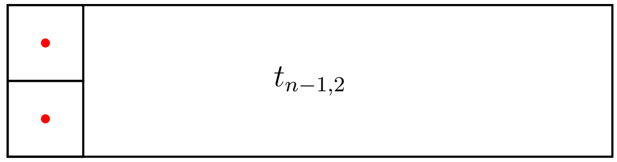


Figure 2: Representation of S^c

where the red dot in the left most cells indicate any marking. For this paper, we will refer to a cell that can have any marking as an *open*. We can conclude that $|S^c| = 6^2 t_{n-1,2}$. Combining S and S^c gives the recurrence relation. Standard techniques then give the generating function and formula. □

Theorem 2. *Setting* $m = 3$ *gives*

$$\sum_{n\geq 2} t_{n,3}x^n = \frac{(73-414x)x^2}{(1-216x)(1-228x+2699x^2-7758x^3)} \tag{3}$$

Proof. Similarly to $t_{n,2}$, we prove a recurrence relation. This time, we prove

$$t_{3,n} = 6^3 t_{n-1,3} + \sum_{i=0}^{n}(6^{3(n-i)} - t_{n-i,3})f_{i,3}$$

where $f_{n,m}$ is the number of mosaics in an $n \times m$ grid that contain just one SAP that is n wide and m tall.

EXAMPLE FIGURE

Notice that $f_{n,2} = 1$ for $n \geq 2$, which is why this 'weight' factor does not appear in the proof of Theorem **??**. It can be shown that

$$F_3(x) = \sum_{n\geq 2} f_{n,3}x^n = \frac{73-414x}{1-12x+43x^2},$$

and similar techniques prove the generating function.

PROOF OF F RECURRENCE RELATION

□

Theorem 3. *Setting* $m = 4$ *gives*

$$\sum_{n\geq 2} t_{n,4}x^n = ? \tag{4}$$