Enumeration of Messy Polygon Mosaics

3rd author?

Jack Hanke
Northwestern University

Michael Maltenfort
Northwestern University

Abstract

Hong and Oh introduced a model for multiple ring polymers in physics in which an $m \times n$ matrix is constructed from a selection of 7 distinct tiles. These matrices are called *mosaics.* The authors provide bounds on a subset of these mosaics that have the property of being suitably connected. These mosaics are called *polygon mosaics* because the tiles in a suitably connected mosaic form polygons. We introduce and enumerate mosaics with the related property of containing at least one polygon, which we call messy polygon mosaics.

1 Introduction

Hong and Oh [2] introduced a model for multiple ring polymers in which an $m \times n$ matrix is constructed using 7 distinct symbols callet *tiles.* These tiles, diagrammed in Figure 1, are composed of unit squares with dotted lines connecting 2 sides at their midpoint, as well as the "blank" tile T_0.

midpoints

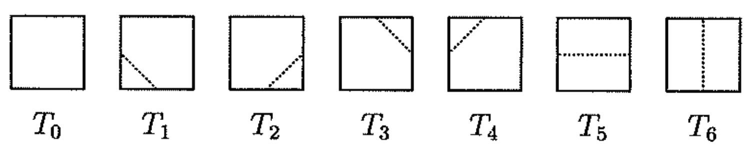


Figure 1: The tile set $\mathbb{T}$

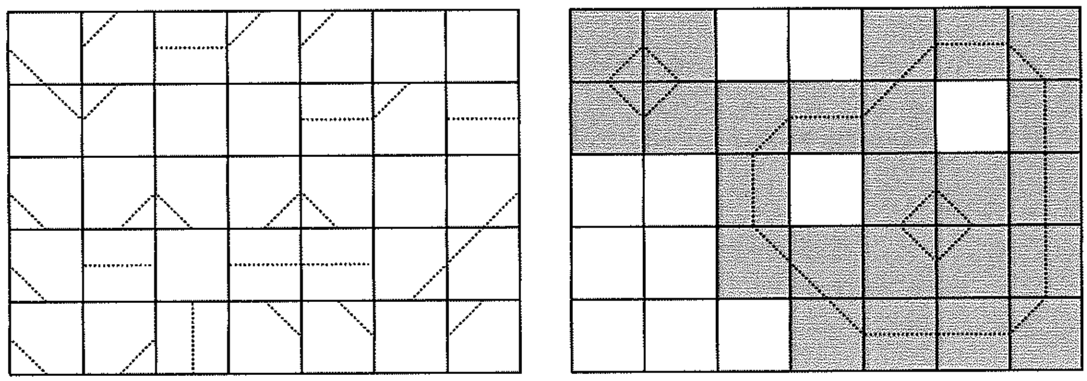
We denote the set of tiles $\mathbb{T} = \{T_0, \ldots, T_6\}$. An (m, n) *mosaic* is an $m \times n$ matrix made up of elements from $\mathbb{T}$. Figure 2a shows a $(5, 7)$ mosaic. We denote the set of all (m, n) mosaics as $\mathbb{M}^{(m,n)}$. As there are 7 elements in $\mathbb{T}$, there are 7^{mn} mosaics in $\mathbb{M}^{(m,n)}$.

The authors in [2] were interested in mosaics with the property of being *suitably connected,* which is defined as follows. Consider an edge shared between two tiles in Figure 2a. The edge has either 0, 1, or 2 dotted lines drawn from its midpoint. Also note that the outer edges of the tiles on the boundary of the matrix are not shared by another tile. Therefore these edges only have 0 or 1 dotted lines drawn from their midpoint. A mosaic is suitably connected if all edges have 0 or 2 dotted lines drawn from their midpoint. The authors call mosaics that are suitably connected and that contain at least one tile in $\mathbb{T} \setminus T_0$ *polygon mosaics* because

Technically: $\mathbb{T} \setminus \{T_0\}$

"Perhaps better: "...contain at least one tile that is not T_0..."

~~and are not entirely composed~~



(a) A mosaic (b) A polygon mosaic

Figure 2: Examples of mosaics of size $(5,7)$ made of tiles in $\mathbb{T}$

the dotted lines form *polygons*[1]. Following [2], when we use the term polygon in this work, we specifically refer to the set of tiles for which the dotted lines form a polygon when part of a mosaic.

Example 1.1. Figure 2b shows a polygon $(5,7)$ mosaic that contains 3 polygons, with the tiles that make up the polygons shaded in gray. Note that a mosaic can contain polygons that surround other polygons, ~~such~~ as in Figure 2b.

As in Fig 2b, a mosaic can...

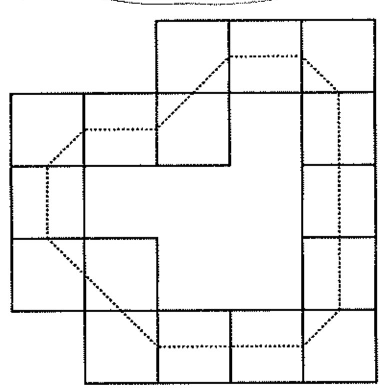


Figure 3: A polygon in Figure 2b

How about minimizing notation and never defining $\mathbb{P}^{(m,n)}$?

~~Using the notation~~ conventions in [9], we denote the subset of (m,n) ~~mosaics that are polygon mosaics as $\mathbb{P}^{(m,n)}$~~.

Theorem 1.1 ([2]). *The number of polygon (m,n) mosaics $|\cancel{\mathbb{P}^{(m,n)}}|$ for $m,n \geq 3$ is bounded by*

$$2^{m+n-3}\left(\frac{17}{10}\right)^{(m-2)(n-2)} \leq |\mathbb{P}^{(m,n)}| \leq 2^{m+n-3}\left(\frac{31}{16}\right)^{(m-2)(n-2)}.$$

[1]Polygons are more commonly refered to as "self-avoiding polygons" in the literature to highlight their connection with self-avoiding walks.

