**Proposition 1** (The inclusion exclusion principle.)**.** *For finite sets $A_1$, $A_2$, ..., $A_M$ finite sets,*

$$\left|\bigcup_{j\in 1}^{M} A_j\right| = \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1}\left|\bigcap_{j\in I} A_j\right|.$$

*Proof.* The case of one set is trivial, and the case of two sets is the familiar and easily justified statement that $|A_1 \cup A_2| = |A_1|+|A_2|-|A_1\cap A_2|$. Proceeding inductively, for $M \geq 3$, assume the result for $M-1$ sets, where and let us prove it for $M$ sets. Let us split up the sum on the right into those sets $I$ that do not contain $M$ and those that do. By our inductive hypothesis, the first sum equals $\left|\bigcup_{j\in 1}^{M-1} A_j\right|$ and the second sum can be written as $\sum_{\{M\}\subseteq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1}\left|\bigcap_{j\in I} A_j\right|$. Separating out the term with $I = \{M\}$, this equals $|A_M|-\sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M-1\}} (-1)^{|I|}\left|\bigcap_{j\in I}(A_j\cap A_M)\right|$. Again using our inductive hypothesis, the sum here $\left|\cup_{j=1}^{M-1}(A_j\cap A_M)\right| = \left|\left(\cup_{j=1}^{M-1} A_j\right)\cap A_M\right|$. The entire right side of our desired equation therefore equals $\left|\bigcup_{j\in 1}^{M-1} A_j\right|+|A_M| - \left|\left(\cup_{j=1}^{M-1} A_j\right)\cap A_M\right|$. By the case of our proposition with two sets, this equals $\left|\bigcup_{j\in 1}^{M} A_j\right|$, as desired. □

*Proof.* The case of one set is trivial, and the case of two sets is the familiar and easily justified statement that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. Proceeding by induction, for $M \geq 2$, we prove the case of $M + 1$ sets using the case of 2 sets and the case of $M$ sets.

$$\begin{aligned}
\left|\bigcup_{j\in 1}^{M+1} A_j\right| &= \left|\bigcup_{j\in 1}^{M} A_j\right| + |A_{M+1}| - \left|\left(\bigcup_{j\in 1}^{M} A_j\right)\cap A_{M+1}\right| \\
&= \left|\bigcup_{j\in 1}^{M} A_j\right| + |A_{M+1}| - \left|\bigcup_{j\in 1}^{M}(A_j\cap A_{M+1})\right| \\
&= \left|\bigcup_{j\in 1}^{M} A_j\right| + |A_{M+1}| - \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1}\left|\bigcap_{j\in I}(A_j\cap A_{M+1})\right| \\
&= \left|\bigcup_{j\in 1}^{M} A_j\right| + |A_{M+1}| - \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1}\left|\left(\bigcap_{j\in I} A_j\right)\cap A_{M+1}\right| \\
&= \left|\bigcup_{j\in 1}^{M} A_j\right| + \sum_{\{M+1\}\subseteq I\subseteq\{1,2,\ldots,M+1\}} (-1)^{|I|-1}\left|\bigcap_{j\in I}(A_j\cap A_{M+1})\right|
\end{aligned}$$

Proceeding by induction, for $M \geq 2$, we prove the case of $M + 1$ sets by first using the case of 2 sets to get

$$(*)\qquad \left|\bigcup_{j\in 1}^{M+1} A_j\right| = \left|\bigcup_{j\in 1}^{M} A_j\right| + |A_{M+1}| - \left|\left(\bigcup_{j\in 1}^{M} A_j\right)\cap A_{M+1}\right|.$$

Noticing that $\left(\bigcup_{j\in 1}^{M} A_j\right) \cap A_{M+1} = \bigcup_{j\in 1}^{M} (A_j \cap A_{M+1})$, the case of $M$ sets gives us

$$-\left|\left(\bigcup_{j\in 1}^{M} A_j\right) \cap A_{M+1}\right| = \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|} \left|\bigcap_{j\in I} (A_j \cap A_{M+1})\right|.$$