~~The array of values of $|\mathbb{P}^{(m,n)}|+1$~~ is Sequence A181245 on the OEIS [4, OEIS].

is the array of 1 + the number of (n,m) polygon mosaics

In related work, Lomonaco and Kauffman [3] introduced mosaics constructed from a tile set of 11 distinct tiles, of which $\mathbb{T}$ is a subset. The authors call mosaics constructed from this tile set that are suitably connected *knot mosaics.* Oh et al. [9] enumerated the number of knot mosaics.

Theorem 1.2 ([9]). *The number of knot mosaics of size (m,n) for $m,n \geq 2$ is $2\,\|(X_{m-2}+O_{m-2})^{n-2}\|$, where $X_0 = O_0 = [1]$ and X_{m-2} and O_{m-2} are $2^{m-2} \times 2^{m-2}$ matrices defined as*

$$X_{k+1} = \begin{pmatrix} X_k & O_k \\ O_k & X_k \end{pmatrix} \text{ and } O_{k+1} = \begin{pmatrix} O_k & X_k \\ X_k & 4O_k \end{pmatrix},$$

for $k = 0, 1, \ldots, m-3$. Here $\|N\|$ denotes the sum of elements of matrix N.

Oh and colleagues refer to these matrices X_k and O_k as *state matrices.* The authors utilize this state matrix recursion to bound the growth rate of knot mosaics [6, 8, 1], and Oh further adapts the method to solve problems in monomer and dimer tilings [5, 7]. An unexamined direction in this research program is modifying the suitably connected property. This motivates us to introduce *messy polygon mosaics* using the tiles set $\mathbb{T}$ from [2], and enumerate them with state matrices as in [9].

$*$ See A

2 Messy Polygon Mosaics

Definition 2.1. A *messy polygon mosaic* is a mosaic that contains at least one polygon.

Figure 4 shows two examples of messy polygon $(5,7)$ mosaics, both of which contain 3 polygons. In both examples, we shade the tiles that make up each polygon in gray.

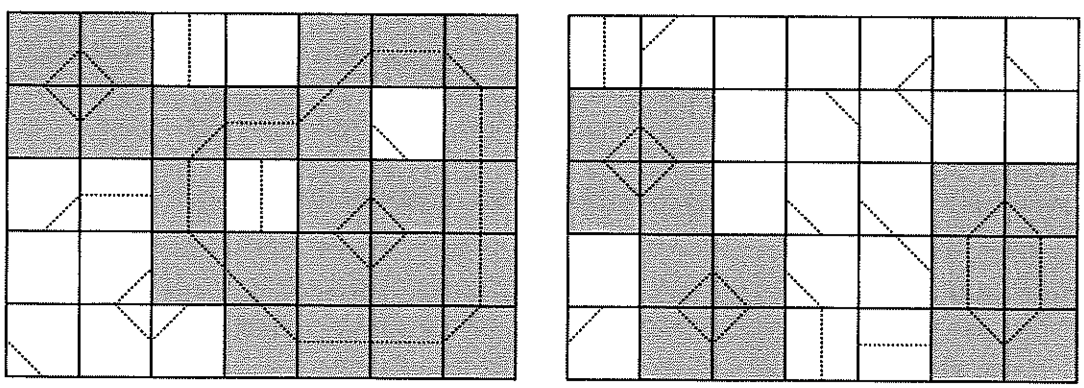


Figure 4: Examples of $(5,7)$ messy polygon mosaics

The main goal of this paper is to enumerate messy polygon mosaics, but it turns out to be simpler to enumerate the number of mosaics that *do not* contain a polygon. Therefore, let $\mathbb{S}^{(m,n)}$ be the subset of (m,n) mosaics that do not contain a polygon. Clearly the number of messy polygon (m,n) mosaics is then $7^{mn} - |\mathbb{S}^{(m,n)}|$.

notation: why 'S'? what does it stand for?

m, n

From the fact that the smallest polygon is made of 4 tiles, shown in Figure 5, we can conclude that $|\mathbb{S}^{(n,1)}| = 7^n$, and $|\mathbb{S}^{(2,2)}| = 7^4 - 1$. For $n, m \geq 2$, we first define the state matrices for messy polygon mosaics.

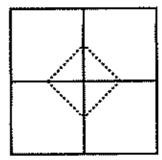


Figure 5: The smallest polygon

Perhaps not worth doing this here.

Definition 2.2. For integers $k \geq 1$ let A_k, B_k, C_k, D_k be $2^{k-1} \times 2^{k-1}$ matrices with integer entries, where $A_1 = (7), B_1 = (-1), C_1 = (1), D_1 = (1)$ and

$$A_{k+1} = \begin{pmatrix} 7A_k & B_k \\ C_k & D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} -A_k & B_k \\ 0C_k & D_k \end{pmatrix}$$

$$C_{k+1} = \begin{pmatrix} A_k & 0B_k \\ C_k & D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} A_k & -B_k \\ C_k & 7D_k \end{pmatrix}.$$

Here is our ~~Our~~ main result ~~is Theorem 2.1.~~

Theorem 2.1. *The number of (m, n) mosaics that do not contain a polygon $|\mathbb{S}^{(m,n)}|$ is the $(0, 0)$ entry of A_n^m.*

Either say "upper left" or explain our convention that indexing rows and colums starts with zero. (Before or after statement of theorem.)

3 Preliminaries

"a map that takes an (m,n) mosaic and gives an (m,n) binary lattice."

We begin by defining a mapping f between a mosaic in $\mathbb{M}^{(m,n)}$ to an (m, n) *binary lattice.* An (m, n) binary lattice is a rectangular lattice of $m + 1$ by $n + 1$ vertices, with each vertex labeled 0 or 1. We also define a *framed* binary lattice to be a binary lattice in which the boundary vertices are labeled 0. An example of a $(5, 7)$ framed binary lattice is shown on the right of Figure 6. Also let $\mathbb{L}^{(m,n)}$ be the set of all (m, n) binary lattices and $\mathbb{F}^{(m,n)}$ be the set of all (m, n) framed binary lattices. We immediately have $|\mathbb{L}^{(m,n)}| = 2^{(m+1)(n+1)}$, $|\mathbb{F}^{(m,n)}| = 2^{(m-1)(n-1)}$.

Italicize just occurring when it's defined.

XX See B

Maybe unclear, let's discuss polygon= tiles is not respected here.

Definition 3.1. $f : \mathbb{M}^{(m,n)} \to \mathbb{F}^{(m,n)}$ takes a mosaic and labels each vertex according to ~~with the~~ following rule. If the vertex is surrounded by an even number of polygons (including 0 polygons), label it 0. If the vertex is surrounded by an odd number of polygons, label it 1. Removing the red dotted lines from the tiles then gives the framed binary lattice.

Notice that by the definition of f, in a binary lattice polygons draw out the boundary between edge-connected vertices with the same label, excluding the vertices that are edge-connected to the boundary 0's. For example in Figure 6 the three polygons correspond with the boundary of the three edge-connected regions of vertices that are not edge-connected to the boundary 0's.

not really unique, is it? Need to say better.

To enumerate $|\mathbb{S}^{(m,n)}|$, it will be useful to consider how f maps individual tiles in $\mathbb{T}$ to individual *cells* in a binary lattice.

unitalicize.

Not well said. The edge-connected surrounding 0 has both an inner & outer edge.

shad if the tile is changed to T_2

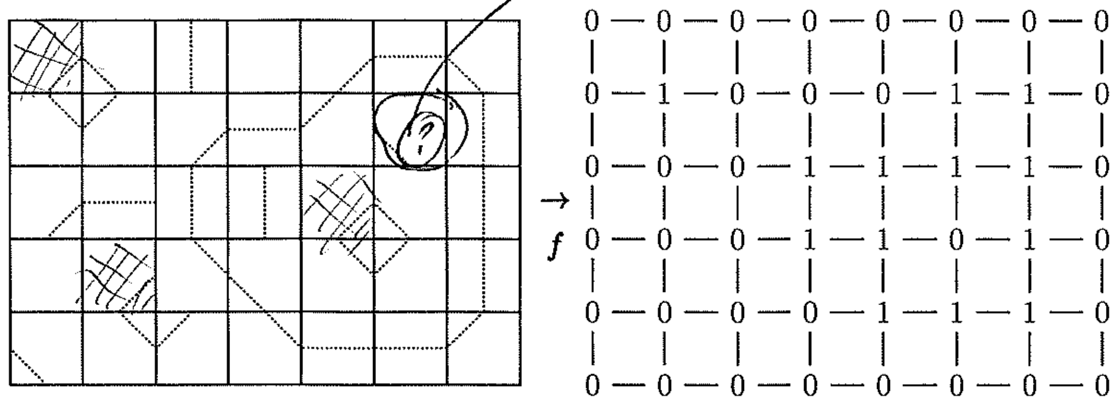
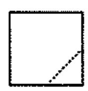


Figure 6: f applied to the left mosaic in Figure 4, resulting in a binary lattice

Definition 3.2. Let a *cell* be a $(1, 1)$ binary lattice.

Example 3.1. Applying f to the mosaic in Figure 6 results in the top-left tile T_2 mapping to the top-left cell diagrammed below.

Perhaps shade the cells as above to show that T_2 can map to $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$, $\begin{smallmatrix}1&1\\1&0\end{smallmatrix}$, $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$, or $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$

 $\rightarrow$ $\begin{array}{ccc}0 & - & 0\\ | & & | \\ 0 & - & 1\end{array}$

For convenience, we denote a cell by the 2×2 matrix of its vertex labels. For example, we denote the cell in Example 3.1 as $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$. ~~The set of all unique cells is~~ There are the sixteen cells:

$$\{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}0&0\\0&1\end{smallmatrix}, \begin{smallmatrix}0&0\\1&0\end{smallmatrix}, \begin{smallmatrix}0&0\\1&1\end{smallmatrix}, \begin{smallmatrix}0&1\\0&0\end{smallmatrix}, \begin{smallmatrix}0&1\\0&1\end{smallmatrix}, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, \begin{smallmatrix}0&1\\1&1\end{smallmatrix}, \begin{smallmatrix}1&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, \begin{smallmatrix}1&0\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\1&1\end{smallmatrix}, \begin{smallmatrix}1&1\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\0&1\end{smallmatrix}, \begin{smallmatrix}1&1\\1&0\end{smallmatrix}, \begin{smallmatrix}1&1\\1&1\end{smallmatrix}\}.$$

" and "

Definition 3.3. For a given (m, n), let ℓ^* be the framed binary lattice made of only $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells.

Therefore, as all mosaics that do not contain a polygon map to ℓ^* under f, we have

$$\mathbb{S}^{(m,n)} = f^{-1}(\{\ell^*\}), \tag{1}$$

and so it suffices to compute $|f^{-1}(\{\ell^*\})|$.

$f^{-1}(\{\ell\})$ framed (?)