This right side of this equals $\sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|} \left|\left(\bigcap_{j\in I} A_j\right) \cap A_{M+1}\right|$ and therefore equals $\sum_{\{M+1\}\neq I\subseteq\{1,2,\ldots,M+1\}} (-1)^{|I|-1} \left|\bigcap_{j\in I} A_j\right|$. Now again using our inductive hypothesis to get $\left|\bigcup_{j\in 1}^{M} A_j\right| = \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1} \left|\bigcap_{j\in I} A_j\right|$, the right side of (*) equals $\sum_{\emptyset\neq I\subseteq\{1,2,\ldots,M\}} (-1)^{|I|-1} \left|\bigcap_{j\in I} A_j\right|$, as desired. □

**Proposition 2** (The inclusion exclusion principle.). *Consider finitely many subsets $A_j$, where $j \in J$, of a finite set $S$. Then*

$$\left|S \setminus \left(\bigcup_{j\in J} A_j\right)\right| = \sum_{I\subseteq J} (-1)^{|I|} \left|\bigcap_{j\in I} A_j\right|,$$

*where we consider $\bigcap_{j\in I} A_j$ to be equal to $S$ when $I$ is the empty set.*

*Proof.* Fix $s \in S$. We consider which sets $s$ is in, and how this contributes to the expressions on the left and right side of our desired equation.

If $s \notin \bigcup_{j\in J} A_j$, i.e., $s \in S \setminus \left(\bigcup_{j\in J} A_j\right)$, then $s$ contributes 1 to the left side. Since $s \notin \bigcap_{j\in I} A_j$ for all nonempty $I \subseteq J$, we have that $s$ contributes to the right side only when $I$ is the empty set, and since the size of the empty set is zero and $(-1)^0 = 1$, $s$ contributes 1 to the right side.

Now suppose $s \in \bigcup_{j\in J} A_j$, so that $s$ contributes 0 to the left side. Let $A_{j_1}$, $A_{j_2}$, $\ldots$, $A_{j_n}$ be the distinct sets $A_j$ that contain $s$. We then have $n \geq 1$. To see that $s$ contributes 0 to the sum on the right side, first notice that $s \in \bigcap_{j\in I} A_j$ if and only if $I \subseteq \{j_1, j_2, \ldots, j_n\}$. For each subset $I \subseteq \{j_1, j_2, \ldots, j_n\}$, the contribution due to $s$ is positive when $|I|$ is even, and negative when $|I|$ is odd. For all nonnegative integers $k \leq n$, there are $\binom{n}{k}$ subsets of $\{j_1, j_2, \ldots, j_n\}$ of size $k$, so the contribution of $s$ to the right side is $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$. □

The other way...

**Proposition 3** (The inclusion exclusion principle.). *Let $J$ be a finite set, and for each $j \in J$, let $A_j$ be a finite set. Then*

$$\left|\bigcup_{j\in J} A_j\right| = \sum_{I\subseteq J, I\neq\emptyset} (-1)^{|I|-1} \left|\bigcap_{j\in I} A_j\right|.$$

*Proof.* Every set appearing in this equation is contained in $\bigcup_{j\in J} A_j$, so let $s \in \bigcup_{j\in J} A_j$. Clearly $s$ contributes 1 to the left side of the equation. To see that it also contributes 1 to the right side, let $\{j_1, j_2, \ldots, j_n\}$ be the set of all $j \in J$ such that $s \in A_j$. We have $n \geq 1$ since $s \in \bigcup_{j\in J} A_j$. Also, for any nonempty $I$ with $I \subseteq J$, $s \in \bigcap_{j\in I} A_j$ if and only if $I \subseteq \{j_1, j_2, \ldots, j_n\}$. For each nonempty $I \subseteq \{j_1, j_2, \ldots, j_n\}$, the contribution on the right side of the equation due to $s$ is negative when $|I|$ is even, and positive when $|I|$ is odd. For all positive integers

$k \le n$, there are $\binom{n}{k}$ subsets of $\{j_1, j_2, \ldots, j_n\}$ of size $k$, so the contribution of $s$ to the right side is $\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1 - \sum_{k=0}^{n} (-1)^{k} \binom{n}{k}$. The binomial theorem tells us $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} = (1-1)^{n} = 0$, so the contribution of $s$ to the right side is also 1. $\square$