It would be convenient to compute $f^{-1}(\ell)$ for any, binary lattice ℓ "cell-by-cell". ~~Let $u: \mathbb{L}^{(m,n)} \to \mathbb{N}$ map a binary lattice ℓ to~~ the product over all cells in ℓ, with each term being

For a binary lattice ℓ, let $u(\ell)$ be the product...

add comma

$$\begin{cases} 7 & \text{for cells } \begin{smallmatrix}0&0\\0&0\end{smallmatrix}, \begin{smallmatrix}1&1\\1&1\end{smallmatrix} \\ 0 & \text{for cells } \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \\ 1 & \text{otherwise}, \end{cases}$$

move comma

not a new paragraph

where these terms come from the number of tiles in $\mathbb{T}$ that can map to a specific cell under f. For a given binary lattice ℓ that does not contain $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells, the function $u(\ell)$ ~~enumerates a set~~ counts the number of mosaics of a specific form, in which at each square a tile is either

See page $\boxed{E}$

General suggestions: ① Define $u(\ell)$ for a cell ℓ. ② Explain reasoning (i.e. number of elements in $\mathbb{T}$). ③ Define $u(\ell)$ for lattice ℓ ④ Explain.

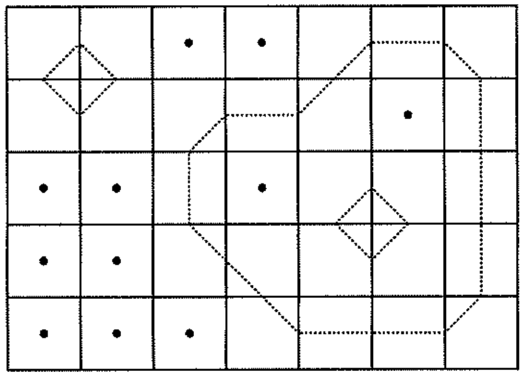


Figure 7: Mosaic Diagram of Binary Lattice in Figure 6

Are these useful? Fig 7 seems to be the only one. Even if necessary, lets not define a name (in italics)

Don't leave reader wondering if u is useless.

uniquely specified, or can be any tile in $\mathbb{T}$. We can depict this set with a *mosaic diagram*, where we introduce a notational tile with a ~~red~~ dot at the center to indicate that any tile in $\mathbb{T}$ can be in that location (to avoid confusion with the T_0 tile). For example, the mosaic diagram for the binary lattice in Figure 6 is depicted in Figure 7. Consequently, the mosaic in Figure 6 is included in the set represented by the mosaic diagram in Figure 7.

One would hope that the function u could be used to compute $|\mathbb{S}^{(m,n)}|$ in some way. However for the (m,n) binary lattice ℓ^*, we have

$$u(\ell^*) = 7^{mn} = |\mathbb{M}^{(m,n)}| > |\mathbb{S}^{(m,n)}|.$$

not a strict inequality e.g. (m=1 or n=1)

not a new paragraph

Clearly, this approach overcounts $|\mathbb{S}^{(m,n)}|$. We can examine this phenomena more closely by first definining g.

Definition 3.4. $g : \mathbb{F}^{(m,n)} \to \mathbb{M}^{(m,n)}$ takes a framed binary lattice that does not contain the cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$, and constructs a mosaic by replacing all $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ and $\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$ cells with T_0, and replacing all other cells with the unique tile that maps to that cell under f.

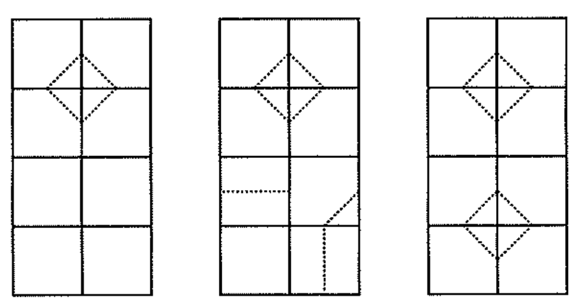
invalid domain (what if 01 or 10 / 10 01 ?)

Notice that for some binary lattice ℓ, $g(\ell)$ is not necessarily equal to $f^{-1}(\ell)$, as g only returns polygon mosaics and not messy polygon mosaics. We can now examine the overcounting more closely.

(XXX) See page B

Example 3.2. Consider the following binary lattices and mosaics for $(m,n) = (4,2)$.

$$\begin{array}{ccc}
0 - 0 - 0 \\ 0 - 1 - 0 \\ 0 - 0 - 0 \\ 0 - 0 - 0 \\ 0 - 0 - 0 \\ \ell_1
\end{array}
\qquad
\begin{array}{c}
0 - 0 - 0 \\ 0 - 0 - 0 \\ 0 - 0 - 0 \\ 0 - 1 - 0 \\ 0 - 0 - 0 \\ \ell_2
\end{array}
\qquad
\begin{array}{c}
0 - 0 - 0 \\ 0 - 1 - 0 \\ 0 - 0 - 0 \\ 0 - 1 - 0 \\ 0 - 0 - 0 \\ \ell_3
\end{array}$$



Mosaic 1 Mosaic 2 Mosaic 3

What map do we mean here? maybe "is mapped to"?? Wouldn't it be better to also say $f(\text{Mosaic } 1) = f(\text{Mosaic } 2) = \ell_1$?

We have $u(\ell_1) = 7^4, u(\ell_2) = 7^4$, and $u(\ell_3) = 1$. We also have $f(\text{Mosaic } 3) = \ell_3$. Each cell in the bottom two rows of ℓ_1 map to 7 possible tiles. However, 1 of the 7^4 combinations forms a new polygon ~~in these~~ bottom two rows. This is also true for the top two rows of ℓ_2. Therefore, $u(\ell_1)$, $u(\ell_2)$, and $u(\ell_3)$ all count ~~Mosaic 3.~~

Nothing wrong per se but very confusing.

Example 3.2 leads us to conclude the following.

I know what we're saying and am getting confused. We must say better.

Proposition 3.1. *For a framed binary lattice ℓ that does not contain the cells $\substack{0\,1\\1\,0}$ or $\substack{1\,0\\0\,1}$, $u(\ell)$ counts the number of mosaics that have at least the polygons in $g(\ell)$.*

Proof. Choose an framed binary lattice ℓ that does not contain the cells $\substack{0\,1\\1\,0}$ or $\substack{1\,0\\0\,1}$. By the definition of g, the function $u(\ell)$ counts the number of mosaics that have exactly the polygons in $g(\ell)$. Additionally, as u counts each possible combination of tiles for $\substack{0\,0\\0\,0}$ and $\substack{1\,1\\1\,1}$ cells, this includes all polygons that do not conflict with the polygons in $g(\ell)$. □

undefined

Surprisingly, this overcounting phenomena can be remedied by a small modificiation to u, which we call v.

Definition 3.5. Let $v : \mathbb{L}^{(m,n)} \to \mathbb{Z}$ map a binary lattice ℓ to the product over all cells in ℓ with each term being

$$\begin{cases} 7 & \text{for cells } \substack{0\,0\\0\,0}, \substack{1\,1\\1\,1} \\ 0 & \text{for cells } \substack{0\,1\\1\,0}, \substack{1\,0\\0\,1} \\ -1 & \text{for cells } \substack{0\,0\\1\,0}, \substack{1\,0\\1\,1} \\ 1 & \text{otherwise} \end{cases}$$

and the empty product being 1. do this also for u (right?)

Example 3.3. If we let ℓ be the framed binary lattice on the right of Figure 6, we have $v(\ell) = -7^{11}$. (Help the reader) "Notice there are 2 $\substack{0\,0\\1\,0}$ cells, 1 $\substack{1\,1\\0\,1}$ cell, 9 cells $\substack{0\,0\\0\,0}$, 2 cells $\substack{1\,1\\1\,1}$.

Helpful or too confusing? I found it easiest to look at Fig. 7.

We will first show that $v(\ell)$, which is computed "cell-by-cell" from ℓ, recovers global information about the number of polygons in $g(\ell)$ if ℓ is framed.

Definition 3.6. For a framed binary lattice ℓ, let $P(\ell)$ to be the number of polygons in the polygon mosaic $g(\ell)$.

What about when ℓ contains $\substack{0\,1\\1\,0}$ or $\substack{1\,0\\0\,1}$?

Proposition 3.2. *If ℓ is a framed binary lattice that does not contain the cells $\substack{0\,1\\1\,0}$ or $\substack{1\,0\\0\,1}$, then*

OK, but start proof by showing it's trivial for $m=0$

$$sign(v(\ell)) = (-1)^{P(\ell)}. \tag{2}$$

text mode

UGH!! I can help you rewrite to eliminate this.

You mean $(0,n)$

Proof. For this proof, we use LHS and RHS to abbreviate the left and right hand side of Equation 2. We prove the result by induction. For the base case, construct the $(1,n)$ framed binary lattice ℓ for some $n \geq 1$. As there are no cells in ℓ, from the definition of v the LHS is $v(\ell) = 1$. As there no polygons in $g(\ell)$, we have the RHS is 1.

You mean to say $sign(v(\ell)) = 1$

For the induction step, fix an (m,n) framed binary strip ℓ for ~~$m \geq 1$~~ $m \geq 0$. Then consider ~~an~~ any $(m+1,n)$ framed binary strip ℓ' that shares the top $m-1$ rows of vertices with ℓ and an

Probably clearer: First take ℓ', then define ℓ based on ℓ'.

arbitrarily-labeled m-th row, such that there are no $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells. We show that we can construct ℓ' from ℓ with a procedure that preserves Equation 2 with each intermediate step.

Procedure:

Step a. Add a bottom row to ℓ of $n+1$ vertices, all labeled 0. This results in a new $(m+1, n)$ framed binary lattice we denote ℓ_a.

Step b. Scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}1\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}1\\0\end{smallmatrix}$ column in ℓ_a to $\begin{smallmatrix}1\\1\end{smallmatrix}$. Completing this scan results in a new framed binary lattice denoted ℓ_b.

"Again –"

Step c. Scanning rows $m-1$ and m of ℓ' left to right, if there exists a column of the form $\begin{smallmatrix}0\\1\end{smallmatrix}$, change the associated $\begin{smallmatrix}0\\0\end{smallmatrix}$ column in ℓ_b to $\begin{smallmatrix}0\\1\end{smallmatrix}$. Completing this scan results in the framed binary lattice ℓ'.

I'm unconvinced this is helpful throughout the paper. Can we say better or rephrase?

For step a, only $\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ cells are added. As $\mathrm{sign}(v(\begin{smallmatrix}0&0\\0&0\end{smallmatrix})) = 1$, the LHS is unchanged. For the RHS, no new edge-connected regions are created, no new polygons are created in $g(\ell_a)$ so Equation 2 is preserved by step a.

For steps b and c, we encounter two distinct cases which we diagram in Figure 8. These are portions of the bottom 3 rows. These diagrams have $a, b, c, d \in [0, 1]$, and use the # symbol to indicate the vertex being changed from a 0 to a 1.

You mean $\{0, 1\}$

$$\begin{array}{ccccc} a & — & 1 & — & b \\ | & & | & & | \\ c & — & \# & — & 0 \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array}$$

(a) Step b. Diagram

$$\begin{array}{ccccc} a & — & 0 & — & b \\ | & & | & & | \\ c & — & \# & — & d \\ | & & | & & | \\ 0 & — & 0 & — & 0 \end{array}$$

(b) Step c. Diagram

Figure 8: Step Diagrams

If a, b, c, d are used as variables here, it's a bad idea to say "step a" "step b" "step c". Worse: ℓ_a, ℓ_b. Also fix. Maybe ℓ_1, ℓ_2, ℓ_3?

Wrong reason because # changes from 0 to 1. Correct reason: the change in vertex c from 0 to 1 happens in step c, and here we are in step b.

Unclear this is only step b. Combine paragraphs?

For step b, Figure 8a depicts all possible cases one can encounter scanning left to right on ℓ_a. Note that the vertex right of the # symbol must be 0, as the procedure moving left to right on ℓ_a gives that this vertex must be 0.

Notice that no assignment of a, b, and c can create or destroy cells $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$. Additionally, it cannot be the case that $a = 0$ and $c = 1$, ~~as that would imply that ℓ' contains the cell $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$~~. Therefore we have $\mathrm{sign}(v(\ell_b)) = \mathrm{sign}(v(\ell_a))$, so the LHS is preserved. Also note that because a polygon in $g(\ell)$ is an edge-connected set of vertices with the same label (not including the boundary 0's), that for all assignments of a, b, c, step b only involves one edge-connected set of vertices. There~~fore, we have $P(\ell_a) = P(\ell_b)$, and so the~~ RHS is preserved.

**** See page [C]

For step c, Figure 8b depicts all possible cases one can encounter scanning left to right on ℓ_b. It cannot be the case that $a = 1$ and $c = 0$, as this would imply ℓ' contains $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. Similarly, it cannot be the cases that $b = 1$ and $d = 0$, as this would imply ℓ_b contains $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$. Additionally, it cannot be the case that $b = 0$ and $d = 1$, as the procedure moves left-to-right. This results in the following 6 cases, which we show each preserve Equation 2.

Confusing phrasing

You mean ℓ'

In Case 9a, the flip add one $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. As the flip creates a new edge-connected region, a new polygon is created, and so the sign of the RHS changes.

"adds" "preserves"

put this at the end of the paragraph after ruling out the other cases.

I like how you list the cases but I think it's visually better: (a) (c) (e) because the upper row differs from the
(b) (d) (f) bottom only in changing the right-hand elements.

$$\begin{array}{ccc}
\begin{array}{ccccc}0&-&0&-&0\\|&&|&&|\\0&-&\#&-&0\\|&&|&&|\\0&-&0&-&0\end{array} &
\begin{array}{ccccc}0&-&0&-&1\\|&&|&&|\\0&-&\#&-&1\\|&&|&&|\\0&-&0&-&0\end{array} &
\begin{array}{ccccc}0&-&0&-&0\\|&&|&&|\\1&-&\#&-&0\\|&&|&&|\\0&-&0&-&0\end{array} \\
\text{(a)} & \text{(b)} & \text{(c)} \\
\begin{array}{ccccc}0&-&0&-&1\\|&&|&&|\\1&-&\#&-&1\\|&&|&&|\\0&-&0&-&0\end{array} &
\begin{array}{ccccc}1&-&0&-&0\\|&&|&&|\\1&-&\#&-&0\\|&&|&&|\\0&-&0&-&0\end{array} &
\begin{array}{ccccc}1&-&0&-&1\\|&&|&&|\\1&-&\#&-&1\\|&&|&&|\\0&-&0&-&0\end{array} \\
\text{(d)} & \text{(e)} & \text{(f)}
\end{array}$$

] Alignment weird since a, b, c closer to bottom row than top

Figure 9: Step c. Cases

In Case 9b, the flip does not add a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell or $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 9c, the flip removes and adds a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 9d, the flip removes a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell, so the sign of the LHS changes. Before the flip, either the two edge-connected regions of 1's are distinct in the full framed binary lattice, or they are joined. After the flip, if the regions were distinct, they are now joined, and so 1 polygon is removed. If the regions were joined, a new edge-connected region of 0's (that is not connected to the boundary) is created, and so 1 polygon is created. Either way, the RHS changes sign. "one"

In Case 9e, the flip adds both a $\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$ cell and a $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$ cell, so the sign of the LHS stays the same. As the flip does not create a new edge-connected region, the RHS stays the same.

In Case 9f, the flip adds $\begin{smallmatrix}1&0\\1&1\end{smallmatrix}$, so the sign of the LHS changes. The same logic from Case 9d applies, as the flip either removes or adds a polygon, so the RHS changes sign.

As all cases preserve Equation 2, the $(m+1, n)$ framed binary lattice ℓ' also follows Equation 2. □

We point out here that the function v if defined for all binary lattices, while Proposition 3.2 only holds for framed binary lattices that do not contain cells $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$.

Proposition 3.3. *The number of (m, n) mosaics that do not contain a polygon has*

probably "Theorem"

$$\left|\mathbb{S}^{(m,n)}\right| = \sum_{\ell \in \mathbb{F}^{(m,n)}} v(\ell).$$

Proof. Choose a mosaic $\mathcal{M}$ that has $P(\ell')$ polygons, where $\ell' = f(\mathcal{M})$. We begin by determining how many times $\mathcal{M}$ is counted in the related sum

I don't really feel like I know what this means.

a) confusing
b) related exposition probably belongs here

not sure this is helpful

too vague! Instead, talk about sums over ~~[illegible]~~ subset of $\mathbb{F}^{(m,n)}$

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} u(\ell).$$

We point out here that as $u\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = u\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = v\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = v\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = 0$, framed binary lattices with these cells don't contribute to either summation, and so can be ignored for the remainder of this argument.

Proposition 3.1 shows that the function $u(\ell)$ counts the mosaics with at least the polygons in $g(\ell)$. Consequently, for a framed binary lattice ℓ, $u(\ell)$ counts $\mathcal{M}$ if the polygons in $g(\ell)$ are a subset of the polygons in $\mathcal{M}$. Therefore, as the number of ways to choose a size p subset of $P(\ell)$ polygons is $\binom{P(\ell)}{p}$, $\mathcal{M}$ is counted $\sum_{p=0}^{P(\ell)}\binom{P(\ell)}{p}$ times. This then gives

was this defined?

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} u(\ell) = \sum_{\mathcal{M}\in\mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} \binom{P(f(\mathcal{M}))}{p}.$$

Following the same logic for $\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell)$, Proposition 3.2 gives that size p subsets where p is odd are subtracted, which gives

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\mathcal{M}\in\mathbb{M}^{(m,n)}} \sum_{p=0}^{P(f(\mathcal{M}))} (-1)^p\binom{P(f(\mathcal{M}))}{p}.$$

Finally, by the binomial theorem, for $P(\ell) > 0$ we have

$$\sum_{p=0}^{P(\ell)}(-1)^p\binom{P(\ell)}{p} = 0,$$

and as the only binary lattice with $P(\ell) = 0$ is ℓ^*, we have

$$\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell) = \sum_{\{\mathcal{M}\in\mathbb{M}^{(m,n)}|P(f(\mathcal{M}))=0\}} \binom{0}{0} = |f^{-1}(\ell^*)| = |\mathbb{S}^{(m,n)}|.$$

□

As in Theorem 1.2 from [9], we can compute $\sum_{\ell\in\mathbb{F}^{(m,n)}} v(\ell)$ efficiently using the state matrix recursion method.

4 Proof of Theorem 2.1

We begin by defining the state matrices.

Definition 4.1. Let A_k, B_k, C_k, D_k be $2^{k-1}\times 2^{k-1}$ matrices with integer entries, where $A_1 = \left(v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)$, $B_1 = \left(v\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right)$, $C_1 = \left(v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right)$, $D_1 = \left(v\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)\right)$, and for integers $k \geq 1$,

$$A_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)D_k \end{pmatrix} \quad B_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)D_k \end{pmatrix}$$

$$C_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)D_k \end{pmatrix} \quad D_{k+1} = \begin{pmatrix} v\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)A_k & v\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)B_k \\ v\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)C_k & v\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)D_k \end{pmatrix}.$$

Very confusing to define something that was defined on page 4! Perhaps: "let us recognize that $A_1 = \left(v\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)$," etc.

How about the same variable as in proving ~~[illegible]~~ Prop. 4.1. I.e., either $k \mapsto n$ here or $n \mapsto k$ there.

No! Should be a string of n zeros. And this should be a remark, not part of the definition.

nonnegative integer $k < 2^n$

Definition 4.2. Let the n digit binary representation of the number k be written as $\beta_n(k)$. If k is 0, $\beta_k(n)$ returns the empty string.

or have you swapped n, k

Better $\beta_n(k)$

For all integers $n \geq 1$

Proposition 4.1. *The (i,j)-th entry of A_n is $v(\ell)$, where ℓ is the $(1,n)$ binary lattice where the top row of vertices has ~~labels~~ $0\beta_{n-1}(i)0$ and the bottom row of vertices ~~has labels~~ $0\beta_{n-1}(j)0$, both read left to right. ~~Furthermore, the same statement is true~~ for B_n ~~with~~ top row $1\beta_{n-1}(i)0$ and bottom row $0\beta_{n-1}(j)0$, for C_n with top row $0\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$, and for D_n with top row $1\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$.*

Similarly except with

typo for $\beta_n(k)$

Proof. We prove by induction. For $n = 1$, the definition of the state matrices gives A_1 is $v(\ell)$ where ℓ is the $(1,1)$ binary lattice with top row $0\beta_0(0)0 = 00$ and bottom row $0\beta_0(0)0 = 00$, as $\beta_0(0)$ is the empty string. The same is true for B_1 with top row $0\beta_0(0)0 = 10$ and bottom row $1\beta_0(0)0 = 00$, for C_1 with top row $1\beta_0(0)0 = 00$ and bottom row $0\beta_0(0)0 = 10$, and for D_1 with top row $1\beta_0(0)0 = 10$ and bottom row $1\beta_0(0)0 = 10$.

Remind the reader that rows and cols are numbered starting at 0 not 1.

We next assume ~~matrices A_n, B_n, C_n, and D_n follow their associated statements in Proposition~~ 4.1. Therefore, the entry (i,j) in, say, B_n is $v(\ell)$ where ℓ is the binary lattice with top row $0\beta_{n-1}(i)0$ and bottom row $1\beta_{n-1}(j)0$. The argument is analagous for any choice of A_n, B_n, C_n, D_n. For $n > 1$ we can depict the $(1,n)$ binary lattice using similar notation to cells, namely

"our results for some fixed $n \geq 1$."

also true for $n=1$

do you mean "assumption"?

$$v(\ell) = v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix}.$$

We show that A_{n+1} also follows Proposition 4.1. From the definition, we have

See $\boxed{G}$

$$A_{n+1} = \begin{pmatrix} v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)A_n & v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)B_n \\ v\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)C_n & v\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)D_n \end{pmatrix}.$$

not a new paragraph

By construction, the (i,j)-th entry in B_n is located in the $(i, j + 2^{n-1})$-th entry of A_{n+1}, as each block matrix A_n, B_n, C_n, D_n are $2^{n-1} \times 2^{n-1}$ matrices. Also by construction, the value in the (i,j)-th entry in B_n which we call $v(\ell)$ is multiplied by $v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The definition of v gives

Better if these parentheses all the same size (e.g. use "phantom" in LaTeX, I think)

$$\cancel{v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)v(\ell) =} v\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ 1 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & \beta_n(i) & 0 \\ 0 & \beta_n(j+2^{n-1}) & 0 \end{pmatrix},$$

Therefore, the $(i, j+2^{n-1})$-th entry of A_{n+1} is $v(\ell)$ where ℓ is the $(1, n+1)$ binary lattice where the top row of vertices is $0\beta_n(i)0$ and the bottom row of vertices is $0\beta_n(j + 2^{n-1})0$, which ~~completes the induction step for A_{n+1}.~~ Similar arguments hold for matrices B_{n+1}, C_{n+1}, D_{n+1}. □

It only completes one quarter of the calculation of A_{n+1}.

For all pos integers m, n

~~For all integers $n \geq 1$,~~

Proposition 4.2. *The (i,j)-th entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m,n) binary lattices with the top row of vertices ~~having labels~~ $0\beta_{n-1}(i)0$ and the bottom row of vertices having labels $0\beta_{n-1}(j)0$, ~~both read left to right~~.*

Proof. We prove by induction. The base case $m = 1$ is Proposition 4.1, as the set L only has ~~the unique $(1,n)$~~ binary lattice.

one $(1,n)$

We next assume that the (i,j)-th entry of A_n^m satisfies the statement in Proposition 4.2, and show the statement also holds for A_n^{m+1}. To begin, consider the product $A_n^m \cdot A_n$, and

fix pos integers m and n and

Notation here needs work. Perhaps better: more words. Do you want me to draft something?

choose an integer $k \in [0, 2^{n-1} - 1]$. From the induction hypothesis~~,~~ the ~~value at the~~ (i, k)-th entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m, n) binary lattices with the top row of vertices having labels $0\beta_{n-1}(i)0$ and the bottom row of vertices having labels $0\beta_{n-1}(k)0$.

By Prop 4.2,

~~Similarly~~, the (k, j)-th entry of A_n is ~~the $(1, n)$ binary lattice~~ $\begin{pmatrix} 0 & \beta_{n-1}(k) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix}$.

~~Therefore,~~ The dot product of the i-th row and the j-th column is (i, j)-th value of A_n^{m+1}, which is

not following from what preceded

We mean the sum of these (a double sum unless you say what $v\begin{pmatrix} 0\beta_{n-1}(i)0 \\ \cdots \\ 0\beta_{n-1}(j)0 \end{pmatrix}$ means.

$$\sum_{k=0}^{2^{n-1}-1} v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(k) & 0 \end{pmatrix} v\begin{pmatrix} 0 & \beta_{n-1}(k) & 0 \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix} = v\begin{pmatrix} 0 & \beta_{n-1}(i) & 0 \\ & \cdots & \\ 0 & \beta_{n-1}(j) & 0 \end{pmatrix},$$

sum

which gives the desired result for A_n^{m+1}. □

Theorem 2.1

~~Proposition 4.3~~. *The number of (m, n) mosaics that do not contain a polygon is the $(0, 0)$ entry of A_n^m.*

Proof. By Proposition 4.2, the $(0, 0)$ entry of A_n^m is $\sum_{\ell \in L} v(\ell)$, where L is the set of (m, n) binary lattices with the top and bottom rows of vertices having labels $0\beta_{n-1}(0)0$. As A has left-most and right-most columns all labeled 0, all binary lattices counted at $(0, 0)$ are framed, so $L = \mathbb{F}^{(m,n)}$. Substituting the values for v from Definition 3.5 gives the sum from Proposition 3.3, which completes the proof. □

5 Acknowledgements

The authors would like to thank Richard Schank for his suggestions, including proposing the definition for f and working to develop the state matrix recursion matrix solution.

References

[1] Dooho Choi et al. "Quantum knot mosaics and bounds of the growth constant". In: *Reviews in Mathematical Physics* 36.10 (2024), p. 2450025. DOI: 10.1142/S0129055X24500259. eprint: https://doi.org/10.1142/S0129055X24500259. URL: https://doi.org/10.1142/S0129055X24500259.

[2] Kyungpyo Hong and Seungsang Oh. "Bounds on Multiple Self-avoiding Polygons". In: *Canadian Mathematical Bulletin* 61.3 (Sept. 2018), pp. 518–530. ISSN: 1496-4287. DOI: 10.4153/cmb-2017-072-x. URL: http://dx.doi.org/10.4153/CMB-2017-072-x.

[3] Samuel J. Lomonaco and Louis H. Kauffman. "Quantum knots and mosaics". In: *Quantum Information Processing* 7.2 (2008), pp. 85–115. DOI: 10.1007/s11128-008-0076-7. URL: https://doi.org/10.1007/s11128-008-0076-7.

[4] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org.

[5] Seungsang Oh. "Domino tilings of the expanded Aztec diamond". In: *DISCRETE MATHEMATICS* 341.4 (Apr. 2018), pp. 1185–1191. ISSN: 0012-365X. DOI: 10.1016/j.disc.2017.10.016.

[6] Seungsang Oh. “Quantum knot mosaics and the growth constant”. In: *Topology and its Applications* 210 (2016), pp. 311–316. ISSN: 0166-8641. DOI: https://doi.org/10.1016/j.topol.2016.08.011. URL: https://www.sciencedirect.com/science/article/pii/S0166864116301857.

[7] Seungsang Oh. “State matrix recursion method and monomer-dimer problem”. In: *DISCRETE MATHEMATICS* 342.5 (May 2019), pp. 1434–1445. ISSN: 0012-365X. DOI: 10.1016/j.disc.2019.01.022.

[8] Seungsang Oh and Youngin Kim. “Growth rate of quantum knot mosaics”. In: *Quantum Information Processing* 18.8 (2019), p. 238. DOI: 10.1007/s11128-019-2353-z. URL: https://doi.org/10.1007/s11128-019-2353-z.

[9] Seungsang Oh et al. “Quantum knots and the number of knot mosaics”. In: *Quantum Information Processing* 14.3 (2015), pp. 801–811. DOI: 10.1007/s11128-014-0895-7. URL: https://doi.org/10.1007/s11128-014-0895-7.